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Tinkertoys for Gaiotto duality

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Tinkertoys for Gaiotto duality

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DISSERTATION

Presented to the Faculty of the Graduate School of

The University of Texas at Austin

in Partial Fulfillment

of the Requirements

for the Degree of

DOCTOR OF PHILOSOPHY

THE UNIVERSITY OF TEXAS AT AUSTIN

August 2011

Dedicated to my parents, José and Sonia.

Acknowledgments

There are many people whose support to attain this new point in my academic life I want to acknowledge.

First, I wish to thank my parents and my sister, for having provided a comfortable, encouraging and positive environment for me. Without that basis, I would most definitely not be here right now.

In Peru, I want to thank Orlando Pereyra, my undergraduate advisor at UNI, for suggesting that I follow graduate studies in the United States, and for his continuous concern about my academic future. I also wish to thank Professor Holger Valqui, a legendary figure at UNI, for providing a role model for us to follow.

In Austin, I had the honor to learn from first-rank physicists and mathematicians, who have not only enriched my knowledge of these disciplines to a level currently impossible to find in Peru, but, more importantly, for having taught me, if not in a classroom, their very own philosophies and methods for doing research. First, I want to thank my advisor, Jacques Distler, for being an excellent friend, teacher, guide and collaborator; for sharing his tremendous insight, and for having encouraged me from the start to come up with my own ideas. I am also very especially grateful to Sonia Paban, for her active support, most of times beyond the call of duty, to help me improve my academic prospects. I also want to thank Vadim Kaplunovsky, from whom I learned the fundamentals of modern quantum field theory, often with insider knowledge impossible to find in the literature, which fill the pages of this dissertation (which doesn't mean he's to blame for any of my mistakes!). I also want to thank Willy Fischler for his objective, sometimes blunt, feedback and criticism in my first few talks at the Theory Group. Having a clear notion of the quality of physics that I was expected to produce is one of the things that most helped me to grow as a physicist. I wish to thank as well Professor Weinberg, with whom I interacted relatively little, but who I had the opportunity to observe closely in the series of cosmology and quantum field theory lectures that he taught, and some of whose insights and ways of thinking I believe to have grasped. I also want to express my gratitude to the Theory Group in general for providing financial support in various occasions.

In the Math department, I want to thank Dan Freed, for being a great geometry teacher and a paradigm of clarity. Enjoying teaching myself, I appreciated the time he took to prepare his lectures, and to communicate his vast knowledge clearly and effectively. I also wish to thank Andy Neitzke, who I met in my last year at UT, for collaboration, for having the patience to explain various points of his papers to me, for never dismissing the simple (and, in hindsight, often silly) questions I often ask, and for having kindly written a letter of recommendation for me despite our having met recently.

I am also indebted to Jan Duffy, for helping me countless times throughout the years, and for doing an excellent job at managing the Theory Group.

I wish to thank also Terry Riley and Abel Ephraim, for keeping everything in our offices running smoothly.

I also want to thank my fellow graduate students, postdocs and visitors at the Theory Group, Anindya, Arnab, Aswin, Dan, Dustin, Elena, Emanuela, Marcus, Matthias, Mohamed, Timm, Tom, Pavel, Sandipan and Walter, for their friendship and for providing continuous fodder for academic discussion.

On the more personal side, I want to thank my friends at UT, especially Rez Khan, for his help during my first years in the United States. I also want to thank my friends at the Abbey House; in particular, I am grateful to Rudy Bauss for consistently caring about my future, and for kindly offering to proofread this manuscript. I also want to thank the close-knit Peruvian community in Austin, for being a great family and a home away from home.

Finally, I wish to thank Mary, for being the greatest partner, friend, and study buddy, whose constant support I will always be grateful for.

Tinkertoys for Gaiotto duality

Publication No. ______________

Oscar Chacaltana Alarcón, Ph.D.
The University of Texas at Austin, 2011

Supervisor: Jacques Distler

We describe a procedure for classifying 4D $\mathcal{N}=2$ superconformal theories of the type introduced by Davide Gaiotto. Any punctured curve, C, on which the 6D $(2,0)$ SCFT is compactified, may be decomposed into 3-punctured spheres, connected by cylinders. The 4D theories, which arise, can be characterized by listing the "matter" theories corresponding to 3-punctured spheres, the simple gauge group factors, corresponding to cylinders, and the rules for connecting these ingredients together. Different pants decompositions of C correspond to different S-duality frames for the same underlying family of 4D $\mathcal{N}=2$ SCFTs. We developed such a classification for the A_{N-1} and the D_N series of 6D $(2,0)$ theories. We outline the procedure for general A_{N-1} and D_N, and construct, in detail, the classification through A_4 and D_4, respectively.

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Chapter 1

Introduction

Gaiotto duality [1–10] identifies a large class of 4D $\mathcal{N} = 2$ SCFTs with compactifications of the 6D $\mathcal{N} = (2, 0)$ SCFT on a punctured Riemann surface, C. The moduli space, $\mathcal{M}_{g,n}$, parametrizes the family of exactly-marginal deformations of the SCFT. For every pants-decomposition of C, there is an $\mathcal{N} = (2, 0)$ gauge-theoretic interpetation, in which each cylinder represents the $\mathcal{N} = 2$ vector multiplets for some (simple) gauge group, and the 3-punctured spheres represent some sort of “matter”, charged under the gauge groups of the attached cylinders. In particular, this construction identifies the boundaries of the moduli space, $\mathcal{M}_{g,n}$, with limits in which some, or all, of the gauge couplings become weak. Different degenerations correspond to different, S-dual, realizations of the same family of SCFTs.

Classifying the theories that arise, in this way, comes down to specifying (for a given 6D (2,0) theory) what all of the 3-punctured spheres are, what gauge groups are associated with the cylinders that connect them, and what are the the rules for gluing these ingredients together. Arbitrarily complicated 4D $\mathcal{N} = 2$ SCFTs can be constructed, in “tinkertoy” fashion, by connecting together these basic ingredients.

For a given (2,0) theory, this is a finite task. In [6], we carried out this program for theories that are obtained from a compactification of the (2,0) theories of type A_{N-1}. In so-doing, we identified a multitude of new interacting, non-Lagrangian SCFTs (generalizing [11]), corresponding to compactifications of the A_{N-1} theory on certain 3-punctured spheres. Their appearance, in the context of Gaiotto duality, is a vast generalization of the classic examples of non-Lagrangian SCFTs appearing in the S-dual description of more-familiar $\mathcal{N} = 2$ gauge theories, discovered by Argyres and Seiberg [12].

While Gaiotto's original arguments relied on the realization of the 6D theory as the low-energy theory of N M5-branes, which necessarily implied working with a 6D theory of A_{N-1} type, the idea can be straightforwardly generalized to the case of N M5 branes in the presence of an orientifold, whose low-energy limit is the 6D theory of type D_N. (There is, by contrast, no realization of the 6D theories of type E as a low-energy theory of M5 branes.) The class of 4D SCFTs arising from the compactification of the D_N 6D theories on Riemann surfaces has been considerably less studied [7–9] than its A_{N-1} analogue.

As for the A_{N-1} theories, the Seiberg-Witten curve of 4D theories arising from the D_N theories can be written in Gaiotto's form, as a polynomial equation in the Seiberg-Witten differential (a 1-form on T^*C), whose coefficients are (the pullbacks of) differentials on C. The differentials descend from protected operators of the 6D theory, and so their degrees are equal to the exponents of $Spin(2N)$.

Just as Gaiotto used the well-known $SU(n)$ linear quivers to test his arguments for the A_{N-1} theory, Tachikawa [7, 8] studied the SO-Sp linear quivers [13, 14] to find the pole structure and flavour symmetry group for punctures in the D_N theory, and discovered a few examples of S-duality. Unfortunately, the SO-Sp linear quivers linear quivers, that arise from the orientifold construction, live in a theory slightly larger than the one we are interested in. The A_{N-1}, D_N and E_6 theories have a $\mathbb{Z}_2$ outer-automorphism (which gets enhanced to S_3 in the case of D_4), and we can consider compactifications of the (2,0) theory, where going around a homologically-nontrivial cycle on C (circumnavigating a handle, or circling a puncture) is accompanied by an outer-automorphism twist.

A proper discussion of the incorporation of outer-automorphism twists should treat the A_{N-1}, D_N and E_6 (2,0) theories in tandem, as all of these Dynkin diagrams have a $\mathbb{Z}_2$ outer automorphism. Instead, in [10] we studied the compactifications of the D_N theory, *without* outer-automorphism twists, and developed a classification precisely analogous to the one we developed for the A_{N-1} theory (also without outer automorphism twists). Nonetheless, at a crucial point, we had recourse to Tachikawa's linear quiver tail analysis which, strictly speaking, embeds the D_N theories without outer automorphism twists in the larger class of D_N theories which *do* include outer automorphism twists.

The analysis in the D_N case introduces several new complications, not seen in the A_{N-1} case. In the A_{N-1} theory, each puncture corresponded to a choice of partition of N (equivalently, to an N-box Young diagram, or a nilpotent orbit in the complexified Lie algebra, $\mathfrak{sl}(N)$). The chosen partition determined the "flavour symmetry" group (essentially, the isometry group of the Higgs branch) associated to a given puncture. At the same time, it (or, more accurately, its transpose) determined the singular behaviour of the Hitchin system at the puncture which, in turn, gave the geometry of the Coulomb branch.

In the present case, that relationship is more complicated. As in the A_{N-1} case, the flavour symmetry group (geometry of the Higgs branch) is determined by a "D-partition" of 2N. Such partitions also label nilpotent orbits in $\mathfrak{so}(2N)$. However, only for a subset of these, the "special" D-partitions [15], is the behaviour of the Hitchin system at the puncture given by (the Spaltenstein dual) nilpotent orbit.

The Coulomb branch of the theory comprises the degrees of freedom associated to a set of meromorphic k-differentials on the Riemann surface which are allowed to have poles of certain orders (determined by the choice of partition) at the punctures. A new feature, of the D_N case, is that the coefficients of the leading poles of these differentials obey certain polynomial constraints. The "true" Coulomb branch is obtained, after imposing the constraints.

These constraints were derived by Tachikawa [7], by considerations involving linear quiver tails. We will present a slightly different, more intrinsic, viewpoint on the origin of these constraints. For the special partitions, we will see that the constraints pop out naturally from requiring that the Higgs field have a simple pole with residue lying in the Spaltenstein-dual nilpotent orbit. For the non-special partitions, our results are less satisfactory. We can determine (using the linear quiver tail analysis) the pole structure of the k-differentials at the puncture, and the associated constraints. But we do not, currently, know how to express this as a boundary condition of the Hitchin system.

A further peculiar feature of the non-special punctures is that the global symmetry group of the puncture contains $Sp(l)_k$ factors, with k odd. This level for the current algebra is that which would be induced by an odd number of half-hypermultiplets in the fundamental $2l$-dimensional representation. In other words, this symmetry is subject to Witten's global anomaly [16] and (in the absence of additional matter) could not be consistently gauged.

Even after having dealt with these new complexities, simply enumerating the *results* in the D_N case is considerably more tedious than it was in the A_{N-1} case. The number of fixtures (3-punctured spheres), and the number of cylinders that connect them, proliferate much more rapidly with N.

We will restrict ourselves to presenting a complete catalogue only for D_4. As a measure of the complexity, there are 99 3-punctured spheres for D_4; we will list all of those. There are 785 4-punctured spheres — theories with a single gauge group factor — it would be prohibitive to list all of those.

Nevertheless D_4 is an interesting case to study. As already mentioned, the outer automorphism group is enhanced to S_3. This group is a symmetry of the D_4 (2,0) theory, and so acts on the set of punctures/fixtures/cylinder, which are naturally organized into multiplets, permuted by the outer automorphisms. As already mentioned, we will *not* consider the inclusion of outer-automorphism *twists*.

For the D_5 and D_6 theories, we will present tables of the regular punctures and their properties, but will refrain from presenting a complete catalogue of fixtures and cylinders.

As in the A_{N-1} series, we discover several new interacting SCFTs — non-Lagrangian fixed points of the renormalization group — and we realize a number of S-dualities predicted by Argyres and Wittig [17]. We also provide formulæ for the conformal-anomaly central charges a, c, and explain how to compute the flavour current-algebra charges k, for interacting SCFTs.

Chapter 2

The $(2,0)$ theories

2.1 Basics

The $\mathcal{N} = (2,0)$ theories [18–24] are maximally superconformal, intrinsically interacting, non-gravitational theories in six dimensions. These theories were initially constructed by Witten in [18] as low-energy limits of IIB string theory compactified on a K3 surface, where the K3 is at a singular point in its moduli space. The resolution of these singularities requires the introduction of exotic massless degrees of freedom, namely tensionless strings. Thus, the (2,0) theories are theories of non-gravitational tensionless strings. Since the K3 moduli-space singularities obey an A-D-E classification, there exist (2,0) theories corresponding to each of the simply-laced Dynkin diagrams: the A_{N-1} series, the D_N series, and the exceptional E_6, E_7, and E_8. There exist no (2,0) theories associated to the non-simply-laced Dynkin diagrams. As we will review shortly, in addition to the interacting A-D-E (2,0) theories, there exist also a free (2,0) theory. The most general (2,0) theory is a tensor product of copies of A-D-E and free (2,0) theories.

The maximal superconformal symmetry in six dimensions has 16 supercharges [25], with superconformal group $OSp(2,6|2)$ in Lorentzian signature.

The bosonic part of the superconformal group is $\mathrm{Spin}(5,1) \times Sp(2)_R$. The 6D $\mathcal{N} = (2,0)$ supersymmetry is chiral, and 6D spinors are symplectic-Majorana-Weyl. The supercharges Q^a_α transform as a $\mathbf{4} \times \mathbf{4}$ of $\mathrm{Spin}(5,1) \times Sp(2)_R$, and the 6D (2,0) supersymmetry algebra is [26]

$$\{Q^a_\alpha, Q^b_\beta\} = 2\omega^{ab}\gamma^\mu_{\alpha\beta}P_\mu + \gamma^\mu_{\alpha\beta}Z^{ab}_\mu \tag{2.1}$$

where ω^{ab} is the Sp(2)-invariant tensor, Z^{ab}_μ is a central charge of the supersymmetry algebra, transforming in the $\mathbf{6} \times \mathbf{5}$ of $\mathrm{Spin}(5,1) \times Sp(2)_R$. Since Z is a vector of $Spin(5,1)$, the corresponding gauge field is a 2-form $B_{\mu\nu}$, which couples to the tensionless strings.

The (2,0) supersymmetry algebra has two massless representations: a tensor multiplet, and a gravity multiplet. Since the (2,0) theories are non-gravitational, we will only be interested in the tensor multiplet. The little group of $Spin(5,1)$ is $Spin(4) \simeq SU(2) \times SU(2)$. The degrees of freedom of the tensor multiplet transform as the

$$(1,3;\mathbf{1}) \oplus (1,1;\mathbf{5}) \oplus (1,2;\mathbf{4}) \tag{2.2}$$

of $SU(2) \times SU(2) \times Sp(2)_R$. The three terms in this expression represent a self-dual 2-form $B_{\mu\nu}$, 5 scalars, and 4 Weyl spinors, respectively.

2.2 M-theory picture and Coulomb branch

In addition to its IIB construction [1], the A_{N-1} series also allows for a definition as the low-energy worldvolume theory on N coincident M5- branes [19] in M-theory on $\mathbb{R}^{10}$. Separating the M5 branes corresponds to giving non-zero VEVs to the 5 scalars in the N tensor multiplets. One of these tensor multiplets corresponds to the center-of-mass mode of the N M5-branes, and can be decoupled.

The space parametrized by the scalars is the *Coulomb branch* of the (2,0) theory, which, as we will see, naturally descends to the more familiar Coulomb branch of 4D $\mathcal{N} = 2$ super Yang-Mills theory after compactification on a torus. For the A_{N-1} series, the Coulomb branch is

$$\mathcal{B} = (\mathbb{R}^5)^{N-1}/S_N. \tag{2.3}$$

Taking the low-energy limit, one gets N independent copies of the free (2,0) theory, or tensor multiplets. We see that the free (2,0) theory also has an M-theory interpretation, as the low-energy theory of a single M5 brane on $\mathbb{R}^{10}$.

Similarly, the D_N series can be defined as the low energy theory of $2N$ M5-branes on the singularity of an M-theory orientifold, $\mathbb{R}^5 \times \mathbb{R}^5/\mathbb{Z}_2$; here

[1]In IIB theory, before we decouple the 6D theory from gravity, there is a gravitational anomaly that vanishes only if there are 21 tensor multiplets present. However, the (2,0) theory that we are interested in is obtained after decoupling gravity, so there is no restriction on the number of tensor multiplets, and we are indeed allowed to consider, say, the A_{N-1} and D_N theories for arbitrary N.

the $\mathbb{Z}_2$ reflects the five coordinates transverse to the M5 branes. On the other hand, no M5-brane construction for the E-type (2,0) theories is known to exist.

More generally, the Coulomb branch $\mathcal{B}$ of a (2,0) theory associated to a simply-laced Lie algebra $\mathfrak{g}$ is [27]

$$\mathcal{B} = (\mathbb{R}^5)^{\text{rank}(\mathfrak{g})}/W_{\mathfrak{g}}, \tag{2.4}$$

where $W_{\mathfrak{g}}$ is the Weyl group of $\mathfrak{g}$.

Now, a more natural way to parametrize the Coulomb branch, instead of giving VEVs to the tensor-multiplet scalars, is to give non-zero VEVs to chiral primary operators of the (2,0) theory. These are operators whose scaling dimensions are protected by supersymmetry. Chiral primary operators are associated to the Casimirs of $\mathfrak{g}$, and so they have mass dimensions equal to the exponents of the Lie group associated to the Lie algebra $\mathfrak{g}$. For A_{N-1}, the exponents are $2, 3, 4, \ldots, N$. For the D_N series, they are $2, 4, 6, \ldots, 2N-2; N$. The last chiral operator of the D_N series is called the Pfaffian.

2.3 S-duality of 4D super Yang-Mills theory

It will be useful to review how the 6D (2,0) theory is the natural setting to describe S-duality of $\mathcal{N} = 4$ super Yang-Mills theory [18], as some aspects of Gaiotto duality will mimic this well-known example. Specifically, $\mathcal{N} = 4$ super Yang-Mills with *simply-laced* gauge group G is the low energy theory of a 6D (2,0) theory of type G compactified on a torus. Since the torus is flat, the low energy theory automatically preserves the original 16 supersymmetries of the (2,0) theory. Furthermore, the $OSp(2,6|2)$ superconformal group in 6D becomes the $PSU(2,2|4)$ superconformal group of 4D $\mathcal{N}=4$ super Yang-Mills. The modular parameter τ of the torus becomes the gauge coupling of the 4D theory,

$$\tau = \frac{\theta}{\pi} + \frac{8\pi i}{g^2}, \tag{2.5}$$

which has zero beta function, and is thus tunable.

The S-duality group of 4D $\mathcal{N}=4$ super Yang-Mills is generated by the following discrete symmetries

$$T : \tau \to \tau + 1$$

$$S : \tau \to -1/\tau$$

The first transformation corresponds to a shift in the theta angle, and the second to trading the theory with gauge group G and gauge coupling τ by the theory with gauge group ${}^LG$ and coupling $\tau' = -1/\tau$, where ${}^LG$ is the Langlands dual group. Thus, the S transformation exchanges weak and strong coupling . In the case of 4D theories obtained from 6D (2,0) theories by compactification on a torus, the gauge group G is simply laced, so, ignoring relatively inocuous $\mathbb{Z}_2$ quotients, we have ${}^LG \equiv G$. (See the last paragraph of this section for comments on the non-simply-laced case.)

The Coulomb branch of the 6D theory descends to the more familiar Coulomb branch of 4D $\mathcal{N}=4$ super Yang-Mills. The superconformal point sits at the origin of the Coulomb branch, and at a generic point, accessed by giving non-zero VEVs to the scalars in the vector multiplets, the gauge group gets broken to $U(1)^{\text{rank}(G)}$, while some photons acquire non-zero masses and become W-bosons.

In fact, the way in which the 6D point of view makes S-duality clear is most easily seen at a generic point on the Coulomb branch. In this case W-bosons and monopoles have finite masses. Specifically, as we compactify on a torus $S^1 \times S^1$, whose radii are R and R' respectively, the limit $R/R' \to 0$ corresponds to pinching one of the cycles of the torus, and equivalently, to the weakly coupled limit of the theory. Here the W-bosons acquire masses $\alpha \cdot \langle \Phi \rangle R$, where α is a root of G, and $\langle \Phi \rangle$ are the scalar VEVs, while monopoles have masses $\alpha \cdot \langle \Phi \rangle R'$. The invariance under electric-magnetic duality, $\tau \to -1/\tau$, is equivalent to exchanging the cycles of the torus, $R \leftrightarrow R'$.

We stressed above that we only get $\mathcal{N} = 4$ super Yang-Mills with simply-laced gauge group G by this procedure. How about non-simply-laced groups? To get these, one can introduce a twist line [28] wrapped around one of the cycles of the torus. This basically means that one sets an odd boundary condition for the fields as we loop around one of the torus cycles. The twist line has the effect of collapsing the Dynkin diagram of the Lie algebra for the gauge group, and we thus obtain a quotient of the gauge group by one of its outer automorphisms. Thus, we can get $\mathcal{N} = 4$ super Yang-Mills with non-simply laced gauge groups, i.e., Lie groups of the type B_N, C_N, G_2 and F_4. On the other hand, S-duality exchanges the cycle on which the twist line is wrapped. At the same time, Hence, 6D engineering allows us to get $\mathcal{N} = 4$ super Yang-Mills for both simply-laced and non-simply laced gauge groups.

2.4 Compactifications and defect operators

In this section we want to take a first look at the compactifications that will occupy us in the following chapters. See Figure 2.1. Compactifications on circles or tori, which are flat, always preserve all the supersymmetry, but compactifying on arbitrary manifolds, even if they are Riemann surfaces (again, the torus is the exception), will generically not preserve *any* supersymmetry at all. Thus, since we want to preserve some of the supersymmetry (usually half) after compactifying, we will compactify on a Riemann surfaces and impose a *twist*. The twist relevant to us will be reviewed later. So, in the diagram above, to go from the 6D (2,0) theory to 4D super Yang-Mills theory we do not need any twist, since the torus is flat. However, to go to a 2D N=(2,2) theory, we do need a twist. Also, a twist is crucial to go from the 6D (2,0) theory to the 4D $\mathcal{N} = 2$ SCFT, and to compactify 5D $\mathcal{N} = 2$ super Yang-Mills theory on a circle to obtain a 3D $\mathcal{N} = 4$ sigma model.

Generically, a twist corresponds to replacing the embedding of a subgroup of the bosonic symmetry group of the theory by a different one. Thus, there may exist more than one way of twisting. For instance, there are 3 ways to twist $\mathcal{N} = 4$ super Yang-Mills [29]. In our case we are interested in the so-called GL twist [30], relevant to geometric Langlands. In the more recent context of Gaiotto duality, i.e, 4D $\mathcal{N} = 2$ theories obtained from compactification of 6D $\mathcal{N} = (2, 0)$ theories, the appropriate twist has been written in $[1, 2, 31]$. We will explain the twist relevant to us in Section 5.1.1.

On the other hand, one of the most important ingredients in obtaining

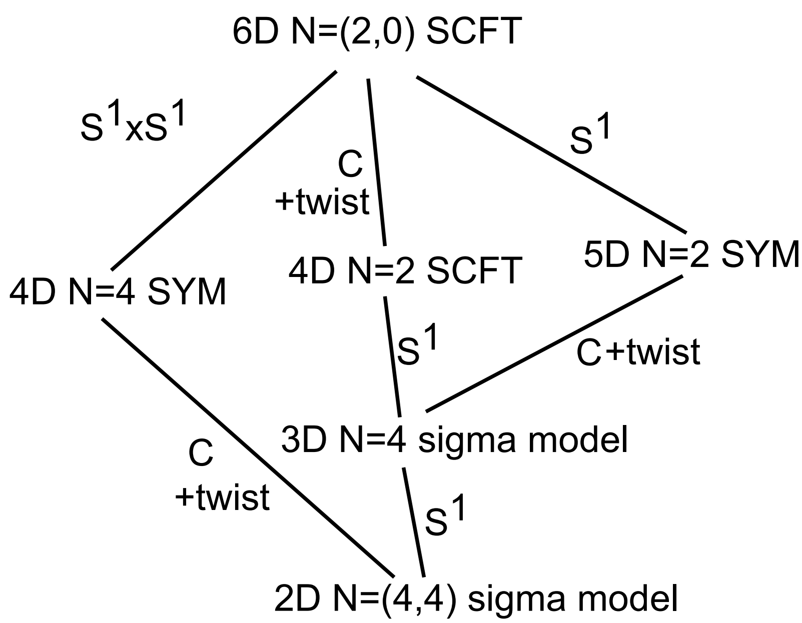


Figure 2.1: A roadmap of compactifications. While we are mainly interested in the 4D $\mathcal{N} = 2$ theories in the middle of the diagram, we will find the other compactifications to be very useful. Surface operators of 4D $\mathcal{N} = 4$ super Yang-Mills, which we will review in Chapter 3, are close relatives of the punctures on Riemann surfaces that appear in the 4D $\mathcal{N} = 2$ context. 5D $\mathcal{N} = 2$ super Yang-Mills will provide BPS equations, from which we will derive Hitchin's equations to describe the punctures. Finally, the 3D $\mathcal{N} = 4$ sigma model and the 2D $\mathcal{N} = (4, 4)$ sigma model share the same target space, which is furthermore equal to the Seiberg-Witten fibration for the 4D $\mathcal{N} = 2$ theory.

a large family of 4D $\mathcal{N} = 2$ theories will be to compactify the 6D $\mathcal{N} = (2,0)$ theory in the presence of a number of *codimension-two defect operators* [1, 32–34]. In fact, it will later become clear that if we compactified the 6D theory *without* any defect operators on a Riemann surface, we would obtain a much smaller class of 4D $\mathcal{N} = 2$[2], which are furthermore intrinsically interacting. The presence of the defect operators is what actually allows Gaiotto's procedure to yield standard 4D Lagrangian $\mathcal{N} = 2$ gauge theories.

Thus, in compactifying the 6D (2,0) theory on the torus to get 4D super Yang-Mills theory, the codimension-two (four-dimensional) defects of the 6D theory are wrapping the torus, so they descend to codimension-two (2-dimensional) defects of 4D super Yang-Mills. These are called *surface operators*, and we review them in Chapter 3. They can be defined by imposing a singular behavior of the fields on the support of the surface (as we will do in Chapter 3), or one can construct a 2D sigma model living on the defect, coupled to 4D $\mathcal{N} = 4$ super Yang-Mills.

Similarly, when we compactify the 6D (2,0) theory on a Riemann surface to obtain a 4D $\mathcal{N} = 2$ theory, the codimension-two defects of the 6D theory are wrapping the four-dimensional spacetime of the 4D theory, so they appear as a *puncture* on the Riemann surface.

Finally, when compactifying the 6D (2,0) theory to get 5D $\mathcal{N} = 2$ super Yang-Mills, we are wrapping the codimension-two defect on the S^1, so we are

[2]Namely, the so-called T_N theories, and surfaces constructed from them. In the language of the following Chapters, this is the same as Riemann surfaces with only maximal punctures.

left with codimension-two (three-dimensional) defects of 5D super Yang-Mills. One expects a 3D $\mathcal{N} = 4$ sigma model living on the defect, coupled to 5D super Yang-Mills[3].

[3]The first attempts to describe aspects of the 6D (2,0) theory in terms of 5D $\mathcal{N} = 2$ super Yang-Mills can be found in [35, 36].

Chapter 3

Surface operators

As we reviewed in Section 2.4, the compactification of a 6D $(2, 0)$ theory on a torus in the presence of a codimension-two (four-dimensional) defect operator, itself also wrapping the torus, leads to 4D $\mathcal{N} = 4$ super Yang-Mills theory in the presence of a *surface operator* [32–34, 37]. While we are actually interested in 4D $\mathcal{N} = 2$ theories, which are reached by a different compactification of the 6D (2,0) theories (namely, on an arbitrary Riemann surface), one of the main ingredients in the 4D $\mathcal{N} = 2$ story will be the *punctures* on the Riemann surface, which descend from the codimension-two defects of the 6D theory. We have no way to study the defects directly in the (2,0) setting, so surface operators are, for now, our only handle on them. Furthermore, we will see that these defects can be understood as singular boundary conditions for a Hitchin system in *both* contexts: 4D $\mathcal{N} = 4$ super Yang-Mills, and 4D $\mathcal{N} = 2$ theories. Thus, Hitchin's equations govern the defects in both pictures. Finally, statements about S-duality of surface operators, thoroughly studied by Gukov and Witten in [32, 33], may provide clues about certain not-well-understood $\mathcal{N} = 2$ punctures, namely the non-special punctures (which we will introduce in Chapter 5) and the sectors of punctures that are odd under outer automorphisms.

3.1 Definition

Surface operators [32–34, 37] are defined by specifying a singularity on a codimension-two submanifold of spacetime. In this sense, they are defined in a way analogous to 't Hooft line operators, rather than Wilson line operators. Also, the presence of a surface operator modifies the Hilbert space of the quantum theory, i.e., it restricts the evaluation of the path integral to fields that have the prescribed singularity. By contrast, a Wilson line operator modifies the integrand, by introducing the holonomy operator in it, instead of altering the Hilbert space.

Let us consider $\mathcal{N} = 4$ super Yang-Mills theory on $D \times C \simeq \mathbb{R}^4$, where D and C are both planes, isomorphic to $\mathbb{R}^2$. D will be the support of our surface operator. From the point of view of C, the surface operator will be located at the origin. We introduce the following coordinates on D and C:

$$D : x^0, x^1, \qquad C : x^2, x^3$$

A half-BPS surface operator preserves 2D (4,4) supersymmetry. A 4D $\mathcal{N} = 4$ vector supermultiplet decomposes into a (4,4) vector and hyper-multiplet. The hypermultiplet lives on the plane C. The surface operator is defined by demanding a singular behavior for the (4,4) hypermultiplet along D, i.e., at the origin of C. We choose the (4,4) hypermultiplet fields to be (A, ϕ), with

$$A = A_2 dx^2 + A_3 dx^3, \qquad \phi = \phi_2 dx^2 + \phi_3 dx^3, \tag{3.1}$$

and where A^2, A^3, ϕ_2, ϕ_3 are, respectively, gauge field and scalar components of an $\mathcal{N} = 4$ vector multiplet.

Thus, dimensionally reducing the $\mathcal{N} = 4$ super Yang-Mills BPS equations on C, we arrive at *Hitchin's equations* on C,

$$\begin{aligned} F_A - \phi \wedge \phi = 0, \\ d_A \phi = 0, \qquad d_A * \phi = 0, \end{aligned} \tag{3.2}$$

where $d_A = d + A$ is the covariant derivative, and F is the curvature of A.

It is reasonable to require solutions to also be rotation invariant on C. The most general ansatz compatible with rotation invariance is

$$A = a(r)d\theta + f(r)\frac{dr}{r}, \qquad \phi = b(r)\frac{dr}{r} - c(r)d\theta, \tag{3.3}$$

where we have introduced a complex coordinate in C, $x^2 + ix^3 = re^{i\theta}$. We eliminate the parameter $f(r)$ by a gauge transformation. Replacing this ansatz in Hitchin's equations, we get *Nahm's equations*:

$$\frac{da}{ds} = [b, c], \qquad \frac{db}{ds} = [c, a], \qquad \frac{dc}{ds} = [a, b] \tag{3.4}$$

where $s = -\ln r$.

We want to find solutions to (3.4) that preserve conformal symmetry. These should simply be independent of s. To satisfy the equations $[a, b] = [b, c] = [c, a] = 0$ one can take $a = \alpha$, $b = \beta$, $c = \gamma$, for any constant elements $\alpha, \beta, \gamma \in \mathfrak{t}$, where $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$. The solution is then

$$A = \alpha d\theta, \qquad \phi = \beta\frac{dr}{r} - \gamma d\theta. \tag{3.5}$$

Actually, in the path integral, the fields (A, ϕ) need only have this form *near* the singularity. Generically, these fields are allowed to have additional terms *less singular* than $1/r$. In Section 3.4 we will actually compute a limit $\alpha, \beta, \gamma \to 0$ to expose these less-singular terms, and define new surface operators associated to them.

One can also turn on another $\mathfrak{t}$-valued parameter, η, which is roughly a two-dimensional theta angle.

Also, in quantizing, one should divide by gauge transformations that, along D, take values in the subgroup L of G that commutes with $\alpha, \beta, \gamma, \eta$. The subgroup L, called *Levi subgroup*, always contains the maximal torus T; moreover, if we take generic $\alpha, \beta, \gamma, \eta$, we have $L \simeq T$.

So, instead of defining a conformal surface operator by $\alpha, \beta, \gamma, \eta$, we can also define it by a choice of Levi subgroup L, and then choose $\alpha, \beta, \gamma, \eta$ such that the subgroup of G that commutes with them is exactly L. This point of view has the convenience that it allows us to vary $\alpha, \beta, \gamma, \eta$ in a space of sets of matrices whose commutant in G is L.

For simplicity, we will set the theta angle η to be zero in what follows, and work only with α, β, γ.

3.2 Complex structures

The moduli space of *smooth* solutions of Hitchin's equations are well-known to have a hyper-Kahler structure. Similarly, the moduli space of *singular* solutions of Hitchin's equations with the *singular* behavior discussed above, also possesses a hyper-Kahler structure.

A hyper-Kahler structure means that there is a 2-sphere worth of possible complex structures for the moduli space of solutions of Hitchin's equations with our prescribed singularity. Different choices of complex structures provide a different point of view on Hitchin's equations. Let us see what this means more precisely.

First, we will describe the complex structure most important to us. In a certain, distinguished complex structure, which we call I, a solution of Hitchin's equations on a Riemann surface C describes a *Higgs bundle*. (A Higgs bundle is a pair (E, φ), where E is a holomorphic G-bundle and φ, called the *Higgs field*, is a holomorphic section of $K_C \otimes \mathrm{ad}(E)$, where K_C is the canonical bundle of C. Basically, the Higgs field φ is a global holomorphic function of C that takes values in the adjoint representation. In particular, φ is *not* gauge invariant.) In our case, this Higgs bundle is constructed from the fields (A, ϕ) of Hitchin's equations. We define an operator $\bar{\partial}_A$ as the (0,1) part of the covariant exterior derivative $d_A = d + A$. We use $\bar{\partial}_A$ to give the bundle E a holomorphic structure. On the other hand, the Higgs field φ is defined as the (1,0) part of ϕ. (Since ϕ is a 1-form, it decomposes as $\phi = \varphi + \bar{\varphi}$, where φ is of type (1,0) and $\bar{\varphi}$ is of type (0,1). Hitchin's equations then mean that φ is holomorphic, that the pair (E, φ) is a Higgs bundle, and that φ has a simple pole,

$$\varphi = \frac{1}{2}(\beta + i\gamma)\frac{dz}{z} \tag{3.6}$$

In a different complex structure, which we call J, the natural variable is instead the connection $\mathcal{A} = A + i\phi = (\alpha - i\gamma)d\theta$, which takes values in the complexified gauge group $G_{\mathbb{C}}$. In this complex structure, Hitchin's equations mean that $\mathcal{A}$ is a flat connection, whose monodromy around the singularity is

$$U = \exp(-2\pi(\alpha - i\gamma)) \tag{3.7}$$

Finally, the complex structure $K = IJ$ is qualitatively similar to J, and also describes a flat $G_{\mathbb{C}}$-connection.

3.3 Relation to 2D (4,4) sigma models

We can understand our derivation of Hitchin's equations in a different way [2]. Let C be now a Riemann surface instead of a plane, and let $D \times C$, with $D \simeq \mathbb{R}^2$ be 4D spacetime. To preserve supersymmetry after the compactification on a Riemann surface we need to perform the GL twist [30]. So, compactifying GL-twisted $\mathcal{N} = 4$ super Yang-Mills theory on C will yield a 2D (4,4) sigma model on D. The target space of the 2D sigma model is a hyper-Kahler manifold $\mathcal{M}$. Being a space of vacua, $\mathcal{M}$ can be identified with the space of solutions of the 4D BPS equations that are furthermore Poincaré invariant on the plane D. But we have seen in Section 3.1 that this procedure yields precisely Hitchin's equations. Thus, the target space $\mathcal{M}$ of the 2D (4,4) sigma model can be identified with the moduli space of solutions to Hitchin's equations. As we saw in Section 3.2, in the complex structure I, a solution to Hitchin's equations is a Higgs bundle, so we also say that $\mathcal{M}$ is the Hitchin moduli space of Higgs bundles.

When we deal with the 4D $\mathcal{N} = 2$ theories, we will arrive at $\mathcal{M}$ by a different path. Namely, we will compactify (twisted) 5D $\mathcal{N}=2$ super Yang-Mills on a Riemann surface, to obtain a 3D $\mathcal{N}=4$ sigma model with target space $\mathcal{M}$. The compactification of the 3D sigma model on S^1 yields the *same* 2D sigma model with target space $\mathcal{M}$ that we just found above. Notice that the 3D and 2D sigma models share the same hyper-Kahler target space $\mathcal{M}$.

3.4 $\alpha, \beta, \gamma \to 0$ limit

In the limit $\alpha, \beta, \gamma \to 0$, the solution to Nahm's equations does not become regular, but, rather, becomes less singular than $1/r$. Specifically, the Nahm solution becomes

$$a = -\frac{t_1}{s+\frac{1}{f}}, \qquad b = -\frac{t_2}{s+\frac{1}{f}}, \qquad c = -\frac{t_3}{s+\frac{1}{f}}, \tag{3.8}$$

where t_1, t_2, t_3 are the generators of a certain $su(2)$ embedding into the Lie algebra $\mathfrak{g}$. What $su(2)$ embedding appears depends on the values of α, β, γ. The generators t_i satisfy $[t_1, t_2] = t_3$, etc. Also, f is a non-negative constant, which we allow to fluctuate, as opposed to assigning a specific value to it, and we integrate it later in the path integral. With this caveat, this surface operator is conformally invariant.

So, a surface operator with parameters α, β, γ tends in the limit $\alpha, \beta, \gamma \to 0$ to a surface operator characterized by the Nahm solution above, for some f. Specifically, near $r=0$, the fields behave as

$$A = \frac{t_1}{\ln r} d\theta + \dots \tag{3.9}$$

$$\phi = \frac{t_2}{r \ln r} dr - \frac{t_3}{\ln r} d\theta + \dots \tag{3.10}$$

where the ellipses represent terms less singular than $1/\ln r$.

3.5 Monodromy

The flat connection $\mathcal{A} = A + i\phi$, which is valued in $\mathfrak{t}_{\mathbf{C}}$ (the complexification of the Lie algebra $\mathfrak{t}$ of the maximal torus T of G), is invariant under part of the supersymmetry preserved by the surface operator. Thus, the *conjugacy class* of the monodromy

$$U = P\exp\left(-\int_l \mathcal{A}\right) \in G_{\mathbf{C}} \tag{3.11}$$

is a supersymmetric observable. Here $G_{\mathbf{C}}$ is the complexification of G, and l is a contour surrounding the singularity. Hitchin's equations imply that $\mathcal{A}$ is flat. Thus, the conjugacy class of U is invariant under deformations of l.

For a surface operator with parameters α, β, γ, we have $\mathcal{A} = \xi d\theta$, where $\xi = \alpha - i\gamma \in \mathfrak{t}_{\mathbf{C}}$, and

$$U = \exp(-2\pi\xi) \in G_{\mathbf{C}} \tag{3.12}$$

The conjugacy class $\mathcal{C}_\xi$ of U above tends, in the limit $\xi \to 0$ to the union of two *unipotent* conjugacy classes,

$$\mathcal{C}_\xi \to \mathcal{C}' \cup \mathcal{C}_0 \tag{3.13}$$

Here $\mathcal{C}'$ is the unipotent class of matrices

$$\begin{pmatrix} 1 & 0 \\ w & 1 \end{pmatrix} \tag{3.14}$$

for non-zero w. On the other hand, $\mathcal{C}_0$ consists only of the unit matrix.

Instead of unipotent conjugacy classes of $G_{\mathbb{C}}$, we can express the above in terms of the Lie algebra $\mathfrak{g}_{\mathbb{C}}$. (We refer to [15] and references therein for an introduction to nilpotent orbits.) Specifically, the result above says that the boundary of a semisimple orbit contains two nilpotent orbits, one of which is the trivial nilpotent orbit, which consists just of the zero element. The other nilpotent orbit has actually the same dimension as the semisimple orbit. This is generic. Given a nilpotent orbit in $\mathfrak{g}_{\mathbb{C}}$, we may always compute its boundary to find a disjoint union of nilpotent orbits. The largest nilpotent orbit in this union has the same dimension as the original semisimple orbit. This nilpotent orbit is the correct $\alpha, \beta, \gamma \to 0$ limit of the semisimple orbit. In terms of Hitchin's equations, the Higgs field

$$\varphi(z) = \frac{X}{z} + \ldots, \tag{3.15}$$

should have X be in a semisimple or nilpotent orbit of $\mathfrak{g}_{\mathbb{C}}$.

3.6 S-duality

We are interested in understanding the action of S-duality on surface operators. Our working assumption will be that S-duality maps surface operators to surface operators. When S-duality exchanges weak and strong coupling, a surface operator in $\mathcal{N} = 4$ super Yang-Mills theory with gauge group G is expected to be mapped to a surface operator in $\mathcal{N} = 4$ super Yang-Mills theory with gauge group ${}^LG$. For a general S-duality transformation, if a surface operator is parametrized by $(\alpha, \beta, \gamma, \eta)$, we want to be able to compute the parameters $({}^L\alpha, {}^L\beta, {}^L\gamma, {}^L\eta)$ of the dual surface operator.

The S-duality group is generated by two transformations: 1) the electric-magnetic duality $S : \tau \to -1/n_{\mathfrak{g}}\tau$ (where $n_{\mathfrak{g}}$ is 1 for simply-laced G, and is otherwise 2 or 3), which exchanges strong and weak coupling, and 2) $T : \tau \to \tau+1$, which shifts the theta angle of the theory. Our plan will be to write down the actions of S and T separately, and then, at least for simply-laced G, write down the map for a general element of the S-duality group.

First, let us see how β and γ transform under S. Since the combination $\beta + i\gamma$ appears in the residue of the Higgs field φ, it is convenient to think first about how the scalar field ϕ in the 2D hypermultiplet transforms under S. Under this transformation, fields in the original theory do *not* map to fields in the dual theory; rather, gauge invariant quantities are mapped to gauge invariant quantities.

However, we can do the computation at a *generic* point of the Coulomb branch, where the gauge group G gets broken to an Abelian torus $\mathbb{T}$, and things simplify considerably. In this vacuum, the (remaining) scalar fields ϕ take values in $\mathfrak{t}$, while ${}^L\phi$ of the dual theory takes values in ${}^L\mathfrak{t}$. We take advantage of the fact that the Lie algebras $\mathfrak{t}$ and ${}^L\mathfrak{t}$ are dual as *vector spaces*, so choosing a Weyl-invariant metric directly provides a Weyl-invariant identification between them. We choose the metric in $\mathfrak{t}$ to be $\langle x, y\rangle = \mathrm{Tr}\ xy$, for $x, y \in \mathfrak{t}$. The S transformation then acts linearly on ϕ, which means that ${}^L\phi$ is a multiple of ϕ^*, where * stands for duality in the vector-space sense. Imposing that the kinetic energy of the scalars be preserved, we are able to find this map explicitly,

$$S : \phi \to |\tau|\phi^*. \tag{3.16}$$

From this expression, one can deduce that β and γ, both of which are in $\mathfrak{t}$, transform under S into their duals in ${}^L\mathfrak{t}$,

$$S : (\beta, \gamma) \to |\tau|(\beta^*, \gamma^*), \tag{3.17}$$

This last relation should be independent of the vacuum, so it should be true also at the superconformal point of the Coulomb branch, where the gauge group does not get broken to an Abelian group.

On the other hand, β and γ are not changed by the shift $T : \tau \to \tau + 1$. Thus, for β and γ, it is only the action of S that matters.

The effect of a general S-duality transformation, generated by S and T, on β and γ is easiest to write for a *simply-laced* gauge group G. In this case one can identify (β, γ) with their duals (β^*, γ^*), and so for a general $SL(2, \mathbb{Z})$ transformation $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we have

$$(\beta, \gamma) \to |c\tau + d|(\beta, \gamma). \tag{3.18}$$

Let us now see how α and η transform under S. Recall that $\alpha \in \mathbb{T}$ and $\eta \in {}^L\mathbb{T}$. Since S exchanges $\mathbb{T}$ and ${}^L\mathbb{T}$, one can guess that α and η will get exchanged by S. More precisely, since the transformation S^2 is a central element of the duality group Γ, one must have, up to sign,

$$S : (\alpha, \eta) \to (\eta, -\alpha). \tag{3.19}$$

It is harder to argue what the transformation rule is for a T. We direct the reader to Gukov-Witten [32, 33] for more details. Here we will simply write the answer:

$$T : (\alpha, \eta) \to (\alpha, \eta - \alpha). \tag{3.20}$$

The effect of a general S-duality transformation, again more simply written for simply-laced G, is

$$(\alpha, \eta) \to (\alpha, \eta)\mathcal{M}^{-1}, \tag{3.21}$$

where $\mathcal{M}$ is an element of $SL(2, \mathbb{Z})$.

Chapter 4

Gaiotto duality

In this Chapter we review relevant aspects of 4D $\mathcal{N} = 2$ SCFTs, which we will need to be familiar with to describe the intrinsic Hitchin picture in Chapter 5. In particular, we want to understand one of two pioneering examples of S-duality of $\mathcal{N} = 2$ SCFTs, which are known as *Argyres-Seiberg dualities* [12], and constitute our first examples of the more general Gaiotto duality. Our example involves a 4D $\mathcal{N} = 2$ SCFT with $SU(3)$ gauge group and $N_f = 6$ fundamental hypermultiplets. The fundamental region of the marginal gauge-coupling moduli space exhibits an infinitely strongly-coupled point. At this strongly-coupled point, a new weakly-coupled, S-dual picture emerges. In this example, the weakly-coupled S-dual theory involves an *interacting SCFT*, i.e., an isolated fixed point of the renormalization group, which has no conventional Lagrangian description and no gauge couplings, but which possesses a Coulomb branch and a conventional low-energy description in terms of a Seiberg-Witten curve. For the $SU(3)$ $N_f = 6$ theory, this weakly coupled S-dual frame is an SU(2) gauging of the interacting E_6 SCFT [11], coupled to one fundamental hypermultiplet. In this chapter we also familiarize ourselves with S-duality invariant quantities needed to identify our candidate S-dual pairs, which will be crucial tools to verify our predictions for S-duality.

4.1 Argyres-Seiberg duality

4.1.1 Strongly-coupled cusps of $SU(N)$ $N_f = 2N$

The 4D $\mathcal{N} = 2$ SCFTs with gauge group $SU(N)$ and $N_f = 2N$ fundamental hypermultiplets provide good examples of superconformal theories that can be described in Gaiotto's picture, as we will see in Chapter *A-series*. The case $N = 3$ is also the first of the two examples of Argyres-Seiberg duality [12], now understood to be a particular case of Gaiotto duality. Thus, it will be quite useful for us to study S-duality related aspects of this series of SCFTs and develop some intuition.

The 4D $\mathcal{N} = 2$ $SU(N)$ $N_f = 2N$ theories enjoy superconformal symmetry with a single marginal deformation, i.e., one gauge coupling

$$\tau = \frac{\theta}{\pi} + \frac{8\pi i}{g^2} \tag{4.1}$$

whose beta function is exactly zero. Thus, the gauge coupling τ is tunable. Just like $\mathcal{N} = 4$ super Yang-Mills theory, the $SU(N)$ $N_f = 2N$ theories have a discrete S-duality group, generated by the S and T transformations,

$$T : \tau \to \tau' = \tau + 1 \tag{4.2}$$

$$S : \tau \to \tau' = \begin{cases} -\frac{1}{\tau} \; (N = 2), \\ -\frac{1}{4\tau} \; (N \geq 3) \end{cases} \tag{4.3}$$

The S transformation takes us to a $SU(N)$ $N_f = 2N$ theory with gauge group ${}^{L}G = SU(N)$ (ignoring an inocuous quotient by a discrete group) and marginal gauge coupling τ'.

The S-duality groups for $N = 2$ and for $N \geq 3$ are thus different. For $N = 2$, the S-duality group is $SL(2, \mathbf{Z})$, whereas for $N \geq 3$ the S-duality group Γ_0 is strictly smaller,

$$\Gamma_0 = \left\{ \begin{pmatrix} a & 2k \\ c & d \end{pmatrix}, a, k, c, d \in \mathbf{Z}; ad - 2ck = 1 \right\} \subset SL(2, \mathbf{Z})$$

The existence of the S-duality group implies that the parameter τ actually lives in the *fundamental region* H/Γ, where H is the upper-half complex plane (since $g^2 > 0$), and Γ is the S-duality group, i.e., either $SL(2, \mathbf{Z})$ or Γ_0.

The fundamental region for $SU(2)$ $N_f = 4$ has only *one* cusp, at $\tau \to i\infty$, which corresponds to the point where the theory is weakly coupled. Hence, we never really have to deal with an infinitely coupled theory: S-duality transformations always allow us to go to finite τ.

The case $N \geq 3$ is what we are really after. In this case the fundamental region has *two* cusps: one is the weakly-coupled point similar to the one just discussed, and the other is an infinitely strongly-coupled cusp. The existence of the latter means that, at this point, we cannot appeal to S-duality to get a version of the theory with finite coupling.

For the specific case $N = 3$, i.e., the $SU(3)$ $N_f = 6$ SCFT, Argyres and Seiberg [12] proposed that this strongly-coupled cusp in gauge-coupling moduli space can be described in terms of a *different*, weakly-coupled theory, namely, an $SU(2)$ gauging of the interacting E_6 $\mathcal{N} = 2$ SCFT, coupled to one fundamental hypermultiplet. (See Section 4.1.2 for a quick review of interacting SCFTs.) And conversely, at the cusp where the $SU(2)$ theory becomes infinitely strongly coupled, the $SU(3)$ theory becomes weakly coupled, and so it is the more natural description of the underlying theory at this cusp. None of the two descriptions is more fundamental than the other; simply, there exists a different weakly-coupled description of the underlying theory at each cusp. We call each of these two weakly-coupled descriptions, corresponding to the two cusps in this example, *S-dual frames* of the underlying theory.

4.1.2 A first look at interacting SCFTs

Having stumbled upon an interacting theory, the E_6 SCFT, let us pause to recall a few facts about these theories [11]. By *interacting* or *non-Lagrangian* $\mathcal{N} = 2$ SCFTs we refer to superconformal theories that are *isolated fixed points* of the renormalization group, i.e., they do not possess a moduli space of marginal deformations, and so they are *not* our familiar superconformal gauge theories. In particular, they do not have a gauge group, gauge couplings, or known Lagrangian description. Still, interacting $\mathcal{N} = 2$ SCFTs enjoy many of the properties of Lagrangian SCFTs:

- Interacting SCFTs have a Coulomb branch parametrized by a set of (dimensionful) VEVs of relevant operators. The dimension of the Coulomb branch is also known as the *rank* of the interacting SCFT, by analogy with the rank of the gauge group in a Lagrangian SCFT. The superconformal point sits at the origin of the Coulomb branch, whereas, at a generic point, where the VEVs are not zero, the theory becomes asymptotically free. In particular, the low-energy theory at such a generic point of the Coulomb branch can be described by a Seiberg-Witten curve, just as in a Lagrangian theory.

- Interacting SCFTs have a *global symmetry group*, which is furthermore customarily used to label the theories. For instance, the E_6 SCFT in the previous paragraph has global symmetry E_6. In a Lagrangian theory, the global symmetry group rotates flavors of matter hypermultiplets that transform under the same representation of the gauge group. See also Section 4.5.1.

- Interacting SCFTs allow also for relevant deformations corresponding to *mass-deformation parameters* that break the global symmetry group to its maximal torus. In a Lagrangian theory, these mass deformations would give masses to the matter hypermultiplets.

- As we will see in Section 4.5.2, each non-abelian subgroup of the global symmetry group of a 4D $\mathcal{N} = 2$ SCFT, including interacting SCFTs, has a *central charge* k, which is related to OPEs of flavor currents corresponding to this subgroup. In practice, if we gauge a subgroup G of the global symmetry group of the SCFT, k is related to the contribution of the SCFT to the beta function of the gauge coupling associated to G. By analogy with the contribution of matter hypermultiplets in a certain gauge-group representation to the beta function, we can use an interacting SCFT as "matter", and couple it to hypermultiplets or to other interacting SCFTs via a gauge group.

- Also in Section 4.5.2, we will see that interacting SCFTs, like any other 4D $\mathcal{N} = 2$ SCFT, have *anomaly coefficients* a and c, which are obtained from the VEV of the energy-momentum tensor of the theory, when put in a gravitational background.

Thus, in the introduced nomenclature, the E_6 theory has a one-dimensional Coulomb branch (i.e., it is rank-one theory) parametrized by a VEV of dimension 3; a central charge $k = 6$ for the E_6 global symmetry group; and anomaly coefficients $a = 41/24$ and $c = 13/6$. It is also conventional to specify the central charge k of each non-abelian factor G of the global symmetry group, in the form $(G)_k$, to label an interacting theory; in this case, we refer to this theory as the $(E_6)_6$ interacting SCFT. We will see how to compute all the quantities mentioned in this paragraph in Gaiotto's picture in Section 4.5 and Chapters 6 and 7.

Similarly, Minahan and Nemeschansky [11] found $(E_7)_8$ and $(E_8)_{12}$ interacting SCFTs. All these theories have rank one. See Table 4.1.

$(G_{global})_k$	Coulomb branch dimensions	(a, c)
$(E_6)_6$	3	(41/24, 13/6)
$(E_7)_8$	4	(59/24, 19/6)
$(E_8)_{12}$	6	(95/24, 31/6)

Table 4.1: Properties of the Minahan-Nemeschansky interacting SCFTs.

The $(E_7)_8$ theory figures in the second example of Argyres-Seiberg duality, which involves an $Sp(2)$ gauge theory with $N_f = 6$. In this case, the fundamental region of gauge-coupling space has again two cusps, one corresponding to a Lagrangian $Sp(2)$ gauge theory with $N_f = 6$, and the other corresponding to an $SU(2)$ gauging of the $(E_7)_8$ interacting SCFT coupled to no matter. We refer to Argyres and Seiberg [12] for more details.

4.1.3 Checks of Argyres-Seiberg duality

Let us review some of the checks provided in [12] to prove the claims of S-duality. First, the $SU(3)$ gauge theory has rank 2; the SU(2) gauging of the $(E_6)_6$ theory also has rank 2 because the $SU(2)$ gauge group has rank 1 and the $(E_6)_6$ theory has rank 1 as well. Furthermore, the Coulomb branch of the $SU(3)$ gauge theory is parametrized by VEVs that are Casimirs of SU(3), and so their mass dimensions are equal to the exponents of $SU(3)$, i.e., 2 and 3. On the other hand, the Coulomb branch for the $SU(2)$ gauge group is parametrized by a VEV of dimension 2, while the $(E_6)_6$ SCFT has a Coulomb branch parametrized by a VEV of dimension 3.

Similarly, the global symmetry group of the $SU(3)$ gauge theory is a $U(6)$ that rotates the six fundamental hypermultiplets. On the S-dual side, we have a $SU(6)$, which is the commutant of the gauged $SU(2)$ in the original global symmetry group E_6 of the $(E_6)_6$ SCFT, and we have an additional $U(1)$ that rotates the fundamental hyper coupled to the $(E_6)_6$ theory.

Also, the gauge coupling of the $SU(3)$ gauge theory is marginal. To see that the $SU(2)$ theory also has a marginal coupling, one must compute the contribution of the $(E_6)_6$ SCFT to the beta function. This contribution is given by the central charge k, which we will discuss in Section 4.5.2. Here we just need to know that k can be computed using group theory, and that the result is that, indeed, the coupling of the $SU(2)$ theory is marginal.

Next, the two S-dual theories can be compared by studying the low-energy theories at generic points on the moduli space. First, one can go to a generic point on the Coulomb branch. The low-energy theory is described by a Seiberg-Witten curve. First, the Seiberg-Witten curve for the $SU(3)$ $N_f = 6$ theory at any point in the gauge-coupling moduli space is known. One can evaluate this expression at the point where the $SU(3)$ theory becomes very strongly coupled. At this point, the $SU(2)$ theory should become very weakly coupled. And indeed, at this point one finds the Seiberg-Witten curve for the $(E_6)_6$ SCFT.

This check of the Seiberg-Witten curves on both sides of the duality can be done again considering mass deformations. This involves breaking the global symmetry group of the theory to its maximal torus. At the level of the Seiberg-Witten curves, one typically finds mass deformations in the form of Casimirs of the global symmetry group. The mass-deformed Seiberg-Witten curves for both the $SU(3)$ $N_f = 6$ and the $(E_6)_6$ SCFT are known. So, in [12], one indeed checks that the $(E_6)_6$ Seiberg-Witten curve arises in the very-strongly coupled limit of the Seiberg-Witten curve for the $SU(3)$ gauge theory.

Two mutually-related S-duality invariants not discussed in [12] are the anomaly charges (a, c), which we review in Section 4.5.2. Again, agreement on both sides of the duality is found.

We will see how all the quantities, including Seiberg-Witten curves, in this Section can be constructed using Gaiotto's picture.

4.2 Comparison with the moduli space of punctured spheres

One of Gaiotto's key observations is that the one-dimensional moduli space of marginal deformations of the SU(3) $N_f = 6$ theory can be identified with the moduli space of complex structures of a sphere with four marked punctures; two punctures being of a certain type, and the other two of a second type. We will see later how the two types of punctures in this example extend to a larger, but finite, class of punctures, which in turn descend from a class of codimension-two defects of the (2,0) theory of type A_2, after being compactified on the 4-punctured sphere, yields the S-duality frames of the SU(3) $N_f = 6$ theory.

Since the SU(N) $N_f = 2N$ theory is not too different from the special case $N = 3$, we might hope to identify the one-dimensional marginal-coupling moduli space of this gauge theory with the moduli space of complex structures of a certain punctured sphere. First, we saw that in the case of of $SU(2)$ $N_f = 4$ there is only one S-dual frame. The moduli space of this theory is the same as the moduli space of complex structures of a sphere with four identical punctures.

For SU(N) $N_f = 2N$, the picture is different, the picture is similar to the case $N = 3$. The marginal-coupling moduli space is isomorphic to the moduli space of complex structures of a sphere with four punctures, that are not identical. Instead, two of them should be identical to each other, and we call them "minimal"; the other two punctures are similarly identical to each other, and we call them "maximal".

Actually, for $N = 3$, the two types of punctures in this example are all the kinds of punctures one can introduce to construct 4D theories from the compactification of a (2,0) theory of type A_2[1]. For the A_{N-1} theory, punctures correspond to partitions of N, and so there are $P(N)$ punctures. (Actually, one of these punctures will be trivial, so there are really $P(N)-1$ punctures.) So, the two punctures here are just two members of a bigger, but finite class of punctures of the theory.

4.3 Seiberg-Witten curves and k-differentials

The Seiberg-Witten curves of linear quivers, both in the massless and mass-deformed versions, is well known [38]. All these will correspond to compactifications of the (2,0) theory on punctured *spheres*. The $SU(3)$ $N_f = 6$ theory is an example of a linear quiver. Following Gaiotto, the Seiberg-Witten curve (without mass deformations) for this theory can be written in the form

$$x^3 - \phi_2(z)x - \phi_3(z) = 0 \tag{4.4}$$

[1]We want to understand the irregular puncture of the A_2 theory in Chapter 6 as a constrained version of the maximal puncture.

and

$$\phi_2(z) = \frac{u_2}{(z-a)(z-b)(z-c)(z-d)}(dz)^2 \tag{4.5}$$

$$\phi_3(z) = \frac{u_3}{(z-a)(z-b)(z-c)^2(z-d)^2}(dz)^3, \tag{4.6}$$

where a, b, c, d are complex numbers representing the positions of the punctures. The maps ϕ_2 and ϕ_3 are a meromorphic 2-differential and a holomorphic 3-differential, respectively, on a sphere, where z is a patch covering the sphere but one point. (The point at infinity in the complex plane represents the single point not covered by the patch.) Also, x can be locally interpreted as a coordinate along the fiber of the cotangent bundle. Since (4.4) is a polynomial of degree 3, the Seiberg-Witten curve is a triple cover of the sphere. Intuitively, we cannot find global roots for (4.4), but instead, the three branches are really a single one that wraps the sphere three times. Still, locally, if we restrict to a small enough chart of the sphere, we can see the three roots as three disjoint sheets. Thus, in this case the Seiberg-Witten is a 3-sheet cover Σ of the sphere. The complex parameters u_2, u_3 should be interpreted as Coulomb branch parameters. The Seiberg-Witten differential is the one-form $\lambda = x\ dz$ on Σ (not C), where x is a root of the Seiberg-Witten equation (4.4). Naively, in terms of the parameter z on C, we would seem to have 3 different one-forms on C (for each of the three roots x_i, $i = 1, 2, 3$ of (4.4)), but this is *not* so; it is really a single one-form on the triple cover Σ[2].

[2]The "three" Seiberg Witten-differentials here can be interpreted as the eigenvalues of the Higgs field, in the language of Chapter 5. We will see that the Higgs field is a one-form on C, rather than Σ, but taking values in the adjoint representation of the simply-laced Lie algebra $\mathfrak{g}$.

We mentioned in Section 4.2 that we expected two types of punctures for this theory. Here it's clear that one way to differentiate between them is the order of the poles in ϕ_2, ϕ_3. For *minimal* punctures, the poles orders of the punctures for ϕ_2, ϕ_3 are $\{1, 1\}$. For a *maximal* puncture, these pole orders are $\{1, 2\}$. We call this list of pole orders the *pole structure* of the puncture.

In one degeneration of the sphere, a minimal puncture collides with a maximal puncture. This cusp of moduli space should correspond to the $SU(3)$ $N_f = 6$ theory. The parameters u_2, u_3 should be understood as VEVs that break the $SU(3)$ gauge group to its maximal torus $U(1)^2$. Upon complete degeneration, the sphere breaks into two identical 3-punctured spheres. Each of these two spheres has one minimal and two maximal punctures, and should represent 9 hypers, or 3 hypers in the fundamental of the $SU(3)$. If we see the procedure in reverse, we are connecting these two spheres by weakly gauging an $SU(3)$ flavor subgroup in both.

It is then natural to assume that the maximal puncture provides an $SU(3)$ global symmetry, and that the minimal puncture provides a $U(1)$.

In the second degeneration, the two minimal punctures collide (or, equivalently, the two maximal punctures collide). This cusp corresponds to the $SU(2)$ gauging of the E_6 SCFT coupled to one fundamental hyper. The parameter u_2 should be a Coulomb branch parameter corresponding to the $SU(2)$ gauge group. Again, the original sphere breaks into two 3-punctured spheres. These two spheres are not equal to each other, and, in fact, they are both different from the 3-punctured spheres found in the previous degeneration. Indeed, one sphere has 3 maximal punctures, and corresponds to the E_6 SCFT. The other 3-punctured sphere has 2 minimal punctures, and one new type of puncture, which later we will call an *irregular* puncture. Roughly, the irregular puncture is a Higgsed version of a maximal puncture. This 3-punctured sphere should correspond to 2 free hypermultiplets, or 1 hypermultiplet in the fundamental of the weakly-gauged $SU(2)$. At any rate, we can revert the procedure by connecting the irregular puncture with a maximal puncture in the E_6 3-punctured sphere.

We will call 3-punctured spheres *fixtures* in what follows.

On the other hand, mass-deforming the theory means allowing the poles of *all* types of punctures to be $p_k = k$, where p_k is the leading pole order for ϕ_k at the puncture. Thus, in our A_2 example, both minimal and maximal punctures have pole structure $\{2, 3\}$ for ϕ_2, ϕ_3. We have

$$\phi_2(z) = \frac{P_4(z)}{(z-a)^2(z-b)^2(z-c)^2(z-d)^2}(dz)^2 \tag{4.7}$$

$$\phi_3(z) = \frac{P_6(z)}{(z-a)^3(z-b)^3(z-c)^3(z-d)^3}(dz)^3, \tag{4.8}$$

where $P_4(z)$ and $P_6(z)$ are polynomials in z of degree 4 and 6, respectively. We determine the degree of these polynomials by the condition that all our punctures are at a, b, c, d, and we do not have any puncture at $z = \infty$. This means that, as $z \to \infty$, $\phi_k(z)$ should go as $1/z^{2k}(dz)^k$. For a k-differential on a sphere, this means that the degree of the polynomial in the numerator should be $-2k + \sum -i = 1^n p_k^{(i)}$, where $\{p_k^{(i)}\}$ is the pole structure for the i-th puncture, and there are n punctures labeled by i.

Now, since the pole orders at the punctures are higher in the mass-deformed case than in the massless case, we need more parameters to parametrize the k-differentials. These additional parameters are precisely the mass deformations. But the minimal puncture should introduce fewer mass deformation parameters than the maximal puncture. This is so, because the A_2 minimal puncture corresponds to a $U(1)$ flavor group, which has rank 1, whereas the A_2 maximal puncture has $SU(3)$ flavor group, which has rank 2. Thus, we expect one mass deformation parameter for the minimal puncture, and two mass deformation parameters for the maximal puncture.

The point is that if we solve the Seiberg-Witten equation (4.4) for x locally around an A_2 maximal puncture, we find three roots $x_{1,2,3}$ of the form

$$x_1 = \frac{m}{z} + \ldots, \qquad x_2 = \frac{n}{z} + \ldots, \qquad x_3 = \frac{-m-n}{z} + \ldots \tag{4.9}$$

Instead, at an A_2 minimal puncture, we should find

$$x_1 = \frac{p}{z} + \ldots, \qquad x_2 = \frac{p}{z} + \ldots, \qquad x_3 = \frac{-2p}{z} + \ldots \tag{4.10}$$

So, locally, we have two roots that are equal up to next-to-leading order[3]. This should be detected by the discriminant,

$$\Delta(z) = 4\phi_2(z)^3 - 27\phi_3(z)^2, \tag{4.11}$$

which, if expanded around an A_2 maximal puncture would give

$$\Delta(z) = \frac{c}{z^6}, \tag{4.12}$$

[3]In fact, in the language of Chapter 5, since the local roots of the Seiberg-Witten equation are the eigenvalues of the Higgs field, the expressions (4.9) and (4.10) give us directly the mass-deformed (semisimple) orbits for these punctures.

whereas if expanded around the A_2 minimal puncture, it would give

$$\Delta(z) = \frac{c}{z^4}. \tag{4.13}$$

To deduce this we have used the expression for the determinant in terms of the roots, $\Delta(z) = (x_1 - x_2)^2(x_2 - x_3)^2(x_3 - x_1)^2$.

This condition that we should have a smaller pole order for $\Delta(z)$ around a minimal puncture compared to that for a maximal puncture is what makes us have fewer mass deformations, and so, although the pole structures of the mass-deformed punctures are the same, we are still able to differentiate between them through $\Delta(z)$.

We should note that we have been able to write global expressions for the ϕ_k because we have chosen C to be a sphere. Had we chosen a higher genus Riemann surface, we could only write local expansions around a puncture. Still, while in general we do not have a polynomial, there is a definite number of parameters needed to parametrize k-differentials on a Riemann surface C of genus g; this number is $(1-2k)(1-g) + \sum_{i=1}^{n} p_k^{(i)}$, where $\{p_k^{(i)}\}$ is the pole structure of the i-th puncture, and there are n punctures on C.

4.4 Gaiotto duality

Here we summarize the observations of Argyres-Seiberg duality, and gather the examples of linear quivers studied by Gaiotto, to state Gaiotto's proposal for S-duality. A more intrinsic understanding of Gaiotto duality, from the point of view of Hitchin's equations, will be explained in Chapter 5.

Gaiotto's proposal refers to 4D $\mathcal{N} = 2$ SCFTs that arise from the (twisted) compactification of a 6D (2,0) theory on a (possibly punctured) Riemann surface C. Apparently, not every 4D $\mathcal{N} = 2$ SCFT can be derived from the compactification of a 6D (2,0) theory. Finally, Gaiotto's proposal can be extended to 4D $\mathcal{N} = 2$ theories that are asymptotically free [39, 40].

The twisting necessary to preserve half of the supersymmetry will be discussed in Section 5.1.1.

The punctures correspond to codimension-two defect operators of the (2,0) theory, wrapped on 4D spacetime (and which thus are located at points on the Riemann surface). In this sense, the punctures in Gaiotto's picture are closely related to the surface operators in 4D $\mathcal{N} = 4$ super Yang-Mills theory studied in Chapter 3, since both descend from the (2,0) defects through different paths of compactification.

The gauge-coupling moduli space of the 4D $\mathcal{N} = 2$ SCFT is identified with the complex-structure moduli space of a possibly-punctured Riemann surface C. Each cusp in gauge-coupling moduli space corresponds to a degeneration limit of C. In a degeneration limit of C, a long cylinder is produced, which corresponds to a weakly-coupled gauge group. Thus, each degeneration limit corresponds to a weakly-coupled theory, which we refer to as *S-dual frame*.

The S-duality group is the mapping-class group of the family of curves on which the (2,0) theory is compactified.

Furthermore, we have holomorphic k-differentials ϕ_k on C, which descend from chiral operators in the (2,0) theory. Here, k takes on values on the exponents of the Lie algebra $\mathfrak{g}$ that defines the (2,0) theory. For the A_{N-1} series, these exponents are $2, 3, 4, \ldots, N$. For the D_N series, the exponents are $2, 4, 6, 8, 2N-2; N$. The last exponent corresponds to a the Pfaffian $\tilde{\phi}$, which is an N-differential.

The ϕ_k have singularities at each puncture. The set of leading poles of the ϕ_k at a fixed puncture is the *pole structure* of such puncture.

The SW curves for the A_{N-1} and D_N theries are, respectively,

$$x^N - \phi_2 x^{N-2} - \phi_3 x^{N-3} - \cdots - \phi_{N-1} x - \phi_N = 0 \qquad (4.14)$$

$$x^{2N} - \phi_2 x^{N-2} - \phi_4 x^{N-4} - \cdots - \phi_{2N-2} x^2 - \tilde{\phi}^2 = 0 \qquad (4.15)$$

Now we discuss what *kinds* of punctures there exist. For A_{N-1}, punctures correspond to partitions of N. Sometimes we refer to partitions as A-partitions, for clarity. On the other hand, for D_N, punctures correspond to D-partitions of $2N$. D-partitions are defined in Chapter 7. Moreover, in Chapter 5, we will identify punctures with nilpotent orbits of the A_{N-1} and D_N Lie algebras. A_{N-1} and D_N nilpotent orbits are classified precisely by A- and D-partitions.

We will also see in Chapter 5 that Hitchin's equations govern all local properties of the punctures, and at least some global properties of Gaiotto's picture. The punctures, both massless and mass-deformed, correspond to various boundary conditions for Hitchin's equations, and the Seiberg-Witten curve is associated with the Hitchin fibration. Furthermore, the k-differentials ϕ_k correspond to invariant polynomials of the Higgs field in Hitchin's equations.

4.5 S-duality invariants

4.5.1 Global symmetry groups

Given a punctured curve, with punctures labelled by $i = 1, \ldots, n$, we have a rule that associates a global symmetry group G_i to the i-th puncture. The global symmetry group G_{global} associated to the theory on the punctured curve is a Lie group

$$G_{global} \supset \prod_{i=1}^{n} G_i \tag{4.16}$$

such that

$$\mathrm{rank}(G_{global}) = \sum_{i=1}^{n} \mathrm{rank}(G_i) \tag{4.17}$$

In other words, only a maximal subgroup $\prod_{i=1}^{n} G_i$ of G_{global} is made manifest by the punctures. When $\prod_{i=1}^{n} G_i$ is a proper subgroup of G_{global}, we say that we have "enhanced" global symmetry.

The group G_{global} is independent of the S-dual frame in which it is computed, which means it can be used to check our proposed S-dualities. The independence of G_{global} of the S-dual frame is also clear from (4.16) and (4.17).

Typically, in studying specific examples, one discovers that the global symmetry group G_{global} has to be strictly bigger than the naive $\prod_{i=1}^{n} G_i$ when one glues curves with the property that each curve contributes hypers in a certain representation of the gauge group that connects the curves. Thus, there exists a bigger symmetry group rotating these hypers bigger than the product of the groups that rotate hypers in the individual curves before gluing. Sometimes the enhanced global symmetry group of a Lagrangian theory implies that an interacting SCFT, which appears in an S-dual frame of the theory, must itself have an enhanced symmetry.

One way to compute/check G_{global} for a fixture is to compactify the theory on a circle to obtain a low-energy 3D sigma model, and then compute the mirror 3D theory. A 3D theory that descends from a 4D $\mathcal{N} = 2$ theory in Gaiotto's class has been shown to be mirror dual to a 3D linear quiver in the shape of an extended Dynkin diagram [9, 40, 41]. This extended Dynkin diagram immediately reveals the Lie algebra of G_{global} for the original 4D theory. We will see examples of the use of 3D mirrors to check global symmetries in Section 6.5.

4.5.2 Central charges

Each nonabelian factor G_i of the global symmetry group G_{global} has a *central charge* k_{G_i}, defined via the current algebra

$$J^a_\mu(x) J^b_\nu(0) = \frac{3k_G}{4\pi^4} \delta^{ab} \frac{g_{\mu\nu} x^2 - 2x_\mu x_\nu}{(x^2)^4} + \frac{2}{\pi^2} f^{ab}{}_c \frac{x_\mu x_\nu x \cdot J^c}{(x^2)^3} \tag{4.18}$$

The conformal anomaly coefficients a and c appear in the conformal anomaly of the trace of the energy-momentum tensor that arises when we put the 4D $\mathcal{N} = 2$ theory in a gravitational background [42],

$$\langle T^\mu_\mu \rangle = \frac{c}{16\pi^2} (\text{Weyl})^2 - \frac{a}{16\pi^2} (\text{Euler}), \tag{4.19}$$

where

$$(\mathrm{Weyl})^2 = R^2_{\mu\nu\rho\sigma} - 2R^2_{\mu\nu} + \frac{1}{3}R^2$$

$$\mathrm{Euler} = R^2_{\mu\nu\rho\sigma} - 4R^2_{\mu\nu} + R^2$$

The central charges k_G, a and c are straightforwardly calculable in an $\mathcal{N} = 2$ gauge theory with a Lagrangian description, and are constant over the whole family of SCFTs parametrized by $\mathcal{M}_{g,n}$ [43].

The trace anomaly coefficients, a and c, of the SCFT, can be computed [3, 6, 10] from two auxiliary *integer* quantities: the effective number of hypermultiplets, n_h, and the effective number of vector multiplets, n_v,

$$\begin{aligned} a &= \tfrac{5n_v + n_h}{24} \\ c &= \tfrac{2n_v + n_h}{12}. \end{aligned} \tag{4.20}$$

The integers n_h and n_v are the actual number of hypermultiplets and vector multiplets in a *Lagrangian* S-duality frame of the theory, provided such frame exists. As a consequence, the n_h of a free-field fixture (for which $n_v = 0$) is equal to the number of free hypermultiplets in this fixture. For an interacting SCFT, these should be simply regarded as auxiliary quantities used to compute a and c, which do have a sensible meaning in all cases. For a mixed fixture, i.e., one that represents an interacting SCFT together with free hypers, the difference between n_h for the mixed fixture and n_h for the SCFT alone is equal to the number of free hypers in the mixed fixture.

We will give formulæ to compute n_h and n_v for regular and irregular punctures in the A_{N-1} and D_N series in the following chapters. These formulas are heavily dependent on properties of the nilpotent orbits of each Lie algebra.

Chapter 5

The Hitchin system

In this chapter we show how the Hitchin equation provides an intrinsic understanding of the Coulomb branch properties of the punctures discussed in Chapter 4. In Section 5.1, we argue that the Hitchin system is the BPS equation for the $4D$ $\mathcal{N} = 2$ theories that arise from compactification of a (2,0) theory on a Riemann surface C. The Hitchin fibration is then identified with the Seiberg-Witten fibration of the 4D theory. We thus realize that the k-differentials ϕ_k of Chapter 4 as well as the Seiberg-Witten curve can easily be constructed from the Higgs field. In Section 5.2.1, we discuss the punctures, which, in the present context, are identified with codimension-one defect operators of the Hitchin system on C. The Hitchin system and a class of codimension-one defect operators of Hitchin's equations were already discussed in Chapter 3 in the context of surface operators of 4D $\mathcal{N} = 4$ super Yang-Mills. In this dissertation we focus on defects that respect superconformal symmetry, which are the relevant defects to study 4D $\mathcal{N} = 2$ SCFTs.

We saw in Chapter 3, in the context of surface operators, that superconformal codimension-one defects of Hitchin's equations with Lie algebra $\mathfrak{g}_{\mathbb{C}}$ obey Nahm's equations, and correspond essentially to semisimple and nilpotent orbits of $\mathfrak{g}_{\mathbb{C}}$. In the context of 4D $\mathcal{N} = 2$ SCFTs, semisimple orbits correspond to mass-deformed punctures, whereas nilpotent orbits correspond to massless ones. The $\alpha, \beta, \gamma \to 0$ limit of Section 3.4 corresponds to a *zero-mass limit* for a family of semisimple orbit, which yields a nilpotent orbit, i.e., the massless version of the puncture. Nonetheless, not all massless punctures arise as zero-mass limits of mass-deformed punctures. Punctures with trivial flavor group exist only in their massless version, which corresponds to a *rigid* nilpotent orbit.

However, it is not clear that *all* the superconformal codimension-one defect operators of Hitchin's equations studied in Chapter 3 are relevant to 4D $\mathcal{N} = 2$ SCFTs. At least, we are sure that certain defects, corresponding to *special* nilpotent orbits, are relevant. Special nilpotent orbits are defined as those lying in the range of a certain map, called the *Spaltenstein map*, which takes nilpotent orbits in $\mathfrak{g}_{\mathbb{C}}$ to nilpotent orbits in $\mathfrak{g}_{\mathbb{C}}$. The interpretation of non-special orbits for 4D $\mathcal{N} = 2$ SCFTs is not currently understood. Fortunately, the methods of Chapter 4 can be used to find the properties of non-special punctures, which allows us to perform our classification of the A_{N-1} and D_N 4D theories in Chapters 6 and 7. In what comes to our intrinsic understanding via Hitchin's equations, we will have to content ourselves with understanding only the special punctures.

In Section 5.2.2 we give the complete picture for special punctures. The local form of the Higgs field near the special puncture yields the *Coulomb branch* information for the puncture, in particular, the pole structure and the constraints. On the other hand, the *Higgs branch* properties of the puncture, in particular the global symmetry group, are naturally given by the Spaltenstein dual nilpotent orbit.

5.1 Hitchin's equations and Seiberg-Witten theory

5.1.1 Topological twist

As we mentioned in Section 2.4, compactifying the 6D (2,0) theory on an arbitrary two-dimensional surface, even a generic Riemann surface, will not preserve any supersymmetry. However, when we compactify on a generic Riemann surface C, one can perform a topological twist, in addition to the compactification, to preserve *half* of the supersymmetry. Hence, this twist allows us to obtain 4D $\mathcal{N} = 2$ theories. As we will see in Section 5.2.1, we can also incorporate defects, and if these defects respect superconformal symmetry, the resulting 4D $\mathcal{N} = 2$ theory will be a SCFT. Otherwise, we will get a 4D $\mathcal{N} = 2$ asymptotically-free theory.

To define the twist, we need to recall how the supersymmetry charges transform under the bosonic symmetry of the 6D (2,0) theory, which includes the Lorentz symmetry and the R-symmetry. Specifically, this bosonic symmetry is $SO(5,1) \times SO(5)_R$. The (2,0) supersymmetry charges Q transform as the $\mathbf{4} \times \mathbf{4}$ representation, and obey a symplectic-Majorana reality condition. When we compactify on a Riemann surface C, the new bosonic symmetry is

$$SO(3,1) \times SO(2)_C \times SO(3)_R \times SO(2)_R \tag{5.1}$$

and so Q now transforms as

$$Q : (2_{1/2} + 2'_{-1/2}) \times (2_{1/2} + 2_{-1/2}) \tag{5.2}$$

and the previous symplectic Majorana condition reduces to a relation between the 2 and 2' of $SO(3,1)$.

Now, to preserve 4D $\mathcal{N} = 2$ supersymmetry, we twist [1, 2, 31] the spin connection $SO(2)_C$ by $SO(2)_C \to SO(2)_C - SO(2)_R$. The supercharges now transform as

$$\mathbf{2}_0 \times \mathbf{2}_{1/2} + \mathbf{2}_1 \times \mathbf{2}_{-1/2} + \mathbf{2}'_{-1} \times \mathbf{2}_{1/2} + \mathbf{2}'_0 \times \mathbf{2}_{-1/2} \tag{5.3}$$

The preserved supercharges must be covariantly constant on C, so they are $2_0 \times 2_{1/2}$. These generate an $\mathcal{N} = 2$ superalgebra.

5.1.2 The Hitchin system

Consider for a moment a *generic* 4D $\mathcal{N} = 2$ theory; that is, one not necessarily obtained from the compactification of a (2,0) theory on C. If we compactify this 4D theory on a circle, the low-energy theory is a 3D $\mathcal{N} = 4$ sigma model with target space $\mathcal{M}$. The target space $\mathcal{M}$ has the structure of a hyper-Kahler manifold, which means that we have a sphere worth of complex structures for $\mathcal{M}$. What is interesting is that in a distinguished complex structure, $\mathcal{M}$ is equivalent to a torus fibration, which moreover has a physical interpretation for the 4D theory [44]. Specifically, $\mathcal{M}$ can be identified, in this complex structure, with the *Seiberg-Witten fibration* of the 4D theory, i.e., the fibration over the Coulomb branch $\mathcal{B}$ of the 4D $\mathcal{N} = 2$ theory whose generic fiber is a compact torus $U(1)^r$, with r the rank of the gauge group. Thus, the Seiberg-Witten curve of the 4D theory is encoded in the target space $\mathcal{M}$ of the 3D theory.

Now, let us go back strictly to 4D $\mathcal{N} = 2$ theories obtained from the compactification of a (2,0) theory on C. In this case, the target space $\mathcal{M}$ of the 3D $\mathcal{N} = 4$ theory has an *additional* interpretation [2]; namely, $\mathcal{M}$ can be identified with the moduli space of solutions to a *Hitchin system*. Let us see why this is true.

We previously compactified the 6D (2,0) theory on a Riemann surface C and performed a twist to get a 4D $\mathcal{N} = 2$ theory, and then compactified it on a circle to get the 3D $\mathcal{N} = 4$ theory. From Fig. 2.1, we can reverse the order of the compactifications [2]; namely, we compactify the (2,0) theory on a circle to get 5D $\mathcal{N} = 2$ super Yang-Mills, and then compactify on C, with a twist, to arrive at the same 3D theory as before.

Let us now try to understand what $\mathcal{M}$ means in terms of the 5D theory compactified on $C \times \mathbb{R}^3$. The target space $\mathcal{M}$ is the space of vacua of the 3D $\mathcal{N} = 4$ theory. These vacua are *constant* over the $\mathbb{R}^3$ of the 3D theory, and preserve half of the supersymmetry of the 5D theory. Thus, every solution of the half-BPS equations of the 5D theory which is furthermore *independent* of the $\mathbb{R}^3$ of the 3D theory should yield a 3D vacuum.

Thus, we start with the 5D half-BPS equations,

$$(F_{\mu\nu}\gamma^{\mu\nu} + D_\mu Y_I \gamma^{\mu I} + [Y_I, Y_J]\gamma^{IJ})\epsilon = 0 \tag{5.4}$$

where the Y^I $(I = 1, \ldots, 5)$ are the adjoint scalars of 5D super Yang-Mills, perform a twist, and declare the solutions to be independent of the $\mathbb{R}^3$ of the 3D theory. Thus, we are left with equations on the Riemann surface C, every solution of which represents a vacuum of the 3D theory. Furthermore, because of the twist, we have a choice of complex structure, and so our equations are given in terms of complex fields on C. The resulting system is Hitchin's equations on C,

$$\begin{gathered} F + [\varphi, \bar{\varphi}] = 0 \\ d\bar{z}(\bar{\partial}\varphi + [\bar{A}(\bar{z}), \varphi]) = 0, \qquad dz(\partial\bar{\varphi} + [A(z), \bar{\varphi}]) = 0 \end{gathered} \tag{5.5}$$

and the moduli space of solutions to Hitchin's equations is precisely the target space $\mathcal{M}$. In (5.5), we have

$$\varphi = \frac{1}{2}(Y^1 + iY^2)dz \tag{5.6}$$

is a holomorphic adjoint-representation-valued 1-form on the Riemann surface C, and $A = A(z)dz + \bar{A}(\bar{z})d\bar{z}$ is the gauge field cotangent to C.

In retrospect, we see why it was convenient to reverse the order of the compactifications. We obtained Hitchin's equations (5.5) from the half-BPS equations (5.4) of 5D $\mathcal{N} = 2$ super Yang-Mills, which is a *Lagrangian* field theory, and so these equations are easy to compute. On the other hand, it is not always easy to compute the BPS equations for a 4D $\mathcal{N}=2$ SCFT because it is not necessarily a Lagrangian theory.

Therefore, the target space $\mathcal{M}$ of the 3D theory can be identified *both* with the Seiberg-Witten fibration $\mathcal{M} \to \mathcal{B}$ for the 4D theory and with the moduli space of solutions to Hitchin's equations on C. From the point of view of the Hitchin system, the projection $\mathcal{M} \to \mathcal{B}$ can be seen as follows. $\mathcal{B}$ is parametrized by the gauge-invariant VEVs $\langle \mathcal{O}_k \rangle$. In 5D super Yang-Mills, $\langle \mathcal{O}_k \rangle$ are identified with the Casimirs of φ. Thus, if (A, φ) is a solution to Hitchin's equations (5.5), the projection $\mathcal{M} \to \mathcal{B}$ is given by $(A, \varphi) \mapsto \{\text{Casimirs of } \varphi\}$.

Hence, the Seiberg-Witten curve is given by the characteristic equation for the Higgs field, which for the A_{N-1} series has the expansion

$$\det(\varphi - \lambda x \mathbf{1}) = x^N - \phi_2 x^{N-2} - \cdots - \phi_{N-1} x - \phi_N = 0, \tag{5.7}$$

where the ϕ_k are k-differentials on C, and, from these equations, are equal to the Casimirs of φ. The analog of (5.7) for the D_N series is (4.15).

Now, the punctures represent singular *boundary conditions* for Hitchin's equations at specific points. We can now recycle our work on surface operators in Chapter 3, where we arrived at the same Hitchin system on C with singularities. To produce 4D $\mathcal{N}=2$ *superconformal* theories, which is the main object of study of this dissertation, we are interested in solutions that are singular at the punctures, but such that they respect superconformal symmetry and are invariant under the $U(1)$ isometry group of the Riemann surface. The latter condition just means that the nature of the punctures should be independent of the choice of complex structure on the Riemann surface. This is because, following Gaiotto, we want to identify the moduli space of complex structures of C with the moduli space of marginal gauge couplings of the 4D $\mathcal{N} = 2$ theory. Physically, we want the defects to be independent of how we tune the gauge couplings.

5.2 Punctures and Hitchin's equations

5.2.1 Superconformal punctures

In Section 5.1 we have seen that Hitchin's equations on C are the BPS equations for 4D $\mathcal{N} = 2$ theories derived from the compactification of a (2,0) theory on C. We anticipated that punctures should provide singular boundary conditions for the Hitchin system on C. Another way to say this is that punctures are complex codimension-one (i.e., zero-dimensional) defects of the Hitchin system on C.

We emphasize that, in the derivation above, our Hitchin system is defined for fields in a representation of the same gauge group G of 5D super Yang-Mills, which in turn corresponds to the simply-laced Lie algebra $\mathfrak{g}$ associated to the original 6D (2,0) theory. Thus, in the context of 4D $\mathcal{N} = 2$ theories, we are only interested in Hitchin systems with *simply-laced* gauge group $G_{\mathbb{C}}$.

Now, if we restrict to punctures, or boundary conditions, that respect superconformal symmetry, which is the main topic of this dissertation[1], then, borrowing from Chapter 3, we reach the conclusion that punctures correspond to *simple* poles for the Higgs field, and whose residue lies in a semisimple or nilpotent orbit in $\mathfrak{g}_{\mathbb{C}}$. Locally, we have for the Higgs field,

$$\varphi(z) = \frac{X}{z} + \dots \tag{5.8}$$

where the puncture we are studying at is located at $z = 0$, the ellipsis denotes a generic matrix in $\mathfrak{g}_{\mathbb{C}}$ (in the fundamental representation), and X is a representative of a nilpotent/semisimple orbit in $\mathfrak{g}_{\mathbb{C}}$.

If we choose a semisimple orbit, then computing the k-differentials ϕ_k, we obtain

$$\phi_k(z) = \frac{m_k}{z^k} + \dots, \tag{5.9}$$

where k are the exponents of our simply-laced Lie group G. Thus, semisimple orbits are always associated to *mass-deformed* punctures.

On the other hand, if we choose a nilpotent orbit, we obtain

$$\phi_k(z) = \frac{u_k}{z^{p_k}} + \dots, \tag{5.10}$$

where $1 \leq p_k \leq k - 1$. Thus, nilpotent orbits are associated to *massless* punctures. For the minimal nilpotent orbit, we get $p_k = 1$ for every k, whereas for the maximal nilpotent orbit we get $p_k = k - 1$ for every k.

[1]Punctures that do not respect superconformal symmetry have been studied in [2, 39, 40]. Including these punctures on C yields 4D $\mathcal{N} = 2$ theories that are not SCFTs, but rather asymptotically free. From the Hitchin point of view, the Higgs field for these punctures is allowed to have a pole of order greater than one. In the geometric Langlands language, this corresponds to the problem of *wild* ramification, whereas the superconformal case corresponds to *tame* ramification.

How should we interpret the $\alpha, \beta, \gamma \to 0$ limit of Section 3.4 in the present context? This limit corresponds to a *zero-mass limit*, in the sense that if the mass deformations of a mass-deformed puncture are taken to zero, one should be left with the *massless* version of the puncture, so one should be able to go from an expansion of the form (5.9) to one of the form (5.10) by a limiting procedure. An element of a chosen semisimple orbit can always be diagonalized; the eigenvalues are precisely the mass deformations of the puncture. But naively taking these eigenvalues directly to zero leads us not to a nilpotent orbit, but to the zero element of $\mathfrak{g}_{\mathbb{C}}$. A more careful limiting procedure leads us actually to a nilpotent orbit whose dimension as a manifold is equal to the dimension of the semisimple orbit. This nilpotent orbit is the biggest contained in the semisimple orbit. So, there are various ways to take the limit where the mass deformations go to zero, and generically we obtain various nilpotent orbits, but the nilpotent orbit that should be identified with the massless puncture is the biggest of all these, and we can identify it because its dimension should be equal to that of the semisimple orbit. The correct limiting procedure that yields the biggest nilpotent orbit is precisely the $\alpha, \beta, \gamma \to 0$ limit.

Nonetheless, it is possible to find nilpotent orbits that *cannot* possibly arise as a zero-mass limit of any semisimple orbit. There is simply no semisimple orbit with the same dimension as these nilpotent orbits. These orbits are *rigid*. We will see that rigid nilpotent orbits can be of two types: special and non-special. Special nilpotent orbits that are rigid (and different from the zero orbit) correspond to punctures with trivial flavor group. There are no examples of this kind of nilpotent orbit in the A_{N-1} series, but there are in the D_N. So, punctures with trivial flavor group (different from the trivial puncture) are associated to a special rigid nilpotent orbit, and they do not have a mass-deformed version.

On the other hand, non-special nilpotent orbits are currently not well understood. Again, there exist no examples of non-special nilpotent orbits in the A_{N-1} series, but we have several in the D_N series. Requiring the residue of the Higgs field φ on a non-special orbit seems to be a consistent boundary condition for Hitchin's equations, and one indeed produces a pole structure, but it is not clear if this orbit should correspond to a puncture. Specifically, the pole structure, or, more generally, the Coulomb branch properties, are not everything it takes to make a puncture. A puncture should also have Higgs branch properties, in particular a Higgs branch. Getting a little ahead of ourselves, when we discuss the Spaltenstein map in more detail in Section 5.2.2, we will see that one can also take a non-special orbit to correspond to the "Higgs branch" of a puncture, but then we do not understand what orbit should give the Coulomb branch information. But this time, by indirect methods, we are sure that this Higgs-branch non-special orbit *must* correspond to a puncture. We cannot somehow forbid the existence of this non-special puncture, because, as we will see in Chapter 7, this non-special puncture naturally appears in degenerations of surfaces exclusively involving special punctures. We can even compute the pole structure (as well as the other Coulomb branch properties) of these non-special punctures by indirect techniques (i.e., not using the Hitchin system), and it does not correspond to the pole structure of any Coulomb branch non-special orbit. Thus, we currently do not understand what boundary condition corresponds to these non-special punctures. We cannot simply match Coulomb branch non-special nilpotent orbits and Higgs branch non-special orbits, as one would naively think. The resolution to this puzzle is currently being investigated [45].

There are also non-special *semisimple* orbits of the *gauge group* $G_{\mathbb{C}}$, in contrast to orbits of the *Lie algebra* $\mathfrak{g}_{\mathbb{C}}$, which is what we have been discussing so far. Orbits of the Lie algebra can be mapped to orbits of the Lie group, but the converse is not true. These non-special semisimple orbits of $G_{\mathbb{C}}$, which turn out to be rigid, have been discussed in [33]. We presently do not know if there should or should not be an intepretation as punctures for non-special semisimple orbits of $G_{\mathbb{C}}$.

Finally, we should note that, in the context of surface operators, one is led to study the Hitchin system on *any* semisimple Lie algebra $\mathfrak{g}_{\mathbb{C}}$, whereas in the context of 4D $\mathcal{N} = 2$ we are restricted to Hitchin systems on *simply-laced* Lie groups. If we recall our discussion in Section 2.3, we were able to get 4D $\mathcal{N} = 4$ super Yang-Mills with non-simply-laced gauge group when we wrapped a twist line on one of the non-trivial cycles of the torus on which the (2,0) theory is compactified. These twist loops are associated with a (sub)group of outer automorphisms of a simply-laced Lie algebra $\mathfrak{g}$. So, one may wonder what happens if we wrap twist loops on non-trivial cycles of C, in the context of 4D $\mathcal{N} = 2$ theories. Actually, since we are allowing for the possibility of punctured Riemann surfaces, we may also have twist lines, as opposed to twist loops, connecting two punctures. More explicitly said, these twist lines do not have to wrap a non-trivial cycle of the Riemann surface. However, the punctures are not the punctures we have been discussing so far. They should belong to a different sector, which should be odd under the outer automorphism group of $\mathfrak{g}$ that we are discussing. However, unlike $\mathcal{N} = 4$ super Yang-Mills, we are still studying the Hitchin system with the original simply-laced Lie algebra $\mathfrak{g}_{\mathbb{C}}$, not on the quotient Lie algebra by the outer automorphism. Instead, the new punctures in the odd sector have a particular pole structure, where certain poles have *half-integer* values. So, we have another problem to investigate. We do not know the Hitchin boundary conditions for the punctures in odd sectors under outer automorphisms of a simply-laced Lie algebra, and we do not know if there is a puncture interpretation for the Hitchin boundary conditions for the Hitchin system for a non-simply-laced Lie algebra. It should be quite interesting to resolve these issues.

5.2.2 The Spaltenstein map

Since nilpotent orbits will enter the scene soon, there is a result we need to discuss. It is a map that takes nilpotent orbits of $\mathfrak{g}_{\mathbb{C}}$ to nilpotent orbits of $\mathfrak{g}_{\mathbb{C}}$, called the *Spaltenstein map* [15]. We will find it important to concentrate on nilpotent orbits that lie in the *range* of the Spaltenstein map; these are called *special* nilpotent orbits. For punctures corresponding to special nilpotent orbits, the Spaltenstein map roughly relates Higgs branch information and Coulomb branch information of a puncture. The correct picture for *non-special* punctures, which do not lie in the range of the Spaltenstein map, is not well understood yet [45]. This will not be a problem when we deal with the (2,0) theories of type A_{N-1}, since in that case *all* punctures are special, but we will run into non-special punctures when we study the D_N (2,0) theories. We should emphasize that we do know how to compute the properties of non-special punctures by other methods, but we do not have an intrinsic understanding from the point of view of the Hitchin equation.

Another noteworthy property of the Spaltenstein map is that, if restricted to the set of special nilpotent orbits, is an order-reversing involution. To understand in what sense the Spaltenstein map is order-reversing, we need to recall the notion of partial ordering on the set of nilpotent orbits.

Nilpotent orbits are manifolds, and they admit a hyper-Kahler structure. They generically have different dimensions as manifolds. The nilpotent orbit with the greatest dimension is called *maximal*; the trivial nilpotent orbit, which consists only of the zero element, has dimension zero. The smallest non-trivial nilpotent orbit is called *minimal*.

While nilpotent orbits must be by definition disjoint, it is possible that a nilpotent orbit $\mathcal{O}_1$ be contained in the *closure* of another, $\mathcal{O}_2$. If that is the case, we denote this by $\mathcal{O}_1 \leq \mathcal{O}_2$. The maximal nilpotent orbit is strictly bigger than any other, and the minimal nilpotent orbit is strictly smaller than any other non-trivial nilpotent orbit. A diagram showing the partial ordering in the set of nilpotent orbits for a given Lie algebra $\mathfrak{g}_{\mathbb{C}}$ is called *Hasse diagram.* Now, the statement that the Spaltenstein map s, when restricted to special orbits, is an order-reversing involution means that $s^2|_{\text{special}} = id$ and that $s(\mathcal{O}_1) \geq s(\mathcal{O}_2)$ for nilpotent orbits O_1, O_2 such that $\mathcal{O}_1 \leq \mathcal{O}_2$.

For A_{N-1}, the Spaltenstein map is a bijection (equal to the transpose of the partition), so every nilpotent orbit is special. On the other hand, the Spaltenstein map for the other Lie algebras is generically *not* a bijection. The Spaltenstein map for the D_N theories will be explicitly defined in Chapter 7.

5.2.3 Puncture properties and nilpotent orbits

Let $\mathfrak{g}$ be a simply-laced Lie algebra. Massless punctures for the 4D theories that arise from the compactification of a (2,0) theory of type $\mathfrak{g}$ on a Riemann surface are classified by nilpotent orbits O^{Higgs} in $\mathfrak{g}_{\mathbb{C}}$. We call these *Higgs branch* nilpotent orbits.

Let us see how $\mathcal{O}^{\text{Higgs}}$ encodes Higgs-branch properties of a puncture. Let p be a puncture, whose Higgs branch nilpotent orbit is $\mathcal{O}^{\text{Higgs}}$. The nilpotent orbit $\mathcal{O}^{\text{Higgs}}$ determines an embedding $\rho : \mathfrak{sl}(2) \to \mathfrak{g}_{\mathbb{C}}$. The centralizer of ρ in $\mathfrak{g}_{\mathbb{C}}$ provides a dim($\mathfrak{g}$)-dimensional representation of the Lie algebra $(\mathfrak{g}_{\text{flavor}})_{\mathbb{C}}$ of the complexified global symmetry group $(G_{\text{flavor}})_{\mathbb{C}}$ for the puncture. The Cartan subalgebra of $(\mathfrak{g}_{\text{flavor}})_{\mathbb{C}}$ in this representation is precisely the mass-deformed (semisimple) orbit for the puncture.

Also, the difference between the effective numbers of hypermultiplets and vector multiplets, $\delta n_h - \delta n_v$, that the puncture provides can be computed from $\mathcal{O}^{\text{Higgs}}$,

$$\delta n_h - \delta n_v = \frac{1}{2}(\dim(\mathfrak{g}) - \operatorname{rank}(\mathfrak{g}) - \dim(\mathcal{O}^{\text{Higgs}})) \tag{5.11}$$

Now we discuss Coulomb branch properties of the puncture p. If $\mathcal{O}^{\text{Higgs}}$ is *non-special*, we already mentioned that we do not understand how to compute Coulomb branch properties from the Hitchin point of view[2]. Thus, let us restrict to the case where $\mathcal{O}^{\text{Higgs}}$ is *special*. Let $\mathcal{O}^{\text{Coulomb}} = s(\mathcal{O}^{\text{Higgs}})$, where s is the Spaltenstein map. To find the pole structure of p, we put the residue of the Higgs field φ at p on the nilpotent orbit $\mathcal{O}^{\text{Coulomb}}$, and compute the k-differentials ϕ_k near p. Looking at the relations between the expansions of the various ϕ_k, we can deduce the constraints. After determining exactly how many independent parameters there are and what their dimensions are, we can also compute n_v. The explicit formulas for the A_{N-1} and D_N series are given in Chapters 6 and 7.

[2]However, we can always resort to the linear quiver associated to the non-special puncture to compute its Coulomb branch properties. Thus, one can compute, e.g., the pole structure and the constraints for non-special punctures in Chapter 7.

Chapter 6

The A_{N-1} series

6.1 Setup

We study the A_{N-1} (2,0) 6d theory compactified on a Riemann surface C of genus g with n punctures [1, 4, 6] located at points $y_i \in C$, $i = 1, \ldots, n$. We closely follow [6].

The Seiberg-Witten curve, $\Sigma \subset T^*C$ of the 4d low-energy A_{N-1} theory is given by

$$0 = \lambda^N + (-1)^N \sum_{k=2}^{N} \lambda^{N-k} \phi_k(y), \tag{6.1}$$

where λ is the Seiberg-Witten differential, and the $\phi_k(y)$ are k-differentials on C (pulled back to T^*C). The ϕ_k are allowed to have poles of various orders at the y_i.

The theory possesses a set of relevant operators, whose vacuum expectation values parametrize the Coulomb branch of the theory. At a generic point on the Coulomb branch, the theory is infrared-free; at the origin, it is superconformal. The tangent space at the origin of the Coulomb branch is a *graded vector space*[1],

$$V = \bigoplus_{k=2}^{N} V_k. \tag{6.2}$$

where $V_k = H^0\Big(C, K^k\left(\sum_{i=1}^n p_k^{(i)} y_i\right)\Big)$ is the vector space of meromorphic of k-differentials, ϕ_k, with poles of order *at most* p_k^i at the punctures y_i. The graded dimension of V_k is given by

$$d_k = \dim(V_k) = (2k-1)(g-1) + \sum_{i=1}^{n} p_k^{(i)}.$$

As we vary the gauge couplings, the graded vector spaces, V, fit together to form the fibers of a graded vector bundle over the moduli space, $\mathcal{M}_{g,n}$, of marginal-deformations. Our main guiding principle is that this vector bundle should extend to the boundary of $\mathcal{M}_{g,n}$. What naturally extends, over $\overline{\mathcal{M}}_{g,n}$, are the virtual bundles whose fibers are [2]

$$H^0\Big(C, K^k\big(\sum_{i=1}^{n} p_k^{(i)} y_i\big)\Big) \ominus H^1\Big(C, K^k\big(\sum_{i=1}^{n} p_k^{(i)} y_i\big)\Big) \quad .$$

We will arrange for the H^1s to *vanish*, so that the virtual bundle is an honest bundle, which extends to the boundary of the moduli space $\mathcal{M}_{g,n}$. At the boundary of $\mathcal{M}_{g,n}$, the Coulomb branch has components associated to the

[1]We will see in Chapter 7 that in the D_N case the Coulomb branch is actually a complex *variety*, determined as the zero-locus of certain polynomial equations, and that the vector-space structure appears only at the tangent space at the origin, where the Coulomb branch is smooth. This more general picture of the Coulomb branch as a variety is the one that should apply, say, to the exceptional (2,0) theories.

[2]This picture does not take *mass-deformed* punctures into account. These are relevant because upon degeneration of a surface, even if all punctures in the original surface are massless, the new punctures that appear when the cylinder becomes infinitely long will necessarily be mass deformed [1]. Thus, the actual picture for the bundles is probably slightly more complicated than the one discussed here.

irreducible components of C (i.e., 3-punctured spheres) and components associated to the gauge groups on the degenerating cylinders.

The 3-punctured spheres that appear at the boundary of $\mathcal{M}_{g,n}$ will be called *fixtures*. To each cylinder connecting these fixtures we associate a plumbing parameter[3] , $s \sim 16q^{1/2} + \dots$, with $q = e^{2\pi i\tau}$, which controls the strength of the gauge coupling for that factor of the gauge group,

$$\tau = \frac{\theta}{\pi} + \frac{8\pi i}{g^2}.$$

Since there are various different degeneration limits of C, there are various different gauge theory presentations of the same underlying SCFT. These are related by S-duality.

What we would like to do is understand the taxonomy of gauge theory presentations which arise in this way from compactifying a given (2,0) SCFT. To do this, we need a catalogue of what the allowed fixtures (3-punctured spheres) are, and what cylinders (gauge groups) connect them. We can then build up the surface C, in a degeneration limit, as a “tinkertoy”, by connecting fixtures together with cylinders, according to the allowed rules.

In what follows, for the most part, we will restrict ourselves to the case of the sphere, $g = 0$, so that the only degenerations come from the collisions of (multiple) punctures.

[3]In the limit that the other gauge couplings are turned off,

$$s = \frac{\theta_{10}^4(\tau)}{\theta_{00}^4(\tau)}$$

6.1.1 Regular Punctures

In the A_{N-1} series, punctures are labeled by partitions of N. To each such partition, $[h_1, h_2, \ldots, h_p]$, with

$$h_1 \geq h_2 \geq \cdots \geq h_p,$$
$$\sum_{i=1}^{p} h_i = N,$$

we associate a Young diagram, whose i^{th} column has height h_i. The corresponding flavour symmetry group is

$$G = S\left(\prod_h U(n^{(h)})\right) \tag{6.3}$$

where $n^{(h)}$ is the number of columns of height h. Of course, a Young diagram with N boxes determines a second partition of N, given by the row-lengths, $[r_1, r_2, \ldots, r_q]$. The two partitions are said to be transposes of each other, as the map between them consists of taking the transpose of the Young diagram.

This second partition determines a nilpotent orbit [15], $o_{[r_1,r_2,\ldots,r_q]} \subset \mathfrak{sl}(N)$, which determines the pole structure of the $\phi_k(y)$ at the puncture. Specifically, the Higgs field of the Hitchin system (obtained upon further compactifying the theory on a circle) has a simple pole, with residue $X \in o_{[r_1,r_2,\ldots,r_q]}$ at the puncture [2, 5, 32]. There's a fairly simple algorithm for choosing such a representative, X:

- Let X be a block-diagonal matrix, where the i^{th} block is $r_i \times r_i$.
- Within each block, let X be strictly upper-triangular.

The characteristic equation

$$\det(\varphi(y) - q\mathbf{1}) = (-q)^N + \sum_{k=2}^{N} q^{N-k}\phi_k(y) \tag{6.4}$$

(for generic finite part of φ) determines the allowed pole orders of the ϕ_k. The result is easily-expressed in terms of the corresponding Young diagram:

- Starting with 0 in the first box, number the boxes in the first row with successive positive integers.
- When you get to the end of a row, repeat that integer as the number assigned to the first box of the succeeding row. Continue numbering the boxes of that row with successive integers.
- The integers inscribed in boxes $2, \ldots, N$ are, respectively, the pole orders of $\phi_2, \ldots, \phi_N$ at the puncture.

For example, for $N = 6$, the Young diagram with two columns of height 3 corresponds to the pole structure $\{1, 1, 2, 2, 3\}$ and global symmetry group $SU(2)$. In general, for even N, the Young diagram with two columns of the same height will correspond to the pole structure $\{1, 1, 2, 2, 3, 3, \ldots, N-1, N-1, N\}$ and global symmetry group $SU(2)$.

By construction, partitions of N (or Young diagrams) and pole structures in the A_{N-1} theory are in 1:1 correspondence. So, for the A_{N-1} series, we are allowed to use the pole structures to label punctures. This will not be true for the punctures of the other (2,0) theories[4].

There are two regular punctures that deserve special names. The regular puncture with $p_k = k-1$, for $k = 2,\dots,N$, will be called a "maximal puncture". It corresponds to the situation with N different residues of the mass-deformed Seiberg-Witten differential, so its associated Young diagram consists of one row with N boxes, and its associated global symmetry group is $SU(N)$. On the other hand, the regular puncture with $p_k = 1, \forall k$, will be called "minimal"; it corresponds to having $(N-1)$ equal residues of the mass-deformed Seiberg-Witten differential, its Young Diagram consists of one row with two boxes, and $N-2$ rows with one box, and its associated global symmetry group is $U(1)$.

Also, there is always a trivial A_{N-1} nilpotent orbit, of zero-dimension, "pole structure" $\{0,0,\dots,0,0\}$ and trivial global symmetry. It corresponds to the "absence" of a puncture. Thus, we will simply ignore it.

Thus, for the A_{N-1} theory, ignoring the trivial orbit, we will have $P(N)-1$ punctures. A colliding pair of regular punctures will give rise to a fixture connected by a cylinder to the rest of the surface. Our job will be to characterize the fixtures that arise as well as the cylinders that connect them.

[4]For the other simply-laced Lie algebras, there exist different punctures with the same pole structures (but with other physical properties, such as global symmetry group, or constraints, that are different). Also, the exceptional nilpotent orbits are *not* classified by partitions. In general, we should use the *nilpotent orbits*, rather than pole structures or partitions, of our simply-laced Lie algebra to classify punctures.

6.1.2 Irregular punctures

As we will see below, when two regular punctures collide, the resulting fixture will correspond to one of three possibilities:

1. a number of free hypermultiplets,
2. an interacting SCFT,
3. an interacting SCFT accompanied by a number of free hypermultiplets.

The first case corresponds to a fixture with no Coulomb branch, while the other two cases correspond to a fixture with a positive-dimensional Coulomb branch.

As we mentioned in the Section 6.1, we want the graded dimension of the Coulomb branch of the degenerate surface (defined as the sum of the graded dimensions of the Coulomb branch of the fixture, the Coulomb branch of gauge theory on the attaching cylinder and the Coulomb branch of the rest of the surface) to agree with the graded dimension of the Coulomb branch of the original surface, C. To achieve this, we would like — as a *bookkeeping device* — for the graded virtual dimension and the actual graded dimension of the Coulomb branch of the fixture to agree. This determines, uniquely, the pole structure at the attaching puncture (the third puncture of the 3-punctured sphere).

For a fixture corresponding to free hypermultiplets, the Coulomb branch is zero-dimensional. To achieve this, we are forced in most cases (the exception being the collision of a minimal and a maximal puncture) to introduce punctures with pole structures that are *not* regular, i.e., that do not arise from the construction detailed in §6.1.1. We call punctures with such pole structures "irregular"[5].

Irregular punctures will also appear in some fixtures associated to interacting SCFTs. There, too, they will be determined by requiring that, when certain d_k are supposed to vanish, the actual and virtual value of d_k agree (and are zero).

We do not have an algorithm to generate the possible irregular punctures in the A_{N-1} series. Instead, we will have to find them by experimenting with degenerations. There turns out to be a finite set of them for every N. They satisfy the following properties:

- From the pole structure $\{p_k\}$, of the irregular puncture, we should be able to construct a regular pole structure $\{p_k^{(\mathrm{reg})}\}$, which corresponds to a puncture with global symmetry group G_{reg}, and such that

[5]This point of view ignores the complication of mass-deformed punctures. In a more complete picture, an irregular puncture will be a constrained versions of the mass-deformed version of a certain regular puncture, and should probably *not* have the "higher poles" described in this section. This picture of an irregular puncture should also clarify what the Hitchin boundary condition for an "irregular puncture" should be: the semisimple version of the regular puncture, some of whose eigenvalues are functions of mass parameters of other punctures present in the surface. In particular, our "irregular" punctures are *not* related to the "irregular singularities" of the Hitchin-system literature (as in, e.g, [2, 39, 46]). Along with the issue of the correct picture for the bundles on C, which we mentioned in a previous footnote, this is still a point currently under investigation. In any case, an improved picture will not change our results about S-duality, but should merely make the local Hitchin boundary condition more precise.

- $p_k^{(\mathrm{reg})} = p_k = k-1$ if k is an exponent of a simple Lie subgroup $G \subset G_{\mathrm{reg}}$.
- $p_k^{(\mathrm{reg})} + p_k = 2k-1$ otherwise.

- We declare the group G to be the global symmetry group of the puncture.
- We denote the irregular puncture, thus constructed, by the Young diagram of the associated regular puncture, with one or more "$*$"s appended.

Thus, every irregular puncture is associated to a specific regular puncture. However, this is not a 1:1 relation. A single regular puncture may have several irregular punctures associated to it.

6.1.3 Fixtures

From (6.1), the dimension d_k of the Coulomb branch subspace V_k for a sphere with n punctures is

$$d_k = 1 - 2k + \left(\sum_{i=1}^{n} p_k^{(i)} \right), \tag{6.5}$$

where $p_k^{(i)}$ $(k = 2, \dots, N)$ represents the pole structure of the i-th puncture, $i = 1, \dots, n$. Moreover, we require that the RHS of (6.5) be *non-negative*, for each k, i.e. that the virtual dimension and the actual dimension agree. Having done this, our bookkeeping rules will ensure that, when C degenerates, the same is true of the d_k of the Coulomb branches associated to each of the component pieces.

For a 3-punctured sphere (a “fixture”), we will require, for each k, that

- if $d_k > 0$, then $p_k^i \leq k - 1$, for $i = 1, 2, 3$.

As a simple consequence any fixture has at most one irregular puncture.

If $d_k = 0$ for all k, we have a free-field fixture. If the three punctures are regular, then necessarily one of them is minimal and the other two are maximal. On the other hand, an interacting SCFT fixture (which could also have free hypermultiplets) consists of three punctures such that $d_k > 0$ for at least one k.

6.1.4 Cylinders

When two or more punctures on a Riemann surface collide, the surface degenerates, and a long cylinder connecting the two pieces appears (which could still be attached somewhere else). When the cylinder becomes infinitely long and thin, a new puncture appears at each of the two pieces of the Riemann surface where the ends of the cylinder were. The long, thin cylinder corresponds to a weakly-coupled gauge group. When the gauge coupling is infinitely weak, we are left with flavor symmetries at each end of the cylinder, corresponding to the two new punctures. Similarly, two punctures on a Riemann surface (or on two initially disconnected Riemann surfaces) can sometimes be glued to each other by a cylinder. In both cases the gauge group corresponding to the cylinder is a subgroup of the flavor groups associated to the punctures. Given two (regular or irregular) punctures, we want to see when they can be connected to each other, and what the arising gauge group is.

We will denote a cylinder connecting a puncture of pole structure $\{p_k\}$ with a puncture of pole structure $\{p'_k\}$ by

$$\{p_k\} \xleftrightarrow{\quad G \quad} \{p'_k\},$$

where G denotes a gauged subgroup of the flavor symmetry groups of the two theories connected by the cylinder.

Let $q_k = \min(p_k, p'_k)$. For the cylinder to be valid, G, $\{p_k\}$ and $\{p'_k\}$ must satisfy the following requirements:

- q_k is a regular pole structure.
- G is a subgroup of the global symmetry group, G_q, where G_q is the symmetry group associated to $\{q_k\}$, following the Young diagram prescription.
- $\mathrm{rank}(G) = N^2 - 1 - \sum_{k=2}^{N}(p_k + p'_k)$.
- For each k, we have either $\begin{cases} p_k = p'_k = k-1 \\ p_k + p'_k = 2k-1 \end{cases}$.
- The exponents of G are the set of k such that $p_k = p'_k = k-1$. (Notice there cannot be repeated exponents.)

In particular, for the A_{N-1} theories, $G = SU(n)$ or $Sp([n/2])$, for some $n \leq N$.

Since we must have $1 \leq \text{rank}(G) \leq N-1$, two regular punctures can be connected by a cylinder if and only if they are maximal, in which case the gauge group is $G = SU(N)$. The vast majority of cylinders will connect a regular and an irregular puncture.

Occasionally, though, cylinders connecting two irregular punctures will appear (see the case A_3 below). These are rare, as the tension between the rank condition and the condition on the exponents is quite restrictive.

We can now explain how the irregular punctures serve as a useful book-keeping device. Consider the collision of two punctures $\{p_k\}$ and $\{p'_k\}$ on a Riemann surface C. They bubble off a sphere S, which is attached by a cylinder T to the rest of C. Let the pole structure of the new puncture to which S is attached by T be $\{p''_k\}$. Before the collision, the contribution of $\{p_k\}$, $\{p'_k\}$ to the total dimension of the Coulomb branch of the theory on C was

$$\sum_{k=2}^{N} p_k + p'_k. \tag{6.6}$$

After the collision, such contribution becomes

$$d_S + \text{rank}(G_T) + \sum_{k=2}^{N} p''_k, \tag{6.7}$$

where $d_S \geq 0$ is the dimension of the Coulomb branch associated to the fixture S, and G_T is the gauge group associated to the cylinder T. The requirements on the cylinder that we listed above ensure that (6.6) and (6.7) agree.

The rules above actually guarantee that the agreement is finer than that. Recall that the Coulomb branch (6.2) is not just a vector space, but a *graded* vector space (with grading given by k). We want to ensure that the *graded* dimensions, $d_k = \dim(V_k)$, agree. In the degeneration limit, certain of the ϕ_k (precisely the ones satisfying the $p_k = p'_k = k-1$ condition) are allowed to have a k-th order pole at the node, with the residues agreeing on the two sides. The residue is the Coulomb-branch parameter for the gauge theory on the cylinder. The degrees of these Coulomb-branch parameters are precisely the exponents of G. In other words, when $p_k = p'_k = k-1$, the dimension of that graded component of the Coulomb branch of G is 1. When $p_k + p'_k = 2k-1$, the dimension (and virtual dimension) of that graded component of the Coulomb branch of G vanishes.

6.2 Symmetries and Central Charges

We already wrote down the formula for the effective number of vector multiplets,

$$n_v = \sum_{k=2}^{N} (2k-1) d_k, \tag{6.8}$$

which is true for a Lagrangian theory. As will be clear from our analysis, (6.8) will provide the correct definition for the effective n_v, even in cases where there is no weakly-coupled Lagrangian dual.

It will be convenient for us to have an expression for the contribution δn_v of each puncture to n_v. Using the expression for d_k, we get

$$n_v = \frac{4N^3 - 4N + 3}{3} \times (g-1) + \sum_i \delta n_v^{(i)}, \tag{6.9}$$

where i runs over the punctures, and

$$\delta n_v = \sum_{k=2}^{N} (2k-1) p_k, \tag{6.10}$$

is the contribution of a single puncture with poles $\{p_k\}$. We take this expression for δn_v to be correct for both regular and irregular punctures.

For the effective number of hypermultiplets, we combine the above with a result of Nanopoulos and Xie [4], to obtain

$$n_h = -(1-g)\frac{4N(N^2-1)}{3} + \sum_{i=1}^{n} \delta n_h^{(i)}, \tag{6.11}$$

where $\delta n_h^{(i)}$ is the contribution of the i^{th} puncture. For a regular puncture,

$$\delta n_h^{(\text{reg})} = \tfrac{1}{2}\left(-N + \sum_r l_r^2\right) + \delta n_v^{(\text{reg})}, \tag{6.12}$$

where l_r is the length of the r^{th} row of the Young diagram, and $\delta n_v^{(\text{reg})}$ is the contribution of this puncture to n_v.

For an irregular puncture, define the pole structure $\{p_k^{(\text{reg})}\}$, as in (6.1.2). $\{p_k^{(\text{reg})}\}$ is, by definition, a regular pole structure, corresponding to a puncture with Young diagram rows $\{l_r\}$, and whose contribution to n_h is $\delta n_h^{(\text{reg})}$. The contribution of an irregular puncture, then, is [6]

$$\delta n_h^{(\text{irreg})} = \frac{4(N^2-1)N}{3} - \delta n_h^{(\text{reg})}. \tag{6.13}$$

[6] The origin of this formula is clear. The irregular puncture, $\{p_k\}$, can be attached to the rest of the surface via a cylinder $\{p_k\} \longleftrightarrow \{p_k^{(\text{reg})}\}$. Cylinders do not contribute any hypermultiplets, and (6.13) is simply the embodiment of the requirement that n_h should be the same, before and after sewing.

Notice that, because of this equation, all irregular punctures associated to a single regular puncture share the same value of $\delta n_h^{(\mathrm{irreg})}$.

Applying this to the case of a sphere with three maximal punctures, one recovers the result for the T_N theories [3, 47],

$$\begin{aligned} k &= 2N, \\ a &= \frac{N^3}{6} - \frac{5N^2}{16} - \frac{N}{16} + \frac{5}{24}, \\ c &= \frac{N^3}{6} - \frac{N^2}{4} - \frac{N}{12} + \frac{1}{6}. \end{aligned}$$

We will check these results for the T_N theories explicitly for the cases up to $N = 5$, as well as identify a host of new theories.

6.3 Identifying fixtures

In this section, for convenience, we will denote punctures by their pole structures instead of their Young diagrams. We already mentioned in Section 6.1.1 that doing this is allowed for the A_{N-1} series.

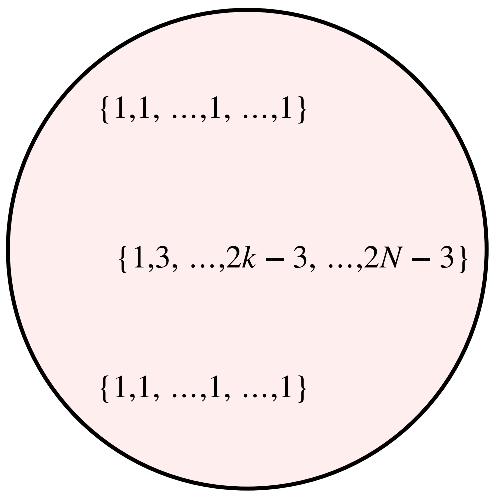
We take as our starting point that

1. The A_{N-1} fixture arising from the collision of a minimal puncture and a maximal puncture corresponds to N^2 free hypermultiplets. (The third puncture in this fixture is then also maximal.)

2. The A_{N-1} fixture arising from the collision of two minimal punctures corresponds to 2 free hypermultiplets. (The third puncture in this fixture is then irregular, of the form $\{1, 3, \ldots, 2k-3, \ldots, 2N-3\}$.)

3. Fixtures corresponding to n_h free hypermultiplets (with n_h given by (6.11)) will have $n_v = 0$, according to (6.8) and have zero-dimensional Coulomb branches.

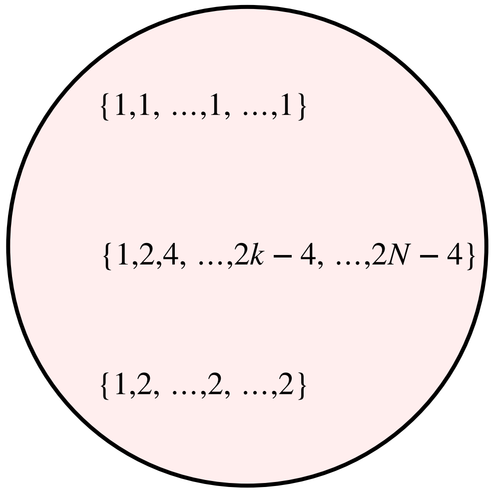
By studying collisions of more regular punctures, we can bootstrap the properties above to identify further fixtures. Consider, for instance, the collision of several minimal punctures. When two of them collide, the fixture



is attached to the rest of the surface with the cylinder

$$\{1,3,\dots,2k-3,\dots,2N-3\} \xleftrightarrow{SU(2)} \{1,2,\dots,2,\dots,2\}.$$

Colliding the $\{1,2,\dots,2,\dots,2\}$ puncture with another minimal puncture produces a free-field fixture



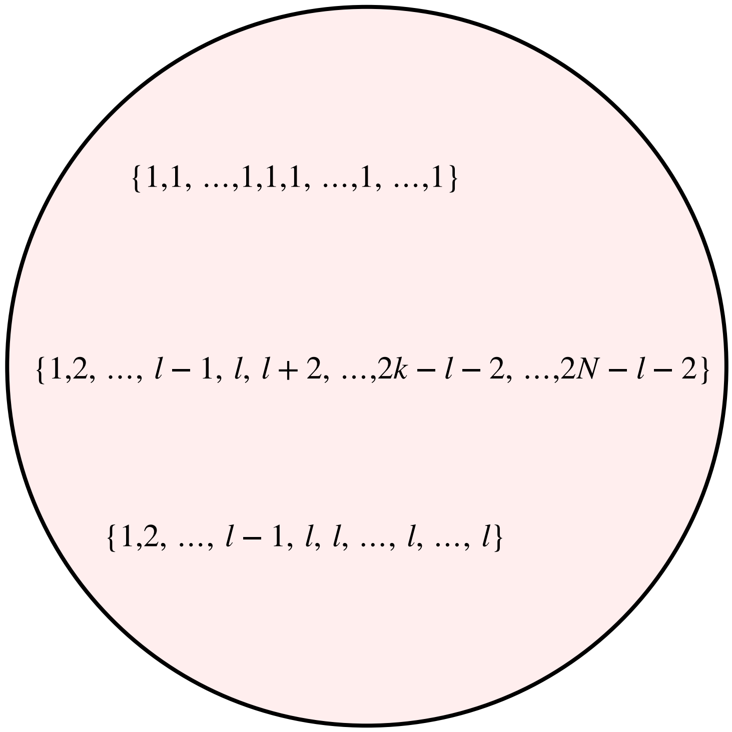
By conformality of the $SU(2)$, this consists of 6 hypermultiplets (transforming as 3 copies of the 2). This fixture, in turn, is attached to the rest of the surface by the cylinder

$$\{1,2,4,\dots,2k-4,\dots,2N-4\} \xleftrightarrow{SU(3)} \{1,2,3,\dots,3,\dots,3\}.$$

Colliding the $\{1,2,3,\dots,3\}$ puncture with another minimal puncture produces a fixture which (by conformality of the $SU(3)$) consists of 12 hypermultiplets, transforming as 4 copies of the 3.

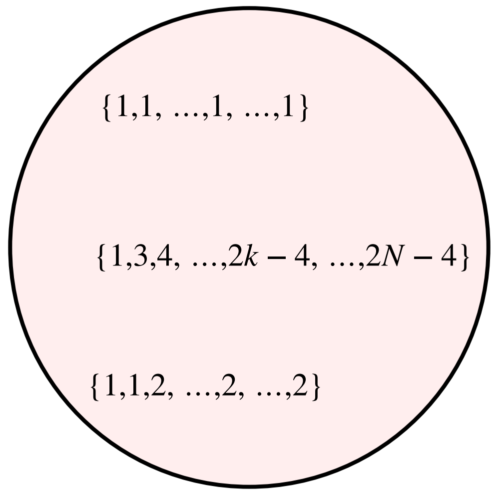
Repeating the process, we deduce a series of fixtures consisting of $l(l+1)$

hypermultiplets, transforming as the bifundamental of $SU(l) \times SU(l+1)$,



The next simplest puncture has pole structure, $\{1, 1, 2, \ldots, 2, \ldots, 2\}$, corresponding to the Young diagram with two rows of length 2, and the rest of length 1.

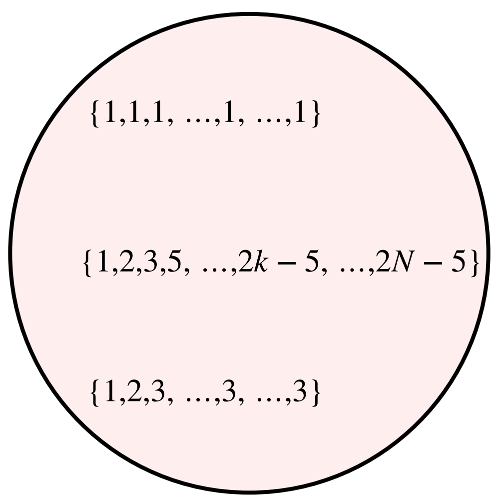
Colliding this with a minimal puncture, we produce the fixture



This attaches to the rest of the surface via the cylinder

$$\{1,3,4,\ldots,2k-4,\ldots,2N-4\} \xleftrightarrow{SU(2)} \{1,2,3,\ldots,3,\ldots,3\}.$$

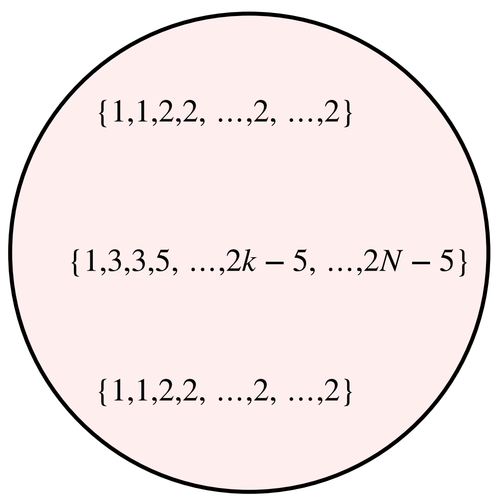
If we collide that puncture with another minimal puncture, we obtain a fixture we have seen before,



which consisted of 12 hypermultiplets, transforming as the $(3,4)$ of $SU(3) \times SU(4)$. Here, we are gauging an $SU(2) \subset SU(3)$. This fixture, by itself, provides enough matter to make the $SU(2)$ conformal. Thus, the fixture in (6.3) consists of zero hypermultiplets.

In similar fashion, we can proceed to identify the free-field fixtures corresponding to the collision of *any* regular puncture with a minimal puncture.

We can then go on to identify other fixtures, which arise as collisions of punctures we have studied already. For instance, colliding two $\{1,1,2,2,\dots,2,\dots,2\}$ punctures, we obtain the fixture



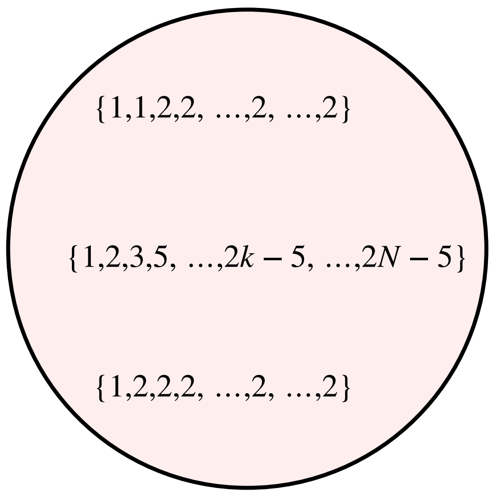
This attaches to the rest of the surface via the cylinder

$$\{1,3,3,5,,\dots,2k-5,\dots,2N-5\} \xleftrightarrow{Sp(2)} \{1,2,3,4,4,\dots,4,\dots,4\}.$$

If we collide that puncture with another minimal puncture, we again obtain a fixture we have seen before: this time, 20 hypermultiplets transforming as the

$(4,5)$ of $SU(4)\times SU(5)$. Here we are gauging $Sp(2)\subset SU(4)$, so conformality of the $Sp(2)$ requires that the fixture in (6.3) consists of 4 hypermultiplets, transforming as the fundamental of $Sp(2)$.

Colliding a $\{1,1,2,2,\dots,2,\dots,2\}$ puncture with a $\{1,2,2,2,\dots,2,\dots,2\}$ puncture, we obtain the free-field fixture



On the one hand, we can gauge this fixture by attaching a

$$\{1,3,5,\dots,2k-3,\dots,2N-3\}\xleftrightarrow{SU(2)}\{1,2,2,\dots,2,\dots,2\}$$

cylinder. On the other, we can attach a

$$\{1,2,3,5,\dots,2k-5,\dots,2N-5\}\xleftrightarrow{SU(4)}\{1,2,3,4,\dots,4,\dots,4\}.$$

To ensure conformality of both the $SU(2)$ and the $SU(4)$, we conclude that this fixture consists of 10 hypermultiplets, transforming as the $(1,4)+\frac{1}{2}(2,6)$ of $SU(2)\times SU(4)$. (Note that the $(2,6)$ representation is pseudo-real, so we can have matter in a half-hypermultiplet, in that representation. Also, $\ell_6 = 2$, which ensures conformality of the $SU(4)$.)

Having proceeded as far as we can, in this fashion, we can then use these "known" fixtures, plus S-duality, to deduce the identity of other fixtures (including the interacting SCFTs). To see how that works, it is perhaps best to proceed by example.

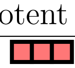
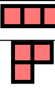
6.4 Taxonomy

6.4.1 A_1

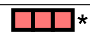
For A_1, there's just one type of regular puncture, $\{1\}$, where the quadratic differential, ϕ_2 is allowed to have a simple pole, and there are no irregular punctures. Correspondingly, there is one type of cylinder, which has gauge group $SU(2)$. Similarly, there is only one fixture, with three $\{1\}$ punctures, which is the free theory of four hypermultiplets, or, in a language more appropriate for the A_1 case, eight half-hypermultiplets, which transform as a $(2, 2, 2)$ representation of the $SU(2) \times SU(2) \times SU(2)$ flavor subgroup of this fixture. As before, half-hypermultiplets are allowed because the fundamental representation of $SU(2)$ is pseudo-real.

6.4.2 A_2

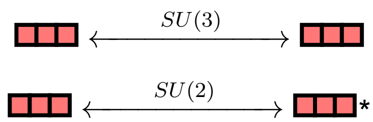
There are now two regular punctures:

Nilpotent orbit	Pole structure	Flavour symmetry	$(\delta n_h, \delta n_v)$
	$\{1, 2\}$	$SU(3)$	$(16, 13)$
	$\{1, 1\}$	$U(1)$	$(9, 8)$

There is one type of irregular puncture:

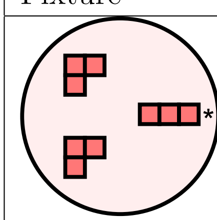
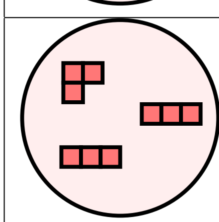
Young diagram	Pole structure	Flavour symmetry	$(\delta n_h, \delta n_v)$
	$\{1,3\}$	$SU(2)$	$(-4,18)$

and two cylinders:



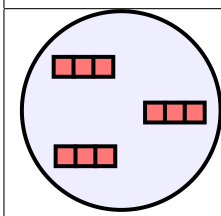
There are three distinct types of collisions giving rise to three different fixtures: the collision of two minimal punctures, a minimal and a maximal puncture, and two maximal punctures. The first two cases yield free-field fixtures. The third yields a fixture with a one-dimensional Coulomb branch, the interacting E_6 SCFT of Minahan and Nemeschansky [11].

The free-field fixtures are:

Fixture	Number of Hypers	Representation
	2	$\mathbf{2}$
	9	$(\mathbf{3}, \mathbf{3})$

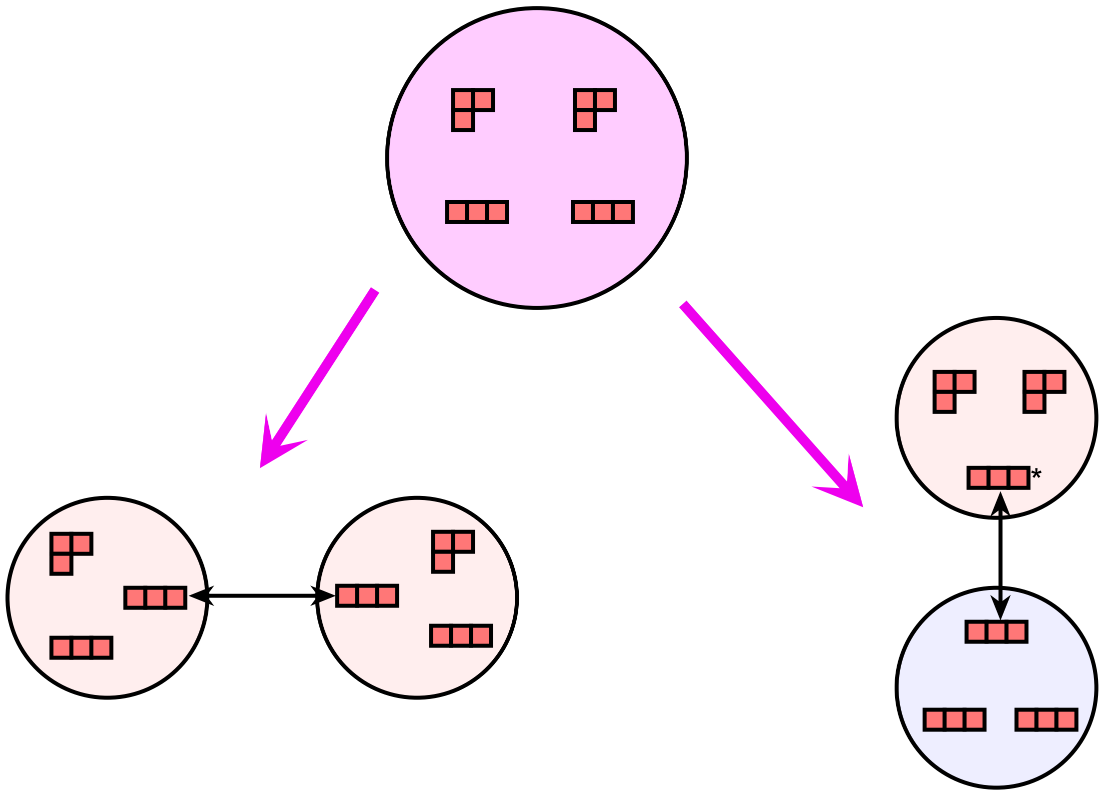
Here we have listed the matter representation of the (non-Abelian) global symmetry group of each puncture (or, in the case of an irregular puncture, of the gauge group of the attaching cylinder).

The interacting fixture is

Fixture	(d_2, d_3)	(a, c)	$(G_{\text{global}})_k$
	$(0, 1)$	$\left(\frac{41}{24}, \frac{13}{6}\right)$	$(E_6)_6$

Here we have listed the graded dimensions d_k of the Coulomb branch (the total dimension is $d = \sum_k d_k$), the central charges, (a, c), the global symmetry group G_{global} of the SCFT, and the central charge k of the G_{global} current algebra.

The basic S-duality of the A_2 theory (discovered by Argyres and Seiberg [12]), can be seen by studying the various degenerations of the 4-punctured sphere.

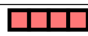
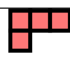
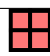
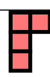


On one side we have an $SU(3)$ gauge theory with 6 hypermultiplets in the fundamental (3 from each fixture). On the other, we have an $SU(2)$ gauge theory coupled to one fundamental hypermultiplet, where the $SU(2)$ is a gauged subgroup of the original $\subset E_6$ flavor symmetry of the interacting E_6 SCFT. The central charge of the E_6 current algebra is such that the β-function of the $SU(2)$ vanishes. In both cases, the global symmetry group is $SU(6) \times U(1)$. In the $SU(2)$ gauge theory, the $SU(6)$ global symmetry arises as the commutant of $SU(2) \subset E_6$.

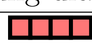
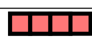
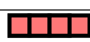
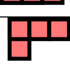
We can use this example of S-duality to compute the (a, c) central charges of the E_6 SCFT. The effective number of vector multiplets and hypermultiplets of the $SU(3)$ $N_f = 6$ theory are $n_v = 8$ and $n_h = 18$, respectively. In the S-dual theory, the $SU(2)$ gauge group and the fundamental hypermultiplet contribute $n_v = 3$ and $n_h = 2$, so the difference gives $n_v = 5$ and $n_h = 16$ for the E_6 theory. From these numbers we compute $a = \frac{41}{24}$ and $c = \frac{13}{6}$. The results, of course, agree with our explicit formulæ, (6.11) and (6.8).

6.4.3 A_3

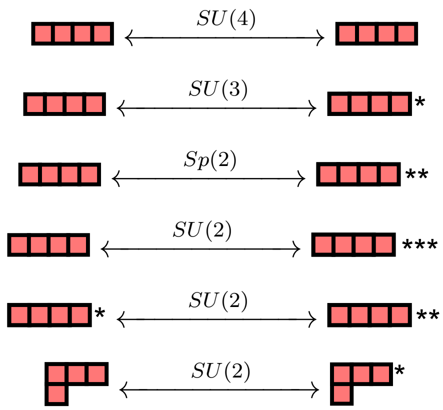
Now we turn to the A_3 theory. There are four regular punctures:

Nilpotent orbit	Pole Structure	Global Symmetry	$(\delta n_h, \delta n_v)$
	$\{1, 2, 3\}$	$SU(4)$	$(40, 34)$
	$\{1, 2, 2\}$	$SU(2) \times U(1)$	$(30, 27)$
	$\{1, 1, 2\}$	$SU(2)$	$(24, 22)$
	$\{1, 1, 1\}$	$U(1)$	$(16, 15)$

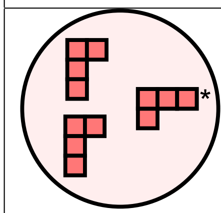
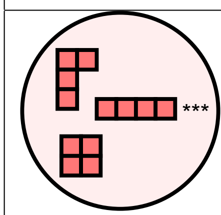
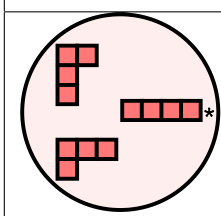
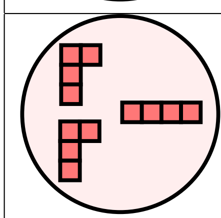
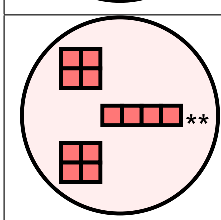
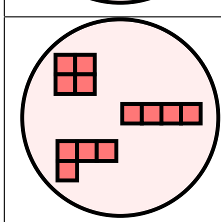
and four irregular punctures:

Young diagram	Pole Structure	Global Symmetry	$(\delta n_h, \delta n_v)$
 *	$\{1, 2, 4\}$	$SU(3)$	$(40, 41)$
 **	$\{1, 3, 3\}$	$Sp(2)$	$(40, 39)$
 ***	$\{1, 3, 4\}$	$SU(2)$	$(40, 46)$
 *	$\{1, 3, 5\}$	$SU(2)$	$(50, 53)$

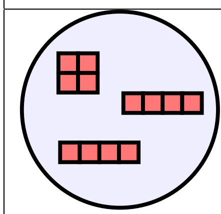
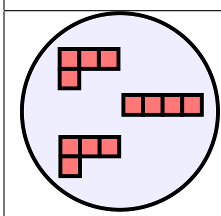
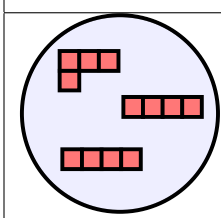
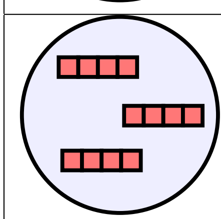
The cylinders are:



To determine the fixtures, we need to consider all possible collisions of pairs of regular punctures. There are ten such collisions; six lead to free-field fixtures, and four to interacting SCFT fixtures. The ones which lead to free-field fixtures are (we draw the pair of punctures that collide on the left):

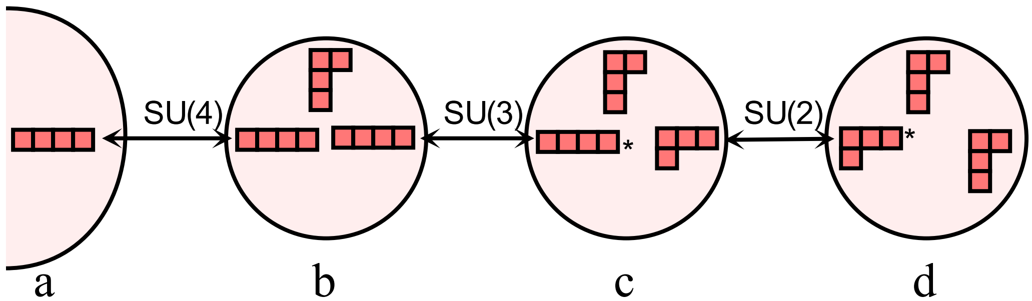
Fixture	Number of Hypers	Representation
	2	2
	0	–
	6	$(2,3)$
	16	$(4,4)$
	8	$\frac{1}{2}(2,2,4)$
	14	$(2,1,4)+\frac{1}{2}(1,2,6)$

The interacting fixtures are:

Fixture	(d_2, d_3, d_4)	(a, c)	$(G_{\text{global}})_k$	Theory
	$(0, 0, 1)$	$\left(\frac{59}{24}, \frac{19}{6}\right)$	$(E_7)_8$	The E_7 SCFT of Minahan-Nemeschansky
	$(0, 1, 0)$	$\left(\frac{15}{8}, \frac{5}{2}\right)$	$(E_6)_6$	The E_6 SCFT plus 4 free hypers
	$(0, 1, 1)$	$\left(\frac{15}{4}, \frac{9}{2}\right)$	$SU(2)_6 \times SU(8)_8$	New. "$R_{0,4}$".
	$(0, 1, 2)$	$\left(\frac{45}{8}, \frac{13}{2}\right)$	$SU(4)_8^3$	"New." T_4.

To understand the free-field fixtures, it is helpful to repeat the analysis that Gaiotto did, of "the ends of linear quivers" [1]. In the present notation, we have a set of punctures colliding, in hierarchical fashion, producing a chain of fixtures, connected to the rest of C.

Consider the following chain, obtained as the collision of four minimal ($\{1,1,1\}$) punctures on C.

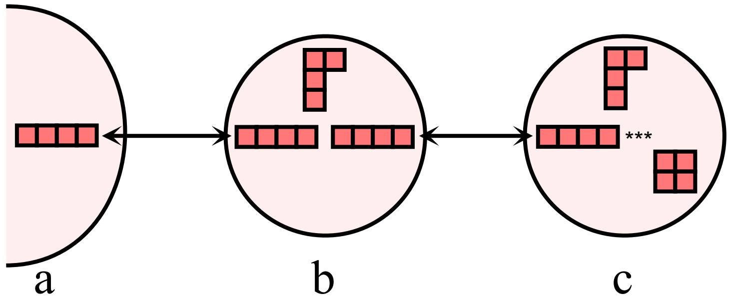


Representing the rest of C as four hypermultiplets in the fundamental of $SU(4)$, the matter content of this theory is

	# hypers	$SU(4)$	$SU(3)$	$SU(2)$
a	4	4	1	1
b	1	4	1	1
	1	4	3	1
c	1	1	3	2
d	1	1	1	2

Each gauge group factor has vanishing β-function. We can obtain the gauge theories correponding to other, related, collisions by lopping fixtures off of the end of the picture. For instance, the gauge theory corresponding to the collision of two minimal punctures and a $\{1,2,2\}$ puncture is obtained by omitting fixture "d" and the $SU(2)$ gauge group factor.

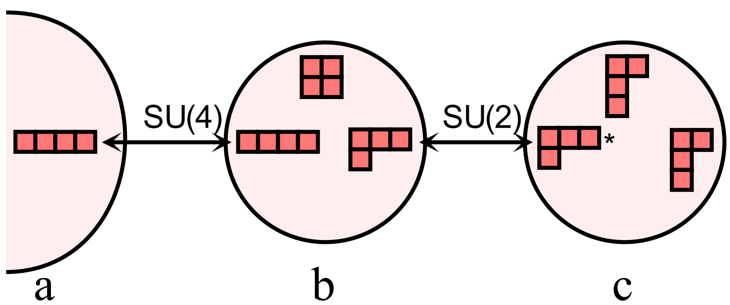
The collision of two minimal punctures and a $\{1,1,2\}$ puncture gives rise to



with matter content

	# hypers	$SU(4)$	$SU(2)$
a	4	4	1
b	1	4	2
	2	4	1
c	–	–	–

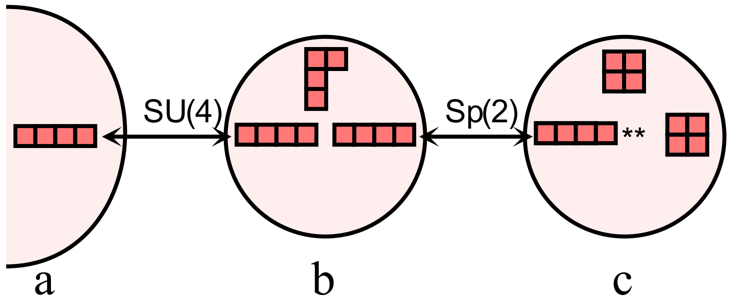
This theory is S-dual to



with matter content

	# hypers	$SU(4)$	$SU(2)$
a	4	4	1
b	2	4	1
	$\frac{1}{2}$	6	2
c	1	1	2

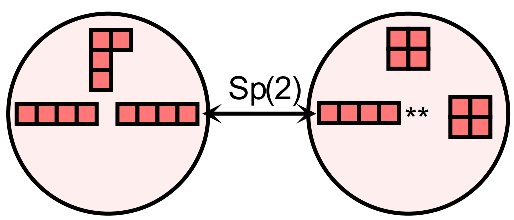
The collision of one minimal puncture and two $\{1,1,2\}$ punctures gives



with matter content

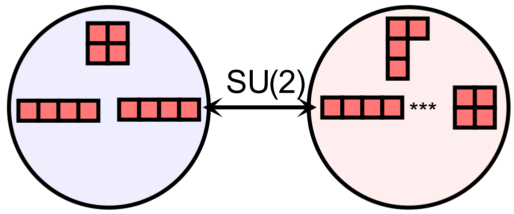
	# hypers	$SU(4)$	$Sp(2)$
a	4	4	1
b	1	4	4
c	2	1	4

If we S-dualize this, we end up with an interacting SCFT fixture. To study that, in its simplest context, let's turn off the $SU(4)$, and consider the simpler theory



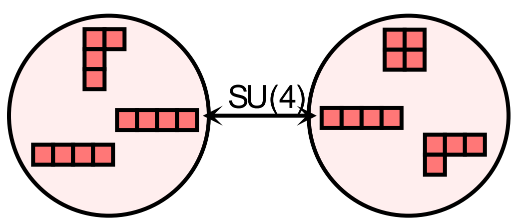
which is an $Sp(2)$ gauge theory with 6 hypers in the fundamental (4 from the fixture on the left, and 2 from the fixture on the right). The global symmetry group is $SO(12)$. The Seiberg-Witten solution can be found in [48].

S-dualizing, we obtain

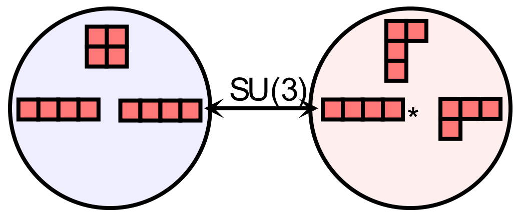


The fixture on the right contains no matter, so the theory is an $SU(2)$ gauging of the interacting fixture on the left. The commutant of $SU(2) \subset G$ must be $SO(12)$, and conformality implies that the central charge $k = 8$. Exactly these considerations led Argyres and Seiberg [12] to identify the SCFT corresponding to this fixture as the E_7 SCFT of Minahan and Nemeschansky [11]. We can use this example to find $n_v = 7$ and $n_h = 24$ for the E_7 SCFT, from which we compute $a = \frac{59}{24}$ and $c = \frac{19}{6}$ (which, again, agree with our explicit formulæ, (6.8),(6.11)).

We can use our rules to find the E_7 theory in a different example, as the strong coupling point of a Lagrangian theory with $SU(4)$ gauge group. Consider

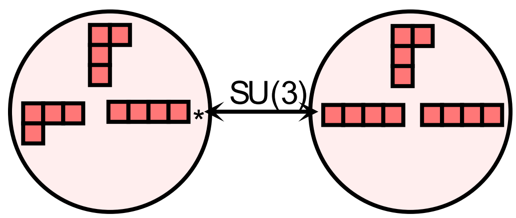


This is an $SU(4)$ gauge theory with 6 fundamental hypermultiplets, and 1 hypermultiplet in the 6 of $SU(4)$. The S-dual frame containing the E_7 theory is

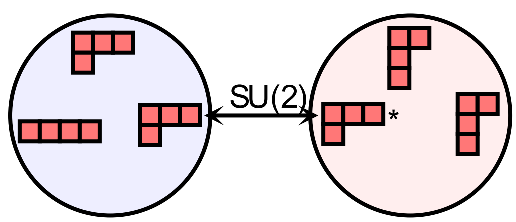


This is an $SU(2)$ gauging of the E_7 theory, coupled to 2 fundamental hypermultiplets. One can also compute $n_v = 7$ and $n_h = 24$ for the E_7 theory from this example, which agrees with what we obtained previously.

Let us study the next in the list of interacting SCFT fixtures. Start with

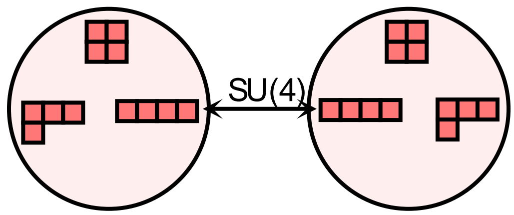


This is $SU(3)$ with 6 fundamental hypers, and 4 free hypers. S-dualizing, we obtain



But we have seen this S-duality before (without the 4 free hypers) when we studied the A_2 theory. The fixture on the right is two hypers (one fundamental of $SU(2)$). So the fixture on the left must be the E_6 SCFT plus 4 free hypers. Indeed, one finds $n_v = 5$ and $n_h = 20$ (and so $a = \frac{15}{8}$ and $c = \frac{5}{2}$) for this fixture, which is what we expected, given the values $n_v = 5$ and $n_h = 16$ for the E_6 SCFT alone.

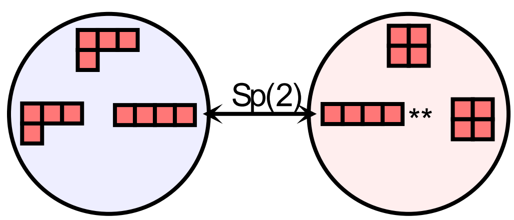
As a further check on this identification, consider



This is an $SU(4)$ gauge theory with 4 hypermultiplets in the fundamental, and 2 hypermultiplets in the 6. The global symmetry group is

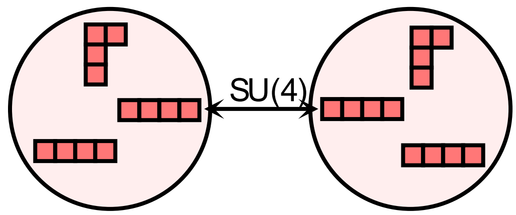
$$G_{\text{global}} = SU(4)_8 \times Sp(2)_6 \times U(1). \tag{6.14}$$

S-dualizing, we obtain

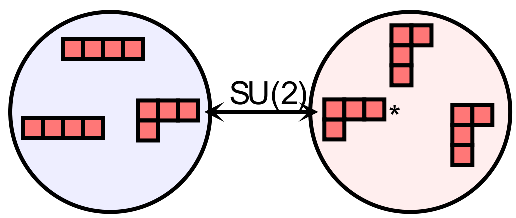


This is an $Sp(2)$ gauge theory. The fixture on the right supplies two hypermultiplets in the fundamental. According to our identification, the fixture on the left provides one more fundamental hypermultiplet, making a total of 3 fundamental hypers. Gauging an $Sp(2) \subset E_6$, with $k = 6$, ensures conformality. The global symmetry group associated to the 3 fundamental hypers is $SO(6) \sim SU(4)$. The commutant of $Sp(2) \subset E_6$ is $Sp(2) \times U(1)$, giving an overall global symmetry group which agrees with (6.14).

Next we turn to

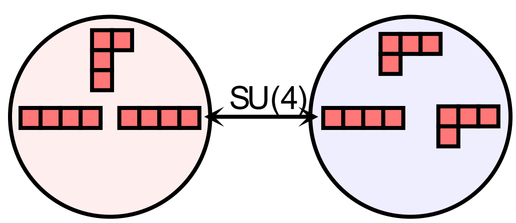


This is $SU(4)$ with 8 fundamental hypers. It is conformal, and has an $SU(8)_8 \times U(1)$ global symmetry. S-dualizing, we obtain



This is $SU(2)$ with one fundamental hyper (from the fixture on the right), coupled to an $SU(2)$ subgroup of the global symmetry group of the interacting SCFT fixture on the left. The commutant of $SU(2)$ must be $SU(8)$, and the central charge of the $SU(2)$ current algebra must be $k = 6$.

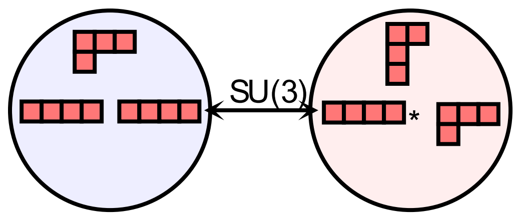
To gain more information, consider



This is an $SU(4)$ gauge theory. The fixture on the left provides 4 hypermultiplets in the fundamental. The free hypers from the fixture on the right provide one more fundamental (making a total of 5 fundamental hypers). Gauging an $SU(4) \subset E_6$, at $k = 6$ makes the theory conformal. The commutant of $SU(4) \subset E_6$ is $SU(2) \times SU(2) \times U(1)$, so the global symmetry group of this gauge theory is

$$G_{\text{global}} = SU(5)_8 \times SU(2)_6^2 \times U(1)^2.$$

S-dualizing, we obtain



The fixture on the right supplies 2 hypermultiplets in the fundamental. These supply an $SU(2) \times U(1)$ subgroup of the global symmetry group.

If we gauge an $SU(3) \subset SU(8)$ of the fixture on the right, we obtain conformality for $k = 8$. Moreover, the commutant of $SU(3) \subset SU(8)$ is $SU(5) \times U(1)$. So we obtain conformality and the correct global symmetry groups for our two examples if

$$G_{\text{SCFT}} = SU(2)_{k=6} \times SU(8)_{k=8}.$$

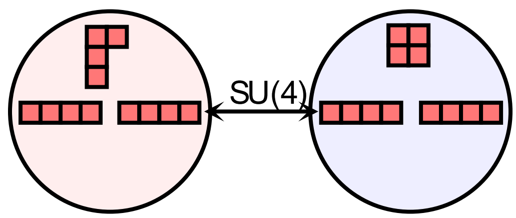
From either of the above two gaugings of this $SU(2)_{k=6} \times SU(8)_{k=8}$ SCFT we can compute $n_v = 12$ and $n_h = 30$, and so its central charges are $a = \frac{15}{4}$ and $c = \frac{9}{2}$.

This SCFT with global symmetry $SU(2)_{k=6} \times SU(8)_{k=8}$ belongs to a series, $R_{0,N}$, of A_{N-1} ($N \geq 3$) interacting SCFTs with global symmetry

$$G_{\text{global}} = SU(2)_{k=6} \times SU(2N)_{k=2N} \ ,$$

which we will discuss in §6.6.

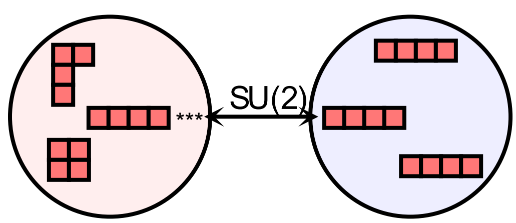
Finally, let us pass to the last of the interacting fixtures on our list. Consider



The fixture on the left provides 4 hypermultiplets in fundamental. Gauging an $SU(4) \subset E_7$ at $k = 8$ achieves conformality. The commutant of $SU(4) \subset E_7$ is $SU(4) \times SU(2)$. So, overall, the global symmetry group is

$$G_{\text{global}} = SU(4)_8^2 \times SU(2)_8 \times U(1). \tag{6.15}$$

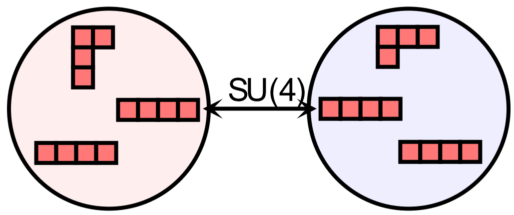
S-dualizing, we obtain



The fixture on the left supplies no matter. To achieve conformality, gauging an $SU(2)$ subgroup of G_{SCFT}, we must have $k = 8$. For the global symmetries to agree with (6.15), the commutant of $SU(2)$ must be $SU(4)^2 \times SU(2) \times U(1)$, which suggests that

$$G_{\text{SCFT}} = SU(4)^3_{k=8}.$$

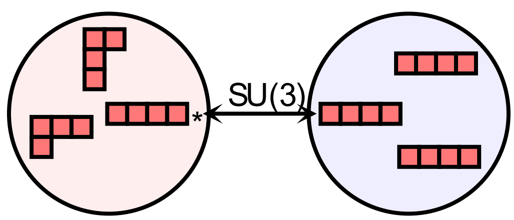
As another check, consider



The fixture on the left supplies 4 hypermultiplets in the fundamental of $SU(4)$, which contribute an $SU(4) \times U(1)$ to G_{global}. On the right, we gauge an $SU(4) \subset SU(2)_{k=6} \times SU(8)_{k=8}$. The commutant is $SU(2) \times SU(4) \times U(1)$. So, overall,

$$G_{\text{global}} = SU(4)_8^2 \times SU(2)_6 \times U(1)^2 \; . \tag{6.16}$$

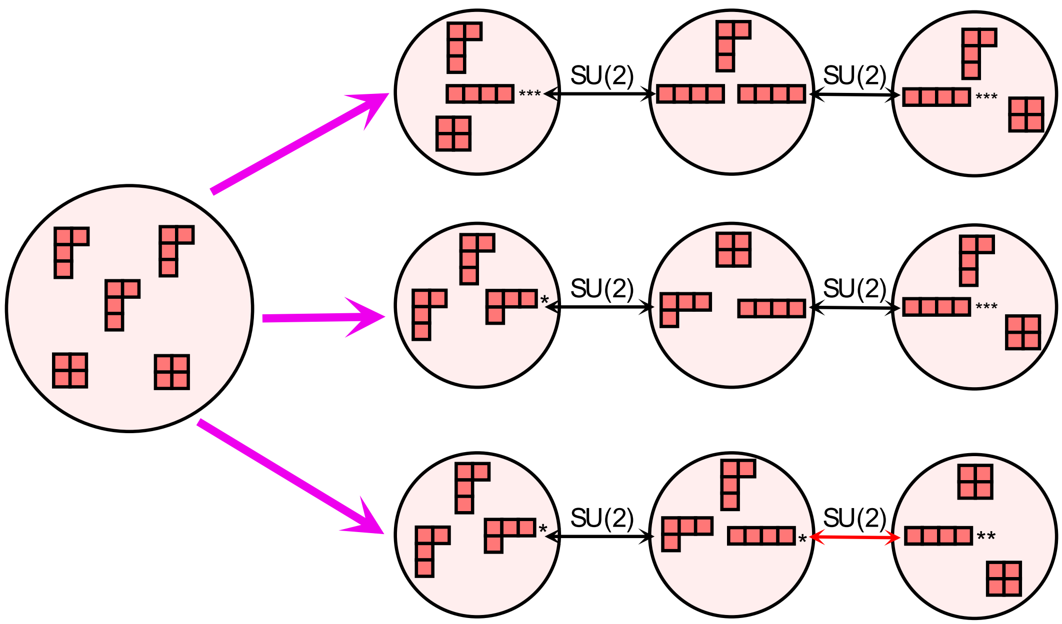
S-dualizing, we obtain



The fixture on the left supplies 2 hypermultiplets in the fundamental of $SU(3)$ (contributing an $SU(2)_6 \times U(1)$ factor to G_{global}). On the right, we gauge an $SU(3) \subset SU(4)^3_{k=8}$, which yields a conformal theory. And the commutant, $SU(4)_8^2 \times U(1)$, combines to give (6.16).

Using any of these gaugings we find $n_v = 19$ and $n_h = 40$, and so $a = \frac{45}{8}$ and $c = \frac{13}{2}$, for the $SU(4)^3_{k=8}$ SCFT. This SCFT is part of the T_N series [1, 3, 47], which for $N \geq 4$ has $SU(N)^3_{k=2N}$ global symmetry.

Finally, let us note that the cylinder between the pair of irregular punctures is crucial to understanding certain S-duality frames. For instance, consider the 5-punctured sphere



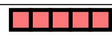
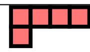
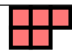
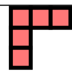
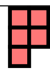
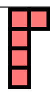
Note that, for each degeneration, we have an $SU(2) \times SU(2)$ gauge theory, with matter in the $(2,2) + 2(2,1) + 2(1,2) + 4(1,1)$, so that

$$G_{\text{global}} = SU(2)^2 \times U(1)^3 + 4 \text{ free hypers.}$$

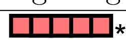
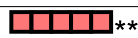
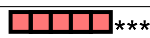
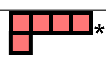
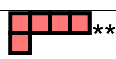
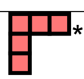
But, to make sense of the last degeneration, we crucially need the cylinder between two irregular punctures.

6.4.4 A_4

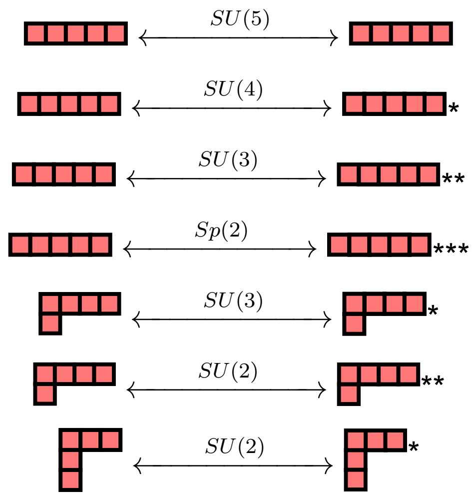
Now we turn to the A_4 theory. There are six regular punctures:

Young Diagram	Pole Structure	Global Symmetry	(n_h, n_v)
	$\{1,2,3,4\}$	$SU(5)$	$(80,70)$
	$\{1,2,3,3\}$	$SU(3)\times U(1)$	$(67,61)$
	$\{1,2,2,3\}$	$SU(2)\times U(1)$	$(58,54)$
	$\{1,2,2,2\}$	$SU(2)\times U(1)$	$(48,45)$
	$\{1,1,2,2\}$	$U(1)$	$(42,40)$
	$\{1,1,1,1\}$	$U(1)$	$(25,24)$

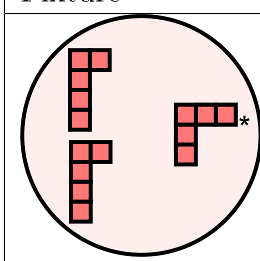
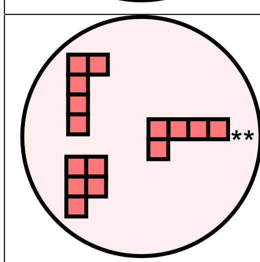
and six irregular punctures:

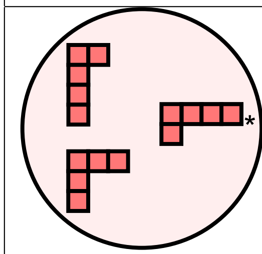
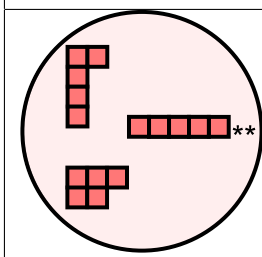
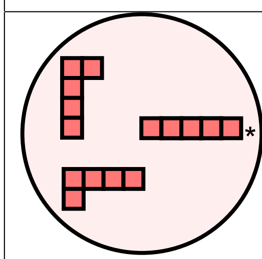
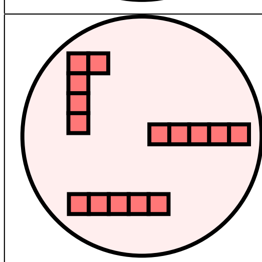
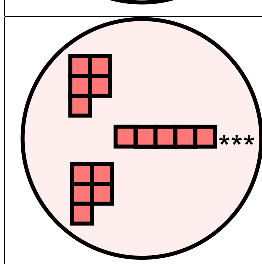
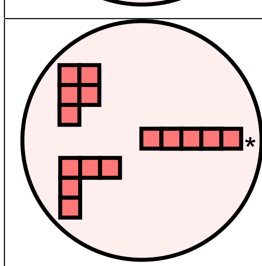
Young Diagram	Pole Structure	Global Symmetry	(n_h, n_v)
	$\{1,2,3,5\}$	$SU(4)$	$(80,79)$
	$\{1,2,4,5\}$	$SU(3)$	$(80,86)$
	$\{1,3,3,5\}$	$Sp(2)$	$(80,84)$
	$\{1,2,4,6\}$	$SU(3)$	$(93,95)$
	$\{1,3,4,6\}$	$SU(2)$	$(93,100)$
	$\{1,3,5,7\}$	$SU(2)$	$(112,116)$

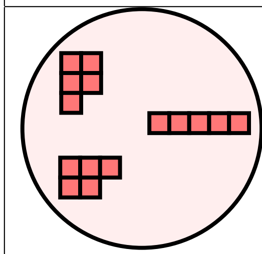
The cylinders are:



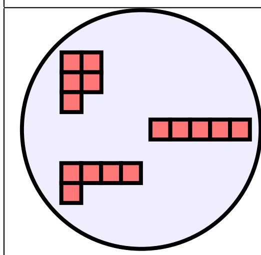
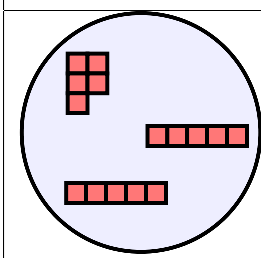
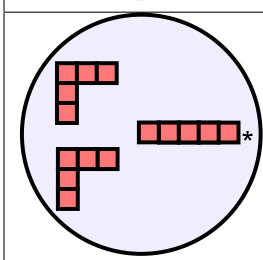
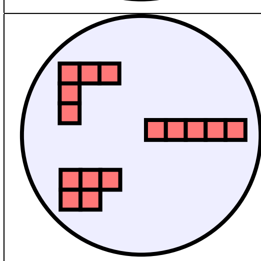
The free-field fixtures are

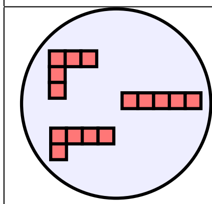
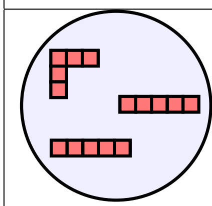
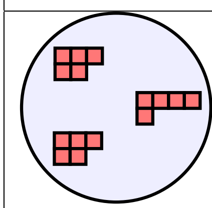
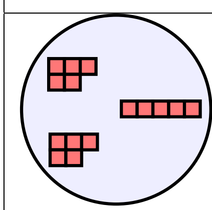
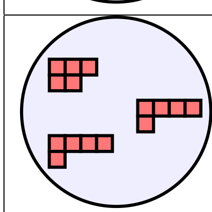
Fixture	Number of Hypers	Representation
	2	2
	0	–

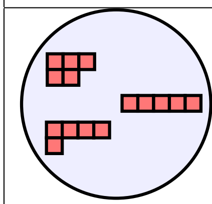
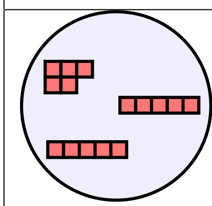
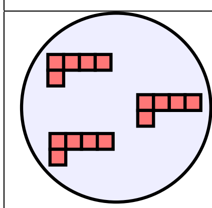
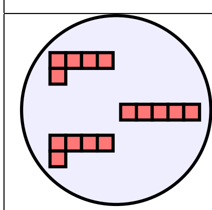
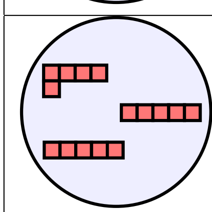
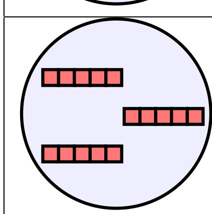
Fixture	Number of Hypers	Representation
	6	$(2,3)$
	3	$(1,3)$
	12	$(3,4)$
	25	$(5,5)$
	4	4
	10	$(1,4)+\frac{1}{2}(2,6)$

Fixture	Number of Hypers	Representation
	20	$(2,5)+(1,10)$

The interacting fixtures are:

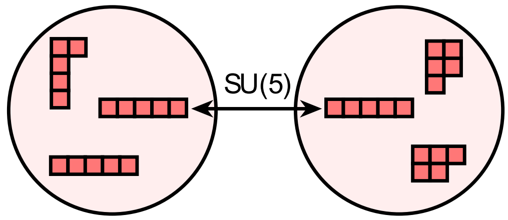
Fixture	(d_2, d_3, d_4, d_5)	(a, c)	$(G_{\text{global}})_k$	Theory
	$(0, 0, 1, 0)$	$\left(\frac{8}{3}, \frac{43}{12}\right)$	$(E_7)_8$	The E_7 SCFT plus 5 hypers
	$(0, 0, 1, 1)$	$\left(\frac{61}{12}, \frac{37}{6}\right)$	$SU(10)_{10}$	New. "S_5".
	$(0, 1, 0, 0)$	$\left(\frac{41}{24}, \frac{13}{6}\right)$	$(E_6)_6$	The E_6 SCFT
	$(0, 1, 0, 0)$	$\left(\frac{17}{8}, 3\right)$	$(E_6)_6$	The E_6 SCFT plus 10 hypers, in the $(1, 2, 5)$

Fixture	(d_2, d_3, d_4, d_5)	(a, c)	$(G_{\text{global}})_k$	Theory
	$(0,1,1,0)$	$\left(\frac{95}{24}, \frac{59}{12}\right)$	$SU(2)_6 \times SU(8)_8$	$SU(2)_6 \times SU(8)_8$ SCFT + 5h
	$(0,1,1,1)$	$\left(\frac{51}{8}, \frac{15}{2}\right)$	$SU(2)_6 \times SU(10)_{10}$	New. "$R_{0,5}$".
	$(0,1,0,0)$	$\left(2, \frac{11}{4}\right)$	$(E_6)_6$	The E_6 SCFT + 7h in the $(2,2,1)+(1,1,3)$
	$(0,1,0,1)$	$\left(\frac{53}{12}, \frac{16}{3}\right)$	$SO(14)_{10} \times U(1)$	New. "$R_{2,5}$".
	$(0,1,1,0)$	$\left(\frac{23}{6}, \frac{14}{3}\right)$	$SU(2)_6 \times SU(8)_8$	The $SU(2)_6 \times SU(8)_8$ SCFT +2h

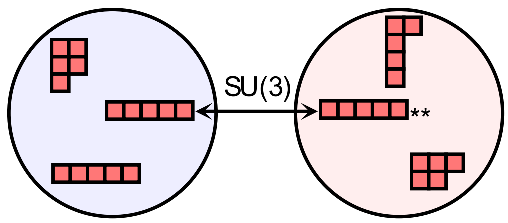
Fixture	(d_2, d_3, d_4, d_5)	(a, c)	$(G_{\text{global}})_k$	Theory
	$(0,1,1,1)$	$\left(\frac{25}{4}, \frac{29}{4}\right)$	$SU(3)_8 \times SU(7)_{10} \times U(1)$	New. "$R_{1,5}$".
	$(0,1,1,2)$	$\left(\frac{26}{3}, \frac{59}{6}\right)$	$SU(5)^2_{10} \times SU(2)_{10} \times U(1)$	New. "V_N"
	$(0,1,2,0)$	$\left(\frac{17}{3}, \frac{79}{12}\right)$	$SU(4)^3_8$	The $SU(4)^3_8$ SCFT +1h
	$(0,1,2,1)$	$\left(\frac{97}{12}, \frac{55}{6}\right)$	$SU(6)_{10} \times SU(3)^2_8 \times U(1)$	New
	$(0,1,2,2)$	$\left(\frac{53}{6}, \frac{47}{4}\right)$	$SU(5)^2_{10} \times SU(3)_8 \times U(1)$	New. "U_5".
	$(0,1,2,3)$	$\left(\frac{155}{12}, \frac{43}{3}\right)$	$SU(5)^3_{10}$	New. "T_5".

Since our procedures should by now be more or less straightforward, let us simply present the A_4 interacting SCFTs as strong coupling points of linear quivers of special unitary groups.

For the $SU(10)$ theory, we study the following theory

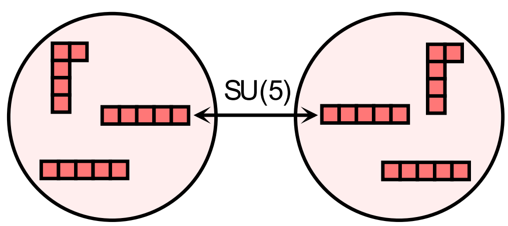


which is a $SU(5)$ gauge theory with 7 fundamental hypermultiplets and one hypermultiplet in the 10 of $SU(5)$. The S-dual frame in which we are interested is

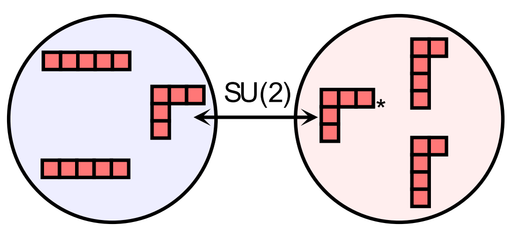


which is a $SU(3)$ gauging of the $SU(10)$ theory coupled to one fundamental hypermultiplet. The $SU(10)$ theory is the first in a series of interacting SCFTs, S_N ($N \geq 5$), which we discuss in §6.6.

For the $SU(2) \times SU(10)$ theory, consider the Lagrangian theory

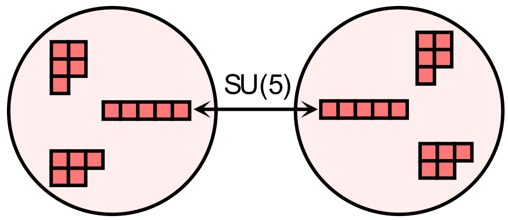


which is the $SU(5)$ $N_f = 10$ gauge theory. The S-dual theory, which we are interested in, is

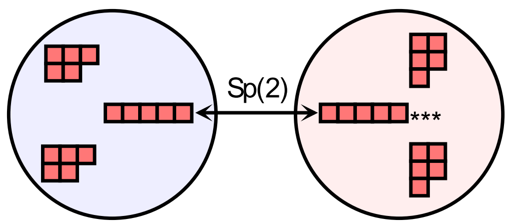


which is a $SU(2)$ gauging of the $SU(2) \times SU(10)$ theory, coupled to one fundamental hypermultiplet.

For the $SO(14) \times U(1)$ theory we consider the Lagrangian theory



which is a $SU(5)$ gauge theory with 4 fundamental hypermultiplets and 2 hypermultiplets in the 10 representation. The S-dual frame in which we are interested is

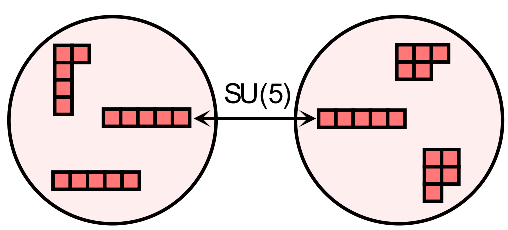


which is an $Sp(2)$ gauging of the $SO(14) \times U(1)$ theory with 1 fundamental hypermultiplet. The $SO(14) \times U(1)$ theory is part of an infinite series of interacting SCFTs we call $R_{2,N}$, for N odd, with global symmetry group

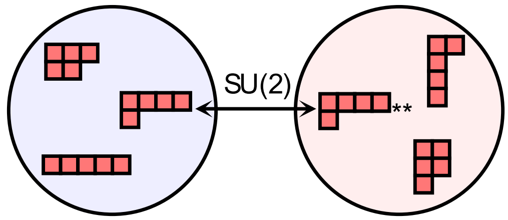
$$G_{\text{global}} = SO(2N+4)_{k=2N} \times U(1).$$

For $N = 3$, the $SO(10)_6 \times U(1)$ is enhanced to $(E_6)_6$, and we identify $R_{2,3} \equiv T_3$.

For the $SU(3) \times SU(7) \times U(1)$ theory, we consider the Lagrangian theory



which is a $SU(5)$ gauge theory with 7 fundamental hypermultiplets and 1 hypermultiplet in the 10 of $SU(5)$. The S-dual frame in which we are interested is

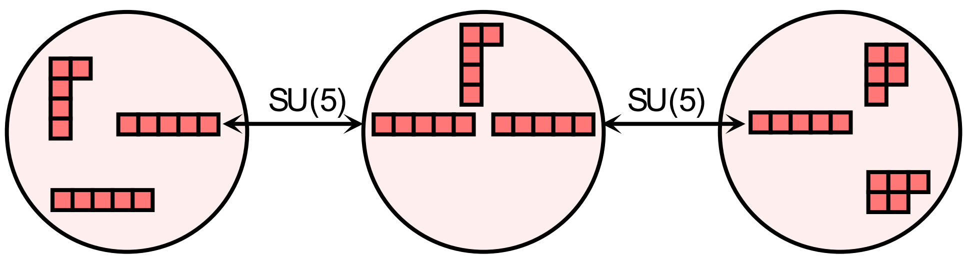


which is an $SU(2)$ gauging of the $SU(3) \times SU(7) \times U(1)$ SCFT.

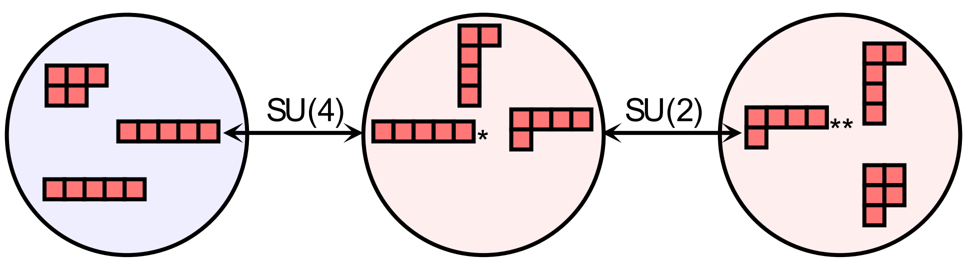
As discussed in §6.6, this theory, too, is part of an infinite series of interacting SCFTs, $R_{1,N}$, for odd N, with global symmetry group

$$G_{\text{global}} = SU(3)_{k=8} \times SU(N+2)_{k=2N} \times U(1).$$

For the $SU(5)^2 \times SU(2) \times U(1)$ theory, consider the Lagrangian theory

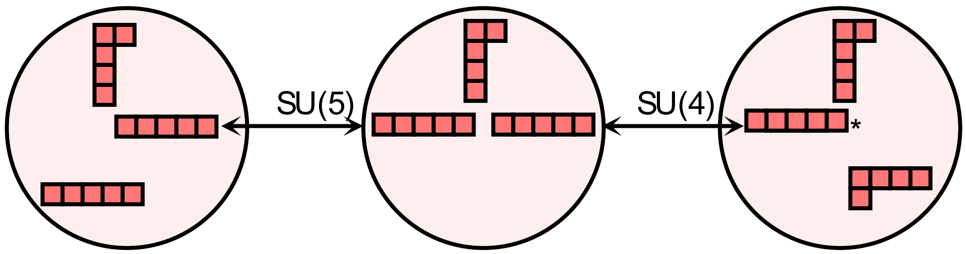


which is an $SU(5) \times SU(5)$ gauge theory with matter in the $5(5,1) + (5,5) + 2(1,5) + (1,10)$. The S-dual frame in which we are interested is

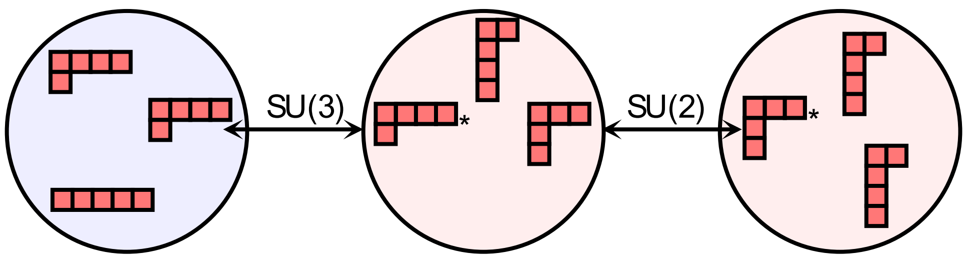


which is an $SU(4)$ gauging of the $SU(5)^2 \times SU(2) \times U(1)$ SCFT coupled to a $SU(2)$ gauge theory with matter in the $(4,2)$ of $SU(4) \times SU(2)$.

For the $SU(6)\times SU(3)^2\times U(1)$ theory, consider the following Lagrangian theory

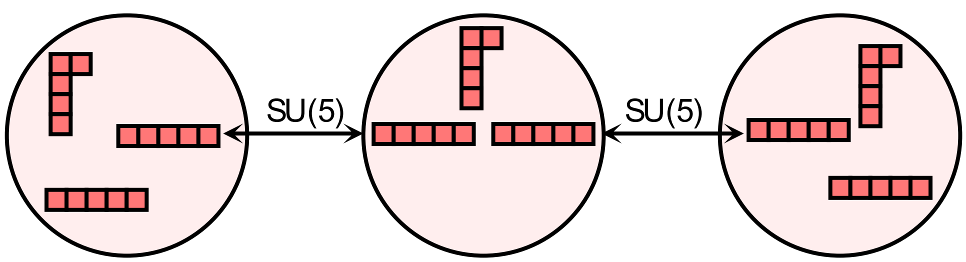


which is an $SU(5) \times SU(4)$ gauge theory with matter in the $6(5,1) + (5,4) + 3(1,4)$ representation of $SU(5) \times SU(4)$. The S-dual frame in which we are interested is

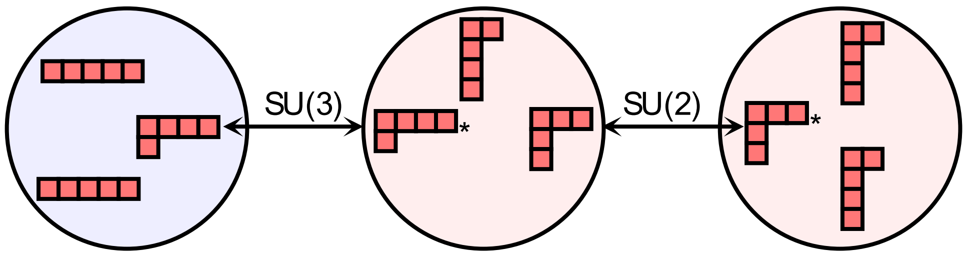


which is an $SU(3)$ gauging of the $SU(6) \times SU(3)^2 \times U(1)$ SCFT coupled to a $SU(2)$ gauge theory with matter in the $(3,2) + (1,2)$ representation of $SU(3) \times SU(2)$.

For the $SU(5)^2 \times SU(3) \times U(1)$ theory, we consider the following Lagrangian theory,



which is an $SU(5) \times SU(5)$ gauge theory with matter in the $5(5,1)+(5,5)+5(1,5)$ representation. The S-dual frame in which we are interested is



which is an $SU(3)$ gauging of the $SU(5)^2 \times SU(3) \times U(1)$ SCFT coupled to a $SU(2)$ gauge theory with matter in the $(3,2)+(1,2)$ representation of $SU(3) \times SU(2)$. This interacting fixture is, again, the first of an infinite series we call U_N.

4-Punctured Spheres

As a concrete test that our enumeration of fixtures and cylinders, in the A_4 theory, didn't miss anything, we decided to systematically study *all* 4-punctured spheres – that is, all theories with a single gauge group factor – which arise from the A_4 theory. There are 90 such spheres, consisting of 4 regular punctures and a positive (graded) dimensional Coulomb branch.

- Three are spheres with 4 identical punctures.
- Twenty-one are spheres with 3 identical punctures.

In each of these cases, the gauge theory is self-dual, and so does not yield much of an interesting check on our predictions.

- Fifty-four are spheres with two identical punctures. These lead to *pairs* of distinct gauge theories, which are related by S-duality.
- Twelve are spheres with four distinct punctures. These lead to *triples* of distinct gauge theories, related by S-duality.

We have checked that our rules reproduce the correct global symmetry groups, Coulomb branch dimension and conformal anomaly coefficients for all 66 theories. Since each fixture, and each cylinder appears multiple times among the

144 distinct degenerations, this provides a powerful check on our methods. We give a brief summary of the results in the Appendix.

6.5 3D Mirrors

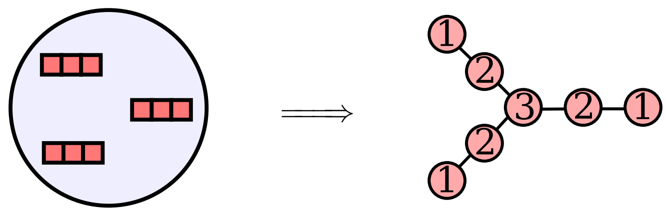
To bolster our identification of the global symmetry groups of the interacting SCFTs that we have found, we will use an approach described by Benini, Tachikawa and Xie [9].

They compactify from four down to three dimension, and construct the mirror of the 3D SCFT. The 3D mirror of the A_{N-1} theory on an n-punctured sphere ($\times S^1$) is a star-shaped quiver gauge theory, with n arms, whose central node is $U(N)$. We will be interested in the case $n = 3$. The other $U(k)$ gauge groups, in each arm of the quiver, are dictated by the Young diagram associated to the puncture. Starting at the central node, we reduce the rank of each successive $U(k)$ gauge group by the *height* of each successive column of the Young diagram. Since all of the matter is in bifundamental hypermultiplets, the mirror gauge group is $\left(\prod_i U(k_i)\right)/U(1)_{\text{diag}}$.

Having constructed the quiver, Gaiotto and Witten [41] tell you how to extract the global symmetry group (by construction, all of our quivers are "good quivers", in the sense of Gaiotto and Witten):

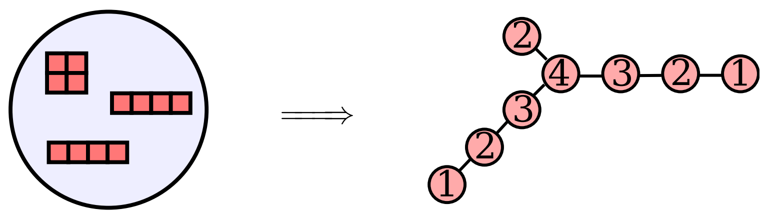
- Mark each "balanced" node of the quiver (one for which $\sum k_i$ for the adjacent nodes is equal to $2k$).
- If all of the nodes of the quiver are balanced, remove one of the $U(1)$ nodes (since we are modding out by the diagonal $U(1)$.
- The marked nodes form the Dynkin diagram of the semi-simple part of G_{global}. The abelian part is $U(1)^{p-1}$, where p is the number of unmarked nodes.

For the A_2 theory, there's just one interacting SCFT, and the quiver corresponding to its 3D mirror has the shape of the E_6 extended Dynkin diagram.



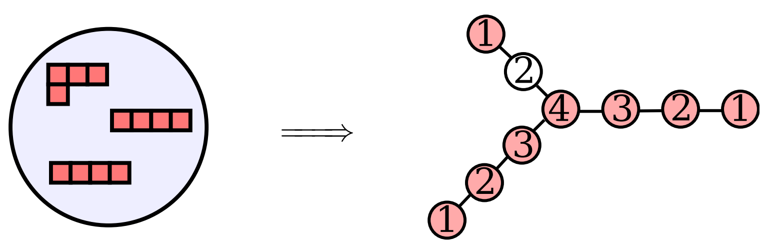
After modding out by the diagonal $U(1)$, we reproduce the global symmetry group, E_6.

In the A_3 theory, there are three "new" interacting SCFTs. The first has a mirror quiver in the shape of the extended Dynkin diagram of E_7.



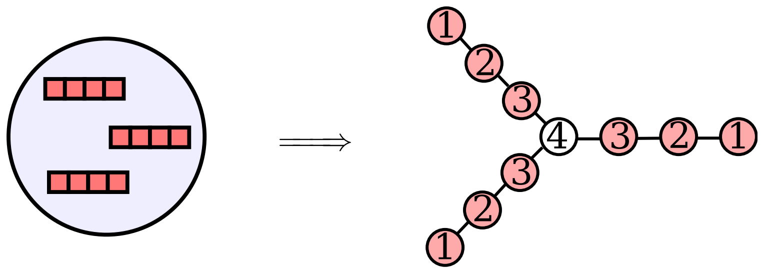
After modding out by the diagonal $U(1)$, this yields the flavour symmetry E_7.

In the 3D mirror of the second SCFT

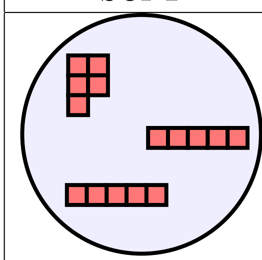
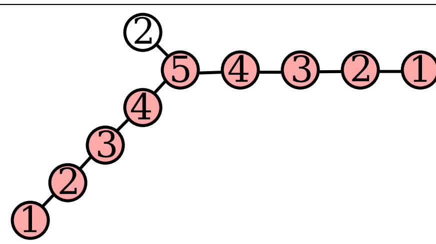
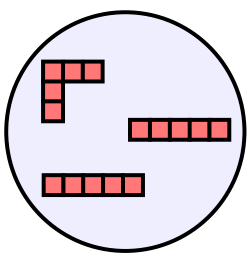
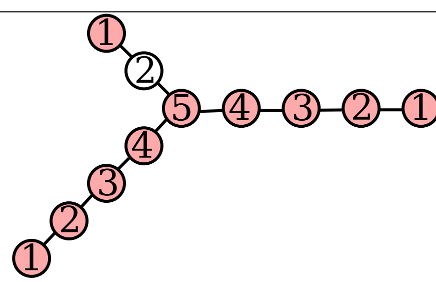


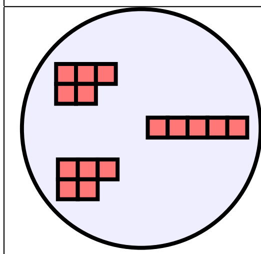
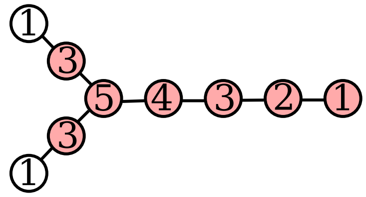
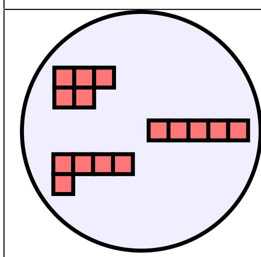
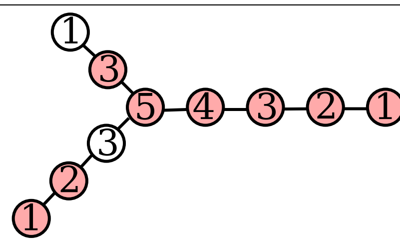
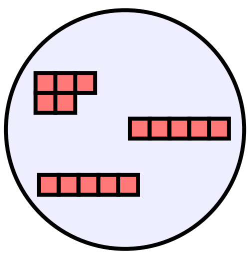
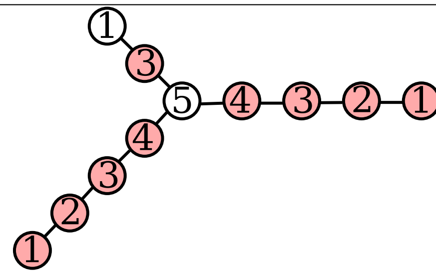
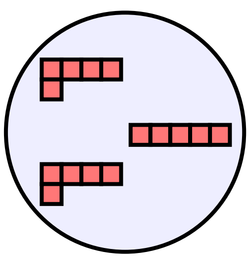
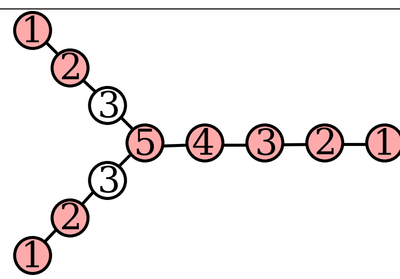
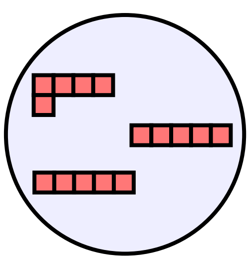
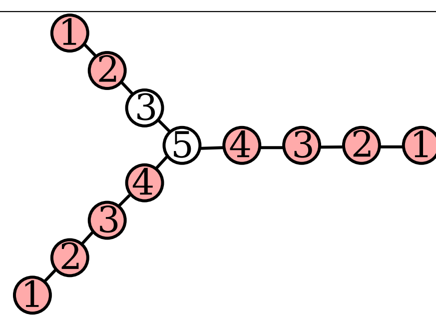
not all the nodes of the quiver are superconformal. Modding out by the diagonal $U(1)$ kills one of the non-superconformal nodes (in this case, there's only one), leaving $SU(2) \times SU(8)$ as the global symmetry group.

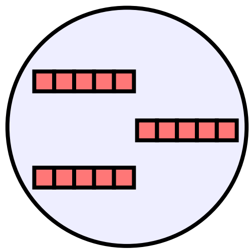
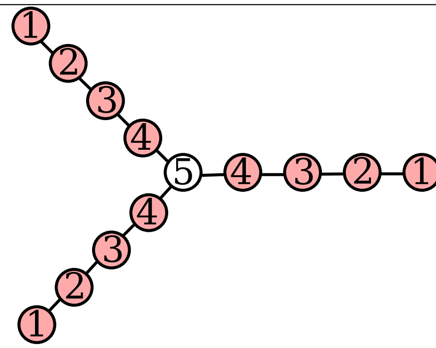
Finally, the T_4 theory has an $SU(4)^3$ global symmetry group.



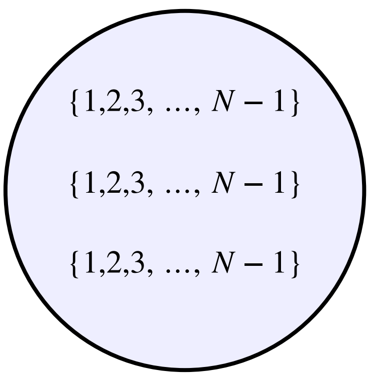
Turning to the A_4 theory, there are 8 new interacting SCFTs which arise. The 3D dual theories each have $l > 0$ nodes of the quiver which are non-superconformal. Modding out by the diagonal $U(1)$ yields a $U(1)^{l-1}$ factor in the global symmetry group.

SCFT	3D Mirror	G_k
		$SU(10)_{10}$
		$SU(2)_6 \times SU(10)_{10}$

SCFT	3D Mirror	G_k
		$SO(14)_{10} \times U(1)$
		$SU(3)_8 \times SU(7)_{10} \times U(1)$
		$SU(5)_{10}^2 \times SU(2)_{10} \times U(1)$
		$SU(6)_{10} \times SU(3)_8^2 \times U(1)$
		$SU(5)_{10}^2 \times SU(3)_8 \times U(1)$

SCFT	3D Mirror	G_k
		$SU(5)^3_{10}$

6.6 Infinite Series



We are already familiar with the T_N series of interacting SCFTs, introduced by Gaiotto, whose fixture consists of three maximal punctures. The global symmetry group is

$$G_{\text{global}} = SU(N)^3_{k=2N}.$$

The graded dimension of the Coulomb branch is

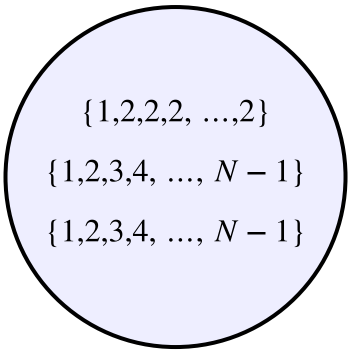
$$(d_2, d_3, d_4, \dots, d_N) = (0, 1, 2, 3, \dots, N-2),$$

and conformal anomaly coefficients are

$$a = \frac{N^3}{6} - \frac{5N^2}{16} - \frac{N}{16} + \frac{5}{24},$$
$$c = \frac{N^3}{6} - \frac{N^2}{4} - \frac{N}{12} + \frac{1}{6}.$$

For $N = 3$, G_{global} is enhanced to $E_{6k=6}$.

In our investigations, we have come across several new series of interacting SCFTs. Below, we will discuss seven of them.



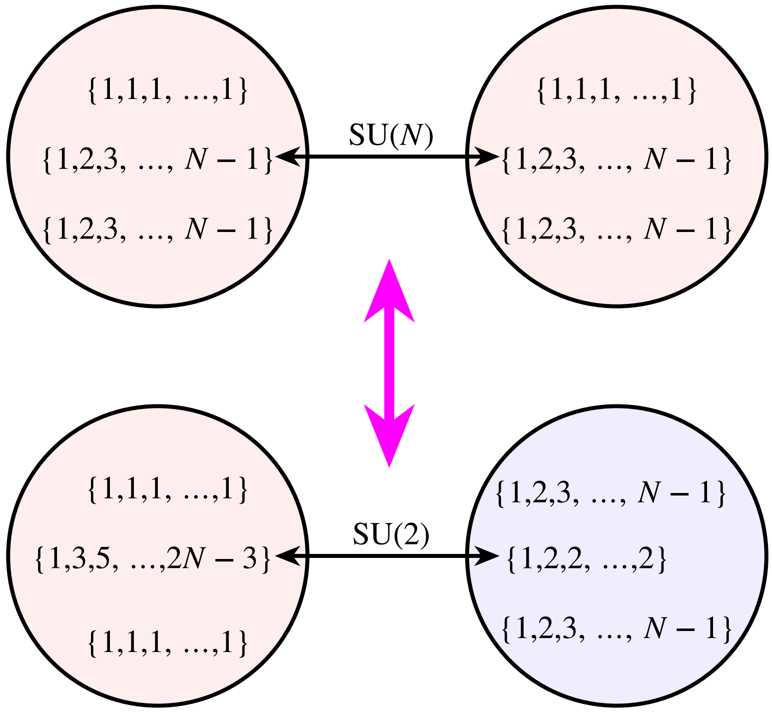
The $R_{0,N}$ series of interacting SCFTs has global symmetry

$$G_{\text{global}} = SU(2)_{k=6} \times SU(2N)_{k=2N},$$

and has a Coulomb branch of graded dimension

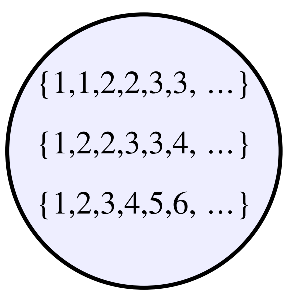
$$(d_2, d_3, d_4, \ldots, d_N) = (0, 1, 1, \ldots, 1).$$

The strong coupling cusp of $SU(N)$, $N_f = 2N$ gauge theory [12, 49] is S-dual to an $SU(2)$ gauging of the $SU(2)_{k=6} \subset G_{\text{global}}$ coupled to a fundamental hypermultiplet.

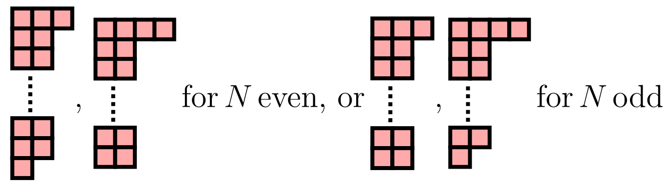


For $R_{0,3}$ ($\equiv T_3$), the $SU(2)_6 \times SU(6)_6$ global symmetry is enhanced to $(E_6)_6$, and we get back the classic example of Argyres-Seiberg duality.) The conformal anomaly coefficients for the $R_{0,N}$ series are

$$a = \frac{7N^2 - 22}{24},$$
$$c = \frac{2N^2 - 5}{6}.$$



The fixture for the $R_{1,N}$ ($N \geq 5$) series has one maximal puncture, and two other punctures, corresponding to Young diagrams of the form



The Coulomb branch has graded dimension

$$(d_2, d_3, d_4, \ldots, d_N) = (0, 1, 1, 1, \ldots, 1),$$

and the conformal anomaly coefficients are

$$a = \frac{13N^2 + 3N - 40}{48},$$
$$c = \frac{7N^2 + 3N - 16}{24}.$$

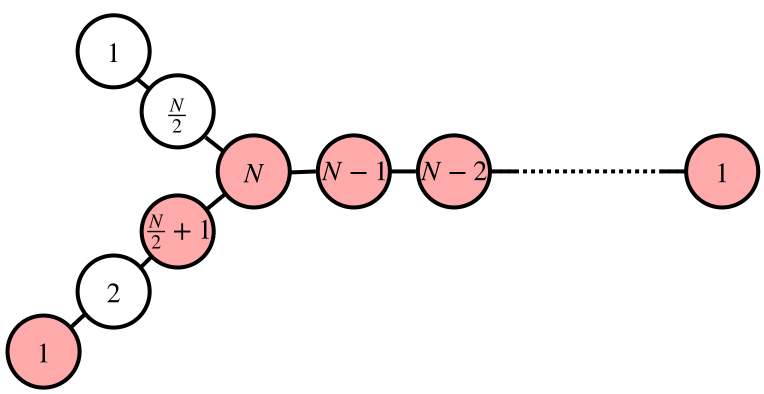
$R_{1,N}$ has global symmetry group

$$G_{\text{global}} = SU(2)_{k=8} \times SU(N+2)_{k=2N} \times U(1)^2$$

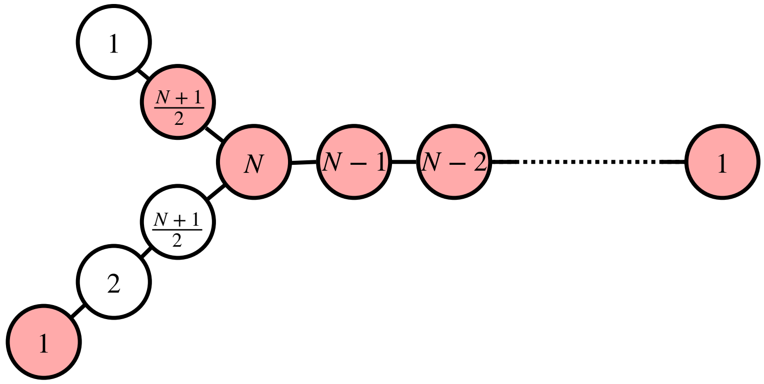
(enhanced to $SU(3)_{k=8} \times SU(7)_{k=10} \times U(1)$ for $N = 5$).

However, the realization differs slightly in the N even versus N odd cases. This is easily seen by examining the 3D mirrors

N even :

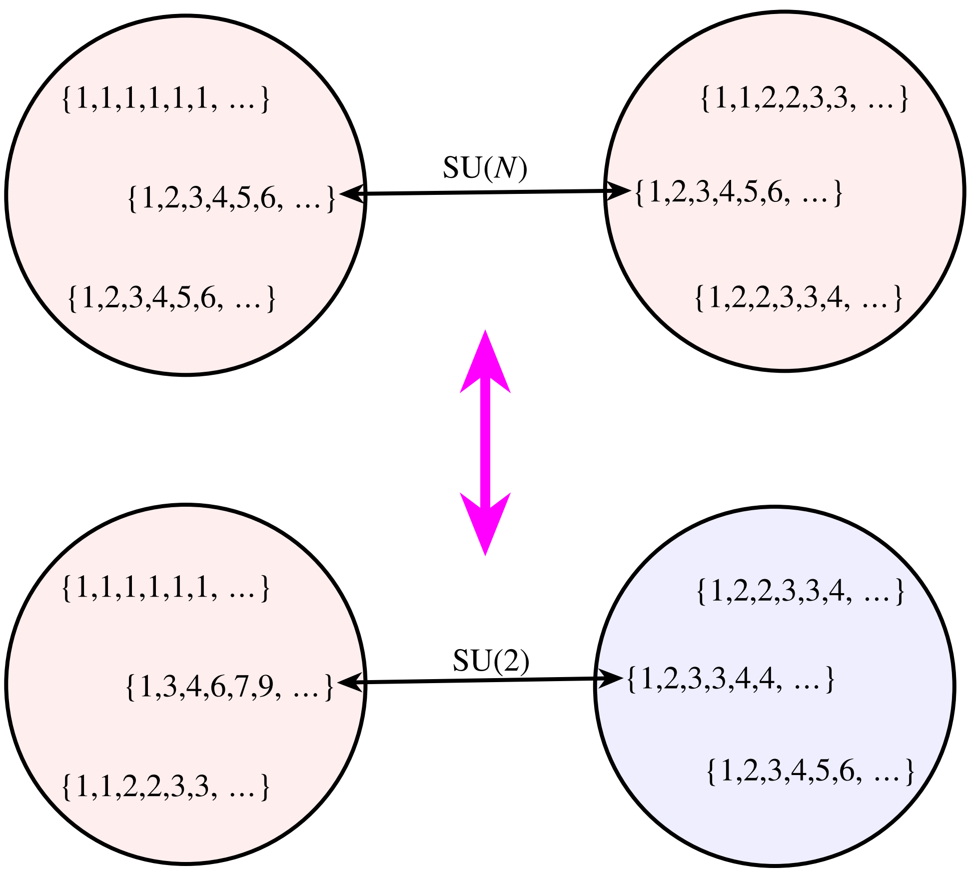


N odd :



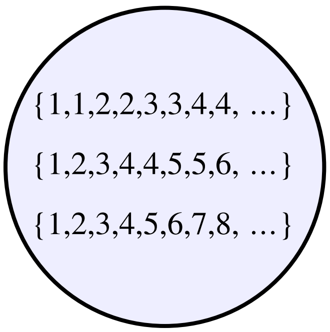
From the 3D mirrors, one also readily sees the enhancement in G_{global} for $N = 5$.

(One of) the S-duals of $SU(N)$ with matter in the $(N+2)(\square) + \boxminus$ is a gauging of the $SU(2)_8 \subset G_{\text{global}}$ symmetry of $R_{1,N}$.



In the upper figure, the fixture on the left contributes N fundamentals; the fixture on the right contributes 2 fundamental and one . In the lower figure, the fixture on the left contributes nothing; the fixture on the right is $R_{1,N}$.

Of course, the above 4-punctured sphere has another degeneration, which leads us to our fourth series of interacting SCFTs



The S_N series has global symmetry

$$G_{\text{global}} = SU(N+2)_{k=2N} \times SU(3)_{k=10} \times U(1)$$

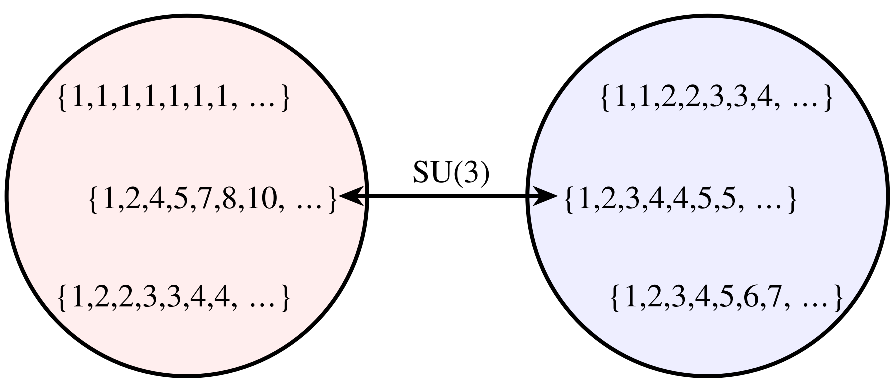
(enhanced to $SU(10)_{10}$, for $N=5$). Its Coulomb branch has graded dimension

$$(d_2, d_3, d_4, d_5, \dots) = (0, 0, 1, 1, 1, \dots, 1).$$

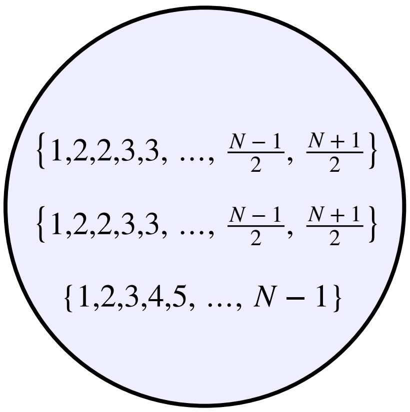
The conformal anomaly coefficients are

$$\begin{aligned} a &= \frac{13N^2 + 3N - 96}{48}, \\ c &= \frac{7N^2 + 3N - 42}{24}. \end{aligned}$$

The third S-duality frame of the $SU(N)$ gauge theory we have been discussing is



an $SU(3)$ gauging of the S_N theory, coupled to a single fundamental hypermultiplet.



Next, we turn to the $R_{2,N}$ theory, which appears, for N odd, as a fixture in the (unique) S-dual of $SU(N)$, with matter in the $4(\square) + 2\left(\begin{smallmatrix}\square\\ \square\end{smallmatrix}\right)$.

The global symmetry group of $R_{2,N}$ is

$$G_{\text{global}} = SO(2N+4)_{k=2N} \times U(1)$$

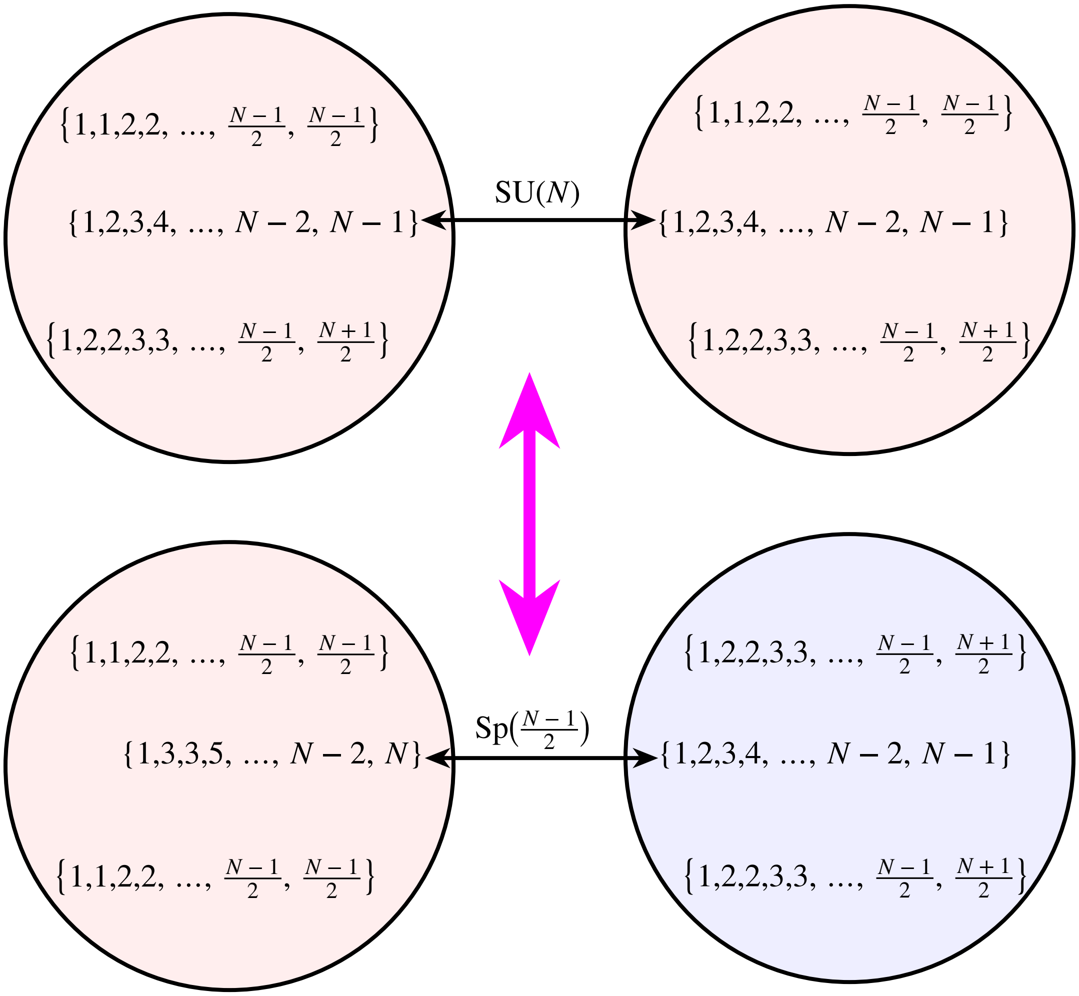
(enhanced to $(E_6)_6$ for $N = 3$, where there is no distinction between a fundamental hypermultiplet and an antisymmetric tensor). The graded dimension of the Coulomb branch is

$$(d_2, d_3, d_4, d_5, d_6, \ldots, d_N) = (0, 1, 0, 1, 0, \ldots, 1).$$

The conformal anomaly coefficients for the $R_{2,N}$ series are

$$\begin{aligned} a &= \frac{7N^2 + 9N - 8}{48}, \\ c &= \frac{2N^2 + 3N - 1}{12}. \end{aligned}$$

The strong coupling S-dual of $SU(N)$ (N odd), with matter in the $4(\square)+2\left(\begin{smallmatrix}\square\\ \square\end{smallmatrix}\right)$ is an $Sp\left(\frac{N-1}{2}\right)$ gauge theory coupled to one fundamental hypermultiplet and gauging an $Sp\left(\frac{N-1}{2}\right) \subset SO(2N+4)_{2N}$ of the $R_{2,N}$ theory[7].



For N even, the S-duality of $SU(N)$, with matter in the $4(\square) + 2\left(\begin{smallmatrix}\square\\ \square\end{smallmatrix}\right)$, looks almost identical to the picture above. The S-dual gauge group is $Sp(N/2)$. The fixture on the left contributes $2N$ hypermultiplets, transforming as 2

[7]Here, and in several other S-dualities discussed in this paper, we use the embedding

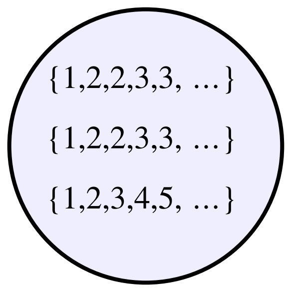
$$SO(4lm+2n)_k \supset Sp(l)_{km} \times Sp(m)_{kl} \times SO(2n)_k$$

under which the fundamental of $SO(4lm+2n)$ decomposes as $(2l, 2m, 1) + (1, 1, 2n)$.

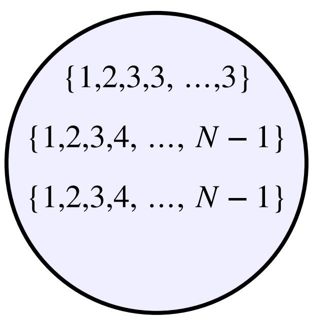
fundamentals of $Sp(N/2)$ (instead of $N-1$ hypermultiplets, transforming as one fundamental of $Sp((N-1)/2)$, as it did, for N odd). The fixture on the right is $R_{2,N-1}$ plus N hypermultiplets (which contribute another fundamental of $Sp(N/2)$).

All together, the S-dual of $SU(N)$ (N even), with matter in the $4(\square)+2\left(\begin{smallmatrix}\square\\ \square\end{smallmatrix}\right)$, is $Sp(N/2)$ with 3 hypermultiplets in the fundamental, gauging the $R_{2,N-1}$ theory.

The fixture



is $R_{2,N}$, for N odd, and $R_{2,N-1}$ plus N hypermultiplets, for N even.



The U_N series has global symmetry

$$G_{\text{global}} = SU(N)^2_{k=2N} \times SU(3)_{k=8} \times U(1)$$

(enhanced to $SU(4)^3_8$ for $S_4 \equiv T_4$). The Coulomb branch has graded dimension

$$(d_2, d_3, d_4, d_5, \dots) = (0, 1, 2, 2, 2, \dots, 2).$$

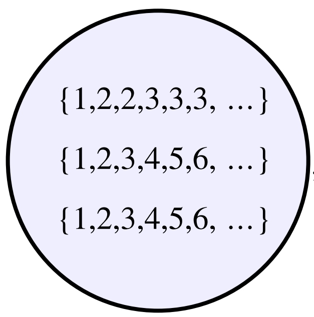
and the conformal anomaly coefficients are

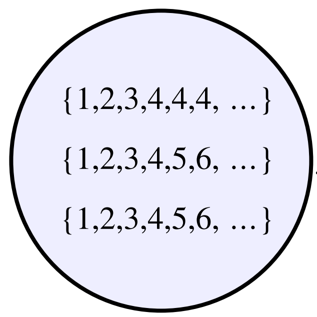
$$a = \frac{13N^2 - 73}{24},$$
$$c = \frac{7N^2 - 34}{12}.$$

Consider an $SU(N)^2$ gauge theory, with matter in the $N(N,1)+(N,N)+N(1,N)$. One S-dual frame is, of course, an $SU(2)\times SU(N)$ gauge theory, with matter in the $(2,1)+(1,N)$, gauging an $SU(2)\times SU(N)\subset SU(2)\times SU(2N)_{2N}$ of the $R_{0,N}$ theory. The other S-dual frame is an $SU(2)\times SU(3)$ gauge theory, with matter in the $(2,1)+(2,3)$, where the $SU(3)$ gauges the $SU(3)_8\subset G_{\text{global}}$ of U_N.

So far, our infinite series have been fixtures which appear in S-dual descriptions of Lagrangian field theories. In light of recent progress, this seems like a quaint restriction.

Let us turn to a pair of infinite series of interacting SCFT fixtures, consisting of a pair of maximal punctures plus a puncture whose Young diagram's first column has a height that grows like N.

$V_N =$ 

, $W_N =$ 

.

The Coulomb branch of V_N has graded dimension

$$(d_2,d_3,d_4,d_5,d_6,d_7,\dots)=(0,1,1,2,2,2,\dots,2).$$

From the 3D mirror, we find its global symmetry group to be

$$G_{\text{global}}=SU(N)^2_{k=2N}\times U(1)^2$$

(enhanced to $SU(5)^2_{10}\times SU(2)_{10}\times U(1)$ for $N=5$). It has $n_v=2N^2-20$, and $n_h=3N^2-17$, or

$$a=\frac{13(N^2-9)}{24},$$
$$c=\frac{7N^2-57}{12}.$$

The Coulomb branch of W_N has graded dimension

$$(d_2,d_3,d_4,d_5,d_6,d_7,\dots)=(0,1,2,3,3,3,\dots,3).$$

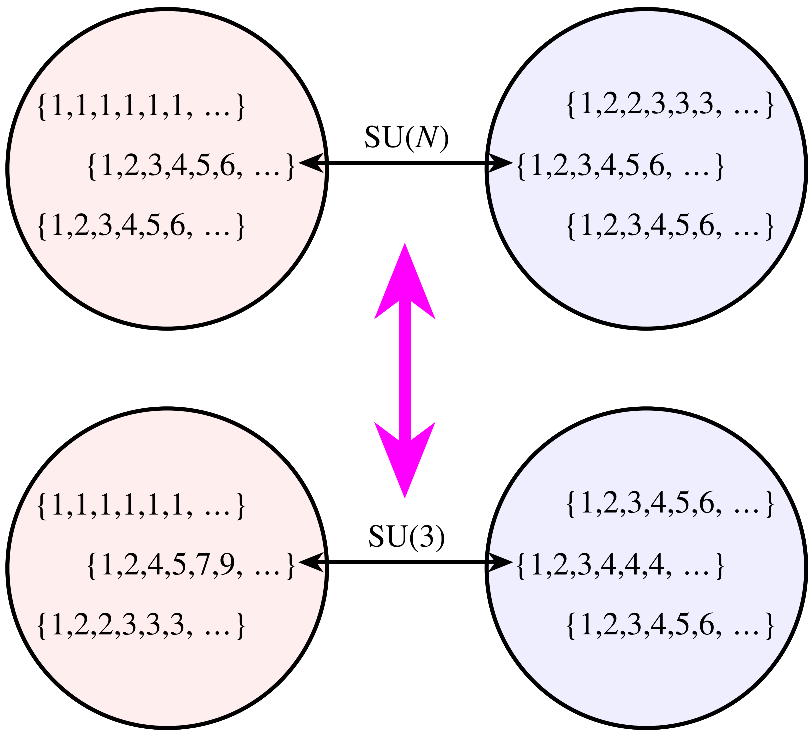
Its global symmetry group is

$$G_{\text{global}} = SU(N)^2_{k=2N} \times SU(4)_{10} \times U(1)$$

(enhanced to $SU(5)^3_{10}$ for $N = 5$). It has $n_v = 3N^2 - 29$, and $n_h = 4N^2 - 20$, or

$$a = \frac{19N^2 - 174}{24},$$
$$c = \frac{10N^2 - 87}{12}.$$

Using these interacting fixtures, we construct a family of S-dual theories



The upper theory is an $SU(N)$ gauge theory, with N fundamentals, coupled to V_N. The lower theory is an $SU(3)$ gauge theory, with one fundamental, coupled to W_N.

Of course, there are an infinite number of arbitrary-N families of Young diagrams, that one can write down, and from there, an infinite number of arbitrary-N families of interacting fixtures. The ones discussed here were those which cropped up in the theories up through $N = 5$, and which gave rise to interesting series of S-dualities.

6.7 Theories with irregular punctures

Having introduced 3-punctured spheres with irregular punctures, we should ask whether — according to our rules — it is possible to construct *connected* curves, C, with $g > 0$ and/or $n > 3$, containing one or more irregular punctures.

It would be most dangerous if we could construct connected surfaces with *two* or more irregular puncture, as we would then have to specify what happens when two irregular punctures collide, and that would take us outside the set of configurations we have allowed.

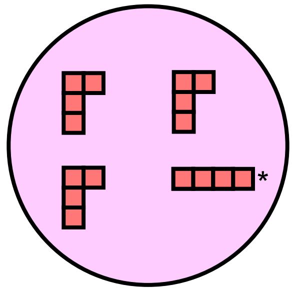
It is easy to see, however, that this complication does not arise. At least up through A_4, we can exhaustively list all the connected surfaces, constructed according to our rules, with one or more irregular punctures. These are a *finite* in number, and contain just one irregular puncture. All have $g = 0$.

More generally, we can argue as follows. Assume there exists a connected surface, C, with two irregular punctures.

- One of the implications of our rules for constructing surfaces is that, for any k, if C had $d_k > 0$, then, for that value of k, $p_k^{(i)} \leq k - 1$, $\forall i$.
- On the other hand, an irregular puncture, by definition, has $p_k \geq k - 1$, $\forall k$ and $> k - 1$ for at least some k. Pick one such value of k.
- We demand $0 \equiv d_k = -(1 - g)(2k - 1) + \sum_{i=1}^{n} p_k^{(i)}$. The second term is manifestly positive, and the two irregular punctures make a contribution $\geq 2k - 1$. The only way to satisfy the equality is to set $g = 0$, with no other punctures.
- But, for $g = 0$, we must have $n \geq 3$ (otherwise, the virtual dimension d_2 is negative).

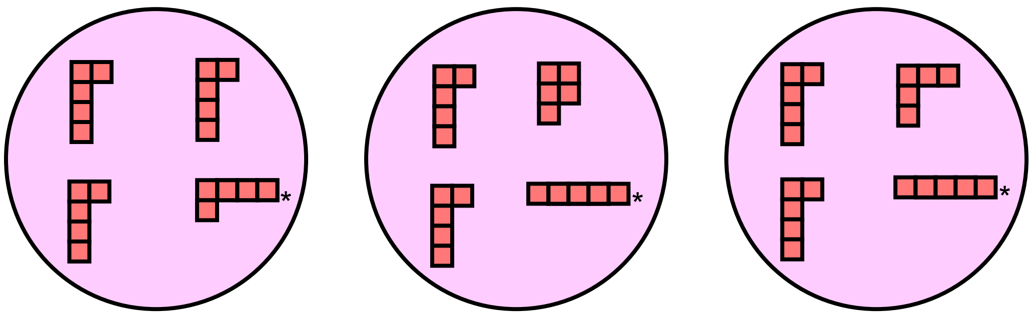
Thus, we reach a contradiction: there can be no connected curves, C, with two (or more) irregular punctures.

It remains to list the finite number of A_{N-1} ($N \leq 5$) theories with $g = 0$, a single irregular puncture and $n > 3$. In the A_2 theory, there is only the 3-punctured sphere, listed above. Starting with A_3, however, we find a 4-punctured sphere



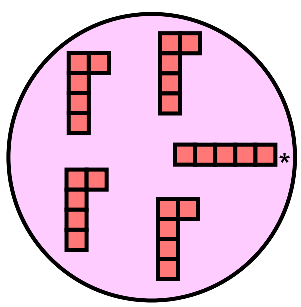
This is the $SU(2)$, $N_f = 4$ theory, as it arises in the A_3 theory.

For A_4, we find three 4-punctured spheres,



The first is, again, the A_4 expression of the $SU(2)$ $N_f = 4$ theory. The second is the $SU(2)$ $N_f = 4$ theory plus 4 free hypers. The third is the $SU(3)$ $N_f = 6$ theory (or its S-dual).

From the latter, we can construct a 5-punctured sphere



which is an $SU(2) \times SU(3)$ gauge theory, with matter in the $(2,1) + (2,3) + 4(1,3)$.

Chapter 7

The D_N series

7.1 The D_N Series

We consider a 6D D_N (2,0) theory compactified on a Riemann surface C of genus g with n punctures (complex codimension-1 defect operators) [7, 8, 10] located at points $y_i \in C$, $i = 1, \ldots, n$. Here we follow [10]. Also, we use the methods of Chapter 6 for the A_{N-1} theories as much as possible.

The Seiberg-Witten curve for the D_N theories takes the form [7]

$$0 = \lambda^{2N} + \sum_{k=1}^{N-1} \lambda^{2(N-k)} \phi_{2k}(y) + \tilde{\phi}^2(y). \tag{7.1}$$

Again, the ϕ_{2k} and $\tilde{\phi}$ are meromorphic differentials on C, with poles of up to the prescribed orders at the punctures. ($\tilde{\phi}$ is the Pfaffian, i.e., an N-differential.)

However, there are some crucial differences between the A_{N-1} and the D_N theories. While in the A_{N-1} case, the coefficients in the Seiberg-Witten equation (6.1) were just linear functions of the Coulomb branch (6.2), in the D_N case, the coefficients in Seiberg-Witten equation (7.1) are, in general, polynomial expressions when expressed in terms of the natural linear coordinates at the origin of the Coulomb branch. We see that, already, in the fact that the Seiberg-Witten equation depends quadratically on $\tilde{\phi}$. But there are further polynomial constraints on the coefficients in the ϕ_{2k}, which need to be solved before one sees the natural linear structure [7].

While the constraints are polynomial, they are always *linear* in (at least) *one* of the variables. Moreover, they are of homogeneous degree in the aforementioned grading. So the space of solutions of the constraints is always smooth at the origin of the Coulomb branch, and hence the tangent space at the origin has the desired structure of a graded vector space.

The other complication in the D_N theories is that, whereas the differentials in the D_N theory have degrees $2, 4, 6, \ldots, 2(N-1); N$, the Coulomb branch has components in other degrees. For instance, in D_4, there is a component of degree 3, in addition to the "expected" components of degrees $2, 4, 6$. In general, the Coulomb branch takes the form

$$E \subset V$$

where

$$V = \bigoplus_{k=1}^{N-1} H^0\left(C, K^{2k}(\sum_{i=1}^{n} p_i^{(k)} y_i)\right) \quad \oplus \quad \bigoplus_{k=3}^{N-1} W_k \quad \oplus \quad H^0\left(C, K^N(\sum_{i=1}^{n} \tilde{p}_i y_i)\right)$$

Here the W_k are vector spaces of degree k and E is the subvariety satifying the collection of polynomial constraints (linear in at least one variable, and of homogeneous degree).

If we denote the coefficient of l^{th}-order pole of ϕ_k, at one of the punctures, by $c_l^{(k)}$, the constraints can roughly be divided into

- polynomials (of homogeneous degree in both k and l) in the $c_l^{(k)}$

- polynomials (again, of appropriately homogeneous degree) involving both the $c_l^{(k)}$ and a basis $a^{(k)}$ for the vector spaces, W_k

In the case of D_4, there is just W_3, and $dim(W_3) = n_o$, the number of punctures, on C, corresponding to a particular special D-partition. At each such puncture, there is a constraint $c_4^{(6)} = \left(a^{(3)}\right)^2$, which says that the coefficient of the leading singularity of ϕ_6 is a perfect square. As we will elaborate in Section 7.1.1, there is a unique non-special nilpotent orbit of D_4. The puncture in question is the image, under the Spaltenstein map, of that non-special nilpotent orbit.

7.1.1 Punctures and the Spaltenstein Map

For the D_N series, punctures are labeled by certain partitions of $2N$. Not all partitions of $2N$ are allowed. The rules are as follows:

- Even integers must occur with even multiplicity.
- When all the integers in the partition are even, such a partition is called *very even*, and we get *two* punctures associated to this partition. Such partitions only occur for N even. These two punctures are exchanged by the $\mathbb{Z}_2$ outer automorphism of D_N which exchanges the two spinor representations. We will colour the corresponding Young diagrams red and blue, to distinguish them.

Such a partition is called a *D-partition* of $2N$. As we shall see in this Section, nilpotent orbits in $\mathfrak{so}(2N)$ are in 1:1 ccorrespondence with D-partitions of $2N$

(with the caveat that there are two *different* nilpotent orbits associated to each very-even partition[1]).

We will also have recourse to *C-partitions* of $2N$ (in 1:1 correspondence with nilpotent orbits in $\mathfrak{sp}(N)$), which are defined as partitions of $2N$ such that odd integers occur with even multiplicity.

From the Young diagram, corresponding to a D-partition, we reconstruct the flavour symmetry group, associated to the puncture.

$$G = \prod_{h\,\text{odd}} \text{Spin}\left(n^{(h)}\right) \times \prod_{h\,\text{even}} Sp\left(\tfrac{n^{(h)}}{2}\right) \tag{7.2}$$

From this, the necessity of the the rule that $n^{(h)}$ be even, for even h, is obvious. The origin of the additional rule (which arises for N even) — that "very even" D-partitions occur twice — has a more subtle origin.

For N odd, the irreducible spinor representation of D_N is complex, and the right-handed spinor representation is the complex-conjugate of the left-handed one. So a "hypermultiplet in the spinor" contains fields transforming as spinors of both chiralities.

For N even, the irreducible spinor representation is real ($N = 4l$) or pseudoreal ($N = 4l+2$), and the left- and right-handed spinor representations are inequivalent. So a "hypermultiplet in the left-handed spinor representation" is *different* from a "hypermultiplet in the right-handed spinor representa-

[1] This phenomenon of having two nilpotent orbits associated to a single (very-even) partition is characteristic of the D_N Lie algebras with N even. The nilpotent orbits in the other classical Lie algebras $\mathfrak{g}$ are in 1:1 correspondence with their respective $\mathfrak{g}$-partitions.

tion." When we discuss fixtures, we will need to keep track of this distinction. Exchanging "red" and "blue" punctures will exchange the roles of left- and right-handed spinors.

Understanding the singularities of the ϕ_k at the puncture is somewhat more involved than in the A_{N-1} case.

As in the A_{N-1} case, we might expect to associate a nilpotent orbit in $\mathfrak{so}(2N)$ to the rows of the Young diagram. Unfortunately, when the *columns* of a $2N$-box Young diagram form a D-partition, the *rows* typically do not. In other words, the transpose does not map D-partitions to D-partitions. Nevertheless, there is a simple modification of the transpose map, called the "Spaltenstein map" which *does* map D-partitions to D-partitions.

This procedure may be described as (row) "D-collapse":

- Given a Young diagram whose columns form a D-partition, take the longest even row, which occurs with odd multiplicity (if the multiplicity is greater than 1, take the *last* row of that length), and remove the last box. Place the box at the end of the next available row, such that the result is a Young diagram.

- Repeat the process with next longest even row, which occurs with odd multiplicity.

- This process eventually terminates, and the result is a "corrected" Young diagram, whose row-lengths form a D-partition.

Conversely, starting with a Young diagram whose rows form a D-partition (thus specifying a nilpotent orbit), we can define a process of *column D-collapse*, which yields a Young diagram whose columns form a D-partition (hence, a flavour symmetry group).

In the A_{N-1} case, the Spaltenstein map was given by transpose (alternatively by reading the partition from the rows/columns instead of columns/rows of the Young diagram). In the D_N case, the Spaltenstein map is defined as the composition of the transpose with the appropriate (row/column) D-collapse. Unfortunately, unlike the transpose, the Spaltenstein map is *not* an involution of the set of D-partitions; in general, it is neither 1-1 nor onto. The set of partitions in the image of the Spaltenstein map are called "special", and the Spaltenstein map, restricted to the special partitions, *is* an involution.

More formally, let s be the Spaltenstein map, and let p be a D-partition. p is called "special" if $s^2(p) = p$. In the A_{N-1} case, all partitions were special $((p^t)^t = p)$. That's not the case for D_N. Instead, we have the theorem

Theorem ([15] Corollary 6.36 and Proposition 6.3.7)

1. For any D-partition, p, $s(p)$ is a special D-partition.

2. A D-partition, p, is special, if and only if p^t is a C-partition.

The boundary conditions for the punctures corresponding to special D-partitions are determined as in the A_{N-1} case. Let f be the D-partition which gives the flavour symmetry. Let $o = s(f)$ be the image of f under the

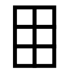
Spaltenstein map. If f is special (which was *always* the case for A_{N-1}), then the Higgs field $\varphi(y)$ has a simple pole, with residue $X \in o$. Under the obvious embedding $\mathfrak{so}(2N) \hookrightarrow \mathfrak{sl}(2N)$, the characteristic equation

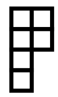
$$\det(\varphi(y) - q\mathbf{1}) = q^{2N} + \sum_{k=1}^{N-1} q^{2(N-k)} \phi_{2k}(y) \;+\; \big(\tilde{\phi}(y)\big)^2 \tag{7.3}$$

yields the pole orders of the k-differentials. These can be read off from the Young diagram for o, just as if it were a Young diagram for A_{2N-1} (see the rule above). Because $\varphi(y)$ lies in the $\mathfrak{so}(2N)$ subalgebra, the ϕ_k vanish for odd k, and $\phi_{2N}(y) = \big(\tilde{\phi}(y)\big)^2$. That, however, does not quite exhaust the constraints on the polar parts of the k-differentials, which follow from restricting to $\mathfrak{so}(2N) \subset \mathfrak{sl}(2N)$. There are additional polynomial constraints among the coefficients of the leading-order poles of the various k-differentials.

These additional constraints were previously found by Tachikawa [7] by applying the restrictions, imposed by M-theory orientifolds [50], to SO-Sp linear quiver tails[2]. As already mentioned, the SO-Sp quivers naturally live in the larger theory, with outer-automorphism twists. From our present perspective it is better to think of the constraints as coming directly from putting the polar part of $\varphi(y)$ in a special nilpotent orbit of $\mathfrak{so}(2N)$. (For our explicit conventions on nilpotent orbits in $\mathfrak{so}(2N)$, see Appendix A.)

[2]These constraints, also from the Hitchin-system perspective, have been found as well in [33] in the context of surface operators of 4D $\mathcal{N} = 4$ super Yang-Mills. Still, the constraints have a much richer interpretation for 4D $\mathcal{N} = 2$ theories, in terms of the parameters in the Seiberg-Witten curve, than for surface operators of $\mathcal{N} = 4$ super Yang-Mills.

As a simple example, consider the minimal D_3 puncture, . To find its pole structure, we put the polar part of the Higgs field in the nilpotent orbit of the Spaltenstein dual,



We write $\varphi(y) = \frac{X}{y} + M$, where $X = X^{-}_{1,2}$ is the canonical nilpotent element in this orbit (see Appendix A for our conventions), and M is a generic matrix in $\mathfrak{so}(2N)$, of the form (A.1). The differentials are thus of the form

$$\phi_2 = \frac{2a}{y} + \dots, \qquad \phi_4 = \frac{a^2}{y^2} + \dots, \qquad \tilde{\phi} = \frac{b}{y} + \dots \tag{7.4}$$

Thus, the pole structure is $\{1, 2; 1\}$, with a constraint $c^{(4)}_2 = \frac{1}{4} c^{(2)}_1$. This pole structure and constraint was computed in [7] from the SO-Sp linear quiver tail for this puncture.

That takes care of the punctures corresponding to special D-partitions. What about the punctures corresponding to non-special D-partitions? Here the situation is a bit more awkward. The Spaltenstein map is not an involution, when applied to non-special partitions, and so the boundary conditions on $\varphi(y)$ are not currently known. (This is currently under investigation [45].) The effect on the pole structure of the k-differentials, however, is easy to find (say, from the linear quiver tail analysis), and amounts to the following. Given a non-special D-partition, f, $f_s = s^2(f)$ is a special D-partition. The pole structure of the $\phi_k(y)$ is precisely that one would find for the puncture f_s. However, f_s has a series of constraints of the form $c^{(2k)}_{2l} = \left(a^{(k)}\right)^2$ on the leading pole coefficients. For the puncture, f, some (or all) of these constraints are relaxed.

To see which constraint(s) are relaxed, notice that f is related to f_s by a process of (row) C-collapse. That is, we remove the last box from a row of odd length (which occurred with odd multiplicity) and place it lower-down on the Young diagram. The box we removed was an odd-numbered box (call it $2k+1$). By removing it, an even-numbered box (box $2k$) becomes the last box in that row. The puncture, f_s, had a constraint of the form $c_{2l}^{(2k)} = \left(a^{(k)}\right)^2$. For each $(2k)^{\text{th}}$ box, thus exposed, we relax the corresponding constraint of f_s.

For D_4, there is just one non-special puncture and, correspondingly, just one constraint that gets relaxed. We will defer a complete discussion to [45].

Finally, let us elaborate on our conventions for "very even" punctures. When N is even, the Pfaffian, $\tilde{\phi}$ has the same degree as ϕ_N. The outer-automorphism of D_N, which exchanges the roles of the two spinor representations, takes

$$\begin{aligned} \tilde{\phi} &\mapsto -\tilde{\phi} \\ \phi_{2k} &\mapsto \phi_{2k}, \qquad k = 1, \dots, N-1 \end{aligned} \tag{7.5}$$

For most punctures, the contraints are such that there is a unique Coulomb branch parameter (the coefficient of the highest-order pole of one of the ϕ_{2k}) which appears linearly. We can take that to be the variable eliminated by the constraint, so for the purpose of counting the graded dimension of the

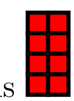
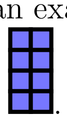
Coulomb branch, it is as if we simply reduced the allowed pole-order for that differential by 1.

The red/blue punctures are an exception. At a (a certain subset of) red/blue punctures, both $\tilde{\phi}$ and ϕ_N are allowed to have poles of some order (say, l) but a linear combination of the coefficients, $c_l^{(N)} \pm 2\tilde{c}_l$, is the variable that appears linearly in the associated constraints. Our convention[3] will be that, at a red regular puncture, the constraint is of the form

$$c_l^{(N)} + 2\tilde{c}_l = \dots \tag{7.6}$$

At the corresponding blue regular puncture, the constraint is

$$c_l^{(N)} - 2\tilde{c}_l = \dots \tag{7.7}$$

As an example, let us look at the punctures with flavour Young diagrams  and . Their nilpotent orbits correspond to these same Young diagrams, and the canonical nilpotent elements (see Appendix A) are $X^{(r)} = X^-_{1,2} + X^-_{3,4}$ and $X^{(b)} = X^-_{1,2} + X^+_{3,4}$, respectively. After writing $\varphi(y) = \frac{X^{(\mathrm{r/b})}}{y} + M$ for the

[3] At red/blue irregular punctures, the convention is reversed. At a red irregular puncture, the constraint is of the form

$$c_l^{(N)} - 2\tilde{c}_l = \dots$$

while, at the corresponding blue irrregular puncture, the constraint is

$$c_l^{(N)} + 2\tilde{c}_l = \dots$$

Higgs field, with M a generic $\mathfrak{so}(2N)$ matrix, we find for the differentials,

$$\begin{aligned} \phi_2 &= \frac{2a}{y} + \dots \\ \phi_4 &= \frac{a^2 \mp 2b}{y^2} + \dots \\ \phi_6 &= \frac{\mp 2ab}{y^3} + \dots \\ \tilde{\phi} &= \frac{b}{y^2} + \dots \end{aligned} \tag{7.8}$$

with the top sign for the red and the lower sign for the blue puncture. So the pole structure for these punctures is $\{1,2,3;2\}$, with constraints $c_2^{(4)} \pm 2\tilde{c}_2 = \frac{1}{4}(c_1^{(2)})^2$ and $c_3^{(6)} = \mp\tilde{c}_2 c_1^{(2)}$. The $\mathbb{Z}_2$ outer automorphism acts as $b \mapsto -b$, and it exchanges the red and blue constraints.

In the presence of red/blue punctures, a little extra care must be taken in computing the graded Coulomb branch dimensions. Too large an excess, of one or the other, over-constrains the differentials and would lead to a difference between the virtual and actual dimension of the Coulomb branch. The dimension of the degree-N component,

$$\dim(V_N) = d_N + \tilde{d} - n_r - n_b \tag{7.9}$$

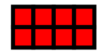
where d_N and $\tilde{d}$ are the dimensions we would obtain from applying Riemann-Roch (suitably-adjusted for the other constraints) to ϕ_N and $\tilde{\phi}$, and $n_{r,b}$ are the number of constraints of the form (7.6), (7.7) respectively. In order that the constraints not be over-determined, it suffices to ensure that either

$$d_N - n_r \geq 0, \quad \tilde{d} - n_b \geq 0 \tag{7.10}$$

or

$$d_N - n_b \geq 0, \quad \tilde{d} - n_r \geq 0 \tag{7.11}$$

holds. Either condition is sufficient to ensure that $\dim(V_N) \geq 0$, but is slightly stronger.

For instance, there is no 3-punctured sphere with three  punctures. The constraints would overconstrain (imply a negative virtual dimension for) the space of sections of the differential $\phi^{(4)} + 2\tilde{\phi}$.

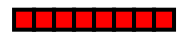
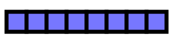
7.1.2 Irregular Punctures

In addition to regular punctures, we will, again, need to introduce a class of "irregular" punctures, which admit higher-order poles. Ignoring, for the moment, the question of constraints, the class of irregular punctures is the one we introduced in [6] for the A_{N-1} series.

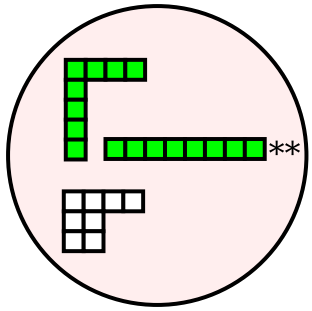
- Each irregular puncture is associated to a simple subgroup $G \subset Spin(2N)$.
- From the pole structure $\{p_k\}$, of the irregular puncture, we construct the "conjugate pole structure," $\{p'_k\}$
  - $p'_k = p_k = k - 1$ if k is an exponent of G.
  - $p'_k + p_k = 2k - 1$ otherwise.
- We demand that the conjugate pole structure be that of a regular puncture, and we denote the irregular puncture, thus constructed, by the

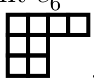
Young diagram of the conjugate regular puncture, with one or more "$*$"s appended.

Incorporating the constraints simply amounts to "correcting" which values of k correspond to exponents of G.

For example, the D_4 puncture, *, has as its conjugate puncture the maximal puncture, . Its pole structure, $\{1, 3, 5; 4\}$, allows for a quartic, rather than merely a cubic pole for $\tilde{\phi}$. Thus, the corresponding symmetry group is a $Spin(7)$ subgroup of $Spin(8)$. There are three inequivalent embeddings of $Spin(7) \hookrightarrow Spin(8)$ (depending on which eight-dimensional representation decomposes as the $7 + 1$). Thus, we also have * and *, which are exchanged by the usual $\mathbb{Z}_2$ outer automorphism. These latter have pole structure $\{1, 4, 5; 4\}$, and impose, respectively, a constraint $c_4^{(4)} \mp 2\tilde{c}_4 = 0$. This constraint is consequence of using $\phi^{(4)}$, $\tilde{\phi}$ as our basis of 4-differentials (rather than the linear combination that appears more naturally at a red/blue puncture).

Similarly, the puncture ** corresponds to an $SU(4)$ subgroup of $Spin(8)$, and has poles $\{1, 3, 6; 4\}$. There are again blue and red versions of this puncture corresponding to the other two embeddings of $SU(4)$ related by triality to the green one. The exponent 3 in $SU(4)$ (as opposed to 6) means that we need a constraint $c_6^{(6)} = -(a^{(3)})^2$ that appropriately corrects the dimensions of the Coulomb branch. In a free-field fixture, e.g.,



the constraint $c_6^{(6)} = -(a^{(3)})^2$ from ** offsets the constraint $c_6^{(6)} = (a^{(3)})^2$ from , so the virtual dimension of the Coulomb branch is indeed equal to its actual dimension (zero).

The red and blue versions of this puncture, ** and **, have poles $\{1, 4, 6; 4\}$, and have the same constraint as the green one, $c_6^{(6)} = -(a^{(3)})^2$, plus an additional constraint $c_4^{(4)} \mp 2\tilde{c}_4 = 0$ as usual.

Finally, we can assign a level, k, to the G symmetry of the irregular puncture. It is simply defined such that the G gauge group on the cylinder, $p \xleftrightarrow{G} p'$ between p and its conjugate regular puncture p', is conformal.

7.1.3 Central charges

Having explained the definition of the central charges previously in Section 4.5.2, we simply mention facts specific to the D_N case, as well as show formulas for n_h and n_v.

The central charge, k, for each simple factor in the flavour symmetry group associated to a regular puncture can be computed directly from the

Young diagram. Denote the length of the i^{th} row by r_i. In the A_{N-1} case, the flavour symmetry group was given by (6.3) and each $SU(r_i - r_{i+1})$ factor had level

$$k = 2\sum_{j=1}^{i} r_j \tag{7.12}$$

For the D_N case, the flavour symmetry group is given by (7.2), and

- For i odd, this gives a $Spin(r_i - r_{i+1})_k$ factor in the flavour symmetry group, where

$$k = \begin{cases} 2\left(\sum_{j=1}^{i} r_j\right) - 4 & r_i - r_{i+1} \geq 4 \\ 4\left(\sum_{j=1}^{i} r_j\right) - 8 & r_i - r_{i+1} = 3 \end{cases} \tag{7.13a}$$

- For i even, this gives an $Sp\left(\frac{r_i - r_{i+1}}{2}\right)_k$ in the flavour symmetry group, where

$$k = \sum_{j=1}^{i} r_j \tag{7.13b}$$

From Theorem 7.1.1, a non-special puncture corresponds to a $2N$-box Young diagram, whose columns form a D-partition, with at least one (in fact, at least two) odd-length row(s) which appears with odd multiplicity. With a little more work, one can show that at least one of these rows is an even-numbered row. By (7.13b), this gives an $Sp(l)_k$ factor, in the flavour symmetry group, with k odd. As mentioned in the introduction, this poses an obstruction to gauging: without additional matter to cancel the anomaly, the $Sp(l)$ gauge theory would suffer from Witten's global anomaly [16].

The trace anomaly coefficients, a and c, of the SCFT, can be computed (as we did [6], for the A_{N-1} series) from two auxiliary quantities: the effective number of hypermultiplets, n_h, and the effective number of vector multiplets, n_v,

$$\begin{aligned} a &= \tfrac{5n_v+n_h}{24} \\ c &= \tfrac{2n_v+n_h}{12}. \end{aligned} \tag{7.14}$$

In the previous chapter we gave formulæ to compute n_h and n_v for regular and irregular punctures in the A_{N-1} series. As before, n_h and n_v are the actual number of hypermultiplets and vector multiplets in a *Lagrangian* S-duality frame of the theory, provided such frame exists. As a consequence, the n_h of a free-field fixture (for which $n_v = 0$) is equal to the number of free hypermultiplets in this fixture.

To compute n_v for a D_N theory on a curve of genus g, one should first calculate the graded dimensions of the Coulomb branch. Then

$$\begin{aligned} n_v &= \sum_k (2k-1)d_k \\ &= \sum_{k=1}^{N-1}(4k-1)d_{2k} + \sum_{k=1}^{[\frac{N-1}{2}]}(4k+1)d_{2k+1}. \end{aligned} \tag{7.15}$$

For example, in the D_4 theory, the possible non-zero Coulomb branch dimensions are d_2, d_3, d_4, d_6, while in the D_5 theory, they are $d_2, d_3, d_4, d_5, d_6, d_8$. The odd-degree components of the Coulomb branch of the D_N theory appear only up to degree $2[\frac{N-1}{2}]+1$. We will discuss below how to compute the d_{2k} and d_{2k+1}, but we will treat the case of d_N separately, since it involves the pole orders of the Pfaffian $\tilde{\phi}$.

As we saw before, the even-degree sectors of the Coulomb branch, with dimensions d_{2k} $(2k \neq N)$, arise from $2k$-differentials, and so

$$d_{2k} = (1-4k)(1-g) + \sum_{\alpha}(p^{\alpha}_{2k} - s^{\alpha}_{2k} + t^{\alpha}_{2k}) \qquad (7.16)$$

where α runs over the punctures on the curve, p^{α}_{2k} is the pole order of ϕ_{2k} at the α^{th} puncture, s^{α}_{2k} is the number of constraints of homogeneous degree $2k$ (i.e., polynomial constraints of the form $c^{(2k)}_{l} = \dots$), and t^{α}_{2k} is the number of $a^{(2k)}$ parameters (i.e., parameters arising from constraints of the form $c^{(4k)}_{l} = (a^{(2k)})^2$) that the α^{th} puncture contributes.

On the other hand, since there are no ϕ_{2k+1} differentials (except for the Pfaffian, when N is odd), these odd-degree sectors of the Coulomb branch receive contributions *only* from the $a^{(2k+1)}$ parameters (i.e., parameters arising from constraints of the form $c^{(4k+2)}_{l} = (a^{(2k+1)})^2$). We write

$$d_{2k+1} = \sum_{\alpha} t^{\alpha}_{2k+1}, \qquad (7.17)$$

Notice that this expression is independent of the genus (in contrast to the contributions, to the d_{2k}, from the Riemann-Roch Theorem).

As for d_N, if N is even, then d_N gets a contribution from both ϕ_N and from the Pfaffian $\tilde{\phi}$. The formula for d_N is almost the same as for the d_{2k} case,

$$d_N = 2(1-2N)(1-g) + \sum_{\alpha}(p^{\alpha}_{N} - s^{\alpha}_{N}) + \tilde{p}^{\alpha}. \qquad (7.18)$$

Notice that there is no t^{α}_{N} term, since we do not have a $2N$-differential.

Similarly, if N is odd, only the Pfaffian (the unique odd-degree differential) contributes to d_N, and so,

$$d_N = (1-2N)(1-g) + \sum_\alpha \tilde{p}^\alpha. \tag{7.19}$$

Adding up the global, genus-dependent contribution from the $2k$-differentials and the Pfaffian, we obtain

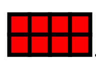
$$n_v = -\tfrac{1}{3}(1-g)N(16N^2 - 24N + 11) + \sum_\alpha \delta n_v^{(\alpha)}, \tag{7.20}$$

where α runs over the punctures on the curve, and the contribution $\delta n_v^{(\alpha)}$ of the α^{th} puncture to n_v is

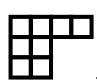
$$\delta n_v^{(\alpha)} = \sum_{k=1}^{N-1} (4k-1)(p_{2k}^\alpha - s_{2k}^\alpha + t_{2k}^\alpha) + \sum_{k=1}^{[\frac{N-1}{2}]} (4k+1)t_{2k+1}^\alpha + (2N-1)\tilde{p}^\alpha \tag{7.21}$$

Let us see a few examples of how to compute δn_v. First, consider the maximal D_3 puncture, which has poles $\{1,3;2\}$, and no constraints. One gets

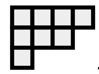
$$\delta n_v = 3(1) + 7(3) + 5(2) = 34. \tag{7.22}$$

Next, consider the D_4 puncture, . The poles are $\{1,3,4;3\}$ and there is one constraint ($c_3^{(4)} + 2\tilde{c}_3 = 0$), so $s_4 = 1$. We then have

$$\delta n_v = 3(1) + 7(3-1) + 11(4) + 7(3) = 82. \tag{7.23}$$

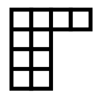
Now consider the D_4 puncture . The poles are $\{1,2,4;2\}$ and there is one constraint ($c_4^{(6)} = \left(a^{(3)}\right)^2$), so $s_6 = 1$ and $t_3 = 1$. Thus,

$$\delta n_v = 3(1) + 7(2) + 11(4-1) + 7(2) + 5(1) = 69. \tag{7.24}$$

Now look at the non-special D_4 puncture . Its poles are $\{1, 2, 4; 2\}$, and it has no constraints. This means that

$$\delta n_v = 3(1) + 7(2) + 11(4) + 7(2) = 75. \tag{7.25}$$

Finally, let us look at the D_5 puncture



(7.26)

which has poles $\{1, 2, 4, 5; 3\}$. The two constraints ($c_4^{(6)} = (a^{(3)})^2$ and $c_5^{(8)} = 2a^{(3)}\tilde{c}_3$) imply that $t_6 = 1$, $t_8 = 1$, and $s_3 = 1$. Hence,

$$\delta n_v = 3(1) + 7(2) + 11(4 - 1) + 15(5 - 1) + 9(3) + 5(1) = 142. \tag{7.27}$$

Let us now go on to discuss n_h. Just like n_v, n_h is a sum of a global piece and contributions from each puncture,

$$n_h = -\tfrac{8}{3}(1 - g)N(N - 1)(2N - 1) + \sum_\alpha \delta n_h^{(\alpha)} \tag{7.28}$$

where α runs over the punctures, and

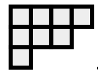
$$\delta n_h^{(\alpha)} = \delta n_v^{(\alpha)} + f^{(\alpha)} \tag{7.29}$$

is the contribution of the α^{th} puncture to n_h. We will see below how to compute $f^{(\alpha)}$ for regular and irregular punctures.

For a *regular* puncture, $f^{(\alpha)}$ can be found from the row-lengths $r_1 \geq r_2 \geq \dots$ of the flavour Young diagram,

$$f^{(reg)} = \frac{1}{4}\sum r_i^2 - \frac{1}{2}\sum r_{odd}, \tag{7.30}$$

where the first sum is over all rows, and the second is restricted to odd-numbered rows $(r_1, r_3, r_5, r_7, \ldots)$.

For example, the D_4 puncture, , has $f = \frac{1}{4}[4^2+3^2+1^2]-\frac{1}{2}[4+1] = 4$. Since we previously computed $n_v = 75$ for this puncture, we have $n_h = 79$.

The $f^{(irreg)}$ for an irregular puncture, p, follows from consistency with degeneration,

$$f^{(irreg)} = -N + \dim G - f^{(reg)}, \tag{7.31}$$

where $f^{(reg)}$ is the contribution of the regular puncture, p', conjugate to p. G is the flavour symmetry group we ascribe to the irregular puncture, p (equivalently, the gauge group on the cylinder $p \xleftrightarrow{G} p'$).

7.1.4 Regular Punctures (up through D_6)

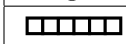
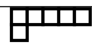
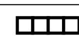
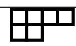
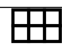
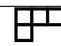
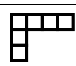
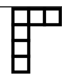
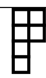
We list below the properties of regular punctures for D_3, D_4, D_5, and D_6. In writing down the global symmetry groups, it will be convenient to use the isomorphisms

$$\begin{aligned} Spin(2) &\simeq U(1) \\ Spin(3) &\simeq Sp(1) \simeq SU(2) \\ Spin(4) &\simeq SU(2)^2 \\ Spin(5) &\simeq Sp(2) \\ Spin(6) &\simeq SU(4) \end{aligned} \tag{7.32}$$

As in the A_{N-1} case, there's a Young diagram (this time, with a column of height $2N-1$ and a column of height 1), which corresponds to a regular point on the curve C, so we exclude it from our discussion.

7.1.4.1 D_3

Since $D_3 \simeq A_3$, the results for D_3 were already reported in our previous paper. However, as a warm-up, it will be convenient to repeat them here, recast in the notation we will use for the higher entries in the D_N series.

Young Diagram	Nilpotent Orbit	Pole structure	Constraints	A_3 Young Diagram	Flavour Symmetry	$(\delta n_h, \delta n_v)$
		$\{1,3;2\}$	–		$SU(4)_8$	$(40,34)$
		$\{1,2;2\}$	–		$SU(2)_6 \times U(1)$	$(30,27)$
		$\{1,2;1\}$	–		$SU(2)_8$	$(24,22)$
		$\{1,2;1\}$	$c_2^{(4)} = \frac{1}{4}\left(c_1^{(2)}\right)^2$		$U(1)$	$(16,15)$

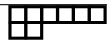
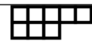
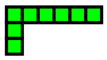
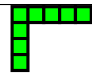
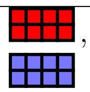
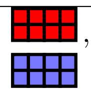
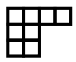
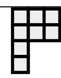
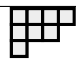
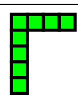
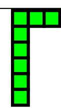
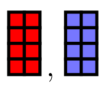
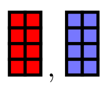
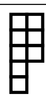
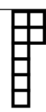
Note that, in the D_3 description, the quartic differential is allowed to have a double pole at the minimal puncture, instead of only a simple pole (as in the A_3 description). However, the coefficient of the double pole is constrained, so that the Coulomb branch has the same graded dimension as before.

7.1.4.2 D_4

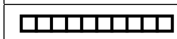
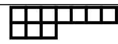
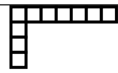
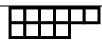
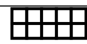
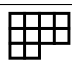
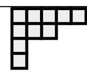
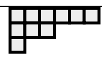
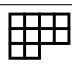
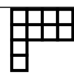
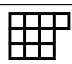
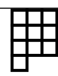
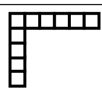
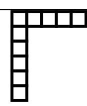
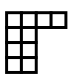
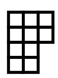
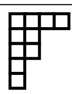
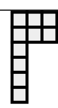
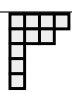
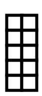
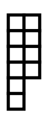
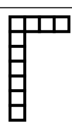
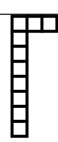
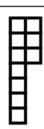
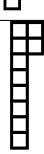
For D_4, the outer automorphism group is enhanced from $\mathbb{Z}_2$ to S_3. Hence, the pairs of punctures, which were related by exchanging $8_s \leftrightarrow 8_c$, are actually organized into triples, under permutations of $8_s, 8_c, 8_v$. We indicate this by colouring the Young diagram, corresponding to the other puncture in the triple, green.

The fact that the nilpotent orbits in a triple are related by triality becomes particularly clear if one looks at their weighted Dynkin diagrams ([15]). More practical evidence comes from the fact that the punctures in a triple exhibit the same flavour group and $(\delta n_h, \delta n_v)$.

In this table, and in the D_5, D_6 tables below, we've shaded each non-special flavour Young diagram and the (special) nilpotent orbit which is its image under the Spaltenstein map.

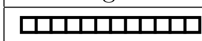
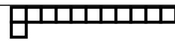
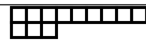
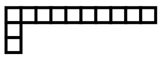
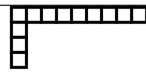
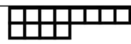
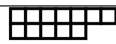
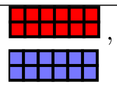
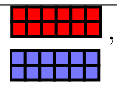
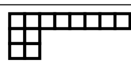
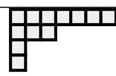
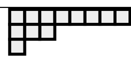
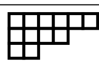
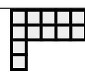
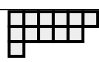
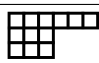
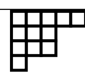
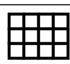
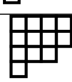
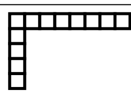
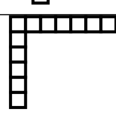
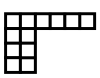
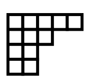
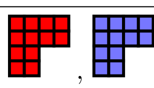
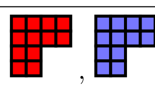
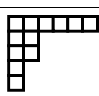
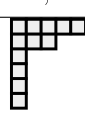
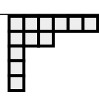
Young Diagram	Nilpotent orbit	Pole structure	Constraints	Flavour Symmetry	$(\delta n_h, \delta n_v)$
		$\{1,3,5;3\}$	–	$Spin(8)_{12}$	$(112,100)$
		$\{1,3,4;3\}$	–	$SU(2)_8^3$	$(96,89)$
		$\{1,3,4;2\}$	–	$Sp(2)_8$	$(88,82)$
		$\{1,3,4;3\}$	$c_3^{(4)} \pm 2\tilde{c}_3 = 0$	$Sp(2)_8$	$(88,82)$
		$\{1,2,4;2\}$	$c_4^{(6)} = \left(a^{(3)}\right)^2$	$U(1)^2$	$(72,69)$
	–	$\{1,2,4;2\}$	–	$SU(2)_7$	$(79,75)$
		$\{1,2,2;1\}$	–	$SU(2)_8$	$(48,46)$
		$\{1,2,3;2\}$	$c_2^{(4)} \pm 2\tilde{c}_2 = \frac{1}{4}\left(c_1^{(2)}\right)^2$ $c_3^{(6)} = \mp\tilde{c}_2 c_1^{(2)}$	$SU(2)_8$	$(48,46)$
		$\{1,2,2;1\}$	$c_2^{(4)} = \frac{1}{4}\left(c_1^{(2)}\right)^2$	none	$(40,39)$

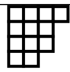
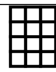
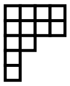
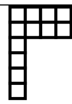
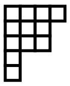
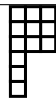
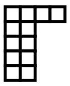
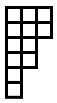
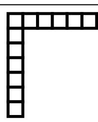
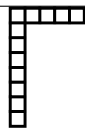
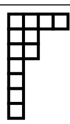
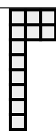
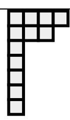
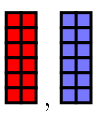
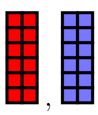
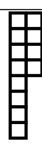
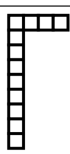
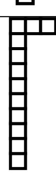
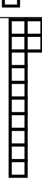
7.1.4.3 D_5

Young Diagram	Nilpotent Orbit	Pole structure	Constraints	Flavour Symmetry	$(\delta n_h, \delta n_v)$
		$\{1,3,5,7;4\}$	$-$	$Spin(10)_{16}$	$(240,220)$
		$\{1,3,5,6;4\}$	$-$	$SU(4)_{12}\times SU(2)_{10}$	$(218,205)$
		$\{1,3,5,6;3\}$	$-$	$Spin(7)_{12}$	$(208,196)$
		$\{1,3,4,6;4\}$	$-$	$Sp(2)_{10}\times U(1)$	$(204,194)$
		$\{1,3,4,6;3\}$	$c_6^{(8)}=\left(a^{(4)}\right)^2$	$SU(2)_8^2\times U(1)$	$(184,177)$
	$-$	$\{1,3,4,6;3\}$	$-$	$SU(2)_{16}\times SU(2)_9$	$(193,185)$
		$\{1,3,4,6;3\}$	$c_6^{(8)}=\frac{1}{4}\left(c_3^{(4)}\right)^2$	$SU(2)_8\times U(1)$	$(176,170)$
		$\{1,2,4,5;3\}$	$-$	$SU(2)_{32}$	$(168,163)$
		$\{1,3,4,4;2\}$	$-$	$Sp(2)_8$	$(152,146)$
		$\{1,2,4,5;3\}$	$c_4^{(6)}=\left(a^{(3)}\right)^2$ $c_5^{(8)}=2a^{(3)}\tilde{c}_3$	$SU(2)_{10}\times U(1)$	$(146,142)$
		$\{1,2,4,4;2\}$	$c_4^{(6)}=\left(a^{(3)}\right)^2$	$U(1)$	$(136,133)$
	$-$	$\{1,2,4,4;2\}$	$-$	$SU(2)_7$	$(143,139)$
		$\{1,2,3,4;2\}$	$c_3^{(6)}=\frac{1}{2}c_1^{(2)}$ $\times\left(c_2^{(4)}-\frac{1}{4}\left(c_1^{(2)}\right)^2\right)$ $c_4^{(8)}=$ $\frac{1}{4}\times\left(c_2^{(4)}-\frac{1}{4}\left(c_1^{(2)}\right)^2\right)^2$	$U(1)$	$(104,102)$
		$\{1,2,2,2;1\}$	$-$	$SU(2)_8$	$(80,78)$
		$\{1,2,2,2;1\}$	$c_2^{(4)}=\frac{1}{4}\left(c_1^{(2)}\right)^2$	none	$(72,71)$

7.1.4.4 D_6

Again, in D_6, we have very-even partitions, which correspond to two *distinct* punctures, which we have coloured red and blue.

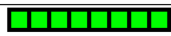
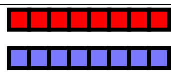
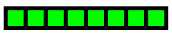
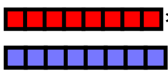
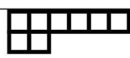
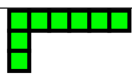
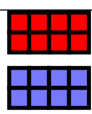
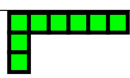
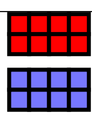
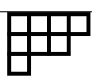
Young Diagram	Nilpotent orbit	Pole structure	Constraints	Flavour Symmetry	$(\delta n_h, \delta n_v)$
		$\{1,3,5,7,9;5\}$	–	$Spin(12)_{20}$	$(440,410)$
		$\{1,3,5,7,8;5\}$	–	$Spin(8)_{16}\times SU(2)_{12}$	$(412,391)$
		$\{1,3,5,7,8;4\}$	–	$Spin(9)_{16}$	$(400,380)$
		$\{1,3,5,6,8;5\}$	–	$Sp(2)_{12}\times SU(2)_{12}^2$	$(392,376)$
		$\{1,3,5,6,8;5\}$	$c_5^{(6)}\pm 2\tilde{c}_5=0$	$Sp(3)_{12}$	$(380,365)$
		$\{1,3,5,6,8;4\}$	$c_8^{(10)}=\left(a^{(5)}\right)^2$	$SU(4)_{12}\times U(1)$	$(368,355)$
	–	$\{1,3,5,6,8;4\}$	–	$Sp(2)_{12}\times SU(2)_{11}$	$(379,365)$
		$\{1,3,4,6,8;4\}$	$c_8^{(10)}=\left(a^{(5)}\right)^2$	$SU(2)_{10}\times U(1)^2$	$(354,344)$
	–	$\{1,3,4,6,8;4\}$	–	$Sp(2)_{11}$	$(366,354)$
		$\{1,3,4,6,7;4\}$	–	$SU(2)_{40}\times SU(2)_{16}$	$(344,335)$
		$\{1,3,4,6,7;4\}$	$c_6^{(8)}=\frac{1}{4}\left(c_3^{(4)}\right)^2$	$SU(2)_{20}^2$	$(328,320)$
		$\{1,3,5,6,6;3\}$	–	$Spin(7)_{12}$	$(328,316)$
		$\{1,3,4,6,7;4\}$	$c_6^{(8)}=\left(a^{(4)}\right)^2$ $c_7^{(10)}=a^{(4)}\tilde{c}_4$	$SU(2)_{12}\times SU(2)_8^2$	$(316,308)$
		$\{1,3,4,6,7;4\}$	$c_6^{(8)}=\frac{1}{4}\left(c_3^{(4)}\right)^2$ $c_7^{(10)}=\pm\tilde{c}_4c_3^{(4)}$	$SU(2)_{12}\times SU(2)_8$	$(308,301)$
		$\{1,3,4,6,6;3\}$	$c_6^{(8)}=\left(a^{(4)}\right)^2$	$SU(2)_8^2$	$(304,297)$
	–	$\{1,3,4,6,6;3\}$	–	$SU(2)_{16}\times SU(2)_9$	$(313,305)$

		$\{1,2,4,5,6;4\}$	–	$SU(2)_{12}$	$(300,294)$
		$\{1,3,4,6,6;3\}$	$c_6^{(8)} = \frac{1}{4}(c_3^{(4)})^2$	$SU(2)_8$	$(296,290)$
		$\{1,2,4,5,6;3\}$	–	$U(1)$	$(288,283)$
		$\{1,2,4,5,6;3\}$	$c_4^{(6)} = \left(a^{(3)}\right)^2$ $c_6^{(10)} = \left(a^{(5)}\right)^2$ $c_4^{(8)} = 2a^{(3)}a^{(5)}$	$U(1)^2$	$(256,252)$
		$\{1,3,4,4,4;2\}$	–	$Sp(2)_8$	$(232,226)$
		$\{1,2,4,4,4;2\}$	$c_4^{(6)} = \left(a^{(3)}\right)^2$	$U(1)$	$(216,213)$
	–	$\{1,2,4,4,4;2\}$	–	$SU(2)_7$	$(223,219)$
		$\{1,2,3,4,5;3\}$	$c_3^{(6)} \pm 2\tilde{c}_3 = \frac{1}{2}c_1^{(2)}\left(c_2^{(4)} - \frac{1}{3}\left(c_1^{(2)}\right)^2\right)$ $c_4^{(8)} = \frac{1}{4}\left(c_2^{(4)} - \frac{1}{3}\left(c_1^{(2)}\right)^2\right)^2 \mp \tilde{c}_3 c_1^{(2)}$ $c_5^{(10)} = \mp\tilde{c}_3\left(c_2^{(4)} - \frac{1}{3}\left(c_1^{(2)}\right)^2\right)$	$SU(2)_{12}$	$(196,193)$
		$\{1,2,3,4,4;2\}$	$c_3^{(6)} = \frac{1}{2}c_1^{(2)}\left(c_2^{(4)} - \frac{1}{3}\left(c_1^{(2)}\right)^2\right)$ $c_4^{(8)} = \frac{1}{4}\left(c_2^{(4)} - \frac{1}{3}\left(c_1^{(2)}\right)^2\right)^2$	none	$(184,182)$
		$\{1,2,2,2,2;1\}$	–	$SU(2)_8$	$(120,118)$
		$\{1,2,2,2,2;1\}$	$c_2^{(4)} = \frac{1}{4}\left(c_1^{(2)}\right)^2$	none	$(112,111)$

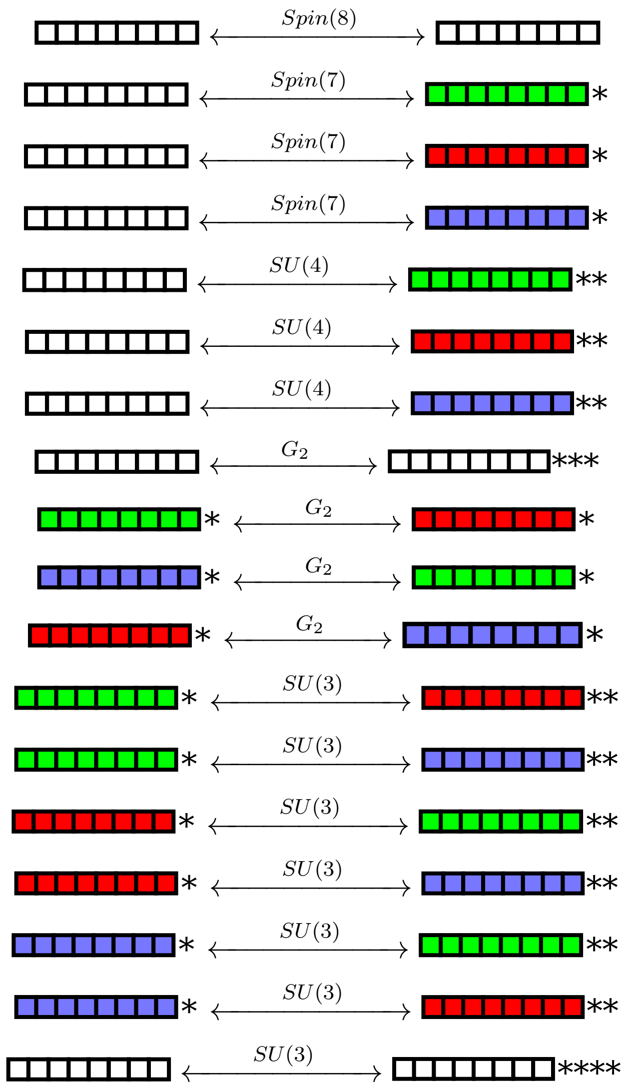
7.2 The D_4 theory

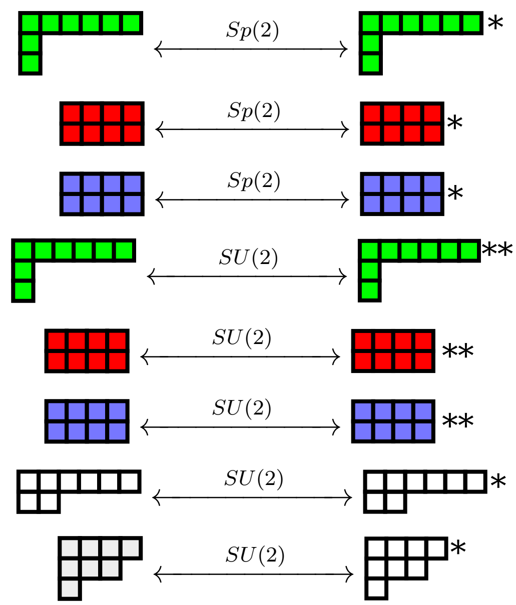
7.2.1 Irregular punctures and cylinders

In this section, we will develop the complete "tinkertoy" catalogue for the D_4 theory. The regular punctures are listed in §7.1.4.2. In the D_4 case, we have the following list of irregular punctures.

Young Diagram	Pole structure	Constraints	Flavour Symmetry	$(\delta n_h, \delta n_v)$
 *	$\{1,3,5;4\}$	$-$	$Spin(7)_8$	$(112,107)$
 *, *	$\{1,4,5;4\}$	$c_4^{(4)} \mp 2\tilde{c}_4 = 0$	$Spin(7)_8$	$(112,107)$
 **	$\{1,3,6;4\}$	$c_6^{(6)} = -\left(a^{(3)}\right)^2$	$SU(4)_4$	$(112,113)$
 **, **	$\{1,4,6;4\}$	$c_4^{(4)} \mp 2\tilde{c}_4 = 0$ $c_6^{(6)} = -\left(a^{(3)}\right)^2$	$SU(4)_4$	$(112,113)$
 ***	$\{1,4,5;4\}$	$-$	$(G_2)_4$	$(112,114)$
 ****	$\{1,4,6;4\}$	$c_6^{(6)} = -\left(a^{(3)}\right)^2$	$SU(3)_0$	$(112,120)$
 *	$\{1,4,7;4\}$	$-$	$SU(2)_0$	$(128,136)$
 *	$\{1,3,7;5\}$	$-$	$Sp(2)_4$	$(136,136)$
 *, *	$\{1,5,7;5\}$	$c_5^{(4)} \mp \tilde{c}_5 = 0$ $c_4^{(4)} \mp \tilde{c}_4 = 0$	$Sp(2)_4$	$(136,136)$
 **	$\{1,4,7;5\}$	$-$	$SU(2)_0$	$(136,143)$
 **, **	$\{1,5,7;5\}$	$c_5^{(4)} \mp \tilde{c}_5 = 0$	$SU(2)_0$	$(136,143)$
 *	$\{1,5,7;5\}$	$-$	$SU(2)_1$	$(145,150)$

The cylinders in the D_4 theory are





Note that some of the irregular punctures have level $k = 0$. Appropriately, these will appear, below, on "empty" fixtures, with zero hypermultiplets. Also, note that each of the cylinders, $p \xleftrightarrow{G} p'$, satisfies

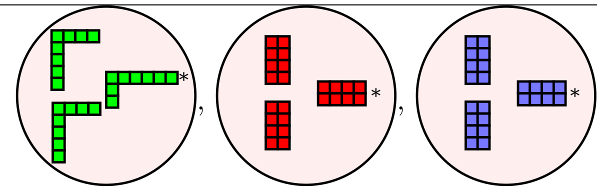
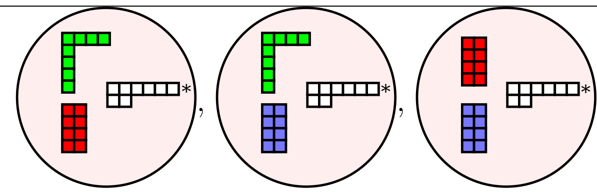
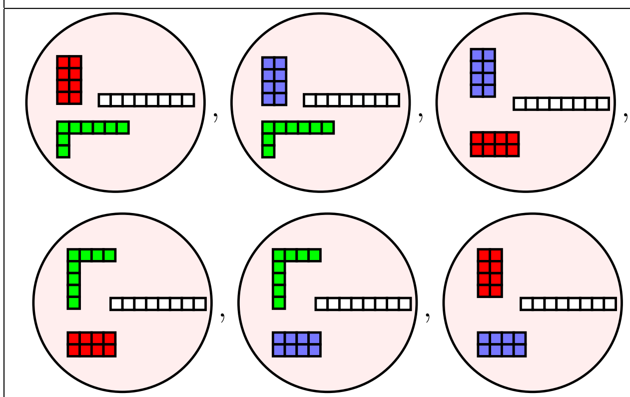
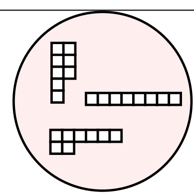
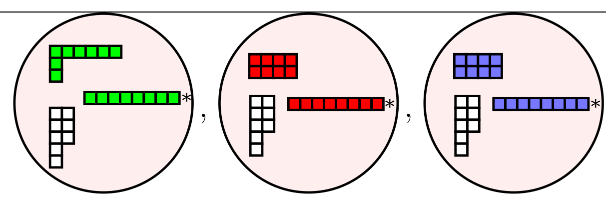
$$\begin{aligned} \delta n_h + \delta n_h{}' - 8N(N-1)(2N-1)/3 &= 0 \\ \delta n_v + \delta n_v{}' - N(16N^2 - 24N + 11)/3 &= dim(G) \\ k + k' &= k_{\text{critical}} \end{aligned} \tag{7.33}$$

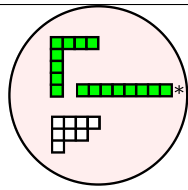
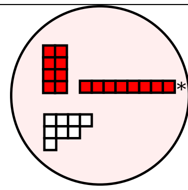
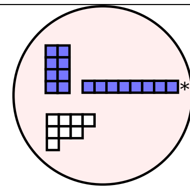
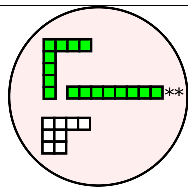
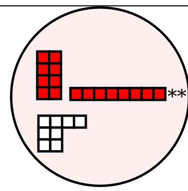
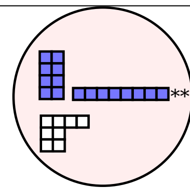
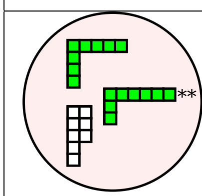
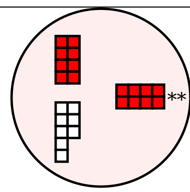
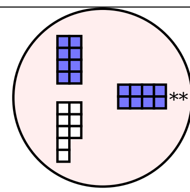
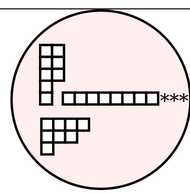
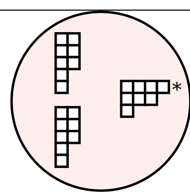
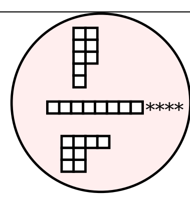
where $k_{\text{critical}} = 2\ell_{\text{adj}}$ is the value of k which gives vanishing β-function for G. While this was true (by construction) when p' is the conjugate regular puncture to p, it's not automatically-satisfied for cylinders between two irregular punctures. In essence, these conditions determine which cylinders between pairs of irregular punctures are allowed.

7.2.2 Fixtures

Here, we list all of the 3-punctured spheres. There are a lot of them, but fortunately, the profusion is partially tamed by the fact that they are organized into multiplets under the outer automorphism group.

7.2.2.1 Free-field fixtures

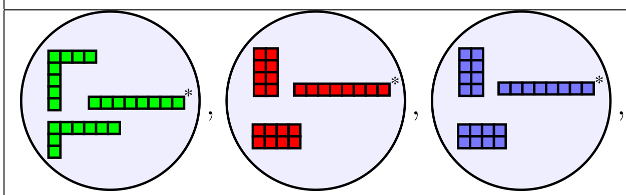
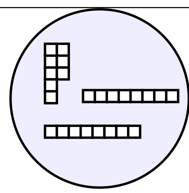
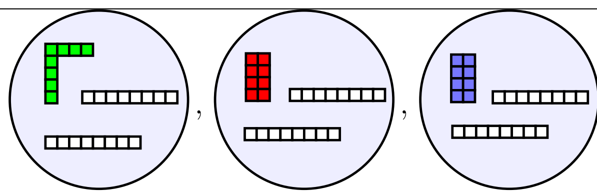
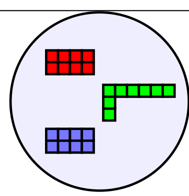
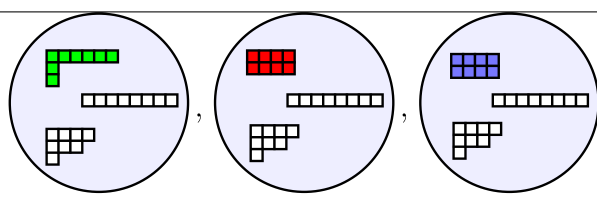
Fixture	Number of Hypers	Representation
	8	$\frac{1}{2}(2,2,4)$
	0	none
	24	$\frac{1}{2}(1,4,8_u)+\frac{1}{2}(2,1,8_d)$, where $8_{u/d}=8_v$, 8_s, or 8_c depending on whether the upper/lower left-hand puncture is coloured green, red, or blue.
	24	$\frac{1}{2}(2,1,1,8_v)$ $+\frac{1}{2}(1,2,1,8_s)$ $+\frac{1}{2}(1,1,2,8_c)$
	16	$\frac{1}{2}(4,8)$

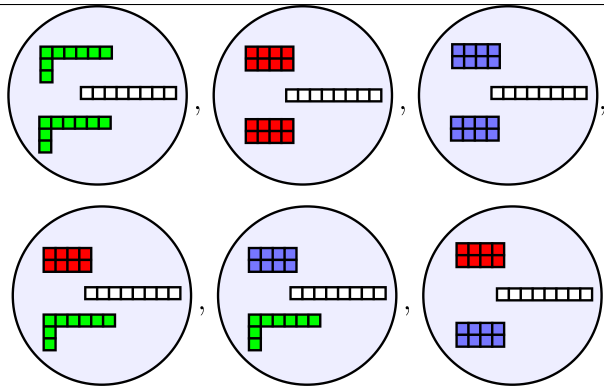
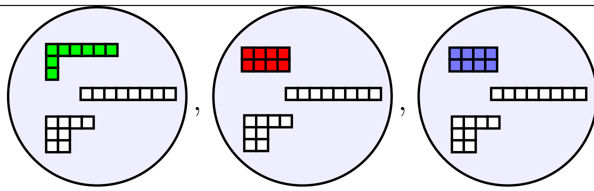
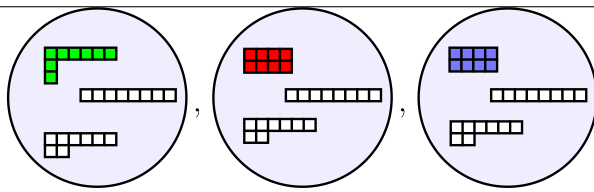
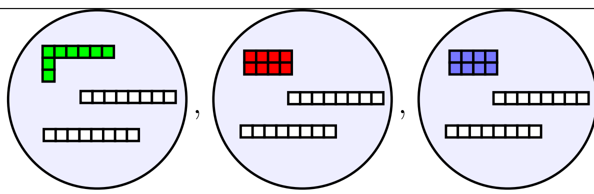
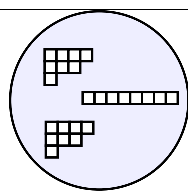
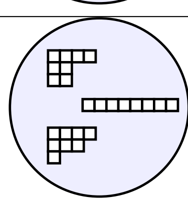
, , 	15	$\frac{1}{2}(2,1,8)+\frac{1}{2}(1,2,7)$
, , 	8	$(2,4)$
, , 	0	none
	7	$\frac{1}{2}(2,7)$
	1	$\frac{1}{2}(2)$
	0	none

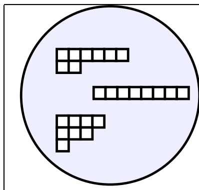
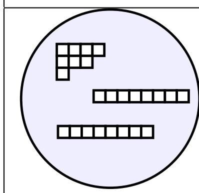
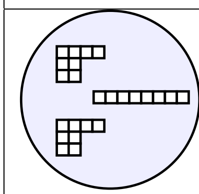
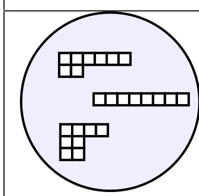
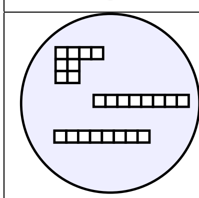
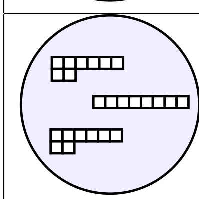
Note that, among the free field fixtures, are six which are empty (zero hypermultiplets). It might, at first blush, seem peculiar to assign global symmetry groups ($SU(2)_8^2$ and $SU(2)_8$, respectively) to the regular punctures on them. However, they are attached to the rest of the surface by an $SU(2)$ cylinder, which gauges an $SU(2)$ subgroup of the global symmetry group of the attaching puncture. The centralizer of that $SU(2)$ is, respectively $SU(2)_8^2$

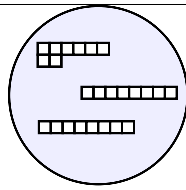
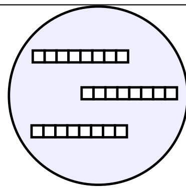
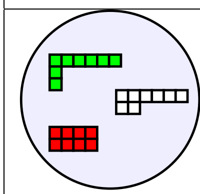
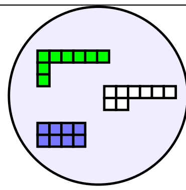
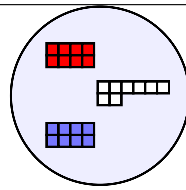
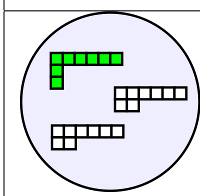
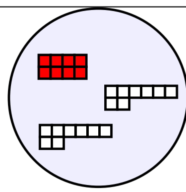
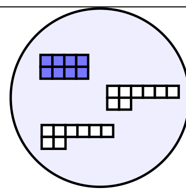
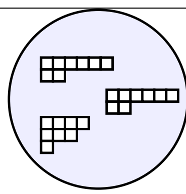
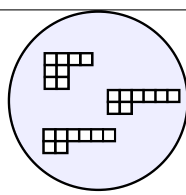
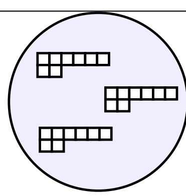
or $SU(2)_8$. That centralizer is what is detected by the punctures on the ostensibly "empty" fixture. Similar remarks applied to the analogous fixtures that we saw in the D_3 and A_{N-1} cases, studied in [6].

7.2.2.2 Interacting fixtures

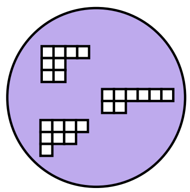
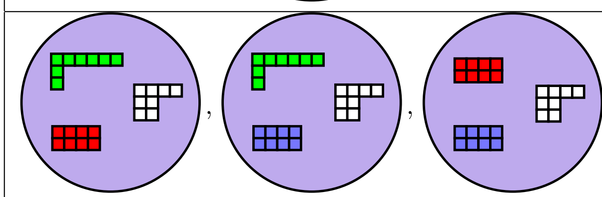
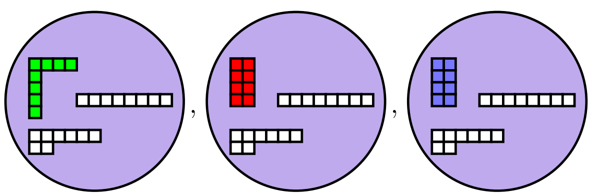
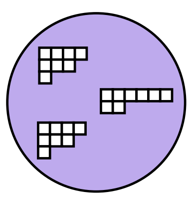
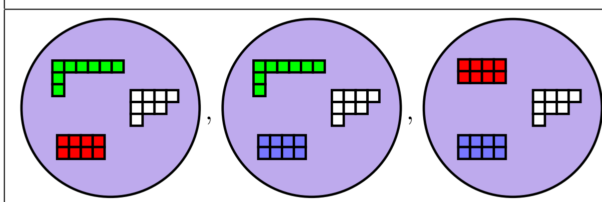
Fixture	$(d_2, d_3, d_4, d_5, d_6)$	(a, c)	$(G_{\text{global}})_k$	Theory
	$(0,0,1,0,0)$	$(\frac{59}{24}, \frac{19}{6})$	$(E_7)_8$	The E_7 SCFT
	$(0,0,0,0,1)$	$(\frac{95}{24}, \frac{31}{6})$	$(E_8)_{12}$	The E_8 SCFT
	$(0,0,1,0,1)$	$(\frac{23}{4}, 7)$	$Spin(8)^2_{12}$ $\times SU(2)_8$	
	$(0,0,1,0,1)$	$(\frac{65}{12}, \frac{19}{3})$	$Sp(6)_8$	
	$(0,0,1,0,2)$	$(\frac{25}{3}, \frac{113}{12})$	$Spin(8)_{12}$ $\times Sp(2)_8$ $\times SU(2)_7$	

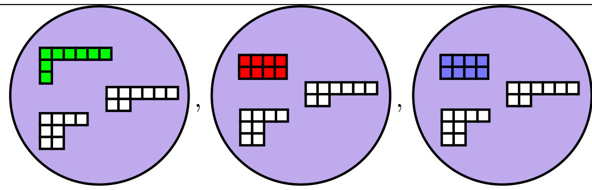
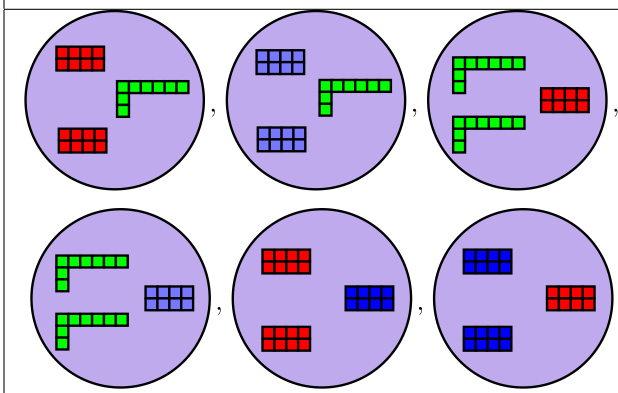
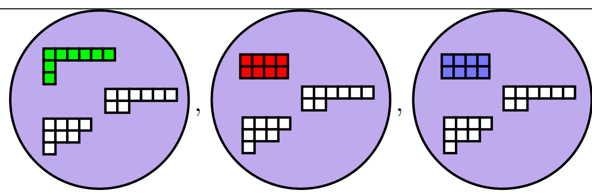
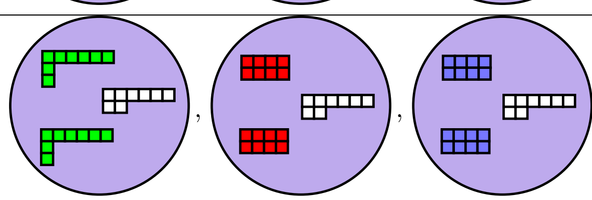
	$(0,0,2,0,2)$	$(\frac{61}{6},\frac{34}{3})$	$Spin(8)_{12}\times Sp(2)_8^2$	
	$(0,1,1,0,1)$	$(\frac{163}{24},\frac{47}{6})$	$Spin(8)_{12}\times Sp(2)_8$ $\times U(1)^2$	
	$(0,0,3,0,2)$	$(\frac{287}{24},\frac{79}{6})$	$Spin(8)_{12}\times Sp(2)_8$ $\times SU(2)_8^3$	
	$(0,0,3,0,3)$	$(\frac{179}{12},\frac{49}{3})$	$Spin(8)_{12}^2\times Sp(2)_8$	
	$(0,0,0,0,2)$	$(\frac{13}{2},\frac{15}{2})$	$Spin(8)_{12}\times SU(2)_7^2$	
	$(0,1,0,0,1)$	$(\frac{119}{24},\frac{71}{12})$	$Spin(8)_{12}\times SU(2)_7$ $\times U(1)^2$	

	$(0,0,2,0,2)$	$(\frac{81}{8},\frac{45}{4})$	$Spin(8)_{12} \times SU(2)_8^3 \times SU(2)_7$	
	$(0,0,2,0,3)$	$(\frac{157}{12},\frac{173}{12})$	$Spin(8)_{12}^2 \times SU(2)_7$	
	$(0,2,0,0,0)$	$(\frac{41}{12},\frac{13}{3})$	$Spin(8)_{12} \times U(1)^4$	
	$(0,1,2,0,1)$	$(\frac{103}{12},\frac{29}{3})$	$Spin(8)_{12} \times SU(2)_8^3 \times U(1)^2$	
	$(0,1,2,0,2)$	$(\frac{277}{24},\frac{77}{6})$	$Spin(8)_{12}^2 \times U(1)^2$	
	$(0,0,4,0,2)$	$(\frac{55}{4},15)$	$Spin(8)_{12} \times SU(2)_8^6$	

	$(0,0,4,0,3)$	$(\frac{401}{24},\frac{109}{6})$	$Spin(8)_{12}^2 \times SU(2)_8^3$	
	$(0,0,4,0,4)$	$(\frac{59}{3},\frac{64}{3})$	$Spin(8)_{12}^3$	
, , 	$(0,0,2,0,1)$	$(\frac{173}{24},\frac{49}{6})$	$Sp(3)_8^2 \times SU(2)_8$	
, , 	$(0,0,3,0,1)$	$(9,10)$	$Sp(2)_8^2 \times SU(2)_8^4$	
	$(0,0,2,0,1)$	$(\frac{43}{6},\frac{97}{12})$	$Sp(2)_8^3 \times SU(2)_7$	
	$(0,1,2,0,0)$	$(\frac{45}{8},\frac{13}{2})$	$SU(4)_8^3$	T_4
	$(0,0,4,0,1)$	$(\frac{259}{24},\frac{71}{6})$	$SU(2)_8^9$	

7.2.2.3 Mixed fixtures

Fixture	$(d_2, d_3, d_4, d_5, d_6)$	(a, c)	SCFT	# Free hypers
	$(0,1,0,0,0)$	$(2, \frac{11}{4})$	$(E_6)_6$	7 hypers, transforming as $\frac{1}{2}(2;1,1,1)$ $+(1;2,1,1)$ $+(1;1,2,1)$ $+(1;1,1,2)$
	$(0,1,0,0,0)$	$(\frac{49}{24}, \frac{17}{6})$	$(E_6)_6$	8 hypers, transforming as $(4;1)+(1;4)$
	$(0,0,1,0,0)$	$(\frac{67}{24}, \frac{23}{6})$	$(E_7)_8$	8 hypers, transforming as the $(1;1,1,1;8_u)$, where $8_u = 8_{v,s,c}$, depending on the colour of the puncture on the upper left
	$(0,0,0,0,1)$	$(\frac{85}{24}, \frac{13}{3})$	$Sp(5)_7$	3 hypers, transforming as $\frac{1}{2}(1;1;2,1,1)$ $+\frac{1}{2}(1;1;1,2,1)$ $+\frac{1}{2}(1;1;1,1,2)$
	$(0,0,0,0,1)$	$(\frac{43}{12}, \frac{53}{12})$	$Sp(5)_7$	4 hypers, transforming as $\frac{1}{2}(1;4;1)$ $+\frac{1}{2}(4;1;1)$

	$(0,1,1,0,0)$	$(\frac{23}{6},\frac{14}{3})$	$SU(2)_6$ $\times SU(8)_8$	2 hypers, transforming as $(1;2,1,1)$
	$(0,0,1,0,1)$	$(\frac{65}{12},\frac{19}{3})$	$Sp(4)_8$ $\times Sp(2)_7$	2 hypers, transforming as $\frac{1}{2}(1;1;4)$
	$(0,0,1,0,1)$	$(\frac{43}{8},\frac{25}{4})$	$Sp(4)_8$ $\times Sp(2)_7$	1 hyper, transforming as $\frac{1}{2}(1;1;2,1,1)$
	$(0,0,2,0,1)$	$(\frac{173}{24},\frac{49}{6})$	$Sp(2)_8^3$ $\times SU(2)_7$	1 hyper, transforming as $\frac{1}{2}(1;1;2,1,1)$

7.2.3 The $Sp(4)_8 \times Sp(2)_7$ and $Sp(5)_7$ SCFTs

A couple of SCFTs make a somewhat unusual appearance in the above list of mixed fixtures. Usually, the mixed fixtures contain SCFTs which have previously appeared elsewhere (without the additional hypermultiplets). In the present case, we find two new ones, which do not appear to arise *in the absence* of accompanying hypermultiplets.

7.2.3.1 $Sp(4)_8 \times Sp(2)_7$ SCFT

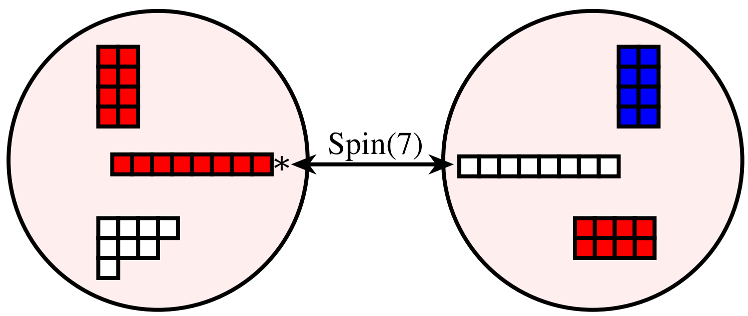
One is the $Sp(4)_8 \times Sp(2)_7$ SCFT. It has $(a,c) = \left(\frac{16}{3}, \frac{37}{6}\right)$, and graded Coulomb branch dimension $(d_2, d_3, d_4, d_5, d_6) = (0,0,1,0,1)$. Its global symmetry group is

$$G_X = Sp(4)_8 \times Sp(2)_7$$

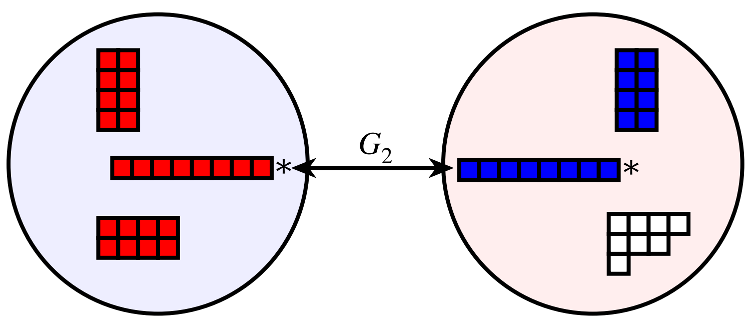
It appears in our table, accompanied by either 1 hypermultiplet (3 fixtures) or 2 hypermultiplets (6 fixtures).

Let's look a couple of examples of its appearance.

Consider a $Spin(7)$ gauge theory, with matter in the $3(8) + 2(7) + 1$.

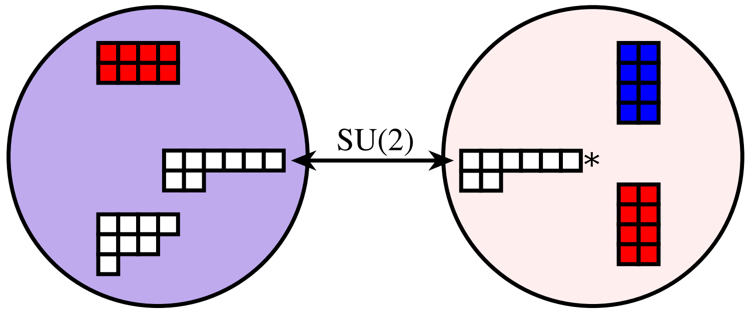


This theory has two distinct strong-coupling points. One,



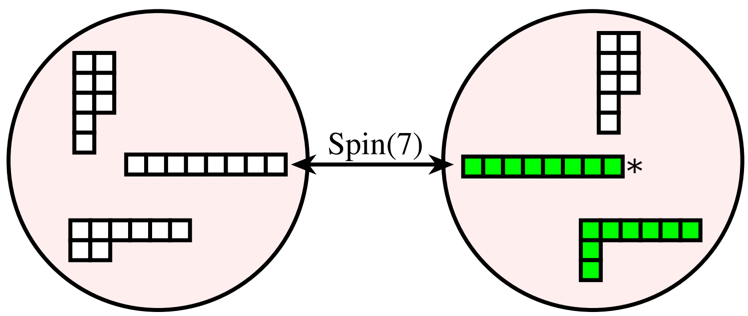
is a G_2 gauge theory, with matter in the $2(7)+1$, coupled to the $(E_7)_8$ SCFT. Aside from the addition of the free hypermultiplet, this was example 10 of Argyres and Wittig [17].

The other strong coupling point of this theory,

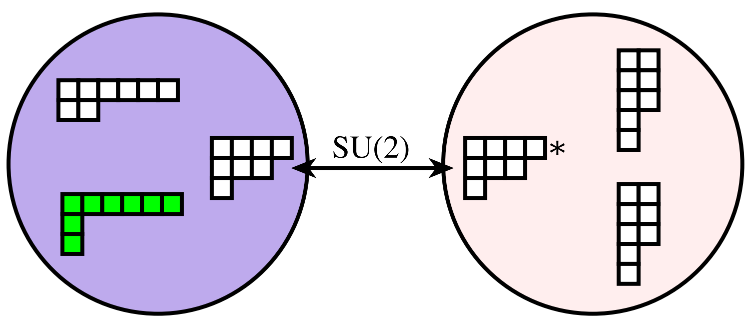


is an $SU(2)$ gauge theory coupled to the $Sp(4)_8 \times Sp(2)_7$ SCFT. The fixture on the right is empty; the mixed-fixture on the left provides both the SCFT and an additional free hypermultiplet.

As a second example, consider



This is a $Spin(7)$ gauge theory, with matter in the $4(8)+(7)+(1)$. The S-dual theory



is an $SU(2)$ gauge theory. The fixture on the right contributes a half-hypermultiplet in the fundamental. The fixture on the left is the $Sp(4)_8 \times Sp(2)_7$ SCFT plus a single *free* hypermultiplet. We weakly gauge an $SU(2)$ subgroup of $Sp(2)_7 \subset G_X$. From both points of view, we reproduce

$$G_{\text{global}} = Sp(4)_8 \times SU(2)_7 + 1\,\text{free hypermultiplet}$$

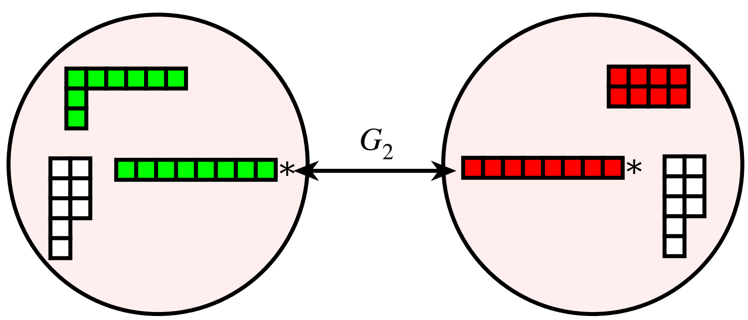
A third example is provided by the S-dual of $Spin(8)$ gauge theory with matter in the $4(8_s) + 2(8_c)$. This is discussed in section §7.3.4.

7.2.3.2 $Sp(5)_7$ SCFT

The other "new" SCFT is the $Sp(5)_7$ SCFT. It has $(a, c) = \left(\frac{41}{12}, \frac{49}{12}\right)$ and a Coulomb branch of graded dimension $(d_2, \ldots, d_6) = (0, 0, 0, 0, 1)$. The global symmetry group is $Sp(5)_7$.

The $Sp(5)_7$ SCFT appears twice on our list, once accompanied accompanied by 3 hypermultiplets (transforming as the $\frac{1}{2}(1;1;2,1,1)+\frac{1}{2}(1;1;1,2,1)+\frac{1}{2}(1;1;1,1,2)$ of the manifest $SU(2) \times SU(2) \times SU(2)^3$ associated to the punctures), and once (3 fixtures) accompanied by 4 hypermultiplets (transforming as the $\frac{1}{2}(1;4;1) + \frac{1}{2}(4;1;1)$ of the manifest $Sp(2) \times Sp(2) \times SU(2)$ associated to the punctures).

Let's look at some examples of the $Sp(5)_7$ SCFT. Consider the 4-punctured sphere

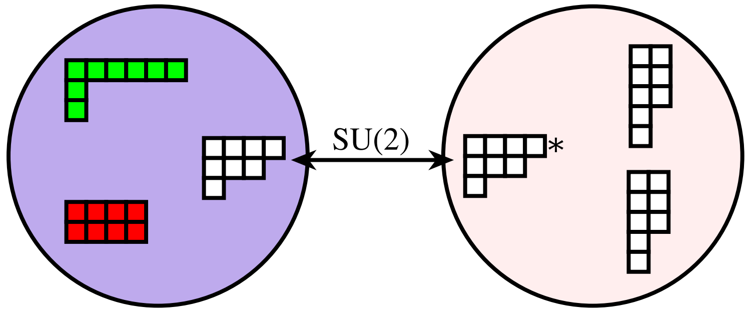


Both fixtures provide 2 hypers in the 7 of G_2, plus 2 free hypers, so the 4-punctured sphere represents the G_2 theory with 4 hypers in the 7, plus 4 free hypers.

$$G_{\text{global}} = Sp(4)_7 + 4 \text{ free hypers}$$

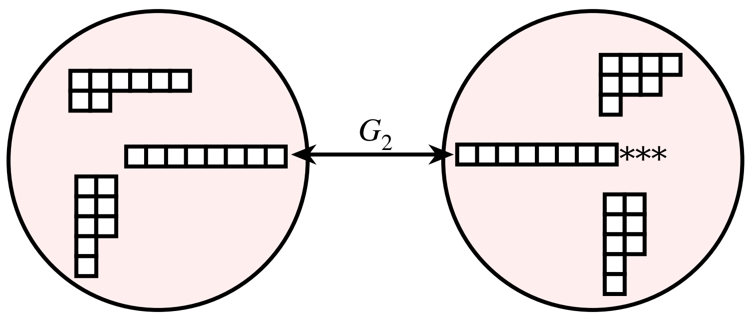
Aside from the 4 free hypers, this is example 4 of Argyres-Wittig [17].

The S-dual theory is



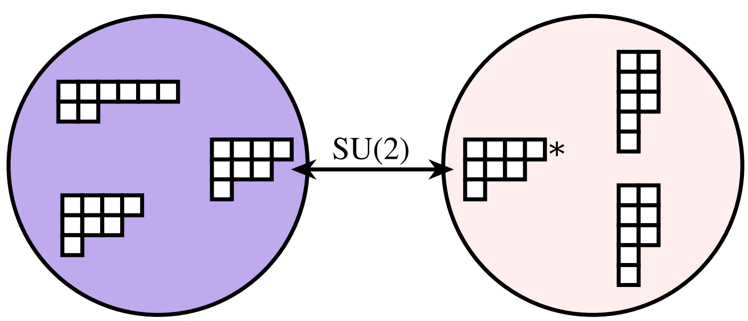
The fixture on the left is the $Sp(5)_7$ SCFT, with 4 free hypers. The fixture on the right contributes a half-hyper in the fundamental of $SU(2)$. Gauging an $SU(2) \subset Sp(5)_7$, yields the expected $Sp(4)_7$ global symmetry group of the S-dual of G_2 with 4 fundamentals.

As another example, consider the 4-punctured sphere



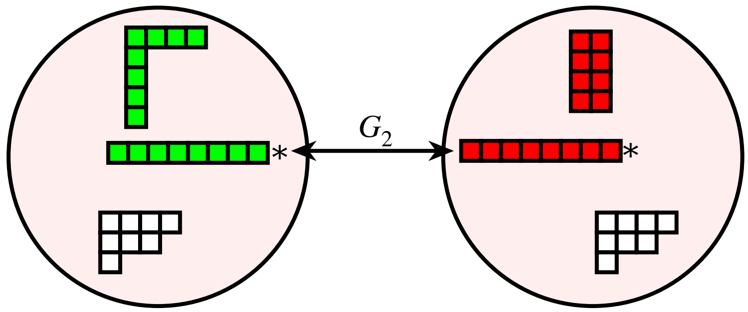
Here the fixture on the left represents 3 hypers in the 7 of G_2 plus 3 free hypers, and the fixture on the right represents 1 hyper in the 7. Notice that the G_2 cylinder in this example is different from the one in the previous example.

S-dualizing, we obtain



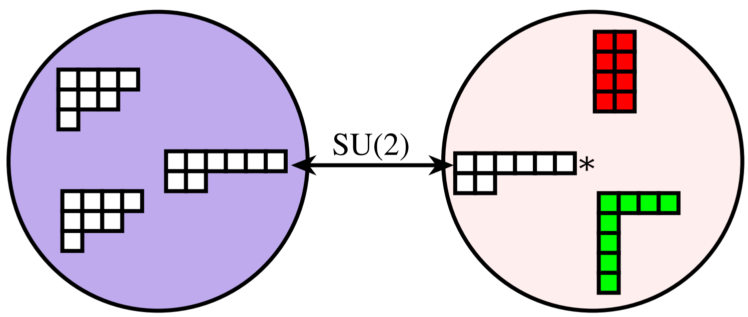
The fixture on the left is the $Sp(5)_7$ SCFT, where we gauge an $SU(2) \subset Sp(5)$, accompanied by 3 free hypers. The fixture on the right contributes 1 fundamental half-hyper.

A third example, also involving G_2, is



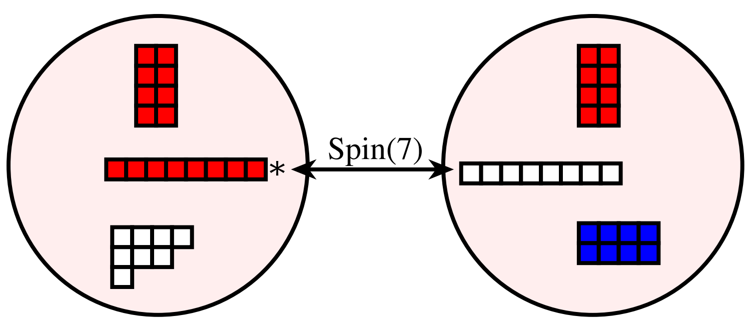
This is G_2 with 4 fundamentals and *two* free hypermultiplets.

The S-dual is



The fixture on the right is empty. The fixture on the left is, again the $Sp(5)_7$ SCFT, with one hypermultiplet transforming as a half-hyper in the fundamental of $SU(2)$ and two free hypermultiplets.

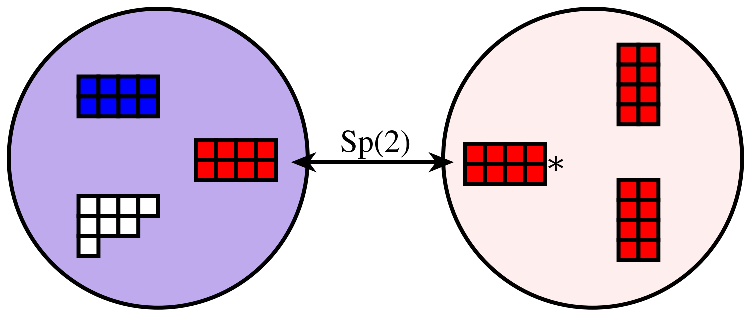
For a non-G_2-related example, consider



The fixture on the left contributes hypermultiplets in the $2(7)+1$. The fixture on the right is an $8_s+2(8_c)$, considered as a representation of $Spin(8)$. Under the chosen embedding of $Spin(7)$, the 8_s decomposes as $7+1$, which the 8_c (and also the 8_v) decomposes as the 8. So, all-in-all, this is a $Spin(7)$ gauge theory, with matter in the $3(7)+2(8)+2(1)$, so

$$G_{\text{global}} = Sp(3)_7 \times Sp(2)_8 + 2 \text{ free hypers}$$

The S-dual theory is



The fixture on the right contribute 2 hypermultiplets in the fundamental of $Sp(2)$. The fixture on the left is the $Sp(5)_7$ SCFT, accompanied by 4 hypermultiplets, two of which form an additional half-hypermultiplet in the fundamental of $Sp(2)$ and two of which are free. Altogether, there are 5 half-hypermultiplets in the fundamental, yielding the $Spin(5) = Sp(2)_8$ factor in G_{global}. Gauging the $Sp(2) \subset Sp(5)_7$ yields the remaining $Sp(3)_7$. This is example 5 of Argyres and Wittig [17].

7.3 $Spin(8)$ Gauge Theory

$Spin(8)$ gauge theory — with n_s hypermultiplets in the 8_s, n_c hypermultiplets in the 8_c and n_v hypermultiplets in the 8_v — has vanishing β-function for $n_s+n_c+n_v=6$. The global symmetry group is

$$G_{\text{global}} = Sp(n_s)_8 \times Sp(n_c)_8 \times Sp(n_v)_8$$

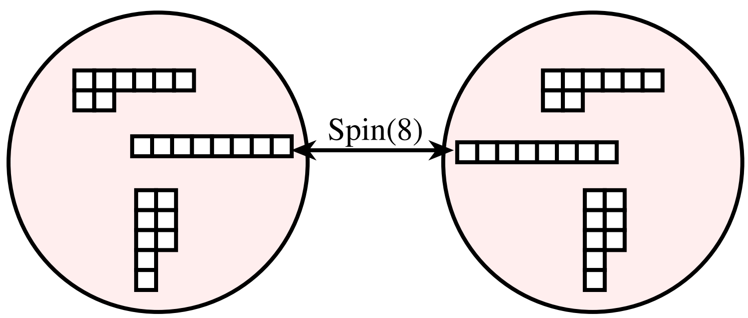
In the D_4 theory, all of the cases, with $n_{s,c,v} \leq 4$, are realized on the 4-punctured sphere. Up to $Spin(8)$ triality, this yields five different cases. We will discuss each of them, in turn, and give the strong-coupling behaviour in each case.

For the cases of $(n_s, n_c, n_v) = (3, 2, 1)$ and $(3, 3, 0)$, Argyres and Wittig [17] conjectured a strong-coupling dual. We find that each of these cases has *two* distinct strong-coupling limits. In each case, the conjecture of Argyres and Wittig corresponds to one of the two strong-coupling limits, that we find.

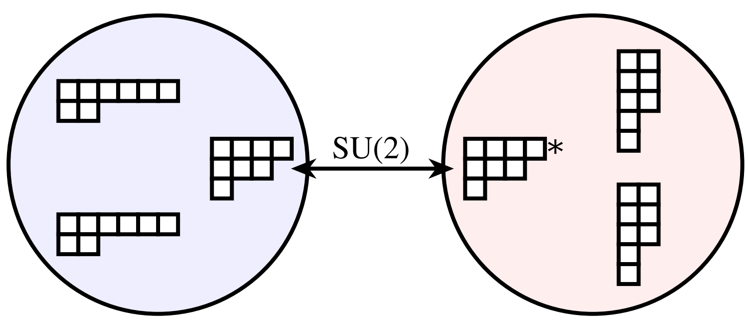
7.3.1 $2(8_s) + 2(8_c) + 2(8_v)$

The dual of $Spin(8)$, with matter in the $2(8_s) + 2(8_c) + 2(8_v)$, is an $SU(2)$ gauge theory, coupled to a half-hypermultiplet in the fundamental, and to the $Sp(2)^3_8 \times SU(2)_7$ SCFT.

One realization is

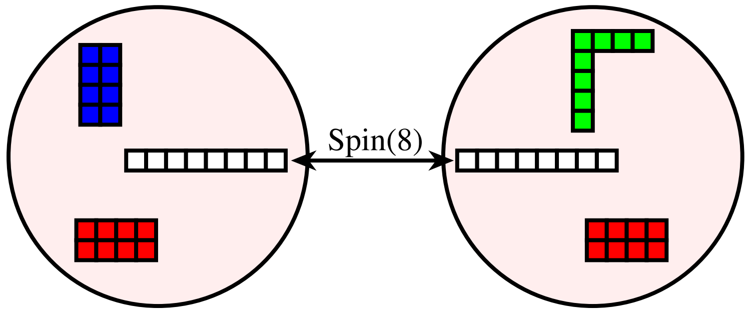


Each fixture contributes one $(8_v + 8_s + 8_c)$. The S-dual theory is

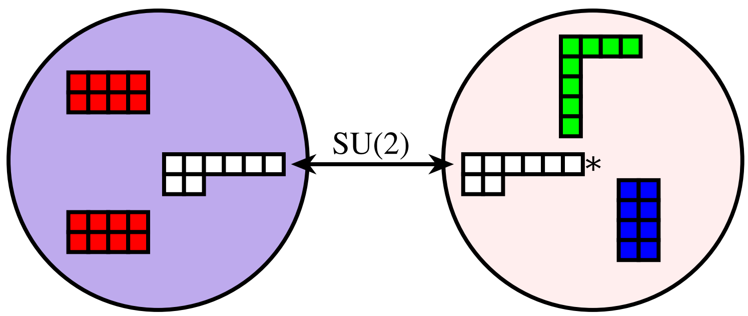


where the fixture on the right is a half-hypermultiplet in the fundamental of $SU(2)$, and the fixture on the left is the $Sp(2)^3_8 \times SU(2)_7$ SCFT.

Another realization of the same theory is



Here, the fixture on the left contributes $8_s + 2(8_c)$, and the fixture on the right contributes $8_s + 2(8_v)$. The S-dual is

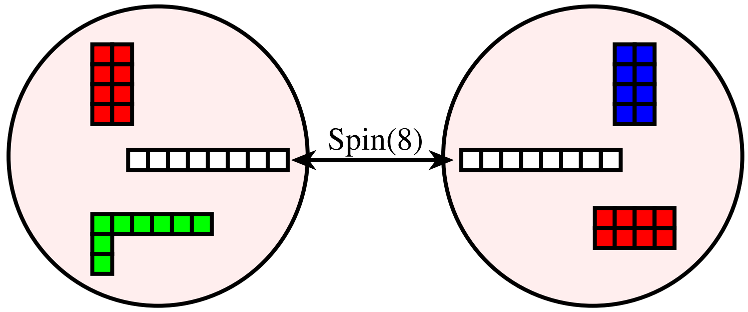


The fixture on the right is empty; the fixture on the left is the $Sp(2)_8^3 \times SU(2)_7$ SCFT plus a half-hypermultiplet in the fundamental of $SU(2)$.

7.3.2 $3(8_s) + 2(8_c) + 8_v$

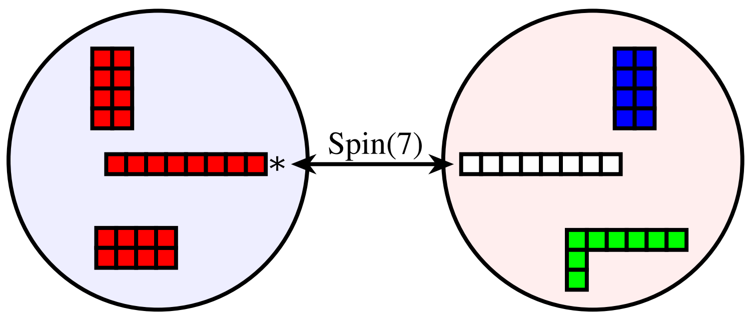
Spin(8) gauge theory, with matter in the $3(8_s) + 2(8_c) + 8_v$, has two *distinct* strong-coupling limits. One is a *Spin*(7) gauge theory, with matter in the 3(8), coupled to the $(E_7)_8$ SCFT. The other strong coupling limit is an $SU(2)$ gauging of the $Sp(3)_8^2 \times SU(2)_8$.

One realization is



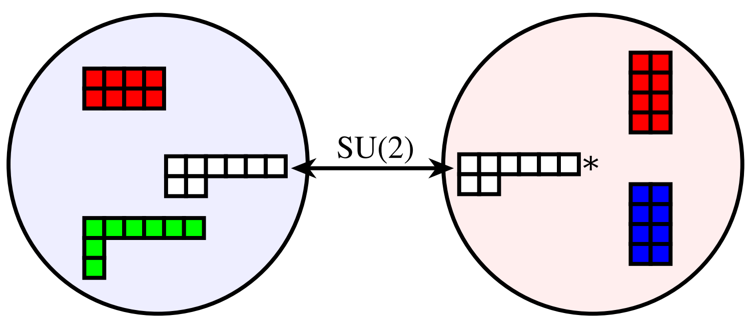
The fixture one the left contributes $2(8_s) + 8_v$, and the fixture on the right contributes $8_s + 2(8_c)$.

One of the corresponding strong-coupling points is given by



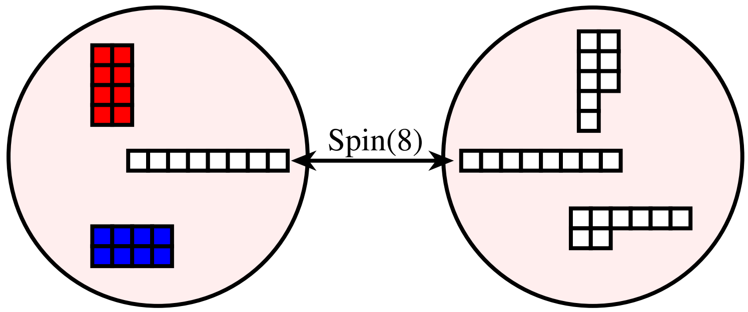
The fixture on the right yields matter in 3 copies of the 8; the fixture on the left is the $(E_7)_8$ SCFT.

The other strong coupling point is

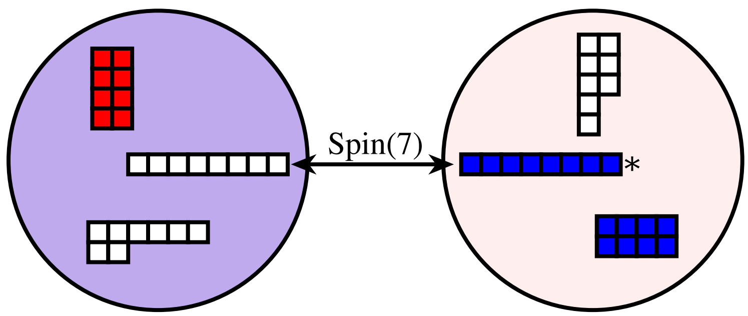


The fixture on the right is empty, while the fixture on the left is the $Sp(3)_8^2 \times SU(2)_8$ SCFT, where we gauge an $SU(2) \subset Sp(3)_8$.

Another realization of the same theory is

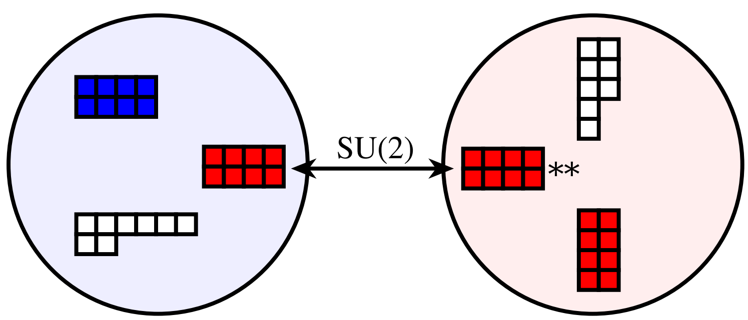


One strong coupling point is given by



The fixture on the right contribute 2 hypermultiplets in the 8 of $Spin(7)$. The fixture on the left is the $(E_7)_8$ SCFT plus an additional hypermultiplet in the 8.

The other strong coupling point is

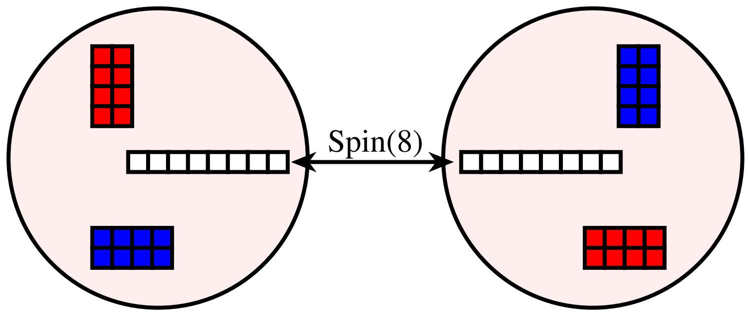


The fixture on the right is empty; the fixture on the left is, again, the $Sp(3)_8^2 \times SU(2)_8$ SCFT.

7.3.3 $3(8_s)+3(8_c)$

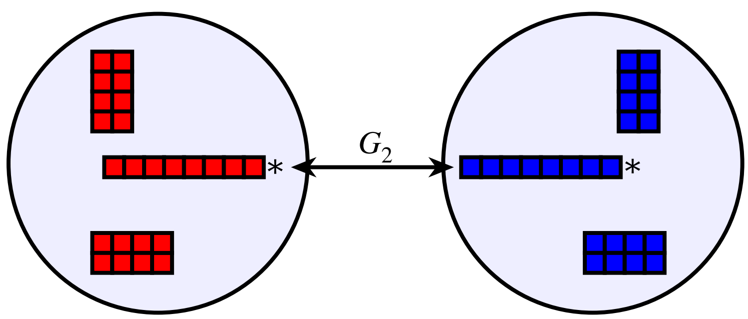
Spin(8) gauge theory, with matter in the $3(8_s)+3(8_c)$ also has two distinct strong coupling points. One is G_2 gauge theory, coupled to two copies of the $(E_7)_8$ SCFT. The other is an $SU(2)$ gauging of the $Sp(3)_8^2\times SU(2)_8$ SCFT.

This is realized via



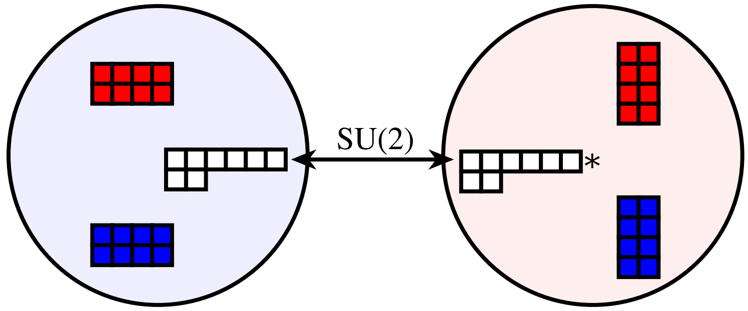
The fixture on the left yields $2(8_s)+8_c$, while the figure on the right yields $8_S+2(8_c)$.

One strong-coupling point is given by



Here, each fixture is a copy of the $(E_7)_8$ SCFT.

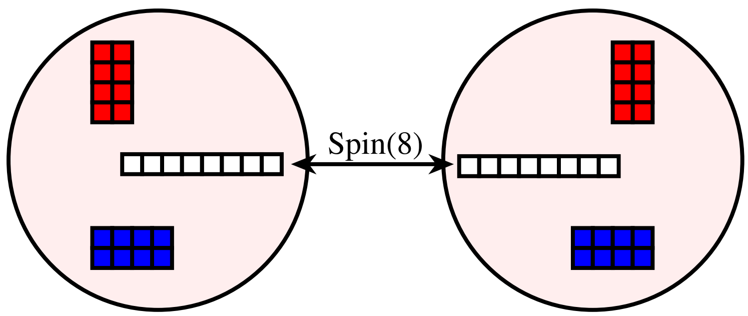
The other strong coupling point is



The fixture on the right is empty. The fixture on the left is the $Sp(3)_8^2 \times SU(2)_8$ SCFT where, this time, we gauge the $SU(2)_8$.

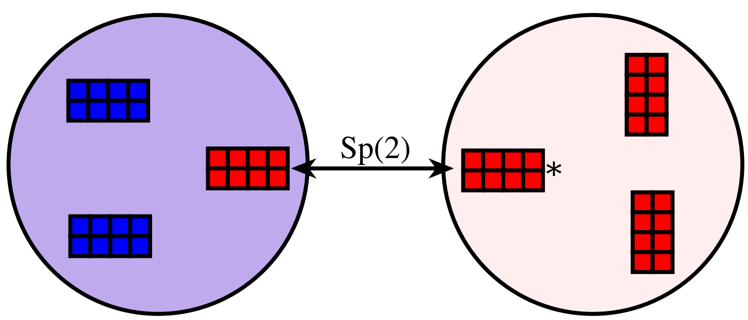
7.3.4 $4(8_s) + 2(8_c)$

$Spin(8)$, with matter in the $4(8_s) + 2(8_c)$ has, as its S-dual, an $Sp(2)$ gauge theory, with 5 half-hypermultiplets in the fundamental, coupled to the $Sp(4)_8 \times Sp(2)_7$ SCFT.



yields a $Spin(8)$ gauge theory, with matter in the $4(8_s) + 2(8_c)$.

The S-dual theory is

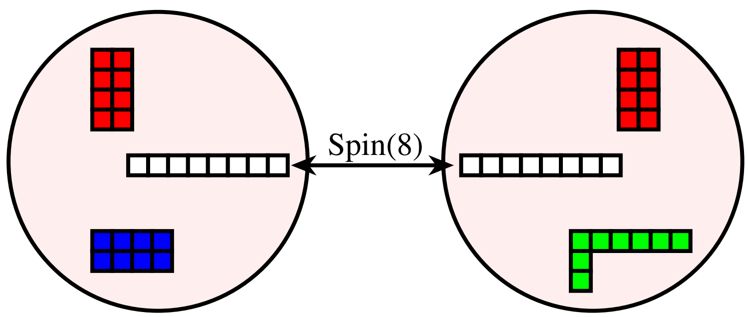


The fixture on the right contributes two hypermultiplets in the fundamental. The fixture on the left is the $Sp(4)_8 \times Sp(2)_7$ with an additional half-hypermultiplet in the fundamental of $Sp(2)$. Since there are, in total, five half-hypermultiplets in the fundamental, the flavour symmetry associated to the matter is $Spin(5) = Sp(2)_8$; the rest of G_{global} comes from the $Sp(4)_8 \subset Sp(4)_8 \times Sp(2)_7$.

7.3.5 $4(8_s) + 8_c + 8_v$

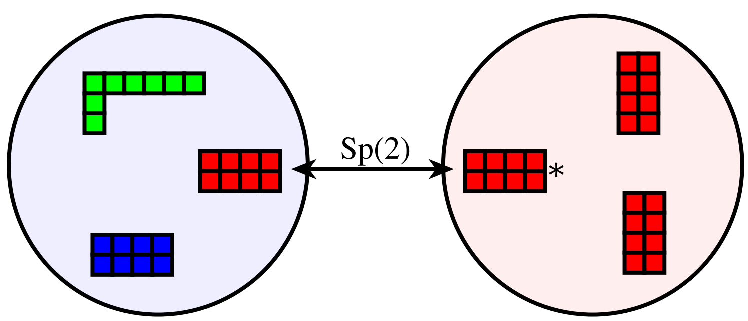
Finally, $Spin(8)$ gauge theory, with matter in the $4(8_s)+8_c+8_v$ has, as its S-dual, an $Sp(2)$ gauge theory, with 2 hypermultiplets in the fundamental, coupled to the $Sp(6)_8$ SCFT.

The $Spin(8)$ gauge theory can be realized as



where the fixture on the left gives matter in the $2(8_s) + 8_c$ and the fixture on the right gives matter in the $2(8_s) + 8_v$.

The S-dual is



The fixture on the right is 2 fundamental hypermultiplets of $Sp(2)$, which contribute the $Spin(4) = SU(2)_8^2$ factor to the global symmetry group. The fixture on the left is the $Sp(6)_8$ SCFT.

7.3.6 Seiberg-Witten curves

It is straightforward to compute the Seiberg-Witten curves, associated to any of these theories, in the form (7.1)

$$0 = \lambda^{2N} + \sum_{k=1}^{N-1} \lambda^{2(N-k)} \phi_{2k}(y) + \tilde{\phi}^2(y)$$

For instance, for $Spin(8)$ gauge theory, with hypermultiplets in the $3(8_v) + 3(8_s)$, imposing the constraints, at each of the punctures, yields

$$\begin{aligned}
\phi_2(y) &= \frac{u_2\,(dy)^2}{(y-y_1)(y-y_2)(y-y_3)(y-y_4)} \\
\phi_4(y) &= \frac{[u_4\,(y-y_2)(y-y_3) - 2\tilde{u}\,(y-y_1)(y-y_4) + u_2^2\,(y-y_1)(y-y_3)/4](dy)^4}{(y-y_1)^3(y-y_2)^2(y-y_3)^3(y-y_4)^2} \\
\phi_6(y) &= \frac{[u_6\,(y-y_2) + u_2\tilde{u}\,(y_1-y_2)](dy)^6}{(y-y_1)^4(y-y_2)^3(y-y_3)^4(y-y_4)^2} \\
\tilde{\phi}(y) &= \frac{\tilde{u}\,(dy)^6}{(y-y_1)^2(y-y_2)^2(y-y_3)^3(y-y_4)}
\end{aligned} \tag{7.34}$$

Here u_2, u_4, u_6 and $\tilde{u}$ are the Coulomb branch parameters. The obvious $SL(2,\mathbb{C})$ symmetry means that the physics depends only on the cross-ratio

$$e(\tau) = \frac{(y_1-y_2)(y_3-y_4)}{(y_1-y_3)(y_2-y_4)}$$

The $e(\tau) \to 0$ limit is the weakly-coupled $Spin(8)$ gauge theory; $e(\tau) \to \infty$ is the weakly-coupled $SU(2)$ gauge theory and $e(\tau) \to 1$ yields the weakly-coupled G_2 gauge theory.

The other cases are equally-easy to write down. It would be interesting to compare these results with the Seiberg-Witten curves obtained in [51, 52].

7.4 $Spin(7)$ Gauge Theory

$Spin(7)$, with n hypermultiplets in the 8 and $(5-n)$ in the 7, also has vanishing β-function. Perhaps with the addition of some free hypermultiplets, we can realize the cases $n = 2, 3, 4, 5$ in the D_4 theory.

7.4.1 $2(8) + 3(7)$

This theory (with the addition of two free hypermultiplets) was one of the examples discussed in §7.2.3.2. The theory has two strong-coupling points.

- One is a G_2 gauge theory, with two hypermultiplets in the 7, coupled to the $(E_7)_8$ SCFT.
- The other is an $SU(2)$ gauge theory coupled to the $Sp(4)_8 \times Sp(2)_7$ SCFT.

7.4.2 $3(8) + 2(7)$

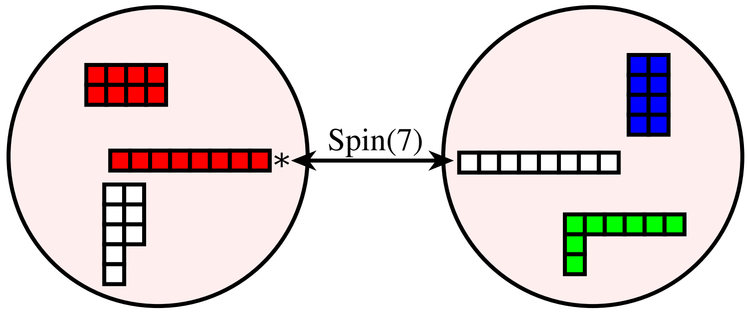
This theory (with the addition of two free hypermultiplets) was discussed in §7.2.3.1. The S-dual theory is an $Sp(2)$ gauge theory with 5 half-hypermultiplets in the 4, coupled to the $Sp(5)_7$ SCFT.

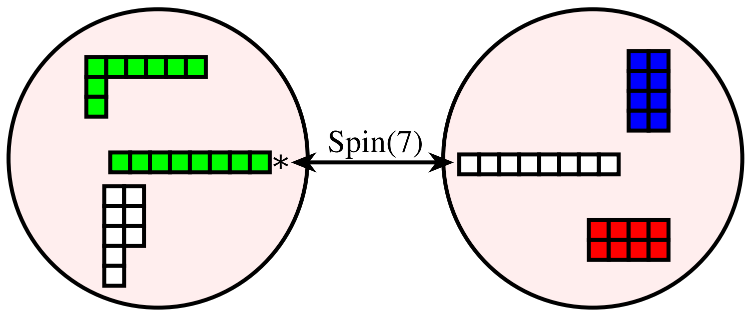
7.4.3 $4(8) + 1(7)$

This theory (with the addition of one free hypermultiplet) was alos discussed in §7.2.3.1. The S-dual theory is an $SU(2)$ gauge theory with a half-hypermultiplet in the 2, coupled to the $Sp(4)_8 \times Sp(2)_7$ SCFT.

7.4.4 5(8)

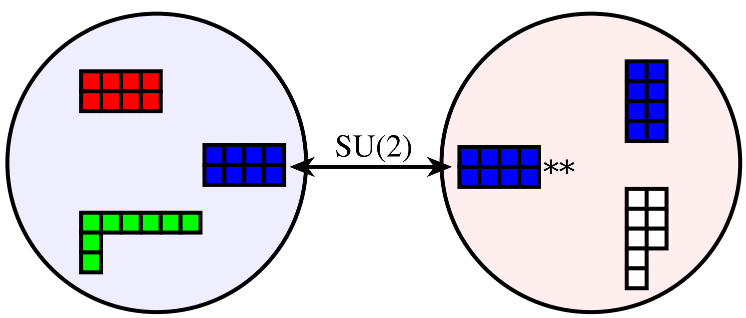
This theory has three degeneration limits, two of which





are $Spin(7)$ gauge theories with matter in the 5(8). The fixture on the left contributes 2(8); the fixture on the right contributes 3(8).

The other degeneration,

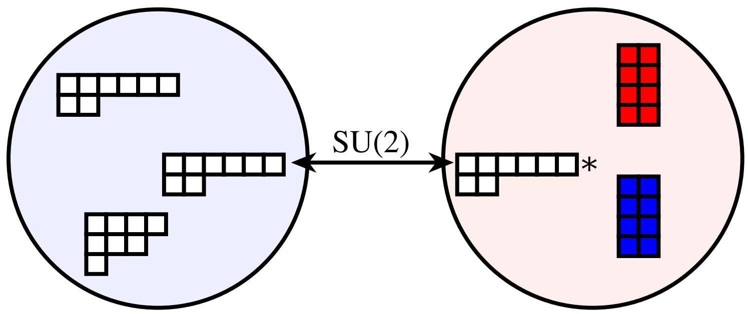


is an $SU(2)$ gauge theory coupled to the $Sp(6)_8$ SCFT (the fixture on the right is empty).

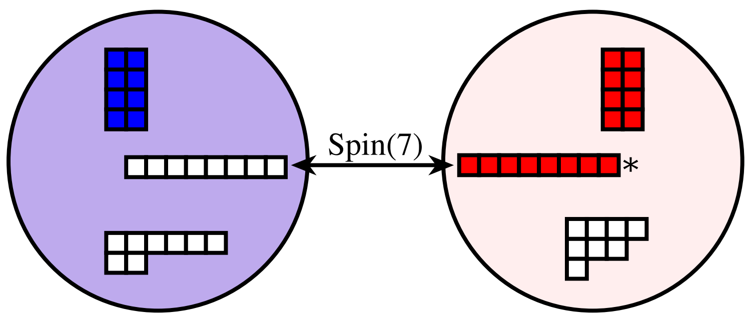
7.5 Other Interesting Examples

7.5.1 Fun with interacting SCFTs

Let's take the $Sp(2)_8^3 \times SU(2)_7$ SCFT and gauge an $SU(2)_8$ subgroup (the fixture on the right is empty):

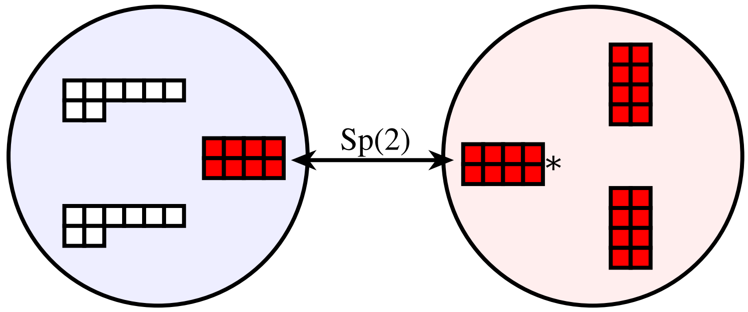


The S-dual theory is

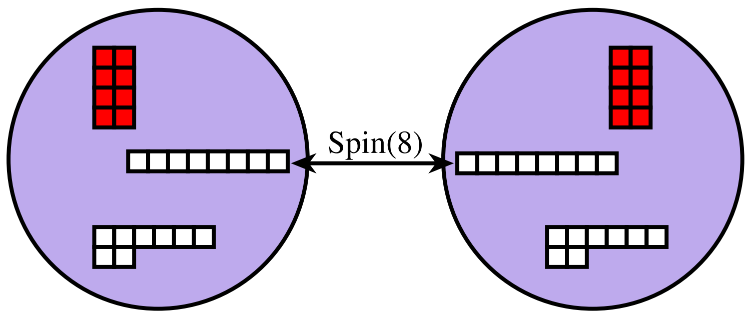


The fixture on the right contributes hypermultiplets in the $7 + 8$. The fixture on the left is the $(E_7)_8$ SCFT with matter in the 8_c of $Spin(8)$. Under the given embedding of $Spin(7)$, this matter transforms as an additional 8. So the matter contributes an $Sp(2)_8 \times SU(2)_7$ to the global symmetry group of the theory. The rest, $Sp(2)_8 \times SU(2)_8$, is the centralizer of $Spin(7) \subset E_7$.

As another example of our methods, let us consider various gaugings of the $Sp(2)_8^2 \times SU(2)_8^4$ SCFT. We can gauge an $Sp(2)$ subgroup,

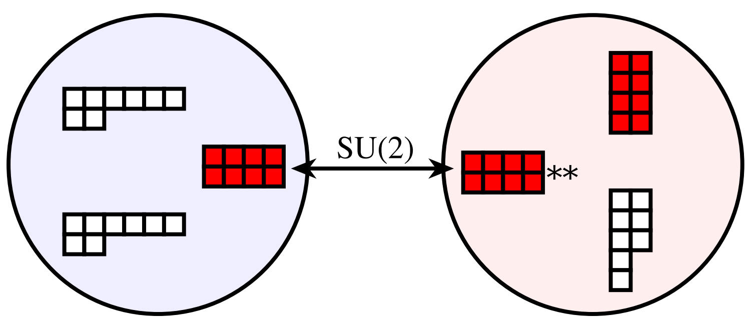


where the fixture on the right provides two hypermultiplets in the fundamental of $Sp(2)$. The S-dual theory,

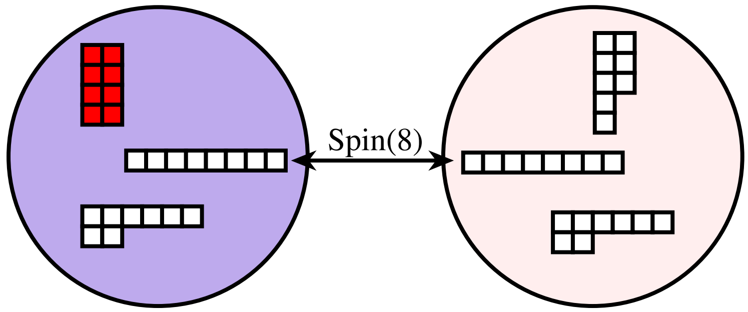


is a $Spin(8)$ gauge theory, with matter in the $2(8_s)$, coupled to two copies of the $(E_7)_8$ SCFT.

Instead, we can gauge an $SU(2)$ subgroup

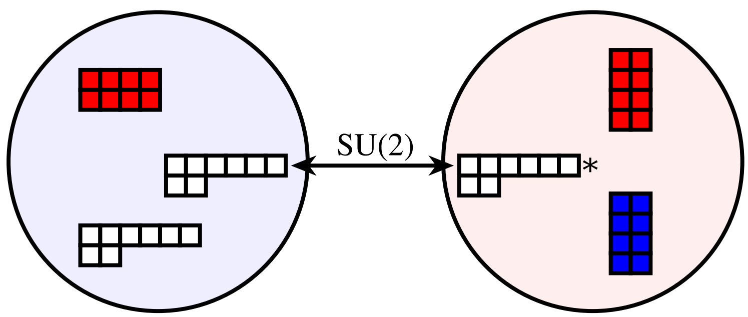


where the fixture on the right is empty. The S-dual

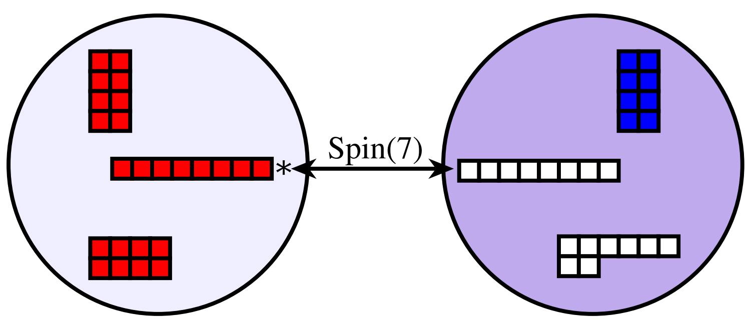


is a $Spin(8)$ gauge theory, with matter in the $2(8_s) + 8_c + 8_v$, coupled to one copy of the $(E_7)_8$ SCFT.

A different $SU(2)$ gauging of the $Sp(2)_8^2 \times SU(2)_8^4$ SCFT

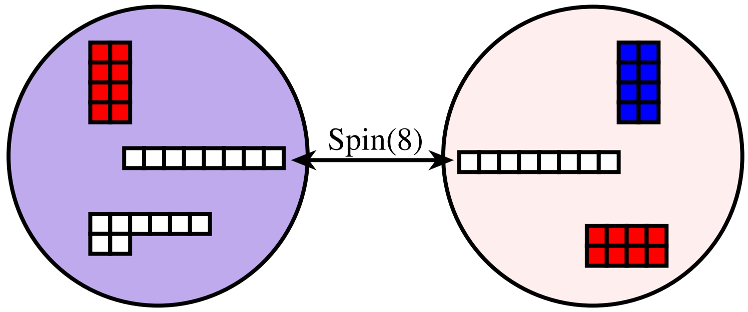


has two distinct strong-coupling points. One,



is a $Spin(7)$ gauge theory, with matter in the 8, coupled to two copies of the

$(E_7)_8$ SCFT. The other,

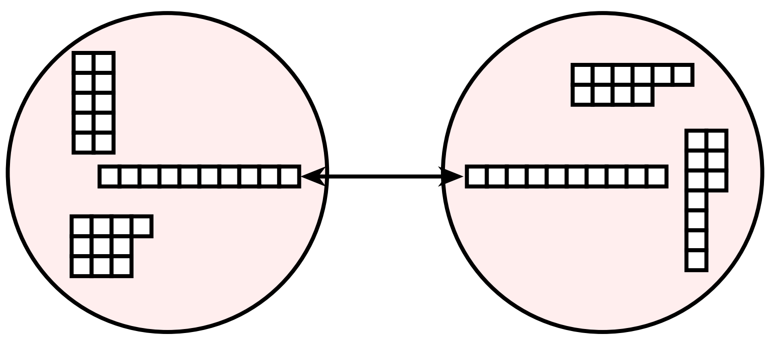


is a $Spin(8)$ gauge theory, with matter in the $2(8_s)+2(8_c)$, coupled to a single copy of the $(E_7)_8$ SCFT.

7.5.2 D_5 example: $Spin(10)$ gauge theory

To further illustrate our methods, let us study *one* example from the D_5 theory, involving a $Spin(10)$ gauge theory with matter in the $3(16)+2(10)$.

Start with the 4-punctured sphere

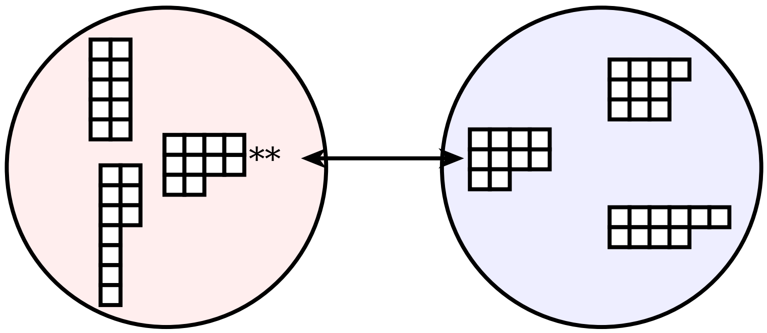


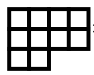
This is a Spin(10) Lagrangian field theory with matter in the $3(16) + 2(10)$ representation. The left fixture provides 32 free hypermultiplets in the $(16, 2)$ of $Spin(10) \times SU(2)$, and the right fixture, 36 free hypermultiplets in the $(16, 1) + \frac{1}{2}(10, 4)$ of $Spin(10) \times Sp(2)$.

The global symmetry group of the theory is, thus,

$$G_{global} = SU(3)_{32} \times Sp(2)_{10} \times U(1),$$

This theory has two distinct strong coupling cusp points. One appears in the degeneration

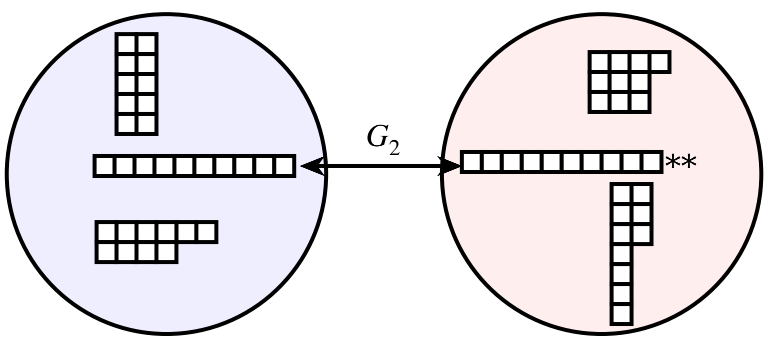


Here the left fixture is empty. The ** irregular puncture has pole structure $\{1,5,7,10;6\}$, and imposes the constraint $c_{10}^{(8)} = \left(c_5^{(4)}\right)^2$. The right fixture is an interacting SCFT with graded Coulomb branch dimension $d = (0,0,1,1,1,0,1)$ and global symmetry group

$$G_{SCFT} = Sp(2)_{10} \times SU(3)_{32} \times SU(2)_8 \times U(1),$$

and we gauge the $SU(2)_8$ subgroup.

The second strong coupling point appears in the remaining degeneration,



Here the fixture on the left is an SCFT with graded Coulomb branch dimension $d = (0,0,1,1,0,0,1)$ and global symmetry group

$$G_{SCFT} = (E_6)_{16} \times Sp(2)_{10} \times U(1),$$

and the fixture on the right is empty. The $\square\square\square\square\square\square\square\square\square\square{**}$ irregular puncture has pole structure $\{1, 4, 5, 8; 5\}$. Under the decomposition $(E_6)_k \supset (G_2)_k \times SU(3)_{2k}$, we gauge a $(G_2)_{16} \subset (E_6)_{16}$.

Appendices

Appendix A

Nilpotent orbits in $\mathfrak{so}(2N)$

Here we lay out our conventions for nilpotent orbits in $\mathfrak{so}(2N)$. For more details, see [15]. We take $\mathfrak{so}(2N)$ to consist of block matrices of the form

$$\begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \tag{A.1}$$

where A, B, C are $n \times n$ matrices and $B^t = -B$, $C^t = -C$. Nilpotent orbits are in 1-1 correspondence with embeddings $\rho : \mathfrak{sl}(2) \hookrightarrow \mathfrak{so}(2N)$, up to conjugation. Here, $\mathfrak{sl}(2)$ is generated by $\{H, X, Y\}$ satisfying

$$[H, X] = 2X, \quad [H, Y] = -2Y, \quad [X, Y] = H \tag{A.2}$$

and we take $\rho(X)$ (which we will, henceforth, simply denote by X) as our representative element of the nilpotent orbit.

As noted in Chapter 7, a nilpotent orbit, in $\mathfrak{so}(2N)$, is specified by a D-partition of $2N$. Here, we will give our convention for assigning a triple of matrices of the form (A.1), satisfying (A.2), to such a partition.

Let $e_1, e_2, \ldots e_n$ be the standard basis for $\mathbb{C}^N$. Let $E_{i,j}$ be the $2N \times 2N$ matrix with a 1 in the $(i, j)^{\text{th}}$ position and zeroes everywhere else. To the root, $e_i - e_j$, assign the matrix, of the form (A.1),

$$X^-_{i,j} = E_{i,j} - E_{j+N,i+N}$$

To the root $e_i + e_j$ (for $i < j$), assign

$$X^+_{i,j} = E_{i,j+N} - E_{j,i+N}, \quad i < j$$

Also, let

$$H_i = E_{i,i} - E_{i+N,i+N}$$

- Take the D-partition, $[r_1, r_2, \dots]$, and divide it into pairs of the form $[r, r]$ and $[2s + 1, 2t + 1]$ $(s > t)$. This is not quite unique: the D_6 partition, $[3, 3, 2, 2, 1, 1]$ can be divided into $[3, 3]$, $[2, 2]$, $[1, 1]$ or into $[2, 2]$, $[3, 1]$, $[3, 1]$. Different choices will result in different representatives of the same nilpotent orbit.

- To each pair of the form $[r, r]$, assign a block of r consecutive basis vectors of $\mathbb{C}^N$. We'll denote those by $(e_1, e_2, \dots, e_r)$, but they might be, say, $(e_{17}, e_{18}, \dots, e_{16+r})$. To each pair of the form $[2s+1, 2t+1]$, assign a block of $s+t$ consecutive basis vectors of $\mathbb{C}^N$. The blocks, thus assigned, must be non-overlapping, and will exhaust $e_1, \dots, e_N$.

- For each pair of the form $[r, r]$, let

$$
\begin{aligned}
H &= \sum_{k=1}^{r} (r+1-2k) H_k \\
X &= \sum_{k=1}^{r-1} \sqrt{k(r-k)}\, X^-_{k,k+1} \\
Y &= X^t
\end{aligned}
$$

- For pairs of the form $[2s+1, 2t+1]$, the general formula can be found in [15]. We'll just need the first few, for small values of t.

  - For pairs of the form $[2s+1, 1]$, let

$$
\begin{aligned}
H &= \sum_{k=1}^{s} 2(s+1-k) H_k \\
X &= \sum_{k=1}^{s-1} \sqrt{k(2s+1-k)}\, X^-_{k,k+1} + \sqrt{s(s+1)/2}\left(X^-_{s,s+1} + X^+_{s,s+1}\right) \\
Y &= X^t
\end{aligned}
$$

  - For pairs of the form $[2s+1, 3]$, let

$$
\begin{aligned}
H &= \sum_{k=1}^{s} 2(s+1-k) H_k \; + 2H_{s+1} \\
X &= \sum_{k=1}^{s-2} \sqrt{k(2s+1-k)}\, X^-_{k,k+1} + \sqrt{(s-1)(s+2)}\, X^-_{s-1,s} \\
&\quad + \sqrt{s(s+1)/2}\left(X^-_{s,s+2} + X^+_{s,s+2}\right) + \left(X^-_{s+1,s+2} - X^+_{s+1,s+2}\right) \\
Y &= X^t
\end{aligned}
$$

- Add up the contributions to H, X, Y from each pair. The resulting triple, $\{H, X, Y\}$, will be our embedding of $\mathfrak{sl}(2)$ and X will be our representative of the nilpotent orbit, corresponding to this partition.

The one exception to this rule has to do with "very even" partitions and our red/blue nilpotent orbits.

- For the red orbit, follow the prescription above.
- For the blue orbit, replace every instance of $X^{\mp}_{i,N}$ with $X^{\pm}_{i,N}$ and replace every instance of H_N with $-H_N$. This has the effect of exchanging the roles of the two irreducible spinor representations and flips the sign of the Pfaffian, $\tilde{\phi}(y) \to -\tilde{\phi}(y)$.

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