# Modular Representations of Symmetric Groups

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## Abstract

I have studied representation theory of finite groups, in particular of the symmetric group over fields of prime characteristic. Over $\mathbb{C}$, there is a nice classification of the simple representations of symmetric groups. Here I give a description of how the standard representation behaves in prime characteristic, and I study the structure of the group algebras of small symmetric groups in more detail.

The general subject of representation theory sits at the crossroads of a vast array of subjects in mathematics, including algebraic geometry, module theory, analytic number theory, differential geometry, operator theory, algebraic combinatorics, topology, fourier analysis, and harmonic analysis. Modular Representation theory, the study of representations of finite groups over a field of positive characteristic, has in particular been used in the classification of finite simple groups, and itself finds applications in a variety of areas of mathematics.

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# Chapter 1

# Basic Definitions and Theorems

In this paper, I will be studying representations of groups and group algebras, particularly the symmetric groups over fields of prime characteristic. We start off by giving definitions and theorems relating to algebras and representations in a more general context [1].

**Definition 1.** An **associative algebra with identity** over a field $k$ is a vector space $A$ over $k$ with an associative bilinear map, denoted by multiplication $a, b \mapsto a \cdot b$, with a unit $1 \in A$ such that $1 \cdot a = a \cdot 1 = a$ for $a \in A$.

Equivalently, an associative algebra with identity over $k$ is a (possibly noncommutative) ring $A$ which contains a selected copy of $k$ in its center (namely, the span $k1$, where $1 \in A$). In this paper when I use the term "algebra", I will be referring to an associative algebra with identity. An important example of an algebra is End$(V)$, for $V$ a vector space over $k$. When $V = k^d$, this algebra is denoted Mat$_d(k)$, and referred to as a "matrix algebra".

**Definition 2.** Let $A$ be an algebra over $k$. A **representation** of $A$ is a vector space $V$ together with a homomorphism of rings $\rho : A \to$ End$(V)$ (which maps the identity to the identity). A **subrepresentation** of a representation $\rho : A \to$ End$(V)$ is a subspace $W \subset V$ with $\rho(a)W \subset W$ for every $a \in A$. If $v \in V$, then $Av$ defines the subrepresentation of $V$ spanned by $v$. If $V$ is any vector space over $k$, then we can define the **trivial representation** on $V$ by mapping all of $A$ to $Id \in$ End$(V)$.

For $a \in A, v \in V$, We use the shorthand $av$ to denote $\rho(a)v$. This notation suggests that a representation is nothing other than a left $A$-module, and indeed this is the case. We say that $A$ "acts linearly" on $V$. When a representation $V$ is fixed, we associate with each $a \in A$ the endomorphism $\rho(a)$.

**Definition 3.** Let $V, W$ be representations of $A$. Then the **direct sum of representations** $V \oplus W$ is the direct sum of vector spaces with the action given by $a(v + w) = (av) + (aw)$. The **tensor product of representations** $V \otimes W$ is the tensor product of vector spaces with the action given by $a(v \otimes w) = (av) \otimes (aw)$. The **dual representation** $V^*$ is a right module of $A$: for $a \in A, f \in V^*, v \in V$, we define $fa$ to be the functional given by $(fa)v = f(av)$. If we identify $a$ with its associated endomorphism on $V$, then the right action of $A$ on $V^*$ is given by composition of maps. Finally, $\text{End}(V)$ is a representation of $A$ with the action given by $(aE)v = a(Ev)$, similarly defining a composition of maps. In fact, $\text{End}(V)$ has a two-sided module structure, with $aEb = a \circ E \circ b$ for $a, b \in A$, $E \in \text{End}(V)$, and these two module structures commute with each other in the sense that $(aE)b = a(Eb)$.

Note that a right $A$-module is the same thing as a left $A^{\text{op}}$-module, where $A^{\text{op}}$ is the opposite ring. Also note that we have given $\text{End}(V)$ both the structure of a representation and that of an algebra itself.

**Definition 4.** Let $V \subset W$ be nested representations of $A$. Then the **quotient representation** is the set of cosets $W/V = \{w + V \mid w \in W\}$ with the action given by $a(w + V) = aw + V$ for $a \in A, w \in W$.

We now consider representations of groups. First, we give the classical definition.

**Definition 5.** Let $G$ be a group. A **representation** of $G$ over a field $k$ is a vector space $V$ over $k$ together with a homomorphism of groups $\phi : G \to GL(V)$.

For $g \in G, v \in V$, we use the shorthand $gv$ for $\phi(g)v$, and we say that $G$ acts linearly on $V$.

**Definition 6.** Define the set of formal sums $kG = \{\sum_{g \in G} \alpha_g g : \alpha_g \in k\}$, together with componentwise addition, and multiplication given by the group law for $g, h \in G \subset kG$ and extended by distributivity.

**Theorem 1.** *The above definition for $kG$ makes it an algebra over $k$, known as the "group algebra" or "group ring" of $G$.*

Now we make explicit the correspondence between representations of the group $G$ and representations of the algebra $kG$.

**Theorem 2.** *Let $G$ be a group and $\varphi : G \to GL(V)$ be a representation of $G$ over the field $k$. The map $\varphi : G \to End(V)$ extends uniquely to a map of rings $kG \to End(V)$, giving a representation of $kG$. Conversely, a representation $\rho : kG \to End(V)$ of the algebra $kG$ defines a representation of the group $G$, with the homomorphism $G \to GL(V)$ given by restriction of the ring map to $G \subset kG$.*

Notice that $\rho(g)$ must be invertible, so $\rho(G) \subset \text{End}(V)^* = GL(V)$. This shows that representations of a group $G$ and representations of the group algebra $kG$ are in one-to-one correspondence. All definitions given for representations of algebras extend accordingly to representations of a group.

**Definition 7.** The **regular representation** of an algebra $A$ is $A$ considered as a left-module over itself.

When $A$ is a group algebra, this representation has a $k$-basis given by the elements of $G$, and $G$ permutes this basis in the obvious manner.

**Definition 8.** A nonzero representation $V$ of an algebra $A$ is called **simple** or **irreducible** if the only subrepresentations of $V$ are $\{0\}$ and $V$. A nonzero representation is called **indecomposible** if $V \cong W \oplus W'$ implies either $W = \{0\}$ or $W' = \{0\}$. A representation is **semisimple** if it is a direct sum of irreducible representations.

We remarked earlier that if $v \in V$, then we can consider the subrepresentation generated by $v$, which is simply $Av$, the image under left multiplication. From this, the following theorem is almost immediate:

**Theorem 3.** *A representation $V$ is irreducible if and only if every nonzero $v \in V$ generates $V$, i.e., $Av = V$.*

*Proof.* If $V$ is irreducible and $v \in V$ is nonzero, then $Av$ gives a nonzero subrepresentation of $V$, which must hence be all of $V$.

For the other direction, if a representation $V$ fails to be irreducible, then let $W \subset V$ be a proper subrepresentation, let $v \in W$ be nonzero, and consider $Av \subseteq W \subsetneq V$. □

**Definition 9.** A **homomorphism** of representations $\phi : V \to W$ over an algebra $A$ is a homomorphism of modules. The kernel and image, $\ker \phi$ and $\phi(V)$, define subrepresentations of $V$ and $W$. More generally, if $V' \subset V$ is a subrepresentation, then $\phi(V')$ is a subrepresentation of $W$, and if $W' \subset W$ is a subrepresentation, then $\phi^{-1}(W')$ is a subrepresentation of $V$.

A homomorphism of group representations is one which "commutes" with the action of $G$, i.e., $\phi(gv) = g\phi(v)$.

**Theorem 4.** *(**Schur's Lemma**) Let $V$ and $W$ be irreducible representations of an algebra $A$, and let $\phi : V \to W$ be a homomorphism. Then $\phi$ is either the zero map or an isomorphism.*

*Proof.* If $\phi$ is not the zero map, then $\ker \phi$ is a proper subrepresentation of $V$, which by irreducibility must be $\{0\}$, and $\phi(V)$ is a nonzero subrepresentation of $W$, which must be all of $W$. This shows that $\phi$ is an isomorphism. □

We will only be concerned with algebraically closed fields, so the following version will be relevant:

**Theorem 5.** *(***Schur's Lemma for Algebraically closed fields***) If $A$ is an algebra over the algebraically closed field $k$, and $\phi : V \to V$ is an endomorphism of the irreducible representation $V$, then $\phi = \lambda Id$ for some $\lambda \in k$.*

*Proof.* Let $\lambda$ be an eigenvalue of $\phi$, and apply Schur's Lemma to $\phi - \lambda \mathrm{Id}$. □

All remaining theorems in this section require that $k$ be algebraically closed.

**Theorem 6.** *(***Density Theorem***) Let $V$ be an irreducible representation of $A$ via $\rho : A \to End(V)$. For every $E \in End(V)$ (endomorphisms of the vector space), there exists $a \in A$ with $av = Ev$ for $v \in V$. That is, $\rho$ is surjective.*

The group algebras we will be considering here are, in particular, finite-dimensional algebras, so the following definitions and theorems for finite-dimensional algebras will be relevant.

**Definition 10.** Let $A$ be a finite-dimensional algebra. Then Rad $A$ is defined as the elements of $A$ which act on every irreducible representation of $A$ by $0$.

An ideal $I$ in $A$ is called nilpotent if $I^n = 0$ for some $n$.

**Theorem 7.** *Rad $A$ is the largest nilpotent two-sided ideal of $A$.*

**Theorem 8.** *There are finitely many simple representations of $A$, they are all finite-dimensional, and we have an isomorphism of rings*

$$A/Rad\, A \cong \bigoplus_{V \text{ simple}} End(V).$$

Moreover, the isomorphism in theorem 8 is natural: for any fixed simple representation $V_i$, the projection

$$A \twoheadrightarrow A/\mathrm{Rad}\, A \xrightarrow{\sim} \bigoplus_{V \text{ simple}} \mathrm{End}(V) \twoheadrightarrow \mathrm{End}(V_i)$$

is given by

$$\rho : A \to \mathrm{End}(V_i),$$

the map which defines the representation $V_i$ in the first place.

Finally, we see how the structure of $A$ reflects the structure of all finite-dimensional representations of $A$.

**Theorem 9.** *(***Artin-Wedderburn***) The following are equivalent:*

*(1)* $Rad\, A = 0$

*(2)* $\sum_{V \text{ simple}} (\dim V)^2 = \dim A$

(3) $A \cong \bigoplus_{V \text{ simple}} Mat_{\dim V}(k)$ *as algebras.*

(4) *Any finite-dimensional representation of $A$ is semisimple.*

(5) *The regular representation of $A$ is semisimple.*

We give the theorem which determines whether or not a group algebra is semisimple.

**Theorem 10.** **(Maschke)** *Let $k$ be an algebraically closed field and $G$ a finite group. The group algebra $kG$ is semisimple if and only if the characteristic of $k$ does not divide $|G|$.*

# Chapter 2

# Simple Representations in the Regular Representation

Let $G$ be any finite group, $k$ any algebraically closed field, and $V$ any irreducible finite-dimensional representation of $G$ over $k$ via a left action $kG \times V \to V$. We will identify exactly how $V$ occurs as a subrepresentation of the regular representation $kG$. Consider the representation $V \otimes V^*$; for now $G$ acts on $V^*$ trivially. In this manner, if we let $d$ be the dimension of $V$, then as representations,

$$V \otimes V^* \cong dV = \bigoplus_{i=1}^{d} V.$$

We will give a natural injection of representations

$$\Phi : V \otimes V^* \hookrightarrow kG.$$

Recall that the left action of $G$ on $V$ induces a right action of $G$ on the dual space $V^*$, via $(fg)v = f(gv)$, where $f \in V^*, g \in G, v \in V$, and that under the identification of $g$ with its corresponding endomorphism on $V$, the right action of $G$ on $V^*$ is just composition of functions. Putting these together, we get that $V \otimes V^*$ has a left action and a right action which commute with each other, given by $g(v \otimes f)h = (gv) \otimes (fh)$, where $g, h \in G, v \in V, f \in V^*$. Similarly, $kG$ has a left and a right action by $G$ given by left and right multiplication by $G$ considered as a subset of $kG$.

In general, if the group $G$ acts on a set $S$ on the left, and the group $H$ acts on $S$ on the right, and these actions commute with each other, then this induces a single left action on $S$ by $G \times H$ given by $(g, h) \cdot s = gsh^{-1}$, or similarly a right action given by $s \cdot (g, h) = g^{-1}sh$. In this manner, we see that the two-sided action of $G$ on $V \otimes V^*$ makes $V \otimes V^*$ into a $G \times G$ representation.

**Theorem 11.** *The map*

$$\begin{aligned}\Phi : V \otimes V^* &\to kG \\ v \otimes f &\mapsto \sum_{g \in G} [f(g^{-1}v)]g\end{aligned}$$

*defines an injective homomorphism of $G \times G$ representations, and $V \otimes V^*$ is an irreducible representation of $G \times G$.*

To understand the map more explicitly, $g^{-1}v \in V$, and $f(g^{-1}v)$ is the scalar coefficient of $g$ in $\Phi(v \otimes f) \in kG$.

*Proof.* The map $\Phi$ is bilinear in $v$ and $f$, so is well-defined. For $x, y \in G$, consider:

$$\begin{aligned}\Phi(x(v \otimes f)y) &= \Phi((xv) \otimes (fy)) \\ &= \sum_g (fy(g^{-1}xv))g \\ &= \sum_g (fyg^{-1}xv)g\end{aligned}$$

and making the substitution $yg^{-1}x = h^{-1}$, we get

$$\begin{aligned}\Phi(x(v \otimes f)y) &= \sum_h (fh^{-1}v)xhy \\ &= x\left(\sum_h f(h^{-1}v)h\right)y \\ &= x\Phi(v \otimes f)y.\end{aligned}$$

Thus $\Phi$ defines a homomorphism of representations of $G \times G$. I now show that $V \otimes V^*$ defines an irreducible representation of $G \times G$. This is equivalent to showing that every nonzero vector in $V \otimes V^*$ is a generator for $V \otimes V^*$. By Schur's lemma, it will follow that $\Phi$ is either the zero map or injective, and I will show it is not the zero map.

It will be easier to show that $V \otimes V^*$ is irreducible as a $G \times G$ representation via the standard isomorphism $V \otimes V^* \cong \mathrm{End}(V)$ which identifies $v \otimes f$ with the map

$$\begin{aligned}V &\to V \\ w &\mapsto f(w)v.\end{aligned}$$

By this identification, the left and right action of $G$ on $\mathrm{End}(V)$ are given by left and right composition $gEh = g \circ E \circ h$ for $g, h \in G, E \in \mathrm{End}(V)$. Also under this identification, we get $\Phi(E) = \sum_{g \in G} \mathrm{Tr}(g^{-1}E)g$. Thus, to show that End$(V)$ defines an irreducible representation for $G \times G$, we need only show that if $E \in$ End$(V)$ is not the zero map, then the span of $\{gEh : g, h \in G\}$ is all of End$(V)$. By the Density Theorem, the map $kG \to$ End$(V)$ which sends each $g \in G$ to its associated endomorphism is surjective. So we need only show that $\{AEB : A, B \in$ End$(V)\}$ spans End$(V)$. This follows because the only two-sided ideals of End$(V)$ are the zero ideal and all of End$(V)$.

Finally, we show that there exists $v \otimes f \in V \otimes V^*$ with $\Phi(v \otimes f) \neq 0$. For any choice of nonzero $v$ and $f$, since $\{gv : g \in G\}$ spans $V$, there exists a $g$ such that $gv \notin \ker f$, for otherwise $f$ would have to be the zero map. Therefore the coefficient of $g^{-1}$ in $\Phi(v \otimes f)$ for such a $g$ is nonzero, so $\Phi(v \otimes f) \neq 0$. This completes the proof that $\Phi$ is an inclusion of representations of $G \times G$. $\square$

If the group algebra is semisimple, then the domain ($V \otimes V^* \cong$ End$(V)$) and image ($\Phi$(End$(V)$)) are both rings. We will see, however, that $\Phi$ itself is not necessarily a map of rings, because it may not map identity to identity. Since we assume $k$ is algebraically closed, Schurr's Lemma applied to $\Phi \circ \rho$, where $\rho : kG \to$ End$(V)$ defines the representation $V$, implies that some multiple of $\Phi(I)$ is an idempotent of $kG$, where $I$ is the identity endomorphism on $V$.

We make one last observation: under $\Phi$, $I \mapsto \sum_{g \in G} \mathrm{Tr}(g^{-1})g$, which is the character of $V^*$ considered as a representation of $G$ via $gf = f \circ g^{-1}$.

# Chapter 3

# Classification of Simple Representations of the Symmetric Groups over the Complex Numbers

Fulton and Harris' [2] Theorem 4.5 gives a classification of the irreducible representations of $S_n$ over $\mathbb{C}$ in the following manner: Each irreducible representation, $V$, can be identified with many subrepresentations of $\mathbb{C}S_n$. Using a Young Tableau, $\lambda$, one is able to choose a generator, $c_\lambda$ (defined via two factors $c_\lambda = b_\lambda a_\lambda$), so that $V$ can be identified with the specific subrepresentation $\mathbb{C}S_n \cdot c_\lambda \subset \mathbb{C}S_n$. The process of using $\lambda$ to identify a single copy of $V$ in $\mathbb{C}S_n$ is asymmetric, unlike injecting $V \otimes V^* \cong \mathrm{End}(V)$ into $kG$, which does not involve making any choices.

Here we examine the nature of the classification theorem from this perspective in some special cases. For the trivial and sign representations, since they are 1-dimensional representations of $S_n$, they only occur once in the regular representation, and the $c_\lambda$ given by Fulton and Harris is just the character. The first interesting case is the standard representation $V$ of $S_3$.

**Definition 11.** Let $k$ be any field (not necessarily algebraically closed). We define the **standard representation** of $S_n$ over $k$ as follows: let $k^n$ be the representation of $S_n$ given by permuting the standard basis. This has a trivial subrepresentation spanned by $\begin{pmatrix}1\\ \vdots \\ 1\end{pmatrix}$. The standard representation is the quotient of $k^n$ by this subrepresentation, i.e.,

$$k^n \Big/ k\begin{pmatrix}1\\ \vdots \\ 1\end{pmatrix}.$$

If the characteristic of $k$ does not divide $n$ (as in the case $k = \mathbb{C}$), there is a subrepresentation complementary to the trivial one, characterized by the sum of the entries of the vector being 0, which identifies naturally with the quotient. In general, regardless of whether a complement can be found in $k^n$, this standard representation is spanned by vectors $v_1, \cdots, v_n$ whose only linear dependence is given by $v_1 + \cdots + v_n = 0$, and $S_n$ acts directly on these vectors by permutation. Here $v_i = e_i + k \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$.

In the case when $k = \mathbb{C}$, this representation is the symmetries of an $n$-symplex in $n-1$-dimensional space. In order to give concrete matrices, I will choose the basis $v_1, \cdots, v_{n-1}$ for $V$.

Now we look at the standard representation $V$ of $S_3$. Here the identity map $I \in \mathrm{End}(V)$ is associated by $\Phi : \mathrm{End}(V) \to \mathbb{C}S_3$ from Theorem 8 with the character of $V$, which is $2 - (1\,2\,3) - (1\,3\,2)$. The $c_\lambda$ given by Fulton and Harris here is $c_\lambda = b_\lambda a_\lambda = (1 + (1\,2))(1 - (1\,3)) = 1 + (1\,2) - (1\,3) - (1\,3\,2)$.

We have two ways of understanding $c_\lambda$: first, in terms of the surjection $\rho : \mathbb{C}S_3 \twoheadrightarrow \mathrm{End}(V)$, and second, in terms of the inclusion $\Phi : \mathrm{End}(V) \hookrightarrow \mathbb{C}S_3$ that I have given above. The endomorphism induced by $c_\lambda \in \mathbb{C}S_3$ on $V$ carries $v_1 \mapsto v_1 + v_2 - v_3 - v_3 = -3v_3, v_2 \mapsto 0, v_3 \mapsto v_3 + v_3 - v_1 - v_2 = 3v_3$, which is given in terms of the above named basis, $v_1, v_2$, by the matrix $\begin{pmatrix} 3 & 0 \\ 3 & 0 \end{pmatrix}$. In terms of $\Phi$, the endomorphism which is associated to $c_\lambda$ is given by $\Phi^{-1}(c_\lambda) = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$. The observation to make here is that a 2-dimensional subrepresentation of the 4-dimensional representation $\Phi(\mathrm{End}(V)) \cong \mathrm{End}(V)$ is a choice of a $2 - 1 = 1$-dimensional kernel, and in this case the kernel is spanned by $v_2$. In fact, this is the kernel for the right factor $a_\lambda$.

The more general observation, for an irreducible $d$-dimensional representation $V$ of $G$ over $\mathbb{C}$, is that we have seen that $V$ occurs in $\mathbb{C}G$ as the representation $\mathrm{End}(V)$. A single copy of $V$ inside $\mathrm{End}(V)$ is the same thing as a minimal left ideal of the ring $\mathrm{End}(V)$, which is in turn simply a selection of a $d-1$-dimensional subspace $U$ of $V$, so that the left ideal is given by $\{f : V \to V \mid U \subset \ker f\}$, i.e., $U$ is contained in the kernel of every endomorphism in the subrepresentation. We will see what this common kernel is for the standard representation of $S_n$ for all $n$.

For the standard representation of $S_n$,

$$c_\lambda = b_\lambda a_\lambda = \left( \sum_{g \in S_{n-1}} g \right) \left( 1 - (1\,n) \right)$$

where $S_{n-1} \subset S_n$ is the set of permutations which fix $n$. Here we can see from $a_\lambda$ that the common kernel of $\mathbb{C}S_n(c_\lambda)$ includes $v_2, \cdots, v_{n-1}$, because $b_\lambda v_i = v_i - (1\,n)v_i$ is zero for $1 < i < n$. Since this spans an $(n-2)$-dimensional subspace of the standard representation of dimension $n-1$, this is the full kernel.

# Chapter 4

# The Standard Representation of the Symmetric Groups in Prime Characteristic

We study the standard representation of $S_n$ over $k = \overline{\mathbb{F}}_p$, for a prime $p$, and some particular choices of $n \geq p$ (i.e., when $p \mid n!$). In the representation $k^n$ of $S_n$, the matrix of an $n$-cycle is just a permutation matrix satisfying the polynomial equation $A^n = I$, i.e., $x^n - 1 = 0$. We will examine the structure of $k^n$ in the extreme cases when $x^n - 1$ is purely inseparable (i.e., $n = p^m$), and when $x^n - 1$ is separable (i.e., when $p \nmid n$).

Consider $n = p^m$. An $n$-cycle $A : k^n \to k^n$ in characteristic $p$ satisfies the polynomial $A^n = I$, so the minimal polynomial divides $x^{p^m} - 1 = (x-1)^{p^m} = 0$. This shows that 1 is the only eigenvalue for an $n$-cycle. An eigenvector under the $n$-cycle with eigenvalue 1 must have adjacent entries equal to each other, so the eigenspace is simply the span of $\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$. It follows that the Jordan form of an $n$-cycle is one full block with eigenvalue 1:

$$\begin{pmatrix} 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

If we let $e_1, \cdots, e_n$ be a basis with respect to which the $n$-cycle is identified with this matrix, then the subspaces which are invariant under this $n$-cycle are exactly the span of the first $i$ vectors $e_1, \cdots, e_i$ for each $i \leq n$ (and we refer to the $e_i$ as "generalized eigenvectors"). This tells us how to get a restricted set of subspaces to look for subrepresentations, since a subrepresentation must, in particular, be invariant under the $n$-cycles. Furthermore, since the invariant subspaces of $k^n$ under the $n$-cycle are nested inside each other, it follows that $k^n$ is always indecomposable, as is every subspace and quotient of subspaces of $k^n$. We will be able to describe the structure of $k^n$ in this context generally, but first we consider some special cases of small order.

For $p = n = 2$, $S_2$ is the group of order 2. The unique non-identity element acts on $k^2$ by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. As noted already, the Jordan form of this matrix is $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, where $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector, and $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is a generalized eigenvector. The only subrepresentation is the trivial one spanned by $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. The standard representation is given by permuting the two vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix} + k\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix} + k\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ in the quotient space; these are $v_1$ and $v_2$. But these are the same vector, because the linear dependence $v_1 + v_2 = 0$ expresses $v_1 = v_2$ in characteristic 2. From this perspective, we can see that in characteristic $p \neq 2$, $v_1 = -v_2$ are distinct and the standard representation of $S_2$ is the sign representation. In characteristic 2, however, we see that the standard representation is the trivial representation.

For $p = n = 3$, the 3-cycles have generalized eigenvectors $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$, and $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, in order. The intermediate subspace spanned by the first two vectors is invariant under $S_3$. I describe the structure of $k^3$ in terms of a filtration: $0 \subset V_1 \subset V_2 \subset V_3$ where $V_3 = k^3$. Define $V_1 = k\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ to be the trivial subrepresentation of $k^3$, and $V_2 = k\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + k\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$, which is the subspace of $k^3$ consisting of vectors whose entries in $k$ add up to 0. These are all $S_3$-invariant subspaces ($V_2$ because permuting the entries preserves the property of adding up to 0), and it remains to see what the intermediate quotients are, i.e., what the simple representations $V_1/\{0\}$, $V_2/V_1$, and $V_3/V_2$ are. Note that these are all degree 1 representations, and the only degree 1 representations of $S_3$ (over any field) are trivial and sign. We've already seen that $V_1$ is the trivial representation. To see what $V_2/V_1$ is, we see what the action of a 2-cycle on $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + V_1$ is: permuting the lower two entries gives $\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + V_1$, which is the negative of the original vector, so it's the sign representation. Finally, permuting the first two entries of $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ fixes the vector, so $V_3/V_2$ must be trivial.

**Theorem 12.** *Let $p$ be a prime, $n = p^m$ for some $m \geq 0$, $k = \overline{\mathbb{F}}_p$, and assume $n > 2$. The subrepresentations of $k^n$ are exactly $\{0\} \subset V_1 \subset V_2 \subset k^n$, where $V_1$ is the trivial subrepresentation spanned by $\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$, and $V_2$ is the subrepresentation of vectors whose entries add up to $0$. The intermediate quotients are $V_1/\{0\}$, which is the trivial representation, $V_2/V_1$, an irreducible $(n-2)$-dimensional representation, and $k^n/V_2$, which is the trivial representation.*

Note that $p = n = 2$ is a degenerate case.

*Proof.* It is clear that these are invariant subspaces, and that we need only show that there is no representation strictly between $V_1$ and $V_2$. By the observation about the subspaces invariant under an $n$-cycle, any such representation $V$ must contain every generalized eigenvector $v$ with $(A - I)v \neq 0$ but $(A - I)^2 v = 0$, when $A$ is an $n$-cycle. For the $n$-cycle $(1\,2\,3 \cdots n)$, such a vector is given by $\begin{pmatrix} 0 \\ 1 \\ \vdots \\ n-1 \end{pmatrix}$.
The subrepresentation generated by this vector includes all vectors obtained by permuting the entries. (Note that if $m > 1$, then the entries of this vector are not all distinct. This does not pose a problem to the proof.) In particular, we can permute any two adjacent entries, and subtracting gives $\begin{pmatrix} 0 \\ \vdots \\ 1 \\ -1 \\ \vdots \\ 0 \end{pmatrix} \in V$, with a 1 and a $-1$ in two adjacent entries, and 0's elsewhere. These generate the $(n-1)$-dimensional subspace $V_2$, which completes the proof that there is no properly intermediate subrepresentation.

To see that $k^n/V_2$ is the trivial representation, consider the standard basis vectors $e_i \in k^n$, for $i = 1, \cdots, n$, and note that $e_i \notin V_2$, as the entries of these vectors add up to $1 \neq 0$. Every 2-cycle $(a\,b)$ fixes $e_i$ for $i \notin \{a, b\}$. Since we assume $n > 2$, and $k^n/V_2$ is 1-dimensional, it follows that every 2-cycle acts trivially on $k^n/V_2$, which proves the representation is trivial. □

Now we consider when $p \nmid n$. In this case, we already know that $k^n$ decomposes into the trivial subrepresentation and the standard subrepresentation, where the standard subrepresentation consists of vectors whose entries add up to zero. I first illustrate the structure of the group algebra in the case $p = 2, n = 3$.

Let $\omega \in \overline{\mathbb{F}}_2$ be a primitive cube root of unity (i.e., an element of $\mathbb{F}_4 \setminus \mathbb{F}_2$). The three cycle $(1\,2\,3)$ has eigenvalues equal to the powers of $\omega$, with eigenvectors

$$\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\\omega\\\omega^2\end{pmatrix}, \begin{pmatrix}1\\\omega^2\\\omega\end{pmatrix}.$$

Subsets of this set of three vectors span all of the subspaces of $k^3$ which are invariant under a three-cycle. But the 2-cycle $(2\,3)$ interchanges the latter two. Since these span the standard representation, it follows that the standard representation is simple.

In general, when $p \nmid n$, we find that the roots of $x^n - 1$ in $k$, the eigenvalues of an $n$-cycle, are all distinct, so an $n$-cycle diagonalizes. More explicitly, let $\zeta$ be a primitive $n^{\text{th}}$ root of unity in $k$. Then the eigenvalues of any $n$-cycle are exactly the powers of $\zeta$, and the eigenvectors for the $n$-cycle $(1\,2\,3 \cdots n)$ corresponding to $\zeta^{-i}$ can be given explicitly by $\begin{pmatrix}1\\\zeta^i\\\zeta^{2i}\\\vdots\\\zeta^{(n-1)i}\end{pmatrix}$. So the subspaces which are invariant under the $n$-cycle $(1\,2\,3 \cdots n)$ are given by spans of these eigenvectors.

**Theorem 13.** *Assume $n > p$ and $n, p$ are both prime. Then the standard representation of $S_n$ over $\overline{\mathbb{F}}_p$ is irreducible.*

*Proof.* In this case, all of the eigenvectors of the standard representation are just permutations of $\begin{pmatrix}1\\\zeta\\\zeta^2\\\vdots\\\zeta^{n-1}\end{pmatrix}$, and therefore any one of them generates the rest as an $S_n$-representation. This implies the standard representation is irreducible. □

# Chapter 5

# Structure of the Group Algebra of Symmetric groups and Morita Equivalence

**Definition 12.** Let $A$ be a finite-dimensional algebra. A **projective module** is a direct summand of a free module

$$\bigoplus_{1}^{n} A,$$

That is, $P$ is a projective module if there exists a module $Q$ and a positive integer $n$ such that

$$P \oplus Q \cong A^n.$$

A projective module can be decomposed into indecomposible projectives (as long the pieces are not all indecomposable, keep splitting into direct sums). In representation theory in characteristic 0, one studies the simple representations. In characteristic $p$, one studies their projective covers [4]:

**Definition 13.** Let $k$ be an algebraically closed field, $G$ a finite group, and $V$ a simple representation of $kG$. A **projective cover** of $V$ is an indecomposable, projective representation $P$ such that $P \twoheadrightarrow V$.

**Theorem 14.** *Let $k$ be an algebraically closed field, and $G$ a finite group. Each simple representation $V$ has a unique projective cover $P$, and this association gives a 1-1 correspondence between simple representations and indecomposible projective representations.*

In our context, the indecomposible projectives always occur as a summand of the regular representation [3]. Here we study some cases of small order.

We have already seen the full structure of $kS_2$ when $k$ is characteristic 2. Here the projective cover of the trivial representation is the full regular representation $kS_2$.

For $kS_3$ when $k$ has characteristic 2, we saw that the standard representation $V$ is irreducible, which implies by our work in chapter 2 that $\mathrm{End}(V)$ occurs as a subrepresentation of $kS_3$. In fact, $kS_3 \cong kS_2 \oplus \mathrm{End}(V)$ as rings, where the first summand is generated by $1 + (123) + (132)$, and the second by $(123) + (132)$.

For $kS_3$ when $k$ has characteristic 3, we saw that $k^3 \twoheadrightarrow V_{\text{triv}}$, the trivial representation, so it is natural to think that $k^3$ may be the projective cover of the trivial representation. It happens that this is indeed the case, and the projective cover for the sign representation is given by $V_{\text{sign}} \otimes k^3$. This comes from the decomposition of the regular representation $kS_3 \cong k^3 \oplus k^3_-$ as representations, not as algebras. This decomposition is not canonical; an example of a pair of generators for one decomposition is $1 + (23)$ and $1 - (23)$. This shows that the indecomposable projective representations are $k^3$ and $k^3_-$.

I now illustrate an alternative way of understanding these representations. We have been describing them as left modules of $kG$. Let $k$ be any algebraically closed field and $G$ any finite group. Let $s_1, \cdots, s_n$ be the pairwise nonisomorphic simple modules of $kG$, and $P_1, \cdots, P_n$ be their projective covers. Denote by $\mathbf{P}$ the direct sum

$$\mathbf{P} = \bigoplus_i P_i$$

of the projective indecomposable modules. Let $A = \mathrm{End}_{kG}(\mathbf{P})$ be the set of $kG$-endomorphisms of $\mathbf{P}$, which is also an algebra over $k$ (with multiplication given by composition of maps). Let "$kG$-mod" denote the left modules of $kG$, and "mod-$A$" denote the *right* modules of $A$.

**Theorem 15.** *The functors*

$$\begin{aligned} F : kG\text{-}mod &\to mod\text{-}A \\ M &\mapsto Hom_{kG}(\mathbf{P}, M) \end{aligned}$$

*and*

$$\begin{aligned} H : mod\text{-}A &\to kG\text{-}mod \\ N &\mapsto N \otimes_A \mathbf{P} \end{aligned}$$

*where*

$$N \otimes_A \mathbf{P} = (N \otimes \mathbf{P})/(na \otimes p - n \otimes ap)$$

*are inverse equivalences of categories.*

The right module structure of $\mathrm{Hom}_{kG}(\mathbf{P}, M)$ is given by composition of maps $n\varphi$ for $n \in \mathrm{Hom}_{kG}(\mathbf{P}, M)$ and $\varphi \in A = \mathrm{End}_{kG}(\mathbf{P})$. The left module structure of $N \otimes_A \mathbf{P}$ is given by $b(n \otimes p) = n \otimes (bp)$ for $b \in kG, n \in N, p \in \mathbf{P}$. This is an instance of Morita equivalence which we leave unproved in this paper. Instead, we look at it in detail for a few cases.

## 5.1 Over a semisimple group algebra

When the regular representation of $kG$ is semisimple, the projective covers of simple modules $P_i = s_i$ are equal, and

$$\mathbf{P} = \mathrm{End}\left(\bigoplus_i s_i\right) = \bigoplus_i \mathrm{End}_{kG}(s_i) \cong \bigoplus_i k = k^n$$

follows from Schur's Lemma. By theorem 9, the representations of $kG$ are all of the form $c_1 s_1 \oplus \cdots \oplus c_n s_n$, $c_i \geq 0$. These map by $F$ to $(c_1 k) \oplus \cdots \oplus (c_n k)$ (also by Schur's Lemma), where the $i^{\text{th}}$ copy of $k$ acts directly on $c_i k$. As this computation suggests, a representation of $A = k^n$ is simply a sequence of $k$-vector spaces $V_1, \cdots, V_n$, with the $i^{\text{th}}$ copy of $k$ acting on $V_i$ [1]. We can interpret this result in terms of Quivers.

**Definition 14.** A **Quiver** is a directed graph. A **representation of a quiver** is a vector space for each vertex together with a linear map of vector spaces for each directed edge.

Using Quivers, we see that a representation of $A$ is the same as a representation of the Quiver with $n$ isolated vertices:

$$\underbrace{\bullet\bullet\cdots\bullet}_{n}$$

How the inverse map $G$ works is a bit more interesting. Let $V_1 \oplus \cdots \oplus V_n$ be a representation of $A$, with $\dim(V_i) = c_i$. Then

$$\begin{aligned} H(V_1 \oplus \cdots \oplus V_n) &= (V_1 \oplus \cdots \oplus V_n) \otimes (s_1 \oplus \cdots \oplus s_n) /\!\sim \\ &= \left(\bigoplus_{i,j} V_i \otimes s_j\right) /\!\sim , \end{aligned}$$

and for $i \neq j, a \in \mathrm{End}(s_i), v \in V_i$, we have $va \neq 0$ but $as_j = 0$, so the equivalence relation quotients out all but the diagonal entries of the direct sum. It is easy to verify that

$$H(V_1 \oplus \cdots \oplus V_n) = \bigoplus_i V_i \otimes s_i$$

satisfies the relations induced by $A$.

Now we consider the more interesting cases in characteristic $p$. This time we use the inverse map $H$ to discover what the $kG$-modules are.

## 5.2 Over the symmetric group on 2 elements in characteristic 2

Let $k = \overline{\mathbb{F}}_2$ and $G = S_2$. Recall that $kG$ is itself the unique projective cover for the single trivial simple representation $s_1$, so $P_1 = kG$. The endomorphisms of $\mathbf{P} = P_1$ include the identity endomorphism, and $\epsilon : P_1 \to P_1$ which carries a generator to the trivial subrepresentation:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

This latter map satisfies $\epsilon^2 = 0$. These two maps generate $A$ as a $k$-vector space (since $A$ itself is actually only 2-dimensional, and a $kG$-map is determined by where it sends a generator). So in fact $A \cong k[x]/(x^2) = k[\epsilon]$. A representation of $kG$ is the same as a representation of the following quiver:

$$\begin{tikzcd} \bullet \arrow[loop left, "\epsilon"] \end{tikzcd}$$

with the linear map associated to $\epsilon$ squaring to the zero map. Such a map is given in Jordan Canonical Form by a direct block sum of zero maps $\begin{pmatrix} 0 \end{pmatrix}$ and the block of size 2, $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. We wish to see what $kG$ modules these map to. Since the tensor product over $A$ distributes over direct sum, it is sufficient to see what $H$ does to the two summands.

For the zero map on $k$,

$$H(k) = k \otimes_A \mathbf{P}/\sim,$$

where $k \otimes_A \epsilon\mathbf{P} = k\epsilon \otimes_A \mathbf{P} = 0$ implies we take the quotient of $\mathbf{P}$ by the image $\epsilon\mathbf{P}$, which is the trivial subrepresentation, and the quotient is the trivial representation, so

$$H(k) = k \otimes k = k$$

is the trivial representation of $kG$.

For the block matrix, we have $A$ acting on $k^2$. Here, we get

$$H(k^2) = k^2 \otimes_A \mathbf{P}/\sim.$$

Recall that on $k^2$, $\epsilon$ carries

$$\begin{pmatrix}0\\1\end{pmatrix} \mapsto \begin{pmatrix}1\\0\end{pmatrix} \mapsto \begin{pmatrix}0\\0\end{pmatrix},$$

and on $\mathbf{P}$, $\epsilon$ carries

$$\begin{pmatrix}1\\0\end{pmatrix} \mapsto \begin{pmatrix}1\\1\end{pmatrix} \mapsto \begin{pmatrix}0\\0\end{pmatrix}.$$

So $\sim$ identifies

$$\begin{pmatrix}1\\0\end{pmatrix} \otimes \epsilon \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\1\end{pmatrix}$$

with

$$\begin{pmatrix}1\\0\end{pmatrix} \epsilon \otimes \begin{pmatrix}1\\0\end{pmatrix} = 0,$$

and further identifies

$$\begin{pmatrix}0\\1\end{pmatrix} \epsilon \otimes \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix}$$

with

$$\begin{pmatrix}0\\1\end{pmatrix} \otimes \epsilon \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}1\\1\end{pmatrix}.$$

In this way, every element of $H(k^2)$ can be expressed in the form

$$\begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}a\\b\end{pmatrix}$$

with $a, b \in k$, and the relations are all accounted for. So we get the projective cover of the trivial representation in $kS_2$.

Note that we saw before that $kS_3 \cong kS_2 \oplus \mathrm{End}(k^2)$ as algebras, where $\mathrm{End}(k^2)$ represents the $k$-linear maps on the standard representation, which is irreducible in characteristic 2. So representations of $kS_3$ are the same as representations of the quiver

$$\epsilon \circlearrowright \bullet \qquad \bullet$$

i.e., they are given by direct sums of representations of $kS_2$ and of $\mathrm{End}(k^2)$.

## 5.3 Over the symmetric group on 3 elements in characteristic 3

Let $k = \overline{\mathbb{F}}_3$ and $G = S_3$. Let $s_1$ and $s_2$ be the trivial and sign representations, respectively, and let $P_1 = k^3$ and $P_2 = k^3_-$ be their projective covers. To avoid confusing the 3-vectors in $P_1$ and in $P_2$, I will mark vectors in each with a plus or a minus on top respectively. Now we look at endomorphisms of $\mathbf{P}$.

Define $\epsilon_1 : \mathbf{P} \to \mathbf{P}$ which sends $P_2$ to zero, and maps a generator of $P_1$

$$\begin{pmatrix}1\\0\\0\end{pmatrix}^{+} \mapsto \begin{pmatrix}1\\1\\1\end{pmatrix}^{+}$$

to the trivial subrepresentation of $P_1$ (this expresses the fact that $P_1$ has a maximal quotient and a subrepresentation both isomorphic to the trivial representation). Define $\epsilon_2 : \mathbf{P} \to \mathbf{P}$ similarly by

$$\begin{pmatrix}1\\0\\0\end{pmatrix}^{-} \mapsto \begin{pmatrix}1\\1\\1\end{pmatrix}^{-},$$

carrying a generator to the sign subrepresentation. Both of these maps satisfy $\epsilon_1^2 = \epsilon_2^2 = 0$.

There are also endomorphisms between $P_1$ and $P_2$. Let $\eta_{1,2}$ be a map which vanishes on $P_2$, and carries

$$\begin{pmatrix}1\\0\\0\end{pmatrix}^{+} \mapsto \begin{pmatrix}0\\1\\-1\end{pmatrix}^{-},$$

and similarly let $\eta_{2,1}$ carry

$$\begin{pmatrix}1\\0\\0\end{pmatrix}^{-} \mapsto \begin{pmatrix}0\\1\\-1\end{pmatrix}^{+}.$$

The interested reader can recall the filtration I have given in Chapter 4 on $P_1$, and determine the very similar filtration that can be given for $P_2$, to see an easy plausibility argument for why these four maps are well-defined homomorphisms of representations. For a proof, note that $\mathbf{P} \cong kG$, and see where $1 \in kG$ is sent.

These maps satisfy $\eta_{1,2}\eta_{2,1} = \epsilon_2$ and $\eta_{2,1}\eta_{1,2} = \epsilon_1$, and it can be verified that all other pairwise compositions of these four maps give the zero map. These four maps, together with the identity maps on $P_1$ and $P_2$, generate $\operatorname{End}(\mathbf{P})$ as a $k$-vector space. For $V$ a representation of $kG$, the functor $F$ gives vector spaces $M_1 = \operatorname{Hom}(P_1 \oplus \{0\}, V)$ and $M_2 = \operatorname{Hom}(\{0\} \oplus P_2, V)$ with the linear maps induced by $\epsilon_1, \epsilon_2, \eta_{1,2}, \eta_{2,1}$ between them. If one looks back at the definition of $F$, one can see that, for example, $\epsilon_{1,2}$ induces a map from $M_2$ to $M_1$, which is why I have reversed the labels in the following quiver:

$$\begin{tikzcd}
\bullet \arrow[loop left, "\epsilon_1"] \arrow[r, bend left, "\eta_{2,1}"] & \bullet \arrow[l, bend left, "\eta_{1,2}"] \arrow[loop right, "\epsilon_2"]
\end{tikzcd}$$

A representation of $A$ is a representation of the quiver with the restriction that the linear maps satisfy the relations I have stated in the previous paragraph, but with the multiplication in the reverse order.

More explicitly, if $V$ is an irreducible representation of $kG$, then $F(V) = M_1 \oplus M_2$, where the elements of $M_1$ and $M_2$ are $kG$-maps $m_1 : P_1 \to V$ and $m_2 : P_2 \to V$. We can put all of this together in a diagram indicating all the relevant maps:

$$\begin{tikzcd} P_1 \arrow[loop left, "\epsilon_1"] \arrow[rr, bend right, "\eta_{1,2}"'] \arrow[dr, "m_1"'] & & P_2 \arrow[ll, bend right, "\eta_{2,1}"'] \arrow[loop right, "\epsilon_2"] \arrow[dl, "m_2"] \\ & V & \end{tikzcd}$$

This quiver structure is significantly more complicated. Here we will see how $F$ behaves on three specific representations, namely the three quotient representations of $P_1$, from the filtration $\{0\} \subset V_1 \subset V_2 \subset P_1$ seen in Chapter 4. That is, we consider $P_1/V_2$, which is the trivial representation, $P_1/V_1$, a 2-dimensional representation which contains a sign subrepresentation and surjects onto a trivial representation, and all of $P_1$. Here I have only stated what $M_1$ and $M_2$ together with the induced maps are. The interested reader can draw a diagram of the structure of $P_1$ and $P_2$ together with the four maps between them, and see how each of the three quotient representations interact with these maps, in order to verify these claims.

For the trivial representation $P_1/V_2$, there is a surjection $m_1 : P_1 \twoheadrightarrow P_1/V_2$, so $M_1$ is a 1-dimensional vector space. There can be no nonzero map from $P_2$ to $P_1/V_2$, for that would imply $P_2$ surjects onto the trivial representation, which contradicts $P_2$ being the projective cover of the nonisomorphic sign representation. So $M_1$ is a 1-dimensional vector space and $M_2$ is the zero space. The induced maps by $\eta_{1,2}$, $\eta_{2,1}$ and $\epsilon_2$ must be zero as a consequence. The map induced by $\epsilon_1$ is also the zero map, since $\epsilon_1$ carries the generator of $P_1$ to the trivial subrepresentation of $P_1$, which is in the kernel of $m_1$. So all four induced maps are zero.

For the next quotient $P_1/V_1$, there is again a surjection $m_1 : P_1 \twoheadrightarrow P_1/V_1$ which spans $M_1$. If $M_1$ were 2-dimensional, then $P_1$ would have to surject onto the sign part of $P_1/V_1$, which it doesn't. However, there is a map $m_2 : P_2 \to P_1/V_1$ which surjects onto the sign part of $P_1/V_1$, so $M_2$ is 1-dimensional as well. As before, the maps induced by $\epsilon_1$ and $\epsilon_2$ are zero, because the images of $\epsilon_1$ and $\epsilon_2$ are in the kernel of $m_1$ and $m_2$. The map induced by $\eta_{1,2}$ is zero, because $\eta_{1,2}$ maps into $\ker m_2$. The map induced by $\eta_{2,1}$, on the other hand, is nonzero (so an isomorphism): $\eta_{2,1}$ maps $P_2$ into the sign part of $P_1$, which is then mapped by $m_1$ into the sign subrepresentation of $P_1/V_1$.

Finally, for all of $P_1$, we have already seen what the maps from $P_1$ to itself are and the maps from $P_2$ to $P_1$. Namely, these are the identity map $m_{1,id} : P_1 \to P_1$ and $m_{1,\epsilon} = \epsilon_1$, which span $M_1$, and $m_2 = \eta_{2,1}$ spans $M_2$. The map induced by $\epsilon_2$ is still zero, but the map induced by $\epsilon_1$, perhaps unsurprisingly, carries $m_{1,id}$ to $m_{1,\epsilon}$

and $m_{1,\epsilon}$ to 0. The map induced by $\eta_{1,2}$ carries $m_2$ to $m_{1,\epsilon}$. The map induced by $\eta_{2,1}$ carries $m_{1,id}$ to $m_2$, and $m_{1,\epsilon}$ to 0.

I end my analysis here, but there is plenty more work that can be done towards getting a full understanding of the representations of $kS_3$; here we've only seen how some representations of $kS_3$ are converted into representations of the quiver, but one could also go in the reverse direction, determining how the functor $H$ converts representations of $A$ into representations of $kS_3$, and work towards giving a classification of representations of both algebras. Beyond that, one could continue examining the representations of larger groups in prime characteristic. In all of our examples, $A$ was actually isomorphic to $kG$ (since $kG \cong \mathbf{P}$, End($\mathbf{P}$) $\cong$ $\text{Hom}_{kG}(kG, kG) \cong kG$), but nevertheless provided an alternative perspective for us to understand the representations. For larger groups, the algebra $A$ may turn out to be much simpler to study.

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