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Gravitation and Electromagnetism

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To my parents

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Gravitation and Electromagnetism

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The theory of general relativity unifies gravitation with the geometry of space-time by replacing the scalar Newtonian gravitational potential with the symmetric metric tensor $g_{\mu\nu}$ of a four-dimensional general Riemannian manifold by means of the equivalence principle. As is well known, the electromagnetic field has resisted all efforts to be interpreted in terms of the geometrical properties of space-time as well. In this investigation, we show that the electromagnetic field may indeed be given a geometrical interpretation in the framework of a modified version of general relativity - unimodular relativity. According to the theory of unimodular relativity developed by Anderson and Finkelstein, the equations of general relativity with a cosmological constant are composed of two independent equations, one which determines the null-cone structure of space-time, another which determines the measure structure of space-time. The field equations that follow from the restricted variational principle of this version of general relativity only determine the null-cone structure and are globally scale-invariant and scale-free. We show that the electromagnetic field

may be viewed as a compensating gauge field that guarantees local scale invariance of these field equations. In this way, Weyl's geometry is revived. However, the two principle objections to Weyl's theory do not apply to the present formulation: the Lagrangian remains first order in the curvature scalar and the non-integrability of length only applies to the null-cone structure.

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Chapter 1

Introduction

Einstein's theory of general relativity [16, 21] is considered one of the greatest intellectual achievements of twentieth century physics. It is a theory of gravity that resulted from Einstein's deep philosophical convictions about the nature of physical laws. Throughout his arduous path to formulate a generally covariant theory Einstein was guided by a simple experimental fact: the equality of gravitational and inertial mass. Gravitation and the space-time manifold became inextricably linked in the natural language of Riemannian geometry as Einstein yet again revolutionized our conception of space and time. Even Einstein's own revolutionary postulate of the constancy of the speed of light was revised in the wake of his new theory. General relativity has enjoyed overwhelming success since its inception.

However, the theory of general relativity does not offer a geometrical interpretation in terms of the space-time manifold of the other fundamental forces of nature: the electromagnetic, weak, and strong forces. The electromagnetic field, for example, also manifests itself in our everyday experience; it also possesses an infinite range of influence as well as a propagation speed equal to that of gravitation. The field equations of electromagnetism, Maxwell's equations, may be generalized in order to accommodate a gravitational field and the Maxwell stress-energy tensor enters the gravitational field equations naturally as the source for the gravitational field. Yet, the electromagnetic field may not be interpreted in terms of the geometrical properties of space-time. This difficulty has led many, including Einstein himself, to generalize Riemannian geometry in order to cast the electromagnetic potentials into the underlying geometrical framework [41, 43]. The problem was attacked primarily from a mathematical point of view, as various authors identified particular restrictions imposed by the choice of a four-dimensional Riemannian manifold and examined the consequences of their mitigation.

For example, in 1918 Weyl [56, 57, 58, 59, 60] generalized Riemannian geometry by introducing a change of scale to supplement the coordinate transformations of general relativity. By relaxing the requirement of the constancy of length under the parallel displacement of vectors he observed that a four-vector potential emerged, which he identified with the electromagnetic four-vector potential. Therefore, according to this theory, the electromagnetic potentials determined the path-rescaling of length in an analogous manner that the metric potentials determined the path dependence of direction. Weyl thus developed an elegant mathematical structure that treated electromagnetism as a manifestation of geometry. This theory even predicted the perihelion precession of Mercury in addition to the bending of light rays in a gravitational field [43]. However, the physical interpretation of Weyl's scale change was proven unreasonable by Einstein [39, 43].

With the same goal in mind, Kaluza [31] proposed a five-dimensional generalization of Einstein's theory by adding to the 4D space-time manifold an additional coordinate x^5. The fifth dimension assumed a distinct role via the "cylinder condition" by which all derivatives with respect to x^5 vanished. Interpreting four of the additional metric components as the electromagnetic potentials he showed that in weak fields and for non-relativistic velocities his theory reproduced the Einstein-Maxwell field equations and the equations of motion of a charged particle in a combined gravitational-electromagnetic field. A few years later Klein [33] showed that these approximations were unnecessary and in addition discussed a connection between this formalism and quantum theory [34]. Subsequent work has incorporated additional dimensions in order to include the strong and weak forces as well (for reviews see Appelquist et al. [4] and Overduin and Wesson [40]).

Eddington [15], Einstein [41, 43], and Schrodinger [52], also participated in the unification program. Eddington and Schrodinger focused primarily on the notion of the affine connection and attempted to develop purely affine theories; they treated the metric tensor as a secondary item. Einstein, on the other hand, worked incessantly on many different theories. For example, he considered five-dimensional theories, purely affine theories, and nonsymmetric metrico-affine theories, among others. Unfortunately, the lack of a physical principle resembling the equality of gravitational and inertial mass, as in the gravitational problem, rendered the number of such unified theories unlimited.

It is safe to say that a satisfactory understanding of the connection between gravitation and electromagnetism did not emerge from the early unification program. Of course modern gauge theory has brought investigators a step closer. It is now generally believed that the gravitational and electromagnetic fields both possess a common gauge structure. As is well known the electromagnetic field may be derived by demanding invariance of the Lagrangian for a complex scalar field under local $U(1)$ phase transformations [50]. Similarly, the U_4 theory of gravity [27], which is a generalization of Einstein's general relativity, emerges by demanding invariance under the local Poincaré group. While the Poincaré group is the symmetry group of space-time, the group $U(1)$ is an internal symmetry group and is unrelated to the space-time manifold. Therefore, modern gauge theory does not succeed in casting the electromagnetic potentials into the space-time manifold even though it is indeed a great step forward in the problem of unification.

The purpose of this investigation is to return to the original problem of unification. Our goal is to cast the electromagnetic potentials into the geometrical framework. We begin, in Chapter 2, with a brief review of the traditional gravitational and electromagnetic fields. Each field is first discussed separately. Einstein's general theory of relativity is reviewed and it is followed by a review of Maxwell's equations of electrodynamics. Gravitation and electromagnetism are then discussed simultaneously; we examine how the equations of each theory may be changed in order to accommodate the presence of the other field. Thus, Einstein's field equations are modified in order to include an electromagnetic field and the equations of electromagnetism are written in the presence of a gravitational field. We also discuss alternate formulations of general relativity, in particular, Rosen's bimetric theory and Brans-Dicke theory.

In Chapter 3 we review some of the previous attempts to unify the gravitational and electromagnetic fields. Since it is beyond the scope of this work to discuss all of the previously proposed theories, we focus on the important contributions that shaped the manner in which the problem was subsequently attacked. We begin with a review of Weyl's proposal which generalizes the Riemannian manifold by relaxing the restriction of length preservation under vector transplantation. Weyl's theory inspired the purely affine approach of Eddington, which in turn, was further developed by Einstein and Schrodinger. Next we turn our attention to five-dimensional Kaluza-Klein theory and examine gravitational theory in five dimensions. By restricting the set of transformations to a special subset of the full five-dimensional coordinate transformations we show how this theory reproduces the usual four-dimensional gravitational field equations in the presence of an electromagnetic field. We also discuss generalizing these field equations in order to include a scalar field. This chapter is concluded with a brief discussion about the common gauge structure of gravitation and electromagnetism.

In Chapter 4 we lay the groundwork for a new unified theory of gravitation and electromagnetism that is associated with the group of scale transformations of the metric tensor. First we show that general coordinate invariance and scale invariance of the action are fundamentally incompatible in gravitational theory. Since the gravitational action is not invariant under global scale transformations one cannot proceed in the usual manner of gauge theory and introduce a compensating gauge field that cancels the offending terms when the parameters of the conformal group are permitted to depend on the space-time coordinates. Therefore, we develop a new method of incorporating local scale invariance into general relativity. First, we reformulate general relativity so that the scale dependence is removed from the field equations themselves. This is accomplished by a well-known procedure developed by Anderson and Finkelstein [3] for introducing the cosmological constant into general relativity. Thus, general relativity with a cosmological constant may be viewed as a union of two independent equations. One of these equations determines the null-cone or causal structure of space-time; the other equation determines the measure structure of space-time. We show that this bifurcation of general relativity into two independent components suggests a new relationship between geometry and space-time measurements. This naturally leads to the introduction of a Weyl manifold $\mathcal{W}$ that is not immediately identifiable with space-time, but nonetheless permits a correspondence with space-time measurements.

In Chapter 5 we derive a generalized set of field equations. Since the field equations that determine the null-cone structure are globally scale-invariant and scale-free, and are furthermore *independent* of the measure equation, we consider them the dynamical equations of a globally scale-invariant theory. We demand local scale invariance of this theory and see that the electromagnetic field may indeed be treated as a compensating gauge field associated with the group of local scale transformations. The measure structure is left undetermined by the field equations and is introduced as an external field which is treated as an absolute object. This theory shares similarities with Weyl's unified theory but does not yield to the same criticisms, since the Lagrangian is first order in the curvature scalar and Einstein's objection does not apply.

In Chapter 6 we summarize and discuss the results of this investigation.

In this work we use Gaussian cgs units and the following notation: Greek indices run from $0 \ldots 3$, lower-case Latin indices run from $1 \ldots 3$, and upper-case Latin indices denote $0 \ldots 3, 5$. The signature of 4D space-time is $+---$ and 5D space-time is $+----$. In addition, the Einstein summation convention applies for repeated indices.

Chapter 2

The Gravitational and Electromagnetic Fields

We begin our investigation by first introducing the gravitational and electromagnetic fields separately. We then discuss these fields together by considering the changes incurred in Einstein's field equations due to the presence of an electromagnetic field as well as the equations of electromagnetism in the presence of a gravitational field. Any theory that purportedly unifies gravitation with electromagnetism must be able to reproduce these equations identically, or at least in some well-defined limit. This is not intended to be a complete review; we highlight the essential features that are relevant for the developments to follow.

2.1 Gravitation

As is well known Einstein [16] replaced the scalar Newtonian gravitational potential with the symmetric metric tensor $g_{\mu\nu}$ of a four-dimensional general Riemannian manifold. Hence, he suggested that the gravitational field determines the square of the distance in a coordinate system $S : \{x^\alpha\}$ between two infinitesimally separated space-time points x^α and $x^\alpha + dx^\alpha$ by the relation:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu. \tag{2.1}$$

The differential quadratic form ds^2 is invariant under an arbitrary transformation to a new system of coordinates $S' : \{x'^\alpha\}$ related to the first system by a coordinate transformation of the form:

$$x^\mu = f^\mu(x'^\alpha). \tag{2.2}$$

Infinitesimally small test particles and light propagation are governed by geodesic equations in this Riemannian space, which are derived by minimizing the quantity:

$$\int ds \tag{2.3}$$

between two fixed points. In an inertial system of coordinates the line element (2.1) becomes:

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu, \tag{2.4}$$

where $\eta_{\mu\nu}$ is the special-relativistic Minkowski metric:

$$\eta_{\mu\nu} = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{2.5}$$

The use of a general Riemannian space requires one to generalize many of the mathematical operations that are trivial in Minkowski space (see References [1] and [11] for excellent introductions to the mathematical formulation of general relativity). For example, the components of a vector do not necessarily remain constant under an infinitesimal displacement between two neighboring points. Rather, given an arbitrary vector A^μ at a point $P(x^\mu)$, one obtains a new vector $A^\mu + dA^\mu$ upon displacing the vector from the point $P(x^\mu)$ to a new point $Q(x^\mu + dx^\mu)$ in infinite proximity. The change dA^μ is determined by the law of parallel displacement. For a Riemann space this is derived by first assuming that the change in the vector is a bilinear function of the displacement and the vector components, and then demanding that the scalar product between two arbitrary vectors remains constant under such a displacement. Thus, the change in an arbitrary vector A^μ under an infinitesimal displacement dx^μ in a Riemann space is given by the expression:

$$dA^\mu = -\Gamma^\mu_{\rho\sigma} A^\rho dx^\sigma, \tag{2.6}$$

where $\Gamma^\mu_{\rho\sigma}$ is the Christoffel symbol of the second kind:

$$\Gamma^\mu_{\rho\sigma} = \frac{g^{\mu\alpha}}{2}\left(\frac{\partial g_{\alpha\rho}}{\partial x^\sigma} + \frac{\partial g_{\alpha\sigma}}{\partial x^\rho} - \frac{\partial g_{\rho\sigma}}{\partial x^\alpha}\right). \tag{2.7}$$

The above expression is symmetric in the two lower indices of $\Gamma^\mu_{\rho\sigma}$. Note that $dA^\mu = 0$ for the case $g_{\mu\nu} = \eta_{\mu\nu}$, as expected.

The covariant derivative of a contravariant vector field in a Riemann space is defined as:

$$A^\mu_{;\nu} = \frac{\partial A^\mu}{\partial x^\nu} + \Gamma^\mu_{\nu\rho} A^\rho. \tag{2.8}$$

This definition may be generalized to tensors with an arbitrary number of covariant and contravariant indices. For example, the covariant derivative of the tensor $T^{\alpha\beta}$ is given by:

$$T^{\alpha\beta}_{;\nu} = \frac{\partial T^{\alpha\beta}}{\partial x^\nu} + \Gamma^\alpha_{\tau\nu} T^{\tau\beta} + \Gamma^\beta_{\tau\nu} T^{\alpha\tau}. \tag{2.9}$$

In a general Riemannian space covariant derivatives do not necessarily commute. We may characterize this non-commutability by the Riemann curvature tensor $R^{\alpha}{}_{\nu\beta\gamma}$. It is defined by the following expression:

$$A^{\alpha}_{;\beta;\gamma} - A^{\alpha}_{;\gamma;\beta} = R^{\alpha}{}_{\nu\beta\gamma} A^{\nu}. \tag{2.10}$$

Using equation (2.8) this gives explicitly:

$$R^{\rho}{}_{\alpha\beta\gamma} = \frac{\partial \Gamma^{\rho}_{\alpha\gamma}}{\partial x^{\beta}} - \frac{\partial \Gamma^{\rho}_{\alpha\beta}}{\partial x^{\gamma}} + \Gamma^{\delta}_{\alpha\gamma}\Gamma^{\rho}_{\beta\delta} - \Gamma^{\delta}_{\alpha\beta}\Gamma^{\rho}_{\gamma\delta}. \tag{2.11}$$

General relativity unifies the so-called "fictitious forces" which are observed in noninertial reference frames with the real or permanent gravitational fields that are due to local masses. Indeed, the principle of equivalence states that a noninertial system of reference is equivalent to an inertial system that is permeated by a gravitational field; such fields are also called non-permanent gravitational fields. For example, a uniformly accelerated system of reference is identical to an inertial frame in the presence of a constant, external gravitational field.

While fundamentally related, permanent and non-permanent gravitational fields may be distinguished by their behavior at infinity. Non-permanent gravitational fields remain finite or may even increase without limit at infinity; real gravitational fields due to local masses, on the other hand, disappear at infinite distances from the bodies producing these fields. For example, while the gravitational field of a localized mass disappears at infinite distances from the mass distribution, an external gravitational field due to a uniformly accelerated system of reference is the same over all space and hence non-zero at infinity.

In addition, non-permanent gravitational fields may be distinguished from real gravitational fields with the use of the Riemann curvature tensor. Non-permanent fields to which noninertial reference systems are equivalent are characterized by a vanishing curvature tensor. In such cases space-time is said to be flat, or equivalently, Galilean. This implies that a coordinate system exists such that the metric potentials assume the simple diagonal form $\eta_{\mu\nu}$ of special relativity. This is easily understood by calculating the curvature tensor for a flat space-time. Since the components $\eta_{\mu\nu}$ are constant we obtain:

$$R^{\rho}{}_{\alpha\beta\gamma} = 0. \tag{2.12}$$

The above equation is a tensor equation and therefore it is valid for all frames connected to the flat-space metric by a transformation of the form (2.2). Thus, a vanishing curvature tensor indicates that the metric potentials $g_{\mu\nu}$ are connected to the Minkowski metric $\eta_{\mu\nu}$ by a general transformation of the coordinates. In other words, an inertial system may be defined globally. In such a manifold one may establish a contravariant vector field A^{α} by parallel displacement from an initial point to all other points in the manifold independent of the path. A real gravitational field, on the other hand, possesses a non-zero Riemann curvature tensor and therefore may not be transformed away globally by a general transformation of the coordinates. Such a space-time is said to be curved. When this is the case the change of a vector under parallel displacement is non-integrable and therefore path-dependent. Hence, in a permanent gravitational field, parallel displacement of a vector from an initial point $P(x^\mu)$ around a closed curve may produce a new vector upon return to the same initial point $P(x^\mu)$. The net change in an arbitrary contravariant vector A^μ from the parallel displacement around a closed curve enclosed by the two infinitesimal displacement vectors $d\mathbf{x}_{(1)}$ and $d\mathbf{x}_{(2)}$ is determined by the Riemann curvature tensor according to the relation:

$$\Delta A^\alpha = R^\alpha{}_{\beta\eta\gamma} A^\beta dx^\eta_{(1)} dx^\gamma_{(2)}. \tag{2.13}$$

For an actual gravitational field the Riemann curvature tensor may not be specified arbitrarily over all of space-time. The metric potentials must satisfy a certain set of differential field equations supplemented by appropriate boundary conditions. The field equations for a permanent gravitational field are called Einstein's field equations and follow naturally from a variational principle. The action I_G for the pure gravitational field may be determined uniquely from reasonable mathematical restrictions. First, we note that the action must be expressed in terms of a scalar integral over a given domain D of space-time:

$$I_G = \int_D L_G \sqrt{-g}\, d^4x, \tag{2.14}$$

where L_G is the Lagrangian for the gravitational field and g is the determinant of the metric tensor. Since the four-dimensional volume element $\sqrt{-g}\, d^4x$ is an invariant the quantity L_G must be a scalar. Of course it is not possible to form a scalar from the metric potentials $g_{\mu\nu}$ and the Christoffel symbols $\Gamma^\mu_{\rho\sigma}$ alone because one may always choose a coordinate system such that the quantities $\Gamma^{\mu}_{\rho\sigma}$ vanish at a given point. Thus, it is easy to see that L_G must contain second derivatives of the metric potentials in order to remain invariant under general coordinate transformations. However, if we demand that the resulting field equations contain derivatives of the metric tensor no higher than the second, then the second derivatives must enter the variational principle linearly in order not to introduce third-derivative terms into the field equations. In order to satisfy this demand, we consider the doubly contracted form of the curvature tensor itself. The first contracted form of the Riemann curvature tensor is called the Ricci tensor:

$$R^{\rho}{}_{\alpha\rho\beta} = R_{\alpha\beta}. \tag{2.15}$$

The second contracted form of the Riemann tensor is obtained by contracting the Ricci tensor:

$$R = R^{\alpha}{}_{\alpha} = g^{\alpha\beta} R_{\alpha\beta}. \tag{2.16}$$

This is the Ricci scalar curvature. The scalar curvature R is the only quantity constructed from the metric tensor and its first and second derivatives alone, linear in the latter, that is an invariant. Therefore, we see that the choice of the Lagrangian for the pure gravitational field is unique:

$$L_G = R. \tag{2.17}$$

Thus the action for the pure gravitational field is given by:

$$I_G = \int_D R\sqrt{-g}\, d^4x. \tag{2.18}$$

The field equations are determined by the variational principle:

$$\delta I_G = 0. \tag{2.19}$$

The components $g_{\mu\nu}$ themselves are varied independently subject only to the requirement that their variations $\delta g_{\mu\nu}$ as well as the variations of their first derivatives $\delta\left(\frac{\partial g_{\mu\nu}}{\partial x^\lambda}\right)$ vanish on the boundary of integration. The resulting field equations are:

$$G_{\mu\nu} = 0, \tag{2.20}$$

where $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$ is the divergenceless Einstein tensor. These equations are known as the Einstein free-field equations for the metric tensor $g_{\mu\nu}$. Contracting the above equation we see that the Ricci scalar also vanishes for the free-field case; consequently, the free-field equations may also be written as:

$$R_{\mu\nu} = 0. \tag{2.21}$$

If additional fields are present then they may be included in the action by adding terms representing these matter fields to the pure gravitational Lagrangian:

$$I = \int_D (R - 2\kappa L_F)\sqrt{-g}\, d^4x, \tag{2.22}$$

where κ is a constant and L_F describes all fields except the gravitational field. Again, the variational principle:

$$\delta I = 0, \tag{2.23}$$

yields the field equations:

$$G_{\mu\nu} = \kappa T_{\mu\nu}, \tag{2.24}$$

where $T_{\mu\nu}$ is the energy-momentum tensor of all the other fields. These are Einstein's field equations in non-empty space. The constant κ is determined by demanding that the field equations reproduce the equations of Newtonian gravitational theory in the appropriate limit. Thus, by considering a source with a low proper density moving at a low velocity, one may show that equation (2.24) yields the classical equation of the gravitational field if we make the identification:

$$\kappa = \frac{8\pi G}{c^4} = 2.08 \times 10^{-48}\mathrm{cm}^{-1}\mathrm{g}^{-1}\mathrm{s}^2. \tag{2.25}$$

2.1.1 The Cosmological Constant

General relativity provides a partial realization of Mach's principle in physics. Mach's principle states that inertia cannot be defined relative to absolute space, but must be defined relative to the entire matter content of the universe. Thus, guided by Mach's principle, Einstein believed that the quantities $T_{\mu\nu}$ should determine the gravitational field uniquely and in a generally covariant manner. Indeed, the matter tensor $T_{\mu\nu}$ serves as the source of the field $g_{\mu\nu}$ in equation (2.24). However, it also follows from Mach's principle that the quantities $g_{\mu\nu}$ should vanish in the absence of matter, i.e. $g_{\mu\nu} = 0$ when $T_{\mu\nu} = 0$. Since Einstein's equations (2.24) admit the solution $g_{\mu\nu}$ = constant for matter-free space, there exists an apparent contradiction with Mach's principle. Therefore, Einstein [17] proposed a modified set of equations that were consistent with general covariance:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}, \tag{2.26}$$

where Λ is the cosmological constant. Einstein hoped that his reformulation of the field equations with a cosmological constant would eliminate solutions in the absence of mass, giving $g_{\mu\nu} = 0$ when $T_{\mu\nu} = 0$. Also, he wanted to obtain static cosmological solutions from the gravitational field equations because, he observed, the relative velocities of the stars were small compared to the velocity of light. Soon afterward, de Sitter [12] showed that a solution to (2.26) with non-zero $g_{\mu\nu}$ existed even in the absence of matter [54]:

$$ds^2 = \frac{1}{\cosh^2 Hr}\left[c^2dt^2 - dr^2 - \frac{1}{H^2}\tanh^2 Hr(d\theta^2 + \sin^2\theta d\phi^2)\right], \tag{2.27}$$

where $H = \sqrt{\Lambda/3}$. Subsequently, Hubble discovered the expansion of the universe. Thus, Einstein retracted his proposal.

However, the cosmological constant has reappeared frequently ever since, for there is no principle that prohibits its inclusion in the field equations. Nowadays the cosmological constant is interpreted as a contribution to the total effective vacuum energy density ρ_V:

$$\rho_V = \rho_0 + \frac{\Lambda}{8\pi G}, \tag{2.28}$$

where ρ_0 is the vacuum energy density. Current measurements of the cosmological expansion indicate an extremely small cosmological constant:

$$|\rho_V| < 10^{-29}\text{gcm}^3 \sim (10^{-11}\text{GeV})^4, \tag{2.29}$$

much smaller than the underlying field zero-point energies of particle physics.

2.2 Electromagnetism

We now discuss the electromagnetic field in the absence of gravitation; hence, we assume that space-time possesses the metric $\eta_{\mu\nu}$. The electromagnetic field is described by an antisymmetric second-rank tensor, the electromagnetic field-strength tensor $F_{\mu\nu}$, which satisfies [1, 30]:

$$\frac{\partial F_{\mu\nu}}{\partial x^{\lambda}} + \frac{\partial F_{\lambda\mu}}{\partial x^{\nu}} + \frac{\partial F_{\nu\lambda}}{\partial x^{\mu}} = 0. \tag{2.30}$$

Equation (2.30) is simply the necessary and sufficient condition that $F_{\mu\nu}$ is closed and has a tensor potential; hence the field strength tensor $F_{\mu\nu}$ may be written as a curl of a covariant vector ϕ_μ:

$$F_{\mu\nu} = \frac{\partial \phi_\mu}{\partial x^{\nu}} - \frac{\partial \phi_\nu}{\partial x^{\mu}}. \tag{2.31}$$

The above relation is invariant under the so-called gauge transformations:

$$\phi_\mu \rightarrow \phi_\mu + \frac{\partial f}{\partial x^{\mu}}, \tag{2.32}$$

where $f = f(x^\alpha)$ is an arbitrary function of the space-time coordinates. The quantities ϕ_μ are the covariant components of the electromagnetic four-vector potential defined such that in rectangular Galilean coordinates:

$$\phi_\mu = (V, -A_x, -A_y, -A_z)\,, \tag{2.33}$$

where V is the scalar potential and the A_i are the components of the magnetic vector potential. As with the gravitational field we may also associate with the electromagnetic field an invariant differential form which we call $d\omega$. In contrast to the gravitational differential form, however, the electromagnetic differential form is linear:

$$d\omega = \phi_\mu dx^\mu, \tag{2.34}$$

and does not have a simple interpretation in terms of the metrical properties of the continuum. Equation (2.34) is invariant under arbitrary four-dimensional coordinate transformations but it is not invariant under gauge transformations; it acquires an additional exact differential term.

The field equations for the external electromagnetic field may also be derived from a variational principle. We write the action for the electromagnetic field in the absence of gravitation as:

$$I_{EM} = \int_D L\, d^4x, \tag{2.35}$$

where L is the Lagrangian density for the electromagnetic field. If we include a source four-vector s_μ, then the electromagnetic Lagrangian is:

$$L = -\frac{1}{16\pi}\eta^{\mu\alpha}\eta^{\nu\beta}F_{\mu\nu}F_{\alpha\beta} + \frac{1}{c}\eta^{\mu\nu}\phi_\mu s_\nu, \tag{2.36}$$

where $s_\mu = (c\rho, -\mathbf{j})$, ρ is the charge density, and $\mathbf{j}$ is the vector current density. Variation of the quantities ϕ_μ in (2.35) yields:

$$\frac{\partial F^{\mu\nu}}{\partial x^\nu} = \frac{4\pi}{c}s^\mu. \tag{2.37}$$

Equations (2.30) and (2.37) are the Maxwell field equations in flat space-time.

The electromagnetic field-strength tensor is related to the electric field

$\mathbf{E}$ and the magnetic field $\mathbf{B}$ by the following identification:

$$F_{\alpha\beta} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}. \tag{2.38}$$

Thus, the Maxwell system of equations may also be written:

$$\nabla\times\mathbf{B} - \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t} = \frac{4\pi}{c}\mathbf{j} \tag{2.39}$$

$$\nabla\cdot\mathbf{E} = 4\pi\rho \tag{2.40}$$

$$\nabla\times\mathbf{E} + \frac{1}{c}\frac{\partial\mathbf{B}}{\partial t} = 0 \tag{2.41}$$

$$\nabla\cdot\mathbf{B} = 0, \tag{2.42}$$

where $\nabla \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$.

We may also distinguish between permanent and non-permanent electromagnetic fields. If the electromagnetic field tensor $F_{\mu\nu}$ vanishes, or equivalently, if $d\omega$ is an exact differential, then one may transform using a gauge transformation (2.32) to a "frame" in which the electromagnetic potentials vanish identically: $\phi_\mu = 0$. This "frame" is analogous to the frame in the gravitational case where the metric potentials assume the Galilean form $\eta_{\mu\nu}$; however, it does not possess a similar physical meaning. Unlike the four-dimensional coordinate transformations, gauge transformations have no simple physical interpretation in terms of real observers. Therefore, there has been no reason to recognize electromagnetic fields with $F_{\mu\nu} = 0$ for they are believed to possess no physical significance.

2.3 Gravitation and Electromagnetism

2.3.1 Einstein's Equations in the Presence of an Electromagnetic Field

We have already shown how to modify Einstein's free-field equations if additional fields L_F are present. Thus, the electromagnetic field may be included in the gravitational action by adding the Lagrangian for the electromagnetic field into equation (2.22). The Lagrangian for the electromagnetic field L_F that enters equation (2.22) is obtained by replacing $\eta^{\mu\nu}$ in (2.36) with the curved space-time metric tensor $g^{\mu\nu}$:

$$L_F = -\frac{1}{16\pi} g^{\mu\alpha} g^{\nu\beta} F_{\mu\nu} F_{\alpha\beta} + \frac{1}{c} g^{\mu\nu} \phi_\mu s_\nu. \tag{2.43}$$

Let us consider the addition of the electromagnetic field with a vanishing source four-vector, $s_\mu = 0$. Using this Lagrangian, variation of the action (2.22) yields Einstein's equations for non-empty space (2.24), with the electromagnetic stress-energy tensor serving as the source:

$$T_{\mu\nu} = \frac{1}{4\pi} \left(\frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} g_{\mu\nu} - F_{\mu\alpha} F_\nu{}^\alpha \right). \tag{2.44}$$

Because the electromagnetic field strength tensor is antisymmetric, the trace of the electromagnetic stress-energy tensor vanishes. Thus, the gravitational field equations (2.24) in the presence of the electromagnetic field (2.44) simplify to:

$$R_{\mu\nu} = \kappa T_{\mu\nu}. \tag{2.45}$$

2.3.2 Maxwell's Equations in the Presence of a Gravitational Field

In the presence of a gravitational field Maxwell's system of equations (2.30) and (2.37) must be generalized in order to accommodate a curved space-time. As if often the case, this may be accomplished by replacing partial derivatives with the corresponding covariant derivatives. First we note that equation (2.31) is not changed if the partial derivatives are replaced by covariant derivatives:

$$F_{\mu\nu} = \phi_{\mu;\nu} - \phi_{\nu;\mu} = \frac{\partial \phi_\mu}{\partial x^\nu} - \frac{\partial \phi_\nu}{\partial x^\mu}. \tag{2.46}$$

Therefore, the relation between the electromagnetic potentials and the electric and magnetic fields is not changed due to the presence of a gravitational field.

Next we consider the first set of Maxwell's equations (2.30) in the presence of a gravitational field. The proper generalization is again achieved by replacing the partial derivatives with covariant derivatives; due to the anti-symmetry of $F_{\mu\nu}$ these equations are also unchanged in a curved space-time:

$$F_{\mu\nu;\lambda} + F_{\lambda\mu;\nu} + F_{\nu\lambda;\mu} = \frac{\partial F_{\mu\nu}}{\partial x^\lambda} + \frac{\partial F_{\lambda\mu}}{\partial x^\nu} + \frac{\partial F_{\nu\lambda}}{\partial x^\mu} = 0. \tag{2.47}$$

The second set of Maxwell's equations (2.37) may be generalized in either of two ways. We may replace the partial derivative in equation (2.37) with a covariant derivative, or we may treat the electromagnetic potentials as independent variables in a variational principle. The electromagnetic Lagrangian in a curved space-time is given by equation (2.43). Either method produces the second set of Maxwell's equations generalized to accommodate a gravitational field:

$$F^{\alpha\beta}{}_{;\beta} = \frac{1}{\sqrt{-g}} \frac{\partial \left(\sqrt{-g} F^{\alpha\beta}\right)}{\partial x^{\beta}} = \frac{4\pi}{c} s^{\alpha}. \tag{2.48}$$

This set of Maxwell's equations in curved space-time depends on the metric potentials $g_{\mu\nu}$. Thus, in order to solve the above system of equations one must know the metric tensor. However, as we have shown above, the metric components themselves depend on the electromagnetic field; the electromagnetic stress-energy tensor serves as the source for the gravitational field. Therefore, Maxwell's equations and Einstein's equations must be solved simultaneously. The system of equations (2.45) and (2.48) constitute the coupled Einstein-Maxwell system of equations.

2.3.3 The Equations of Motion of Test Particles

Guided by the equivalence principle, Einstein originally postulated that the motion of all infinitesimal test particles in a pure gravitational field was governed by the geodesic equations of a four-dimensional general Riemannian space. This postulate was originally an axiomatic addition to the field equations of general relativity. Because the gravitational field equations are non-linear and satisfy a set of four differential identities, the Bianchi identities:

$$R^{\mu}{}_{\nu\alpha\beta;\gamma} + R^{\mu}{}_{\nu\gamma\alpha;\beta} + R^{\mu}{}_{\nu\beta\gamma;\alpha} = 0, \tag{2.49}$$

it was subsequently observed that the geodesic postulate need not be assumed separately [22, 23, 25, 29]. In fact, the contracted Bianchi identities provide the covariant conservation law of energy and momentum:

$$T^{\mu\nu}{}_{;\nu} = 0, \tag{2.50}$$

which in turn gives the motion and distribution of the matter producing the gravitational field.

Hence, the Einstein field equations contain the equations of motion of matter. In electrodynamic theory, on the other hand, Maxwell's equations do not determine the equations of motion of charges. As Bergmann states [5]:

> In electrodynamics, the law of motion of charges does not follow from the field equations; in other words, it may happen that the charges do not obey Lorentz's force law, although Maxwell's field equations are satisfied. In fact, this is the case - according to classical electrodynamics - whenever the charges are subject to nonelectric forces in addition to the Lorentz forces.

Of course, the fact that the distribution and the motion of charges may be prescribed arbitrarily in electrodynamic theory is due to the linearity of Maxwell's equations. Indeed, a linear superposition of solutions to Maxwell's equations is also a solution, and therefore, the Maxwell field equations do not restrict the dynamics of point singularities of the field. However, in the context of general relativity, the equations of motion of charged particles may be obtained from the covariant conservation laws of energy and momentum as well [11].

2.4 Alternate Formulations of General Relativity

Einstein's general relativity is the accepted theory of gravitation. However, over the years a number of alternate theories and formulations have appeared (see Wesson [55] for a summary of many of the alternate proposals). While it is out of the scope of this investigation to review all of these alternate formulations, we include two important developments: Rosen's bimetric theory and Brans-Dicke theory.

2.4.1 Rosen's Bimetric Theory

In order to improve upon the formalism of general relativity Rosen [44, 45] introduced a second metric tensor $\gamma_{\mu\nu}$ at each point in space-time. Thus, in addition to the usual line element:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \tag{2.51}$$

Rosen considered the Euclidean line element:

$$d\sigma^2 = \gamma_{\mu\nu} dx^\mu dx^\nu, \tag{2.52}$$

for which the corresponding Riemann curvature tensor vanishes. This leads to two different forms of covariant differentiation. Rosen refers to the usual covariant differentiation based on the metric $g_{\mu\nu}$ (2.8) as g-differentiation. Similarly, he calls the covariant differentiation based on the Euclidean metric γ-differentiation and designates it with a comma. The second metric tensor $\gamma_{\mu\nu}$ along with the new form of covariant differentiation permits one to rewrite the

Christoffel three-index symbol $\Gamma^{\lambda}_{\mu\nu}$ as a sum of two terms:

$$\Gamma^{\lambda}_{\mu\nu} = S^{\lambda}_{\mu\nu} + \Theta^{\lambda}_{\mu\nu}, \tag{2.53}$$

where $\Theta^{\lambda}_{\mu\nu}$ is the Christoffel three-index symbol formed from the quantities $\gamma_{\mu\nu}$, and $S^{\lambda}_{\mu\nu}$ is a tensor given by:

$$S^{\lambda}_{\mu\nu} = \frac{g^{\lambda\alpha}}{2}(g_{\mu\alpha,\nu} + g_{\nu\alpha,\mu} - g_{\mu\nu,\alpha}). \tag{2.54}$$

Note that $S^{\lambda}_{\mu\nu}$ is identical to $\Gamma^{\lambda}_{\mu\nu}$ but with ordinary partial derivatives replaced by γ-derivatives. One may now rewrite the Riemann curvature tensor as:

$$R^{\lambda}{}_{\mu\nu\omega} = K^{\lambda}{}_{\mu\nu\omega} + P^{\lambda}{}_{\mu\nu\omega}, \tag{2.55}$$

where $K^{\lambda}{}_{\mu\nu\omega}$ has the same form as $R^{\lambda}{}_{\mu\nu\omega}$, but with ordinary partial derivatives replaced by γ-derivatives, and $P^{\lambda}{}_{\mu\nu\omega}$ is the Riemann tensor formed from the tensor $\gamma_{\mu\nu}$. Since the quantities $\gamma_{\mu\nu}$ describe flat space-time their corresponding Riemann tensor vanishes:

$$P^{\lambda}{}_{\mu\nu\omega} = 0, \tag{2.56}$$

and therefore the Riemann curvature tensor $R^{\lambda}{}_{\mu\nu\omega}$ may be written as:

$$R^{\lambda}{}_{\mu\nu\omega} = S^{\lambda}_{\mu\omega,\nu} - S^{\lambda}_{\mu\nu,\omega} + S^{\lambda}_{\beta\nu}S^{\beta}_{\mu\omega} - S^{\lambda}_{\beta\omega}S^{\beta}_{\mu\nu}. \tag{2.57}$$

Thus, one concludes that the Riemann curvature tensor may be rewritten with the quantities $S^{\lambda}_{\mu\nu}$ replacing the Christoffel three-index symbols $\Gamma^{\lambda}_{\mu\nu}$ and γ-differentiation replacing ordinary differentiation. Interestingly enough, each term on the right hand side of (2.57) is a tensor and therefore ${R^\lambda}_{\mu\nu\omega}$ is a tensor as well.

We may now use this new formalism to rewrite the equations of general relativity. This may be accomplished as above by replacing $\Gamma^\lambda_{\mu\nu}$ with $S^\lambda_{\mu\nu}$ and ordinary differentiation with γ-differentiation in the equations of Einstein's theory. Furthermore, upon integration one must also make the substitutions:

$$\sqrt{-g} \rightarrow \left(\frac{g}{\gamma}\right)^{1/2} \tag{2.58}$$

and:

$$d^4x \rightarrow \sqrt{-\gamma}\, d^4x, \tag{2.59}$$

where g and γ are the determinants of $g_{\mu\nu}$ and $\gamma_{\mu\nu}$ respectively. Thus, Einstein's free-field equations become:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 0, \tag{2.60}$$

where the first and second contracted forms of the Riemann curvature tensor are now obtained from (2.57); and, the corresponding action is:

$$I_G = \int \bar{L}_G \sqrt{-\gamma}\, d^4x, \tag{2.61}$$

where:

$$\bar{L}_G = \left(\frac{g}{\gamma}\right)^{1/2} g^{\mu\nu}(S^\alpha_{\beta\mu}S^\beta_{\alpha\nu} - S^\alpha_{\alpha\beta}S^\beta_{\mu\nu}). \tag{2.62}$$

One varies the quantities $g_{\mu\nu}$, not the quantities $\gamma_{\mu\nu}$, in the variational principle.

The use of two metrics improves upon the formalism of Einstein's theory [35, 36, 37, 42, 44, 45]. For example, in Rosen's bimetric general relativity one may distinguish between a permanent gravitational field and a non-permanent gravitational field in the presence of both types of fields. Since the quantities $S^{\lambda}_{\mu\nu}$ constitute a tensor and the quantities $\Theta^{\lambda}_{\mu\nu}$ may always be made to vanish by a suitable coordinate transformation, Rosen identified $S^{\lambda}_{\mu\nu}$ with the permanent field and $\Theta^{\lambda}_{\mu\nu}$ with the inertial field. Furthermore, Rosen's formalism provides an energy-momentum density complex that transforms as a tensor which may replace the usual energy-momentum pseudo-tensor of general relativity; and, it permits four additional covariant conditions on the gravitational field that may be used to restrict the form of the solution for a given physical system.

In Rosen's formalism the covariance and equivalence principles by themselves do not single out Einstein's theory uniquely because the use of the second metric $\gamma_{\mu\nu}$ increases the number of tensors and scalars that may be used in formulating the field equations. Thus, Rosen's formalism accommodates additional theories of gravity which satisfy the covariance and equivalence principles. Einstein's general relativity is just one specific theory that results from postulating, in addition to the covariance and equivalence principles, invariance of the field equations with respect to changes in the tensor $\gamma_{\mu\nu}$. Rosen [46, 47, 48, 49] suggested an alternate theory of gravity that satisfies the covariance and equivalence principles as well, but has a simpler structure than general relativity. He obtained these field equations from an action principle and postulated that the Lagrangian for the gravitational field is a homogeneous function of degree zero in the $g_{\mu\nu}$ and its derivatives, and that the quantities $\gamma^{\mu\nu}$ appear only in order to provide contraction of all of the differentiation indices. In place of Einstein's free-field equations (2.60) Rosen discovered:

$$N_{\mu\nu} = 0, \tag{2.63}$$

where $N_{\mu\nu} = \frac{1}{2}\gamma^{\alpha\beta}g_{\mu\nu,\alpha\beta} - \frac{1}{2}\gamma^{\alpha\beta}g^{\lambda\rho}g_{\mu\lambda,\alpha}g_{\nu\rho,\beta}$. He found that for the case of a static, spherically-symmetric field the theory gave results in agreement with observation [46, 47]. Furthermore, he found that his new theory did not predict black hole solutions at a time when the existence of black holes was still greatly questioned. Lee et al. [24] also showed that it agrees with general relativity in the post-Newtonian limit and therefore is indistinguishable from general relativity according to intra-solar system experimental tests. However, it was subsequently observed that this alternate theory of gravity predicts the emission of dipole gravitational radiation from binary systems containing neutron stars that is incompatible with observation [61] (see also [9]). We emphasize that the failure of Rosen's alternate theory of gravity, which is not uniquely determined by the bimetric formalism, does not prohibit the use of a second metric tensor in general relativity. In fact, there are certain advantages in reformulating Einstein's theory with a second metric tensor.

2.4.2 Brans-Dicke Theory

As we discussed above, general relativity only provides a partial realization of Mach's principle in physics; a non-vanishing gravitational field exists even in the absence of matter. Furthermore, deSitter [12] showed that Einstein's field equations with a cosmological constant possess a non-zero solution. In an attempt to rectify this situation, Brans and Dicke postulated that the gravitational constant $\kappa = \frac{8\pi G}{c^4}$ is related to a new scalar field determined by the total mass-energy of the universe. Because the absolute inertial mass of a particle can only be measured by measuring its gravitational acceleration GM/r^2, it follows from this postulate that the entire mass-energy content of the universe would determine inertial masses. Therefore, gravitational theory would be compatible with Mach's principle.

Brans and Dicke start with the gravitational action of general relativity:

$$\delta \int [R - \left(\frac{16\pi G}{c^4}\right) L]\sqrt{-g}\, d^4x = 0. \tag{2.64}$$

Dividing by G and adding the Lagrangian density of a new scalar field ϕ gives:

$$\delta \int \left[\phi R - \left(\frac{16\pi}{c^4}\right) L - \omega \left(\frac{\phi_{,\mu}\phi^{,\mu}}{\phi}\right)\right] \sqrt{-g}\, d^4x = 0, \tag{2.65}$$

where ω is a dimensionless constant. The field ϕ is a new scalar field that determines the local value of the gravitational constant. The Lagrangian density of matter is assumed to be the same as that in general relativity, and since its action is invariant under arbitrary four-dimensional coordinate transformations, the divergence of its Hamiltonian derivative vanishes:

$$T^{\mu\nu}{}_{;\nu} = 0. \tag{2.66}$$

Consequently, the equations of motion of matter in a given field $g_{\mu\nu}$ are the same as those in general relativity.

Varying the quantities ϕ and $\phi_{,\mu}$ in the usual manner gives the dynamical equations for the scalar field ϕ:

$$2\omega\left(\frac{\triangle\phi}{\phi}\right)-\frac{\omega}{\phi^2}\phi^{,\mu}\phi_{,\mu}+R=0, \tag{2.67}$$

where we use the notation $\triangle$ for the generally covariant d'Alembertian operator:

$$\triangle\phi\equiv\phi^{,\mu}{}_{;\mu}=\frac{1}{\sqrt{-g}}[\sqrt{-g}\phi^{,\mu}]_{,\mu}. \tag{2.68}$$

The field equations for the gravitational field $g_{\mu\nu}$ are obtained by varying the metric potentials and their first derivatives, producing:

$$R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R=-\left(\frac{8\pi}{\phi c^4}\right)T_{\mu\nu}+\frac{\omega}{\phi^2}\left(\phi_{,\mu}\phi_{,\nu}-\frac{1}{2}g_{\mu\nu}\phi_{,\alpha}\phi^{,\alpha}\right)+\frac{1}{\phi}\left(\phi_{,\mu;\nu}-g_{\mu\nu}\triangle\phi\right). \tag{2.69}$$

Contracting the above equation gives:

$$-R=-\left(\frac{8\pi}{\phi c^4}\right)T-\frac{\omega}{\phi^2}\phi_{,\alpha}\phi^{,\alpha}-\frac{3}{\phi}\triangle\phi. \tag{2.70}$$

Combining this equation with equation (2.67) produces a new wave equation for ϕ:

$$\triangle\phi=-\frac{8\pi}{(3+2\omega)c^4}T. \tag{2.71}$$

Equations (2.69) and (2.71) are the Brans-Dicke field equations. Since the gravitational coupling is a function of space and time, Brans-Dicke theory contains no fundamental length scale. Furthermore, when $\omega=-\frac{3}{2}$ and the trace of the energy-momentum tensor vanishes, the equations are invariant under local scale transformations.

In the limit $\omega \to \infty$ the field equations of Brans-Dicke theory reduce to Einstein's theory. In their original paper, Brans and Dicke stated that ω must be on the order of unity. Using the weak field approximation and the data available at the time on the perihelion precession of Mercury, they concluded:

$$\omega \geq 6. \tag{2.72}$$

However, recent solar system observations indicate:

$$\omega > 500. \tag{2.73}$$

Thus, even if Brans-Dicke theory is valid, it does not differ significantly from Einstein's relativity in its consequences.

Chapter 3

Attempts to Unify the Gravitational and Electromagnetic Fields

Einstein successfully unified gravitation with geometry by replacing the flat Minkowski space-time of special relativity with a general Riemannian manifold. After the advent of general relativity Einstein and others turned their attention to generalizations of Riemannian geometry itself, hoping to incorporate the electromagnetic field into the space-time manifold as well. In addition to his desire to associate electromagnetism with geometry Einstein was motivated by the belief that a unified field theory of gravitation and electromagnetism would lead to a greater understanding of quantum theory. Furthermore, he had hoped that a new theory would provide singularity-free particle-like solutions which he had failed to derive from the free-field equations of general relativity [41]. This program immediately encountered much difficulty for there was no principle, analogous to the equivalence principle, for the electromagnetic force. As a result, the unification of gravitation and electromagnetism was attacked primarily from a mathematical point of view; fundamental restrictions of Riemannian geometry were identified and then relaxed with the hope that the number of components determining the space-time geometry would increase just enough in order to accommodate the electromagnetic four-vector potential.

The two pioneers of unified field theory were the mathematicians Weyl [60] and Kaluza [31]. Weyl's theory inspired the purely affine theory of Eddington [15], which was further developed by Einstein [20] and Schrodinger [52]. Kaluza's theory was independently improved upon by Klein [33], who also discussed quantum theory in this context. In this chapter we review these important contributions.

3.1 Weyl's Theory

In a series of papers beginning in 1918 Weyl [56, 57, 58, 59, 60] developed a generalization of Riemannian geometry based on the notion of the affine connection. He suggested that a true infinitesimal geometry should not only subject the direction but also the magnitude of an arbitrary vector to a set of integrability conditions. He claimed that it was due to the "accidental development of Riemannian geometry from Euclidean geometry" [56] that we are able to compare the magnitudes of two vectors situated at any two arbitrarily separated points in a Riemannian manifold. Therefore, Weyl relaxed the Riemannian condition that the length of a vector and the scalar product of two vectors remain unchanged under infinitesimal parallel displacements.

Let us consider an arbitrary contravariant vector A^μ. Weyl retains the basic form of the parallel displacement law for the components A^μ:

$$dA^\mu = -\Gamma^\mu_{\rho\sigma} A^\rho dx^\sigma, \tag{3.1}$$

which is motivated by considering the change in the components of the vector in an arbitrary coordinate system when it is constant in one system. However, $\Gamma^{\mu}_{\rho\sigma}$ no longer represents the Christoffel connection. We recall that the Christoffel connection is unique to a Riemann space and is derived by demanding length preservation under parallel displacements. In order to determine an expression for the generalized connection components $\Gamma^{\mu}_{\rho\sigma}$ in Weyl's geometry one must postulate a law of parallel displacement for the length of the vector as well. The length of the vector A^{μ} is defined in the usual manner as:

$$l^2 \equiv \|A\|^2 = g_{\alpha\beta}A^{\alpha}A^{\beta}. \tag{3.2}$$

Weyl assumed, in analogy with the parallel displacement law (3.1), that the increment in length under parallel displacement is a bilinear function of the displacement and the length itself:

$$dl = -\varphi_{\beta}dx^{\beta}l, \tag{3.3}$$

where the vector φ_{β} serves as the connection coefficient for the parallel displacement of length. We may now solve for the generalized connection coefficients $\Gamma^{\mu}_{\rho\sigma}$ in terms of the metric tensor $g_{\mu\nu}$ and the vector φ_{β}. Using equations (3.1), (3.2), and (3.3), it is easy to show that:

$$(\frac{\partial g_{\alpha\beta}}{\partial x^{\gamma}} - 2g_{\alpha\beta}\varphi_{\gamma}) + g_{\sigma\beta}\Gamma^{\sigma}_{\alpha\gamma} + g_{\sigma\alpha}\Gamma^{\sigma}_{\beta\gamma} = 0, \tag{3.4}$$

for an arbitrary vector A^{α} and an arbitrary displacement dx^{α}. By a cyclical permutation of the indices one obtains three equations that may be solved simultaneously for the quantities $\Gamma_{\alpha,\rho\sigma}$. This produces:

$$\Gamma_{\alpha,\rho\sigma} = \Gamma^{*}_{\alpha,\rho\sigma} + (g_{\alpha\rho}\varphi_{\sigma} + g_{\alpha\sigma}\varphi_{\rho} - g_{\rho\sigma}\varphi_{\alpha}), \tag{3.5}$$

where $\Gamma_{\alpha,\rho\sigma} = g_{\alpha\lambda}\Gamma^{\lambda}_{\rho\sigma}$ and $\Gamma^{*}_{\alpha,\rho\sigma}$ is the usual Christoffel connection of Riemannian geometry. Notice that Weyl's geometry reduces to Riemannian geometry when the vector φ_{α} vanishes. This is to be expected, for in this case the change in the length under parallel displacement vanishes according to equation (3.3).

By generalizing the geometry to include changes in vector lengths under parallel displacement, Weyl's formalism places scale transformations of the metric potentials on an equal footing with the arbitrary four-dimensional coordinate transformations of general relativity; at each space-time point we may multiply all elements of length by an arbitrary factor. Thus, the scale of the metric is now arbitrary. Let us consider the transformation:

$$g'_{\alpha\beta} = f(x^{\mu})g_{\alpha\beta}, \tag{3.6}$$

where $f(x^{\mu})$ is an arbitrary function of the space-time coordinates. Under this transformation, the vector field φ_{α} transforms according to:

$$\varphi'_{\alpha} = \varphi_{\alpha} - \frac{1}{2}\frac{\partial \log f}{\partial x^{\alpha}} = \varphi_{\alpha} - \frac{1}{2f}\frac{\partial f}{\partial x^{\alpha}}, \tag{3.7}$$

because of equation (3.3). Since transformation (3.6) may be interpreted as a change of scale at each point of the manifold the transformation (3.6) is called a gauge transformation and the vector φ_{α} is called a gauge vector field.

Let us consider a vector of length l_P at a point P and displace it to an arbitrary point Q. Its length l_Q at the point Q is obtained by integrating equation (3.3):

$$l_Q = l_P \exp\left(-\int_P^Q \varphi_\mu dx^\mu\right). \tag{3.8}$$

Thus, if the linear form $\varphi_\mu dx^\mu$ is a total differential then the length of a vector is independent of the path along which it is transferred. Therefore, Weyl's geometry may be reduced to Riemannian geometry by a transformation of the form (3.6) if the vector field φ_α is a gradient vector field, that is, if the quantities defined by the expression:

$$f_{\mu\nu} \equiv \frac{\partial\varphi_\mu}{\partial x^\nu} - \frac{\partial\varphi_\nu}{\partial x^\mu} \tag{3.9}$$

vanish. For a general Weyl geometry, however, the quantities $f_{\mu\nu}$ do not necessarily vanish. Therefore, a non-trivial Weyl geometry is defined by a non-zero antisymmetric tensor $f_{\mu\nu}$ that resembles the electromagnetic field strength tensor (2.31). Furthermore, these quantities are scale (gauge) invariant; and, because of equation (3.9) they also satisfy the first set of Maxwell's equations:

$$\frac{\partial f_{\mu\nu}}{\partial x^\lambda} + \frac{\partial f_{\lambda\mu}}{\partial x^\nu} + \frac{\partial f_{\nu\lambda}}{\partial x^\mu} = 0. \tag{3.10}$$

Weyl concluded that the quantities φ_α are proportional to the electromagnetic four-vector potential ϕ_α and that the transformation (3.6) is an electromagnetic gauge transformation (2.32) of the electromagnetic potentials. Indeed, this is how transformation (2.32) acquired its name. Thus, Weyl's proposal placed the electromagnetic field-strength tensor $F_{\alpha\beta}$ on an equal geometrical footing as the Riemann curvature tensor $R_{\alpha\mu\beta\nu}$: just as the quantities $R_{\alpha\mu\beta\nu}$ determine the path dependence of direction of an arbitrary vector, the quantities $F_{\alpha\beta}$ determine the path dependence of its length.

Weyl's geometry naturally accommodates all operations of Riemannian geometry that are defined solely by the concept of parallel displacement. Thus, the covariant derivative of a contravariant vector field is again defined by:

$$A^{\mu}_{;\nu} = \frac{\partial A^{\mu}}{\partial x^{\nu}} + \Gamma^{\mu}_{\nu\rho} A^{\rho}. \tag{3.11}$$

However, now the connection coefficients $\Gamma^{\mu}_{\nu\rho}$ are defined by the more general expression (3.5). As before, the curvature tensor follows by interchanging the order of covariant differentiation:

$$A^{\alpha}_{;\beta;\gamma} - A^{\alpha}_{;\gamma;\beta} = R^{\alpha}{}_{\nu\beta\gamma} A^{\nu}. \tag{3.12}$$

Again, this procedure yields the expression:

$$R^{\alpha}{}_{\nu\beta\gamma} = \frac{\partial \Gamma^{\alpha}_{\beta\nu}}{\partial x^{\gamma}} - \frac{\partial \Gamma^{\alpha}_{\nu\gamma}}{\partial x^{\beta}} + \Gamma^{\alpha}_{\tau\gamma} \Gamma^{\tau}_{\beta\nu} - \Gamma^{\alpha}_{\tau\beta} \Gamma^{\tau}_{\gamma\nu}. \tag{3.13}$$

Because the quantities $\Gamma^{\alpha}_{\beta\nu}$ are now defined by equation (3.5) a complete expression for $R^{\alpha}{}_{\nu\beta\gamma}$ in terms of the metric tensor $g_{\mu\nu}$ and the vector φ_{μ} is somewhat involved.

In Weyl's geometry tensors are assigned a weight. The weight of a tensor is defined by the factor of $f(x^{\alpha})$ it acquires under the transformation (3.6). Thus, the metric tensor $g_{\mu\nu}$ by definition is weight $+1$; and, the inverse metric tensor $g^{\mu\nu}$ is weight -1. Similarly, the quantity $\sqrt{-g}$ is weight $+2$. Since the quantities $\Gamma^{\alpha}_{\mu\beta}$ are gauge invariant (weight 0) according to equation (3.5) the Riemann curvature tensor $R^{\alpha}{}_{\nu\beta\gamma}$ is also gauge invariant. Consequently, the Ricci scalar curvature R, which enters the action for the pure gravitational field in conventional relativity is weight -1.

Weyl derived the field equations of his theory by postulating invariance of the action under scale transformations (3.6) in addition to the general four-dimensional coordinate transformations of general relativity. However, the action from which Einstein's equations are produced (2.18) is weight $+1$ and therefore is not gauge invariant. Thus, the demand for gauge invariant physical laws forces one to use the square of the scalar curvature rather than the scalar curvature itself in the action principle. This is an undesirable property for quadratic Lagrangians lead to fourth-order differential equations [28]. Nevertheless, Pauli [43] showed that this theory reproduces the usual classical results of general relativity, namely, the perihelion precession of Mercury as well as the bending of light rays by a gravitational field. However, Einstein pointed out that according to equation (3.8) the readings of clocks would depend on their prehistory, which is in conflict with the well-defined electromagnetic spectrum of chemical elements [39, 43]. As a result, Weyl was forced to abandon his original proposal despite its mathematical elegance.

3.2 Eddington, Einstein, and Schrodinger

Eddington [15], Einstein [41, 43], and Schrodinger [52], also offered fundamental investigations into the mathematical foundations of general relativity in an attempt to incorporate the electromagnetic field into the space-time manifold. Inspired by the work of Weyl, Eddington focused on the notion of the affine connection and attempted to develop a purely affine theory, relegating the metric to a derived, rather than a postulated quantity. Eddington called this work a generalization of Weyl's theory; however, Schrodinger states [51] that this is a modest characterization, for it was a great step forward in the development of the subject. Beginning with a *symmetric* affine connection as the only primary quantity, Eddington showed that the Ricci tensor is not necessarily symmetric and therefore may be separated into a sum of symmetric and antisymmetric parts:

$$R_{\mu\nu} = \bar{R}_{\mu\nu} + \check{R}_{\mu\nu}, \tag{3.14}$$

where $\bar{R}_{\mu\nu}$ is symmetric and $\check{R}_{\mu\nu}$ is antisymmetric. Eddington hoped to identify the antisymmetric term with the electromagnetic field strength tensor. Indeed, he recognized that $\check{R}_{\mu\nu}$ could be written as a curl of a four-vector. Since the affine connections were the only primary quantities, the metric tensor was not postulated from the beginning, but was derived as a secondary quantity. Eddington derived the metric from the following scalar:

$$ds^2 \equiv \frac{1}{\lambda} R_{\mu\nu} dx^\mu dx^\nu = \frac{1}{\lambda} \bar{R}_{\mu\nu} dx^\mu dx^\nu, \tag{3.15}$$

where lambda is a constant. This suggests that the symmetric part of the Ricci tensor may be identified with the metric tensor according to:

$$\bar{R}_{\mu\nu} = \lambda g_{\mu\nu}, \tag{3.16}$$

which is similar to Einstein's field equations with a cosmological constant. Einstein further elaborated on this theory by deriving the field equations for these primary affine connections [20], using Eddington's proposal that the square-root of the determinant of the curvature tensor may serve as the Lagrangian density. This approach was developed by Schrodinger [52] who also considered a purely affine theory with a *non-symmetric* affine connection. While mathematically appealing none of these proposals were capable of reproducing Maxwell's field equations.

As we mentioned above, Einstein participated in almost every aspect of the unification program. He proposed a number of original ideas and also followed many of the proposals of his colleagues to their logical conclusions [41]. Much of his early work was similar to the work of Eddington and Schrodinger; as we mentioned above he elaborated on Eddington's purely affine approach. In later theories, Einstein treated both the metric tensor and the affine connection as independent variables [41, 43]. In these attempts, Einstein relaxed the symmetry of both the metric tensor and the affine connection. In so doing he hoped to identify the electromagnetic field as an independent antisymmetric entity within the resulting nonsymmetric terms. While he was able to recover traditional gravitational theory in the usual symmetric limit, Einstein was not able to duplicate Maxwell's free-field equations in any of these attempts [41].

3.3 The Kaluza-Klein Decomposition of 5D Relativity

Kaluza-Klein theory takes a different approach towards unification. Rather than introducing a new class of connections or abandoning the symmetry of the metric tensor Kaluza-Klein theory retains the basic elements of Riemannian geometry, but increases the dimension of the manifold from four to five dimensions. Kaluza-Klein theory stands alone among all of the attempts at a unification for it is the only theory capable of reproducing the coupled Einstein-Maxwell system of equations exactly. We now proceed with a review of the original Kaluza-Klein proposal, following Klein's original presentation [33].

Let us consider a five-dimensional space-time manifold with coordinates $\{x^A\} = \{x^\mu, x^5\}$. The five-dimensional line element is given by:

$$d\sigma^2 = \gamma_{AB} dx^A dx^B, \tag{3.17}$$

and is invariant under general five-dimensional coordinate transformations:

$$x^A = f^A(x'^B). \tag{3.18}$$

If we demand that the coordinates x^μ denote the usual coordinates of space-time then their transformations are restricted to the 4D coordinate transformations of general relativity:

$$x^\mu = f^\mu(x'^\alpha). \tag{3.19}$$

Furthermore, if we demand that Kaluza's cylinder condition [31] $\frac{\partial \gamma_{AB}}{\partial x^5} = 0$ is satisfied in all frames then the fifth coordinate is restricted to transform as:

$$x^5 = x'^5 + f^5(x'^\alpha), \tag{3.20}$$

ignoring an irrelevant constant factor. The component γ_{55} is invariant under the above transformation, therefore we are free to set $\gamma_{55} = -1$. Note that the fifth dimension is assumed to be space-like. Under these assumptions the five-dimensional line element may be written as a sum of two terms each independently invariant under the subset of transformations defined by equations (3.19) and (3.20):

$$d\sigma^2 = ds^2 + \gamma_{55} d\theta^2, \tag{3.21}$$

where

$$ds^2 \equiv \left(\gamma_{\mu\nu} - \frac{\gamma_{5\mu}\gamma_{5\nu}}{\gamma_{55}}\right) dx^\mu dx^\nu \qquad \text{and} \qquad d\theta \equiv dx^5 + \frac{\gamma_{5\mu}}{\gamma_{55}} dx^\mu. \tag{3.22}$$

The quantity ds^2 is identified as the usual line element of the 4D space-time manifold and $g_{\mu\nu} \equiv \gamma_{\mu\nu} - \frac{\gamma_{5\mu}\gamma_{5\nu}}{\gamma_{55}}$ is the four-dimensional metric.

Under the four-dimensional coordinate transformations (3.19) the quantities $\frac{\gamma_{5\mu}}{\gamma_{55}}$ transform as the covariant components of a four-vector. Furthermore, under the transformation (3.20) the components $\frac{\gamma_{5\mu}}{\gamma_{55}}$ acquire a four-dimensional gradient of an arbitrary function $f^5(x'^\alpha)$ of the space-time coordinates:

$$\left(\frac{\gamma_{5\mu}}{\gamma_{55}}\right)' = \frac{\gamma_{5\mu}}{\gamma_{55}} + \frac{\partial f^5}{\partial x'^\mu}. \tag{3.23}$$

Hence, one assumes that the components $\frac{\gamma_{5\mu}}{\gamma_{55}}$ are proportional to the covariant components of the electromagnetic four-vector potential ϕ_μ:

$$\frac{\gamma_{5\mu}}{\gamma_{55}} = \beta\phi_\mu, \tag{3.24}$$

where β is a constant. As we mentioned above the quantities ϕ_μ are defined such that in rectangular Galilean coordinates $\phi_\mu = (V, -A_x, -A_y, -A_z)$, where V is the scalar potential and the A_i are the components of the magnetic vector potential. The electromagnetic field-strength tensor is defined as:

$$F_{\mu\nu} = \frac{\partial \phi_\mu}{\partial x^\nu} - \frac{\partial \phi_\nu}{\partial x^\mu}. \tag{3.25}$$

As a result of identification (3.24) this expression transforms as a covariant second-rank tensor under transformation (3.19) and is invariant under transformation (3.20).

The constant β is easily determined by computing the field equations resulting from the variation of the pure gravitational action in five dimensions:

$$I_G^{(5)} = \int_{D^{(5)}} R^{(5)} \sqrt{-\gamma}\, d^5x, \tag{3.26}$$

where γ is the determinant of γ_{AB} and $R^{(5)}$ is the five-dimensional curvature scalar derived from the metric corresponding to the line element (3.21). The variation is performed by varying the quantities γ_{AB} and $\frac{\partial \gamma_{AB}}{\partial x^\mu}$, assuming that the variations vanish on the boundary of $D^{(5)}$ and γ_{55} remains constant during the process of variation. The resulting field equations are the coupled Einstein-Maxwell system of equations, (2.45) and (2.48), with a vanishing source four-vector s_μ, provided one makes the identification:

$$\beta = \frac{\sqrt{16\pi G}}{c^2}. \tag{3.27}$$

The assumption of a space-like fifth dimension is crucial in order to obtain the correct sign for the electromagnetic stress-energy tensor in these resultant field equations.

The equations of motion of charged particles are the equations of geodesics belonging to the five-dimensional line element (3.21). This conclusion follows naturally from the five-dimensional conservation laws, just as in the four-dimensional case. Charge is identified as the momentum along the fifth dimension. Thus, given the Lagrangian:

$$L = \frac{1}{2} m \left(\frac{d\sigma}{d\tau} \right)^2, \tag{3.28}$$

where $d\tau$ is the differential of proper time, and the usual definition of momentum:

$$p_A = \frac{\partial L}{\partial (\frac{dx^A}{d\tau})}, \tag{3.29}$$

one obtains the correct equations of motion for a particle of charge q in a combined gravitational-electromagnetic field provided one makes the identification:

$$p_5 = \frac{q}{\beta c}. \tag{3.30}$$

Leibowitz and Rosen [38] showed that in order to accommodate charged particles with arbitrary values of q/m it is necessary to consider time-like, space-like, and null geodesics in five dimensions.

Of course the proposal that space-time is a five-dimensional manifold introduces five additional metric components, not only four, and therefore the theory outlined above may easily be generalized in order to include an additional scalar field [40]. Indeed, one degree of freedom was suppressed by the assumption $\gamma_{55} = -1$. Let us therefore relax the restriction $\gamma_{55} = -1$ and replace it with the identification:

$$\gamma_{55} \equiv -\psi^2, \tag{3.31}$$

where $\psi(x^\mu)$ is a scalar field. The variation of the five-dimensional action (3.26) now produces the following set of equations:

$$G_{\alpha\beta} = \frac{\beta^2\psi^2}{2} T_{\alpha\beta} - \frac{1}{\psi}\left[\left(\frac{\partial\psi}{\partial x^\beta}\right)_{;\alpha} - g_{\alpha\beta}\Box\psi\right], \tag{3.32}$$

$$F^{\alpha\beta}_{\ \ ;\beta} = -\frac{3}{\psi}\frac{\partial\psi}{\partial x^\beta}F^{\alpha\beta}, \tag{3.33}$$

$$\Box\psi = \frac{\beta^2\psi^3}{4}F^{\alpha\beta}F_{\alpha\beta}, \tag{3.34}$$

where $\Box$ is the four-dimensional Laplacian. The field ψ may be identified as a Brans-Dicke-type scalar field [40]. Of course the Einstein-Maxwell system of equations emerge when ψ is taken to be constant.

The above procedure for extracting four-dimensional gravity along with electromagnetism from a subset of five-dimensional coordinate transformations is now well known. As we stated above, five-dimensional Kaluza-Klein theory is the only unified theory of gravitation and electromagnetism that reproduces the coupled Einstein-Maxwell system of equations exactly. However, the mere reproduction of the equations of gravitation and electromagnetism is not satisfactory for no new effects are predicted that may confirm or refute this theory. Furthermore, Kaluza-Klein theory is subject to the same criticism of the other unified field theories of the same period, namely, that there is no physical motivation for the proposed mathematical operations.

3.4 Postscript

It is now generally believed that the gauge principle governs the four fundamental interactions. The general procedure for introducing a gauge field is well known [39]: if a Lagrangian is invariant under some global transformation group, then the properties of the gauge field may be deduced by demanding that it enters the Lagrangian in such a way that guarantees invariance when the parameters of the group become arbitrary functions of the independent variables. The field emerges through a covariant derivative and its transformation properties are uniquely defined by the local symmetry group under investigation. The interaction of the field with matter follows naturally from the new invariant Lagrangian; and, the free Lagrangian for the new field is usually postulated to be the lowest order covariant combination of the new gauge potentials.

Electromagnetism is the quintessential example of a gauge theory [50]. Consider the Lagrangian for a complex scalar field ψ of mass m:

$$L = (\partial_\mu \psi)(\partial^\mu \psi^*) - m^2 \psi^* \psi, \tag{3.35}$$

where $\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$, $\partial^\mu = g^{\mu\nu}\partial_\nu$, and ψ^* is the complex conjugate. This expression is invariant under the global transformation group:

$$\psi \rightarrow e^{-i\Lambda}\psi \qquad\qquad \psi^* \rightarrow e^{i\Lambda}\psi^*, \tag{3.36}$$

where Λ is a constant. If we demand invariance of the above Lagrangian when Λ is permitted to become a function of the coordinates, that is when

$\Lambda = \Lambda(x^\alpha)$, one must replace the derivatives ∂_μ with the following covariant derivatives:

$$D_\mu \psi \equiv (\partial_\mu + ie\phi_\mu)\psi, \tag{3.37}$$

where ϕ_μ is a four-vector potential and e is the coupling strength. Under the local transformation group the four-vector ϕ_μ must transform according to:

$$\phi_\mu \rightarrow \phi_\mu + \frac{1}{e}\partial_\mu \Lambda. \tag{3.38}$$

It is easy to see that the lowest order covariant combination of the potentials ϕ_μ is given by the quantity $F^{\mu\nu}F_{\mu\nu}$ where $F_{\mu\nu} = \partial_\mu \phi_\nu - \partial_\nu \phi_\mu$. Therefore, we may identify ϕ_μ as the electromagnetic four-vector potential and e as the electric charge. One concludes that the electromagnetic field is the gauge field that must be introduced in order to guarantee invariance of the Lagrangian under local $U(1)$ phase transformations.

Due to the success of the above procedure for the electromagnetic field one might expect that the gravitational field may be similarly associated with a symmetry group. Indeed, a gauge theory of gravity was first suggested by Utiyama [53], who applied the gauge principle to the homogeneous Lorentz group. Kibble [32] modified Utiyama's proposal by considering the ten-parameter Poincaré group. The Poincaré group is the global symmetry group of space-time; it includes translations in space, displacements in time, rotations, and transitions to systems in uniform relative motion. It is now generally believed that the local gauge theory of the Poincaré group in space-time is the U_4 theory of gravity which admits spin and torsion into relativistic gravitational theory. This subject has received much attention and is thoroughly reviewed by Hehl et al. [27].

The ten parameters of the Poincaré group exhaust the external degrees of freedom of space-time, and therefore any fields postulated in addition to the gravitational field must stem from an internal symmetry of the Lagrangian. For example, we have seen that the electromagnetic field is derived by demanding invariance with respect to local U(1) phase transformations, which is an internal symmetry of the Lagrangian. Therefore, the gravitational and electromagnetic fields remain unconnected, even though they both possess a common gauge structure.

Chapter 4

Global Scale Invariance

The generalizations of the space-time geometry that emerged from the early unification program were mathematically intriguing; however, no theory satisfactorily placed the electromagnetic four-vector potentials ϕ_μ alongside the gravitational potentials $g_{\mu\nu}$ in the space-time manifold. Besides Kaluza-Klein theory, which offers no new physical predictions, none of these theories were capable of reproducing the coupled Einstein-Maxwell system of equations exactly. It is safe to say that a satisfactory understanding of the connection between gravitation and electromagnetism still does not exist today.

In this chapter we lay the groundwork for a new unified theory of gravitation and electromagnetism that treats the electromagnetic field as a compensating gauge field associated with local scale invariance. We show that a well-known procedure developed by Anderson and Finkelstein [3] for introducing the cosmological constant removes the scale dependence from the field equations, leaving a set of scale-free field equations behind. Thus, general relativity with a cosmological constant may be viewed as a union of two independent equations. One equation determines the null-cone or causal structure of space-time; the other equation determines the measure structure of space-time. We will see that the equations for the null-cone structure are both globally scale-invariant and scale-free, and hence permits the application of the standard gauge trick.

4.1 Coordinate Invariance vs. Scale Invariance

Consider an arbitrary action:

$$I = \int W \sqrt{-g}\, d^4x, \tag{4.1}$$

where W is an arbitrary function of the metric tensor and its derivatives. The variational derivative of I with respect to the metric is defined as:

$$\frac{\delta I}{\delta g_{\mu\nu}} = \mathcal{W}^{\mu\nu}, \tag{4.2}$$

where $\mathcal{W}^{\mu\nu}$ is a symmetrical contravariant density of the second rank. As is well known, if the action is invariant under an arbitrary infinitesimal coordinate transformation that vanishes on the boundary:

$$x'^{\mu} = x^{\mu} - \xi^{\mu}, \tag{4.3}$$

where ξ^{μ} are arbitrary infinitesimal functions of the space-time coordinates, then the covariant divergence of $\mathcal{W}^{\mu\nu}$ vanishes identically:

$$\mathcal{W}^{\mu\nu}{}_{;\nu} = 0. \tag{4.4}$$

This follows from equations (4.1) and (4.2), noting that the transformation (4.3) produces a variation in the metric:

$$\delta g_{\mu\nu} = \xi_{\mu;\nu} + \xi_{\nu;\mu}. \tag{4.5}$$

Similarly, if the action is invariant under an infinitesimal scale transformation of the metric tensor that vanishes on the boundary:

$$g_{\mu\nu} \to \lambda g_{\mu\nu} = (1+\epsilon) g_{\mu\nu}, \tag{4.6}$$

where $\lambda = \lambda(x^\alpha)$ is an arbitrary function of the space-time variables and $\epsilon \ll 1$, then the trace of $\mathcal{W}^{\mu\nu}$ vanishes identically:

$$\mathcal{W}^{\mu}{}_{\mu} = 0. \tag{4.7}$$

This also follows from equations (4.1) and (4.2), noting that the transformation (4.6) produces a variation in the metric:

$$\delta g_{\mu\nu} = \epsilon g_{\mu\nu}. \tag{4.8}$$

The action for the gravitational field in the absence of matter is obtained by setting $W = g^{\mu\nu} R_{\mu\nu}$ in (4.1):

$$I_G = \int g^{\mu\nu} R_{\mu\nu} \sqrt{-g}\, d^4x, \tag{4.9}$$

where $R_{\mu\nu}$ is the Ricci tensor. The variational derivative of equation (4.9) with respect to the metric is:

$$\frac{\delta I_G}{\delta g_{\mu\nu}} = G^{\mu\nu}\sqrt{-g} \equiv \left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right) \sqrt{-g}, \tag{4.10}$$

Since $R\sqrt{-g}$ is a scalar density, the action (4.9) is invariant under the transformation (4.3). Therefore, the covariant divergence of $G^{\mu\nu}$ vanishes:

$$G^{\mu\nu}{}_{;\nu} = 0. \tag{4.11}$$

Note that this equation is a consequence of the invariance of the action and is therefore valid for any reasonable metric field distribution $g_{\mu\nu}$, regardless of whether or not $g_{\mu\nu}$ satisifes the field equations. Equation (4.11) also follows from the Bianchi identities.

While the action (4.9) is invariant under general coordinate transformations it is not invariant under the scale transformation (4.6). I_G is not even invariant under a global scale transformation for which $\lambda =$ constant; R and $\sqrt{-g}$ transform under a global scale transformation with Weyl weights -1 and $+2$, respectively. However, the scalar curvature is the only quantity constructed from the metric tensor and its first and second derivatives alone, linear in the latter, that is an invariant under general coordinate transformations. Therefore, we see that general coordinate invariance and scale invariance of the action are fundamentally incompatible in general relativity. This is further supported by the fact that the trace of the divergenceless quantity $G^{\mu\nu}$ does not vanish. Of course, one may proceed as Weyl [60] and consider Lagrangians quadratic in the curvature scalar in order to guarantee scale invariance of the action. However, the resulting field equations necessarily contain derivatives of the metric tensor higher than the second. Alternatively, one may proceed as Dirac [14] and introduce a new scalar field that transforms under a scale transformation with Weyl weight -1. This theory has enjoyed only limited success [55].

Rather than formulating an action principle that is invariant with respect to both coordinate transformations and scale transformations simultaneously, we reformulate general relativity so that the scale-dependent quantity, the Ricci scalar curvature, remains undetermined by the field equations themselves. As a result, the remaining field equations become scale-free. This allows us to treat these equations as the dynamical equations of a globally scale invariant theory that can be gauged locally.

First, let us consider Einstein's equations in the absence of matter:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 0, \tag{4.12}$$

which follow from the variational principle $\delta I_G = 0$, in which the metric components are varied independently. While the gravitational action is not invariant under a global scale transformation defined by equation (4.6) with $\lambda =$ constant, Einstein's free-field equations are invariant with respect to global scale transformations. This follows because $R_{\mu\nu}$, R, and $g_{\mu\nu}$ transform under a global scale transformation with Weyl weights 0, -1, and $+1$ respectively. The fact that the equations are globally scale invariant does not imply that the theory is also scale-free. This follows by taking the trace of (4.12), giving:

$$R = 0. \tag{4.13}$$

Because R vanishes, pure gravity is also scale-free: pure gravity contains no intrinsic length scale. Note that equation (4.13) is not independent of (4.12); rather, it is a consequence of the field equations.

We stress that the terms scale-free and scale-invariant are similar but not identical. A theory is scale-free if it does not contain any constant fundamental length scale. A theory is (globally) locally scale-invariant if, in addition to the absence of any fundamental length scale, the dynamical equations are covariant with respect to (global) local scale transformations. Note that a theory may be scale-free and not scale-invariant. As we saw above, pure gravity is globally scale-invariant: the equations of pure gravity are covariant, in fact invariant, with respect to global scale transformations, and since R vanishes pure gravity is also scale-free.

Once matter is introduced, global scale invariance of the theory is lost. The action for the gravitational field in the presence of matter is:

$$I = I_G + I_M, \tag{4.14}$$

where

$$I_M = -2\kappa \int L_M \sqrt{-g}\, d^4x, \tag{4.15}$$

is the matter action and $\kappa = \frac{8\pi G}{c^4}$ is the Einstein gravitational constant. The resulting field equations are:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \kappa T_{\mu\nu}. \tag{4.16}$$

Taking the trace of the above equation yields: $R = -\kappa T$. Again, this equation is contained in the field equations. The equations (4.16) may be considered globally scale invariant if one assumes that the product $\kappa T_{\mu\nu}$ is scale-invariant [10], regardless of the manner in which each term transforms individually. However, since R does not vanish the theory is no longer globally scale invariant. Rest masses introduce an intrinsic length scale.

4.2 Unimodular Relativity

There is a way of reformulating the theory so that the scale dependence remains undetermined by the field equations themselves. This is accomplished by a well-known procedure developed by Anderson and Finkelstein [3] for introducing the cosmological constant into Einstein's equations, not as a predetermined coefficient of the action, but as an arbitrary integration constant. Indeed, if one introduces the constraint in the variational principle:

$$\sqrt{-g} = \sigma(x), \tag{4.17}$$

where $\sigma(x)$ is a scalar density of weight $+1$, an external field provided by nature, then the components of the metric tensor cannot be varied independently in the action principle, but must satisfy:

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu} = 0. \tag{4.18}$$

The resulting field equations express the equality of the traceless parts of equation (4.16):

$$R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R = \kappa\left(T_{\mu\nu} - \frac{1}{4}g_{\mu\nu}T\right). \tag{4.19}$$

Because of equation (4.11) and

$$T^{\mu\nu}{}_{;\nu} = 0, \tag{4.20}$$

one obtains:

$$R + \kappa T = 4\Lambda, \tag{4.21}$$

where Λ is an integration constant and the factor four is introduced for convenience. Substituting this back into the field equations (4.19) we recover Einstein's field equations with a cosmological constant:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}. \tag{4.22}$$

Einstein [19] examined the field equations (4.19) with $T_{\mu\nu}$ representing only the stress-energy tensor of the electromagnetic field, and similarly recovered the cosmological constant as a constant of integration. Anderson and Finkelstein [3] were the first to propose the above general procedure in their theory of unimodular relativity. This formulation has the attractive property that the contribution of vacuum fluctuations automatically cancels on the right hand side of equation (4.19) [54]. The full theory is contained in either equations (4.17) and (4.19) or equations (4.17) and (4.22). The full theory is not scale-invariant, because it contains the constraint (4.17), which manifests itself in the field equations by the presence of the fundamental length $\Lambda^{-1/2}$. However, the set of equations (4.19) are scale-free. Equation (4.22) is valid for any value of Λ which is an arbitrary constant of integration. The condition (4.17) does not determine the value of Λ, which must be determined by external conditions.

The ability to remove the scale dependence from the field equations is a consequence of the ability to reduce the metric tensor into two nontrivial geometric objects [3]: g the determinant of $g_{\mu\nu}$, and $\gamma_{\mu\nu}$ the relative tensor $g_{\mu\nu}/(\sqrt{-g})^{1/2}$ of determinant -1. The determinant determines entirely the measure structure of space-time, while the relative tensor alone determines the null-cone or causal structure. In unimodular relativity, the irreducible relative tensor $\gamma_{\mu\nu}$ is the fundamental geometric object of space-time. The metric tensor:

$$g_{\mu\nu} = (\sqrt{-g})^{1/2}\gamma_{\mu\nu}, \tag{4.23}$$

is treated as an artificial construct of two independent entities, the fundamental object $\gamma_{\mu\nu}$ and the measure field $\sqrt{-g}$. The measure field is only included in the formulation of the action principle in order to maintain general covariance. Because of the constraint (4.17), the invariance group of unimodular relativity is the subgroup of the Einstein group with unit determinant:

$$\det\left|\frac{\partial x'^{\mu}}{\partial x^{\nu}}\right| = 1. \tag{4.24}$$

(See Reference [2] for a lucid discussion of the terms "invariance" and "covariance" as they are used here.)

4.3 Geometry and Space-Time Measurements

The bifurcation of general relativity into two independent parts suggests a new way of looking at the connection between geometry and space-time measurements. In general relativity, actual space-time is represented geometrically by a Riemannian manifold $\mathcal{R}$: there exists a transparent correspondence between geometrical quantities on the one hand and physical space-time measurements on the other hand. The square of the length of an arbitrary vector

A^μ in $\mathcal{R}$ is:

$$A^2 = g_{\alpha\beta} A^\alpha A^\beta, \tag{4.25}$$

and the change of an arbitrary vector A^μ under an infinitesimal displacement dx^α in $\mathcal{R}$ is:

$$dA^\mu = -\Gamma^\mu_{\rho\sigma} A^\rho dx^\sigma, \tag{4.26}$$

where $\Gamma^\mu_{\rho\sigma}$ is the Christoffel symbol of the second kind:

$$\Gamma^\mu_{\rho\sigma} = \frac{g^{\mu\alpha}}{2}\left(\frac{\partial g_{\alpha\rho}}{\partial x^\sigma} + \frac{\partial g_{\alpha\sigma}}{\partial x^\rho} - \frac{\partial g_{\rho\sigma}}{\partial x^\alpha}\right). \tag{4.27}$$

Equation (4.27) is obtained from the condition:

$$g_{\mu\nu;\lambda} = 0, \tag{4.28}$$

which follows from the requirement that the length of an arbitrary vector is preserved under parallel displacement in $\mathcal{R}$. Because we identify $\mathcal{R}$ with physical space-time in general relativity it follows that the quantity A^2 may be identified with the result of a physical space-time measurement, that is, a length or time interval. Furthermore, it follows that the parallel displacement of a vector A^μ in $\mathcal{R}$ may be equated with the transfer of the corresponding physical rods and clocks in actual space-time. Moreover, any generalization of the geometrical manifold $\mathcal{R}$ will presumably manifest itself as a generalization of the behavior of physical rods and clocks. These assumptions are fundamental to Einstein's theory.

Unimodular relativity, owing to the bifurcation of the metric tensor, admits a substructure to the manifold $\mathcal{R}$, and hence permits the introduction of another geometrical manifold that is not directly related to space-time measurements. According to unimodular relativity the metric tensor $g_{\mu\nu}$ is an artificial construct of two independent quantities, $\sigma(x)$ and $\gamma_{\mu\nu}$. From this viewpoint, the Riemannian manifold of general relativity is constructed by first determining the null-cone structure and then multiplying it by the measure structure. Therefore, we may define a geometry only associated with the null-cone structure that permits a correspondence with physical space-time measurements via the measure structure.

Let us rewrite equation (4.25) in terms of the quantities $\gamma_{\alpha\beta}$ and $\sqrt{-g}$ of unimodular relativity:

$$A^2 = (\sqrt{-g})^{1/2}\gamma_{\alpha\beta}A^\alpha A^\beta. \tag{4.29}$$

We define the "length":

$$a^2 \equiv \gamma_{\alpha\beta}A^\alpha A^\beta, \tag{4.30}$$

so that equation (4.29) becomes:

$$A^2 = (\sqrt{-g})^{1/2}a^2 = \sigma^{1/2}a^2. \tag{4.31}$$

We see that a physical space-time measurement is obtained by multiplying two independent quantities, $\sigma^{1/2}$ and a^2. In general relativity, both of these quantities are obtained from the geometrical manifold $\mathcal{R}$. However, since $\sigma(x)$ and $\gamma_{\mu\nu}$ are completely independent in unimodular relativity, we may construct a sub-geometry that is only associated with the fundamental object $\gamma_{\mu\nu}$. Only when we multiply the quantities $\gamma_{\mu\nu}$ by the measure structure do we recover the Riemannian manifold $\mathcal{R}$ of general relativity.

Thus, we define a manifold $\mathcal{M}$. On this manifold we define a metric tensor $\tilde{g}_{\mu\nu}$ and an affine connection $\tilde{\Gamma}^{\mu}_{\rho\sigma}$. We do not assume that $\tilde{g}_{\mu\nu}$ satisfies equation (4.28); thus, the quantities $\tilde{\Gamma}^{\mu}_{\rho\sigma}$ are not necessarily Christoffel symbols. Furthermore, the measure field $\sqrt{-\tilde{g}}$ is not identified with σ. A correspondence with physical space-time measurements may be obtained from $\mathcal{M}$ from the following procedure. We first determine from $\mathcal{M}$ the null-cone structure via the relationship:

$$\gamma_{\mu\nu} \equiv \frac{\tilde{g}_{\mu\nu}}{(\sqrt{-\tilde{g}})^{1/2}}. \tag{4.32}$$

Then, from these quantities we construct the Riemannian manifold $\mathcal{R}$. This is accomplished by multiplying the quantities $\gamma_{\mu\nu}$ by the measure structure $\sigma^{1/2}$:

$$g_{\mu\nu} = \sigma^{1/2}\gamma_{\mu\nu} = \sigma^{1/2}\frac{\tilde{g}_{\mu\nu}}{(\sqrt{-\tilde{g}})^{1/2}}. \tag{4.33}$$

We see that the null-cone structure defines a geometry that is once removed from the actual geometry of space-time, but nevertheless permits a correspondence with physical space-time measurements.

In the original formulation of unimodular relativity [3], Anderson and Finkelstein tacitly assumed that the manifold $\mathcal{M}$ was also a Riemannian manifold. However, a physical measurement defined in this manner admits a natural generalization, for the only geometrical quantity obtained from $\mathcal{M}$ is the scale-independent quantity a^2. Consequently, the choice of the scale of the metric tensor on $\mathcal{M}$ is arbitrary. Therefore, we may choose a Weyl manifold $\mathcal{W}$ for $\mathcal{M}$. The law of parallel displacement of an arbitrary vector A^{μ} in $\mathcal{W}$ is:

$$dA^{\mu} = -\tilde{\Gamma}^{\mu}_{\rho\sigma}A^{\rho}dx^{\sigma}, \tag{4.34}$$

where $\tilde{\Gamma}^{\alpha}_{\beta\gamma}$ is the Weyl affine connection:

$$\tilde{\Gamma}^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\beta\gamma} - \tilde{g}^{\sigma\alpha}[\tilde{g}_{\sigma\beta}\varphi_{\gamma} + \tilde{g}_{\sigma\gamma}\varphi_{\beta} - \tilde{g}_{\beta\gamma}\varphi_{\sigma}]. \tag{4.35}$$

$\Gamma^{\alpha}_{\beta\gamma}$ now represents the Christoffel symbol constructed from the quantities $\tilde{g}_{\mu\nu}$. The Weyl affine connection $\tilde{\Gamma}^{\alpha}_{\beta\gamma}$ follows from $\Gamma^{\alpha}_{\beta\gamma}$ by the substitution:

$$\partial_{\gamma} \rightarrow \partial_{\gamma} - 2\varphi_{\gamma}. \tag{4.36}$$

The vector φ_{μ} serves as the connection coefficient for the parallel displacement of length in $\mathcal{W}$:

$$dl = -\varphi_{\beta} dx^{\beta} l, \tag{4.37}$$

where l is the length of an arbitrary vector in $\mathcal{W}$. As long as physically measured quantities are associated with the manifold $\mathcal{W}$ via equation (4.33), the comparison of physical lengths at different points in space-time is an unambiguous procedure that is not to be confused with the comparison of vector lengths at different points in the manifold $\mathcal{W}$. Consequently, the identification $\mathcal{M} = \mathcal{W}$ is not incompatible with the existence of the well-defined electromagnetic spectrum observed from chemical elements and Einstein's objection does not apply to *this* use of a Weyl geometry. Both Weyl [60] and Eddington [15] envisioned that such a geometry could be constructed which is not immediately identifiable with actual space-time but could be associated with physical measurements. We see that unimodular relativity provides a natural framework for the realization of this vision.

Chapter 5

Local Scale Invariance and the Unification of Gravitation and Electromagnetism

We now show that the electromagnetic field can be introduced as a compensating gauge field that guarantees local scale invariance in unimodular relativity. There have been a number of scale-invariant theories of gravity proposed in the past. The first scale-invariant theory of gravity, due to Weyl [56, 57, 58, 59, 60], was also an attempt to incorporate electromagnetism into general relativity. Weyl's theory is based on an elegant generalization of Riemannian geometry that is covariant with respect to both coordinate transformations and local scale transformations. Since the action that produces Einstein's field equations is only invariant with respect to the former group, Weyl proposed a new action that is invariant with respect to the latter group as well. This, however, requires a Lagrangian quadratic in the curvature scalar, and therefore leads to field equations that are fourth-order differential equations. Consequently, Weyl's theory does not reduce to general relativity in the absence of electromagnetism. Furthermore, Einstein [18] showed that the reading of an atomic clock would depend on its prehistory according to Weyl's theory, which is in conflict with the well-defined electromagnetic spectrum observed from chemical elements. As a result, Weyl's theory was ultimately rejected. Years later, Dirac [14] (see also Canuto et al. [10]) revived Weyl's geometry in an attempt to reconcile general relativity with his Large Numbers hypothesis [13]. Dirac maintains second-order differential equations at the expense of introducing a new scalar field and avoids Einstein's objection with his postulate of a second metric, independent of the gravitational potentials, that determines the interval ds measured by an atomic apparatus. This theory belongs to a wider class of theories, named variable-gravity theories, that predict a time-dependent variation in the strength of the gravitational interaction. The advantages and drawbacks of such theories are reviewed by Wesson [55]. Other attempts at incorporating scale invariance in general relativity (see, for example, Hehl et al. [26]) have been motivated by developments in particle physics. Since approximate scale invariance has been observed in deep inelastic electron-nucleon scattering [6, 7] many believe, in accordance with grand unification, that gravitation must also exhibit approximate scale invariance at very high energies.

5.1 The Field Equations

The field equations of unimodular relativity (4.19) are the equations that determine the quantities $\gamma_{\mu\nu}$ corresponding to the special case $\mathcal{M} = \mathcal{R}$. These equations are globally scale-invariant and scale-free, and are furthermore independent of the measure equation. Therefore, we view these equations as the dynamical set of equations of a globally scale-invariant theory. This interpretation is further supported by the fact that equation (4.19) is traceless (see equation (4.7)).

We now demand local scale invariance of this globally scale-invariant theory. We replace the Ricci tensor $R_{\mu\nu}$ in the action (4.9) by the scale-invariant Ricci tensor $\tilde{R}_{\mu\nu}$ of Weyl's theory:

$$\tilde{R}_{\alpha\beta} = \frac{\partial\tilde{\Gamma}^{\rho}_{\alpha\beta}}{\partial x^{\rho}} - \frac{\partial\tilde{\Gamma}^{\rho}_{\alpha\rho}}{\partial x^{\beta}} + \tilde{\Gamma}^{\sigma}_{\alpha\beta}\tilde{\Gamma}^{\rho}_{\rho\sigma} - \tilde{\Gamma}^{\sigma}_{\alpha\rho}\tilde{\Gamma}^{\rho}_{\beta\sigma}, \tag{5.1}$$

where the quantities $\tilde{\Gamma}^{\alpha}_{\beta\gamma}$ are constructed from the metric tensor and the vector field φ_α according to equation (4.35). Under the transformation (4.6), φ_α transforms according to:

$$\varphi_\alpha \rightarrow \varphi_\alpha + \frac{1}{2}(\log\lambda)_{,\alpha} = \varphi_\alpha + \frac{1}{2}\frac{\epsilon_{,\alpha}}{\epsilon}, \tag{5.2}$$

where a comma denotes ordinary differentiation. Equation (5.2) guarantees the invariance of $\tilde{\Gamma}^{\alpha}_{\beta\gamma}$ under local scale transformations. Thus, φ_α may be considered a compensating gauge field that guarantees local scale invariance. Similarly, $\tilde{R}_{\alpha\beta}$ is an invariant under transformation (4.6) and the scalar curvature $\tilde{R}$:

$$\tilde{R} = R + 6(\varphi^\alpha\varphi_\alpha) - 6\varphi^{\alpha}_{;\alpha}, \tag{5.3}$$

transforms with Weyl weight -1. The free Lagrangian L_0 for the gauge field φ_μ is the lowest order covariant combination of the gauge potentials:

$$L_0 = -\frac{1}{16\pi}f_{\mu\nu}f^{\mu\nu}, \tag{5.4}$$

where $f_{\mu\nu} = \varphi_{\mu,\nu} - \varphi_{\nu,\mu}$ and the indices are raised with the metric $\tilde{g}^{\mu\nu}$. Consequently, the action is:

$$\int \left[R + 6(\varphi^\alpha\varphi_\alpha) - 6\varphi^{\alpha}_{;\alpha} - \frac{k}{16\pi}f_{\mu\nu}f^{\mu\nu}\right]\sqrt{-\tilde{g}}\,d^4x + I_M, \tag{5.5}$$

where k is a constant that transforms under a local scale transformation with Weyl weight $+1$.

We obtain the field equations for the quantities $\tilde{g}_{\mu\nu}$ by the usual variational method; however, the theory of unimodular relativity dictates that the only the traceless part of the Hamiltonian derivative must vanish. The components $\tilde{g}^{\mu\nu}$ themselves are varied independently subject only to the requirement that their variations $\delta g_{\mu\nu}$ as well as the variations of their first derivatives $\delta\left(\frac{\partial\tilde{g}^{\mu\nu}}{\partial x^{\lambda}}\right)$ vanish on the boundary of integration. Calculation of the Hamiltonian derivative of the action (5.5) is a little more complicated than in the Riemannian case because the covariant derivative of the metric tensor no longer vanishes. Instead, the following relationship holds:

$$\tilde{g}_{\mu\nu;\alpha} = 2\varphi_{\alpha}\tilde{g}_{\mu\nu}. \tag{5.6}$$

Therefore, the integral over $\delta R_{\mu\nu}$ does not reduce to a surface integral. Furthermore, one must keep in mind that the Riemann tensor is constructed from the Weyl affine connection:

$$\tilde{\Gamma}^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\beta\gamma} - \tilde{g}^{\sigma\alpha}[\tilde{g}_{\sigma\beta}\varphi_{\gamma} + \tilde{g}_{\sigma\gamma}\varphi_{\beta} - \tilde{g}_{\beta\gamma}\varphi_{\sigma}], \tag{5.7}$$

and not the Christoffel connection $\Gamma^{\alpha}_{\beta\gamma}$.

Variation of the third term in (5.5) gives:

$$\int \kappa T_{\mu\nu}\delta\tilde{g}^{\mu\nu}\sqrt{-\tilde{g}}\, d^4x, \tag{5.8}$$

where $T_{\mu\nu}$ is the stress-energy tensor of matter.

Variation of the fourth term in (5.5) gives:

$$\int kT^{(EM)}\delta\tilde{g}^{\mu\nu}\sqrt{-\tilde{g}}\,d^4x, \tag{5.9}$$

where $T^{(EM)}_{\mu\nu} \equiv \frac{1}{8\pi}\left(f_{\mu\alpha}f_{\nu}{}^{\alpha} - \frac{1}{4}\tilde{g}_{\mu\nu}f^{\alpha\beta}f_{\alpha\beta}\right)$ is proportional to the stress-energy tensor of the electromagnetic field.

Setting the traceless part of the Hamiltonian derivative to zero gives:

$$\tilde{R}_{\mu\nu} - \frac{1}{4}\tilde{R}\tilde{g}_{\mu\nu} = kT^{(EM)}_{\mu\nu} + \kappa\left(T_{\mu\nu} - \frac{1}{4}g_{\mu\nu}T\right). \tag{5.10}$$

These field equations are invariant under local scale transformations (4.6). They are similar, but not identical to the Maxwell-Einstein system of equations. The correction terms are on the order of the cosmological constant. Bergmann and Einstein [5] have examined the set of equations: $\tilde{R}_{\mu\nu} - \frac{1}{2}\tilde{g}_{\mu\nu}\tilde{R} = 0$, and found that its solutions do not satisfy reasonable boundary conditions. However, they identified $\tilde{g}_{\mu\nu}$ with the scale-dependent metric tensor of space-time and also failed to include the term $kT^{(EM)}_{\mu\nu}$.

The field equations for the quantities φ_μ are obtained by a variation of φ_μ in (5.5). This produces Maxwell's free-field equations:

$$f^{\mu\nu}{}_{;\nu} = 0, \tag{5.11}$$

only for the case of a vanishing cosmological constant. Therefore, we may identify φ_μ as being proportional to the electromagnetic four-vector potential. This gives a physical interpretation to the electromagetic gauge potentials. They introduce a non-integrability of length into the null-cone structure of space-time. Note the important condition that exact gauge invariance of electromagnetism is connected to a vanishing cosmological constant. This provides possible theoretical evidence that the cosmological constant is identically zero.

Chapter 6

Summary and Conclusions

Einstein's general theory of relativity unifies geometry and gravitation by treating space-time as a four-dimensional general Riemannian manifold. At its foundation rests the principles of equivalence and general covariance, which were motivated by Einstein's reflections concerning mass and acceleration. Immediately after the advent of general relativity Einstein turned his attention to electromagnetism, hoping to incorporate the electromagnetic potentials ϕ_μ into the space-time geometry as well. As a result, the unification program was launched. This program subsequently attracted the attention of many of the leading mathematicians and physicists of the time. Einstein was a major force behind this program, contributing his own original research in addition to critical examinations of other theories. Unfortunately, no principles analogous to the principles of equivalence and general covariance suggested how to generalize Riemannian geometry in order to account for the electromagnetic field. This was due to the lack of a clear physical picture for charged phenomena. Hence, all attempts to incorporate electromagnetism into the geometry of space-time were guided primarily by mathematical considerations.

As we have seen, in the work of Weyl, Kaluza and Klein, Eddington, Einstein, and Schrodinger, the problem was approached by first identifying basic assumptions of four-dimensional general Riemannian geometry and then by relaxing these restrictions. For example, Weyl suggested that the conservation of length under the parallel displacement of vectors is an unnecessary assumption in a "true infinitesimal geometry" and consequently relaxed this restriction in order to develop a new non-Riemannian geometry. Excellent discussions of Weyl's theory are in Adler et al. [1], Pauli [43], and Weyl's own book [60]. Kaluza and Klein both took the bold step of increasing the dimension of space-time to five dimensions, thus relaxing the restriction of a four-dimensional manifold. The interested reader may find the collection of articles by Appelquist et al. [4] very informative. Eddington, Einstein, and Schrodinger developed purely affine theories and attempted to formulate field equations under a minimum number of assumptions for the action principle. In this approach the symmetry properties of the Riemann curvature tensor, the metric tensor, and the affine connections were relaxed in a variety of ways. Schrodinger's book [52] is an excellent exposition of this approach, outlining clearly the fundamental assumptions associated with general relativity and Riemannian geometry and its subsequent non-symmetric generalizations. While all of these attempts were mathematically intriguing, no single theory satisfactorily incorporated the electromagnetic potentials into the four-dimensional space-time geometry.

This problem remains unsolved today. To be sure, modern gauge theory has provided some insight into the fundamental structure of the two fields. According to modern gauge theory, the gravitational and electromagnetic fields are both gauge fields, meaning their existence can be tied to a local symmetry group. However, the symmetry group of the electromagnetic field is the group of phase transformations of the wave function, a group that does not enjoy a space-time interpretation, while the symmetry group of the gauge theory of gravity is the Poincaré group, the fundamental symmetry group of space-time. Therefore, the gravitational and electromagnetic fields remain unconnected even to this day, despite the great insight gained from modern gauge theory

In this investigation we have returned to the original problem of unification. We began with a careful examination of global scale invariance in general relativity. We discovered that coordinate invariance and scale invariance are fundamentally incompatible in the gravitational action. In Weyl's original theory [60], invariance under *local* scale transformations was imposed on the action. However, since the gravitational action itself is not even invariant under the group of *global* scale transformations because the quantities $\sqrt{-g}$ and R transform with Weyl weights $+2$ and -1, respectively, Weyl was forced to reformulate general relativity and consider Lagrangians quadratic in the curvature tensor. This led to field equations of the fourth differential order. Years later, Dirac [14] revived Weyl's geometry and introduced a new scalar field that transforms under scale transformations with Weyl weight -1. He also introduced a second metric, independent of the gravitational metric, that determines the interval ds measured by an atomic apparatus. In this way, he maintained second-order differential equations and avoided Einstein's objection. However, Dirac's theory has enjoyed only limited success [55].

In the same tradition of Weyl and Dirac, we have introduced a new method of incorporating local scale invariance into general relativity. Unlike Weyl and Dirac, however, we do not demand full local scale invariance of the action. Rather, we only demand that the theory of the null-cone structure of space-time is locally scale invariant. This is accomplished by reviving the theory of unimodular relativity developed by Anderson and Finkelstein [3], which is a modified version of general relativity. The field equations of unimodular relativity only determine the null-cone structure of space-time, leaving the measure structure determined by independent, external conditions. These field equations are equivalent to Einstein's equations with a cosmological constant. In unimodular relativity, however, the cosmological constant enters as an arbitrary integration constant and not as a predetermined coefficient in the action.

The authors' original intent in their theory of unimodular relativity was to formulate a theory of gravity that treated the irreducible relative tensor $\gamma_{\mu\nu}$ as the fundamental geometrical object of space-time. This was motivated by the observation that the metric tensor $g_{\mu\nu}$ is reducible into two irreducible objects, the relative tensor $\gamma_{\mu\nu}$ and the determinant g. **However, we have observed that the properties of the unimodular field equations point beyond the original intent of its authors**. Indeed, the equations of unimodular relativity *in the presence of matter* are globally scale-invariant and scale-free, in contrast to the field equations of general relativity, which are globally scale-invariant and scale-free only in the absence of matter. Therefore, one may proceed in the usual manner of gauge theory and demand local scale invariance of the unimodular field equations. Consequently, the electromagnetic field may be viewed as a compensating gauge field that guarantees local scale invariance in unimodular relativity. This procedure avoids the fourth-order differential equations that plagued Weyl's theory and avoids the introduction of Dirac's new scalar field.

We have seen that the space-time manifold $\mathcal{R}$ of general relativity becomes an artifical construct defined from the fundamental manifold $\mathcal{M}$ that we introduced for unimodular relativity. The manifold $\mathcal{M}$ may be identified with the manifolds defined by Weyl and Eddington in their attempts to respond to Einstein's objection to Weyl's theory because the only quantity obtained from the fundamental manifold $\mathcal{M}$ is the scale-independent quantity a defined by equation (4.30). Thus, the scale of the metric $\tilde{g}_{\mu\nu}$ on $\mathcal{M}$ is arbitrary and consequently permits the generalization of geometry first introduced by Weyl. Note, in our theory, $g_{\mu\nu}$ is still identified with the gravitational potentials and the geometry of space-time. The quantities $\tilde{g}_{\mu\nu}$, on the other hand, are only identifiable with the geometry of space-time via equations (4.23) and (4.32).

We contend that general relativity is an artifical melding of two independent theories. A theory that determines the null-cone structure of space-time and a theory that determines the measure structure of space-time are combined naturally by the mathematical formalism of general relativity chosen by Einstein. However, this union may have hidden an important substructure that prevented the incorporation of local scale invariance first envisioned by Weyl. Indeed, the failure of Weyl's theory may be attributed to the fact that the union of these two theories, the null-cone theory and the measure theory, is not globally scale-invariant, while the theory defining the null-cone structure alone is globally scale-invariant. Only when general relativity is separated into its two fundamental parts may local scale invariance be imposed. Once this is imposed the electromagnetic field emerges as a gauge field. Then, and only then, may the theories be combined into the unified formalism of general relativity.

We emphasize that this work adopts a new approach in the unification of gravitation and electromagnetism. We have not generalized Einstein's theory of general relativity; we have generalized unimodular relativity, a *reformulation* of general relativity that differs from Einstein's orginal theory only by the cosmological constant term. On the solar system scale unimodular relativity and general relativity are indistinguishable. We hope that this work stimulates other efforts in this direction, for different formulations of general relativity that are indistinguishable on the solar system scale may possess different advantages for incorporating electromagnetism into the space-time manifold. Rosen's bimetric reformulation of general relativity and Brans-Dicke theory, for example, are to some extent indistinguishable from general relativity and may also be amenable to electromagnetic generalizations. This is the reason they have been included in our discussion.

However, there is an interesting connection between our theory and Brans-Dicke theory because a new solution to the problem of Mach's principle in general relativity emerges. The source of the measure field $\sigma(x)$ was not specified in the original paper [3]; its value was simply provided by an external condition. We postulate that the source of the measure field is the background mass distribution of the distant stars, and consequently the tensor $T_{\mu\nu}$ only represents local matter. Note that in Einstein's original formulation of general relativity there is no distinction between a local mass distribution, such as a planet or the sun, and the background mass distribution of the distant stars. The entire matter content of the universe is contained in the matrix $T_{\mu\nu}$. However, if we postulate that the distant stars are the source of the field $\sigma(x)$ then such a distinction can be made. According to this postulate, distant matter would determine the measure structure of space-time and local matter would detemine the null-cone structure of space-time. Since the volume of space-time within the interval d^4x is $\sigma(x)d^4x$, the very existence of the volume element would then be tied into the boundary conditions defined by the distant stars.

This postulate solves an important problem concerning Mach's principle in general relativity. Mach's principle states that inertia cannot be defined relative to absolute space, but must be defined relative to the entire matter content of the universe. As is well known, one of the reasons Einstein [17] introduced the cosmological constant into the gravitational field equations was to accommodate Mach's principle. Einstein hoped that his reformulation of the field equations with a cosmological constant would eliminate solutions in the absence of mass, giving $g_{\mu\nu} = 0$ when $T_{\mu\nu} = 0$. Soon afterward, de Sitter [12] showed that this was not the case; a solution with a non-zero $g_{\mu\nu}$ existed even in the absence of matter. However, if the above postulate is adopted then de Sitter's solution is not in conflict with Mach's principle, for then the condition $T_{\mu\nu} = 0$ would only indicate the absence of a local mass distribution. The existence of de Sitter's solution would then be connected to a non-zero $\sigma(x)$, which would presumably vanish if the background mass distribution of the distant stars were to disappear. This solution to the problem of Mach's principle in general relativity is similar to that provided by Brans and Dicke [8], who supplied a scalar field in addition to the metric tensor. However, instead of adding an additional degree of freedom, we identify one of the ten degrees of freedom of the metric tensor as a scalar field connected to the boundary conditions of space-time.

In the limit that the measure field may be ignored, viz. small length scales, the full theory is scale invariant, in accordance with the belief that general relativity should exhibit approximate scale-invariance at high energies. In this limit, space-time possesses no volume: for time there is no duration, and for space there is no extension.

The theory outlined above may also provide insight into the strange behavior of quantum particles. As is well known, singularities of the field $g_{\mu\nu}$ traverse geodesics of the manifold, which are identified with the trajectories of material particles. However, according to the theory described above, this is just an approximation, for material particles should be viewed as singularities of the field $\gamma_{\mu\nu}$, not $g_{\mu\nu}$.

Bibliography

[1] R. Adler, M. Bazin, and M. Schiffer. *Introduction to General Relativity.* McGraw-Hill, New York, 1965.

[2] J. Anderson. *Principles of Relativity Physics.* Academic, New York, 1968.

[3] J. Anderson and D. Finkelstein. *AJP*, 39:901, 1971.

[4] T. Appelquist, A. Chodos, and P. Freund, editors. *Modern Kaluza-Klein Theories.* Addison-Wesley, Reading, 1987.

[5] P. Bergmann. *Introduction to the Theory of Relativity.* Prentice-Hall, Englewood Cliffs, 1966.

[6] J. D. Bjorken. *Phys. Rev.*, 163:1767, 1967.

[7] J. D. Bjorken. *Phys. Rev.*, 179:1547, 1969.

[8] C. Brans and R. Dicke. *Physical Review*, 124:925, 1961.

[9] G. Callegari and L. Baroni. *Acta Physica Hungarica*, 54:97, 1983.

[10] V. Canuto, P. J. Adams, S. H. Hsieh, and E. Tsiang. *Phys. Rev. D*, 16:1643, 1977.

[11] M. Carmeli. *Classical Fields: General Relativity and Gauge Theory.* John Wiley and Sons, New York, 1982.

[12] W. de Sitter. *Mon. Not. R. Astron. Soc.*, 78:3, 1917.

[13] P. A. M. Dirac. *Proc. R. Soc. Lond. A*, 165:199, 1938.

[14] P. A. M. Dirac. *Proc. R. Soc. Lond. A*, 333:403, 1973.

[15] A. E. Eddington. *The Mathematical Theory of Relativity.* Cambridge University Press, Cambridge, 1954.

[16] A. Einstein. *Ann. d. Physik*, 49:769, 1916.

[17] A. Einstein. *Sitz. Ber. Preuss. Akad. Wiss.*, K1:142, 1917.

[18] A. Einstein. *Sitz. Ber. Preuss. Akad. Wiss.*, K1:478, 1918.

[19] A. Einstein. *Sitz. Ber. Preuss. Akad. Wiss.*, K1:433, 1919.

[20] A. Einstein. *Nature*, 112:448, 1923.

[21] A. Einstein. *The Meaning of Relativity.* Princeton University Press, Princeton, fourth edition, 1945.

[22] A. Einstein and L. Infeld. *Ann. Math.*, 41:455, 1940.

[23] A. Einstein, L. Infeld, and B. Hoffman. *Ann. Math.*, 39:65, 1938.

[24] D. Lee et al. *Ap. J.*, 206:555, 1976.

[25] V. Fock. *The Theory of Space, Time, and Gravitation.* Pergamon Press, Elmsford, second edition, 1976.

[26] F. W. Hehl, J. D. McCrea, E. W. Mielke, and Y. Ne'eman. *Foundations of Physics*, 19:1075, 1989.

[27] F. W. Hehl, P. von der Heyde, and G. D. Kerlick. *Rev. Mod. Phys.*, 48:393, 1976.

[28] D. Howard and J. Stachel, editors. *Einstein and the History of General Relativity*, Boston, 1989. Birkhauser.

[29] L. Infeld and J. Plebanski. *Motion and Relativity.* Pergamon Press, London, 1960.

[30] J. D. Jackson. *Classical Electrodynamics.* John Wiley and Sons, New York, second edition, 1975.

[31] T. Kaluza. *Sitz. Ber. Preuss. Akad. Wiss. Berlin*, 54:966, 1921.

[32] T. W. B. Kibble. *J. Math. Phys.*, 2:212, 1961.

[33] O. Klein. *Zeit. f. Physik*, 37:895, 1926.

[34] O. Klein. *Nature*, 118:516, 1926.

[35] M. Kohler. *Z. Physik.*, 131:571, 1952.

[36] M. Kohler. *Z. Physik.*, 134:286, 1953.

[37] M. Kohler. *Z. Physik.*, 134:306, 1953.

[38] E. Leibowitz and N. Rosen. *J. Gen. Rel. and Grav.*, 4:449, 1973.

[39] L. O'Raifeartaigh. *The Dawning of Gauge Theory.* Princeton University Press, Princeton, 1997.

[40] J. M. Overduin and P. S. Wesson. *Phys. Rep.*, 283:303, 1997.

[41] A. Pais. *"Subtle is the Lord...": The Science and the Life of Albert Einstein.* Clarendon Press, Oxford, 1982.

[42] A. Papapetrou. *Proc. Roy. Irish Acad.*, A52:11, 1948.

[43] W. Pauli. *Theory of Relativity.* Pergamon Press, London, 1958.

[44] N. Rosen. *Phys. Rev.*, 57:147, 1940.

[45] N. Rosen. *Ann. Phys. (N.Y.)*, 22:1, 1963.

[46] N. Rosen. *J. Gen. Rel. and Grav.*, 4:435, 1973.

[47] N. Rosen. *Ann. Phys. (N.Y.)*, 84:455, 1974.

[48] N. Rosen. *Ap. J.*, 211:357, 1977.

[49] N. Rosen. *J. Gen. Rel. and Grav.*, 9:339, 1978.

[50] L. Ryder. *Quantum Field Theory.* Cambridge University Press, Cambridge, 1985.

[51] E. Schrodinger. *Nature*, 153:572, 1944.

[52] E. Schrodinger. *Space-Time Structure.* Cambridge University Press, Cambridge, 1950.

[53] R. Utiyama. *Phys. Rev.*, 101:1597, 1956.

[54] S. Weinberg. *Rev. Mod. Phys.*, 61:1, 1989.

[55] P. Wesson. *Gravity, Particles, and Astrophysics.* Reidel, Dordrecht, 1980.

[56] H. Weyl. *Sitz. Ber. Preuss. Akad. Wiss.*, page 465, 1918.

[57] H. Weyl. *Math. Z.*, 2:384, 1918.

[58] H. Weyl. *Ann. Phys. Lpz.*, 59:101, 1919.

[59] H. Weyl. *Phys. Z.*, 21:649, 1920.

[60] H. Weyl. *Space, Time, Matter.* Springer, London, 1922.

[61] C. Will and D. Eardley. *Ap. J. (Letters)*, 212:L91, 1977.

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