# When is a Graph Knotted?

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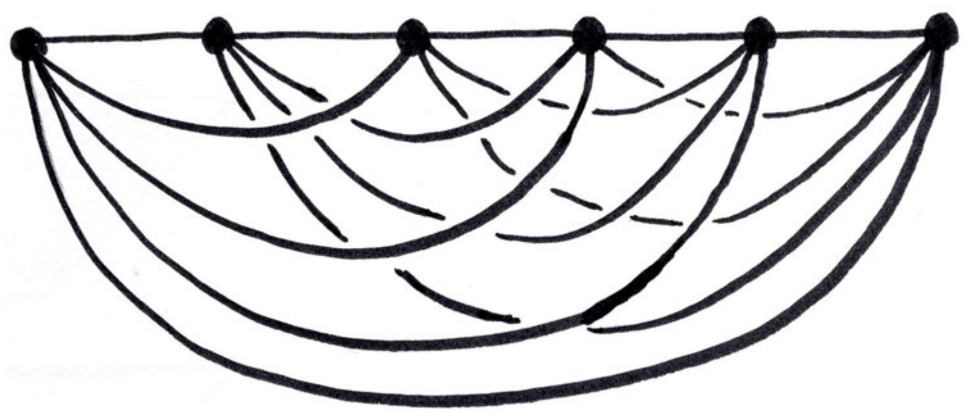


**Abstract**

Knot theory, as traditionally studied, asks whether or not a loop of string is knotted. That is, can we deform the loop in question into a circle without cutting or breaking it. In this thesis, I take a less traditional approach, studying networks of points connected together by string (i.e. a graph) instead of loops. By tracing different paths through this network we can identify many loops (i.e. cycles) in the network, each of which may or may not be knotted. Perhaps surprisingly, there will always be some knotted loop in a sufficiently complicated network. Such "sufficiently complicated" networks are called *intrinsically knotted* graphs. Very complicated graphs are always intrinsically knotted, and very simple graphs are always not, but graphs in between may be harder to identify. In this thesis, I present a method to reduce the question "Is the graph G intrinsically knotted?" to a linear algebra problem mod 2. Using this method I present a computer program that systematizes intrinsic knotting proofs and subsumes previous proof techniques. This program may lead to a conjecture for the intrinsic knotting obstruction set.

# Contents


**Preface** **iv**

**Acknowledgments** **vi**

**1 Introduction** **1**
- 1.1 Abstract Graphs . . . . . . . . . . . . . . . . . . . . . . . . . 1
- 1.2 Planar Graphs . . . . . . . . . . . . . . . . . . . . . . . . . 2
- 1.3 Links and Knots . . . . . . . . . . . . . . . . . . . . . . . . 4
- 1.4 Linked and Knotted Graphs . . . . . . . . . . . . . . . . . . 5
- 1.5 Progress on this and related problems . . . . . . . . . . . . 7
- 1.6 Endnote 1 on a graph $A$ "containing" a graph $B$ . . . . . . 8
- 1.7 Endnote 2 on checking whether a diagram is linked . . . . 8

**2 Background** **10**
- 2.1 Knot Theory . . . . . . . . . . . . . . . . . . . . . . . . . . 10
- 2.2 Graph Theory . . . . . . . . . . . . . . . . . . . . . . . . . 20
- 2.3 Knotted Graphs . . . . . . . . . . . . . . . . . . . . . . . . 28
- 2.4 Linear Algebra (mod 2) and the Edge Space . . . . . . . . . 42

**3 When is a Graph Intrinsically Knotted?** **54**
- 3.1 The Linking Number (mod 2) is Bilinear . . . . . . . . . . . 57
- 3.2 Crossing Moves are Symmetric Bilinear Forms . . . . . . . 59
- 3.3 Reduction to Linear Algebra . . . . . . . . . . . . . . . . . 66
- 3.4 A Program for Intrinsically Linked Graphs . . . . . . . . . 68

**A Open Questions** **69**

# Preface

About a year ago, I went up to professor Cameron Gordon after class with an objection: "$K_9 - C_9$ might actually be intrinsically knotted." It was the end of the semester and we had just begun studying intrinsically knotted graphs in Dr. Gordon's undergraduate knot theory class. As a class we were trying to show that there were no new minor-minimal intrinsically knotted graphs on 9 vertices, following (so and so and so) who had cleaned out the 8 vertex case. Dr. Gordon would start listing graphs on the board and we would go through and make arguments for why they had to be intrinsically knotted, or not. When I tried to reproduce the hasty argument that $K_9 - C_9$ was not intrinsically knotted, I hit a snag. It's actually intrinsically knotted. Dr. Gordon, George Todd and myself proved this fact by exploding the problem into 81 different cases. The proof was rather nasty, and very tedious to produce and check. It had 81 different involved cases. What do you expect?

But it gets worse. We found out $K_9 - C_9$ isn't even minor minimal. Using the same proof techniques, we were looking at even more cases (more than 81) to prove that any proper minor was intrinsically knotted. This was not going to work. I became convinced that we had to systematize the proof process, discover some essential properties, algebraicize the problem (if that's even a word).

The principal aim of this thesis is to demonstrate how to reduce the problem of proving "$G$ is intrinsically knotted" to a tractable linear algebra computation over $\mathbb{Z}_2$. As such, actually proving that any *specific* graph $G$ is intrinsically knotted "lies beyond the scope of this thesis." Ok, so the real scoop is that my computer program is too slow right now. It's the end of the semester and I had to write *something* for my thesis. I should have some specific intrinsic knotting results once I get past a few algorithmic and coding issues. However, the mathematical tools constructed in this thesis should (hopefully) be of more interest going forward. For instance, a good number of the proposed open questions at the end of the thesis aim to sidestep the computer program entirely. Alright, I'm done making excuses now.

## Audience and Structure of the Document

The introduction of this thesis should be accessible to most people with a college level education. The rest of the thesis should be sufficiently self-contained for a mathematics undergraduate to follow in its entirety. That said, if you're really shaky on linear algebra, or first order predicate logic, you might find the main chapter a bit turbulent.

Mathematicians (professional, graduate, and cocky sonovabitch) may be tempted to skip the background entirely. Ehhh…maybe a bad idea. You might want to skim it slower than a page a second. Everything's pretty basic, but there's a very high probability that you haven't seen a good fraction of it before. Some highlights include:

- computing/defining the linking number mod 2 via overcrossing number
- edge disjoint vs. vertex disjoint cycles
- Foisy's proof that $K_{3,3,1,1}$ is intrinsically knotted
- representing a vector space mod 2 using a powerset algebra

All of the material in the background was previously known, and has been cited there as seems appropriate. All of the results in the main chapter are new.

# Acknowledgments

Thanks! To my roommates over the past 3 years (Justin, Gil, Mark, Tarun, Mickey and Jeremy) for being big enough nerds to go buy giant whiteboards for our walls, and being mature enough to cover them with drawings of penises. To Justin Hilburn in particular for near infinite patience in conversation, and for suggesting looking at the space of all linking forms. To my advisor in computer science, Don Fussell for his moral support even though I wasn't working on my computer graphics research. To Geir Helleloid for listening to an early version of this work. To all of my friends at UT for being awesome and making college a blast. To my collaborators Cameron Gordon and George Todd for—duh!

I'd particularly like to thank my advisor, Dr. Gordon for making the effort to find a good open problem for undergraduates to research. I hope this thesis might serve as an example that undergraduate research in mathematics can produce real results not just "experiences."

# Chapter 1

# Introduction

The following introduction is written for an educated layperson[1] who may not have taken any proof-based mathematics in college, but is willing to learn and not math-phobic. If you already know what's up... well, then you know what's up. Skip ahead.

My aim is to explain all of the terms and significance of the following diagram:

Ultimately, I will introduce the questions "When is a graph $G$ planar? linked? knotted?"

I will begin by introducing abstract graphs, and then examine the relatively simple question "when is a graph planar?" From there we will discuss mathematical knots and links. Finally, I will show how the study of linked and knotted graphs mimics the study of planar graphs.

## 1.1 Abstract Graphs

Combinatorics studies the way we put together and construct things, abstractly. A **graph** is one sort of combinatorial object, composed out of **points** (aka. **nodes** or **vertices**) connected together by **edges**. Here is a picture of a graph:

Picture here: A graph

However, a graph doesn't have to have a picture. Suppose we wanted to make a graph representing "friends" on Facebook.

[1] whoever that is

The Facebook Friend Graph:
Let every user on Facebook be a **node**. Then "draw" an **edge** between every pair of friends.

Here are some other examples of graphs:

- Internet Servers and connections between them
- Airports and flights
- Cities and roads

Biologists, particularly epidemiologists have begun using graphs under the name of (contact) network theory.

Two very important "families" of graphs that we will refer back to are the **complete** and **complete, bipartite** graphs.

Picture here: Complete Graph family

The **complete graph** on $n$ vertices, denoted $K_n$, is a graph with $n$ vertices and all possible edges connecting those $n$ vertices.

Picture here: Complete Bipartite Graph family

The **complete bipartite graph** on $n$ and $m$ vertices, denoted $K_{n,m}$ is a graph with $n + m$ vertices, in two groups ($n$ "boxes" and $m$ "circles") and all edges connecting two members of different groups. (Note that $K_{n,m} = K_{m,n}$ since circles/boxes are arbitrary)

Although graphs have been studied since (at least) Euler's famous Königsberg bridge problem, they have only found widespread applications in the last 50–60 years, largely thanks to prosthelytization by computer scientists.

## 1.2 Planar Graphs

Now, we'll look at graphs topologically.

Topology studies the way things are connected. It is often contrasted with geometry, earning the monicker "rubber sheet geometry." In geometry we can rotate and translate our objects without changing them. In topology we can go further, bending, stretching, and squishing our objects. However, we are never allowed to cut, break or smash them. Thus a square, triangle, and a circle are all the "same" topological object, even though they are geometrically distinct.

One simple, topological question we can ask about graphs is

> Can we draw a graph $G$, in the plane, such that no two edges ever cross?

If we can, we call the graph $G$ **planar**, and if not, we call it **non-planar**. Clearly, the graph $K_3$ (picture) is planar. Perhaps less obviously, $K_4$ and $K_{2,3}$ are both planar too.

Picture here:

Notice that $K_4$ is planar because there is *some* planar drawing. It is a much harder matter to prove that a graph is non-planar.

**Theorem 1.2.1.** *The graphs $K_5$ and $K_{3,3}$ are non-planar.*

Although we will not prove this theorem, you can convince yourself that it is plausible by trying to draw planar diagrams of $K_5$ and $K_{3,3}$. For instance,

Picture here: non-planar $K_5$ and $K_{3,3}$

In 1930, the Polish mathematician Kazimierz Kuratowski showed, rather surprisingly, that these two graphs ($K_5$ and $K_{3,3}$) completely characterize the planarity/non-planarity property in the following way:

**Theorem 1.2.2** (Kuratowski's Theorem)**.** *A graph $G$ is non-planar if and only if it "contains" $K_5$ or $K_{3,3}$*

In order to precisely understand the statement that a graph $B$ contains a graph $A$, one must meditate on the topic for a while(ENDNOTE). For present purposes it suffices to get the flavor of the concept. We write $A \leq B$ to mean $A$ is "contained" in $B$, and draw the following conceptual diagrams:

Picture here: lattice-y picture

Thus, Kuratowski's theorem is visualized as

Picture here: Kuratowski's as minima

What's so significant about Kuratowski's theorem? We are deriving an "intrinsic" property of abstract graphs based on how we can "realize" them topologically. For example, the facebook graph is (almost certainly) non-planar, independently of our choice to try and draw it.

## 1.3 Links and Knots

Colloquially, we think of a knot as a tied piece of string. For example,

Picture here: untied knot

However, mathematically, we require that the string forms a closed loop. This way we can't untie the knot without breaking the string.

Picture here: trefoil

Because knots are topological objects we can bend and twist them as we please. By pulling the bottom strand of the trefoil up and over, we get a very different diagram.

Picture here: trefoil to the other trefoil pic

This new diagram is also the trefoil. Here are two other knots:

Picture here: other knots

Sometimes we can have very complicated pictures of very simple knots. Here is a nasty picture of the unknot:

Picture here: nasty unknot

Therefore, the most natural question in knot theory is "when are two knot diagrams actually diagrams of the same knot?" or even more simply, "when is a knot actually knotted?" (i.e. not the unknot) This simpler question is called the **unknotting problem**. For the time being, we will skip over both these questions.

Once you know what a knot is, links are fairly straightforward. Sometimes a diagram will use more than one piece of string.

Picture here: link

In such a case, we call the depicted object a **link**. When $k$ loops are used, we call the object a $k$**-component link**. For instance, here is the unlink (aka. the "trivial" link) on $k$ components. Given a 2-component link, we say that the two components, that is the two loops, the two cycles are **linked** if we cannot disentangle them into two disjoint knots. For instance,

Picture here: unlinked vs. non-trivial link (two trefoils)

the two cycles of the Hopf link are linked, whereas the two trefoils are not[2,3]. It is relatively simple to check whether two components of a diagram are linked(ENDNOTE), although we will avoid that digression.

Instead, we will look at a different perspective on what a knot or link is. We have been thinking of a knot as an object in 3-dimensional space. However, all of these different objects, these different knots, are the same in another sense. They are all circles/loops/cycles.

Taking this perspective, we can think of every knot as some **embedding** of "the" (canonical) circle/loop/cycle into space.

Picture here: knots as embeddings

Similarly, we can think of every 2-component link as some **embedding** of two canonical circles/loops/cycles into space.

Picture here: links as embeddings

## 1.4 Linked and Knotted Graphs

Besides embedding circles (knots) or pairs of circles (links) into space, we can also embed graphs! For instance, we couldn't "embed" $K_5$ into the plane, but there's plenty of space in 3D.

[2]I was tempted to write knot, but once you've been in the knot theory business long enough you realize that knot puns are really knot that funny.

[3]Note that the two trefoils are not equivalent to the 2-component unlink. That is, not all unlinked links are trivial.

Picture here: embedded graph

In fact, it's easy to see that any graph, no matter how big and complicated, can be embedded into 3 dimensions. (Unlike the plane, which only admits embeddings of planar graphs.) It's far less clear whether every graph can be embedded into space in an "unknotted" or "unlinked" way. It's not even immediately clear that such a phrasing is well defined. Consider the following diagram of $K_6$:

Picture here: embedding of $K_6$

Contained inside it, there's a Hopf link!

Picture here: repeat with Hopf link highlighted

This embedding of $K_6$ contains a link, but maybe we can find some other **linkless embedding**. Actually, for $K_6$ we can't.

**Theorem 1.4.1** (Sachs 198x)**.** *Every embedding of $K_6$ contains at least one pair of linked cycles.*

We therefore say that $K_6$ is **intrinsically linked** (**IL**), and similarly for other graphs. That is, a graph $G$ is **intrinsically linked** if it contains some non-trivial link. If on the other hand, we can find some embedding of a graph $G$ where every pair of cycles is trivial (unlinked and unknotted), then we say $G$ is **not intrinsically linked** (**NIL**). For example, consider the previous picture of $K_5$. It contains no links, a fact easily certified by observing that each non-trivial link must contain at least two crossings.

We may define the notion of a "knotted" graph similarly. If every embedding of a graph $G$ contains at least one knotted cycle, then we call the graph **intrinsically knotted** (**IK**). Conversely, if we can find some embedding of the graph $G$ where every cycle is unknotted, then we say that $G$ is **not intrinsically knotted** (**NIK**). As an example, consider $K_6$. In this embedding

Picture here: $K_6$ without knots

there are no knotted cycles (even though there is a pair of linked cycles). Therefore, we knot that $K_6$ is NIK (not intrinsically knotted). By way of contrast,

**Theorem 1.4.2.** *$K_7$ is IK (intrinsically knotted).*

In all of these cases, we have said "intrinsic" to emphasize that IK/NIK and IL/NIL are properties of the abstract graph $G$ in question. As we did with planarity, we are again using topological realizations of graphs to derive intrinsic properties of the abstract graphs. In fact, the properties of non-planarity, intrinsic linking, and intrinsic knotting are all very similar. In the same way that non-planarity is characterized by two minimal graphs ($K_5$ and $K_{3,3}$), intrinsic linking is characterized by a set of minimal IL graphs.

Picture here: minimal IL graphs?

Furthermore every intrinsically knotted graph is necessarily intrinsically linked. This justifies our final conceptual picture depicting three strata of classification. Setting the final stone we call the set of minimal graphs for a property (like IL or IK) an **obstruction set**.

<ins>Our Problem</ins>
What is the obstruction set for the intrinsic knotting property?

## 1.5 Progress on this and related problems

The complete obstruction set is known for the IL/NIL property.

**Theorem 1.5.1** (Conjecture by Sachs 1981; proof by Roberson, Seymour and Thomas 1995)**.** *A graph $G$ is intrinsically linked if and only if it "contains" some graph in the Petersen family of graphs*

Picture here: Petersen Family

the graph labeled $G_{10}$ here is often called the **Petersen graph**.

The obstruction set for IK, by contrast, is only partially known.

**Theorem 1.5.2** (Gordon and Conway 1984?, Foisy 200x,200x)**.** *A graph $G$ is intrinsically knotted if it "contains $K_7$, $K_{3,3,1,1}$, or $H$. Furthermore, any graph "contained" in one of these three (but not one of these three) is not intrinsically knotted.*

NEED picture of 3 IK graphs and mention of results.

## 1.6 Endnote 1 on a graph $A$ "containing" a graph $B$

Recall that Kuratowski's theorem said that a graph is non-planar if and only if it "contains" $K_5$ or $K_{3,3}$. The nicest fully rigorous statement is

**Theorem 1.6.1** (Kuratowski's theorem as stated by Wagner)**.** *A graph G is non-planar if and only if $K_5$ or $K_{3,3}$ is a minor of G.*

**Definition 1.6.2.** A graph $M$ is a **minor** of a graph $G$, written $M \leq G$, if we can reduce $G$ to $M$ by

1. deleting edges
2. contracting edges
3. removing isolated vertices

*Example* 1.6.3. Picture here: $K_5$ as minor, but not subgraph

This graph is non-planar because we can contract the two black vertices into the two bottom vertices to yield $K_5$: $K_5 \leq G$

## 1.7 Endnote 2 on checking whether a diagram is linked

In order to determine whether two components in a diagram are linked, we will count how many times they are linked. For instance,

Picture here

the Hopf link is linked once, whereas the Whitehead link is linked twice. In order to compute this **linking number** we will first compute something called the **writhe**.

Given a diagram with two components, construct an **oriented diagram** by drawing little arrows on each component, thus "orienting" it.

Picture here

Once we have an oriented diagram, we can distinguish between "positive" and "negative" crossings in the following way:

Picture here

Picture here

Here are annotated diagrams of our two links:

We define the **writhe** of an oriented diagram as the sum of these numbers across all crossings between the two different components[4].

$$w(A,B) = \sum \varepsilon$$

Therefore, we compute that the writhe of the Hopf link diagram above is $-2$ and $-4$ for the Whitehead link diagram. In fact, this number will always be even, and the sign will vary depending on our choice of orientation, so we take the absolute value over two.

**Definition 1.7.1.** The **linking number** between two components, $A$ and $B$ of a diagram is

$$lk(A,B) = \frac{|w(A,B)|}{2}$$

[4]note that there are no crossings between, say, the two trefoils because the two different components never cross with each other, although they do cross with themselves.

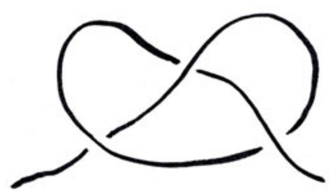
# Chapter 2

# Background

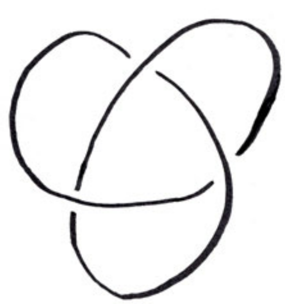
## 2.1 Knot Theory

### Definitions

In mathematics, the following object is not (topologically) a knot,

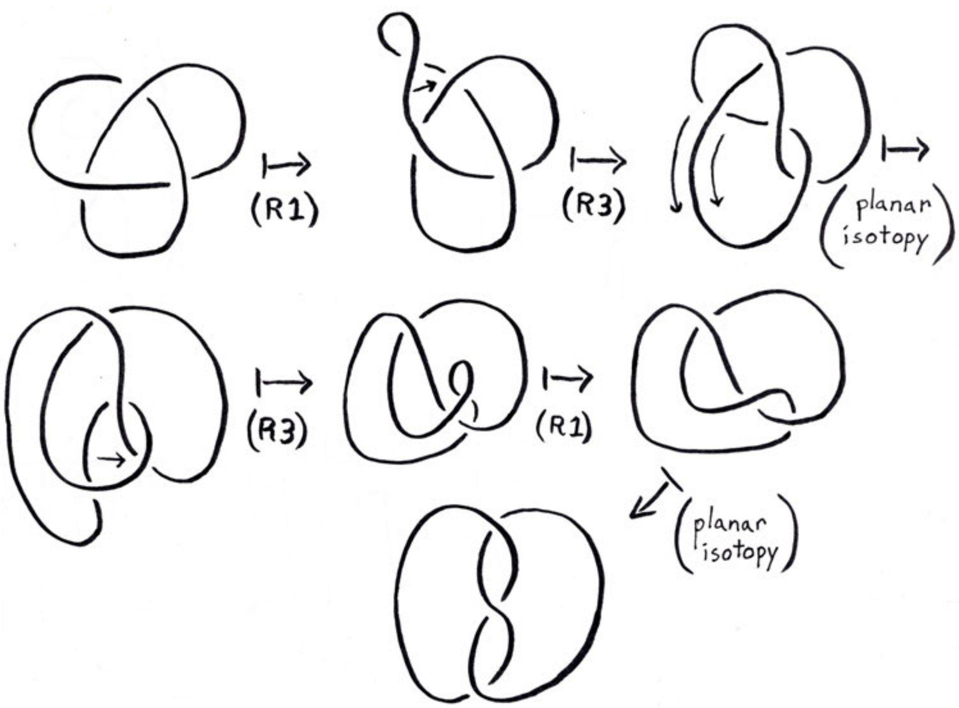


because it can be untied. To prevent such travesties, we will only consider loops (of string). For knots like the former, we can just join the ends.



Furthermore, we want to consider two knots to be equivalent if we can deform one into the other. For instance,

Such a change is called a **planar isotopy**. In addition to planar isotopies, we'd also like to capture equivalences which involve deforming the knot in the full 3-dimensional space, such as:

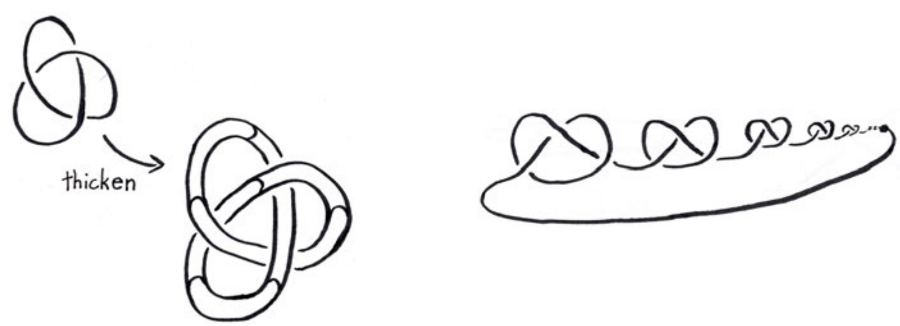



In this manipulation we've used certain natural, local modifications called **Reidemeister Moves** (labeled R1 and R3).

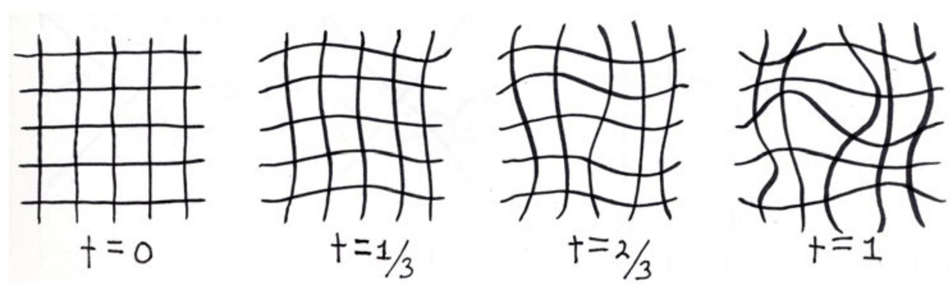
Although this description captures knot theory as it is practiced, we ought to at least sketch a formal foundation for completeness sake. Such an undertaking is usually begun with something like the following incantation:

**Definition 2.1.1.** A **knot** is a tame embedding of the circle into space $\mathbb{R}^3$ (i.e. a tame loop in space). Two knots are considered equivalent if there is an ambient isotopy carrying one into the other.

Perhaps surprisingly, this definition is not complete gibberish. By **embedding**, we mean a continuous 1-1 map, and by **tame** we mean that the embedding of the circle may be safely extended to an embedding of the solid torus. This outlaws so called **wild** knots.




So now, we can assume all of our knots are "nice." The definition is completed by using **ambient isotopy** as our notion of equivalence. An ambient isotopy[1] is a continuous family of continuous 1-1 deformations of $\mathbb{R}^3$. We may visualize two-dimensional ambient isotopies more easily than three-dimensional ones. Just picture the deformation of a grid over "time."




[1]Formally, let $f, g : S^1 \to \mathbb{R}^3$ be knots. Then $f$ and $g$ are related by ambient isotopy if there exists a continuous family of homeomorphisms $\Phi_t : \mathbb{R}^3 \to \mathbb{R}^3$ (that is $\Phi : [0,1] \times \mathbb{R}^3 \to \mathbb{R}^3$ is continuous) such that $\Phi_0 = id$ and $\Phi_1 \circ f = g$.

Using ambient isotopies, we may deform any one embedding of a knot into any other by *deforming the ambient space*. This technicality effectively prevents us not only from breaking the knot, but also from passing it through itself, as this would break the ambient space.

In practice, we may avoid this technicality entirely by dealing with knots through their diagrams.

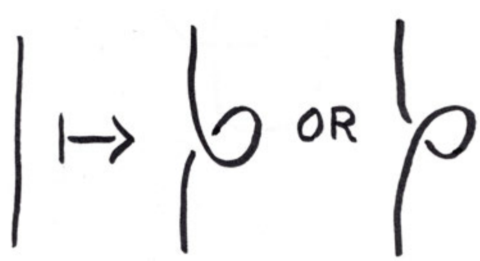
**Definition 2.1.2.** A **knot diagram** is a planar projection of a knot with a finite number of doubly covered points, all of which are transverse crossings. These crossings are drawn with gaps to indicate which "strand" of the diagram lies on top.

Under ambient isotopy we are free to deform this diagram as we please short of creating, destroying or rearranging the crossings. As mentioned earlier, such changes are called **planar isotopies** since they are accomplished by deforming the plane.

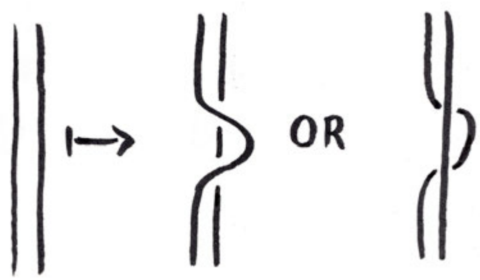
In addition to planar isotopies a number of local crossing modifications appear intuitively admissible:

**Definition 2.1.3.** The Reidemeister moves

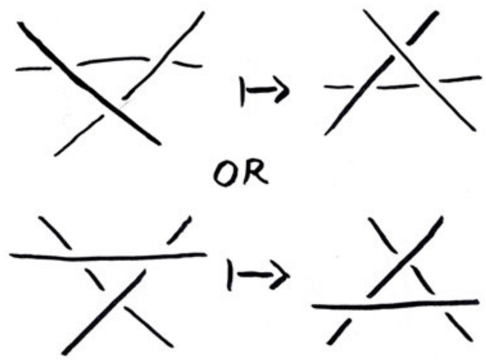
**R1**:



**R2**:



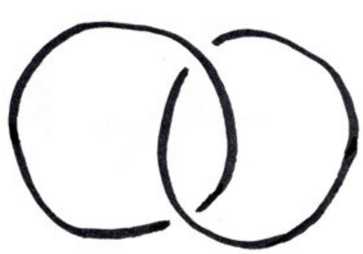

**R3**:



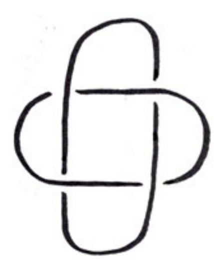

These moves are named in honor of Kurt Reidemeister, who proved the following theorem about them in 1926.

**Theorem 2.1.4.** *Given two diagrams $D_1, D_2$ of knots $K_1, K_2$, $K_1$ and $K_2$ are equivalent if and only if $D_1$ and $D_2$ are related by some sequence of planar isotopies, $R1$, $R2$ and $R3$ moves.*

In effect, this theorem allows us to approach the study of knots purely through their diagrams. However, given arbitrary "knot-like" diagrams composed of oriented transverse crossings and strands running between them, we will soon run into diagrams such as the following,



The Hopf Link



The Whitehead Link

which contain more than one loop. A diagram with $k$ loops is a diagram of a **k-component link**. The formal definition of **link** is directly analogous to and generalizes knots (which are just 1-component links).

**Definition 2.1.5.** A **k-component link** is a tame embedding of $k$ circles into space. Two links are considered equivalent if there is an ambient isotopy between them.

**Definition 2.1.6.** A **link diagram** is a projection of a link with a finite number of transverse crossings.

**Theorem 2.1.7.** *Two link diagrams are of equivalent links if and only if they are related by planar isotopies and Reidemeister moves.*

Big, Hairy, Knotty Questions

1. Given a diagram $D$, is it a diagrams of the unknot? (unknotting problem)

2. Given two diagrams, do they represent equivalent knots?

We will not address these questions in this thesis, but they're useful for getting a flavor of what knot theory is (sometimes) about.

## Invariants

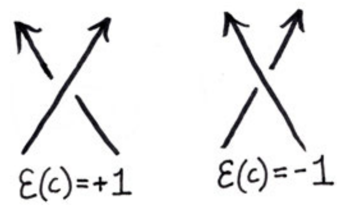
Reidemeister moves are very effective in the event that two diagrams happen to be of equivalent knots. We can prove as much by exhibiting a sequence of Reidemeister moves. However, it's a much trickier matter to prove that two diagrams are actually different! To do so, we will develop some (albeit very lightweight) knot invariants.

An **invariant** is a function $f$ which "eats" a diagram and "secretes" a number (or sometimes a group, ring, etc.) st. for two equivalent diagrams $D_1 \sim D_2$, $f(D_1) = f(D_2)$. An invariant is said to be **complete** if the converse holds: $f(D_1) = f(D_2) \implies D_1 \sim D_2$.

We will begin our discussion with a quantity called the **writhe** which is a NOT invariant[2].

**Definition 2.1.8.** An **oriented diagram** is a link diagram where arrows have been consistently attached to each component. Every crossing in an oriented diagram is now positive or negative. Define $\varepsilon(c) = +1$ if $c$ is positive and $\varepsilon(c) = -1$ if $c$ is negative.

[2] This pun is decidedly knot funny.



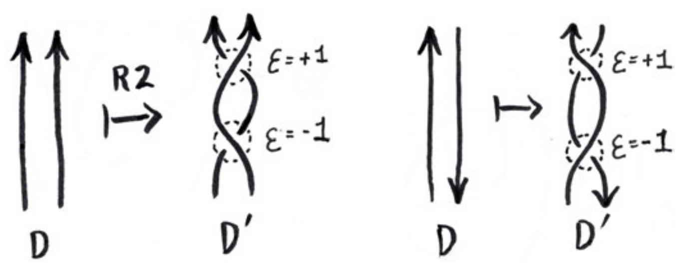

The **writhe** of an oriented diagram is the sum of these values.

$$w(D) = \sum_{c\text{:crossing}} \varepsilon(c)$$

Certainly the writhe is unaffected by planar isotopy. Furthermore, R2 and R3 moves have no effect.

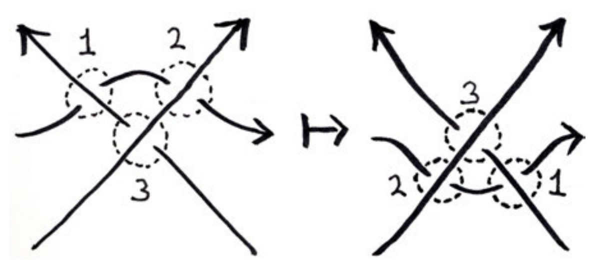
**Theorem 2.1.9.** *Given two oriented diagrams $D_1, D_2$ related by planar isotopy, R2 and R3, $w(D_1) = w(D_2)$.*

*Proof.* invariance under R2:

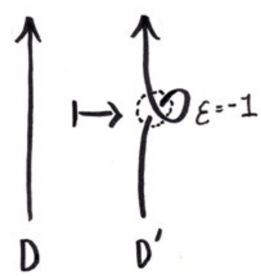



$$w(D') = w(D) + 1 - 1 = w(D)$$

invariance under R3:




Notice that the R3 move (in all variations) just rearranges the existing crossings. Therefore the writhe is left unchanged as well. □




$$w(D') = w(D) - 1$$

The R1 move, however, does change the writhe. (Dang!)

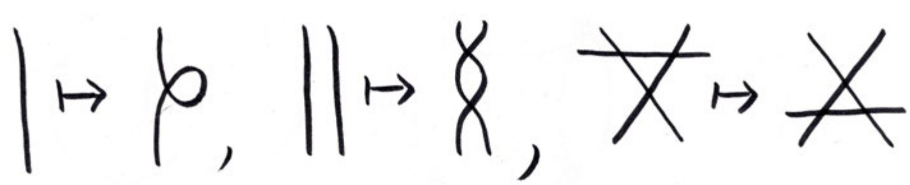
For this reason the writhe is not an invariant of a knot diagram. But not all hope is lost! We can use the writhe to define a link invariant called the **linking number**.

**Definition 2.1.10.** Consider an unoriented 2-component link diagram consisting of components $L_1, L_2$. Choosing orientations for $L_1, L_2$ we get an oriented diagram. Let $w(L_1, L_2)$ denote the writhe only *between* the two components $L_1, L_2$:

$$w(L_1, L_2) = \sum_{\substack{c \text{ crossing} \\ \text{between} \\ L_1 \& L_2}} \varepsilon(c)$$

**Theorem 2.1.11.** *$w(L_1, L_2)$ is always even and is invariant up to choice of sign.*

*Proof.* If we consider 2-component link diagrams without considering the over-under arrangement of each crossing then any two diagrams are related by a sequence of unoriented Reidemeister moves:



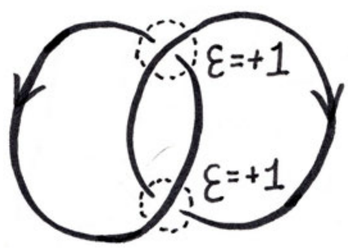
The first of these does not apply to crossings between two distinct components. The second changes the number of crossings by 2. The third leaves the number of crossings invariant. Therefore there are always an even number of crossings between 2 distinct link components. Furthermore, given an even length sequence of increments/decrements, the total increment/decrement must be even.

Now note that any change of orientation must change the orientation of *all* crossings between $L_1$ and $L_2$. Therefore such a change may change $w(L_1, L_2)$ by at most its sign. Finally, by the preceding theorem on the writhe, $w(L_1, L_2)$ is invariant under R2 and R3. Since R1 does not apply to/affect crossings between $L_1$ and $L_2$, $w(L_1, L_2)$ is invariant under R1 as well, making it a true invariant. □

**Definition 2.1.12.** We therefore define the linking number between two components $L_1, L_2$ as

$$lk(L_1, L_2) = \frac{|w(L_1, L_2)|}{2}$$

We may easily compute that the Hopf link has linking number 1.




$$\begin{aligned} w(L_1, L_2) &= 1 + 1 = 2 \\ lk(L_1, L_2) &= |w(L_1, L_2)|/2 \\ &= |2|/2 = 1 \end{aligned}$$

In this thesis, we will only ever make use of the linking form reduced mod 2. In this form the linking number is always either 0 or 1. We may develop this mod 2 invariant a second and ultimately more convenient way.

**Definition 2.1.13.** Let $\omega(L_1, L_2)$ denote the **over-crossing number mod 2**. $\omega(L_1, L_2)$ counts the number of times $L_1$ passes over $L_2$ mod 2.

**Theorem 2.1.14.** $\omega(L_1, L_2) \equiv lk(L_1, L_2) (\text{mod } 2)$

*Proof.* Let $\omega'(L_1, L_2)$ denote the number of times $L_1$ passes over $L_2$, not mod 2. If $\omega'(L_1, L_2) = 0$, then $L_1$ lies entirely beneath $L_2$ so $L_1$ and $L_2$ are unlinked; $lk(L_1, L_2) = 0$. If $\omega'(L_1, L_2) > 0$, then we can get to this diagram starting from the diagram with $L_1$ entirely beneath $L_2$ by making $\omega'(L_1, L_2)$ crossing changes. Each such crossing change changes $\varepsilon$ by $\pm 2$, $w(L_1, L_2)$ by $\pm 2$ and therefore $lk(L_1, L_2)$ by $\pm 1$. If $\omega'(L_1, L_2)$ is odd ($\omega(L_1, L_2) = 1$) then $lk(L_1, L_2)$ is some odd number of increments and decrements away from 0; $lk(L_1, L_2) \equiv 1 (\text{mod } 2)$. If $\omega'(L_1, L_2)$ is even ($\omega(L_1, L_2) = 0$) then $lk(L_1, L_2) \equiv 0 (\text{mod } 2)$. □

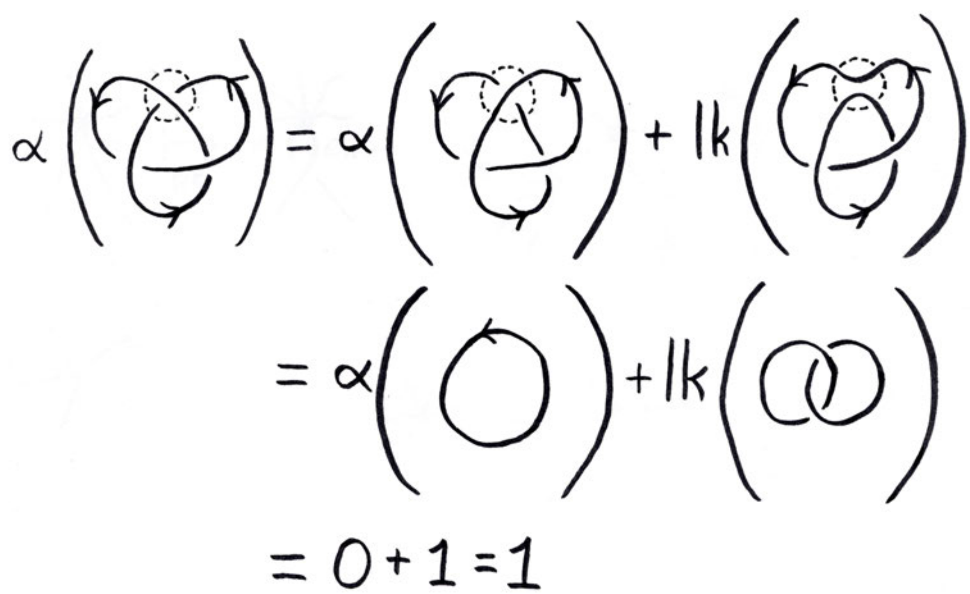
Although it is not at all obvious from the definition, this proves that $\omega$ is symmetric.

**Corollary 2.1.15.** $\omega(L_1, L_2) = \omega(L_2, L_1)$

Although there are many fascinating knot/link invariants, we will only need one more (relatively simple) one, called the Arf invariant (after Cahit Arf). Like our doctored linking number it assumes values mod 2. Unfortunately, all known developments of the Arf invariant require detours into either knot polynomials (CITATION) or Seifert surfaces (CITATION). We will therefore forgo a proof that the invariant is well defined, and instead settle for stating the relevant properties.

**Definition 2.1.16.** The **Arf invariant** of a knot $K$ is written $\alpha(K)$. It assumes the value 0 on the unknot and obeys the following skein relation (mod 2):


From this definition we may compute

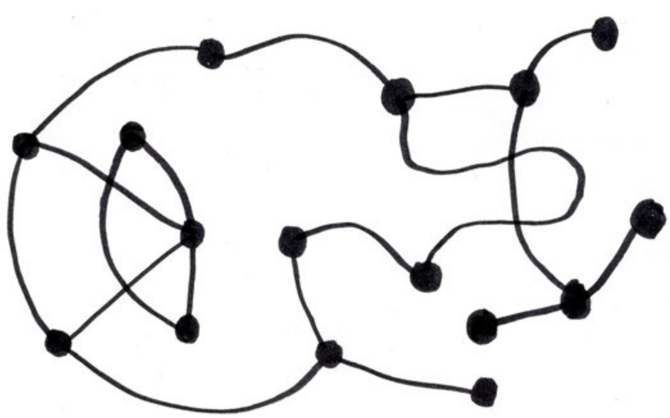



demonstrating that the invariant is non-trivial.

## 2.2 Graph Theory

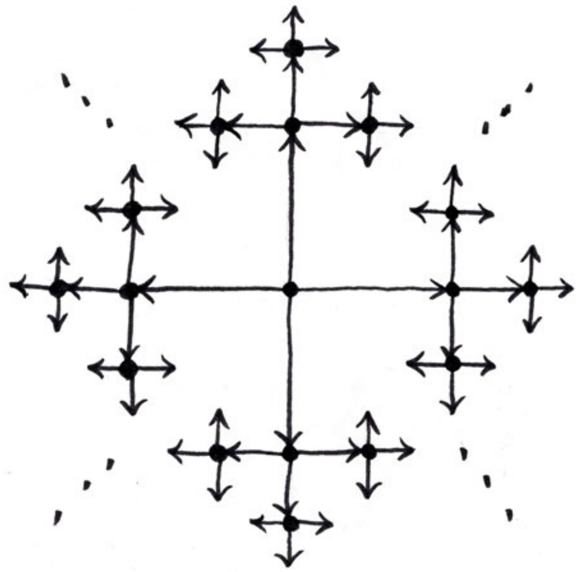
A **graph** is the most basic mathematical object capturing the notion of objects in a pairwise relationship. They are particularly popular because they may also be drawn:

In this drawing there are **points** (also called **nodes** or **vertices**) connected by paths (which are nearly always called **edges**). However, the particular drawing of a graph given is irrelevant. Edges may bend and cross arbitrarily, so long as the "connectivity" of the graph remains constant. For instance, here is a different picture of the same graph just shown:



**Definition 2.2.1.** A **graph** G consists of a finite set of **vertices** $V = V(G)$ and a finite set of edges $E = E(G)$ where each edge is an unordered pair of distinct vertices $e = (a, b) = (b, a)$ with $b \neq a$.

This definition implies that we will restrict our attention to **undirected** and **finite** graphs, outlawing cases such as the following:



loop

double edge

Furthermore, we will generally exclude graphs with **loops** and/or **double edges** from consideration, although we will occasionally break this rule when they crop up. Which restrictions hold should be clear from context.

Although the concepts of paths, or connected graphs are straightforward when pictured, they too need a more formal definition to work with.

a path

a connected graph

a disconnected graph

**Definition 2.2.2.** A **path** in a graph is a sequence of distinct vertices $v_1, v_2, \ldots, v_n$, such that every successive pair of vertices is joined by an edge: $(v_i, v_{i+1}) \in E$. A graph is said to be **connected** if there exists a path between any two vertices of the graph.

The formal definition of a path may be extended to treat the concept of a cycle.

a cycle an acyclic graph

**Definition 2.2.3.** A **cycle** is a path $v_1, v_2, \ldots, v_n$ with an extra repeated vertex $v_0 = v_n$. Thus $(v_0 = v_n, v_1)$ must also be an edge in the graph. A graph that doesn't have any cycles is called **acyclic**.

If a graph is both connected and acyclic, then we call it a **tree**.

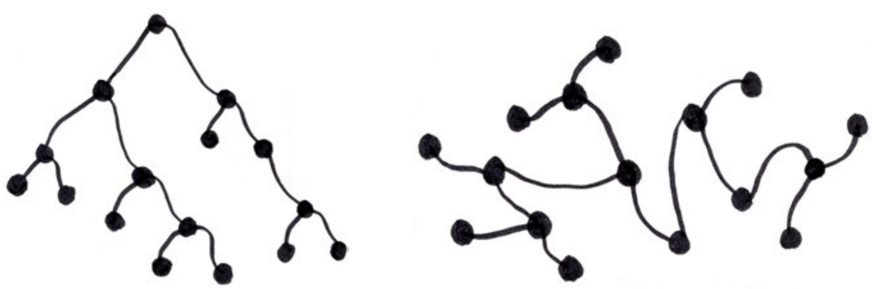
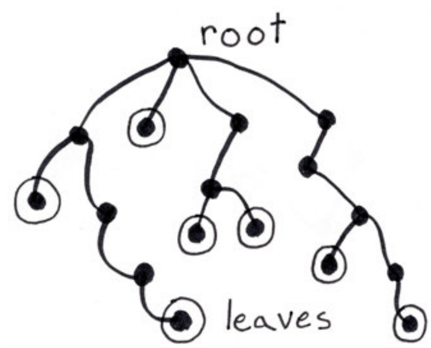


Figure 2.1: Both of these graphs are trees (the same in fact)

**Theorem 2.2.4.** *A tree with n vertices has* $n - 1$ *edges.*

In order to see this result, we show that every tree may be arranged into a nice, somewhat standard diagram. First, pick any node and call it the **root**. Because the tree is connected, there exists a path from the root to every node. Because a tree is acyclic, every such path is unique (or else we would have created a cycle). This is another nice property of trees by the way. This argument allows for drawing any tree in the following way:




Every node $v$ in the tree now has a **depth** $d(v)$, the length of the unique path from the root to that node. Furthermore, every node can have at most one **parent**, a node of depth $d(v) - 1$ connected to $v$, for if not, then we could find two different paths from the root to $v$. Finally, because the tree is finite, there must be a maximum depth node, with no **children** (nodes of which $v$ is a parent). Such a maximum depth node must have degree 1, where **degree** refers to the number of edges emanating from a node. We call a degree 1 node in a tree a **leaf**.

*Proof.* Since any tree has at least one leaf, we may remove this leaf, along with its lone connecting edge, without leaving the remaining tree disconnected. By induction, we may remove vertices and edges in pairs until we are left with a tree on one vertex which contains no edges. □

It's worth noting that any two of the following properties imply the third:

- acyclic
- connected
- has exactly one more vertex than edges

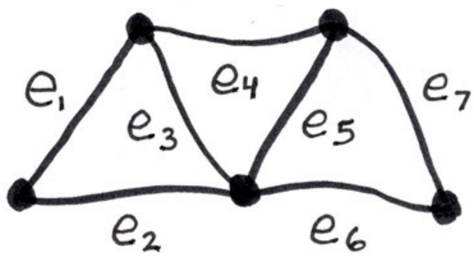
Cool, huh?

When talking Topology, we will often call trees simply connected graphs. And while we're talking Topology, how about those cycles? Aren't those just loops? And aren't loops the stuff knots are made of? Curious...

We will discuss cycles quite a bit in this thesis, so it's best that we get our language straight from the get-go. One powerful perspective on cycles treats them as subgraphs.

**Definition 2.2.5.** A **subgraph** $G' \subseteq G$ is a graph on some subset of $G$'s vertices ($V(G') \subseteq V(G)$) using some subset of $G$'s edges ($E(G') \subseteq E(G)$). A **vertex-induced subgraph** is specified by a subset of vertices, while retaining all edges in $E$ between those vertices. An **edge-induced subgraph** is specified by a subset of edges, retaining all vertices those edges touch.

We will be working primarily with edge-induced subgraphs in this thesis, since cycles can be easily specified as sets of edges. For instance,




| cycles |
|---|
| $\{1,2,3\}$ |
| $\{3,4,5\}$ |
| $\{5,6,7\}$ |
| $\{1,2,4,5\}$ |
| $\{3,4,6,7\}$ |
| $\{1,2,4,6,7\}$ |

Every cycle in a graph may be realized in this way as a subgraph. Viewed in this light, a number of useful properties become apparent.

**Theorem 2.2.6.** *Let $C \subseteq G$ be an edge-induced subgraph. If $C$ is a cycle, then every vertex of $C$ has exactly degree* 2*.*

*Proof.* A cyclic path may pass through a vertex $v$ at most once, in which case it "enters" via one edge and "leaves" via another. That is, $v$ has degree 2. Otherwise $v$ is not passed through and therefore not in $C$. $\square$

According to our current definition of cycle the converse does not hold. Rather than puzzle this curious fact, we barge ahead and redefine our notion of cycle until it stops misbehaving. (Note: this is generally a good policy.)

**Definition 2.2.7.** A **generalized cycle** (or just **cycle**) is an edge-induced subgraph $C \subseteq G$ such that every vertex of $C$ has even degree.

**Definition 2.2.8.** Two subgraphs $(V_1, E_1), (V_2, E_2)$ are said to be **edge disjoint** if $E_1 \cap E_2 = \emptyset$. They are said to be **vertex disjoint** if $V_1 \cap V_2 = \emptyset$.

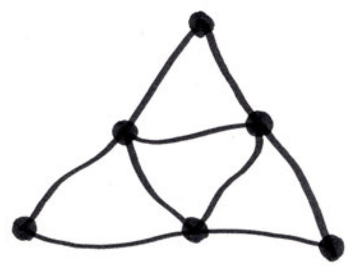


Figure 2.2: a "cycle" in the generalized sense

**Theorem 2.2.9.** *Every generalized cycle may be decomposed into a union of edge disjoint cycles as previously defined via cyclic paths.*

*Proof.* First, decompose the given generalized cycle into a union of vertex disjoint connected components. Consider one such component. Starting at any vertex of this component, trace out a path until some vertex is revisited, forming a cycle out of some subset of the traced path.

Can this always be done? Suppose that tracing a path does not eventually form a cycle. Then the path must terminate at some vertex. However, every vertex has at least degree 2, so the path could have continued, or must be visiting this vertex for the second time. Contradiction. Therefore, tracing a path will always identify some cycle.

Now subtract the identified cycle path from the generalized cycle to get an edge induced subgraph. Since the cycle path touches every vertex 0 or 2 times, this subtraction will decrement the degree of every vertex by 0 or 2, yielding a new, smaller generalized cycle. By repeated iteration we will be left with an edge disjoint union of cycles as defined via cyclic paths. □

Note that this decomposition is not unique in general.

**Corollary 2.2.10.** *If a generalized cycle has maximum vertex degree* 2*, then it may be decomposed into a vertex disjoint union of cycles as defined via cyclic paths.*

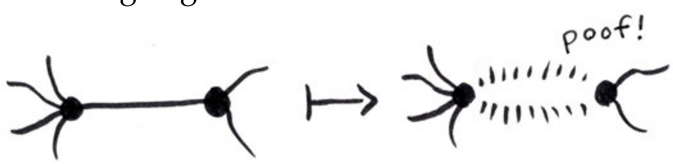
"Cycle as defined via cyclic paths" is quite a mouthful, so we will instead use the term **simple cycle**[3].

The previous development has given us a rather *simple* way to identify simple cycles. A simple cycle is just a connected subgraph in which every vertex has degree 2.

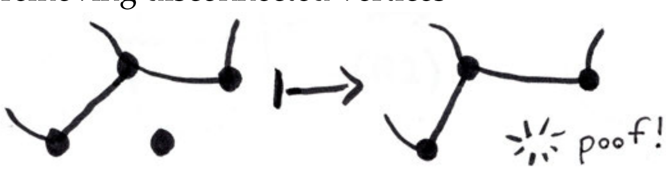
Although subgraphs are very effective for studying structures within a given graph, the structure of all graphs may yield more readily to a different notion, that of a **graph minor**. This notion is meant to resolve the following conundrum. In some sense (a topological sense) we would like to think of the triangular graph as a "subgraph" of any graph with cycles. Unfortunately, it's not a subgraph of all cyclic graphs.

We begin by re-examining our definition of subgraph, so that we may generalize. Suppose we are given two graphs $A, B$ st. $A \subseteq B$. Then we may produce $A$ from $B$ by

- removing edges

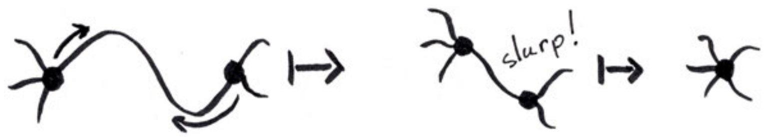



- removing disconnected vertices




[3]Why do mathematicians call so many different things simple? Because they like to make everything complicated. Ba Dum Chh!

If we further allow one to **contract edges**, then we get the notion of graph minor.




**Definition 2.2.11.** Given graphs $A, B$ we say $A$ is a **minor** of $B$ (written $A \leq B$) if we may obtain $A$ from $B$ via **edge deletions**, **vertex deletions**, and **edge contractions**.

This notion of edge contraction gives us yet another perspective on **simply connected** graphs (aka. **trees**). A graph is simply connected if and only if it may be contracted into a point!

At this point it's probably a good idea to note that edge contractions may lead to double edges or loops. In a few cases we will need to pay attention to them.

The minor relationship defines a partial order on all graphs. This partial order plays "nice" with a number of properties. For instance,

**Definition 2.2.12.** A graph is **planar** if it may be drawn in the plane without crossing edges.

**Theorem 2.2.13.** *If a graph $A$ is planar, written $P(A)$, then any minor $B \leq A$ is also planar $P(B)$.*

*Proof.* Given that $A$ is planar, construct a planar diagram. Since $B \leq A$, we may arrive at a diagram for $B$ by contracting and deleting edges. Since such operations will always result in another planar diagram, we get a planar diagram for $B$; $B$ is planar. □

In general any such property is said to be minor-preserved.

**Definition 2.2.14.** A **minor preserved** property $P$ of a graph is some predicate such that $(A \leq B \text{ and } P(B)) \implies P(A)$. Alternatively, we sometimes say that "not $P$" is minor preserved for convenience.

There is a famous theorem of Kuratowski capturing the concept of planarity with two graphs.

**Theorem 2.2.15** (Kuratowski)**.** *A graph $A$ is non-planar if and only if $K_5 \leq A$ or $K_{3,3} \leq A$.*

Using one set of terminology, $K_5$ and $K_{3,3}$ are called **forbidden minors** because any planar graph is forbidden from having either of them as minors. Alternatively, one says that the set of $K_5$ and $K_{3,3}$ is the **obstruction set** for planarity/non-planarity.

**Definition 2.2.16.** An **obstruction set** for a minor preserved property $P$ is a minimally sized set of graphs $\mathcal{O}$ such that not $P(A)$ if and only if $\exists G \in \mathcal{O} : G \leq A$.

Robertson and Seymour proved a ridiculously powerful conjecture of Wagner on this topic, a conjecture which helps motivate the pursuit of this thesis.

**Theorem 2.2.17** (Robertson-Seymour)**.** *For any minor-preserved property $P$, there is a unique finite obstruction set $\mathcal{O}$.*

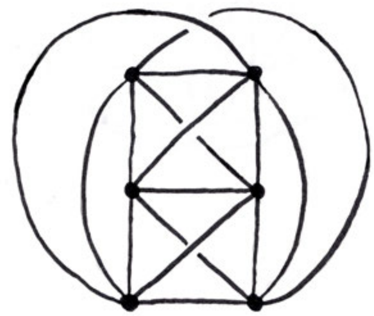
Crazy!

## 2.3 Knotted Graphs

When we introduced knot theory, we defined a knot as an embedding of the circle into space. What about other 1-dimensional objects, like. . . a graph?

**Definition 2.3.1.** A **graph knot** or **knotted graph** is a tame embedding of some graph $G$ into space. Such embeddings are considered equivalent up to ambient isotopy.

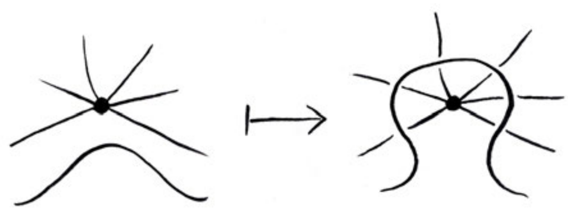
As with knots we may consider diagrams of knotted graphs, such as the following diagram of $K_6$.



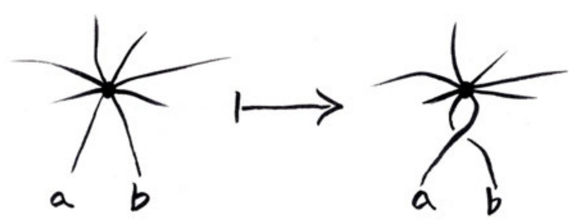
And, just as with knots, we have a Reidemeister-like theorem for manipulating knotted graphs.

**Theorem 2.3.2.** *Two graph embeddings are equivalent if and only if their diagrams are related by a sequence of planar isotopies, R1, R2, R3, R4, and R5 moves.*

An **R4** move allows one to pull a strand over a vertex.



An **R5** move allows one to "twist" strands emanating from a vertex.




Within the theory of knotted graphs, we recover traditional knot theory by considering embeddings of the triangle $K_3$. For this reason it would appear that we have made our lives harder. However, in the process of this "complexification" we have made some previously trivial questions non-trivial. For instance, given a graph $G$, can we embed $G$ into space such that all simple cycles in $G$ are unknotted?

**Definition 2.3.3.** A graph $G$ is said to be **intrinsically knotted** if for all embeddings of $G$, there exists some knotted (simple) cycle of $G$ in the given embedding. We say $G$ is **IK** for short or **NIK** (**not intrinsically knotted**) in the opposite case.

$K_3$ is clearly NIK, confirming our suspicion that intrinsic knotting is not an interesting question in traditional (non-graph) knot theory. Furthermore, some graphs are almost certainly IK, such as $K_{1000}$. The far more interesting graphs to ponder are the simplest ones that are still knotted. For instance, it turns out that $K_7$ is intrinsically knotted, while all of its proper minors are not.

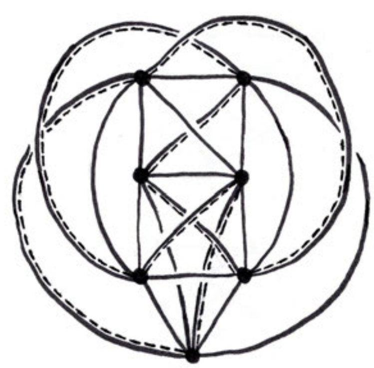
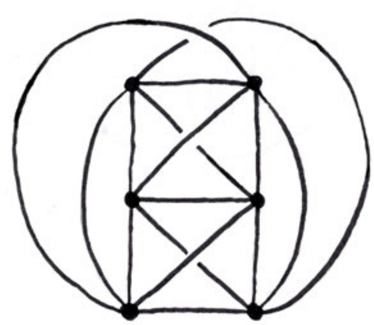


Figure 2.3: embedding of $K_7$ with exactly one knotted cycle (indicated)

By way of contrast the following picture of $K_6$ contains no knots.



Therefore, we may conclude that $K_6$ is NIK. This demonstrates our first key observation about intrinsic knotting: It is far easier to prove that a graph is NIK than IK, just as it's easier to prove that a particular diagram is unknotted (just show a sequence of Reidemeister moves) than it is to prove that it's knotted.

Even though $K_6$ is not intrinsically knotted, it is **intrinsically linked**.

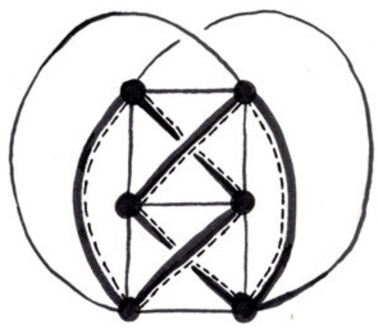


Figure 2.4: embedding of $K_6$ with exactly one pair of linked cycles (indicated)

**Definition 2.3.4.** A graph $G$ is **intrinsically linked** (**IL**) if for all embeddings of $G$ there exists some non-trivial link in the given embedding (on a vertex disjoint union of simple cycles). We say a graph is **not intrinsically linked** (**NIL**) otherwise.

In 198x, Sachs (full name?) proved that $K_6$ is intrinsically linked[4]. Later, (so and so?) showed that a certain kind of graph modification, called a **triangle-star** or **Δ-Y** move would produce yet more intrinsically linked graphs. Applying this move repeatedly to $K_6$ yields a family of 7 graphs called the **Petersen family** after the "largest" graph in the family, the **Petersen graph**.

It's a still more curious fact that none of the Petersen family graphs are minors of each other and that no proper minor of the Petersen family is intrinsically linked. What makes this an even more magnificent coincidence is that the Petersen family forms the obstruction set for the IL/NIL property.

[4]although, he used the term discatenable

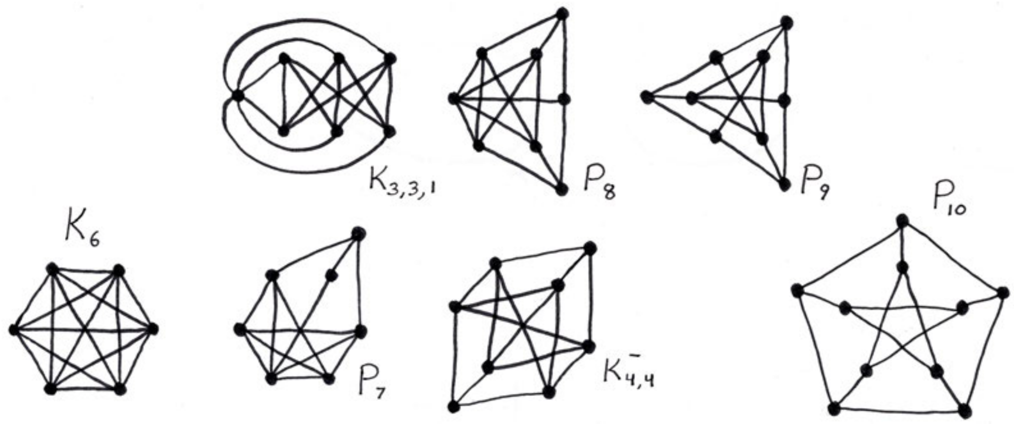



Figure 2.5: The Petersen Family Portrait

**Theorem 2.3.5.** *IL/NIL and IK/NIK are minor-preserved properties.*

*Proof.* Given a graph $G$ that is NIL or NIK, we may find an embedding without links or knots respectively. Since edge deletions and contractions performed on the embedded graph will not introduce any new linked or knotted cycles, any minor of $G$ must also be NIL or NIK, respectively. □

**Theorem 2.3.6** (Robertson & Seymour 199x)**.** *The obstruction set for the IL/NIL property is precisely the Petersen family of graphs.*

Although we do not want to recapitulate that entire proof, we will have cause to make use of some of the techniques for proving intrinsic linking results. Therefore, we'll review the proof that $K_6$ is intrinsically linked.

**Theorem 2.3.7.** *$K_6$ is IL.*

*Proof.* Given an embedding of $K_6$, $E$ and two vertex disjoint cycles $A, B \subseteq K_6$, let $lk_E(A, B)$ denote the mod 2 linking number of $A$ and $B$ in the embedding $E$. Then define the mod 2 linking number of the embedding as $lk(E) = \sum_{(A,B)} lk_E(A, B)$. This proof will now proceed in two steps. First, we note that there is some embedding $E_0$ of $K_6$ such that $lk(E_0) = 1$. Specifically, $E_0$ is the previously exhibited diagram with a single pair of linked triangles. Second, we will demonstrate that the summed linking number is the same for all embeddings.

Using crossing change moves (CX) and R1-R5 moves we can move to any diagram of any embedding of $K_6$. We already know that $lk$ is invariant under R1-R5 moves because it is a link invariant. Therefore, if we can just show that $lk$ is invariant under crossing changes, we will be done.

Crossing moves between an edge and itself or between two adjacent edges can't affect crossings between two vertex disjoint cycles. Therefore we need not consider them. Furthermore, without loss of generality all other crossing changes may be considered as occurring between the same two edges $v_1v_2, v_3v_4$ up to symmetry of the graph $K_6$. Which cycle pairs contain $v_1v_2$ and $v_3v_4$ in separate cycles? Only $(v_1v_2v_5)(v_3v_4v_6)$ and $(v_1v_2v_6)(v_3v_4v_5)$.

Therefore making a crossing move will change the over crossing number $\omega(A, B)$ for each cycle pair by 1. Since $lk_E(A, B) = \omega_E(A, B)$, the crossing change will change the linking number mod 2 of each of these cycles by 1. Thus we may conclude that the linking number of $E$ and the linking number of $E'$, the embedding resulting from a crossing move, are the same.

$$lk(E') = lk(E) + 2 = lk(E) \mod 2$$

□

In 198x, John Conway and Cameron Gordon showed that $K_7$ is intrinsically knotted by a similar argument. However instead of summing the linking number over all pairs of cycles, they summed the Arf invariant over all Hamiltonian cycles of $K_7$. In xxxx, xxxx et al. applied this same form of argument to the graph $K_{3,3,1,1}$ erroneously, as xxxx pointed out.

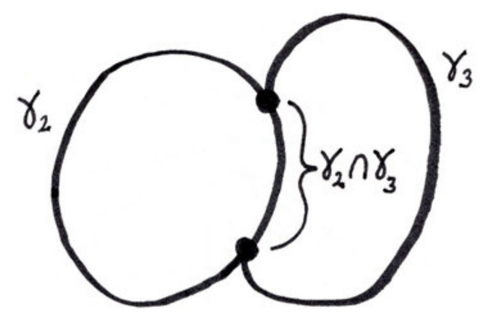
For over (nearly?) a decade the question "Is $K_{3,3,1,1}$ intrinsically knotted?" remained open. Finally, in 200x, Joel Foisy gave a valid proof. In so doing Foisy found the second known member of the IK/NIK obstruction set. Since then, X more members of the obstructions set have been found. However, no conjecture for the complete obstruction set has been advanced.

Because this work builds on Foisy's methods, it's worthwhile to review them. Towards that end, we will show how Foisy proved that $K_{3,3,1,1}$ is intrinsically knotted. He used two critical lemmas which we will start by reviewing.

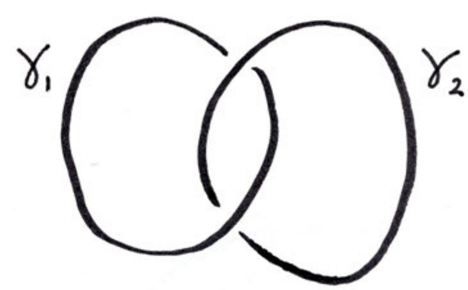
The first is very convoluted and stated in the language of Homology theory. We preserve Foisy's original language here for posterity.

**Lemma 2.3.8** (Foisy's Homological Lemma)**.** *Let $\gamma_1, \gamma_2$ and $\gamma_3$ be simple closed curves in $\mathbb{R}^3$ such that $\gamma_2 \cap \gamma_3$ is an arc, and both $\gamma_2 \cap \gamma_1$ and $\gamma_1 \cap \gamma_3$ are empty. Suppose that $[\gamma_2]$ is non-trivial in $H_1(\mathbb{R}^3 - \gamma_1; \mathbb{Z}_2)$. Then precisely one of $[\gamma_3]$ and $[\gamma_2 + \gamma_3]$ is non-trivial in $H_1(\mathbb{R}^3 - \gamma_1; \mathbb{Z}_2)$.*

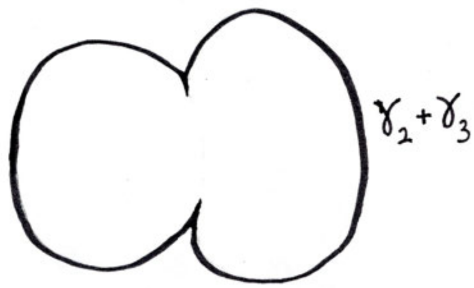
Oy Vey! Let's have some pictures.




$\gamma_2 \cap \gamma_3$ is an arc. Furthermore, they are both disjoint from $\gamma_1$.




$[\gamma_2]$ is non-trivial in $H_1(\mathbb{R}^3 - \gamma_1; \mathbb{Z}_2)$ means that $lk(\gamma_1, \gamma_2) = 1 \bmod 2$. Likewise, $[\gamma_3]$ and $[\gamma_2 + \gamma_3]$ being non-trivial in $H_1(\mathbb{R}^3 - \gamma_1; \mathbb{Z}_2)$ means they are linked with $\gamma_1$ mod 2. But what is $\gamma_2 + \gamma_3$?




$\gamma_2 + \gamma_3 = (\gamma_2 \cup \gamma_3) - (\gamma_2 \cap \gamma_3)$ taken as sets[5].

The lemma syas that the curves $\gamma_1, \gamma_2, \gamma_3$ must "essentially" lie in one of two possible arrangements (from the mod 2 linking number point of view).


$lk(\gamma_1, \gamma_2) = 1$
$lk(\gamma_1, \gamma_3) = 0$
$lk(\gamma_1, \gamma_2 + \gamma_3) = 1$

$lk(\gamma_1, \gamma_2) = 1$
$lk(\gamma_1, \gamma_3) = 1$
$lk(\gamma_1, \gamma_2 + \gamma_3) = 0$

The second lemma uses the following idiosyncratic graph which we'll call **Foisy's graph**.

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[5] Of course we need to take the topological closure to pop back in the cusp points but let's not convolute this even further.

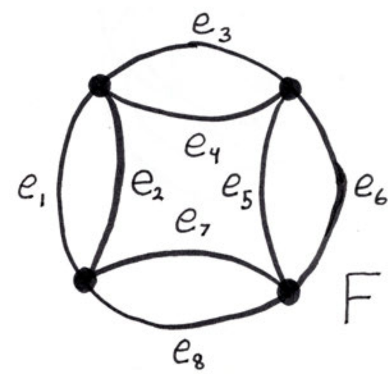



Figure 2.6: Foisy's graph

Let the cycle formed by $e_1$ and $e_2$ be called $C_1$, the one formed by $e_3$ and $e_4$ $C_2$, $e_5$ and $e_6$ $C_3$, $e_7$ and $e_8$ $C_4$. Furthermore, let $\alpha$ denote the sum of the Arf invariants of all simple cycles passing through all vertices. Such cycles are composed of four edges and always of the form

$$C = \{e_1 \text{ or } e_2, e_3 \text{ or } e_4, e_5 \text{ or } e_6, e_7 \text{ or } e_8\}$$

Thus there are $2^4 = 16$ of them.

$$\alpha = \sum_C \alpha(C) \mod 2$$

**Lemma 2.3.9** (Foisy's Lemma)**.** *Given an embedding of the graph $F$, $\alpha = 1$ if and only if $lk(C_1, C_3) = 1$ and $lk(C_2, C_4) = 1$.*

*Proof.* see reference. □

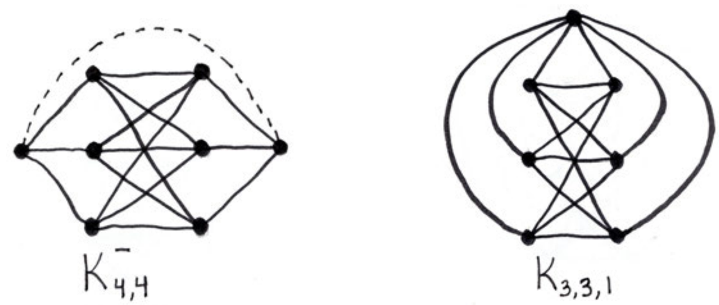
Foisy's lemma lets us prove that there are knotted cycles in a graph using pairs of linked cycles. Thus, we may bootstrap intrinsic knotting proofs using intrinsic linking results.

Foisy's strategy for proving that $K_{3,3,1,1}$ is IK proceeds as follows. First, identify two sets of cycle pairs such that at least one cycle pair in each set is linked an odd number of times, regardless of embedding. Then, by using the homology lemma and these linked pairs, show that $K_{3,3,1,1}$ must always contain a Foisy graph minor where $lk(C_1, C_3) = 1$ and $lk(C_2, C_4) = 1$. Thus every embedding of $K_{3,3,1,1}$ has some cycle with Arf invariant 1, a knotted cycle!

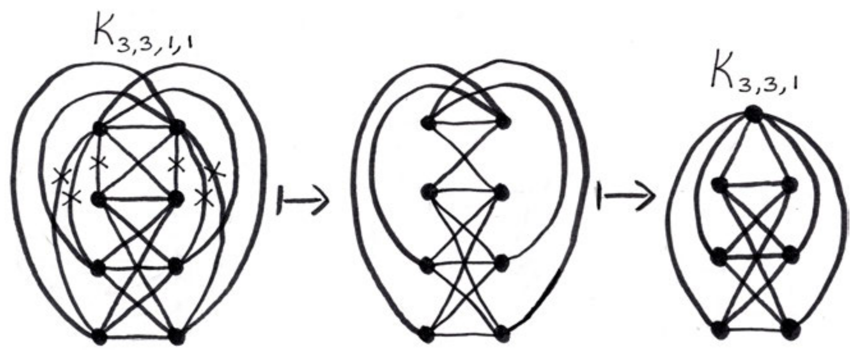
We will now flesh out the details of this approach.

*Proof.* To begin, fix an embedding of $K_{3,3,1,1}$ for the rest of the proof. We will show that regardless of which embedding was chosen, there is a knotted cycle.

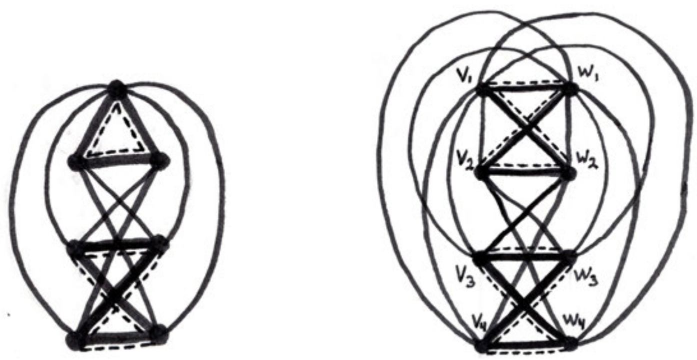
(Step 1) We exhibit two different ways to contract this embedded $K_{3,3,1,1}$ into two different intrinsically linked graphs: $K^-_{4,4}$ and $K_{3,3,1}$.




First, we take $K_{3,3,1,1}$, delete edges carefully and contract the edge between the two degree 7 vertices.

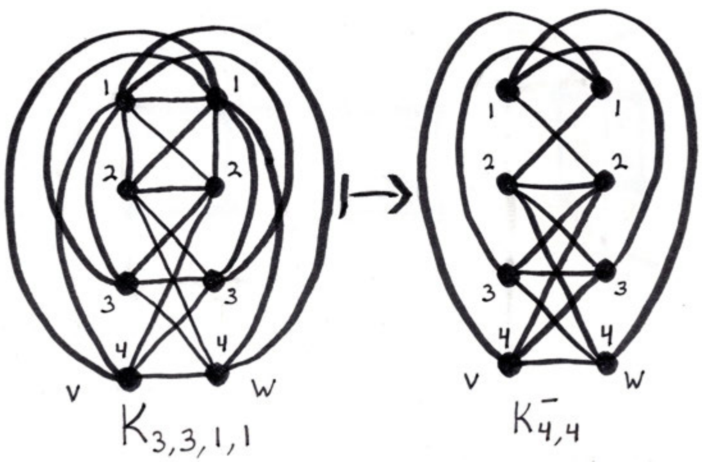



By the intrinsic linking results there must be a pair of cycles in $K_{3,3,1}$ with linking number 1 mod 2. Without loss of generality, this pair of cycles must be composed of a triangle (using the top vertex) and a "square" on the remaining vertices.



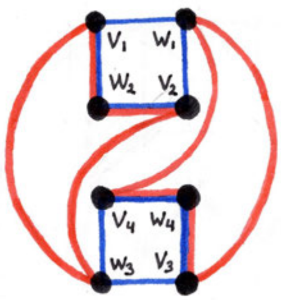

Re-expanding $K_{3,3,1}$ into $K_{3,3,1,1}$ we get two "squares."[6]

We also label the vertices of this diagram for future reference. After having done so, we delete edges to arrive at an embedding of $K^-_{4,4}$

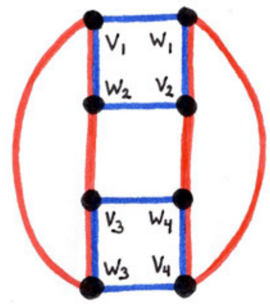



Again, by intrinsic linking results there must be a pair of linked (mod 2) cycles in this embedding of $K^-_{4,4}$. Furthermore, because $v_1, v_2, v_3, v_4$ have no connections between them and because $w_1, w_2, w_3, w_4$ likewise have no connection between them, there are no cycles on 3 vertices; the two linked cycles must be squares, each with two $v$s and two $w$s. Closer inspection reveals 9 possible pairs of cycles. If we draw each pair together with our original pair of cycles $(v_1w_1v_2w_2), (v_3w_3v_4w_4)$, then we get the following diagrams:

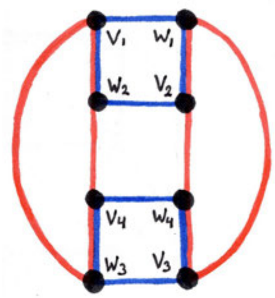
[6]the previous edge deletions ensure that the triangle must expand into a square rather than remain a triangle.



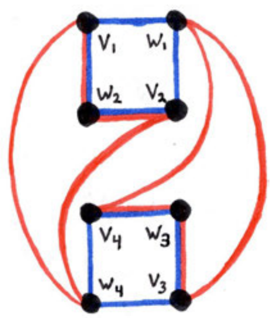

$(v_1 w_2 v_2 w_3)$
$(w_1 v_3 w_4 v_4)$



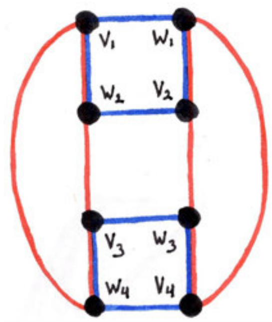

$(v_1 w_2 v_3 w_3)$
$(w_1 v_2 w_4 v_4)$



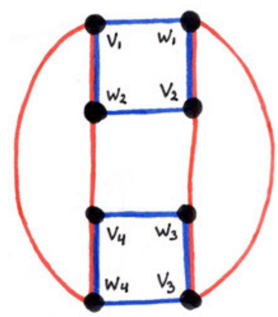

$(v_1 w_2 v_4 w_3)$
$(w_1 v_2 w_4 v_3)$



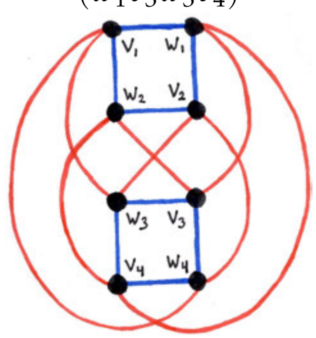

$(v_1 w_2 v_2 w_4)$
$(w_1 v_3 w_3 v_4)$



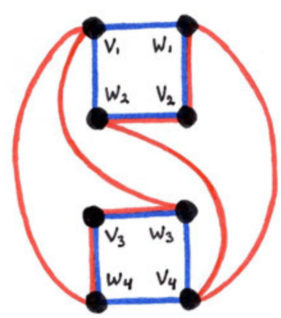

$(v_1 w_2 v_3 w_4)$
$(w_1 v_2 w_3 v_4)$



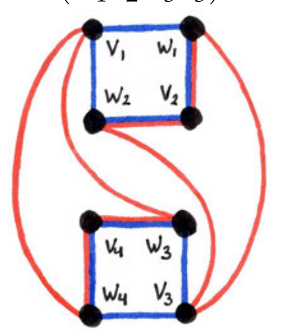

$(v_1 w_2 v_4 w_4)$
$(w_1 v_2 w_3 v_3)$




$(v_1 w_3 v_2 w_4)$
$(w_1 v_3 w_4 v_4)$




$(v_1 w_3 v_3 w_4)$
$(w_1 v_2 w_4 v_4)$




$(v_1 w_3 v_4 w_4)$
$(w_1 v_2 w_4 v_3)$

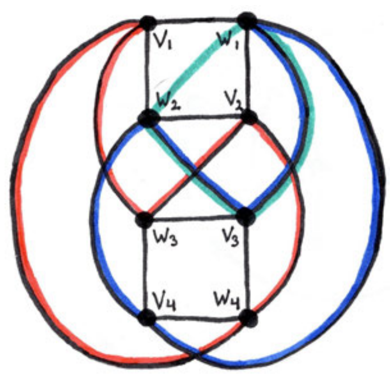
What do these diagrams represent? By the preceding argument (identifying IL minors of $K_{3,3,1,1}$) one of these 9 cases must lie as a subgraph of $K_{3,3,1,1}$ with both pairs of cycles linked (mod 2)[7]. To reiterate, regard-

[7]That is, $lk((v_1 w_1 v_2 w_2), (v_3 w_3 v_4 w_4)) = 1$ and $lk(a, b) = 1$ where $(a, b)$ is the pair of

less of the embedding of $K_{3,3,1,1}$ into space, one of these 9 cases must be "doubly linked."

(Step 2) Now consider the case $(v_1w_2v_3w_3)(w_1v_2w_4v_4)$. By contracting edges $v_1w_2$, $v_3w_3$, $v_2w_1$, and $v_4w_4$ we arrive at Foisy's graph. Not only that, but the cycles are linked mod 2 as required by his lemma! Therefore we may conclude that there is some cycle in the contracted subgraph with Arf invariant 1; there is a knotted cycle. Re-expanding the contracted subgraph we retain the knotted cycle, now lying in $K_{3,3,1,1}$.

If we can make similar arguments for the remaining 8 cases, then we will be done. This is relatively straight forward for all of the cases except $(v_1w_3v_2w_4),(w_1v_3w_2v_4)$. In order to tackle this case we will need to use the homology lemma.




We know that $lk(v_1w_3v_2w_4, w_1v_3w_2v_4) = 1$. Furthermore (noting that $w_1w_2$ is an edge in $K_{3,3,1,1}$) we know that the cycle $w_1v_3w_2$ is disjoint from $v_1w_3v_2w_4$. Therefore we may apply the homology lemma to conclude that either $lk(v_1w_3v_2w_4, w_1v_3w_2) = 1$ or $lk(v_1w_3v_2w_4, w_1v_3w_2v_4 + w_1v_3w_2) = lk(v_1w_3v_2w_4, w_1v_4w_3) = 1$. Without loss of generality, consider the case $lk(v_1w_3v_2w_4, w_1v_3w_2) = 1$.

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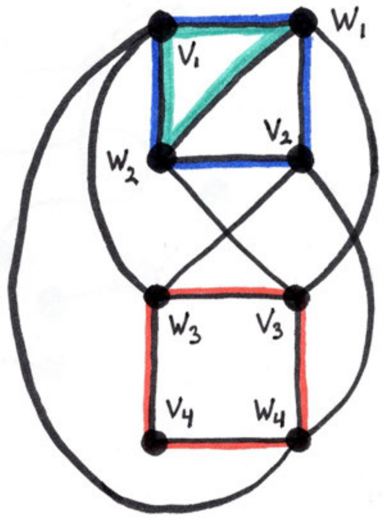
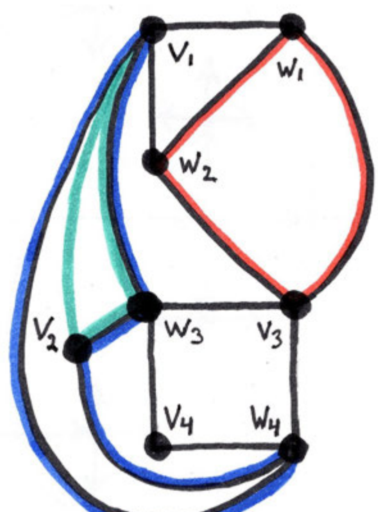
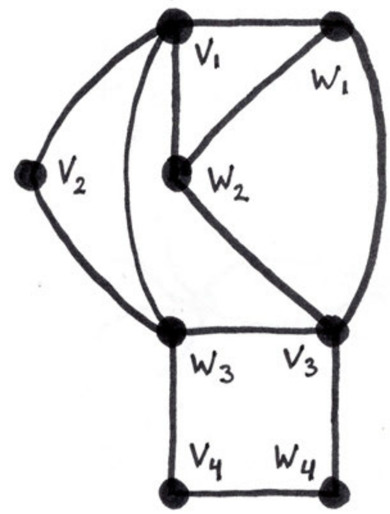
cycles labeling the diagram/case in question.




Now we'll apply the homology lemma again to "split" the top square. We know that $lk(v_3w_3v_4w_4, v_1w_1v_2w_2) = 1$. Furthermore we know that the cycle $w_1v_1w_2$ is disjoint from $v_3w_3v_4w_4$. Therefore we may apply the homology lemma to conclude that either $lk(v_3w_3v_4w_4, w_1v_1w_2) = 1$ or that $lk(v_3w_3v_4w_4, w_1v_1w_2v_2 + w_1v_1w_2) = lk(v_3w_3v_4w_4, w_1v_2w_2) = 1$. Without loss of generality, consider the case $lk(v_3w_3v_4w_4, w_1v_1w_2) = 1$.




Since $v_1v_2$ is an edge in $K_{3,3,1,1}$, we may repeat the exact same argument to split the cycle $v_1w_3v_2w_4$ along $v_1v_2$. Without loss of generality, we may assume that $lk(v_1v_2w_3, w_1w_2v_3) = 1$, yielding the final picture,




which is contractable down to Foisy's graph $F$. Thus we may apply his lemma and conclude our proof that $K_{3,3,1,1}$ is IK! □

This is a long, involved proof. What lessons ought we take away?

- Use links to find knots. (Foisy's lemma)
- Decompose subgraphs in a link preserving manner. (homology lemma)

This thesis will attempt to show how fruitful the second lesson proves when taken to its logical conclusion.

## 2.4 Linear Algebra (mod 2) and the Edge Space

Although many readers are aware that linear algebra may be conducted over the field of integers mod 2, $\mathbb{Z}_2$, fewer may have actually done so. These vector spaces have a natural isomorphism with Boolean algebras and the subset algebra defined on $\mathcal{P}(S)$, the powerset of a set $S$. While these multiple interpretations are often very useful and convenient, they are also very confusing to navigate without a pinch of experience.

**Definition 2.4.1.** Given a finite set $S = \{e_1, e_2, \ldots, e_n\}$, the **mod 2 vector space** on $S$ is $\mathbb{Z}_2 S$.

$$\mathbb{Z}_2 S = \left\{ v = \sum_{i=1}^{n} v_i e_i \,\middle|\, v_i \in \mathbb{Z}_2 \right\}$$

The vector sum is defined as $v + w = \sum_{i=1}^{n} (v_i + w_i) e_i$.

Unlike other vector spaces, linear scaling over $\mathbb{Z}_2$ is dreadfully boring. Either you get $1v = v$ or $0v = 0$.

In order to understand our first connection—with Boolean algebra—we will focus on $\mathbb{Z}_2$ by itself. Looking at all additions of elements, and all multiplications, we get "truth tables" of a sort.

| $x$ | $y$ | $x+y$ |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 1 |
| 1 | 0 | 1 |
| 1 | 1 | 0 |

| $x$ | $y$ | $xy$ | $x$ AND $y$ |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0 | 1 | 0 | 0 |
| 1 | 0 | 0 | 0 |
| 1 | 1 | 1 | 1 |

| $x$ | $y$ | $x$ OR $y$ |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 1 |
| 1 | 0 | 1 |
| 1 | 1 | 1 |

Letting 1 represent true and 0 false, multiplication is the same as **AND**, but addition does not correspond to **OR**. We can fix this shortcoming in one of two ways. First, we could define $x$ OR $y = x + y + xy$. Alternatively we could define a new logical expression, **exclusive or**, meaning either $x$ or $y$ (but not both). We write this statement as $x$ **XOR** $y$, which is just $x + y$. In this way, we've given a logical interpretation to addition in $\mathbb{Z}_2$.

What about $\mathbb{Z}_2 S$? In search of a more perfect analogy we can define a vector multiplication over $\mathbb{Z}_2$ as

$$vw = \sum_{i=1}^{n} (v_i w_i) e_i.$$

Since this multiplication is defined component-wise, our vector algebra $\mathbb{Z}_2 S$ will now behave like $\mathbb{Z}_2$. That is, it will behave like a Boolean algebra. This observation primes us for a connection with the subset algebra on $\mathcal{P}(S)$.

**Definition 2.4.2.** The **subset algebra** on $\mathcal{P}(S)$ consists of the operators union $\cup$, intersection $\cap$ and difference $-$ defined as expected on subsets $a \subseteq S$. That is,

$$\begin{aligned} a \cup b &= \{e_i | e_i \in a \text{ OR } e_i \in b\} \\ a \cap b &= \{e_i | e_i \in a \text{ AND } e_i \in b\} \\ a - b &= \{e_i | e_i \in a \text{ AND } e_i \notin b\} \end{aligned}$$

Although such a definition is the "natural" algebraic picture of $\mathcal{P}(S)$ in some sense, we seem to have lost (and gained) some combinations. To review, we see that a few translations remain undefined.

| $\mathbb{Z}_2 S$ | Boolean Algebra | $\mathcal{P}(S)$ |
|---|---|---|
| $a + b$ | $a$ XOR $b$ | ? |
| $ab$ | $a$ AND $b$ | $a \cap b$ |
| $a + ab + b$ | $a$ OR $b$ | $a \cup b$ |
| ? | ? | $a - b$ |

The set algebra analogue of addition and XOR is called **symmetric difference**. It's written $a \triangle b$.

$$a \triangle b = \{e_i | e_i \in a \text{ XOR } e_i \in B\}.$$

Alternatively, we can write this operator using a combination of set difference and union, $a \triangle b = (a - b) \cup (b - a)$, which is why we call it symmetric difference. Since we won't have any need for set difference itself, we'll forgo back porting it, although the curious reader will find profound "implications."

Our connection $\mathbb{Z}_2 S \cong \mathcal{P}(S)$ (as Boolean algebras) gives us a convenient means of encoding vectors in $\mathbb{Z}_2 S$ as subsets of $S$. Therefore as we develop further machinery on $\mathbb{Z}_2 S$, we will intone the mantra "and what does this mean in $\mathcal{P}(S)$?"

Given that $\mathbb{Z}_2 S$ is a vector space, there must be some **dual vector space** $(Z_2 S)^*$ of **linear functions** (aka. forms) on $\mathbb{Z}_2 S$.

**Definition 2.4.3.** Let $(\mathbb{Z}_2 S)^*$ denote the set of **linear forms** $\varphi : \mathbb{Z}_2 S \to \mathbb{Z}_2$ (Here linear is equivalent to saying $\varphi(0) = 0$ and $\varphi(v + w) = \varphi(v) + \varphi(w)$). $(\mathbb{Z}_2 S)^*$ forms a $\mathbb{Z}_2$ vector space called the **dual space** under the addition $(\varphi + \psi)(v) = \varphi(v) + \psi(v)$.

Unlike our original vector space $\mathbb{Z}_2 S$, which has the natural basis $S$, we don't have a basis for $(\mathbb{Z}_2 S)^*$ readily available. This can lead to a certain amount of disorientation as we have literally lost our bearings. Don't panic! We're prepared—we can use this same standard basis $S = \{e_1, e_2, \ldots, e_n\}$ of $\mathbb{Z}_2 S$ to construct a standard basis of linear forms.

**Definition 2.4.4.** Let $e_i^*$ be the linear form on $\mathbb{Z}_2 S$, which sends $v \in \mathbb{Z}_2 S$ to

$$\begin{aligned} e_i^*(v) = e_i^*\left(\sum_{j=1}^{n} v_j e_j\right) &= \sum_{j=1}^{n} v_j e_i^*(e_j) \\ &= v_i \end{aligned}$$

Alternatively stated,

$$e_i^*(e_j) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

This definition induces a linear bijection $* : \mathbb{Z}_2 S \rightarrow (\mathbb{Z}_2 S)^*$

**Definition 2.4.5.** The **dual vector** $v^*$ of a vector $v$ is the linear form

$$v^* = \sum_{i=0}^{n} v_i e_i^*$$

Be careful! This "duality" mod 2 is not your garden variety duality. $v^*(v)$ is not necessarily 1![8]

"And what does this mean in $\mathcal{P}(S)$?" Using our connection $\mathbb{Z}_2 S \cong \mathcal{P}(S)$, we can encode dual vectors $v^* \in (\mathbb{Z}_2 S)^*$ as subsets of $\mathcal{P}(S)$. Let $v^*$ be represented by the set $a = \{e_i | v_i = 1\}$[9]. Since $*$ is a linear map, we

[8] It's impossible to simultaneously ensure that $v^*(v) = 1$ and that $*$ is an invertible linear map, as this example shows:

$$\begin{aligned} (e_1^* + e_2^*)(e_1 + e_2) &= (e_1^* + e_2^*)(e_1) + (e_1^* + e_2^*)(e_2) \\ &= e_1^*(e_1) + e_2^*(e_1) + e_1^*(e_2) + e_2^*(e_2) \\ &= 1 + 0 + 0 + 1 \\ &= 0 \end{aligned}$$

[9] thus $v^*$ is represented the same way as $v$.

translate $v^* + w^* = (v+w)^*$ into $a \triangle b$, where $a$ and $b$ represent $v^*$ and $w^*$ as prescribed.

Furthermore, we may translate the application of a form $v^*$ to a vector $w$ into operations on their representing sets.

**Theorem 2.4.6.** *Given $v^* \in (\mathbb{Z}_2 S)^*$ and $w \in \mathbb{Z}_2 S$, $v^*(w) = r(vw)$ where $r(x) = \sum_{i=1}^{n} x_i \mod 2$.*

*Proof.*

$$\begin{aligned}
v^*(w) &= \left(\sum_{i=1}^{n} v_i e_i^*\right)\left(\sum_{j=1}^{n} w_j e_j\right) \\
&= \sum_{i=1}^{n} v_i e_i^* \left(\sum_{j=1}^{n} w_j e_j\right) \\
&= \sum_{i=1}^{n} v_i \sum_{j=1}^{n} w_j e_i^*(e_j) \\
&= \sum_{i=1}^{n} v_i w_i \\
&= r(vw)
\end{aligned}$$

□

Let $v^*$ be a dual vector represented by $a$ and $w$ a vector represented by $b$. Then given this theorem, the application $v^*(w) = r(vw)$ is represented by $|a \cap b|$ (mod 2); that is, by the size of the intersection mod 2.

Our accumulated results give us a very convenient way to compute with vectors and linear forms mod 2.

| $v, w \in \mathbb{Z}_2 S$ | is represented by | $a, b \in \mathcal{P}(S)$ |
|---|---|---|
| $v = \sum_{i=1}^{n} v_i e_i$ | | $a = \{e_i \mid v_i = 1\}$ |
| $v + w$ | | $a \triangle b$ |
| $v^*$ | | $a$ |
| $v^*(w)$ | | $\lvert a \cap b \rvert$ (mod 2) |

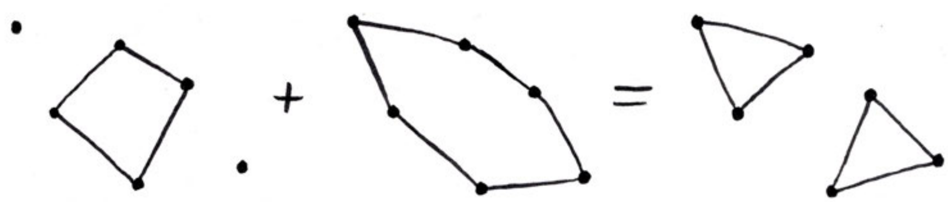
## The Edge Space

We've developed this machinery for handling linear algebra mod 2 in order to treat a particular concept (shamelessly lifted) from matroid theory called the edge space of a graph.

**Definition 2.4.7.** Let $E = \{e_1, e_2, \ldots, e_n\}$ be the set of edges of a graph $G$. Then $\mathbb{Z}_2 E$ is called the **edge space** of $G$.

Note that any subgraph of $G$ specified via some subset of the edges is represented by some vector in $\mathbb{Z}_2 E$. In particular, cycles both simple and generalized manifest as vectors in $\mathbb{Z}_2 E$.

One natural question to ask is the meaning of addition in the edge space. If we think of vectors/subgraphs as subsets of edges then addition translates to symmetric difference of the edge sets. For instance,



Although it's entirely non-obvious, the set of all generalized cycles of $G$ forms a subspace of the edge space. To show this, we will make use of some linear forms on the cycle space. Let $d_v(G)$ denote the degree[10] of the vertex $v$ of $G$ mod 2.

**Lemma 2.4.8.** *$d_i$ is a linear form on $\mathbb{Z}_2 E$.*

*Proof.* Given $d_i$'s definition, we need only consider edges with the vertex $v$ as an endpoint. Relabel these $e_1, e_2, \ldots, e_m$. Then

$$d_v = \sum_{j=1}^{m} e_j^*$$

□

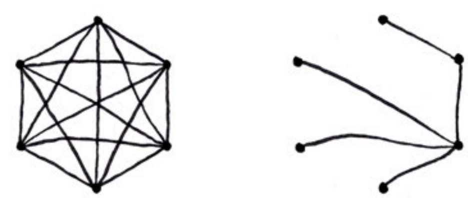
[10] If we think of our graph $G$ as a simplicial complex, then we can think of this map as a differential(?) of 1-chains. The $d$ is a happy coincidence of notation.

We defined a generalized cycle $C$ as an edge-induced subgraph of $G$ such that $d_v(C) = 0$ for all vertices $v$. Therefore if we take two generalized cycles $a$ and $b$, $d_v(a+b) = d_v(a) + d_v(b) = 0 + 0 = 0$ for all vertices $v$. That is, $a + b$ is a generalized cycle too.

**Definition 2.4.9.** The **cycle space** of a graph $G$ is a subspace of the edge space of $G$ comprised exactly of all generalized cycles. It's well defined by the preceding argument and lemma.

It would be nice to have an explicit basis for this cycle space. Surprisingly, we can do so by picking a spanning tree of the graph (assuming the graph's connected).

**Definition 2.4.10.** A **spanning tree** of a connected graph $G$ is a subgraph of $G$ that's both a tree and touches every vertex of $G$.



For instance, here's $K_6$ and a spanning tree of it. If we pick a spanning tree $T$ for some connected graph $G$ then $T$ and all subgraphs of $T$ form a subspace of the edge space with dimension $|V| - 1$, where $|V|$ is the number of vertices of $G$. This means that the cycle space of $G$ can have dimension at most $|E| - |V| + 1$, since $T$ contains no cyclic subgraphs (and is therefore linearly independent from the cycle space).

**Theorem 2.4.11.** *Consider the set of edges not in $T$, $E - T$. For any edge $e \in E - T$, there exists a unique cycle in the graph $T \cup \{e\}$, which we denote $C_e$.*

*Proof.* $T \cup \{e\}$ has more than $|V| - 1$ edges, but is connected. Therefore, $T \cup \{e\}$ must be cyclic, demonstrating existence.

Now, let $C$ and $D$ be any two non-trivial cycles in the graph $T \cup \{e\}$. Because $T$ is acyclic, both $C$ and $D$ must include the edge $e$. Therefore, $e$ is not included in the cycle $C + D$. But $C + D$ is a cycle not including $e$, a cycle in $T$. $T$ is acyclic so $C + D$ must be 0; $C = D$. □

**Corollary 2.4.12.** *The set of cycles $C_e$ for $e \in E - T$ forms a basis for the cycle space of $G$.*

*Proof.* Since there are $|E - T| = |E| - |V| + 1$ cycles in the proposed basis, it's of the maximum possible dimension. Therefore, it suffices to show that the set of cycles $C_e$ is linearly independent.

Each cycle $C_e$ contains an edge (namely $e$) that isn't included in any other cycle $C_e$. Therefore the cycles $C_e$ are linearly independent. □

In this way we've found a basis for the cycle space of $G$: $\{C_e\}_{e \in E-T}$. Although we won't make too much use of this explicit basis in the rest of the thesis, it's nice to know we have it around if we want it.

To close this discussion of the cycle space, we note that given any subset $a \subseteq E - T$, we can construct a unique cycle $C_a$ as

$$C_a = \sum_{e \in a} C_e$$

Nifty!

## Bilinear Forms on $\mathbb{Z}_2 S$ and Tensor Products

Isn't one linear good enough? Occasionally we will want to use functions which compare or assess two vectors relative to each other, which we will write as $\varphi(v, w)$. Because these are two argument functions of vectors, we'd like them to "play nice" with the linear structure of a vector space. To accomplish this effect, we insist that these bilinear functions behave linearly with respect to each argument. That is, if we fix the vector $v$ and vary $w$, $\varphi(v, w)$ is a linear function of $w$, as is $\varphi(w, v)$.

**Definition 2.4.13.** A **bilinear form** $\varphi$ on $\mathbb{Z}_2 S$ is a function $\varphi : \mathbb{Z}_2 S \times \mathbb{Z}_2 S \to \mathbb{Z}_2$ such that

$$\begin{aligned}
\varphi(x, y) + \varphi(x, z) &= \varphi(x, y + z) \\
\varphi(x, z) + \varphi(y, z) &= \varphi(x + y, z) \\
\varphi(x, 0) = \varphi(0, y) &= 0
\end{aligned}$$

We say that a bilinear form is **symmetric** if $\varphi(x, y) = \varphi(y, x)$.

"And what does this mean in $\mathcal{P}(S)$?" That's actually quite a pickle. For linear forms we took advantage of the duality mapping $*$ to represent vectors in $(\mathbb{Z}_2 S)^*$ via representations of vectors in $\mathbb{Z}_2 S$. In fact, we can use the same trick, since the space of bilinear forms is a dual space: $(\mathbb{Z}_2 S \otimes \mathbb{Z}_2 S)^*$.

Here $\otimes$ means the **tensor product** which—colloquially—is the most general form of multiplication between vectors.

**Definition 2.4.14.** The **tensor product** is an operation on two **1-vectors** $x, y \in \mathbb{Z}_2 S$ producing a **2-vector** $x \otimes y \in \mathbb{Z}_2 S \otimes \mathbb{Z}_2 S$ satisfying the relationships:

$$\begin{aligned} x \otimes y + x \otimes z &= x \otimes (y+z) \\ x \otimes z + y \otimes z &= (x+y) \otimes z \\ x \otimes 0 = 0 \otimes y &= 0 \end{aligned}$$

The **tensor product space** $\mathbb{Z}_2 S \otimes \mathbb{Z}_2 S$ is a set consisting of all tensor products of two 1-vectors ($x \otimes y$) and summations thereof ($\sum x_i \otimes y_i$). Elements of the tensor product space are called **2-vectors**. 2-vectors which may be directly written as a tensor product of two 1-vectors ($x \otimes y$) are said to be **decomposable** whereas those only expressible as sums of decomposable 2-vectors are said to be **indecomposable**. Finally, the tensor product space is closed under addition making it a vector space mod 2.

Yikes! Maybe we can be a bit more concrete.

Given two basis vectors $e_i, e_j$ we get the 2-vector $e_i \otimes e_j$. Often we will write this 2-vector using the shorthand $e_{ij}$ or $e_{i,j}$ for brevity. Given these tensor products of basis vectors we can construct a basis for $\mathbb{Z}_2 S \otimes \mathbb{Z}_2 S$: $\{e_{i,j}\}_{i=1,j=1}^{n}$. In this way we can give a more concrete picture of $\mathbb{Z}_2 S \otimes \mathbb{Z}_2 S$.

**Theorem 2.4.15.** *The set of 2-vectors $\{e_i \otimes e_j\}_{i=1,j=1}^{n}$ forms a basis for the tensor product space $\mathbb{Z}_2 S \otimes \mathbb{Z}_2 S$.*

*Proof.* First, note that the set $\{e_{ij}\}$ is linearly independent since no two $e_{ij}$ will interact under the given rules. Therefore it suffices to show that $\{e_{ij}\}$ spans $\mathbb{Z}_2 S \otimes \mathbb{Z}_2 S$. Every 2-vector $v \in \mathbb{Z}_2 S \otimes \mathbb{Z}_2 S$ may be written as a sum of decomposable 2-vectors,

$$v = \sum_{k=1}^{m} x_k \otimes y_k$$

so it suffices to show that every decomposable 2-vector $x \otimes y$ may be written as a linear combination of vectors in $\{e_{ij}\}$. Since every 1-vector may be written as a linear combination of basis vectors from $S = \{e_i\}$, we get

$$\begin{aligned} x \otimes y &= \left(\sum_{i=1}^{n} x_i e_i\right) \otimes \left(\sum_{j=1}^{n} y_j e_j\right) \\ &= \sum_{i=1}^{n} x_i \left(e_i \otimes \left(\sum_{j=1}^{n} y_j e_j\right)\right) \\ &= \sum_{i=1}^{n} x_i \left(\sum_{j=1}^{n} y_j (e_i \otimes e_j)\right) \\ &= \sum_{i=1,j=1}^{n} x_i y_j (e_i \otimes e_j) \end{aligned}$$

□

**Corollary 2.4.16.** $\mathbb{Z}_2 S \otimes \mathbb{Z}_2 S \cong \mathbb{Z}_2(S \times S)$.

*Proof.* Consider the bijection between bases $e_i \otimes e_j \mapsto (e_i, e_j)$ □

Let's stop and regroup. We began by looking at bilinear forms, but quickly switched to talking about the tensor product because of the claim that bilinear forms could be represented as vectors in the dual space $(\mathbb{Z}_2 S \otimes \mathbb{Z}_2 S)^*$. Why does this make any sense?

In effect, we can think of the tensor product as a mechanism for pairing vectors. That is, given a pair of vectors $(x, y)$, we can take their tensor product to form a single 2-vector $x \otimes y$. Let us now pause and reflect on the symmetries in the definitions of bilinear forms and tensor products. Ommmm...

**Theorem 2.4.17.** *$(\mathbb{Z}_2 S \otimes \mathbb{Z}_2 S)^*$ is the space of bilinear forms on $\mathbb{Z}_2 S$.*

*Proof.* We'll begin by showing that dual 2-vectors act like bilinear forms on decomposable 2-vectors. Let $(e_i \otimes e_j)^*$ be a dual basis 2-vector and let $x, y$ be 1-vectors. I claim that the function $f(x, y) = (e_i \otimes e_j)^*(x \otimes y)$ is bilinear. Writing $x \otimes y$ as a linear combination of basis vectors, we get

$$\begin{aligned} f(x,y) &= (e_i \otimes e_j)^* \left( \sum_{k=1,l=1}^{n} x_k y_l (e_k \otimes e_l) \right) \\ &= \sum_{k=1,l=1}^{n} x_k y_l (e_i \otimes e_j)^* (e_k \otimes e_l) \\ &= x_i y_j \end{aligned}$$

Given this identity, $f(x,y) = x_i y_j$, bilinearity of $f$ is immediate:

$$\begin{aligned} f(x,y) + f(x,z) &= x_i y_j + x_i z_j \\ &= x_i (y_j + z_j) \\ &= f(x, y+z) \\ f(x,z) + f(y,z) &= x_i z_j + y_i z_j \\ &= (x_i + y_i) z_j \\ &= f(x+y, z) \\ f(x,0) &= x_i 0 = 0 \\ f(0,y) &= 0 y_j = 0 \end{aligned}$$

Since any other dual 2-vector is just a sum of dual basis 2-vectors, and since the sum of any two bilinear functions is itself bilinear, every dual 2-vector must be a bilinear form in disguise.

Furthermore, every bilinear form is a dual 2-vector in disguise. Let $\varphi$ be an arbitrary bilinear form on $\mathbb{Z}_2 S$. If we apply $\varphi$ to every pair of basis vectors $(e_i, e_j)$, we get the dual 2-vector

$$v = \sum_{\varphi(e_i, e_j) = 1} (e_i \otimes e_j)^*$$

which yields $\varphi$ as a bilinear form. □

But what does this mean in $\mathcal{P}(S)$ for crying out loud!

$$\begin{array}{rcll} (\mathbb{Z}_2 S \otimes \mathbb{Z}_2 S)^* & \cong & \mathbb{Z}_2 S \otimes \mathbb{Z}_2 (S) & \text{(duality)} \\ & \cong & \mathbb{Z}_2 (S \times S) & \text{(corollary)} \\ \text{is represented by} & & \mathcal{P}(S \times S) & \end{array}$$

It appears that our connection between linear algebra mod 2 and the algebra of subsets is translating the tensor product into a cartesian product. To further flesh out this correspondence, recall the formula for expressing a decomposable 2-vector $x \otimes y$ on the standard basis:

$$x \otimes y = \sum_{i=1,j=1}^{n} x_i y_i (e_i \otimes e_j)$$

Given that $x, y \in \mathbb{Z}_2 S$ are represented by subsets $a, b \in \mathcal{P}(S)$, $x \otimes y \in \mathbb{Z}_2 S \otimes \mathbb{Z}_2 S$ is represented by a subset $c \in \mathcal{P}(S \times S)$ where

$$c = \{(e_i, e_j) | e_i \in a \text{ and } e_j \in b\}$$

That is, $c = a \times b$. $x \otimes y$ corresponds to $a \times b$!

As before, we may again summarize our observations in a table. Collectively, they give us a powerful way to compute with vectors and tensor products mod 2.

| $x, y \in \mathbb{Z}_2 S$ | is represented by | $a, b \in \mathcal{P}(S)$ |
|---|---|---|
| $x = \sum_{i=1}^{n} x_i e_i$ | | $a = \{e_i \| x_i = 1\}$ |
| $x + y$ | | $a \triangle b$ |
| $x^*$ | | $a$ |
| $x^*(y)$ | | $\|a \cap b\|$ (mod 2) |
| $x \otimes y \in \mathbb{Z}_2 S \otimes \mathbb{Z}_2 S$ | | $a \times b \in \mathcal{P}(S \times S)$ |
| $x \otimes y + z \otimes w$ | | $(a \times b) \triangle (c \times d)$ |
| $\varphi = \sum_{i=1,j=1}^{n} \varphi_{i,j} (e_i \otimes e_j)^*$ | | $f = \{(e_i, e_j) \| \varphi_{i,j} = 1\}$ |
| $\varphi(x, y) = \varphi(x \otimes y)$ | | $\|f \cap (a \times b)\|$ (mod 2) |

# Chapter 3

# When is a Graph Intrinsically Knotted?

The goal of this chapter is to develop a systematic method for proving that graphs are intrinsically knotted. To understand how we would go about such a systematization, let's first focus on the simpler problem of proving that a graph $G$ is IL.

To begin, let's write our desired statement and unpack it:

> The graph $G$ is IL.

This is equivalent to saying

> $\iff$ For every embedding $E$ of $G$ there is some pair $(A, B)$ of vertex disjoint simple cycles (**VDSC**s) such that $(A, B)$ is a non-trivial link in $E$.

Using logical quantifiers to abbreviate, we get

> $\iff$ $\forall$ embeddings $E$ : $\exists(A, B)$VDSCs: $(A, B)$ is non-trivial in $E$

In the background section we made use of the linking number mod 2 to detect linked cycles. Logically speaking, we used the statement "$lk_E(A, B) = 1 \mod 2 \implies (A, B)$ is a non-trivial link in $E$." Applying this implication here, it suffices to prove that

> $\Longleftarrow \forall$ embeddings $E$ : $\exists(A, B)$VDSCs: $lk_E(A, B) = 1 \mod 2$

Now, observe that the innermost quantified statement "$lk_E(A, B) = 1 \mod 2$" only refers to the embedding $E$ via $lk_E$. This suggests one last rewriting.

$$\iff \forall lk_E : \exists (A, B)\text{VDSCs: } lk_E(A, B) = 1 \mod 2$$

Finally, let's codify this argument into a lemma.

**Lemma 3.0.1.** *Let G be a graph. If*

$$\forall lk_E : \exists (A, B)\text{VDSCs} : lk_E(A, B) = 1 \mod 2$$

*then G is intrinsically linked.*

Although non-obvious, Robertson and Seymour's results on intrinsic linking imply that the converse is also true.

Unsurprisingly, we can execute a similar chain of logic for intrinsic knotting.

The graph $G$ is IK.

Unpacking the definition...

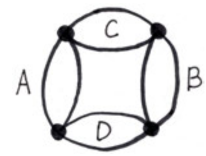
$\iff$ For every embedding $E$ of $G$ there is some simple cycle $C$ such that $C$ is knotted in $E$.

Switching to quantifiers...

$$\iff \forall \text{ embeddings } E : \exists \text{ cycle } C : C \text{ is knotted in } E$$

Let's focus on the inner statement here: $\exists$ cycle $C$ : $C$ is knotted in $E$. In Foisy's proof that $K_{3,3,1,1}$ is IK we didn't look for knotted cycles just anywhere. Rather, we found some Foisy minor of $G$ such that the sum of Arf invariants $\alpha = 1 \mod 2$ in $E$, implying the existence of some knotted cycle.

And we did that by instead showing that $F$ is "doubly linked" ($lk(A,B) = 1$ and $lk(C,D) = 1$), subsequently appealing to Foisy's lemma ($F$ is "doubly linked" $\implies$ knotted cycle in $F$).




Using this argument it suffices to prove that

$\Longleftarrow \forall$ embeddings $E : \exists$ Foisy minor $F : F$ is "doubly linked" in $E$

or if we expand the definitions of these concepts...

$$\begin{aligned} \iff \quad \forall \text{ embeddings } E : \exists \text{ Foisy pair of pairs } ((A,B),(C,D)) :& \\ lk_E(A,B) = 1 \mod 2 \text{ and}& \\ lk_E(C,D) = 1 \mod 2& \end{aligned}$$

Of course we can just replace our quantification over $E$ with a quantification over $lk_E$ as before. Combined with some other tidying, we get

$$\begin{aligned} \iff \quad \forall lk_E : \exists \text{ Foisy pair of pairs } ((A,B),(C,D)) :& \\ lk_E(A,B) \cdot lk_E(C,D) = 1 \mod 2& \end{aligned}$$

Again, we can codify the argument into a lemma.

**Lemma 3.0.2.** *Let G be a graph. If*

$$\begin{aligned} \forall lk_E : \exists \textit{ Foisy pair of pairs } ((A,B),(C,D)) :& \\ lk_E(A,B) \cdot lk_E(C,D) = 1 \mod 2& \end{aligned}$$

*then G is intrinsically knotted.*

The rest of this chapter can be seen in analogy to these two arguments. We will further refine our two current lemmas until their hypotheses become tractably computable statements. At that point the hypotheses will resemble linear algebra problems, and we'll be able to write a computer program that can ascertain their truth for arbitrary graphs $G$.

Because Robertson and Seymour's result implies that the converse of our intrinsic linking lemma is true, the program we develop for intrinsic linking will always decide intrinsic linking for a graph $G$. That is the program will always determine "yes, $G$ is IL" or "no, $G$ is not IL." By contrast, the program we develop for intrinsic knotting will only sometimes decide intrinsic knotting. That is, the program will sometimes answer "yes, $G$ is IK" but might just say "sorry, $G$ may or may not be IK." However, if the converse of the intrinsic knotting lemma is true, then our program always determines intrinsic knotting for intrinsically knotted graphs. This gives us the first open question generated by this thesis.

**Question 3.0.3.** *Does the converse of the intrinsic knotting lemma hold?*

## 3.1 The Linking Number (mod 2) is Bilinear

What else can I say? It should be fairly obvious by now... kind of. We know from the homology lemma that

$$lk_E(A, B) = lk_E(A, B + C) + lk_E(A, C)$$

in very specific cases. Namely all of the pairs $(A, B)$, $(A, B + C)$ and $(A, C)$ must be pairs of vertex disjoint simple cycles. This restriction is rather inconvenient. Perhaps we can define some other form on the *entire* edge space of our graph such that the linking number form and homology lemma are recovered as special cases.

We can. As suggested in the review of knot theory, we may use the over-crossing form $\omega$ (which counts the number of times $\omega(A, B)$ that $A$ crosses over $B$ mod 2). By our earlier review, $\omega(A, B) = lk(A, B)$ for any given diagram. Thus, given diagrams $D, D'$ of the same embedding $E$ of $G$,

$$\omega_D(A, B) = lk_E(A, B) = \omega_{D'}(A, B)$$

for pairs $(A, B)$ of vertex disjoint simple cycles. However, we should note that given arbitrary vectors $C, D \in \mathbb{Z}_2E$, $\omega_D(C, D)$ does not necessarily agree with $\omega_{D'}(C, D)$. Without getting too far ahead of ourselves, this suggests that we should look at diagrams instead of embeddings and forms $\omega_D$ instead of $lk_E$.

Still, we're getting ahead of ourselves. We haven't certified that $\omega$ is a bilinear form yet. In fact $\omega$ isn't even defined on graph diagrams.

**Definition 3.1.1.** Let $G$ be a graph and $D$ a diagram of some embedding of $G$. The **overcrossing number mod 2** of a diagram $D$, $\omega_D(x,y)$ is the number of times the subgraph $x$ crosses over the subgraph $y$ mod 2.

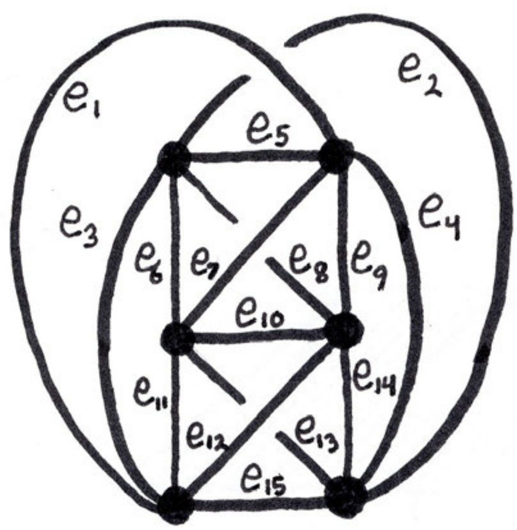
**Theorem 3.1.2.** *$\omega_D(x,y)$ is bilinear on the edge space $\mathbb{Z}_2E$ of $G$.*

*Proof.* How many times does $x$ cross over $y$? Decomposing $x$ into a sum of $e_i$ st. $x_i = 1$, we see that we can count the number of times each $e_i$ crosses over $y$ and then sum these counts up to get $\omega_D(x,y)$. Meanwhile by decomposing $y$ into a sum of $e_j$ st. $y_j = 1$, we see that by counting the number of times $e_i$ passes over $e_j$ for each $e_j$ and summing we get $\omega_D(e_i,y)$. Of course, we can take everything mod 2, and summarize…

$$\omega_D(x,y) = \sum_{x_i=1,y_j=1} \omega_D(e_i,e_j) = \sum_{i=1,j=1}^{n} x_i y_j \omega_D(e_i,e_j)$$

Therefore, $\omega_D$ is bilinear. □

To see this computation in action, consider this diagram of $K_6$.



| $\omega(e_i, e_j)$ | $j = 1$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $i = 1$ | 0 | **1** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 5 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 6 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 7 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **1** | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 8 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 9 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 10 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 11 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 12 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **1** | 0 | 0 |
| 13 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 14 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 15 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

This construction completely subsumes the homology lemma. Whenever $(A, B)$, $(B, C)$ and $(A, B)$ are vertex disjoint simple cycles (VDSCs),

$$\begin{aligned} lk_E(A, B) + lk_E(A, C) &= \omega_D(A, B) + \omega_D(A, C) \\ &= \omega_D(A, B + C) \\ &= \omega_D(A, B + C) \end{aligned}$$

But the middle derivation holds for any pair of vectors from the edge space, provided that a diagram $D$ is fixed. This dependence on diagrams prompts our next investigation.

## 3.2 Crossing Moves are Symmetric Bilinear Forms

In the two lemmas from the beginning of this chapter we replaced a universal quantification over embeddings with a universal quantification over linking numbers:

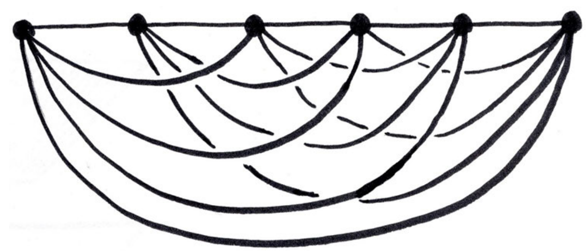
$$\forall \text{ embeddings } E \longrightarrow \forall lk_E$$

However, now we want to work with overcrossing forms, making the substitution

$$\forall \text{ diagrams } D \in \mathcal{D} \longrightarrow \forall \omega_D$$

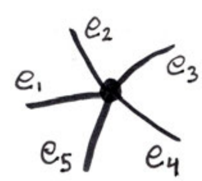
In order to do so we need to find some set of diagrams $\mathcal{D}$ containing at least one diagram of $D$ of every embedding $E$. This is the question we will now address.

To begin choose a diagram $D_0$ of some embedding of $G$. For instance we could choose a $D_0$ of $K_6$:



Now define the set of diagrams $\mathcal{D}$ as all those diagrams reachable from $D_0$ by some combination of planar isotopies, R1, R2, R3 and crossing (CX) moves, but not R4 or R5 moves. I claim that with a suitable choice of $D_0$, $\mathcal{D}$ contains at least one diagram $D$ of every embedding of our graph $G$. We prove so by picking out a canonical-ish diagram for every embedding.

Consider the local neighborhood of a vertex $v$ in a diagram $D$ of our graph $G$. We may record the **radial order** of edges around $v$ in $D$. In this example, the radial ordering is $(e_1 e_2 e_3 e_4 e_5)$.

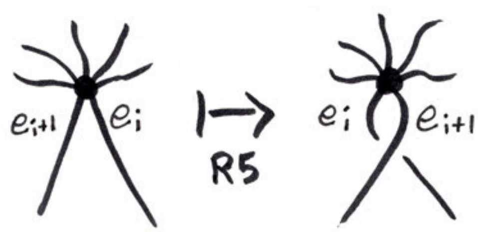



In our initial diagram $D_0$, we may record the radial ordering around every vertex $v$. Call these radial orderings arising from $D_0$ the **canonical radial orderings** (with respect to $D_0$).

**Lemma 3.2.1.** *Fix an inital diagram $D_0$ of $G$. For every embedding of $G$, there exists a diagram $D$ with the same radial orderings as $D_0$ (i.e. with the canonical radial ordering)*

*Proof.* Consider an embedding. Let $D$ be any diagram of this embedding. Now look at vertex $v$ of $D$. I claim that I can arbitrarily modify the radial ordering around $v$ without changing the embedding or affecting the radial orderings of other vertices.

To do so, it suffices to show how to transpose any two adjacent edges in the radial ordering. Just use an R5 move.

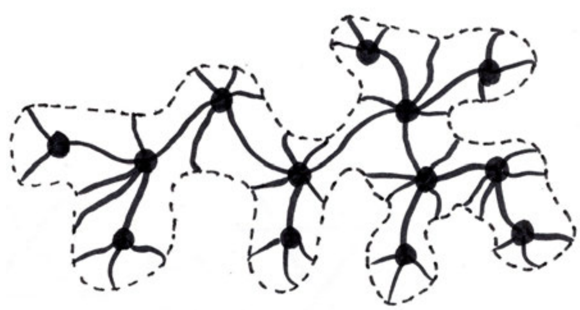



$$(\cdots e_i e_{i+1} \cdots) \mapsto (\cdots e_{i+1} e_i \cdots)$$

□

This lemma allows us to make statements like "Choose a diagram $D$ of the embedding $E$ which has canonical radial ordering."

The next step we take is to choose a **canonical spanning tree** $T_0$ of $G$, in conjunction with $D_0$ so that $D_0$ contains no crossings in a neighborhood of $T_0$. That is, the neighborhood of $T_0$ in $D_0$ looks like

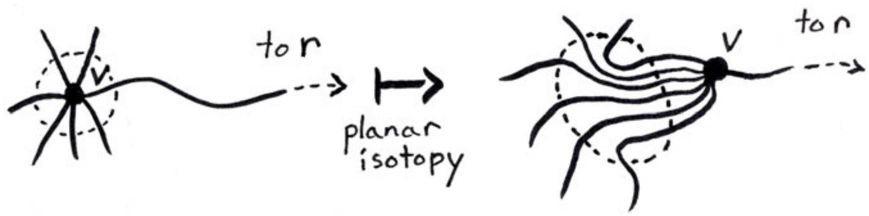


Just like we found a diagram of every embedding with a canonical radial ordering, we can now find a diagram of every embedding with a canonical neighborhood of $T_0$, namely a neighborhood with no crossings *and* the canonical radial orderings.

**Lemma 3.2.2.** *Fix an initial diagram $D_0$ of $G$ and a spanning tree $T_0$ of $G$. Then for every embedding of $G$ there exists a diagram $D$ with canonical radial orderings and no crossings in a neighborhood of $T_0$.*

*Proof.* Consider an embedding. By the preceding lemma, there exists some diagram $D$ of the embedding with canonical radial ordering. I claim that I can modify this diagram until there are no crossings in a neighborhood of $T_0$, without changing radial ordering or embedding.

To see this, we will contract $T_0$ in $D$ towards some arbitrary root node $r$ of $T_0$. Certainly $r$ alone has a neighborhood with no crossing edges. Given as much, can we add one of $r$'s children $v$ to this no-crossing neighborhood? We may "drag" $v$ along the edge between $r$ and $v$ using a combination of planar isotopy and R4 moves.



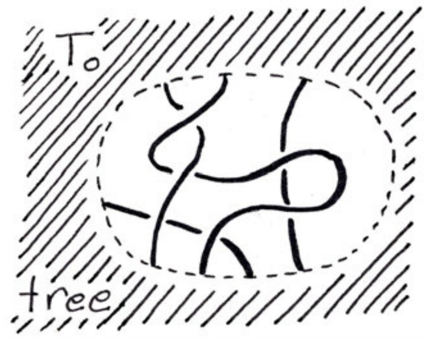


Eventually $v$ will come close enough to $r$ that the edge between them won't have any more crossings. Doing this for all nodes will remove all crossings with edges in $T_0$ from the diagram. Since we didn't use any R5 moves the radial ordering is preserved. □

**Corollary 3.2.3.** *There exists a diagram $D_0$ with no crossings in the neighborhood of a spanning tree $T_0$.*

Together, our two lemmas tell us that we can find a "canonical" picture of every embedding, and thus a set of diagrams which look identical in a neighborhood of $T_0$. Since a neighborhood of a tree can be confined to a topological disk, the only salient variations in these diagrams are what edges do outside this disk. In principal, this obviates us from thinking about vertices and thus from considering R4/R5 moves.

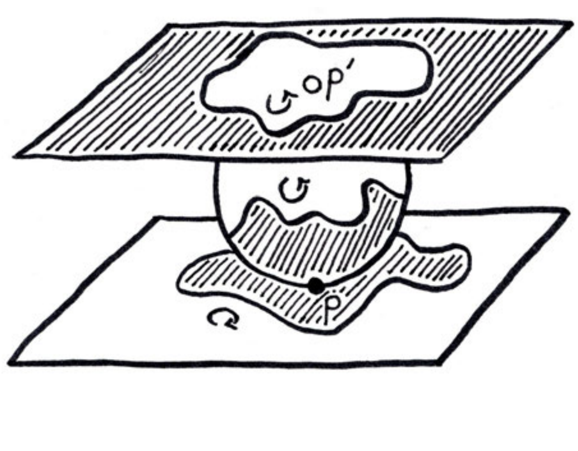
However, we need one more tiny adjustment to get our formal result. Instead of thinking about the canonical neighborhood of our tree as a disk, we want to think of it as the complement of a disk.


**Lemma 3.2.4.** *Let $D$ be a diagram of an embedding of the graph $G$ with no crossings in a neighborhood of $T_0$, a spanning tree of $G$. Then there exists a diagram $D'$ of the same embedding with*

- *the same radial orderings as $D$*
- *no crossings in a neighborhood of $T_0$*
- *the complement of this neighborhood of $T_0$ being a compact disk*

*Proof.* We use two spherical projections. Pick any point $p$ in the original neighborhood of $T_0$ in $D$ not on an edge or vertex. Construct a sphere ($S^2$) tangent to the diagram plane at this point $p$. Let $p'$ be $p$'s antipode on the sphere. Next, take the stereographic projection of the diagram $D$ to the sphere using $p'$ as the center of projection. Note that by perturbing the diagram lying on the surface of the sphere as indicated at crossings, we recover the embedding $E$ in 3-space.



Now, we construct a plane tangent to $p'$ in which to draw the diagram $D'$. We project the spherical diagram onto this plane $D'$ using stereograph projection through $p$. Because a neighborhood of $p$ lies entirely in the no-crossing neighborhood of $T_0$, the complement of the no-crossing neighborhood in $D'$ must be a compact disk. Furthermore if we look at $D'$ from the side of the sphere (rotate the above picture upside down) then the orientation is preserved from $D$ to $D'$. □

**Corollary 3.2.5.** *There exists a diagram $D_0$ of any connected graph $G$ with a neighborhood of $T_0$ (i) containing no crossings and (ii) whose complement is a compact disk.*

*Furthermore, for all embeddings of $G$, there exists a diagram $D$ with a similar neighborhood of $T_0$ and canonical radial ordering relative to $D_0$.*

Given a choice of $D_0$ and $T_0$, call diagrams of the above form **canonical diagrams**. The corollary says there is a canonical diagram of every embedding.

**Theorem 3.2.6.** *Any two canonical diagrams (relative to $D_0, T_0$) of a graph $G$ are related by a series of planar isotopies, R1, R2, R3 and crossing (CX) moves.*

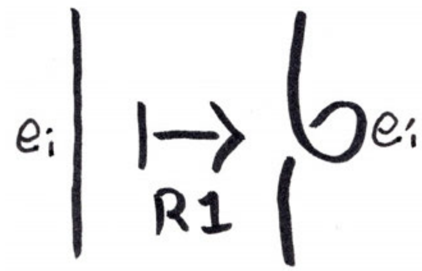
*Proof.* Any two canonical diagrams are identical (up to planar isotopy) everywhere but in a compact disk, forming a "tangle." Using Reidemeister's theorem and crossing moves we can get to any tangle with the same endpoints. □

**Corollary 3.2.7.** *Let $\mathcal{D}$ denote the set of all diagrams reachable from an initial canonical diagram $D_0$ by planar isotopies, R1, R2, R3 and CX moves. $\mathcal{D}$ contains at least one diagram of every embedding of $G$.*

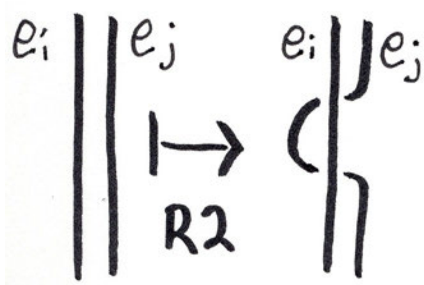
This last corollary was our original goal and so we're through topologizing now. The next theorem folds this topological result into the algebraic picture we've been assembling.

**Theorem 3.2.8.** *Let $D$ be a diagram of a graph $G$ and $D'$ a second diagram reachable from $D$ by planar isotopies, R1, R2, R3, and CX moves. Then $\omega_{D'} - \omega_D$ is a symmetric bilinear form.*

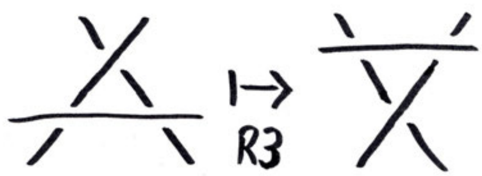
*Proof.* Planar isotopy does not change $\omega$. How about the other moves?



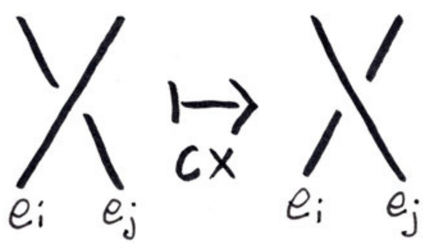

An R1 move just changes the value $\omega(e_i, e_i)$ and so corresponds to offset by the form $(e_i \otimes e_i)^*$ which is symmetric.




An R2 move increments the value $\omega(e_i, e_j)$ by 2, which is just 0 mod 2.




An R3 move just rearranges crossings, but doesn't change the number of them.




CX moves are more interesting. While $\omega(e_i, e_j)$ decreases by 1, $\omega(e_j, e_i)$ increases by 1. Taken mod 2, the signs drop out, making the total change offset by the form $(e_i \otimes e_j + e_j \otimes e_i)^*$.

□

## 3.3 Reduction to Linear Algebra

We began this chapter by reducing the statements "the graph $G$ is IL" and "$G$ is IK" to the respective formal, logically quantified statements

$$\forall lk_E : \exists (A, B)\text{VDSCs} : lk_E(A, B) = 1 \mod 2$$

and

$$\begin{gathered}\forall lk_E : \exists \text{ Foisy pair of pairs } ((A, B), (C, D)) : \\ lk_E(A, B) \cdot lk_E(C, D) = 1 \mod 2\end{gathered}$$

In this section we will show how to reduce these statements even further, into linear algebra (mod 2). In the last two sections, we showed that

- we can substitute the bilinear form $\omega_D$ for $lk_E$, and
- we can quantify $\forall D \in \mathcal{D}$ instead of $\forall$ embeddings

Putting these two results together and turning the key, we get

$$\forall D \in \mathcal{D} : \exists (A, B)\text{VDSCs} : \omega_D(A, B) = 1 \mod 2$$

and

$$\begin{gathered}\forall D \in \mathcal{D} : \exists \text{ Foisy pair of pairs } ((A, B), (C, D)) : \\ \omega_D(A, B) \cdot \omega_D(C, D) = 1 \mod 2\end{gathered}$$

which are logically equivalent to our opening lemmas by the results (need specific references?) of the last two sections. This is great, but we can get more mileage out of this substitution.

First, we should think of the quantification over diagrams $D$ as a quantification over bilinear forms $\omega_D$. Second, we should think of the quantification over pairs of cycles $(A, B)$ and pairs of pairs of cycles $((A, B), (C, D))$ as a quantification over (2- and 4-) vectors $A \otimes B$ and $A \otimes B \otimes C \otimes D$. Finally, since we're quantifying over sets of vectors we should move to quantifying over vector *spaces*. Now, in slow motion (at a theater near you!).

**Lemma 3.3.1.** *Let $\varphi$ be a linear form on $\mathbb{Z}_2 S$ and $V \subseteq \mathbb{Z}_2 S$ an arbitrary subset of $\mathbb{Z}_2 S$. Then*

$$\exists x \in V : \varphi(x) = 1 \iff \exists x \in \operatorname{span}(V) : \varphi(x) = 1$$

*Proof.* Certainly $V$ spans $\operatorname{span}(V)$. Therefore, there must be some basis $B \subseteq V$ of $\operatorname{span}(V)$, a maximal linearly independent subset of $V$. Now suppose $\exists x \in \operatorname{span}(V) : \varphi(x) = 1$. Writing $x$ on the basis $B$, we get

$$1 = \varphi(x) = \varphi\left(\sum_{i=1}^{m} x_i b_i\right) = \sum_{i=1}^{m} x_i \varphi(b_i)$$

So some $\varphi(b_i) = 1$ or we have a contradiction. The opposite direction is trivial. □

Looking at our existential quantifiers, we describe two sets, now written as subsets of tensor product spaces.

$$\text{VDSC} = \left\{ a \otimes b \,\middle|\, \begin{array}{r} a \text{ and } b \text{ are vertex disjoint simple cycles in } G, \\ \text{written as vectors in } \mathbb{Z}_2 E \end{array} \right\}$$

$$\begin{aligned} \text{Foisy} &= \text{Foisy minors} = \text{Foisy pair of pairs} \\ &= \left\{ x \otimes y \otimes z \otimes w \,\middle|\, \begin{array}{r} (x \otimes y), (z \otimes w) \in \text{VDSC and} \\ x \cup y \cup z \cup w \text{ is contractible} \\ \text{to a Foisy graph} \end{array} \right\} \end{aligned}$$

Applying the lemma we march our logical behemoths forward

$$\forall D \in \mathcal{D} : \exists x \in \operatorname{span}(\text{VDSC}) : \omega_D(x) = 1$$
$$\forall D \in \mathcal{D} : \exists x \in \operatorname{span}(\text{Foisy}) : (\omega_D \otimes \omega_D)(x) = 1$$

where $(\omega \otimes \omega)(x \otimes y) = \omega(x) \cdot \omega(y)$[1]

Now, we'll attack the universal quantifier. By the final result of the last section, we know that the set of bilinear forms arising from diagrams in $\mathcal{D}$ is

$$\omega_0 + S = \{\omega_0 + s | s \text{ is a symmetric bilinear form on } \mathbb{Z}_2 E\}$$

[1]This statement may be taken as a form of shorthand so as to avoid a digression on tensor products of forms.

an affine subspace of $(\mathbb{Z}_2 E \otimes \mathbb{Z}_2 E)^*$ where $\omega_0 = \omega_{D_0}$ and $S$ denotes the set of symmetric bilinear forms. Splicing in this perspective we arrive at our final declaration.

**Theorem 3.3.2.** *Let G be a (connected) graph. If*

$$\forall \omega \in \omega_0 + S : \exists x \in \operatorname{span}(\mathrm{VDSC}) : \omega(x) = 1$$

*then G is IL. If*

$$\forall \omega \in \omega_0 + S : \exists x \in \operatorname{span}(\mathrm{Foisy}) : (\omega \otimes \omega)(x) = 1$$

*then G is IK.*

## 3.4 A Program for Intrinsically Linked Graphs

Determining the truth of the statements at the end of the previous section amounts to a linear algebra problem over $\mathbb{Z}_2$. As a prototype for the intrinsic knotting problem, we may construct a computer program to decide the intrinsic linking predicate

$$\forall \omega \in \omega_0 + S : \exists x \in \operatorname{span}(\mathrm{VDSC}) : \omega(x) = 1$$

for an arbitrary graph $G$. This section describes one way to go about doing that.

Or it doesn't at the moment.

# Appendix A

# Open Questions

This thesis has left me with a few questions. I thought I'd indulge myself a bit and share them here at the end.

Robertson and Seymour's argument that the Petersen family is the intrinsic linking obstruction set generated an ancillary result. If a graph is IL, then not only does it contain some pair of linked cycles, but also some pair of cycles linked mod 2. This means that our intrinsic linking program computes a complete invariant. That is, it will always decide whether a graph is IL or NIL correctly. Does a similar result hold for intrinsic knotting?

**Question A.0.1.** *If a graph $G$ is IK, then is it true that $G$ always contains a doubly linked Foisy graph?*

The idea of looking at the edge space $\mathbb{Z}_2 E$ comes from matroid theory. In this thesis, we showed how to abstract most of the topological particulars of embedded graphs to intrinsic graph structures, mainly using this concept of the edge space. Can the idea be further abstracted to a general matroid setting?

**Question A.0.2.** *Is there a nice definition of an intrinsically linked matroid and/or an intrinsically knotted matroid such that the matroid derived from a graph $G$ is IK or IL if and only if the graph is.*

The following questions are motivated by a desire to obviate the computer program part of IK/IL proofs proposed. Graph theoretically, they could be very interesting because they provide a link (no pun intended) between the extremal graph minor problems of finding obstruction sets for IL/IK and the matroid theoretic edge space. Many proofs related to graph minors rely on case analysis or computer programs (e.g. Kuratowski's theorem and the four color theorem). Any strategy for replacing case analysis/computers in IK and IL proofs with more linear algebraic arguments might yield interesting insights into graph structure and/or extremal problems.

**Question A.0.3.** *Given a graph $G$ is there a nice basis for the space spanned by all vertex disjoint simple cycle pairs in $G$, $\text{span}(VDSC)$? Perhaps such a basis could be found by decomposing $\mathbb{Z}_2E \otimes \mathbb{Z}_2E = V \oplus W$ similar to the decomposition of $\mathbb{Z}_2E$ into a spanning tree and the cycle space.*

**Question A.0.4.** *Similarly, is there a nice basis for the space spanned by all Foisy minors?*

Finally, it's somewhat unsatisfying to use Foisy's lemma to bootstrap intrinsic knotting proofs. It's strange to hunt for doubly linked Foisy minors when you're trying to prove intrinsic knotting of a graph. Perhaps there's a more direct way to look for knotted cycles.

**Question A.0.5.** *We showed that the linking number of vertex disjoint simple cycle pairs can be extended to a bilinear function on the edge space of a graph $G$. Can the Arf invariant be extended to a well behaved algebraic function on the entire cycle space of a graph $G$? Is it a quadratic form? Perhaps it's even cubic or quartic.*