Publications Committee

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The Texas Mathematics Teachers' Bulletin

(Vol. 2, No. 2, December 15, 1916)

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The Texas Mathematics Teachers' Bulletin

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This Bulletin is open to the teachers of mathematics in Texas for the expression of their views. The editors assume no responsibility for statements of facts or opinions in articles not written by them.

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The benefits of education and of useful knowledge, generally diffused through a community, are essential to the preservation of a free government.

Sam Houston.

Cultivated mind is the guardian genius of democracy. . . . It is the only dictator that freemen acknowledge and the only security that freemen desire.

Mirabeau B. Lamar.

AN INVITATION

It is the desire of the editors of this Bulletin that it shall render to the teachers of mathematics in Texas the greatest possible amount of service. To this end they invite their readers to suggesct any topics in connection with secondary mathematics that they desire to have discussed, to mention any particular difficulties they wish to have removed. The editors do not agree to discuss any topic proposed or to remove any difficulty mentioned—they scarcely wish to place so large an order,—but they do agree to do the best they can to secure the desired discussion or to suggest a remedy for the difficulty.

Any reader having any good ideas of his own or special methods that he has found helpful is invited to send a paper dealing with them, and if its character seems to warrant the publication of the whole, any part, or a synopsis, the editors will be glad to give it space.

THE ORIGINAL EXERCISE IN GEOMETRY

J. G. Dunlap, Principal Cleburne High School

The "original exercise" in this paper means that group of geometric principles whose truth must be established and problems to be solved as distinguished from the theorems demonstrated in the text. The use of the original exercise in impressing and emphasizing geometric truths is of comparatively recent date. The experienced teacher will find it an extremely fertile field for developing accurate thinking. The earliest manuscripts in geometry were, of course, very primary, and the necessity of some means of fastening in mind the fundamental principles was not so great. As the science was gradually developed from the old Euclidean scroll, the field being extended, the application of the principles then known called for and brought into use the exercise.

The demonstration of a theorem must in form be essentially deductive or inductive—synthetic or analytic. Each has its peculiar use, and, to some extent at least, involves the other; analysis to discover and synthesis to demonstrate the truth or falsity of the exercise. In the solution of practical exercises and problems the ability to investigate and reason for one's self is the necessary prerequisite to success. This ability is not inherent in the pupil, but must be acquired by long and earnest hours of application to study. Happy that teacher who can inspire enthusiasm in his pupils and make them fond of the task of solving the tedious original. And at this point I believe the pupil gets his most lasting benefit—a doggedness of purpose, a determination to win. It has been my observation that most of the failures in geometry are due to a lack of tenacity.

Nothing in the field of secondary mathematics is quite as good in developing tenacity of purpose as the mastery of the original —nothing quite so good in developing the ability to concentrate the mental faculties. I am convinced that, if there is a superiority of the German over the American child in mathematical development, it is due largely to the preponderance of the exercise in the German text. Too many pupils memorize the proof, if given in full in the text, thus relying entirely on the memory and neglecting the reason. Nothing could be more harmful than this process, and yet I realize that it is one of the very things we have to fight, and one that is especially prevalent among beginners. Nothing helps like the knowledge that one must depend entirely upon himself. The original furnishes this field of activity as, in my opinion, no other does. Smith says: "The great value of teaching originals is in developing the power to think along correct lines of logical thought; if properly handled they make the pupil think more intensely and interestedly than any other subject fitted to pupils of the same age."

The subject matter of the original appears in so many forms that its mastery involves a many-sided view of the subject. The pupil must be resourceful, and if blocked in one avenue of attack, try another. This brings confidence in himself and the ability to do something for himself, rather than a dependence on the text. When once the pupil feels the joy of having accomplished something for himself, he has, indeed, a stimulus of no mean account. Too many pupils feel overwhelmed with the apparently impossible task and surrender. Sisson says: "Without enthusiasm no mathematics. Geometry is a human book—not divine—therefore a very imperfect book. Geometry is the product of the human mind and not of the hand; therefore the subject concerns the intellect and is not mechanical." With this viewpoint, which is undoubtedly correct, the aims, which are several, may be reduced to one all important one—the training of the mind to habits of correct thinking and to reasoning logically and accurately to a correct conclusion. Nothing in all the high school course is so well adapted to developing the power to express one's thoughts concisely and elegantly as the original of geometry. I daresay the student of English has found few agencies in the correct and direct expression of his thoughts more helpful. Three-fourths of its value is disciplinary. The mind should direct the hand. Mathematical reasoning is that process which step by step arrives at a definite conclusion. The original, as its name indicates, has within itself the suggestions of the line of reasoning that lead to a solution. At this point are brought to play those powers of observation which must suggest to the beginner the tools needed and the method of attack. To the beginner this is the most difficult point—"getting started"; "finding out what you want to do." These and kindred statements express the pupil's conception of the task before him. In this step-by-step process each is made up of two distinct parts—the statement of a fact that leads the mind in the proper direction toward a conclusion, and the authority for such step. When the beginner has recognized this truth, he has made at least one step in solving the exercise. One pupil who can and does master with fair accuracy the originals as the class proceeds soon becomes the leader. And in this the value of the original is recognized by the class. If this leadership is properly directed by a skillful teacher, growth and development of a friendly rivalry will assert itself and the effect on class work will be beneficial.

The grouping of originals immediately after the basic theorem is a step in the right directon and is productive of splendid results. It enables the student to center-fire, as it were, and holds him on the principle until it is mastered. Many pupils fail utterly because they are unable to translate English into geometric terms, thereby losing the meaning of the exercise. Again, many lose sight of the all-important motive for studying geometry—the training of the reasoning faculties. They take for granted certain relations because it looks that way and are unable to pick up the train of thought when the statement is challenged. This weakness can be corrected by persistent efforts on the part of the teacher in having the pupil attack the problem from another point. Our work in geometry is impaired and our success often disappointing because we omit the originals. The original furnishes the material for practice in the clinic and many originals with a few well-digested theorems will bring better results of a permanent nature than many theorems and few applications of them.

While it is true, as Loomis says, that problems do not constitute a necessary part of the science of geometry, forming no part of the chain of connected truths embodied therein, yet because of their importance as applications of geometric principles, they are of the utmost educational value and should be studied in connection with the theorems upon which they depend. After all, may it not be truthfully said that a pupil's knowledge of geometry is measured by his ability to solve the original?

It is doubtful whether it is best to demand that the proof shall be submitted in a set form without variation. One of the chief ends to be attained is clearness of expression. However, it cannot be attained at once. Would it not be better to be master of facts than a slave to the form? As the pupil advances, the polished form of proof should appear. If the demand for correctness of form is urged too far, the pupil loses the unfolding of the exercise in his effort to be correct in form. Young says: "Not all elegance and verbal accuracy that are to be attained later need be inflexibly required at first." Much of our trouble in teaching this part of geometry is caused by rushing the pupil over a mass of truths without time for digestion on his part. "Better plod at first and rush later," should be our motto. Many times we have heard the expression, "I just know it is so," and kindred expressions, given as reasons for certain steps taken in demonstration. To accept such statements is not wholly bad, for it shows on its face that the pupil has within him the capacity for reasoning. The reason will come later, if he is properly directed. Much of our labor has been lost because we have not directed the mind of the pupil in the proper way. Nothing, in my opinion, in the high school course calls for more patience on the part of the teacher than the direction of a slow student in attacking the original exercise. Again, no fixed rule of attack can be laid down, though some suggestions may be helpful. The first thing, of course, is to get a clear conception of the exercise. Skillful questioning on the part of the teacher will give the student a start, as he calls it, and will help him discover the relation between the hypothesis and some theorem already learned. At this point the imagination is brought into use. Unless the pupil catches the spirit of geometric analysis, he will not succeed in finding the proof. Analysis is the soul of interpretation and interpretation is the key to the solution of the original exercise.

Slaught says: "Many a high-school pupil who can play the game of hypothesis and conclusion—and likes it—has never recognized geometry as a basic fact in the tiles on the floor, in the decorations on the walls, in the arches and windows of great buildings, in fact in all the mechanical and architectural developments of this and of every age. When once he comprehends this, when he takes his 'originals' from actual conditions about him, then the 'game' assumes a new significance to him and geometry becomes a fact instead of a theory, a part of life instead of a mere school creation." Whether or not the student is able to reach the end just mentioned depends largely on the teacher—whether or not we recognize several distinct periods in the mental development of the child and the adaptation of subject matter and methods of teaching. This means in geometry a readjustment of subject matter, a more accurate classification of basic truths and a multiplication of originals for the application of these truths. In fact, the humanizing of the subject will do much to improve the success in teaching it—make the originals as practical as possible, thus appealing to his observation and experience for material.

Plato's school of geometry was for mature men. Plato and Euclid, if living, would be astonished at our methods of teaching geometry and at the personnel of the average class. Yet with all the advancement of the subject, it can be made more practical without losing prestige as a subject which develops and trains the mind in accurate thinking and logical reasoning. At this point in the pupil's development the teacher must do more than ask questions. He must bear in mind that the pupil in the beginning of a subject does not receive and understand as quickly as the teacher. He must remember that the pupil does not generalize, but must learn to do so; that the subject is not developed, but is in the process of being developed. We as teachers are much given to "overshooting," as it were, the pupil—forgetting that he is not a man and therefore does not reason as one. After all, "the pupil must learn to do by doing," and will succeed if properly directed.

(Read before the Mathematics Section of the Texas State Teachers' Association, December 1, 1916.)

THE MATHEMATICS OF INVESTMENT FROM AN ARITHMETIC VIEWPOINT

EDWARD L. DODD

INTRODUCTION

To solve some of the problems in investment, algebra and calculus of an advanced type are necessary. But many problems are susceptible of an arithmetic treatment.

Algebra is commonly viewed as a higher branch of mathematics than arithmetic. And in a certain sense it is. But a clear arithmtic grasp of a fundamental principle may represent higher intellectual activity than the corresponding algebraic reasoning which leads to a formula. And, in addition, it will probably be immensely more useful.

Much the same relation exists between plane geometry and analytic geometry. The latter is rated as the higher branch of mathematics. But when the same subject is treated both in plane geometry and analytic geometry, the plane geometry usually calls forth the higher intellectual activity.

In this presentation of some principles in the Mathematics of Investment from the arithmetic standpoint, letters will be used from time to time to stand for numbers. This can hardly be called algebra, although a beginner in algebra may think of algebra as the mathematics of letters and of arithmetic as the mathematics of numbers—a very infelicitous conception. The use of letters for numbers is mere short-hand. Algebra really begins when we substitute for real thinking some rule such as transposing with change of sign.

In many texts on Investment and Life Insurance there are "verbal explanations." These are usually attempts to give arithmetic color to what has been proven algebraically.

This paper is an attempt to make clear by an informal and largely arithmetic treatment a few fundamental principles in the Mathematics of Investment.

THE MATHEMATICS OF INVESTMENT

The interest being 4%, a deposit of $100 is treated as follows:

Original deposit, principal, or capital................	$100.00
For 4%, the multiplier is............................	.04
Interest for first year..............................	4.00
This is added to the principal.......................	100.00
Amount at end of first year..........................	104.00
Multiply again by....................................	.04
Interest for second year.............................	4.16
This is added to the amount..........................	104.00
Amount at end of second year.........................	108.16

The computation might have been arranged thus:

Principal	$100.00
As multiplier, use 1 + interest rate..................	1.04
Amount at end of one year..........................	104.00
As multiplier use again............................	1.04
	416
	104
	108.16

It is easy to see that the two methods above must give the same result, for multiplying any number by .04 and then adding the number is obviously equivalent to multiplying the number by 1.04.

It is convenient to write, as abbreviations,

$$1.04\times1.04=(1.04)^2$$
$$1.04\times1.04\times1.04=(1.04)^3$$

etc. Furthermore the multiplication sign is often omitted. $100(1.04) means $100×1.04.

From the above, it is then evident that
The amount of \$100 at end of first year is.........\$100(1.04)
The amount of \$100 at end of second year is....... $100(1.04)^2$
The amount of \$100 at end of third year is........ $100(1.04)^3$
The amount of \$100 at end of fourth year is....... $100(1.04)^4$
etc., the multiplier 1.04 being used once for each year that the money is left on deposit.

If by n is meant any number of years,
The amount of \$100 at 4% at end of n years is $\$100(1.04)^n$
Likewise;
The amount of \$100 at 5% at end of n years is $\$100(1.05)^n$

The standard abbreviation for the interest rate, used by texts on the Mathematics of Investment, is i. It should be noticed that the symbol 4%, read 4 per cent or 4 per centum, means 4 per hundred or 4/100 or .04. Thus if the interest rate is 4%, $i=.04$. The multiplier 1.04 used above is $1+i$. Or, if the rate is 5%, i now would mean .05, and $1+i$ would mean 1.05. Whatever be the interest rate i, the multiplier $(1+i)$ is used once for each year that the money remains on deposit. To get the amount after n years, the multiplier $(1+i)$ is used n times, and this is abbreviated $(1+i)^n$. Instead of \$100, we may take any principal and call it P dollars. Hence, if S denotes the amount of P dollars left for n years at the interest rate, i,

$$S=P(1+i)^n \tag{1}$$

This is the most important formula in the Theory of Compound Interest.

Many banks pay 2% every 6 months. Thus, \$100 would become \$102 at the end of 6 months. The \$102 would now be regarded as principal, and 2% of \$102 is \$2.04; and thus at the end of a year the amount would be \$104.04. *It should be noted carefully that 2% for half a year is not the same as 4% for a year.* The 2% for 6 months gives an annual increase of \$4.04 on \$100. The 2% for 6 months is then equivalent to 4.04% for a year.

Instead of saying 2% for 6 months, it is more customary to say 4% payable semi-annually or 4% convertible semi-annually. Text-books call this a *nominal 4% payable semi-annually.* By saying a *nominal 4%*, we emphasize the fact that this nominal 4% is really not the same as 4%. It has just been shown that this *nominal 4%* is *in effect a 4.04%*. This 4.04% is called the *effective rate* corresponding to the *nominal rate* of 4% payable semi-annually.

It will be noticed that the effective rate just mentioned is, indeed, descriptive of the earning power of money for one year. It is an annual rate, and thus we use i for the effective rate, in conformity with the definition for i given above.

It was shown that the amount of \$100 at the end of one year at a nominal 4% payable semi-annually is \$104.04. This can be written in the form $\$100 \times 1.02 \times 1.02$ or more briefly $\$100 \times (1.02)^2$. For each period of compounding we multiply by 1.02. The amount of \$100

At the end of ½ year is............\$100(1.02)
At the end of 1 year is.............. $100(1.02)^2$
At the end of 1½ years is............ $100(1.02)^3$
At the end of 2 years is.............. $100(1.02)^4$

And so forth.

The letter j is used for the *nominal* rate. Here $j=.04$. The letter m is used for the number of times a year that the interest is compounded. Here $m=2$. The interest rate actually used—that is the rate for 6 months—is .02 or j/m. The multiplier above is 1.02 or $1+j/m$, and this is used once for each period of 6 months.

During *1* year, interest is compounded m times, and thus $1+j/m$ is used as a multiplier m times. In n years, interest is compounded $m \times n$ times, or as it is usually written mn times; and thus $1+j/m$ is used as a multiplier mn times. Hence if S is the amount of P dollars at the end of n years, at the nominal rate j payable m times a year.

$$S=P(1+j/m)^{mn} \tag{2}$$

Now, by definition, the amount of *1* dollar at the end of one year at the effective rate i is $1+i$. Hence from (2), by making the principal, $P=1$, and also the number of years, $n=1$, we get

$$1+i=(1+j/m)^m \tag{3}$$

This formula (3) has thus been derived as a special case of

(2). But a little reflection will show that it follows directly from the definitions of i and j. The principal 1 becomes at the end of the first period of compounding $1+j/m$. The amount of the end of the first period is thus found by using $1+j/m$ as a multiplier once. There being m periods of compounding in the year, the multiplier $1+j/m$ is used m times, giving $(1+j/m)^m$ as the amount of 1 dollar at the end of 1 year; and by definition of i, this amount must also be equal to $1+i$. Having thus obtained (3) directly, we may obtain (2) by replacing $1+i$ in (1) by its value given in (3).

Equations (1), (2), and (3) all involve the same principle (not principal); and this may be expressed as a rule.

RULE FOR COMPUTING THE AMOUNT UNDER COMPOUND INTEREST.

To 1 add the interest rate for the period of compounding. With the principal as multiplicand, use the number just found once as a multiplier for each period that interest is compounded.

In the Mathematics of Investment, reciprocals are used extensively. Thus, if 1 dollar will yield an annual interest of .04, it will take 1/.04 or 25 dollars to yield an annual interest of 1 dollar. Problems like the following are given: If a farm yields a net income of $200 per year after all expenses are paid, what is the value of the farm if money is worth 5%? In an actual case, the possibilities of increase or decrease of the income must be considered together with other factors. But assuming the continuance of the specified income, we may say that inasmuch as it will take 1/.05 or 20 dollars to give an income of one dollar at 5%, it will take 200×20 dollars or $4,000 to yield $200 annually. And thus the value of the farm would be placed at $4,000. In practice we would naturally divide the $200 by .05 and get $4,000 directly.

This is in accord with the following well known rule.

RULE TO FIND THE CAPITAL REQUIRED TO PRODUCE A SPECIFIED ANNUAL INCOME.

Divide the specified annual income in dollars by the interest rate, or what is the same thing, by the fraction representing the interest on one dollar for one year.

This is so simple that it would hardly be worth mentioning here, were it not for the fact that it is a natural introduction to a more complex problem.

If one dollar is deposited at 4%, and the interest is drawn out at the end of each year, this interest will be 4 cents. Now in practice it does not often happen that a man deposits just one dollar and continues to draw out year after year his four cents. But one of the beauties of the Mathematics of Investment is that we may compute everything on the basis of one dollar and then find the result on the basis of P dollars by simply multiplying by P. In getting at the fundamentals of the subject, a constant reference to P dollars or 100 dollars or 1,000 dollars becomes a nuisance; and we simply deal with one dollar.

But suppose now that one dollar is left on deposit for 20 years, no interest being withdrawn in the meantime. The amount of this one dollar at 4% will then be $(1.04)^{20}$,—as may be seen from the Rule given or from Formula (1). The principal being 1, the *increase* will be $(1.04)^{20}—1$; and this may properly be called the *interest* on one dollar for twenty years at 4% compounded annually. If, indeed, from the amount $(1.04)^{20}$, the interest, $[(1.04)^{20}—1]$ is withdrawn, the principal of 1 dollar is left. The same process being repeated, this 1 dollar will produce interest to the extent of $(1.04)^{20}—1$ payable 20 years later, and 1 dollar will be left at that time to serve the purpose of principal. So just as 1 dollar will produce an *annual* income of .04 forever,—the 1 dollar as principal never being altered,—so also will 1 dollar produce an income of $(1.04)^{20}—1$ payable at the end of each *period of 20 years* forever, and at each time that interest is paid the principal will remain exactly 1 dollar.

If, then, 1 dollar will produce an income of $(1.04)^{20}—1$ dollars payable at the end of each period of 20 years, it will take

$$\frac{1}{(1.04)^{20}-1}$$

dollars to produce an income of 1 dollar payable at the end of every period of 20 years. Likewise it will take

$$\frac{10{,}000}{(1.04)^{20}-1}$$

dollars to produce an income of 10,000 dollars payable at the end of each period of 20 years.

PROBLEM.

We have thus solved the following problem:

A building must be reconstructed every twenty years at a cost of $10,000. What sum of money laid aside now at 4% will provide for the required reconstruction of the building for all time?

It has been shown that $(1+i)^n$ is the amount of one dollar at the rate i after n years, this n being a whole number of years. If n is not a whole number the amount can not be *deduced* from any considerations thus far advanced. Some *definition* is necessary. It is perhaps obvious that the *most convenient* definition for the amount would retain the same expression $(1+i)^n$, and allow n to be a fraction, proper or improper,—or indeed any real number.

DEFINITIONS.

For all real values of n, integral, fractional or otherwise, the amount of one dollar at the rate i after the lapse of n years is $(1+i)^n$.

The compound interest on one dollar for n years at the rate i is

$$(1+i)^n-1.$$

This definition will now be utilized to express as a rule the gist of the illustration just given.

RULE *to find the capital that must be invested to yield a specified income payable at the end of each period of a specified number of years, forever.*

Divide the specified income in dollars by the number representing the compound interest on one dollar for the specified number of years.

Example. Find the capital that must be invested at 4% compound interest to yield an income of $100 payable at the end of the 2nd, 4th, 6th, 8th year, etc., forever.

At 4% compound interest, the amount of one dollar at the end of one year is 1.04, and the amount at the end of two years is

$$1.04 \times 1.04 = 1.0816.$$

The *compound* interest on one dollar for 2 years is

$$.0816$$

dollars or 8.16 cents.

In passing, it may be noticed that the *simple* interest is only 8 cents on the dollar; whereas the *compound* interest, as just shown, is 8.16 cents on the dollar.

Now divide $100 by .0816. This gives $1225.50 as the capital that must be invested at 4% *compound* interest to yield an income of $100 payable every two years forever.

If *simple* interest at 4% were used, this would be .08 dollars on one dollar in two years. We would have divided $100 by .08 and obtained $1250 as the capital that should be invested. At *compound* interest, however, as has just been shown, the capital needed is only $1225.50.

This result may be checked as follows:

Capital or principal	$1225.50
Add interest at 4%	49.02
Amount at end of first year	1274.52
Again add interest at 4%	50.98
Amount at end of second year	1325.50
Deduct the specified income	100.00
Capital as at beginning	1225.50

It is thus evident that a principal of $1225.50 will at 4% compound interest yield an income of $100 payable at the end of each period of two years, the prncipal remaining "unimpaired" or undiminished. The *algebraic* verification or check of the general Rule is also interesting, but will be left to the reader.

Definition

The capitalized cost of a structure or an article is the first *cost plus the capital which would provide for an indefinite number of renewals.*

It has just been shown how this capital can be computed. By adding to this the original cost, the capitalized cost can be found.

In an enterprise in which machinery or perishable equipment is involved, the percentage of profit must be based upon the *capitalized cost* and not upon the *original cost.*

If a man buys an automobile for \$800, runs it as a jitney for four years clearing \$100 each year over running expenses, and if the automobile then breaks down, being worth as junk only \$50, the man has *not* made 100/800 or $12\frac{1}{2}\%$ on his investment each year; for his original capital of \$800 has been almost completely absorbed or destroyed.

It did not take long to compute 1.04×1.04 or $(1.04)^2$ as used in a foregoing illustration. But to compute by actual multiplication $(1.04)^{50}$ would take considerable time. For 1.04 would be the multiplicand and there would be forty-nine multiples, each being 1.04. $1.04\times1.04\times1.04\times1.04\times\ldots$ etc., involving forty-nine multiplications. Certain monetary tables have been constructed giving the amount of one dollar for any number of years up to 100 years at the usual rates of interest. For unusual rates of interest interpolation can often be used. But unless the consecutive rates given differ from each other by only a small fraction of a per cent, the interpolation must use second or third differences; the interpolation method used in logarithms will not be adequate. Logarithms may be used directly. The logarithms should be given to at least six decimals. Seven or eight place logarithms are frequently necessary.

The computation side of investment problems is very important. But it is not the purpose of this paper to dwell upon this matter.

In this paper just a few rules have been given, applying to simple cases. The reader is invited to ask himself these questions: *How would the last "Rule" given be altered if a nominal rate of interest was given, payable semi-annually or quarterly? How would the Rule be altered if the income was payable monthly?* Before attempting to answer these questions the reader should note (1) that 2% for a half-year is *not* equivalent to 4% for a year; (2) That an income of \$100 at the end of each month is *not* equivalent to an income of \$1200 at the end of the year. In practice income very frequently takes the form of a rent payable monthly, and nominal rates are commonly used.

SUMMARY

The amount S of P dollars at the rate i for n years is

$$S=P(1+i)^n$$

If n is a whole number, this can be *proven*. If n is any other real number, it is true by *definition*.

The relation between the *nominal rate* j payable m times a year and the *effective rate* i is

$$1+i=(1+j/m)^m \qquad (3)$$

By means of (1) and (3), S can be expressed in terms of P, j, n and m.

The compound *interest* on one dollar for a specified period of time is *defined* to be the amount of one dollar for that time diminished by one dollar.

To find the *capital* needed to produce a *specified income* payable at the end of each period of a specified number of years, forever, divide the specified income in dollars by the number representing the compound interest on one dollar for the specified number of years.

This can be used to obtain the *capitalized cost* of a structure or article, viz. the original cost plus the present value of an indefinite number of renewals.

In the case of perishable equipment the *rate of profit* must be computed on the basis of the capitalized cost, not the original cost.

(TO BE CONTINUED.)

LITERAL ARITHMETIC

In this and the following bulletins we will briefly indicate some simple applications that may be made by the use of the literal notation in grammar grade work before the high school. To the teacher in the high school these applications will seem very elementary and in many cases trivial, but if he would help the grammar grade teacher introduce these applications in the study of arithmetic he would find his classes in the beginning of algebra having fewer difficulties in the first few weeks.

In this bulletin we will briefly take up the use of letters in the study of problems in *interest.*

In all interest problems there are four quantities to be considered:

(1) The principle p.

(2) The rate per cent per annum r.

(3) The time t given in years.

(4) The interest i.

The product of $p \times r$ is the interest for one year, and the product $p \times r \times t$ is the interest for t years. Hence we have the formula

(1) $p\ r\ t = i$.

Dividing both sides of (1) respectively by $p\ r$, $p\ t$, $r\ t$, we have

(2) $t = \dfrac{i}{pr}$

(3) $r = \dfrac{i}{pt}$

(4) $p = \dfrac{i}{rt}$

Formula (1), as we have seen, is an expression of the definition of interest by the use of the four letters *i p r and t.* It is therefore easily remembered and can be written down, at any time, when needed. From (1) the pupil can soon learn to find formulas (2), (3), (4). These four formulas show that when any three of the four numbers i, p, r, t, are given the fourth can be determined. Thus

By (1) the interest i may be found when p, r, t are given.

By (2) the time t may be found when **i**, p, r are given.

By (3) the rate r may be found when i, p, t are given.

By (4) the principal p may be found when i, r, t are given.

It would be well for the teacher to give (1) frequently to the class and have formulas (2) (3) (4) derived from it and state the reason for the process. A few examples will be added.

Example. Given p=\$1200, r=$5\frac{1}{2}$% and t=2 yrs., 5 mo., 10 da. to find i.

By definition we know the time t must be expressed in years. Hence

t=2 yrs., 5 mo., 10 da.=2 4/9 yrs.

By formula (1) we have

$$i = p\ r\ t = \$1200 \times .05\tfrac{1}{2} \times 2\ 4/9$$
$$= \$161.33\tfrac{1}{3}$$

Example. What principal in 2 years time will produce \$30 interest, the rate being 5%?

By formula (4) we have

$$p = \frac{i}{r\,t} = \frac{\$30}{2 \times .05}$$

$$= \$300$$

It is sometimes convenient to use a fifth formula derived as follows:

$$a = p + i$$
$$= p + p\ r\ t$$
$$a = p(1 + r\ t)$$

or

$$(5)\quad p = \frac{a}{1 + vt}$$

The a which is defined to be equal to the sum of the interest and principal is called the *amount*.

Example. What principle will amount to \$356 in two years and eight months, the rate being 7%?

Here we have

$$p=\frac{a}{1+r\,t}$$

$$=\frac{356}{1+2\tfrac{2}{3}\times.07}$$

$$=\$300$$

It is thus seen that by remembering the definition of interest that all rules may be written down at once for any case that may arise. The waste of energy in classifying interest problems into so many cases as found in most arithmetics is saved here and the student has to learn the definition of interest only and not spend his time learning a rule to fit each case. It will be well for the teacher to propose at random problems for the different members of the class to indicate and then find the solution. Do not let the class expect that any two successive problems will fall under the same formula. Thus propose a series something as follows:

(1) Given $p=\$900$, $r=.06$, $t=3\frac{1}{2}$; find i

(2) Given $r=.06$, $p=\$500$, $i=\$15$; find t

(3) Given $a=\$912$, $t=4$, $r=.03\frac{1}{2}$; find p

(4) Given $p=\$1089$, $i=\$200.376$, $t=4$ yrs., 7 mo, 6 da.; find r.

In this way the pupil will learn to know interest in a way that would be impossible in studying separate cases by the old methods.

ON POSTULATIONAL SYSTEMS

1. *Introduction.* In most of the modern fields of investigation in mathematics, the philosopher can hope to find little to arouse his interest or challenge his criticism. Undoubtedly there has never been a time in which the current problems of this science have been so unintelligible to the metaphysician and to the man of affairs. But in the marvelous growth and specialization in mathematics during recent years, the relations of mathematics to logic and to the wider branches of philosophy have not been wholly neglected. The ancient craving for a propositional system that shall be at once simple, consistent and universal, and which, nurtured through generations of the finest of Greek thought, found at last an adequate expression in Euclid's superb masterpiece, this same scientific and esthetic longing has recently, also, been making insistent demands for critical investigation and appreciation of what has been called the Foundations of Mathematics.

The processes of counting the construction of weights and measures, of interest tables and calendars, the staking of fields, and the mapping of the starry heavens, the building of houses and ships, the erection of monuments and fortifications, in fact most of the occupations of the artisan and the trader involve at some stage, specialized notions of number and of space relations. Arithmetic, geometry and even trigonometry, arose as inevitable practical sciences. There was probably a fair amount of collected material forming a rudimentary mathematical theory even as early as 2000 B. C., among the most advanced peoples of that era. Looking back to such a time, one can well believe that it was the substance and not the form of their computational science that interested the learned men of that day. Many times since then mathematical rigor has been forgotten in the excitement of suggestive discovery, and the foundation completely obscured by the splendor of the growing superstructure, and even today this situation obtains for many if not most investigators in mathematics.

The "foundations of mathematics" as a theory instead of being historically essential to the growth of mathematical science, has been an esthetic luxury necessary indeed for an adequate appreciation and rigorous treatment of large fields of mathematics but slighted during many periods of rapid, if insecure, extension. Not so much as a partial understanding of the role and the significance of anxioms is essential to the grasp of geometric facts, nor even to the discovery and formulation of many intricate proofs. Thus one is not surprised to learn that the very comprehension of the problems relating to the axiomatic bases was a rare achievement of Greek thought, and since the Alexandrian era largely lost to the world until recent times.

2. *The Nature of a Mathematical System.* As early acknowledged by the Greeks, a formal mathematical proof must be pure deduction, purged of all accidental and extraneous features such as intuitive or inductive arguments. The very existence of a proof implies not only definitions, but certain propositions regarded as established and certain logical processes admitted as valid. In similar manner no definition can be permitted which does not relate objects regarded as already known. It is clear that if any deductive science is to be developed certain logically basal operations, primitive objects, and elementary propositions must be accepted as initial and not requiring justification by the science about to be dealt with. Thus the mathematician presupposes acceptance of logical laws and of certain terms and propositions. For reasons of convenience principles of logic are only rarely explicitly investigated in connection with a mathematical system but the same remark does not hold for the terms and propositions employed. While for each individual treatment there must exist undefined objects and unproved theorems *it is in no respect essential that a preassigned element or proposition be undefined for all possible discussions.* One geometer may regard "point," "line" and "order" as undefined, and another perhaps, define all of these in terms of "transformation," and "planar element." Even the term "undefined," may be misleading since every axiom or unproved proposition is a partial implicit definition of these objects, which were indeed *initially* undefined.

These commonplaces of a deductive science were understood even in the Middle Ages as is shown by the following quaint quotation from Thomas Aquinas. (Summa Theologia 1, 1. questio 1, art. 1-8.)

"But there are two kinds of sciences. There are those which proceed from the principles known by the natural light of the mind, as arithmetic and geometry. There are others which proceed from principles made known by the light of a superior science; as perspective proceeds from principles made known through geometry, and music from principles made known through arithmetic. . . . One science may be said to be worthier than another by reason of its certitude or the dignity of its matter. . . . It should be said that . . . other sciences do not prove their first principles but argue from them in order to prove other matters. . . . One should bear in mind that in the philosophic sciences the lower science neither proves its own first principles nor disputes with him who denies them, but leaves that to a higher science. But the science which is highest among them, that is, metaphysics, does dispute with him who denies its principles, if the adversary will concede anything; if he concede nothing, it cannot thus argue with him but can only overthrow his arguments."

3. *Euclid's Parallel-Postulate.* The modern revival of interest in the logical substructure of analysis has been an outcome almost exclusively of the discussions and discoveries centering in the so-called "parallel-postulate" of Euclid. Except for the phenomenal excitement awakened by the non-Euclidean geometries born in the last century, it is at least improbable that the modern mathematical philosopher could rival the logical independence which recent scholarship discovers in the remains of the Greeks. Euclid who lived about 300 B. C. was associated with the foundation of what may properly be called the first university, and which was situated in Alexandria.

Today we appreciate Euclid's problem of reducing the science of geometry as then known to a postulational basis, and we understand also his objectors. Not until after the twenty-eighth proposition has been proved does Euclid require the famous proposition which has been rendered into English as follows:

"If two lines are cut by a third and the sum of the interior angles on the same side of the cutting line is less than two right angles, the lines will meet on that side when sufficiently produced." This proposition has given rise to much discussion. Its very order among the axioms has been changed by various commentators. Its length and apparent complexity is in striking distinction to the form of most of Euclid's axioms, nor does its independence appear so obvious. It may be regarded as a converse of a previous proposition and yields the converse of yet another. Then, too, it has appeared as being more probably capable of proof by means of the remaining postulates than any of the others.

4. *Origin of Non-Euclidean Geometry.* Saccheri, an Italian monk developed in 1733 a body of geometric theorems in which the above axiom is denied. It must be admitted that he finally concludes the entire system to be contrary to common sense and therefore worthless, but in view of the apparent attitude of the authorities of the time and the anecdotes concerning Galileo, the sincerity of his ruthless criticism might be suspected. In 1766 Lambert maintained that the parallel-postulate requires proof, and suggested some characteristics of the geometry resulting from its denial. Legendre (1752-1833) tried to prove the above proposition. He continued, of course, to regard a line as infinite in length, and proved independently of the parallel-postulate, that the sum of the angles of a triangle is at most equal to two right angles, and that if a single triangle exists in which the sum of the angles is exactly equal to two right angles, then this is the sum for every triangle. The existence of the one triangle that would complete the proof of the theorem that the sum of the angles of a triangle is equal to two right angles and hence the proof of the equivalent theorem, viz. the "parallel-postulate," he could not establish.

Such was the status of the problem until after 1830, when Euclid was justified in a spectacular manner. Approximately simultaneously, Lobachevsky, a Russian, J. Bolyai, an Hungarian, and Gauss, a German, showed the necessity of the postulate for ordinary geometry, by exhibiting a geometric science, different from that of Euclid, but obtained by denying this single postulate. For the first time in scientific thought, the necessity of a given postulate for an assigned geometry was *proved*, and a new independent type of special theory was developed.

Other types of non-Euclidean geometry followed. Riemann discovered a new form very analogous to that above mentioned. Other postulates have also been varied including some unconsciously assumed by Euclid and first explicitly formulated in recent times. New methods of developing the entire subject of geometry have been suggested such as the metrical postulates of Riemann, Helmholz, and Killing. Special disciplines such as Sundara Row's Paper-folding and Mascheroni's Geometry of the Compasses, have been conceived, until today the forms of geometry are as diversified as the races of man.

5. *The Revival of Interest in Postulates.* Ever since the time of Newton and Leibniz, the power and fertility of the infinitesimal calculus have fascinated the minds of mathematicians. The unbroken series of astounding discoveries and the wealth of physical problems that have yielded to rationalization have so glorified the calculus as an instrument, that for generations few cared to claim for mathematics, a purely deductive role. The great rigorists of the last century, however, turned the tide, and with their insistance upon the arithmetization of all analysis, brought the postulational character of the number concept once more to the fore. Intuition and coincidences have been again subordinated to proof, and such notions as irrational number, complex variable, limit, differentiation, integrability, continuity, and the like have been followed back to explicitly postulated properties of numbers.

Everywhere this same tendency has been shown, not merely in geometries of the general Euclidean type and in classical analysis, but in special fields such as Analysis Situs, in the definition of certain functions, as determinants, the Gamma function and the like, and in non-metrical mathematics such as the Algebra of Logic. This leads us to ask the question, "What is a system of postulates?"

6. *The Criteria of a Postulational System.* A mathematical theory must always pre-suppose some concepts and laws unless

it starts with a logical void and investigates the fundamental processes of thought. Many examples exist of nominally mathematical investigations which discuss the philosophical content of the notion of number at great length but contain no significant mathematical theorem. A safer and more expeditious because more formalistic procedure is to posit without further evaluation the legitimacy of drawing conclusions of certain definite forms from premises expressed in assigned conventional types. Of such is the formal logic of Aristotle and its more sophisticated recent successors which step naturally from set forms to special symbols, yielding an unequivocal compactness that discourages if not precludes verbose quibbling. A consistent use of logical symbols to the exclusion of words characterizes much of the published work of the mathematical philosophers of today.

Once the logical laws are established, and the "undefined terms," that are to be employed are stated, the substance of the postulational system lies in axioms or postulates as they may be called indescriminately. Upon the conditions to which we subject the set of axioms there are esthetic as well as logical restrictions. The most commonly mentioned requisites are *independence* and *consistency*, but many others might be suggested. For example, the axioms are usually expressed in as *simple* a manner as possible. Forms which with the usual interpretation of the elements are *intuitionally obvious* are preferred to startling or paradoxical statements. The set of axioms is frequently expected to prove *categorical.* A *generic* and *psychological order* is usually desired as against an apparently artificial or accidental sequence. *Complete independence* is more highly valued than mere simple independence. The *equivalence* of any new system to each of the previously known systems determining the same science must be demonstrated. The possibility of a *postulate being true only vacuously* should not accidentally arise except when a denial of the hypothesis deprives *ipso facto,* the conclusion of a meaning. *In the widest sense of the terms it is desirable that a system of axioms be adequate in substance, elementary in form, economical in context, and symmetrical in arrangement.*

Any criterion of an esthetic or logical type that does not itself admit of formal and unambiguous definition, can only be tested by individual judgment and relative appreciation and differs wholly in spirit from the axiomatic system that is being examined. Controversy and prejudice can be avoided only with the application of formal conditions for which deductive proofs may be given in each case to be examined. We shall confine our attention therefore to those formal conditions which appear most nearly universal and rigorous. We shall examiue in particular the notions of *simplicity, consistency, simple independence, complete independence* and *categoricity.*

7. *Simplicity.* A statement of the form, "If A then B, and if C then D," might obviously be analyzed into two simpler statements (1) "If A then B," (2) "If C then D." To combine into one statement two unrelated concepts can only result in needless complexity. It is not infrequent that a set of postulates may be shown to be formally independent by use of the obvious device of combining two refractory axioms into one and thus permitting the resulting combination to be denied if either part independently be not satisfied. One may adopt as a standard the principle that every axiom shall express a single idea. Difficulties, however, arise immediately. The statement "All A is both B and C," might be analyzed into (1) "All A is B," (2) "All A is C." If, however, we denote by D the features common to B and C and by E those contained in B but not in C, and by F, those in C but not in B, we have the following: In place of "All A is B," read, "All A is D, or all A is E, or some A is D and some A is E." And in place of "All A is C," read, "All A is D, or all A is F, or some A is D and some A is F," while in place of "All A is both B and C," read, "All A is D." In the new notation the analysis of "All A is both B and C," into (1) "All A is B," (2) "All A is C," becomes the analysis of, "All A is D," into (1) "All A is D, or all A is E, or some A is D and some A is E," (2) "All A is D, or all A is F, or some A is D and some A is F." In a similar manner every statement may be analyzed into two statements, and "simplicity" in this sense must be viewed either as an accident of wording or else as a self-contradictory notion.

8. *Consistency.* A set of postulates is naturally said to be consistent if it does not involve a contradiction. It is conceptually possible that a very trivial set of postulates might be such that every derived proposition could be cast into an assigned form, and where it could be proved that a given combination of terms can arise in at most only one way. In such a system no contradiction could arise, because, as we see, if an assertion be made in connection with a given set of terms, a negation can never arise for the same set, and a contradiction is *a priori*, inconceivable. Whether any important branch of mathematics could be put in such a form is at least highly dubious, and if consistency is to be proved, we expect a different procedure. The usual proof consists in exhibiting a known instance which satisfies all of the postulates. It would be at least difficult to prove that the power of recognizing that a set of postulates is satisfied by a given instance is purely logical. Any such act of immediate perception savors strongly of the intuitive,—but we shall waive this point. There remains the question as to whether the instance cited is itself existent. How can we know that the postulates for geometry are consistent by merely pointing to the space in which we live or rather to our conception of this space? It is undoubted that most of us are conscious of having given credence to numberless mutually contradictory beliefs, and logicians regularly point out such supposed contradictions in each others' assertions. A proof of consistency differs very little in its essence from the rough and illogical argument *ad hominem*.

9. *Independence.* A set of postulates has simple independence if no one postulate can be derived from the others. The situation is analogous to that with regard to consistency. Indeed, an implicit definition of simple independence may be stated as follows: "A set of postulates has simple independence if the set obtained by asserting all of the postulates except one, and negating that one is consistent for every choice of this one postulate from among the set." Every objection that can be cited for the tests of consistency holds immediately for tests of independence and in increased measure. For while the examples cited for consistency are usually fa-

miliar and can hardly contain a secreted contradiction, if we are to value the critical experience of the race, the examples for independence, on the other hand, are frequently peculiar if not indeed wierd. Their mere unfamiliarity might be regarded by some as invalidating their claim to obvious consistency. But the tests for independence involve new objectionable features not found in questions of consistency, for here it is no longer the set of postulates as a whole that is being tested but the set with reference to its articulation in individual postulates. Two systems which are logically equivalent might be such that one has and the other has not simple independence.

Complete independence is obtained when the set of postulates is such that every set derived by asserting some and negating the remaining postulates is consistent. This refinement of the notion of independence is still open to the objections already raised.

10. *Categoricity.* A set of propositions is categorical if any instances of systems of objects and relations which satisfy the postulates are in one to one reciprocal correspondence. For a categorical set no theorem stated wholly in the terms introduced by the postulates can be proved true in one instance and false in another. In a categorical set no new postulate can be adjoined unless it introduces a new term or is itself redundant. With respect to internal relations therefore, a categorical set of postulates completely define its terms. This does not mean that the terms are completely characterized with respect to outside objects. For example, the term "point" in a categorical geometry is regularly such that it is impossible to tell whether a "point" is regarded as a visual particle of imperceptible size, such as a "point of light," or is viewed as a set of three numbers. One can say that the set of three numbers represents the visual particle in position, or that the particle by its position represents the three numbers, and either concept has valid claims to the title of "point." The notion of "line" may be treated in a similar manner. And yet if the geometry be categorically determined, every theorem about points and lines must have precisely the same degree of validity in the two interpretations. We shall not insist upon the difficulties attendant upon a proof of categoricity for a given set of postulates.

11. *Conclusion.* We have seen that the demands of mathematical science have resulted in the growth of a system of formal logical and esthetic criteria. These criteria while not without content are difficult if not impossible to apply in a purely abstract manner. The objection suggests itself immediately that the systemmatization of postulates proposes an ideal which is inherently impossible of attainment, and that the claims of any given postulational system are based in presumptious ignorance or superficiality. Indeed, the utter futility of investigations into the foundations of mathematics is assumed as obvious by many people. The real importance and suggestiveness of such inquiries we shall attempt to show in a subsequent article.

WHAT GREAT MEN SAY ABOUT MATHEMATICS

Nor do I know any study which can compete with mathematics in general in furnishing matter for severe and continued thought. Metaphysical problems may be even more difficult, but then they are far less definite, and, as they rarely lead to any precise conclusion, we miss the power of checking our own operations, and of discovering whether we are thinking and reasoning or merely fancying and dreaming.

TODHUNTER, ISAAC.

Would you have a man reason well, you must use him to it betimes; exercise his mind in observing the connection between ideas, and following them in train. Nothing does this better than mathematics, which, therefore, I think should be taught to all who have the time and opportunity not so much to make them mathematicians as to make them reasonable creatures; for though we all call ourselves so, because we are born to it if we please, and we are carried no farther than industry and application have carried us.

LOCKE, JOHN.

Mathematics in its foreform, as arithmetic, algebra, geometry, and the applications of the analytic method, as well as mathematics applied to matter and force or statics and dynamics, furnishes the peculiar study that gives to us, whether as children or as men, the command of nature in this its quantitative aspect; mathematics furnishes the instrument, the tool of thought, which we wield in this realm.

HARRIS, W. T.

THE STRAIGHT EDGE

Don't expect your students to enjoy studying mathematics if you do not enjoy teaching it.

* * * * * *

Don't complain when your students have not prepared the lesson if you have done no better than they.

* * * * * *

Don't try to fool the class into thinking you are "giving them a chance to think" when you are "stumped" and trying to keep them from finding it out. They know.

* * * * * *

Don't imagine that all your teaching should be done on the class. "Thou that teachest another, teachest thou not thyself?"

* * * * * *

Don't try to make your students think they are better and brighter than they are. Some of them will go to college and may change their minds about themselves and—you.

* * * * * *

Don't try to teach the geometry of stained glass windows in a village where the architecture is of the cracker-box style. The Great Teacher drew his illustrations from objects with which his auditors were familiar.