On the Optimal Stopping of Brownian Motion

by

Christopher Wells Miller

THESIS

Presented to the Department of Mathematics at

The University of Texas at Austin

in Partial Fulfillment

of the Requirements

for the Degree of

BACHELOR OF SCIENCE

THE UNIVERSITY OF TEXAS AT AUSTIN

May 2013

On the Optimal Stopping of Brownian Motion

APPROVED BY

SUPERVISING COMMITTEE:

Dr. Mihai Sîrbu, Supervisor

Dr. Gordan Žitković, Reader

Dr. David Rusin, Honors Advisor

I grant the Dean's Scholars Program permission to post a copy of this thesis on the University of Texas Digital Repository. For more information, visit http://repositories.lib.utexas.edu/about.

On the Optimal Stopping of Brownian Motion

Christopher Miller

Dr. Mihai Sîrbu

On the Optimal Stopping of Brownian Motion

Christopher Wells Miller, B.S.
The University of Texas at Austin, 2013

Supervisor: Dr. Mihai Sîrbu

In this thesis, first we briefly outline the general theory surrounding optimal stopping problems with respect primarily to Brownian motion and other continuous-time stochastic processes. In Chapter 1, we provide motivation for the type of problems encountered in this work, and illustrate their importance both mathematically and in terms of applications in science and engineering. In chapter 2, we briefly outline many of the technical aspects of probability theory and stochastic analysis, highlighting important theorems that will be used throughout. Chapter 3, which is he main part of the thesis, presents an optimal stopping problems related to the maximum of a process. This chapter also illustrates how problems in this field are often transformed into equivalent problems in which standard techniques apply. Finally, in Chapter 4, we provide a new problem along these same lines, outline a solution to it, and discuss the interesting elements of the problem.

Table of Contents

List of Figures

Chapter 1

Introduction

In this thesis, we explore the theory and techniques related to optimal stopping problem involving Brownian motion and related continuous-time stochastic processes. In particular, we emphasize how techniques in stochastic analysis are utilized to reduce optimal stopping problems into free-boundary problems, or reduce it to a question involving simpler objects, such as stochastic integrals or other martingales.

From a mathematical perspective, optimal stopping problems are a simple class of stochastic control problems involving a decision at each point in time about whether to stop the process and collect a reward or keep letting time evolve. This thesis emphasizes techniques for casting an optimal stopping problem into an associated free-boundary partial differential equation (PDE), as well as the verification techniques to ensure that a solution to the PDE corresponds to an optimal stopping time. This is one of many examples of the interplay between stochastic analysis and PDE.

From an applications point of view, the influence of optimal stopping problems and related questions in stochastic control is widespread. For example, in economics, engineering, and the life sciences, the primary goal is often to make optimal decisions when facing uncertainty. For example, the pricing of an American option, which is a stock option which can be exercised at any time prior to its maturity date, is an optimal stopping problem corresponding to when to exercise the option. Alternatively, an engineer may be interested in optimal control given noisy measurements of the state of a system. In general, stochastic control is a field interested in acting optimally given the information available at the current time in a system evolving randomly.

Chapter 2

Technical Introduction

In this section, we will give a brief outline of some major theorems in stochastic analysis, which will largely be assumed to be preliminary knowledge throughout this work. In particular, we assume that the reader is familiar with measure-theoretic probability and martingale theory [1]. For a comprehensive reference on continuous-time stochastic processes and stochastic calculus, we refer the reader to [4]. For information regarding optimal stopping problems and stochastic control, [7, 6] are excellent references.

2.1 Martingale Theory

We begin with the definition and basic properties of martingales. The intuition behind a martingale is that it is a stochastic process X_t, along with a filtration $\mathcal{F}_t$, that does not rise or fall in expectation over time. On the other hand, a sub- (super-)martingale will rise (fall) in expectation over time.

In general, the following definitions and theorems hold in both continuous- and discrete-time. For proofs to the following results, see Durrett [1].

Definition 2.1.1 (Martingale)**.** A (sub/super)martingale is a stochastic process X_t along with a filtration $\mathcal{F}_t$ such that:

- $X_t \in \mathcal{F}_t$ for all $t \geq 0$,
- $X_t \in L^1(\Omega)$ for all $t \geq 0$, and

- $\mathbb{E}\left[X_t|\mathcal{F}_s\right] = X_s$ a.s. (for a submartingale $\geq$, and for a supermartingale $\leq$) for all $0 \leq s \leq t$.

There are two key properties about martingales that will be useful throughout this thesis. First, under very general conditions they will converge. This allows for a consistent definition of X_∞ or X_τ, where τ may be an unbounded stopping time.

Theorem 2.1.1 (Martingale Convergence Theorem)**.** *If X_t is a submartingale with $\sup_t \mathbb{E}\left[X_t^+\right] < \infty$, then as $t \to \infty$, X_t converges almost surely to a measurable limit X with $\mathbb{E}|X| < \infty$.*

Remark 2.1.1. Because a martingale is also a submartingale, this convergence theorem extends to martingales.

The second key result involves the result of stopping a martingale. The intuition behind a stopping time is that it is a random time $\tau : \Omega \to [0, \infty]$ such that it is known whether or not τ has occurred or not at all times with the information in the filtration.

Definition 2.1.2 (Stopping Time)**.** Let $\mathcal{F}_t$ be a filtration. Then a stopping time is a random time $\tau : \Omega \to [0, \infty]$ such that $\{\tau \leq t\} \in \mathcal{F}_t$ for all $t \geq 0$.

One key property of martingales is that a stopped martingale is still a martingale. The Optional Sampling Theorem provides conditions for when a martingale stopped at a stopping time will still be a martingale. There are many variations on the so-called Optional Sampling Theorem, but one of the more general statements follows:

Theorem 2.1.2 (Optional Sampling Theorem)**.** *If $\sigma \leq \tau$ are stopping times and $X_{t\wedge\tau}$ is a uniformly integrable submartingale, then $\mathbb{E}X_\sigma \leq \mathbb{E}X_\tau$ and*

$$X_\sigma \leq \mathbb{E}\left[X_\tau|\mathcal{F}_\sigma\right].$$

2.2 Brownian Motion and Stochastic Integration

In this section, we briefly outline the definition and characterization of Brownian motion, as well as the key computational properties of the Itô stochastic integral. In addition, we state some theorems which are useful in characterizing the running maximum process associated with a Brownian motion, which will be used extensively in Chapter 3.

Definition 2.2.1 (Brownian Motion)**.** Consider a fixed filtration $\mathcal{F}_t$. Brownian motion is the unique (in law) adapted, continuous-time process B_t such that:

- $B_t - B_s$ is distributed as $N(0, t-s)$ for all $0 \leq s < t$,
- the increments $B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}}$ are independent for all $0 \leq t_1 < t_2 < \cdots < t_n$,
- $B_0 = 0$ almost surely, and
- the trajectory $t \mapsto B_t(\omega)$ is continuous.

While the above provides intuition about possible constructions of the Brownian motion, the following theorem is a powerful tool for recognizing when a given continuous-time stochastic process is a Brownian motion.

Theorem 2.2.1 (Lévy's Characterization Theorem)**.** *Let M_t be a continuous local $\mathcal{F}_t$-martingale on such that $M_0 = 0$ almost surely and with quadratic variation, $[M,M]_t = t$. Then M_t is a $\mathcal{F}_t$-Brownian motion.*

Theorem 2.2.2 (Itô's Isometry)**.** *Let M_t, N_t be continuous local martingales with respect to $\mathcal{F}_t$ and X_t, Y_t be continuous adapted processes with*

$$\int_0^t X_s^2 \, d[M,M]_s < \infty \text{ and } \int_0^t Y_s^2 \, d[N,N]_s < \infty \text{ for all } t, \text{ a.s.}$$

Then the Itô integral of X_t and Y_t with respect to M_t and N_t respectively are continuous local martingales with quadratic covariation given by

$$\left[\int_0^t X_s \, dM_s \int_0^t Y_s \, dN_s\right]_t = \int_0^t X_s Y_s \, d\left[M,N\right]_s .$$

Together, Itô's Isometry and Lévy's Characterization Theorem provide a powerful method of verifying when Itô integrals are themselves Brownian motions. This will provide many useful equivalences, such as the famous Lévy transform:

$$\int_0^t \operatorname{sign}(B_s)\, dB_s \overset{(d)}{=} B_t.$$

On the other hand, the following theorem is very useful in connecting differential equations to stochastic analysis.

Theorem 2.2.3 (Itô's Lemma)**.** *Let $f : \mathbb{R} \to \mathbb{R}$ be a C^2 function and X_t be a continuous semimartingale with respect to $\mathcal{F}_t$ which decomposes as*

$$X_t = X_0 + M_t + B_t,$$

where M_t is a local martingale and B_t is the difference of continuous, nondecreasing, adapted processes. Then

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)\, dM_s + \int_0^t f'(X_s)\, dB_s + \frac{1}{2}\int_0^t f''(X_s)\, d\,[M]_s\,.$$

Many issues in this thesis are associated with two important processes associated with a given stochastic process. First, we consider the running maximum process:

Definition 2.2.2 (Running Maximum Process)**.** Let X_t be a continuous stochastic process. We define a new stochastic process by

$$S_t = \sup_{0 \le s \le t} X_s,$$

for each $t \geq 0$. This is called the running maximum process associated with X_t.

Furthermore, we are often interested in the local time process associated with Brownian motion, L_t. This process characterizes the amount of time a Brownian motion spends at $B_t = 0$, and has the following important property:

Proposition 2.2.4 (Local Time)**.** *Let B_t be a Brownian. Then the associated local time L_t is a continuous stochastic process such that*

$$L_t = \lim_{\epsilon \to 0} \frac{1}{4\epsilon} \int_0^t 1_{\{|X_s| \leq \epsilon\}} \, ds,$$

for each $t \geq 0$.

In particular, the following lemma is useful in understanding the distributional properties of the running maximum process associated with B_t:

Lemma 2.2.5 (Reflection Principle)**.** *Let B_t be a Brownian motion and S_t be the associated running maximum process. Then for any $a \geq 0$, we have:*

$$\mathbb{P}\left[S_t \geq a, B_t \geq a\right] = \mathbb{P}\left[S_t \geq a, B_t \leq a\right].$$

The intuition behind this lemma follows from the fact that Brownian motion has symmetric, independent increments. As illustrated below, the idea is that for every Brownian path which hits a certain level, there is a one-to-one correspondence between paths which end below that level and those that end above that level, given explicitly by reflecting across the hitting level.

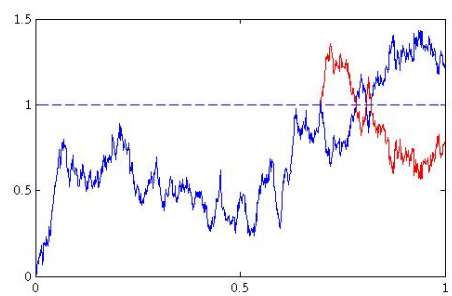

Figure 2.1: Illustration of the reflection principle of a Brownian motion. This shows a particular Brownian path B_t (in blue), as well as its reflected process $\tilde{B}_t$ after hitting the level one (in red).

Proof. Let $a \geq 0$ and consider the reflected Brownian motion:

$$\tilde{B}_t = \begin{cases} B_t, & t \leq \tau_a \\ B_\tau - (B_t - B_\tau), & t > \tau_a \end{cases}$$

where $\tau_a = \inf\{t \geq 0 : B_t = a\}$ is the first hitting time of a by B_t. Then we note that:

$$\tilde{B}_t = \int_0^t \left(2\,1_{\{t \leq \tau\}} - 1\right)\, dB_s,$$

and so by Itô's Isometry and the Lévy Characterization Theorem, we conclude that $\tilde{B}_t$ is a Brownian motion.

Let $\tilde{S}_t$ be the running maximum process associated with $\tilde{B}_t$. Then we have shown that

$$(B_t, S_t) \stackrel{(d)}{=} (\tilde{B}_t, \tilde{S}_t).$$

Now, note that $\{\tau_a \leq t\} = \{S_t \geq a\} = \{\tilde{S}_t \geq a\}$ and also that

$$\{\tau_a \leq t\} \cap \{B_t \geq a\} = \{\tau_a \leq t\} \cap \{\tilde{B}_t \leq a\}.$$

Putting this all together, we conclude that

$$\begin{aligned} \mathbb{P}\left[S_t \geq a, B_t \geq a\right] &= \mathbb{P}\left[\tau_a \leq t, B_t \geq a\right] \\ &= \mathbb{P}\left[\tau_a \leq t, \tilde{B}_t \leq a\right] \\ &= \mathbb{P}\left[\tilde{S}_t \geq a, \tilde{B}_t \leq a\right] \\ &= \mathbb{P}\left[S_t \geq a, B_t \leq a\right]. \end{aligned}$$

□

Using the reflection principle, we can immediately compute the distribution of the running maximum process of a Brownian motion, S_t.

Theorem 2.2.6. *Let B_t be a Brownian motion and S_t be the corresponding running maximum process. Then for each fixed $t \geq 0$, the random variables S_t and $|B_t|$ each are equal in distribution. In particular, S_t is square-integrable.*

Proof. This is trivial at $t = 0$. For $t > 0$, it is sufficient to consider $t = 1$ by scaling because we are looking at individual times. Recall from the Reflection Principle that for $a \geq 0$ we have:

$$\mathbb{P}\left[S_1 \geq a, B_1 \geq a\right] = \mathbb{P}\left[S_1 \geq a, B_1 \leq a\right].$$

Then we can compute immediately that

$$\begin{aligned}\mathbb{P}\left[S_1 \geq a\right] &= \mathbb{P}\left[S_1 \geq a, B_1 \geq a\right] + \mathbb{P}\left[S_1 \geq a, B_1 \leq a\right] \\ &= 2\mathbb{P}\left[S_1 \geq a, B_1 \geq a\right] \\ &= 2\mathbb{P}\left[B_1 \geq a\right] \\ &= 2\int_a^\infty \phi(x)\,dx \\ &= \mathbb{P}\left[|B_1| \geq a\right],\end{aligned}$$

where ϕ is the probability density of a standard normal. Note, in the above computation, we used the fact that for $t > 0$, B_t has zero probability of being at any individual point.

Then we conclude that

$$S_t \overset{(d)}{=} |B_t|.$$

In particular, S_t is then square-integrable, which will be useful in applying the following theorem.

□

The following result is useful in representing square-integrable random variables in terms of an Itô integral with respect to a Brownian motion.

Theorem 2.2.7 (Martingale Representation Theorem)**.** *Let B_t be a Brownian motion and $\mathcal{F}_t$ be its natural filtration, augmented by null sets. If M_t is an L^2 martingale with respect to this same filtration, then there exists a predictable process H_t such that*

$$M_t = M_0 + \int_0^t H_s\,dB_s$$

for all $t \geq 0$.

Finally, the Skorohod equation will prove useful in characterizing the relationship between the running maximum process of a Brownian motion and its local time process.

Lemma 2.2.8 (Skorohod equation)**.** *Let $y \in C^0([0,\infty))$ such that $y(0) = 0$. Then there exists a unique $k \in C^0([0,\infty))$ such that:*

1. $x(t) \triangleq y(t) + k(t) \geq 0$ *for all* $t \in [0,\infty)$,
2. $k(0) = 0$ *and* k *is non-decreasing, and*
3. $\int_0^\infty 1_{\{x(s)>0\}} dk(s) = 0$. *Furthermore, this function is given by*

$$k(t) = \max_{0\leq s\leq t}\{-y(s)\}.$$

For a proof of the Skorohod equation lemma, refer to [4]. An important application of the Skorohod equation is the following result, which will be used in Chapter 3:

Theorem 2.2.9. *Consider a Brownian motion B_t, its running maximum process S_t, and its local time process at zero L_t. Then the following equality in distribution holds:*

$$(S_t - B_t, B_t, S_t) \stackrel{(d)}{=} \left(|B_t|, -\int_0^t signB_s\, dB_s, 2L_t\right).$$

Proof. This is an application of the Skorohod equation. First, note that choosing $y = -B_t$ and $k = S_t$, we satisfy the conditions of the Skorohod equation because

- $S_t - B_t \geq 0$,
- S_t is non-decreasing with $S_0 = 0$ almost surely, and
- S_t only increases when $S_t = B_t$.

On the other hand, if we recall the Tanaka formula that

$$|B_t| = \int_0^t \operatorname{sign} B_s \, dB_s + 2L_t,$$

and choose $y = \int_0^t \operatorname{sign} B_s \, dB_s$ and $k = 2L_t$, then this also satisfies the Skorohod equation because

- $|B_t| = \int_0^t \operatorname{sign} B_s \, dB_s + 2L_t \geq 0$,
- $2L_t$ is non-decreasing with $2L_0 = 0$ almost surely, and
- $2L_t$ only increases when $|B_t| = 0$.

However, by Lévy's characterization theorem, we conclude that $\int_0^t \operatorname{sign} B_s \, dB_s$ is a Brownian motion, so it is equal in distribution to B_t for all t. However, by the uniqueness of the Skorohod equation, we then see that

$$(S_t - B_t, B_t, S_t) \stackrel{(d)}{=} \left(|B_t|, -\int_0^t \operatorname{sign} B_s \, dB_s, 2L_t \right).$$

$\square$

2.3 Optimal Stopping

In this thesis, we consider a particular class of optimal stopping problems that minimize (rather than maximize) a pay-off cost. While the methods used to identify optimal stopping rules vary, the following result is often useful in proving optimality.

Lemma 2.3.1 (Optimal Stopping Condition). *Let X_t be a Markov process with respect to a filtration $\mathcal{F}_t$. Consider the abstract optimal stopping problem:*

$$\inf_{\tau \in \mathcal{M}} \mathbb{E}^x \left[e^{-r\tau} g(X_\tau) + \int_0^\tau e^{-rs} f(X_s) \, ds \right],$$

where $r \geq 0$, $x \in \mathbb{R}$, $\mathcal{M}$ consists of all almost surely finite $\mathcal{F}_t$-stopping times, and $\mathbb{E}^x$ is taken respect to a measure $\mathbb{P}^x$ where $X_0 = x$ almost surely. Consider a function v such that the following conditions hold:

1. $v(x) \leq g(x)$ *for all* $x \in \mathbb{R}$,

2. $e^{-rt}v(X_t) + \int_0^t e^{-rs} f(X_s)\,ds$ *is a submartingale, and*

3. $e^{-rt\wedge\tau_*}v(X_{t\wedge\tau_*}) + \int_0^{t\wedge\tau_*} e^{-rs} f(X_s)\,ds$ *is a uniformly integrable martingale*

where $\tau_* = \inf\{t \geq 0 : v(X_t) = g(X_t)\}$ *is a stopping time and almost surely finite. Then* τ_* *attains the infimum above and*

$$v(x) = \mathbb{E}^x\left[e^{-r\tau_*}g(X_{\tau_*}) + \int_0^{\tau_*} e^{-rs} f(X_s)\,ds\right].$$

Proof. Let τ be any stopping time. Then by conditions 1 and 2, we see that

$$\begin{aligned}
\mathbb{E}^x\left[e^{-r\tau}g(X_\tau) + \int_0^{\tau} e^{-rs} f(X_s)\,ds\right] &\geq \mathbb{E}^x\left[e^{-r\tau}v(X_\tau) + \int_0^{\tau} e^{-rs} f(X_s)\,ds\right] \\
&\geq \mathbb{E}^x\left[v(X_0) + \int_0^{0} e^{-rs} f(X_s)\,ds\right] \\
&= v(x).
\end{aligned}$$

Therefore, it is sufficient to show that τ_* attains the value $v(x)$.

By condition 3 we know that $e^{-rt\wedge\tau_*}v(X_{t\wedge\tau_*}) + \int_0^{t\wedge\tau_*} e^{-rs} f(X_s)\,ds$ is a uniformly integrable martingale. Because it is uniformly integrable, we can apply Optimal Sampling up to time τ_*, which may be unbounded, but is assumed to be almost surely finite. Using the definition of τ_*, we conclude that

$$\begin{aligned}
\mathbb{E}^x\left[e^{-r\tau_*}g(X_{\tau_*}) + \int_0^{\tau_*} e^{-rs} f(X_s)\,ds\right] &= \mathbb{E}^x\left[e^{-r\tau_*}v(X_{\tau_*}) + \int_0^{\tau_*} e^{-rs} f(X_s)\,ds\right] \\
&= \mathbb{E}^x\left[e^{-rt\wedge\tau_*}v(X_{t\wedge\tau_*}) + \int_0^{t\wedge\tau_*} e^{-rs} f(X_s)\,ds\right] \\
&= \mathbb{E}^x\left[v(X_0) + \int_0^{0} e^{-rs} f(X_s)\,ds\right] \\
&= v(x).
\end{aligned}$$

Then τ_* is an optimal stopping time which achieves the optimal value $v(x)$.

$\square$

Chapter 3

Stopping Brownian Motion without Anticipation

This chapter, which is the main part of the thesis, presents the results in [3] related to stopping a process close to its ultimate maximum. Some important techniques and ideas are emphasized in more detail than in the original work.

In [3], the problem considered is to stop a Brownian motion as close as possible to its ultimate maximum on the interval $[0, 1]$. Let B_t be a standard Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathbb{F}_t^B$ be the natural filtration generated by B. If $\mathcal{M}_1$ denotes the set of all stopping times τ such that $0 \leq \tau \leq 1$, then we hope to determine

$$v_* = \inf_{\tau \in \mathcal{M}_1} \mathbb{E} \left(B_\tau - S_1 \right)^2 ,$$

where $S_1 = \sup_{0 \leq t \leq 1} B_t$ is the ultimate maximum of B_t on $[0, 1]$.

The outlined solution to this problem illustrates several important concepts in stochastic control. First, note that while the Brownian motion process is adapted, the ultimate maximum is not adapted. Then the first major challenge is to study the properties of the ultimate maximum random variable

$$S_1 = \sup_{0 \leq t \leq 1} B_t,$$

and, in particular, relate it to a stochastic process which is adapted with respect to $\mathcal{F}_t^B$.

3.1 Ultimate Maximum of the Brownian Motion

In this section, we explore the properties of the ultimate maximum random variable, S_1, which is defined above.

First, we want to explore the distributional properties of the random variable S_1. In particular, if we view this random variable in terms of the running maximum process S_t corresponding to the Brownian motion, we see from the results in the Technical Introduction that $S_1 \overset{(d)}{=} |B_1|$.

Furthermore, S_1 is a square-integrable random variable, and hence by the Martingale Representation Theorem can be written in the form

$$S_1 = \mathbb{E}S_1 + \int_0^1 H_s \, dB_s$$

for some predictable process H_s. However, this theorem does not provide a construction for the process H_t. Fortunately, we have the following theorem:

Theorem 3.1.1. *Let B_t be a Brownian motion and S_t be its associated running maximum process. Then*

$$S_1 = \mathbb{E}S_1 + \int_0^1 H_s \, dB_s$$

for

$$H_t = 2\left(1 - \Phi\left(\frac{S_t - B_t}{\sqrt{1-t}}\right)\right)$$

where

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx$$

is the cumulative distribution function of the unit normal.

Proof. Let $\mathcal{F}_t$ be the natural filtration corresponding to B_t. Using the independent increments of Brownian motion, we note that:

$$\begin{aligned}
\mathbb{E}\left[S_1|\mathcal{F}_t\right] &= S_t + \mathbb{E}\left[\left(\sup_{t\le s\le 1} B_s - B_t\right)^+ |\mathcal{F}_t\right] \\
&= S_t + \mathbb{E}\left[\left(\sup_{t\le s\le 1} (B_s - B_t) - (S_t - B_t)\right)^+ |\mathcal{F}_t\right] \\
&= S_t + \mathbb{E}\left[(S_{1-t} - c)^+\right],
\end{aligned}$$

where $c = S_t - B_t$.

Next, we note that for all $c \geq 0$,

$$\begin{aligned}
\mathbb{E}\left[(S_{1-t} - c)^+\right] &= \int_c^\infty \mathbb{P}\left[S_{1-t} > z\right] dz \\
&= \int_c^\infty \mathbb{P}\left[|B_{1-t}| > z\right] dz \\
&= \int_c^\infty \mathbb{P}\left[|B_1| > \frac{z}{\sqrt{1-t}}\right] dz \\
&= \int_c^\infty 2\left(1 - \Phi\left(\frac{z}{\sqrt{1-t}}\right)\right) dz
\end{aligned}$$

Putting these two results together, we conclude that

$$\mathbb{E}\left[S_1|\mathcal{F}_t\right] = \mathbb{E}S_1 + \int_{S_t - B_t}^\infty 2\left(1 - \Phi\left(\frac{z}{\sqrt{1-t}}\right)\right) dz.$$

Next, we apply Itô's lemma to the function:

$$F(t,x) = \int_x^\infty 2\left(1 - \Phi\left(\frac{z}{\sqrt{1-t}}\right)\right) dz.$$

In particular, we note that

$$\begin{aligned}
\frac{\partial F}{\partial t} &= \int_x^\infty \frac{-z}{(1-t)^{3/2}} \phi\left(\frac{z}{\sqrt{1-t}}\right) dz \\
&= -\frac{1}{\sqrt{1-t}} \phi\left(\frac{x}{\sqrt{1-t}}\right).
\end{aligned}$$

Also, we see that

$$\frac{\partial F}{\partial x} = -2\left(1 - \Phi\left(\frac{x}{\sqrt{1-t}}\right)\right),$$

and

$$\frac{\partial^2 F}{\partial x^2} = \frac{2}{\sqrt{1-t}}\phi\left(\frac{x}{\sqrt{1-t}}\right).$$

Furthermore, because S_t is monotonic, it has finite variation, so we immediately see that $d[S,S]_s = 0$ and $d[S,B]_s = 0$.

Putting this together in Itô's lemma,

$$\begin{aligned} S_1 &= \mathbb{E}\left[S_1 | \mathcal{F}_1\right] \\ &= \mathbb{E}S_1 - \int_0^1 \frac{1}{\sqrt{1-t}}\phi\left(\frac{S_t - B_t}{\sqrt{1-t}}\right) dt - \int_0^1 2\left(1 - \Phi\left(\frac{S_t - B_t}{\sqrt{1-t}}\right)\right) d\left(S - B\right)_t \\ &\quad + \int_0^1 \frac{1}{\sqrt{1-t}}\phi\left(\frac{S_t - B_t}{\sqrt{1-t}}\right) d[S-B, S-B]_t \\ &= \mathbb{E}S_1 - \int_0^1 2\left(1 - \Phi\left(\frac{S_t - B_t}{\sqrt{1-t}}\right)\right) dS_t + \int_0^1 2\left(1 - \Phi\left(\frac{S_t - B_t}{\sqrt{1-t}}\right)\right) dB_t. \end{aligned}$$

Lastly, we note that S_t is flat any time $S_t \neq B_t$. Therefore, the dS_t integral is zero, leaving

$$S_1 = \mathbb{E}S_1 + \int_0^1 2\left(1 - \Phi\left(\frac{S_t - B_t}{\sqrt{1-t}}\right)\right) dB_t.$$

□

3.2 Equivalent problem

The main issue with the original optimal stopping problem is the fact that not all components are adapted to $\mathbb{F}_t$. Now that we have obtained an expression for S_1 in terms of an adapted process, we develop an equivalent problem.

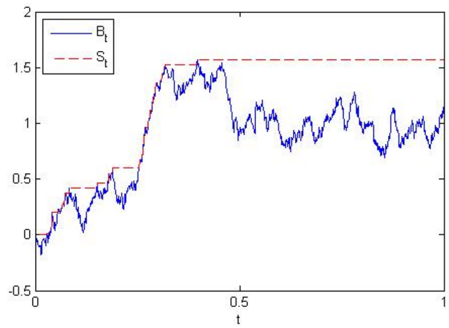

Figure 3.1: Illustration of the ultimate maximum process of Brownian motion, S_t. This illustrates one possible outcome next to its corresponding Brownian path, B_t.

From this point on, we define the process:

$$H_t = 2\left(1 - \Phi\left(\frac{S_t - B_t}{\sqrt{1-t}}\right)\right)$$

and the corresponding integrated process:

$$M_t = \int_0^t H_s\, dB_s,$$

such that $S_1 = \mathbb{E}S_1 + M_1$.

Now, by the martingale properties of stochastic integrals, the Optional Sampling Theorem, and the Itô isometry, we note that for any stopping time τ, we have:

$$\begin{aligned}
\mathbb{E}\left[B_\tau - S_1\right]^2 &= \mathbb{E}\left[B_\tau^2 - 2S_1B_\tau + S_1^2\right] \\
&= \mathbb{E}\left[\tau\right] - 2\mathbb{E}S_1\mathbb{E}\left[B_\tau\right] - 2\mathbb{E}\left[\int_0^1 H_t\, dB_t \int_0^\tau 1\, dB_t\right] + \mathbb{E}|S_1|^2 \\
&= \mathbb{E}\left[\tau\right] - 2\mathbb{E}\left[\int_0^\tau H_t\, dt\right] + 1 \\
&= \mathbb{E}\left[\int_0^\tau \left(1 - 2H_t\right)\, dt + 1\right].
\end{aligned}$$

Then the original optimal stopping problem may be written in terms of an expectation of an integral,

$$v_* = \inf_{\tau \in \mathcal{M}_1} \mathbb{E}\left[\int_0^\tau (1 - 2H_t)\, dt + 1\right] = \inf_{\tau \in \mathcal{M}_1} \mathbb{E}\left[\int_0^\tau F\left(\frac{S_t - B_t}{\sqrt{1-t}}\right) dt\right] + 1,$$

where $F(x) = 4\Phi(x) - 3$.

Recalling that $S_t - B_t \stackrel{(d)}{=} |B_t|$, it is tempting to rewrite the above optimal stopping problem in terms of a Markov process,

$$v_* = \inf_{\tau \in \mathcal{M}_1^{|B|}} \mathbb{E}\left[\int_0^\tau F\left(\frac{|B_t|}{\sqrt{1-t}}\right) dt\right] + 1,$$

where $\mathcal{M}_1^{|B|}$ represents the set of all $\mathcal{F}^{|B|}$-stopping times τ with $0 \leq \tau \leq 1$ almost surely.

$$\begin{aligned} v_* &= \inf_{\tau \in \mathcal{M}_1^{B}} \mathbb{E}\left[\int_0^\tau F\left(\frac{S_t - B_t}{\sqrt{1-t}}\right) dt\right] + 1 \\ &= \inf_{\tau \in \mathcal{M}_1^{S-B}} \mathbb{E}\left[\int_0^\tau F\left(\frac{S_t - B_t}{\sqrt{1-t}}\right) dt\right] + 1 \\ &= \inf_{\tau \in \mathcal{M}_1^{|B|}} \mathbb{E}\left[\int_0^\tau F\left(\frac{|B_t|}{\sqrt{1-t}}\right) dt\right] + 1, \end{aligned}$$

where above we add a superscript to each set of stopping times to emphasize which natural filtration it must correspond to at each point. There are several key subtleties involved in this transformation, however.

The first issue is that we must check that the filtration $\mathcal{F}^B$ and $\mathcal{F}^{S-B}$ coincide.

Lemma 3.2.1. *Let B_t be a Brownian motion and S_t be the corresponding running maximum process. Then the natural augmented filtrations of B and $S - B$ are equal,*

$$\mathcal{F}_t^B = \mathcal{F}_t^{S-B}.$$

Proof. Note that it is clear that $\mathcal{F}_t^{S-B} \subset \mathcal{F}_t^B$ because for each $0 \le s \le t$ we have

$$S_s - B_s = \left(\sup_{0\le r\le s} B_r\right) - B_s \in \mathcal{F}_t^B.$$

The key idea of the converse is to recall from the Technical Introduction that, for a Brownian motion B_t, its running maximum process S_t, and its local time process at zero L_t, we have the following equality in distribution:

$$(S_t - B_t, B_t, S_t) \overset{(d)}{=} \left(|B_t|, -\int_0^t \mathrm{sign} B_s \, dB_s, 2L_t\right).$$

In addition, recall that

$$L_t = \lim_{\epsilon\to 0} \frac{1}{4\epsilon} \int_0^t 1_{\{|B_t|\le\epsilon\}} \, ds,$$

so we claim that by the above equivalence in distribution

$$S_t = \lim_{\epsilon\to 0} \frac{1}{2\epsilon} \int_0^t 1_{\{S_t - B_t\le\epsilon\}} \, ds.$$

But now, we this in hand, we see that

$$B_t = S_t - (S_t - B_t) \in \mathcal{F}_t^{S-B}.$$

□

The second issue is that we have transformed the problem into the form

$$\inf_{\tau\in\mathcal{M}_1^{|B|}} \mathbb{E}\left[\int_0^\tau F\left(\frac{|B_t|}{\sqrt{1-t}}\right) dt\right] + 1,$$

where $\mathcal{M}_1^{|B|}$ is the set of all $\mathcal{F}^{|B|}$-stopping times with $0 \le \tau \le 1$ almost surely. However, the theory of optimal stopping times and their relationship to free-boundary problems is developed for $\mathcal{F}^B$-stopping times, which is a larger set. Therefore,

$$\inf_{\tau\in\mathcal{M}_1^{|B|}} \mathbb{E}\left[\int_0^\tau F\left(\frac{|B_t|}{\sqrt{1-t}}\right) dt\right] \ge \inf_{\tau\in\mathcal{M}_1^{B}} \mathbb{E}\left[\int_0^\tau F\left(\frac{|B_t|}{\sqrt{1-t}}\right) dt\right].$$

However, if we obtain a solution $\tau_* \in \mathcal{M}_1^B$ to this latter problem, and verify that it is in fact a $\mathcal{F}^{|B|}$-stopping time as well, then it corresponds to a solution to the original problem.

3.3 Time rescaling

At this point, the idea is to change the problem from a optimal stopping problem on the time interval $[0,1]$ to an equivalent optimal stopping problem on the time interval $[0,\infty)$ in order to reduce it to a one-dimensional free-boundary problem. This is done by a change of variables.

Consider the process $\{Z_t\}_{t\geq 0}$ defined as

$$Z_t = e^t B_{1-e^{-2t}}.$$

We note, first, that by Itô's formula

$$\begin{aligned} dZ_t &= e^t B_{1-e^{-2t}}\,dt + d\int_0^t e^s\,dB_{1-e^{-2s}} \\ &= Z_t\,dt + \sqrt{2}\,d\beta_t, \end{aligned}$$

where

$$\beta_t = \frac{1}{\sqrt{2}}\int_0^t e^s\,dB_{1-e^{-2s}} = \frac{1}{\sqrt{2}}\int_0^{1-e^{-2t}} \frac{dB_s}{\sqrt{1-s}}.$$

Now, it follows immediately that β_t is a continuous martingale with zero mean. Using Itô's isometry, we verify that

$$\begin{aligned} \mathrm{Var}\left[\beta_t\right] &= \frac{1}{2}\mathbb{E}\left[\left(\int_0^{1-e^{-2t}} \frac{dB_s}{\sqrt{1-s}}\right)^2\right] \\ &= \frac{1}{2}\mathbb{E}\left[\int_0^{1-e^{-2t}} \frac{ds}{1-s}\right] = t. \end{aligned}$$

Then by Lévy's characterization of Brownian motion, we conclude that β_t is a Brownian motion.

Furthermore, we can obtain an equivalent optimal stopping problem in terms of this process Z_t by performing the following change of variables:

$$\begin{aligned} t &= 1-e^{-2s} \\ dt &= 2e^{-2s}\,ds. \end{aligned}$$

Then the integral above becomes:

$$\begin{aligned}\int_0^{\tau} F\left(\frac{|B_t|}{\sqrt{1-t}}\right)\,dt &= \int_0^{-\frac{1}{2}\log(1-\tau)} 2e^{-2s}F\left(|e^s B_{1-e^{-2s}}|\right)\,ds \\ &= \int_0^{\sigma} 2e^{-2s}F(|Z_s|)\,ds,\end{aligned}$$

where $\sigma = -\frac{1}{2}\log(1-\tau)$.

One again, it is necessary to verify that this change of variables respects the filtration. Consider the following result:

Theorem 3.3.1. *Consider a filtration $\mathcal{F}_t$, a random time τ, and*

$$\begin{aligned}\phi &: [0,1) \to [0,\infty) \\ &: t \mapsto -\frac{1}{2}\log(1-t).\end{aligned}$$

Then the random time $\sigma = \phi(\tau)$ is a stopping time with respect to the filtration $\mathcal{F}_{\phi^{-1}(t)}$ if and only if τ is a stopping time with respect to $\mathbb{F}_t$.

Proof. There are two key properties to note about ϕ that allow this change of variables to respect stopping times. First, ϕ is a strictly increasing continuous function, and second, it has a well-defined inverse function ϕ^{-1}. Then the result follows easily from the way ϕ interacts with inequalities.

Let τ be a stopping time with respect to $\mathcal{F}_t$. Then it follows from the definition that

$$\{\phi(\tau) \le t\} = \{\tau \le \phi^{-1}(t)\} \in \mathcal{F}_{\phi^{-1}(t)}.$$

Therefore, $\phi(\tau)$ is a stopping time with respect to $\mathcal{F}_{\phi^{-1}(t)}$.

On the other hand, let $\phi(\tau)$ be a stopping time with respect to $\mathcal{F}_{\phi^{-1}(t)}$. Then it follows similarly that

$$\{\tau \le t\} = \{\phi(\tau) \le \phi(t)\} \in \mathcal{F}_{\phi^{-1}\circ\phi(t)} = \mathcal{F}_t.$$

Therefore, τ is a stopping time with respect to $\mathcal{F}_t$.

$\square$

3.4 Formulation as a free-boundary problem

At this point, we have a related optimal stopping problem:

$$v_* = 2 \inf_{\sigma \in \mathcal{M}} \mathbb{E} \left[\int_0^\sigma e^{-2t} F(|Z_t|) \, dt \right] + 1,$$

where $\mathcal{M}$ is the set of all $\mathcal{F}^B$-stopping times and the expectation is with respect to a Markov process. Then the general theory of optimal stopping problems suggests how to proceed [7].

Now, we define a related family of optimal stopping problem:

$$W(z) = \inf_{\sigma \in \mathcal{M}} \mathbb{E}^z \left[\int_0^\sigma e^{-2t} F(|Z_t|) \, dt \right]$$

for each $z \in \mathbb{R}$. In the previous definition, the expectation is taken with respect to a measure $\mathbb{P}^z$ where $Z_0 = z$ almost surely. This function represents the optimal value value achievable if the process Z_t is started at $Z_0 = z$.

Note that $v_* = 2W(0) + 1$. Then it is sufficient to solve for $W(0)$ in order to determine v_*. We expect that the optimal stopping time will take the form of a hitting time,

$$\sigma_* = \inf\{t > 0 : |Z_t| \geq z_*\},$$

for some $z_* > 0$ to be determined. This assumption will be justified in the following section when we prove that the resulting stopping time is indeed optimal.

A common technique in optimal stopping problems is to convert to a free-boundary problem. We proceed formally, and justify the computations in the following section by proving that the solution obtained is in fact an optimal stopping time.

First, assume that W is sufficiently regular to apply Itô's lemma, i.e. $V \in C^2$. Then for any small $z \in \mathbb{R}$ and any small $h > 0$, it is atleast as good (but possibly worse) to wait for time h and then stop optimally after h. We formalize this by writing that:

$$W(z) \leq \mathbb{E}^z \left[\int_0^h e^{-2t} F(|Z_t|)\, dt + e^{-2h} W(Z_h) \right].$$

Now, recall that Z_t is an Itô process satisfying the stochastic differential equation

$$dZ_t = Z_t\, dt + \sqrt{2}\, d\beta_t,$$

where β_t is a Brownian motion. Then we know that the quadratic variation process is an Itô process satisfying

$$d[Z, Z]_t = 2\, dt.$$

Now, applying Itô's lemma to the inequality above, we note that:

$$\begin{aligned}
W(z) &\leq \mathbb{E}^z \left[\int_0^h e^{-2t} F(|Z_t|)\, dt + e^{-2h} W(Z_h) \right] \\
&= \mathbb{E}^z \left[\int_0^h e^{-2t} \left(F(|Z_t|) - 2W(Z_t) \right)\, dt + W(z) + \int_0^h e^{-2t} W'(Z_t)\, dZ_t \right. \\
&\quad \left. + \int_0^h \frac{1}{2} e^{-2t} W''(Z_t)\, d[Z, Z]_t \right] \\
&= \mathbb{E}^z \left[W(z) + \int_0^h e^{-2t} \left(F(|Z_t|) - 2W(Z_t) + Z_t W'(Z_t) + W''(Z_t) \right)\, dt \right. \\
&\quad \left. + \int_0^h \sqrt{2} e^{-2t} W'(Z_t)\, d\beta_t \right] \\
&= W(z) + \mathbb{E}^z \left[\int_0^h e^{-2t} \left(F(|Z_t|) - 2W(Z_t) + Z_t W'(Z_t) + W''(Z_t) \right)\, dt \right].
\end{aligned}$$

Now, this implies that

$$0 \leq \mathbb{E}^z \left[\int_0^h e^{-2t} \left(F(|Z_t|) - 2W(Z_t) + Z_t W'(Z_t) + W''(Z_t) \right)\, dt \right],$$

but formally at small $h > 0$, this suggests that

$$0 \leq F(|z|) - 2W(z) + zW'(z) + W''(z).$$

From here, we note that if $\pm z_*$ is the value corresponding to the hitting time for an optimal stopping time, then $W(\pm z_*) = 0$. Then we obtain the following free-boundary problem, which will be justified rigorously in the next section after obtaining a solution.

$$\begin{cases} \left(-2 + z\frac{d}{dz} + \frac{d^2}{dz^2}\right) W(z) = -F(|z|) \quad \text{for } z \in (-z_*, z_*) \\ W(\pm z_*) = 0 \\ W'(\pm z_*) = 0 \end{cases}$$

The idea of the derivative condition is to ensure that the extension of the value function to a zero function on $|z| > z_*$ will be as smooth as possible, while still ensuring a unique solution to this free-boundary problem. It will be seen in the next section that a C^2 solution is ideal for application of Itô's lemma, but this particular problem will result in a solution which is C^2 at all but $\pm z_*$. However, the solution is C^1 everywhere, which is called the smooth fit principle.

By the symmetry of the problem, we expect that the solution $W(z)$ will be an even function. Therefore, we can assume that $W'(0) = 0$ and consider the ordinary differential equation:

$$\begin{cases} \left(-2 + z\frac{d}{dz} + \frac{d^2}{dz^2}\right) W(z) = -F(|z|) \quad \text{for } z \in (0, z_*) \\ W'(0) = 0 \\ W(z_*) = 0 \\ W'(z_*) = 0 \end{cases}$$

whose general solution may be verified to be given by [3]:

$$W(z) = C_1\left(1 + z^2\right) + C_2\left(z\phi(z) + (1 + z^2)\Phi(z)\right) + 2\Phi(z) - \frac{3}{2}.$$

To determine the solution with the boundary conditions, we first compute that

$$W'(z) = 2C_1 z + 2C_2\left(\phi(z) + z\Phi(z)\right) + 2\phi(z).$$

Now, we first use the even symmetry condition to find that

$$W'(0) = 2C_2 + 2 = 0,$$

so $C_2 = -1$.

Next, using the smooth fit condition, we see that

$$\begin{aligned} 0 &= W'(z_*) \\ &= 2C_1 z_* - 2\left(\phi(z_*) + z_*\Phi(z_*)\right) + 2\phi(z_*) \\ &= 2C_1 z_* - 2z_*\Phi(z_*), \end{aligned}$$

so $C_1 = \Phi(z_*)$.

Finally, we use the continuity condition to note that

$$\begin{aligned} 0 &= W(z_*) \\ &= \Phi(z_*)\left(1 + z_*^2\right) - \left(z_*\phi(z_*) + (1 + z_*^2)\Phi(z_*)\right) + 2\Phi(z_*) - \frac{3}{2} \\ &= 2\Phi(z_*) - z_*\phi(z_*) - \frac{3}{2}. \end{aligned}$$

We define a function $f : [0, \infty) \to \mathbb{R}$ as

$$f(z) = 2\Phi(z) - z\phi(z) - \frac{3}{2}.$$

It remains to show that f has one unique root.

Note that f is clearly continuous, $f(0) = -\frac{1}{2}$,

$$\lim_{z \to \infty} f(z) = \frac{1}{2},$$

and f is strictly increasing because

$$f'(z) = 2\phi(z) - \phi(z) + z^2\phi(z) = (1 + z^2)\phi(z) > 0.$$

Therefore, we conclude that f has a unique root, z_*, such that $f(z_*) = 0$. Numerically this is given by approximately $z_* \approx 1.1229\cdots$, as illustrated below:

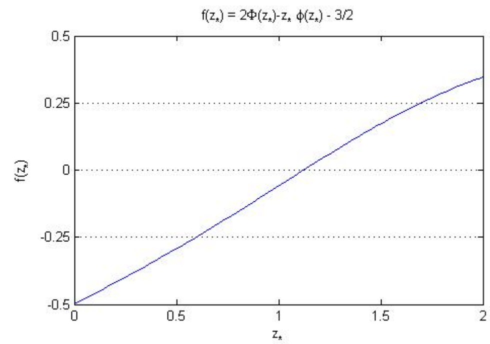

Figure 3.2: Illustration of the function f, defined above, whose zeros correspond to the location of the boundary in this free-boundary problem.

Then we conclude that the unique solution to the free-boundary problem above is given by

$$W(z) = (1 + z^2)\Phi(z_*) - z\phi(z) + (1 - z^2)\Phi(z) - \frac{3}{2}.$$

We verify in the next section that $W(z)$ corresponds to the value obtained by stopping optimally starting from $Z_0 = z$, and the optimal stopping time will consist of stopping at the zeros $\pm z_*$ of $W(z)$.

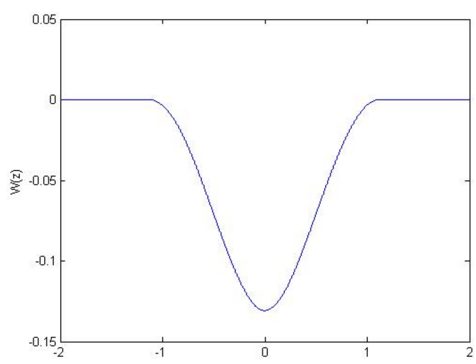

Figure 3.3: Illustration of the solution to the free-boundary problem considered in this chapter, W. Note that W is always non-positive and satisfies the smooth-fit condition at $\pm z_*$.

3.5 Verification

Now that we have obtained a solution to the free-boundary problem corresponding to our optimal stopping problem, the next task is to justify the formal manipulations used in deriving the free-boundary problem. In particular, we prove that even if the obtained solution is not as smooth as initially assumed, it still corresponds to an optimal stopping time.

First, we recall the Optimal Stopping Condition lemma from the Technical Introduction. This states that if we propose a value function v that satisfies three conditions, then the optimal stopping strategy is given by the first hitting time of $g(Z_t)$ by $v(Z_t)$. In particular, we try to show that the solution to our free-boundary problem, W, satisfies the desired conditions.

Theorem 3.5.1. *Let Z_t be the process defined above and W be the solution to the free-boundary problem from above. Then we have that:*

- $W(z) \leq 0$ *for all* $z \in \mathbb{R}$,
- $e^{-2t}W(Z_t) + \int_0^t e^{-2s}F(|Z_s|)\,ds$ *is a submartingale, and*
- $e^{-2t\wedge\sigma_*}W(Z_{t\wedge\sigma_*}) + \int_0^{t\wedge\sigma_*} e^{-2s}F(|Z_s|)\,ds$ *is a uniformly integrable martingale,*

where $\sigma_* = \inf\{t \geq 0 : W(Z_t) = 0\}$*. Then* τ_* *is the optimal stopping strategy and*

$$W(z) = \mathbb{E}^z\left[\int_0^{\sigma_*} e^{-2s}F(|Z_s|)\,ds\right].$$

Proof. Now, we use stochastic calculus and the free-boundary problem from above to show that our solution W solves the optimal stopping problem:

$$\inf_{\tau\in\mathcal{M}} \mathbb{E}^x\left[\int_0^{\tau} e^{-2s}F(|Z_s|)\,ds\right].$$

First, we remark that the solution function is non-positive, so $W(z) \leq 0$ for all $z \in \mathbb{R}$.

Next, we check that $e^{-2t}W(Z_t) + \int_0^t e^{-2s}F(|Z_s|)\,ds$ is a submartingale. The key is to note that W is C^2 at all points but $\pm z_*$, where it is C^1. However, these are a set of points of Lebesgue measure zero, so if we can show that the occupation time of these points by Z_t is measure zero, then we can apply the Itô-Tanaka formula with arbitrary definitions of W'' at $\pm z_*$.

We note that the occupation time of both $\pm z_*$ by Z_t has zero because Z_t was initially defined by $Z_t = e^t B_{1-e^{-2t}}$, and the occupation time of any curve by a Brownian motion is zero.

Then we note by the Itô-Tanaka formula:

$$\begin{aligned}
e^{-2t}W(Z_t) + \int_0^t e^{-2s}F(|Z_s|)\,ds &= W(Z_0) - 2\int_0^t e^{-2s}W(Z_s)\,dt + \int_0^t e^{-2s}W'(Z_s)\,dZ_s \\
&\quad + \int_0^t \frac{1}{2}e^{-2s}W''(Z_s)d[Z,Z]_s + \int_0^t e^{-2s}F(|Z_s|)\,ds \\
&= W(Z_0) + \int_0^t \sqrt{2}e^{-2s}W'(Z_s)\,d\beta_s \\
&\quad + \int_0^t e^{-2s}\left(W''(Z_s) + Z_sW'(Z_s)\right. \\
&\quad \left. -2W(Z_s) + F(|Z_s|)\right)\,ds.
\end{aligned}$$

But then when $|Z_s| < z_*$, the part inside the Lebesgue integral is zero because it is a solution to the differential equation above. When $|Z_s| > z_*$, note that $W(Z_s) = W'(Z_s) = W''(Z_s) = 0$ and the Lebesgue integral is positive because $F(z) \geq 0$. Then because $|Z_s| = z_*$ has zero occupation time almost surely, we conclude that the process above is a local submartingale.

Finally, we need to check that the stopped process

$$e^{-2t\wedge\sigma_*}W(Z_{t\wedge\sigma_*}) + \int_0^{t\wedge\sigma_*} e^{-2s}F(|Z_s|)\,ds$$

is a martingale.

But this is clear by the previous computation and the fact that σ_* is the first time when $|Z_t| \geq z_*$. Then before σ_*, the Lebesgue integral above is zero by satisfying the required differential equation, and after σ_*, the process is stopped, and therefore a local martingale. However, this is a bounded local martingale because $W(z)$ is non-zero only on a compact set. Therefore, this is a true martingale.

$\square$

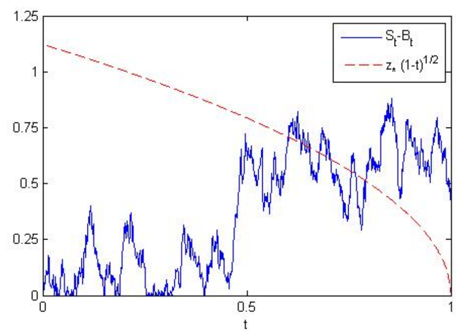

Figure 3.4: Illustration of the optimal stopping strategy in this problem. It is optimal to stop at the first hitting time of the curve $z_*\sqrt{1-t}$ by the process $S_t - B_t$. This illustrates one possible outcome.

Now, we see that the optimal stopping time is

$$\sigma_* = \inf\{t \geq 0 : |Z_t| \geq z_*\} = \inf\{t \geq 0 : S_t - B_t \geq z_*\sqrt{1-t}\}.$$

Now, reversing the time change to obtain the optimal stopping problem τ_* to the problem in Section 3.2, we see:

$$\begin{aligned} \tau_* &= \inf\{t \geq 0 : e^{-\frac{1}{2}\log(1-t)}|B_t| \geq z_*\} \\ &= \inf\{t \geq 0 : |B_t| \geq \sqrt{1-t}z_*\}. \end{aligned}$$

Note, as mentioned before, this is actually the optimal stopping time in a superset of $\mathcal{F}^B$-stopping times. However, this is clearly also a $\mathcal{F}^{|B|}$-stopping time, which is a subset. Therefore, it is also the optimal $\mathcal{F}^{|B|}$-stopping time.

Finally, reversing the equality in distribution of $|B_t|$ and $S_t - B_t$, we obtain the solution to our original problem:

$$\begin{aligned} \tau_* &= \inf\{t \geq 0 : |B_t| \geq \sqrt{1-t}z_*\} \\ &= \inf\{t \geq 0 : S_t - B_t \geq z_*\sqrt{1-t}\}. \end{aligned}$$

The graphic above shows an example of how the stopping rule would play out. In this case, it is optimal to stop the first time that $S_t - B_t$ intersects the curve $z_*\sqrt{1-t}$.

Chapter 4

A New Problem

In this chapter, we apply the methods used in the previous literature with regards to a different optimal stopping problem involving Brownian motion. In particular, we emphasize how different techniques are necessary depending upon the choice of allowed stopping times.

4.1 Specification

Consider the problem of stopping a Brownian motion with its absolute value close to a target value. In this problem we illustrate the effect of the class of stopping times considered on the solution. In particular, we solve this problem with stopping times being bounded by a finite time horizon, being unbounded but having finite expectation, and finally with completely unbounded stopping times.

Consider the following classes of stopping times:

$$\begin{aligned}
\mathcal{M}_T &= \{\text{stopping times } \tau \text{ such that } 0 \leq \tau \leq T \text{ a.s.}\} \\
\mathcal{M} &= \{\text{stopping times } \tau \text{ such that } \mathbb{E}\left[\tau\right] < \infty\} \\
\mathcal{M}_\infty &= \{\text{stopping times } \tau \text{ such that } 0 \leq \tau < \infty\}.
\end{aligned}$$

Let B_t be a Brownian motion. For each $a > 0$, consider the three optimal stopping problems corresponding to each of the above classes of stopping times:

$$\begin{aligned} v_T(x) &= \inf_{\tau \in \mathcal{M}_T} \mathbb{E}^x \left(|B_\tau| - a\right)^2 \\ v(x) &= \inf_{\tau \in \mathcal{M}} \mathbb{E}^x \left(|B_\tau| - a\right)^2 \\ v_\infty(x) &= \inf_{\tau \in \mathcal{M}_\infty} \mathbb{E}^x \left(|B_\tau| - a\right)^2, \end{aligned}$$

where the expectation $\mathbb{E}^x$ is taken with respect to a measure $\mathbb{P}^x$ where the Brownian motion starts at $B_0 = x$ almost surely.

4.2 Solution

We consider these problems in order of increasing restrictions on the stopping times.

4.2.1 Unbounded stopping times

When stopping times τ are unbounded the optimal stopping time is trivial because a one-dimensional Brownian motion will hit any point almost surely. Then it is optimal to wait until the Brownian motion eventually satisfies $|B_t| = a$. The following result formalizes this:

Lemma 4.2.1. *Let B_t be a Brownian motion and $a \in \mathbb{R}$. Let*

$$\tau_a = \inf\{t \geq 0 : B_t = a\}$$

be a the first hitting time a by B_t. Then $\mathbb{P}\left[\tau_a < \infty\right] = 1$.

Proof. If $a = 0$, then this is obvious. Without loss of generality, we assume that $a > 0$. Consider the Brownian motion B_t and its corresponding running maximum process S_t, as well as the stopping time:

$$\tau_a = \inf\{t \geq 0 : B_t = a\}.$$

They key to computing the distribution of τ_a is to note that for any $t \geq 0$, we have:

$$\{\tau_a \geq t\} = \{S_t \leq a\}.$$

However, we showed in the Technical Introduction using the Reflection Principle that $S_t \overset{(d)}{=} |B_t|$, so we can compute the distribution of τ_a as:

$$\begin{aligned}
\mathbb{P}\left[\tau_a \geq t\right] &= \mathbb{P}\left[S_t \leq a\right] \\
&= \sqrt{\frac{2}{t\pi}} \int_0^a \exp\left(-\frac{x^2}{2t}\right) dx.
\end{aligned}$$

But then it is clear that $\mathbb{P}\left[\tau_a = \infty\right] = 0$ because $\lim_{t\to\infty} \mathbb{P}\left[\tau_a \geq t\right] = 0$.

$\square$

From this, it follows immediately that for unbounded stopping times, the optimal stopping problem is trivial:

Theorem 4.2.2. *Let $\mathcal{M}_\infty$ be the set of unbounded stopping times as defined above. Then*

$$v_\infty(x) = \inf_{\tau \in \mathcal{M}_\infty} \mathbb{E}^x \left(|B_\tau| - a\right)^2 = 0.$$

Proof. Consider the stopping times:

$$\begin{aligned}
\tau_+ &= \inf\{t \geq 0 : B_t = +a\} \\
\tau_- &= \inf\{t \geq 0 : B_t = -a\}.
\end{aligned}$$

Then consider the stopping time, $\tau_* = \tau_+ \cap \tau_-$. By the previous theorem, $\tau_* \in \mathcal{M}_\infty$, and $B_{\tau_*} = a$ almost surely. Then

$$\mathbb{E}^x \left(|B_{\tau_*}| - a\right)^2 = 0.$$

Therefore, this is an optimal stopping time.

$\square$

While this is optimal for unbounded stopping times, this stopping time τ_* is not of finite expectation. It is not clear that there is an optimal stopping time in the class $\mathcal{M}$. In fact, it turns out that the solution is non-trivial outside of a region where the Brownian motion is unlikely to return to $\pm a$ in bounded time.

4.2.2 Finite expectation stopping times

In the special case of a time-independent optimal stopping problem over $\mathcal{M}$ involving Brownian motion, there is a common technique for quickly identifying a value function v. In particular, if we identify the maximal convex minorant of the reward function g, it will turn out to be the correct value function.

Definition 4.2.1 (Convex Envelope)**.** Consider a function $f : \mathbb{R} \to \mathbb{R}$. A function $\hat{f} : \mathbb{R} \to \mathbb{R}$ is called the convex envelope of f if:

- $\hat{f}$ is a convex function,
- $\hat{f} \leq f$, and
- for any convex function g such that $g \leq f$, we have $g \leq \hat{f}$.

Convex envelopes are often easy to graphically to determine. Using these, the solution to the associated optimal stopping problem with bounded stopping times may often be obtained.

Proposition 4.2.3. *Let $a > 0$, and consider the reward function $g(x) = (|x| - a)^2$. Then the convex envelope of g is given by*

$$v(x) = \begin{cases} 0 & : |x| \leq a \\ (|x| - a)^2 & : |x| \geq a. \end{cases}$$

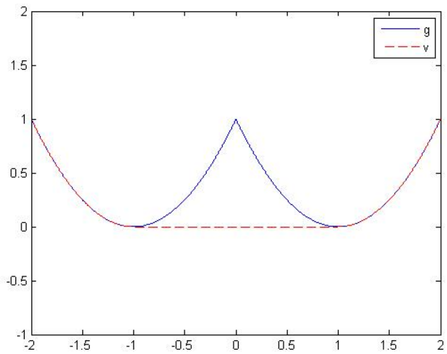

Figure 4.1: Convex envelope $\hat{g}$ of the reward function g considered in this problem.

Proof. First, note that $v \leq g$ for all $x \in \mathbb{R}$ and v is a convex function which is majorized by g. We hope to show that v is, in fact, the largest convex function which is majorized by g.

Suppose that there exists a convex function $\tilde{v}$ such that $v \leq \tilde{v} \leq g$ for all $x \in \mathbb{R}$, but such that there exists $x_0 \in \mathbb{R}$ where $v(x_0) < \tilde{v}(x_0)$. Then $|x_0| < a$ because $v(x) = g(x)$ for all other x_0. Note that $\tilde{v}(\pm a) = g(\pm a) = 0$. Consider the straight line between $(-a, 0)$ and $(a, 0)$. This contains the point $(x_0, 0)$, which is below the graph of $\tilde{v}$. This contradicts the fact that $\tilde{v}$ is convex.

Therefore, v is the convex envelope of g.

□

Note that this convex envelope v is C^1 on $\mathbb{R}$ and C^2 on $\mathbb{R} - \{\pm a\}$. Then we can prove easily that it is a valid value function for this optimal stopping problem. In fact, we invoke the Optimal Stopping Condition lemma from the Technical Introduction.

Theorem 4.2.4. *Let B_t be a Brownian motion and both the reward function g and value function v be as defined above. Then we have that:*

- *$v(x) \leq g(x)$ for all $x \in \mathbb{R}$,*
- *$v(B_t)$ is a supermartingale, and*
- *$v(B_{t\wedge\tau_*})$ is a uniformly integrable martingale,*

where $\tau_* = \inf\{t \geq 0 : v(B_t) = g(B_t)\}$. Then τ_* is the optimal stopping strategy and

$$v(x) = \inf_{\tau \in \mathcal{M}} \mathbb{E}^x \left(|B_\tau| - a\right)^2.$$

Proof. First, note that $v(x) \leq g(x)$ for all $x \in \mathbb{R}$ because v is the convex envelope of g. Furthermore, because v is convex and B_t is a martingale, it follows that $v(B_t)$ is a supermartingale by Jenson's inequality for conditional expectations.

The trick for the last condition is to note that v is C^1 everywhere and C^2 at all points except $\pm a$. Then by Problem 6.24 on page 215 of [4] we see that with respect to a Brownian motion, the Itô-Tanaka formula reduces to

$$v(B_{t\wedge\tau_*}) = v(x) + \int_0^{t\wedge\tau_*} v'(B_s)\, dB_s + \int_0^{t\wedge\tau_*} \frac{1}{2} v''(B_s)\, ds.$$

Now, if $|x| < a$, then $v''(B_s) = 0$ for almost every s, so we conclude that $v(B_{t\wedge\tau_*})$ is a local martingale. However, it is bounded, so it is in fact a uniformly integrable martingale. If $|x| \geq a$, then $v(B_{t\wedge\tau_*}) = v(x)$ almost surely, so $v(B_{t\wedge\tau_*})$ is again a uniformly integrable martingale.

Therefore, we conclude by the Optimal Stopping Condition that v is the correct value function and the optimal stopping strategy is given by:

$$\tau_* = \inf\{t \geq 0 : v(B_t) = g(B_t)\}.$$

Finally, we note that τ_* does in fact lie in the set of finite expectation stopping times, so the optimal stopping time is achieved.

□

Remark 4.2.1. It may be proven in more generality that given a continuous reward function g and a Brownian motion B_t, the value of the optimal stopping strategy is given by the convex envelope of g. For a proof, see [7], for example.

4.2.3 Finitely-bounded stopping times

Finally, we consider the optimal stopping problem on the class of finitely bounded stopping times, $\mathcal{M}_T$ for some $T > 0$.

Consider a hypothetical optimal value function $v : [0,T] \times \mathbb{R} \to \mathbb{R}$ given by:

$$v(t,x) = \inf_{t \le \tau \le T} \mathbb{E}^x \left[g(B_\tau) \right].$$

First, we note that we cannot do any better in finite time than we can with unbounded stopping times. On the other hand, we can do no worse than stopping immediately. Therefore, we conclude immediately that $v(t,x) = g(x)$ for all $|x| \ge a$.

Assuming that this value function is smooth enough to apply Itô's formula, we can follow the same formal derivation of a free-boundary problem as in Chapter 3 and guess the following problem.

Find a function $\gamma : [0,T] \to [0,a]$ and a function $v : \Omega \to \mathbb{R}$ such that:

$$\begin{cases} v_t + \frac{1}{2} v_{xx} = 0 & |x| < \gamma(t), 0 < t < T \\ v(t, \pm\gamma(t)) = g(\pm\gamma(t)) & 0 \le t < T \\ v_x(t, \pm\gamma(t)) = g_x(\pm\gamma(t)) & 0 \le t < T. \end{cases}$$

Furthermore, we ask that we have terminal conditions of

$$v(T,x) = g(x)$$

for all $x \in \mathbb{R}$. This is because there is no choice but to stop immediately at the terminal time.

This is a backwards heat equation with a free-boundary problem, which may seem ill-posed problem. However, because we are working on a finite time horizon, it is not. In particular, we can make a time change as in the previous chapters

$$t \mapsto T - t.$$

Furthermore, if we use the fact that the solution is clearly even in x, we can simplify further to obtain a new free-boundary partial differential equation:

$$\begin{cases} \tilde{v}_t = \frac{1}{2}\tilde{v}_{xx} & 0 < x < \gamma(t), 0 < t < T \\ \tilde{v}(t, \gamma(t)) = g(\gamma(t)) & 0 < t \leq T \\ \tilde{v}_x(t, \gamma(t)) = g_x(\gamma(t)) & 0 < t \leq T \\ \tilde{v}_x(t, 0) = 0 & 0 < t \leq T \\ \tilde{v}(0, x) = g(x) & x \in \mathbb{R}. \end{cases}$$

This is a one-phase Stefan-type free-boundary problem, which is well-characterized in the literature. In particular, for details on this problem and its solutions, [5] is an excellent reference.

Obtaining a solution to this free-boundary problem is a very complicated mathematical theory, which is beyond the scope of this thesis at this point. However, we prove that, if we obtain a solution $\tilde{v}$ which is sufficiently smooth (i.e. C^2 almost everywhere), then this corresponds to a value function for the optimal stopping problem. Because this is a finite time horizon problem, we cannot directly apply the Optimal Stopping Condition theorem, but we prove a result in the same spirit.

Theorem 4.2.5. *Let $v(t, x)$ be a solution to the above free-boundary problem. Suppose that v is C^1 everywhere and C^2 at all points except on the boundary defined by γ. Then we have that:*

- *$v(t, x) \leq g(x)$ for all $(t, x) \in [0, T] \times \mathbb{R}$,*
- *$v(t, B_t)$ is a submartingale, and*

- *$v(t \wedge \tau_*, B_{t \wedge \tau_*})$ is a uniformly integrable martingale,*

where $\tau_* = \inf\{0 \leq t \leq T : v(t, B_t) = g(B_t)\}$. Then τ_* is the optimal stopping strategy and

$$v(t,x) = \inf_{\tau \in \mathcal{M}_{T-t}} \mathbb{E}^x \left(|B_\tau| - a\right)^2.$$

Proof. First, we show that $v(t,x) \leq g(x)$ for all $(t,x) \in [0,T] \times \mathbb{R}$. Consider throughout the region

$$\Omega = \{(t,x) : 0 \leq t \leq T, |x| \leq \gamma(t)\},$$

where γ is the boundary function. Outside of Ω we have equality $v(t,x) = g(x)$ by definition. However, inside Ω, we know that v satisfies the heat equation, which has a strong maximum principle [2]. Therefore, the maximum of $v(t,x)$ on Ω occurs on $\partial\Omega$:

$$\max_{(t,x)\in\Omega} v(t,x) = \max_{(t,x)\in\partial\Omega} v(t,x).$$

However, we can note that because $\gamma(t) < a$, g is always greater in the interior of Ω than anywhere on the boundary $\partial\Omega$:

$$g(x) \geq \max_{\partial\Omega} g \text{ for all } (t,x) \in \Omega.$$

Therefore, because $v = g$ on $\partial\Omega$, we conclude that for all $(t,x) \in [0,T] \times \mathbb{R}$,

$$v(t,x) \leq g(t,x).$$

Next, we note that because v is C^1 everywhere and C^2 at all but a curve of measure zero, we can apply the simplified Itô-Tanaka formula as in the previous chapter and note that:

$$v(t,B_t) = v(0,x) + \int_0^t \left(v_t(s,B_s) + \frac{1}{2}v_{xx}(s,B_s)\right) ds + \int_0^t v_x(s,B_s)\, dB_s.$$

Inside Ω, we satisfy the equation $v_t + \frac{1}{2}v_{xx} = 0$. On the other hand, outside of Ω, $v = g$ identically, so

$$v_t + \frac{1}{2}v_{xx} = \frac{1}{2}g_{xx}.$$

However, g is convex at all but the single point $(0,0)$, so we conclude that $v(t, B_t)$ is a submartingale.

Finally, if we define

$$\tau_* = \inf\{0 \leq t \leq T : v(t, B_t) = g(B_t)\},$$

we can check with the Itô-Tanaka formula and Optional Sampling Theorem that:

$$\begin{aligned} v(t \vee \tau_*, B_{t \vee \tau_*}) &= v(0,x) + \int_0^{t \vee \tau_*} \left(v_t(s, B_s) + \frac{1}{2} v_{xx}(s, B_s) \right) ds + \int_0^{t \vee \tau_*} v_x(s, B_s)\, dB_s \\ &= v(0,x) + v_x(s, B_s)\, dB_s \end{aligned}$$

is a local martingale because for any time $t < \tau_*$, $(t,x) \in \Omega$, where the heat equation is satisfied, and after $t \geq \tau_*$, the process is stopped. But it is a bounded local martingale, so it is a true martingale.

Finally, we note that by taking $t = 0$ and $t = T$ respectively, and using the fact that $v(t \vee \tau_*, B_{t \vee \tau_*})$ is a martingale that:

$$\begin{aligned} v(0,x) &= \mathbb{E}\left[v(0,x)\right] \\ &= \mathbb{E}\left[v(\tau_*, B_{\tau_*})\right] \\ &= \mathbb{E}\left[g(B_{\tau_*})\right], \end{aligned} \tag{4.1}$$

so τ_* achieves the desired optimal stopping value.

□

Below, we illustrate the results of a numerical solution to this problem with $T = a = 1$. Note that the function defining the boundary, γ is non-trivial.

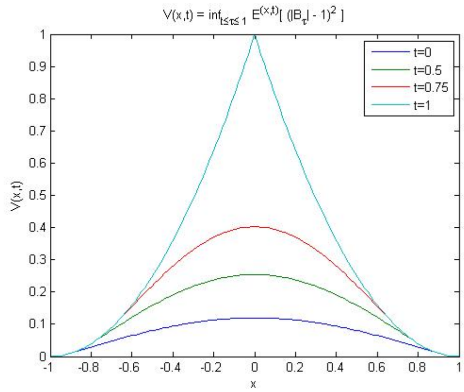

Figure 4.2: Illustration of the numerical solution of this free-boundary problem. Each line illustrates an optimal value function with the time horizon varied between zero and one.

4.3 Discussion

In the section above, we found the solution to one optimal stopping problem involving Brownian motion with respect to three different classes of stopping times. In particular, the solutions utilized an array of techniques, including primarily a probabilistic argument in the unbounded stopping time case, an argument above martingales and convexity in the finite expectation stopping time case, and the specification and solution of a free-boundary partial differential equation in the finitely-bounded stopping time case.

First, note that for any $0 < S \leq T < \infty$, we have:

$$\mathcal{M}_S \subset \mathcal{F}_T \subset \mathcal{M} \subset \mathcal{M}_\infty.$$

This corresponds with the fact that the value function in each class of optimal stopping problems were decreasing:

$$v(S, x) \geq v(T, x) \geq v(x) \geq v_\infty(x).$$

What is very interesting about this particular problem is that for $|x| < a$ all of these inequalities are strict, including $v(x) > v_\infty(x)$.

Bibliography

[1] R. Durrett. *Probability: Theory and Examples.* Cambridge University Press, 2010.

[2] L.C. Evans. *Partial Differential Equations.* American Mathematical Society, 2010.

[3] S.E. Graversen, G. Peskir, and A.N. Shiryaev. Stopping brownian motion without anticipation as close as possible to its ultimate maximum. *Theory Probab. Appl.*, 45(1):41–50, 1999.

[4] I. Karatzas and S.E. Shreve. *Brownian Motion and Stochastic Calculus.* Spring-Verlag, 1988.

[5] D. Kinderlehrer and G. Stampacchia. *An Introduction to Variational Inequalities and Their Applications.* Academic Press, 1980.

[6] J. Sab. *Stochastic Control: With Applications to Financial Mathematics.* 2006.

[7] A.N. Shiryaev. *Optimal Stopping Rules.* Springer-Verlag, 2008.