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The Distribution of Roots of Certain Polynomials

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The Distribution of Roots of Certain Polynomials

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Dedicated to my parents and family for their infinite patience and support. To my mother Cristina González; my father Miguel Rodríguez; sisters Arcelia and Valentina; and brother Marco Aurelio, and stepfather Philip Willan.

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The Distribution of Roots of Certain Polynomials

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We study the distribution of the roots of certain polynomials, among which are certain Hilbert and Ehrhart polynomials. We look at the roots of polynomials of the form $H(x) = \sum_{m=0}^{d} h_m \binom{d+x-m}{d}$. The generating function of the $H(n)$, with $n \in \mathbb{Z}^{\geq 0}$, is the Maclaurin series of $P(t) = \frac{\sum_{m=0}^{d} h_m t^m}{(1-t)^{d+1}}$. For x with $\Re x \in (0,1)$ we represent H as the Mellin transform of P. We then use Laplace's method to study the asymptotic nature of H as its degree grows.

Table of Contents

List of Figures

Chapter 1

Introduction

We study the asymptotics, inside a vertical strip, of certain classes of polynomials. More specifically, we are interested in what happens to the distribution of the roots of these polynomials as their degrees grow. The techniques studied here can be applied to some polynomials that appear in algebra and combinatorics. For example, let A be a graded algebra. Let $H(n)$ be the dimension of the homogenous component of degree n of A. Let $P(t) = \sum_{n\geq 0} H(n)t^n$ be the generating function of the sequence $\{H(n)\}_{n\geq 0}$. We call $P(t)$ the *Poincaré series* associated to A. It can be shown that $H(n)$ is a polynomial in n. H is known as the *Hilbert polynomial* of A. In many cases $P(t)$ is a rational function of the form $\frac{h(t)}{(1-t)^{d+a}}$ where $h(t) \in \mathbb{C}[t]$ is of degree d and $a > 0$. We call this $h(t)$ the *h-polynomial* of A. It is proved by Rodríguez-Villegas in [14] that if the roots of $h(t)$ are all on $\partial\mathbb{D} \setminus \{1\}$ and $a = 1$ then all of the roots of $H(n)$ have real part $-1/2$. This last is inspired in part on [5], where Bumb, Choi, Kurlberg and Vaaler prove that various functions and polynomials related to various algebraic objects have their roots on vertical lines in the complex plane. The objects mentioned in these papers are therefore said to satisfy their own type of Riemann hypotheses. For this last type of polynomials we will be able to produce an asymptotic formula which works in a vertical strip.

Here we will prove that if the coefficients of the polynomial $h(t)$ above are not too large compared to its degree, then the roots of $H(n)$, those in some fixed vertical strip, will accumulate on a curve. The same result will be true if $h(t)$ has its roots in a specific set that contains the unit circle. To achieve this last we will establish certain properties of the map $\mathcal{H}_a : h_0 + h_1 t + \cdots + h_d t^d \mapsto h_0\binom{x+d+a}{d+a} + h_1\binom{x-1+d+a}{d+a} + \cdots + h_d\binom{x+a}{d+a}$, which we consider interesting in their own right. If a polynomial $h(t)$ with roots sufficiently close to the unit circle, $\mathcal{H}_a h$ will have roots on the vertical line $\{\Re = -\frac{a+1}{2}\}$, for example. But we can prove a bit more: if the roots of $h(t)$ are all on one side of the unit circle, then all the roots of $\mathcal{H}_a h$ will be on one side of $\{\Re = -\frac{a+1}{2}\}$. This will imply, by the Hermite-Biehler theorem, that if $h(t)$ and $g(t)$ have roots that interlace on the unit circle, then the non-real roots of $\mathcal{H}_a h$ and $\mathcal{H}_a g$ will interlace on the line $\{\Re = -\frac{a+1}{2}\}$. Finally, the asymptotics $\mathcal{H}_a[h]$, for $h(t)$ of large degree, is studied with the help of Laplace's method. More specifically, we prove that for appropriate h-polynomial, and x inside some compact subset C of $\{z \in \mathbb{C} : \Re z \in (0,1)\}$,

$$\begin{aligned} H(-x)B(x, 1-x) = H(1)^{-x}\Gamma(x) + O\left[H(1)^{-\Re x - 1}\right] + \\ + (-1)^d H(1)^{x-1}\Gamma(1-x) + O\left[H(1)^{-\Re x - 1}\right] \quad (1.1) \end{aligned}$$

This last will allow us to study the distribution, inside a vertical strip and close to the real line, of the roots of H.

Chapter 2

Some analytic properties of Hilbert polynomials

2.1 Rough location of the roots

We study rational functions of the form $P(t) = \frac{h(t)}{(1-t)^{d+a}}$ with $h(t) \in \mathbb{C}[t]$ of degree d and $a > 0$. Let $\sum_{n \geq 0} H(n)t^n$ be the MacLaurin series of $P(t)$. We will always refer to such a series (coming from a rational function of the same type as $P(t)$) as a *Poincaré series*. We will call the polynomial $h(t)$ in the numerator the *h-polynomial*, and, we will call the vector of coefficients of $h(x)$ the *h-vector*. $\mathbb{D}$ will denote the unit disk. To describe certain sets in the complex plane we will use the shorthand $\{\Re \in A, \Im \in B\} := \{z \in \mathbb{C} : \Re z \in A, \Im z \in B\}$. We use traditional asymptotic notation. If $f(z)$ is some complex function and $g(z)$ for all z in the domain of f, we write $f(z) = O(g(z))$, or $f(z) \ll g(z)$, as $z \to a$, if there is some constant $C > 0$ such that $|f(z)| \leq C|g(z)|$ for all z sufficiently close to a (when a is ∞, by all z sufficiently close to a we mean for all sufficiently large $|z|$). By $f(z) = o(g(z))$ as $z \to a$ we mean $\frac{f(z)}{g(z)} \to 0$ as $z \to \infty$. By $f(z) \sim g(z)$ as $z \to \infty$ we understand $\left|\frac{f(z)}{g(z)}\right| \to 1$ as $z \to a$. We will use $B_c(R)$ to denote the ball, in $\mathbb{C}$, of radius R centered at c. It is not hard to prove that $H(n)$ is a polynomial in n. We have the following result from [14]:

Theorem 2.1.1 (Rodríguez-Villegas 2000)**.** *Suppose all the roots of the h-polynomial of $P(t)$ are on $\partial\mathbb{D} \setminus \{1\}$. Then,*

$$H(x) = (x+1)_a p(x)$$

where,

$$(x)_a = \begin{cases} x(x+1)\cdots(x+a-1) & a \geq 1 \\ 1 & a < 1 \end{cases}$$

and p is a polynomial with all of its roots on the line $\{\Re = -a/2\}$.

We prove the theorem below. First, some definitions and lemmas.

Definition 2.1.1. Let $a > 0$. For all $x \in \mathbb{C} \setminus (\mathbb{Z} - a)$,

$$(x)_a = \frac{\Gamma(x+a)}{\Gamma(x)} \tag{2.1}$$

Definition 2.1.2. For $a \geq 1$,

$$S_a = \left\{ f(x) = (x+1)_{a-1}p(x) \;\middle|\; \begin{array}{c} p(x) \text{ is a polynomial with all} \\ \text{of its zeros having real part } -\frac{a}{2} \end{array} \right\} \tag{2.2}$$

Lemma 2.1.2. *Let $a > 0$ and $|\alpha| = 1$. Suppose $f \in S_{a+1}$. Let $g(x) = f(x-1) - \alpha f(x)$. Then, $g(x) \in S_a$ and all of the roots of $g(x)/(x+1)_{a-1}$ are simple.*

Proof. Let $r(x) = f(x-1) - \alpha f(x)$. f is in S_{a+1}, so $f(x) = (x+1)_a p(x)$ where p has all of its roots on the line $\Re = -(a+1)/2$. Then

$$\begin{aligned} r(x) &= f(x-1) - \alpha f(x) \\ &= (x)_a p(x-1) - \alpha(x+1)_a p(x) \\ &= (x+1)_{a-1}\left\{xp(x-1) - \alpha(x+a)p(x)\right\} \end{aligned}$$

Let ρ be a root of $s(x) = xp(x-1) - \alpha(x+a)p(x)$. Then,

$$|\rho||p(\rho-1)| = |\rho+a||p(\rho)| \tag{2.3}$$

First, notice that $p(x)$ and $p(x-1)$ do not have common roots. Suppose $\Re(\rho) < -a/2$ (the case $\Re(\rho) > -\frac{a}{2}$ is identical). Then ρ is closer to $-a$ than to 0. So $|\rho+a| < |\rho|$. Let μ be any root of p. Then, since $\Re(\rho) < -\frac{a}{2}$, ρ is closer to μ than to $\mu+1$ since $\Re\mu + 1 = -(a-1)/2$. It follows that $|\rho-\mu| < |\rho-(\mu+1)|$. It follows that $|\rho+a||p(\rho)| < |\rho||p(\rho-1)|$, contradicting (2.3). It follows that the roots of $s(x)$ have real part $-\frac{a}{2}$.

Now we prove the simplicity of he roots. Let $a > 0$. Let $s(x) = xp(x-1) - \alpha(x+a)p(x)$. If $s(\rho) = 0$ then, as before, $\rho p(\rho-1) = \alpha(\rho+a)p(\rho)$. By the first part of the proof $\rho = -\frac{a}{2} + i\tau$, $\tau \in \mathbb{R}$. Now,

$$\begin{aligned} s'(x) &= p(x-1) + xp'(x-1) - \alpha p(x) - \alpha(x+a)p'(x) \\ &= p(x-1) - \alpha p(x) + xp'(x-1) - \alpha(x+a)p'(x) \\ &= xp(x-1)\left\{\frac{1}{x} + \frac{p'(x-1)}{p(x-1)}\right\} - \alpha(x+a)p(x)\left\{\frac{1}{x+a} + \frac{p'(x)}{p(x)}\right\} \end{aligned} \tag{2.4}$$

Let $\rho_k = -\frac{a+1}{2} + i\tau_k$, $\tau_k \in \mathbb{R}$, for $k = 1, \ldots m$ be the roots of $p(x)$.

Suppose that ρ is a multiple root of $s(x)$. Then $s(\rho) = s'(\rho) = 0$, and,

$$\begin{aligned}
0 &= \rho p(\rho-1)\left\{\frac{1}{\rho} + \frac{p'(\rho-1)}{p(\rho-1)}\right\} - \alpha(\rho+a)p(\rho)\left\{\frac{1}{\rho+a} + \frac{p'(\rho)}{p(\rho)}\right\} \\
&= \rho p(\rho-1)\left\{\frac{1}{\rho} + \frac{p'(\rho-1)}{p(\rho-1)} - \frac{1}{\rho+a} + \frac{p'(\rho)}{p(\rho)}\right\} \\
&= \rho p(\rho-1)\left\{\frac{1}{\rho} - \frac{1}{\rho+a} + \sum_{k=1}^{m}\frac{1}{\rho-1-\rho_k} - \sum_{k=1}^{m}\frac{1}{\rho-\rho_k}\right\} \qquad (2.5) \\
&= \rho p(\rho-1)\left\{\frac{a}{\rho(\rho+a)} + \sum_{k=1}^{m}\frac{1}{(\rho-1-\rho_k)(\rho-\rho_k)}\right\} \\
&= \rho p(\rho-1)\left\{-\frac{a}{|\rho+a|^2} - \sum_{k=1}^{m}\frac{1}{|\rho-\rho_k|^2}\right\}
\end{aligned}$$

In the last line above we used that $(\rho-1-\rho_k)(\rho-\rho_k) = [-\frac{1}{2} + i(\tau-\tau_k)][\frac{1}{2} + i(\tau-\tau_k)] = -|\rho-\rho_k|^2$. We arrive at the contradiction:

$$0 = -\frac{a}{|\rho+a|^2} - \sum_{k=1}^{m}\frac{1}{|\rho-\rho_k|^2} < 0 \qquad (2.6)$$

since $\rho p(\rho-1) \neq 0$. It follows that ρ cannot be a multiple root of $s(x)$. $\square$

Proof of theorem 2.1.1: The Hilbert polynomial of $\frac{h(t)(t-\alpha)}{(1-t)^{d+a}}$, namely $g(x)$, can be obtained from that of $\frac{h(t)}{(1-t)^{d+a}}$, that is from $f(x)$, through the linear recurrence:

$$g(x) = f(x-1) - \alpha f(x) \qquad (2.7)$$

When $\alpha \in \partial\mathbb{D}\backslash\{1\}$ the roots of $f(x) = \binom{x-1+d+a-1}{d+a-1} - \alpha\binom{x+d+a-1}{d+a-1}$ are $-1, \ldots, -(d+a-2)$ and $(d+a-1)\frac{\alpha}{1-\alpha}$. Notice that when $|\alpha| = 1$ and $\alpha \neq 1$ then $\Re(\frac{\alpha}{1-\alpha}) = -\frac{1}{2}$ since

$$\frac{\alpha}{1-\alpha} + \overline{\left(\frac{\alpha}{1-\alpha}\right)} = \frac{\alpha}{1-\alpha} + \frac{1}{\alpha-1} = -1 \qquad (2.8)$$

Then, $\Re[(d+a-1)\frac{\alpha}{1-\alpha}] = -\frac{d+a-1}{2}$. Now, the rest is proved by repeated applications of lemma 2.1.2 and use of the above recurrence formula. $\square$

Remark 2.1.1. There are Poincaré series P, with associated h-polynomials that have roots in the complement of the unit circle, yet the Hilbert polynomial of P, namely H, has roots on a vertical line. For example, the Poincaré series

$$P(t) = \frac{1+t+t^2+t^3+t^4-t^5+t^6+t^7+t^8+t^9}{(1-t)^{10}} \tag{2.9}$$

has an h-polynomial with many roots outside the unit circle, yet its Hilbert polynomial has all of its roots on $\{\Re = -\frac{1}{2}\}$. What seems to be important is that the h-polynomial is self-reciprocal with roots that are not too far from the unit circle.

Remark 2.1.2. Let f and a be as in lemma 2.1.2. Also assume that f has real coefficients. Then, lemma 2.1.2 gives us that, if the degree of f is even then $f(-a/2) \neq 0$; the roots of f are simple, come in conjugate pairs and are even in number. Similar reasoning gives us that $f(-a/2) = 0$ when the degree of f is odd.

We generalize the above result in the lemma below. Let $\psi(t) = \frac{t}{1+t}$. Notice that $\psi(t)$ takes the line $\{\Re = -\frac{1}{2}\}$ to $\partial\mathbb{D} \setminus \{1\}$. Let B be some subset of $\mathbb{C}$. Define

$$S_{B,a} = \{f \in \mathbb{C}[x] : f(x) = (x+1)_{a-1}p(x),\ \text{all roots of } p \text{ are in } B\} \tag{2.10}$$

Lemma 2.1.3. *Let $f(x) \in S_{\{\Re<A\},a}$. Let $|\alpha| > 1$ and $a > 1$. Then, $r(x) = f(x-1) - \alpha f(x) \in S_{\{\Re<\max[A+\frac{1}{2}, -a\frac{|\alpha|}{|\alpha|+1}]\},a-1}$.*

Proof. Let $r(x) = f(x-1) - \alpha f(x)$. $f(x)$ is in $S_{\{\Re<A\},a}$, so $f(x) = (x+1)_a p(x)$ where $p(x)$ has all of its roots in $\{\Re < A\}$. Then,

$$\begin{aligned} r(x) &= f(x-1) - \alpha f(x) \\ &= (x)_a p(x-1) - \alpha(x+1)_a p(x) \qquad (2.11) \\ &= (x+1)_{a-1}\left\{xp(x-1) - \alpha(x+a)p(x)\right\} \end{aligned}$$

Let ρ be a root of $xp(x-1) - \alpha(x+a)p(x)$. Then,

$$|\rho||p(\rho-1)| = |\alpha||\rho+a||p(\rho)| \qquad (2.12)$$

Suppose $\Re(\rho) \geq \max\{A + \frac{1}{2}, -a\frac{|\alpha|}{|\alpha|+1}\}$. Let ρ' be a root of $p(x)$. Then, since $\Re(\rho') < A$, ρ is closer to $\rho' + 1$ than to ρ'. So $|\rho - 1 - \rho'| < |\rho - \rho'|$.

Since $|\alpha| > 1$, the inequality $|x| \geq |\alpha||x+a|$ is equivalent to:

$$\frac{a|\alpha|}{|\alpha|^2-1} \geq \left|x + \frac{a|\alpha|^2}{|\alpha|^2-1}\right| \qquad (2.13)$$

Notice that the disk defined in the above inequality is contained in $\{\Re < -a\psi(|\alpha|)\}$, because its rightmost point is $-\frac{a|\alpha|^2}{|\alpha|^2-1} + \frac{a|\alpha|}{|\alpha|^2-1} = -a\frac{|\alpha|}{|\alpha|+1}$ since we assumed that $\Re(\rho) \geq -a\frac{|\alpha|}{|\alpha|+1}$. It follows that $|\rho| \leq |\alpha||\rho+a|$. From this last and the previous paragraph we get that $|\rho||p(\rho-1)| < |\alpha||\rho+a||p(\rho)|$, contradicting (2.12). The result follows. $\square$

Theorem 2.1.4. *Suppose $h(t) \in \mathbb{C}[t]$ has all of its roots outside the disk of radius $R \geq 1$. Then, the Hilbert polynomial of $\frac{h(t)}{(1-t)^{d+1+a}}$ is in $S_{\{\Re<-a\frac{R}{R+1}\},a}$.*

Proof. As in a previous commentary, the roots of the Hilbert polynomial of $\frac{t-\alpha}{(1-t)^{d+a}}$ are $\{-1, \ldots, -(d+a-2)\}$ and $(d+a-1)\frac{\alpha}{1-\alpha}$. Notice that $\psi^{-1}(t) = \frac{t}{1-t}$.

The circle of radius R is taken by $\psi^{-1}(t)$ to the circle of radius $\frac{1}{1-R^2}$ centered at $-\frac{R^2}{R^2-1}$, which has as its rightmost point $-\psi(R)$. The exterior of the circle of radius R is taken by ψ^{-1} to the interior of the latter. It follows that $(d+a-1)\frac{\alpha}{1-\alpha} \in S_{L_{[-(d+a-1)\psi(R)],d+a-1}}$. The rest follows by repeated application of the lemma. $\square$

Corollary 2.1.5. *Suppose $h(t) \in \mathbb{C}[t]$ has all of its roots in $\{z \in \mathbb{C} : |z| \geq 1\}$ with at least one root outside of the unit circle. Then, the Hilbert polynomial of $\frac{h(t)}{(1-t)^{d+1}}$ has all of its roots in $\{\Re < -\frac{1}{2}\}$.*

Proof. If h has all of its roots outside of the unit circle then this is just theorem 2.1.4. If h has at least one root on the unit circle then we can write $h = h_0 h_1$ where h_0 has all of its roots on the unit circle and degree larger than 0, while h_1 has all of its roots outside of the unit circle and degree larger than 0. By theorem 2.1.1 the Hilbert polynomial of $\frac{h_0(t)}{(1-t)^{d+1}}$, where $d = \deg h$, will have all of its roots on the line $\{\Re = -(d+1-\deg h_0)/2\}$. Let α be a root of h_1. Then, by lemma 2.1.3, the Hilbert polynomial of $\frac{h_0(t)(t-\alpha)}{(1-t)^{d+1}}$ will have all of its roots in $\{\Re < -(d-\deg h_0)/2\}$. The rest of the proof is just a repetition of the previous sentence but using the roots of $h_1(t)/(t-\alpha)$. $\square$

In the following we will use an integral representation of Hilbert polynomials. Let $P(t) = \frac{Q(t)}{R(t)}$ be a rational functions with $Q(t), R(t) \in \mathbb{C}[t]$. We then have:

Theorem 2.1.6. *Let $P(t), Q(t), R(t)$ be as above. Assume $\deg Q(t) < \deg R(t)$ and that $R(t)$ has no zeros in $\mathbb{R}^{\leq 0}$. Assume also that $P(t) = \sum_{n\geq 0} H(n)t^n$ in* *some neighborhood of* 0. *Then,* $H(x)$ *is a quasi-polynomial that for* $\Re(x) \in (0,1)$ *can be written as:*

$$H(-x)B(x,1-x) = \int_0^\infty P(-t)t^{x-1}dt \tag{2.14}$$

Proof. The proof uses the usual method for computing the Mellin transform of rational function through the calculus of residues. Let $\alpha \in \mathbb{C} \setminus \{0\}$. Then,

$$\begin{aligned} \frac{t^{m+x-1}}{\left(1-\frac{t}{\alpha}\right)^M} &= \alpha^{m+x-1}\frac{[(\alpha^{-1}t-1)+1]^{m+x-1}}{\left(1-\frac{t}{\alpha}\right)^M} = \\ &= \alpha^{m+x-1}\sum_{l\geq 0}\binom{m+x-1}{l}(-1)^M\left(\frac{t}{\alpha}-1\right)^{l-M} \end{aligned} \tag{2.15}$$

So,

$$\begin{aligned} \operatorname{Res}\left[\frac{t^{m+x-1}}{\left(1-\frac{t}{\alpha}\right)^M};\alpha\right] &= (-1)^M\alpha^{m+x}\binom{m+x-1}{M-1} = \\ &= \alpha^{m+x}\binom{-x-m+M-1}{M-1} \end{aligned} \tag{2.16}$$

Now, the Maclaurin series of $\frac{t^{m+x-1}}{\left(1-\frac{t}{\alpha}\right)^M}$ is:

$$\frac{t^{m+x-1}}{\left(1-\frac{t}{\alpha}\right)^M} = \sum_{l\geq 0}\alpha^{-l}\binom{l+M-1}{M-1}t^{l+m} = \sum_{l\geq m}\alpha^{m-l}\binom{l-m+M-1}{M-1}t^l \tag{2.17}$$

It follows that if we write $P(t)$ in partial fraction form we get:

$$H(-x) = \int_\Gamma P(t)t^{x-1}dt \tag{2.18}$$

where Γ is a simple, closed curve, oriented counterclockwise, that contains all the poles of $P(t)$ inside the bounded component of its complement, and, $H(x)$

is a *quasi-polynomial* (a finite sum of products of exponentials and polynomials) such that $P(t) = \sum_{m\geq 0} H(m)t^m$.

Since we assumed the poles of $P(t)$ to be non-negative, we can find a simple closed curve Γ that contains the poles of $P(t)$ in its bounded component but does not touch $\mathbb{R}^{\leq 0}$. For example we can take Γ to be a circle with an indentation that avoids the negative real line (see the figure below). We can stretch Γ so that its circular part grows in radius, while the indentation hugs $\mathbb{R}^{\leq 0}$ ever more tightly. It is not hard to see that that the integral over the circular part of Γ will converge to 0 as the radius goes to ∞. The integral over the indentation will approach the Mellin transform of the statement of the theorem. The $B(x, 1-x)$ factor in (2.14), makes its appearance as we tighten the curve around $\mathbb{R}_{\leq 0}$ due to the multi-valued nature of t^{x-1} close to 0.

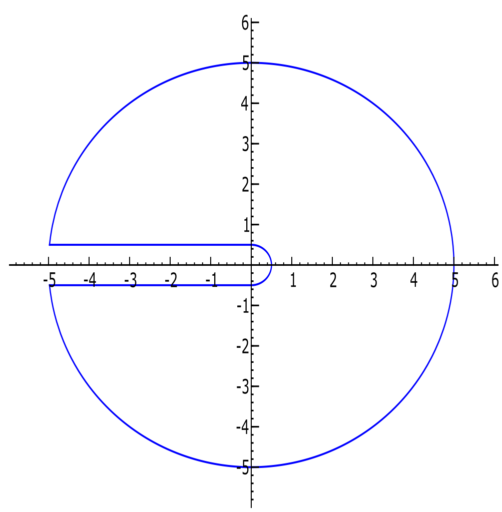

Figure 2.1: Integration contour for theorem 2.1.6.

□

With the above integral representation it is easy to handle certain recurrences and identities of Hilbert polynomials. For example, by making the change of variables $t \mapsto 1/t$ in the integral in (2.14) we get:

$$\begin{aligned} H(-x)B(x,1-x) &= \int_0^\infty \frac{h(-t)}{(1+t)^{d+a}} t^{x-1} dt \\ &= \frac{(-1)^d}{h(0)} \int_0^\infty \frac{h^*(-t)}{(1+t)^{d+a}} t^{a-1-x} dt \\ &= (-1)^d H^*(x-a)B(x,1-x) \quad , \end{aligned} \tag{2.19}$$

where $h^*(t) = t^{\deg h} h(t^{-1})$ is the reciprocal of h and H^* is the Hilbert polynomial of $\frac{h^*(t)}{(1-t)^{d+a}}$. It follows that:

$$H(x) = (-1)^d H^*(-a-x) \tag{2.20}$$

In the case of a self-reciprocal h, we get symmetry around the line $\{\Re = -\frac{a}{2}\}$, or, $H(x) = (-1)^d H(-a-x)$. Combining (2.20) with theorem 2.1.4 we get the following corollary:

Corollary 2.1.7. *Suppose $h(t) \in \mathbb{C}[t]$ has all of its roots inside the disk of radius $R \leq 1$. Then, the Hilbert polynomial of $\frac{h(t)}{(1-t)^{d+a}}$ is in $S_{\{\Re > -\frac{aR}{1+R}\},a}$.*

Definition 2.1.3. Let p and q be polynomials of degree d and $d+1$ respectively. Let $\rho_1, \ldots, \rho_d$ be the roots of p and $\tau_1, \ldots, \tau_{d+1}$ those of q. If these roots alternate in order, $\tau_1 < \rho_1 < \tau_2 \cdots < \tau_k < \rho_k < \tau_{k+1} < \cdots < \tau_d < \rho_d < \tau_{d+1}$, we say that the roots of p and q *interlace strictly.*

The above suggests that "taking Hilbert polynomias" might preserve more properties of the roots of h. In fact, it implies that "taking Hilbert polynomials" preserves root interlacing. To prove this we use the *Hermite-Biehler theorem.*

Theorem 2.1.8 (Hermite-Biehler)**.** *Let $f, g \in \mathbb{R}[x]$ and $p = f + ig$. Then p has either all of its roots in the upper half plane or all of its roots in the lower half plane if, and only if, the roots of f and g are real and interlace.*

We will also use the following partial version of the same theorem.

Theorem 2.1.9. *Let p, f, g be as above. Suppose p has all of its roots in $\mathbb{H}^+$. Then,*

i. The roots of f and g are real and interlace strictly.

ii. $\frac{\partial}{\partial z} \arg p(z) \geq 0$ when $z \in \mathbb{R}$. Also, $\frac{\partial}{\partial z} \arg p(z) = 0$ only at isolated points; i.e. $\arg p(z)$ is strictly monotone on $\mathbb{R}$.

This version of the proof is a modification of a proof of Lagarias' in [10].

Proof. If $z \in \mathbb{H}^-$ and ρ is a root of p then $|z - \rho| > |\overline{z} - \rho|$ since $\rho \in \mathbb{H}^+$. It follows that $|p(z)| > |p(\overline{z})|$ for all $z \in \mathbb{H}^-$. Now, let $z \in \mathbb{C}$. Then $p(\overline{z}) = f(\overline{z}) + ig(\overline{z}) = \overline{f(z)} + i\overline{g(z)} = \overline{f(z) - ig(z)}$. Thus $\overline{p(\overline{z})} = f(z) - ig(z)$. It follows that

$$f(z) = \frac{1}{2}[p(z) + \overline{p(\overline{z})}] \tag{2.21}$$

and

$$g(z) = \frac{1}{2i}[p(z) - \overline{p(\overline{z})}] \tag{2.22}$$

Then $|f(z)| \geq \frac{1}{2}\{|p(z)| - |p(\overline{z})|\} > 0$ if $z \in \mathbb{H}^-$, and it follows that f is real only on the real line. Similarly g is real only on the real line. It follows that the roots of f and g are real.

Now let $x \in \mathbb{R}$ and $h > 0$. Then,

$$\begin{aligned}
0 &< |p(x-ih)|^2 - |p(x+ih)|^2 = \\
&= |f(x-ih) + ig(x-ih)|^2 - |f(x+ih) + ig(x+ih)|^2 \\
&= \{|f(x-ih)|^2 + |g(x-ih)|^2 + 2\Re[\overline{f(x-ih)}ig(x-ih)]\} - \\
&\qquad - \{|f(x+ih)|^2 + |g(x+ih)|^2 + 2\Re[\overline{f(x+ih)}ig(x+ih)]\} \\
&= 2\Re\{i[f(x+ih)g(x-ih) - f(x-ih)g(x+ih)]\} \\
&= 2i\{f(x+ih)g(x-ih) - f(x-ih)g(x+ih)\} \\
&= 2i\{f(x+ih)[g(x-ih) - g(x+ih)] - g(x+ih)[f(x+ih) - f(x-ih)]\}
\end{aligned} \tag{2.23}$$

Dividing the last line above by h and letting it tend to zero we deduce that

$$f(x)g'(x) - f'(x)g(x) \geq 0 \tag{2.24}$$

From this last we get that $\frac{g(x)}{f(x)}$ has non-negative derivative if $x \in \mathbb{R}$ is not a root of f. This last would be enough to give us interlacing if we could prove $|m_\mu(f) - m_\mu(g)| \leq 1$ for any μ root of f or g (just think of the graph of the $\frac{f}{g}$). From the next to last equality in 2.23 we get that

$$-4|g(x+ih)|^2 \Im\left\{\frac{f(x+ih)}{g(x+ih)}\right\} > 0 \tag{2.25}$$

Using this last and the fact that $\frac{f}{g} \in \mathbb{R}(x)$ we get that $\frac{f}{g}$ is real only on the real line. By a standard argument (that is, by using the fact that around any of its zeros an analytic function has a Taylor expansion that begins with a term of order equal to the order the zero...), we get that the order of any zero or pole of $\frac{f}{g}$ is at most one. This gives us the interlacing. Since p has no real roots f and g cannot have a common root and therefore the interlacing of the roots is strict.

Since p has no zeros in some neighborhood of $\mathbb{H}^- \cup \mathbb{R}$, we can define an analytic branch of $\log p$ there. Then for z in this neighborhood

$$\begin{aligned}
\frac{\partial}{\partial z}\log p(z) = & \\
&= \frac{f'(z) + ig'(z)}{f(z) + ig(z)} \\
&= \frac{f'(z)f(z) + g'(z)g(z)}{|p(z)|^2} + i\frac{f(z)g'(z) - f'(z)g(z)}{|p(z)|^2}
\end{aligned} \tag{2.26}$$

Now write $p = u + iv$ where $u = \Re p$ and $v = \Im p$. Then, if $z = x + iy$,

$$\begin{aligned}
\frac{\partial}{\partial z}\log p(z) = & \\
&= \frac{u_x(z) + iv_x(z)}{u(z) + iv(z)} \\
&= \frac{u_x(z)u(z) + v_x(z)v(z)}{|p(z)|^2} + i\frac{u(z)v_x(z) - u_x(z)v(z)}{|p(z)|^2} \\
&= \frac{\partial}{\partial x}\log|p(z)| + i\frac{\partial}{\partial x}\arg p(z)
\end{aligned} \tag{2.27}$$

Then, since $f, g \in \mathbb{R}[x]$ and $\frac{\partial}{\partial x}\arg p(z) \in \mathbb{R}$, it follows that if $z \in \mathbb{R}$ then

$$\frac{\partial}{\partial x}\arg p(z) = \frac{f(z)g'(z) - f'(z)g(z)}{|p(z)|^2} \quad , \tag{2.28}$$

and, $\frac{\partial}{\partial x} \arg p(z) \geq 0$ follows from 2.24. Since f and g are not multiples of each other $\frac{f}{g}$ has only isolated zeros and therefore $f(z)g'(z) - f'(z)g(z) = 0$ only finitely many times. $\square$

Let p, f, g be as above. Then, as in the above proof $f(z) = \Re[p(z)]$ when $z \in \mathbb{R}$. Thus to find the roots of f we just need to solve $\cos(\arg p) = 0$; i.e. $\arg p = \frac{\pi}{2} \mod \pi$. But also, since $\arg p(z) > 0$ is a monotonous function, we just have to know $\arg p(b) - \arg p(a)$ to know how many roots there are in $[a, b]$.

Theorem 2.1.10. *If U and V are polynomials, with respective degrees d and $d+1$, with roots that interlace on the unit circle. Then, the roots the Hilbert polynomials of $\frac{U(t)}{(1-t)^{d+a}}$ and $\frac{V(t)}{(1-t)^{d+1+a}}$ that are on the line $\{\Re = -a/2\}$ will interlace strictly.*

Proof. We will use the following notation:

Definition 2.1.4. Let $U(t) = \sum_{m=0}^{d} h_m t^m \in \mathbb{C}[t]$. Then, we define

$$\begin{aligned} \mathcal{H}_a[U(t)](x) &= \sum_{m=0}^{d} h_m \binom{-x-m+d+a-1}{d+a-1} \\ &= B(-x, 1+x)^{-1} \int_0^\infty \frac{U(-t)}{(1+t)^{d+a}} t^{x-1} dt \quad (2.29) \end{aligned}$$

Assume that the roots of $U(t)$ and $V(t)$ interlace, not necessarily strictly, on the unit circle. The transformation $t \mapsto \frac{t+i}{t-i}$ takes the real line onto the unit circle and 0 to -1. So, $\tilde{U}(t) = (t-i)^d U(\frac{t+i}{t-i})$ and $\tilde{V}(t) = (t-i)^{d+1} V(\frac{t+i}{t-i})$ have real coefficients and roots that interlace on the real line. By the Hermite-Biehler theorem $f(t) = \tilde{U}(t) + i\tilde{V}(t)$ has all of its roots on one side of the real line except for maybe the common roots of $\tilde{U}$ and $\tilde{V}$. The map $t \mapsto i\frac{t+1}{t-1}$ is the inverse of $t \mapsto \frac{t+i}{t-i}$. Then, $F(t) = [-i(t-1)]^{d+1} f(i\frac{t+i}{t-i}) = -i2^d(t-1)U(t) + 2^{d+1}iV(t)$ has all of its roots on one side of the unit circle. Then, by corollary 2.1.5, $\mathcal{H}_a[F(t)](x)$ has all of its roots on one side of $\{\Re = -\frac{a}{2}\}$ except maybe a few in the set $\{-1, \ldots, -a+1\}$. Now $\mathcal{H}_a[F(t)](x) = i2^d\mathcal{H}_a[U(t)](x) + 2^{d+1}i\mathcal{H}_a[V(t)](x)$. The three polynomials in this last sum have $(x+1)_{a-1}$ as a common factor. Let $A(x) = \mathcal{H}_a[F(t)](x)/(x+1)_{a-1}$, $B(x) = i2^d\mathcal{H}_a[U(t)](x)/(x+1)_{a-1}$ and $C(x) = i2^{d+1}\mathcal{H}_a[V(t)](x)/(x+1)_{a-1}$. We end up with the equation $A(x) = iB(x) + C(x)$. Since the roots of $A(x)$ and $B(x)$ are on the line $\{\Re = -\frac{a}{2}\}$, $A(-\frac{a}{2}+iw)$, $iB(-\frac{a}{2}+iw)$ and $C(-\frac{a}{2}+iw)$ have real coefficients and $A(-\frac{a}{2}+iw)$ has all of its roots on one side of the real line except those that $B(-\frac{a}{2}+iw)$ and $C(-\frac{a}{2}+iw)$ might have in common in the set $\{-i(-1+\frac{a}{2}), \ldots, -i(-a+1+\frac{a}{2})\}$. It will then follow, again by the Hermite-Biehler theorem, that $B(-\frac{a}{2}+iw)$ and $C(-\frac{a}{2}+iw)$ have real interlacing roots. From all of this we conclude that the roots of $\mathcal{H}_a[U]$ different from $-1, \ldots, -a+1$ interlace strictly, on the line $\{\Re = -\frac{a}{2}\}$, with those of $\mathcal{H}_a[V]$. □

Remark 2.1.3. We can generate orthogonal sequences of Hilbert polynomials: Let $\{T_d\}_{d\geq 0}$ is a sequence of polynomials such that $\deg T_d = d$ and the leading coefficient of every T_d is positive. Assume the T_d satisfy a recurrence of the form :

$$0 = T_{d+1}(u) + (B_d - A_d a u)T_d(u) - A_d\left(u^2 + \frac{1}{4}\right)T_d'(u) + C_d T_{d-1}(u) \quad , \tag{2.30}$$

where $a > 0$ is fixed, and $\{A_d\}_{d\geq 0}$, $\{B_d\}_{d\geq 0}$ and $\{C_d\}_{d\geq 0}$ are sequences of real numbers with $A_d, C_d > 0$. Assume that for some $d \geq 0$ the roots of T_d and T_{d+1} are real and non strictly interlacing, i.e. if $\tau_{k1} \leq \ldots \leq \tau_{kk}$ are the roots of T_k, then, $\tau_{d+1\ 1} \leq \tau_{d1} \leq \tau_{d+1\ 2} \leq \ldots \leq \tau_{d+1\ d} \leq \tau_{dd} \leq \tau_{d+1\ d+1}$. Then, it can be shown, that the roots of T_k and T_{k+1} are real and interlace for $k = 0, \ldots, d$. Also, given any two polynomials f and g with $\deg g = \deg f + 1$ and real interlacing roots, we can include them in an infinite sequence that satisfies a recurrence of the same type as (2.30). Let,

$$P_d(t) = T_d\left(\frac{1}{2i}\frac{1+t}{1-t}\right)\frac{1}{(1-t)^a} \quad . \tag{2.31}$$

Then, the roots of the numerator of P_d are in $\partial\mathbb{D} \setminus \{1\}$. If H_d is the Hilbert polynomial of P_d, its roots will be on the line $\{\Re = -\frac{a}{2}\}$ by theorem 2.1.1. Define:

$$\phi_d(y) = H_d\left(-\frac{a}{2} + iy\right) \tag{2.32}$$

Then, the sequence $\{\phi_d\}_{d\geq 0}$ satisfies the recurrence:

$$0 = \phi_{d+1}(y) - (A_d y + B_d)\phi_d(y) + C_d\phi_{d-1}(y) \quad , \tag{2.33}$$

and the ϕ_d are real. The map $T_d \mapsto \phi_d$ also takes leading coefficient to leading coefficient. Recurrence (2.33) defines a sequence of orthogonal polynomials (see [15]). This last implies that the roots of ϕ_d and ϕ_{d+1} are real and strictly interlace. We get again:

Theorem 2.1.11. *Let* $P(t) = \frac{h(t)}{(1-t)^{\deg h + a}}$ *and* $Q(t) = \frac{j(t)}{(1-t)^{\deg j + a}}$ *where* h *and* j *are polynomials with roots that interlace on the unit circle and* $\deg j = \deg h + 1$.

Then, the Hilbert polynomials of P and Q have roots that interlace strictly on the line $\{\Re = -\frac{a}{2}\}$.

We can also use theorem 2.1.9 to study the roots of the Hilbert polynomial of $P_d(t) = \frac{\sum_{m=0}^{d} t^m}{(1-t)^{d+1}}$, namely $H_d(x) = \sum_{m=0}^{d} \binom{x-m+d}{d}$. This last is actually the *Ehrhart polynomial* of the simplex

$$S_n(1) = \operatorname{conv}\{e_1, \ldots, e_d, -\sum_{m=1}^{d} e_m\} \tag{2.34}$$

(see chapter 3 or [3]). Since $P_d(t) = \frac{\sum_{m=0}^{d} t^m}{(1-t)^{d+1}} = \frac{1-t^{d+1}}{(1-t)^{d+2}}$ we can write $H_d(x) = \binom{x+d+1}{d+1} - \binom{x}{d+1}$. The roots of $\binom{x+d+1}{d+1}$ are $\{-1, -2, \ldots, -d-1\}$ all to the left of $-\frac{1}{2}$. Also, $\binom{x}{d+1} = (-1)^{d+1}\binom{(-1-x)+d+1}{d+1}$. That is, $\binom{x+d+1}{d+1}$ is the reflection of $\binom{x}{d+1}$ across the line $\{\Re = -\frac{1}{2}\}$. It follows from the Hermite-Biehler theorem that $H_d(x)$ has all of its roots on the line $\{\Re = -\frac{1}{2}\}$, and the roots of $H_d(x)$ can be found by solving, for real τ:

$$\arg\binom{-\frac{1}{2}+i\tau}{d+1} = \begin{cases} \frac{\pi}{2} \pmod{\pi} & \text{if } d \text{ is even} \\ 0 \pmod{\pi} & \text{if } d \text{ is odd} \end{cases} \tag{2.35}$$

Now,

$$(d+1)!\binom{-\frac{1}{2}+i\tau}{d+1} = (-1)^{d+1}\left(d+\frac{1}{2}-i\tau\right)_{d+1} = (-1)^{d+1}\frac{\Gamma\left(d+\frac{1}{2}-i\tau\right)}{\Gamma\left(-\frac{1}{2}-i\tau\right)} \tag{2.36}$$

So, it is enough to study the argument of $\frac{\Gamma\left(d+\frac{1}{2}-i\tau\right)}{\Gamma\left(-\frac{1}{2}-i\tau\right)}$. Using Stirling's formula:

$$\begin{aligned}
&\arg\left\{\frac{\Gamma\left(d+\frac{1}{2}-i\tau\right)}{\Gamma\left(-\frac{1}{2}-i\tau\right)}\right\} = \\
&= \arg\left\{\Gamma\left(d+\frac{1}{2}-i\tau\right)\right\} - \arg\left\{\Gamma\left(-\frac{1}{2}-i\tau\right)\right\} \\
&= \left[-d\arctan\frac{\tau}{d+\frac{1}{2}} - \tau\log\frac{\sqrt{\left(d+\frac{1}{2}\right)^2+\tau^2}}{e} + O\left(\frac{1}{d+\frac{1}{2}-i\tau}\right)\right] - \\
&\qquad - \left[-\arctan 2\tau - \tau\log\frac{\sqrt{\left(\frac{1}{2}\right)^2+\tau^2}}{e} + O\left(\frac{1}{-\frac{1}{2}-i\tau}\right)\right]
\end{aligned} \tag{2.37}$$

We then have:

$$\begin{aligned}
&\arg\left\{\frac{\Gamma\left(d+\frac{1}{2}-i\tau\right)}{\Gamma\left(-\frac{1}{2}-i\tau\right)}\right\} = \\
&= \begin{cases}
-\tau\log d + O[\tau\log(|\tau|+1)] & \text{if } \tau = o(d) \\
-d(\arctan C + C\log\sqrt{1+C^{-2}}) + o(d) & \text{if } \tau \sim Cd \text{ for some } C \in \mathbb{R}^* \\
-\frac{d\pi}{2} + \frac{d^2}{\tau} + O\left(\frac{d^3}{\tau^2}\right) & \text{if } d = o(\tau)
\end{cases}
\end{aligned} \tag{2.38}$$

as $d \to \infty$. For example, let $x = -\frac{1}{2} + i\tau$ be the largest root of H_d with d even, $\tau > 0$. Then, $\arg\left\{\frac{\Gamma\left(d+\frac{1}{2}-i\tau\right)}{\Gamma\left(-\frac{1}{2}-i\tau\right)}\right\} = \frac{d+1}{2}\pi$. By looking at (2.38) we can see that $d = o(\tau)$ and therefore $\frac{d+1}{2}\pi = -\frac{d\pi}{2} + \frac{d^2}{\tau} + O\left(\frac{d^3}{\tau^2}\right)$ as $d \to \infty$. It follows that $x = -\frac{1}{2} + i\left[\frac{2d^2}{\pi} + o(d^2)\right]$. This last estimate can be improved by making a better estimate in (2.38).

In the following we generalize some of the above results to other types of Ehrhart polynomials.

2.2 Asymptotic lemmas

We give two lemmas about the asymptotic properties of certain polynomials on vertical strips. One lemma depends on the size of the coefficients of the h-polynomial while the other depends on its roots.

The proofs of the lemmas depend basically on Laplace's Method, which deals with the asymptotic expansion of integrals of the form:

$$\int_0^a e^{-x\phi(t)} q(t)dt \tag{2.39}$$

as $x \to \infty$, where $\phi(t)$ is of the form $-t + O(t^2)$ and is decreasing on $[0, a]$, and $q(t) \sim t^{\alpha-1}$ as $t \to 0^+$. A typical result for this type of asymtotic method is Watson's Lemma (see [13]).

Theorem 2.2.1. *Let $q(t)$ be a function of a positive real variable t, such that*

$$q(t) \sim \sum_{s\geq 0} a_s t^{(s+\lambda-\mu)/\mu} \quad (t \to 0) \tag{2.40}$$

where λ and μ are positive constants.

$$\int_0^\infty e^{-xt} q(t)dt \sim \sum_{s\geq 0} \Gamma\left(\frac{s+\lambda}{\mu}\right) \frac{a_s}{x^{(s+\lambda)/\mu}} \quad (x \to \infty) \tag{2.41}$$

provided the integral converges throughout its range for all sufficiently large x.

The proof of Watson's lemma is a consequence of the fact that the larger part of the area, under the graph of e^{-xt}, lies over ever smaller intervals of the form $[0, \epsilon)$ as x grows. This implies that the value of the integral in

(2.41) depends mostly on the asymptotic expansion of $q(t)$ when t is close to 0; the main term in the formula comes from

$$\int_0^{\epsilon} e^{-xt} \sum_{s=0}^{M} a_s t^{(s+\lambda)/\mu} dt \sim \int_0^{\infty} e^{-xt} \sum_{s=0}^{M} a_s t^{(s+\lambda)/\mu} dt = \sum_{s=0}^{M} \Gamma\left(\frac{s+\lambda}{\mu}\right) \frac{a_s}{x^{(s+\lambda)/\mu}} \tag{2.42}$$

In our results the functions $\phi(t)$ and $q(t)$ in (2.39) will vary with x. This will require that we be careful about how things change as x grows.

Lemma 2.2.2. *Let $\{B_n\}_{n\geq 0}$ be a sequence of positive numbers such that $B_n \to +\infty$. Let $\{\phi_n(t)\}_{n\geq 0}$ be a sequence of functions, analytic on the unit disk, such that:*

1. *Each $\phi_n(t)$ is increasing on $[0,1)$.*

2. *$\phi_n(t) = t + O(t^2)$ as $t \to 0^+$ and uniformly for all n.*

Let S be some compact subset of the right hand plane $\{\Re > 0\}$. Then, uniformly for all $x \in S$,

$$\int_0^1 e^{-B_n\phi_n(t)} t^{x-1} dt = B_n^{-x}\Gamma(x) + O(B_n^{-\Re x-1}) \sim B_n^{-x}\Gamma(x) \tag{2.43}$$

as $n \to \infty$.

Proof. Since $\phi_n(t) \sim t$ uniformly for all n as $t \to 0$, $\phi_n(\alpha) > \alpha/2$ for all small enough $\alpha \in (0,1)$ and for all n. Then, since all ϕ_n are increasing,

$$E_0 = \int_\alpha^1 e^{-B_n\phi_n(t)} t^{x-1} dt \ll \\ \ll e^{-B_n\phi_n(\alpha)} \int_\alpha^1 t^{\Re x - 1} dt \\ \ll O_{\Re x}\{e^{-\frac{B_n \alpha}{2}}\}, \quad (2.44)$$

as $n \to +\infty$.

$$\begin{aligned} &\int_0^\alpha e^{-B_n\phi_n(t)} t^{x-1} dt = \\ &= \int_0^\alpha e^{-B_n t} e^{B_n[t-\phi_n(t)]} t^{x-1} dt \\ &= \int_0^\infty e^{-B_n t} t^{x-1} dt - \int_\alpha^\infty e^{-B_n t} t^{x-1} dt + \int_0^\alpha e^{-B_n t}\{e^{B_n[t-\phi_n(t)]} - 1\} t^{x-1} dt \\ &= B_n^{-x}\Gamma(x) - E_1 + E_2 \end{aligned} \quad (2.45)$$

Now,

$$|E_1| = \left|\int_\alpha^\infty e^{-B_n t} t^{x-1} dt\right| \le e^{-(B_n-1)\alpha} \int_\alpha^\infty e^{-t} t^{\Re x - 1} dt \ll_{\Re x} e^{-B_n \alpha}, \quad (2.46)$$

as $n \to +\infty$.

By item.2 in the statement of the lemma, $f(t) = e^{B_n O(t^2)}$ for $t = o(\sqrt{B_n})$ as $n \to \infty$, and uniformly for all n. Then, there is $C > 0$ such that $f(t) - 1 \ll e^{CB_n t^2} - 1 = O(B_n t^2)$ for t and n as before. Taking $\alpha = o(\sqrt{B_n})$ as $n \to \infty$,

$$\begin{aligned} E_2 &= \int_0^\alpha e^{-B_n t}\{e^{B_n[t-\phi_n(t)]} - 1\} t^{x-1} dt \\ &\ll \int_0^\alpha e^{-B_n t} B_n t^{\Re x + 1} dt \\ &\ll B_n^{-\Re x - 1}\Gamma(\Re x) \end{aligned} \quad (2.47)$$

as $n \to \infty$.

Finally, take $\alpha = (\Re x + 1)\frac{\log B_n}{B_n}$ in all the above estimates (notice that α is still smaller than $\frac{1}{\sqrt{B_n}}$ for large enough n). We get that, as $n \to \infty$,

$$E_0, E_1, E_2 \ll B_n^{-\Re x - 1} \tag{2.48}$$

Putting all the estimates together:

$$\int_0^1 e^{-B_n \phi_n(t)} t^{x-1} dt = B_n^{-x}\Gamma(x) + O_{\Re x}\{B_n^{-\Re x - 1}\}, \tag{2.49}$$

as $n \to \infty$. Immiadetely, it also follows that:

$$\int_0^1 e^{-B_n \phi_n(t)} t^{x-1} dt \sim B_n^{-x}\Gamma(x) \tag{2.50}$$

as $n \to \infty$, which is (2.43). $\square$

Theorem 2.2.3. *Let $\{U_n(t)\}$ be a sequence of real polynomials with $U_n(t) = 1 + \sum_{m=1}^{d_n} h_{nm} t^m$. Suppose every root of U_n is on $\partial\mathbb{D} \setminus \{1\}$ (the unit circle minus 1). Let $P_n(t) = \frac{U_n(t)}{(1-t)^{d_n+1}} = \sum_{m \geq 0} H_n(m) t^m$. Suppose $h_{n1} > 0$ for all n, and, $d_n \to +\infty$. Let S be a compact subset of $\{\Re \in (0,1)\}$. Then, as $n \to \infty$ and uniformly for all $x \in S$:*

$$\begin{aligned}\int_0^\infty P_n(-t) t^{x-1} dt = H_n(1)^{-x}\Gamma(x) + O[H_n(1)^{-\Re x - 1}] \\ + (-1)^{d_n} H_n(1)^{x-1}\Gamma(1-x) + O[H_n(1)^{\Re x - 2}]\end{aligned} \tag{2.51}$$

Proof. Let α be a root of U_n. Since α is on the unit circle, but different from 1, it can only be -1 or not real. Let $t \in (-1, 0]$. If $\alpha = -1$ then:

$$\frac{d}{dt}\log\frac{t-\alpha}{1-t} = \frac{1}{t-\alpha} + \frac{1}{1-t} = \frac{1}{t+1} + \frac{1}{1-t} = \frac{2}{1-t^2} > 0 \tag{2.52}$$

If $\alpha \neq -1$ then $\overline{\alpha} \neq \alpha$ is also a root since U_n is real. Then,

$$\frac{d}{dt}\log\frac{t-\alpha}{1-t}\frac{t-\overline{\alpha}}{1-t} = \frac{1}{t-\alpha} + \frac{1}{t-\overline{\alpha}} + \frac{1}{1-t} = 2(1-\Re\alpha)\frac{t+1}{|t-\alpha|^2(1-t)} > 0 \tag{2.53}$$

So $P_n(t)$ is increasing on $(-1, 0]$ and positive since it is the product of positive increasing functions. Let $\phi_n(t) = -\frac{\log P_n(-t)}{H_n(1)}$. Now, $H_n(1) = d_n + 1 + h_{n1} > 0$. Then, the $\phi_n(t)$ satisfy condition 1. in the statement of lemma 2.2.2 on $[0, 1)$. Now, for $t \in [0, 1)$,

$$\begin{aligned}
\log P_n(-t) &= \log U_n(-t) - (d_n+1)\log(1+t) \\
&= \log \prod_{\alpha \text{ root of } U_n} \left(1 + \frac{t}{\alpha}\right) - (d_n+1)\log(1+t) \\
&= \sum_{m\geq 1} \frac{(-1)^{m+1}}{m}\left[\sum_\alpha \frac{1}{\alpha^m} - (d_n+1)\right] t^m \\
&= \left[\sum_\alpha \frac{1}{\alpha} - (d_n+1)\right] \sum_{m\geq 1} \frac{(-1)^{m+1}}{m}\left[\frac{\sum_\alpha \frac{1}{\alpha^m} - (d_n+1)}{\sum_\alpha \frac{1}{\alpha} - (d_n+1)}\right] t^m \\
&= -H_n(1) \sum_{m\geq 1} \frac{(-1)^{m+1}}{m}\left[\frac{\sum_\alpha \frac{1}{\alpha^m} - (d_n+1)}{-H_n(1)}\right] t^m
\end{aligned} \tag{2.54}$$

where $H_n(1) = h_{n1} + d_n + 1 \geq d_n + 1$ since $h_{n1} > 0$. Since the roots of $U_n(t)$ are on the unit circle, then, for $m > 1$,

$$\begin{aligned}
\left|\frac{\sum_\alpha \frac{1}{\alpha^m} - (d_n+1)}{-H_n(1)}\right| &\leq \frac{\sum_\alpha \frac{1}{|\alpha|^m} + d_n + 1}{H_n(1)} \\
&< \frac{\sum_\alpha 3^m + d_n + 1}{d_n+1} \\
&< 3^m + 1
\end{aligned} \tag{2.55}$$

Then, for $t \in [0, \frac{1}{3})$, $\phi_n(t) = t + O(\sum_{m\geq 2} \frac{3^m}{m} t^m) = t + O(t^2)$, where the constant in the error term does not depend on n. So $\phi_n(t)$ satisfies condition 2. of lemma

2.2.2. Let $P_n^*(t) = P_n(t^{-1})t^{-1}$. Then, since each U_n is real, with roots on the unit circle and leading coefficient 1,

$$P_n^*(t) = P_n(t^{-1})t^{-1} = (-1)^{d_n+1}\frac{t^{d_n}U_n(t^{-1})}{(1-t)^{d_n+1}} = (-1)^{d_n+1}\frac{U_n(t)}{(1-t)^{d_n+1}} \tag{2.56}$$

where a_n is the multiplicity of -1 as a root of U_n. Then, since the ϕ_n satisfy the hypotheses of 2.2.2,

$$\begin{aligned}
\int_0^\infty P_n(-t)t^{x-1}dt &= \int_0^1 P_n(-t)t^{x-1}dt + \int_1^\infty P_n(-t)t^{x-1}dt \\
&= \int_0^1 P_n(-t)t^{x-1}dt + (-1)^{d_n}\int_0^1 P_n(-t)t^{-x}dt \\
&= \int_0^1 e^{-H_n(1)\phi_n(t)}t^{x-1}dt + (-1)^{d_n}\int_0^1 e^{-H_n(1)\phi_n(t)}t^{-x}dt \\
&= H_n(1)^{-x}\Gamma(x) + O[H_n(1)^{-\Re x-1}]+ \\
&\qquad + (-1)^{d_n}H_n(1)^{x-1}\Gamma(1-x) + O[H_n(1)^{\Re x-2}]
\end{aligned} \tag{2.57}$$

□

Chapter 3

Example: Ehrhart polynomial of the cross-polytope

Ehrhart polynomials provide good examples for testing our estimates; some important families of Ehrhart polynomials satisfy the hypotheses of our theorems in a more or less natural way. Also, although Ehrhart polynomials have a combinatorial origin, we will study their analytic properties. This last allows us to use analytic number theory, specifically the theory of L-functions, as a source of inspiration for new analytic-combinatoric theories.

Let $\mathbb{Z}^n$ be the n-dimensional lattice of integer coordinate points in $\mathbb{R}^n$. A *convex integral polytope*, $\mathcal{P}$, in $\mathbb{R}^n$ is the convex hull of some finite set of points in $\mathbb{Z}^n$. We say $\mathcal{P}$ is d-dimensional if its interior has topological dimension d. For $m \in \mathbb{Z}^{>0}$ let $m\mathcal{P}$ be the polytope obtained by dilating $\mathcal{P}$ by a factor of m. As a reference we mention [11]. We have the following theorem due to Ehrhart:

Theorem 3.0.4 (Ehrhart 1962)**.** *Let $\mathcal{P}$ be a d-dimensional integral lattice polytope in $\mathbb{R}^n$. Then, if $\mathcal{L}_{\mathcal{P}}(m) = \#\{p \in \mathbb{Z}^n : p \in m\mathcal{P}\}$, then, $\mathcal{L}_{\mathcal{P}}(m)$ is a polynomial in m of degree d.*

Definition 3.0.1. We call $\mathcal{L}_{\mathcal{P}}(m)$ the *Ehrhart polynomial* of the polytope $\mathcal{P}$.

More can be said about the Ehrhart polynomial:

Theorem 3.0.5 (Ehrhart 1962). *Let $\mathcal{P}$ be a d-dimensional integral lattice polytope in $\mathbb{R}^n$. Then, the generating function of the $\mathcal{L}_{\mathcal{P}}(m)$ is a rational function of the form:*

$$\mathcal{E}_{\mathcal{P}}(t) = \frac{\sum_{n=0}^{d} h_n t^n}{(1-t)^{d+1}} = \sum_{m \geq 0} \mathcal{L}_{\mathcal{P}}(m) t^m \tag{3.1}$$

where $\{h_0, \ldots, h_d\} \subset \mathbb{Z}$ and $h_0 = 1$.

Definition 3.0.2. We call the rational function $\mathcal{E}_{\mathcal{P}}(t)$ above the *Ehrhart series* of the polytope $\mathcal{P}$. We call the polynomial $h_{\mathcal{P}}(t) = \sum_{n=0}^{d} h_n t^n$ the *h-polynomial* of the polytope $\mathcal{P}$, and, $(h_0, \ldots, h_d)$ its h-polynomial.

We have the following result due to Stanley.

Theorem 3.0.6 (Stanley 1980). $\{h_1, \ldots, h_d\} \subset \mathbb{Z}^{\geq 0}$.

The coefficients of Ehrhart polynomials and their h-polynomials contain interesting information. Write

$$\mathcal{L}_{\mathcal{P}}(z) = \sum_{n=0}^{d} h_d \binom{z-n+d}{d} = \sum_{n=0}^{d} c_n z^n \tag{3.2}$$

We have,

$$\begin{aligned} c_0 &= h_0 = 1 \\ h_1 &= \mathcal{L}_{\mathcal{P}}(1) - d - 1 \\ h_d &= \#\{\text{interior points of } \mathcal{P}\} \\ c_d &= \frac{\sum_{n=0}^{d} h_n}{d!} = \mathrm{Vol}(\mathcal{P}) \end{aligned} \tag{3.3}$$

An important result is:

Theorem 3.0.7 (Ehrhart Reciprocity)**.** *Let* $\mathcal{L}^{\circ}(\mathcal{P}, m) = \#\{\mathbf{p} \in \mathbb{Z}^n : \mathbf{p} \in \mathcal{P}^{\circ}\}$*; that is , the Ehrhart polynomial that counts the interior lattice points of the integral dilates of* $\mathcal{P}$*. Then,*

$$\mathcal{L}_{\mathcal{P}}^{\circ}(m) = (-1)^d \mathcal{L}_{\mathcal{P}}(-m) \tag{3.4}$$

In terms of the Poincaré series of $\mathcal{P}$*:*

$$\mathcal{E}_{\mathcal{P}^{o}}(t) = (-1)^{d+1} \mathcal{E}_{\mathcal{P}} \left(\frac{1}{t} \right) \tag{3.5}$$

Our results about the distribution of roots apply well to certain types of Ehrhart polynomials.

Definition 3.0.3. A d dimensional polytope $\mathcal{P}$ is called reflexive if, and only if, there exists a $d \times d$ integral matrix $\mathbf{A}$ such that $\mathcal{P} = \{\mathbf{z} \in \mathbb{R}^d : \mathbf{Az} \leq 1\}$.

Suppose $\mathcal{P}$ is a reflexive polytope of dimension d with Ehrhart series $\frac{h(t)}{(1-t)^{d+1}}$. We have:

- $L(\mathcal{P}, m) = (-1)^d L(\mathcal{P}, -1-m)$.
- $h_0 = h_1 = h_d = 1$.
- $h_n = h_{d-n}$.

We are interested in studying the roots of Ehrhart polynomials. A general theorem about the location of the roots of an Ehrhart polynomial is:

Theorem 3.0.8 (Braun 2006)**.** *Let $L(x) = \sum_{n=0}^{d} h_n \binom{x-n+d}{d}$ be a polynomial such that $h_n \geq 0$ for all n. Then the roots of $L(x)$ are all inside a circle of radius $d(d - \frac{1}{2})$ centered at $-\frac{1}{2}$. The real roots of $L(x)$ are in the interval $[-d, \lfloor \frac{d}{2} \rfloor)$.*

So the roots of Ehrhart polynomials of polytopes of dimension d are bounded. The theorem also gives a lower bound for the volume of of $\mathcal{P}$: $\mathrm{Vol}(\mathcal{P}) \geq d^{-2d}$.

We also have:

Theorem 3.0.9 (Braun 2006)**.** *Let $\mathcal{P}$ be a polytope of dimension d. If all the roots of $\mathcal{L}_{\mathcal{P}}(z)$ have real part $-\frac{1}{2}$, then, $\mathcal{P}$ is the unimodular image of a reflexive polytope with Vol$(\mathcal{P}) \leq 2^d$.*

The theorems of the previous chapter apply particularly well to the Ehrhart polynomial of the *crosspolytope*. Let $\{e_1, \ldots, e_d\}$ be the canonical basis for $\mathbb{R}^d$; i.e. $e_1 = (1, 0, \ldots, 0), e_2 = (0, 1, 0, \ldots, 0), \ldots$ The *d-dimensional cross-polytope*, $\mathcal{P}_d^{\Diamond}$, is the convex hull of the set $\{\pm e_1, \ldots, \pm e_d\}$. Let $\mathcal{L}_d^{\Diamond}(x)$ be the Ehrhart polynomial of the d-dimensional cross-polytope. Its Ehrhart series is $\mathcal{E}_d^{\Diamond}(t) = \frac{(1+t)^d}{(1-t)^{d+1}}$, and we can write:

$$\mathcal{L}_d^{\Diamond}(x) = \sum_{m=0}^{d} \binom{d}{m} \binom{d+x-m}{d} \tag{3.6}$$

By theorems 3.0.9 and 2.1.1, $\mathcal{P}_d^{\Diamond}$ is reflexive. Let $\delta > 0$. Suppose $x \in S =$

$\left\{\Re=\frac{1}{2},\Im\in[a,b]\right\}$. Then, by formula (2.51) and theorem 2.1.6,

$$\begin{aligned}&\mathfrak{L}_d^{\Diamond}(-x)B(x,1-x)=\\&=\int_0^\infty\frac{(1-t)^d}{(1+t)^{d+1}}t^{x-1}dt\\&=\int_0^1\frac{(1-t)^d}{(1+t)^{d+1}}t^{x-1}dt+(-1)^d\int_0^1\frac{(1-t)^d}{(1+t)^{d+1}}t^{-x}dt\\&=(2d+1)^{-x}\Gamma(x)+g_d(x)+(-1)^d[(2d+1)^{x-1}\Gamma(1-x)+g_d(1-x)],\end{aligned}\tag{3.7}$$

where $g_d(x)=\int_0^1\frac{(1-t)^d}{(1+t)^{d+1}}t^{x-1}dt-(2d+1)^{-x}\Gamma(x)=O_S[(2d+1)^{-\Re x-1}]$. Also, $g_d(x)=O[(2d+1)^{-\frac{3}{2}}]$ and is analytic on S since it is the sum of two analytic functions. By lemma 2.1.2, the roots of $\mathfrak{L}_d^{\Diamond}(x)$ are simple and have real part $-\frac{1}{2}$. Furthermore, for $x\in\left\{\Re=\frac{1}{2}\right\}$:

$$\mathfrak{L}_d^{\Diamond}(-x)B(x,1-x)=\begin{cases}2\Re\left[(2d+1)^{-x}\Gamma(x)+g_d(x)\right] & \text{, if } d \text{ is even}\\ 2i\Im\left[(2d+1)^{-x}\Gamma(x)+g_d(x)\right] & \text{, if } d \text{ is odd}\end{cases}\tag{3.8}$$

$B(x,1-x)=\frac{\pi}{\sin\pi x}$ has no zeros in $\{\Re\in(0,1)\}$. Then, to find roots of $\mathfrak{L}_d^{\Diamond}(-x)$ in S we just have to solve:

$$\arg\left\{(2d+1)^{-x}\Gamma(x)+g_d(x)\right\}=\begin{cases}\frac{\pi}{2}\mod\pi & \text{, if } d \text{ is even}\\ 0\mod\pi & \text{, if } d \text{ is odd}\end{cases}\tag{3.9}$$

Let $x=\frac{1}{2}+i\tau\in S$. Then, since S is compact, there is $C>0$ such that:

$$|(2d+1)^{-x}\Gamma(x)|=\sqrt{\frac{\pi}{(2d+1)\sin\pi x}}\sim\sqrt{\frac{\pi}{(2d+1)e^{\pi|\tau|}}}\geq C\sqrt{\frac{1}{2d+1}}\quad\text{, as } d\to\infty\tag{3.10}$$

Then, using Stirling's formula, as $d \to \infty$,

$$\begin{aligned}
&\arg\left\{(2d+1)^{-x}\Gamma(x) + g_d(x)\right\} = \\
&\qquad = \arg\left\{(2d+1)^{-x}\Gamma(x)\left[1 + \frac{g_d(x)}{(2d+1)^{-x}\Gamma(x)}\right]\right\} \\
&\qquad = \arg\left\{(2d+1)^{-x}\Gamma(x)\right\} + \arg\left\{1 + O[(2d+1)^{-1/2}]\right\} \\
&\qquad = \arg\left\{(2d+1)^{-x}\Gamma(x)\right\} + O[(2d+1)^{-1/2}] \\
&\qquad = \tau \log \frac{|x|}{e(2d+1)} + O[(2d+1)^{-1/2}] \\
&\qquad = -\tau \log d + O_S(1) \quad (3.11)
\end{aligned}$$

Now, if $f = u + iv$ is some analytic function with real and imaginary parts u, v, we can write the partial derivatives of the component functions u and v in terms of the derivative of f. In fact, if f is a function of the complex variable $x = \sigma + i\tau$, with σ and τ in $\mathbb{R}$, then, using the Cauchy-Riemann equations for example,

$$\frac{d}{dt} v(x) = \Re f'(x) \tag{3.12}$$

This means that we can use Cauchy's theorem to bound the partial derivatives of u and v in terms of f. Then, since g_d is analytic and different from 0 inside S, we can use Cauchy's estimate to obtain:

$$\begin{aligned}
&\frac{d}{d\tau}\arg\left\{(2d+1)^{-x}\Gamma(x) + g_d(x)\right\} = \\
&\qquad = \Re \frac{d}{dx} \log\left\{(2d+1)^{-x}\Gamma(x) + g_d(x)\right\} = -\log d + O_S(1) < 0, \quad (3.13)
\end{aligned}$$

for large enough d. Here we took the argument to be the principal branch with $\arg = 0$ on the positive real line. It follows that for large enough d,

d	A_d	$\pi^{-1}\log\mathfrak{L}_d^{\Diamond}(1)$	$A_d/[\pi^{-1}\log\mathfrak{L}_d^{\Diamond}(1)]$
20	5	5.910333509439762583506098820	0.8459759490753250877251994932
40	6	6.993983051321195559949425642	0.8578802602140381808956421261
60	7	7.632737713650990113309660175	0.9171021280451767544900753110
80	7	8.087306225360107642977192040	0.8655539687676791786671237315
100	7	8.440471908411114270043407555	0.8293375152429981303364420175
120	8	8.729325438203072426633897195	0.9164511114442763027284077592
140	8	8.973720165297346507445165627	0.8914920292407979225960235041
160	8	9.185533835099821764157439888	0.8709346831242782922550340430
180	8	9.372440363335864551342398611	0.8535663807790416463254578944
200	9	9.539685898579927602476961794	0.9434272884539872587842239782
220	9	9.691015906357396968868726684	0.9286952046065589389066362241
240	9	9.829198039167814210920382540	0.9156392987643966539059128601

Table 3.1: $A_d = \#\{\text{roots of } \mathfrak{L}_d^{\Diamond} \text{ with } \Im \in [0,1]\}$

$\arg\{(2d+1)^{-x}\Gamma(x) + g_d(x)\}$ is a monotone function of τ for $x \in S$. Then, by (3.9), as $d \to \infty$,

$$\begin{aligned}
&\#\{\text{roots of } \mathfrak{L}_d^{\Diamond} \text{ with } \Im \in [a,b]\} = \\
&\qquad = \frac{1}{\pi}\arg\left\{(2d+1)^{-\frac{1}{2}-ib}\Gamma(\frac{1}{2}+ib) + g_d(\frac{1}{2}+ib)\right\} - \\
&\qquad - \frac{1}{\pi}\arg\left\{(2d+1)^{-\frac{1}{2}-ia}\Gamma(\frac{1}{2}+ia) + g_d(\frac{1}{2}+ia)\right\} + O_S(1) \qquad (3.14)
\end{aligned}$$

Where the error above is actually less than 1. Then,

$$\#\{\text{roots of } \mathfrak{L}_d^{\Diamond} \text{ with } \Im \in [a,b]\} \sim \frac{\log d}{\pi}(b-a) \quad ,\text{as } d \to \infty \qquad (3.15)$$

Chapter 4

Some remarks

- As can be seen in the figure the approximation given by (2.51) is only good on a small segment of the line $\{\Re = \frac{1}{2}\}$. The author claims that he can prove an estimate which works on segments of the form $\{\Re = \frac{1}{2}, \Im \in [0, d)\}$, at least in the case of $\mathcal{L}_d^{\lozenge}$. In fact it seems that the largest root of $\mathcal{L}_d^{\lozenge}$ has absolute value $O(d)$ as $d \to \infty$. This will be part of future work.

- One of the benefits of approximating the roots of $\mathcal{L}_d^{\lozenge}$ is that it might allow us to estimate its coefficients. The coefficients of Ehrhart polynomials are not well understood in many cases. In some particular cases they are known to contain geometrical information. As we saw before, the leading coefficient of an Ehrhart polynomial is the volume of the associated polytope.

- The asymptotic results proven here can be proven for Hilbert polynomials whose h-polynomials have large degree and have roots outside the unit circle. The roots must be contained in an appropriate set containing the unit circle. We can also prove a similar result when the coefficients of the h-polynomial are bounded and its degree is large.

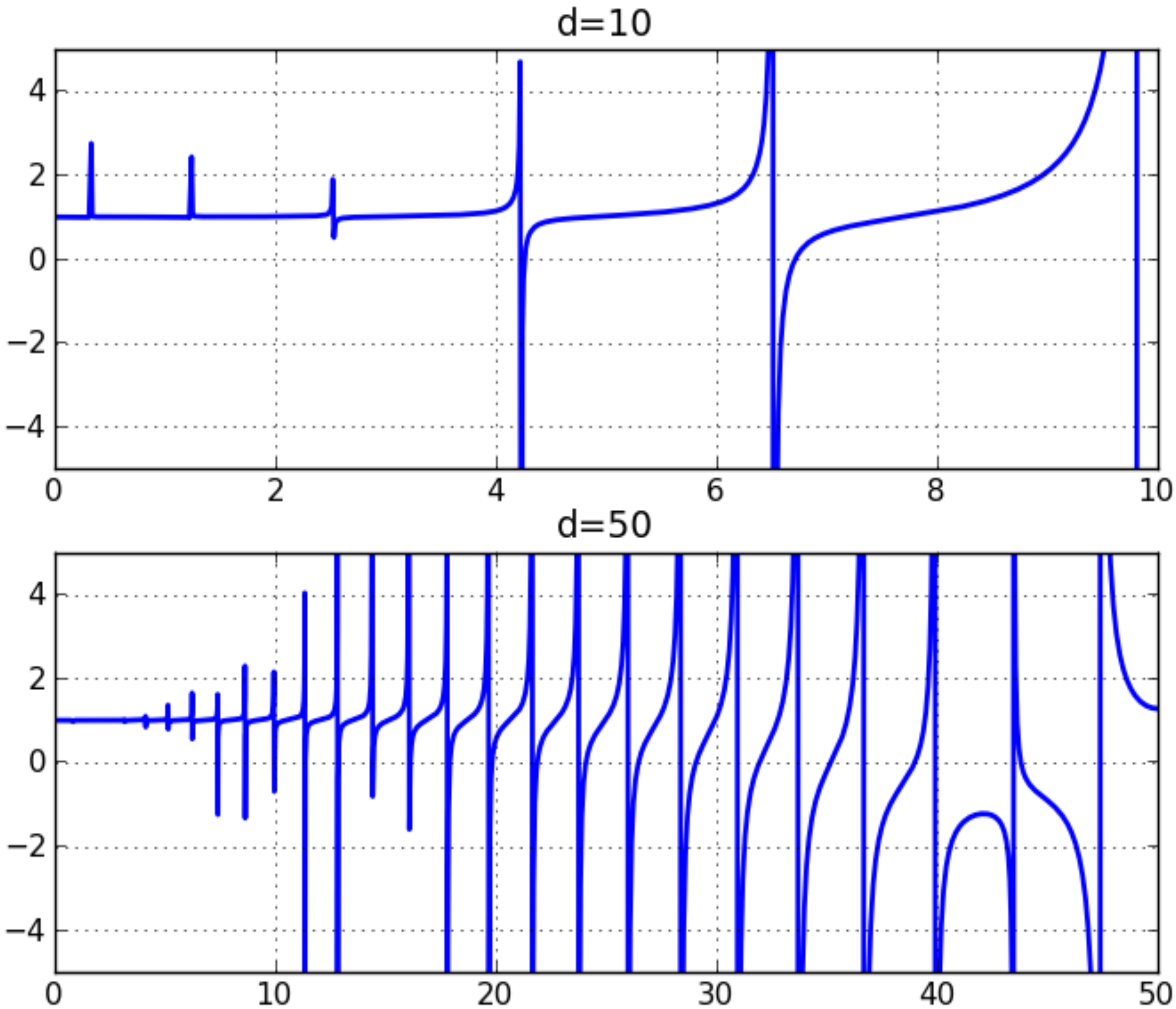

Figure 4.1: $\frac{\mathcal{L}_d^{\Diamond}(x)B(-x,1+x)}{(2d+1)^x\Gamma(-x)+(-1)^d(2d+1)^{-x-1}\Gamma(1+x)}$ for $d = 10, 50$ with $\Re x = -\frac{1}{2}$. There is a vertical asymptote at each zero of $f_{2d+1,2d+1}$. Notice that the approximation is best when $\Im x$ is close to 0.

- If one follows the above proof of the estimate:

$$\#\{\text{roots of } \mathcal{L}_d^{\Diamond} \text{ with } \Im \in [a,b]\} \sim \frac{\log d}{\pi}(b-a) \tag{4.1}$$

  as the dimension d of the cross-polytope goes to ∞ for fixed $a < b$, one can see that the above result is true if we replace the sequence of $\mathcal{L}_d^{\Diamond}$ with any sequence of Hilbert polynomials $\{H_d\}_{d\geq 0}$, if the latter come from Poincaré series with all of their zeros on the unit circle, and positive coefficients for their respective h-polynomials. The estimate is identical to the one above since, in this last case, $d+1 \leq H_d(1) \leq 2d+1$.

- By remark 2.1.3 we can generate families of polynomials orthogonal on the line $\{\Re = \frac{1}{2}\}$. We can generate them by choosing two polynomials with roots that interlace on the unit circle and using the recurrences in remark 2.1.3. We can then use the asymptotic estimates in this dissertation to study the asymptotic properties of the above orthogonal families.

Bibliography

[1] N. Anderson, E. B. Saff, and R. S. Varga. On the Eneström-Kakeya theorem and its sharpness. *Linear Algebra and its Applications*, 28:5–16, 1979.

[2] Matthias Beck and Alan Stapledon. On the log-concavity of hilbert series of veronese subrings and ehrhart series. *Mathematische Zeitschrift*.

[3] Christian Bey, Martin Henk, and Joerg M. Wills. Notes on the roots of ehrhart polynomials, 2006.

[4] Benjamin Braun. An ehrhart series formula for reflexive polytopes. *Electronic Journal of combinatorics*, 13(15), 2006.

[5] D. Bump, K. Choi, P. ar, K. Je, and r Vaaler. A local riemann hypothesis.

[6] Henri Cohen. *Number theory*, volume II: Analytic and Modern tools. Springer Verlag, 2007.

[7] Juan B. Gil and Sinai Robins. Hecke operators on rational functions, 2003.

[8] Paul Gunnells and Fernando Villegas. Lattice polytopes, hecke operators, and the ehrhart polynomial. *Selecta Mathematica, New Series*, 13(2):253–276, 2007.

[9] J. Keating and N. Snaith. Random matrix theory and, 2000.

[10] Jeffrey C. Lagarias. Zero spacing distributions for differenced l-functions. *ACTA ARITHMETICA*, 120:159, 2005.

[11] Sinai Robins Mathias Beck. *Computing the Continuous Discretely.* Undergraduate texts in mathematics. Springer-Verlag, 2007.

[12] Victor H. Moll, Sinai Robins, and K. Soodhalter. The action of hecke operators on hypergeometric functions. Aug 2008.

[13] F. W. J. Olver. *Asymptotics and Special Functions.* Academic Press, 1974.

[14] Fernando Rodriguez-Villegas. On the zeros of certain polynomials. *Proc. Amer. Math. Soc.*, 130(8):2251–2254 (electronic), 2002.

[15] Gábor Szegő. *Orthogonal polynomials.* American Mathematical Society, Providence, R.I., fourth edition, 1975. American Mathematical Society, Colloquium Publications, Vol. XXIII.

[16] N. M. Temme. Uniform asymptotic expansions of confluent hypergeometric functions. *Journal of the Institute of Mathematics and its Applications*, 22(2):215–223, 1978.

Vita

Miguel Antonio Rodríguez González was born in Caracas, Venezuela on 28 November 1972. He received an undergraduate degree in Mathematics from Simón Bolívar University in 2000. He received a Master's degree in Mathematics from the same university in 2003. His Master's thesis was a primality test for numbers of the form $5^p - F_p 5^{\frac{p+1}{2}} + 1$ where p is a prime number and F_p is the p-th Fibonacci number. He applied and was accepted into the doctoral program in mathematics at University of Texas at Austin in 2003. He has spent the last few years studying under the supervision of Professor Fernando Rodríguez-Villegas.

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