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Stackelberg Differential Game Models in Supply Chain Management

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Dissertation

Presented to the Faculty of the Graduate School of

the University of Texas at Austin

in Partial Fulfillment

of the Requirements

for the Degree of

Doctor of Philosophy

The University of Texas at Austin

August 2007

Dedicated to my parents...

Acknowledgments

First and foremost, I would like to express my sincere gratitude to my adviser Prof. Genaro Gutierrez. I have greatly benefited from his patience, wisdom, and encouragement. The process of getting a doctoral degree is like a difficult journey of passing through a dark tunnel. His guidance put me through this journey to see the light at the end of the tunnel.

I would like to thank my committee members Professors Edward Anderson, Steve Gilbert, Vijay Mahajan, and Suresh Sethi for their guidance and support. I enjoyed discussing research with Professors Anderson and Gilbert in their seminars which broadened my perspectives in operations management. Prof. Mahajan encouraged me to find real world examples for motivating my research. I greatly benefited from reading his papers on product diffusion. I learned about Prof.Sethi and his work by reading his optimal control text in 2005. Little did I know that I will meet him at a conference in 2006 and even become his coauthor. I got very inspired by his enthusiasm and energy for research.

I thank my coauthor Prof. Ashutosh Prasad for involving me in marketing research and Prof. Qi Feng for her helpful comments and guidance. I am grateful to my former manager Dr.Yufeng Yan for encouraging me to pursue a doctoral degree.

My special thanks go to my parents and Jianghui Wu for their love and support.

Stackelberg Differential Game Models in Supply Chain Management

Publication No. ________________________

Xiuli He, Ph.D.

The University of Texas at Austin, 2007

Supervisor: Genaro J. Gutierrez

The dissertation consists of three essays. In the first essay, I analyze the dynamic interactions in a decentralized distribution channel, composed of a manufacturer and a retailer, to launch an innovative durable product (IDP) whose underlying retail demand is influenced by word-of-mouth from past adopters and follows a Bass-type diffusion process. The word-of-mouth influence creates a trade-off between immediate and future sales/ profits, resulting in a multi-period dynamic supply chain coordination problem. The analysis shows that the manufacturer and retailer may have conflicts regarding their trade-offs and preferences between immediate and future profits. I characterize equilibrium pricing strategies and the resulting sales and profit trajectories. Surprisingly, I find that the manufacturer, and sometimes even the retailer, is better off with a myopic retailer strategy in some cases. Furthermore, I propose that revenue sharing contracts can coordinate the IDP supply chain throughout the entire planning horizon.

In the second essay, I extend the demand model by considering the impact of shelf space allocation on the retail demand of an IDP. I assume the retail demand to be an increasing and concave function of the merchandise displayed on the shelf. I include a linear cost of shelf space in the retailer's objective function. I characterize the optimal dynamic shelf space allocation and retail pricing policies for the retailer and wholesale pricing policies for the manufacturer. I find that a myopic retailer allocates the constant amount of shelf-space to the IDP over the selling horizon, whereas the shelf space allocated to the IDP by a far-sighted retailer varies over time. Consistent with the first essay, the manufacturer and the retailer have conflict over the retailer's profitability strategy.

In the third essay, I review the Stackelberg differential game models that study such issues in dynamic environments as production and inventory policies, outsourcing decisions, channel coordination, and competitive advertising. I introduce the basic concepts of the basics of the Stackelberg differential games. I focus on the models that derive the Stackelberg equilibria in the area of supply chain management and marketing channels.

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Chapter 1

Introduction

Supply chain management is a subject that has been extensively studied by researchers in operations management, marketing, and economics. Most of the papers have focused on the static models which do not account for the effect of current period decisions of channel members on their future actions. While the demand may evolve gradually over time due to the *word-of-mouth* effect, understanding the dynamic behavior of channel members is important for designing various strategies, including dynamic pricing, production, inventory, outsourcing, shelf space allocation, and production capacity allocation. However, the dynamic nature of the coordination aspects of these various in a distribution channel has received limited attention in the literature. The studies by Elishberg and Steinberg (1987), Desai (1992, 1996), and Kogan and Tapiero (2007) are the notable exceptions.

The focus here is on the dynamic nature of supply chain and marketing channel coordination. In the first essay, we analyze the dynamic interactions in a decentralized distribution channel, composed of a manufacturer and a retailer, to launch an innovative durable product (IDP) whose underlying retail demand is influenced by word-of-mouth from past adopters and follows a Bass-type diffusion process. The retailer (she) has alternative uses for the critical resources that are essential to sell the manufacturer's new product. The word-of-mouth influence creates a trade-off between immediate and future sales/ profits, resulting in a multi-period dynamic supply chain coordination problem.

The maximization of life-cycle profits derived from an IDP presents us with a multi-period, dynamic supply chain coordination problem. We address the following research questions: Is it desirable for the manufacturer, through an up-front fee, to induce the retailer to dedicate a number of her selling resources to the IDP? This type of agreement is analogous to the "store within a store" practice utilized in the retailing of consumer products. What is the benefit of such an agreement for the manufacturer, for the retailer, and for the supply chain? Finally, we ask if it is possible to fully coordinate the supply chain throughout the entire life-cycle of the IDP? If so, what are the terms of such contract? How will the profits be split between the manufacturer and the retailer under a coordinating contract? We provide explicit answers to these questions.

The complexities of the supply chain coordination issues described above cannot be captured by a static or a single period model. In this research we develop and analyze a dynamic multi-period model to address and provide insights into the above questions. Our analysis shows that the manufacturer and retailer may have conflicts regarding their trade-offs and preferences between immediate and future profits. We characterize equilibrium pricing strategies and the resulting sales and profit trajectories, and propose that revenue sharing contracts can coordinate the IDP supply chain throughout the entire planning horizon and arbitrarily allocate the channel profit.

In the second essay, we consider a supply chain in which a manufacturer sells an innovative durable product to an independent retailer over its life-cycle. We assume that the product demand follows a Bass-type diffusion process, and it is determined by the market influences, retail price of the product, and shelf-space allocated to it. We consider the following retailer profit optimization strategies: (i) the "myopic" strategy of maximizing the current-period profit and (ii) the "far-sighted" strategy of maximizing the life-cycle profit. We characterize the optimal dynamic shelf space allocation and retail pricing policies for the retailer and wholesale pricing policies for the manufacturer. We find that a myopic retailer allocates the constant amount of shelf-space to the IDP over the selling horizon, whereas the shelf space allocated to the IDP by a far-sighted retailer varies over time. Surprisingly, we find that the manufacturer, and sometimes even the retailer, is better off with a myopic retailer strategy in some cases.

In the last essay, we investigate the applications of the Stackelberg differential game (DG) models to the supply chain management and marketing channels. Stackelberg differential game models have been used to study such issues in dynamic environments as production and inventory policy, capacity investment, dynamic pricing for new products, shelf-space allocation over the life-cycle of the products, competitive advertising, government's subsidy policy in new technology, and monetary and fiscal policies in economics. This review focuses on these applications. We consider Stackelberg equilibria as the solution concept for the games under consideration. We shall begin our review with an introduction to the basics of the Stackelberg DGs. We then summarize the important managerial insights obtained in each of the studies being reviewed. Finally, we point out future research avenues for applications of the Stackelberg DGs in supply chain and marketing channel.

The rest of the paper is organized as follows. In the next section, we review the related literature. In Section 2, we analyze the life-cycle channel coordinations issues in launching an IDP. In Section 3, we study the dynamic slotting and pricing decisions in a durable product supply chain. In Section 4, we review the Stackelberg DGs in the supply chain and marketing channel management.

Chapter 2

Life-cycle Channel Coordination Issues in Launching an Innovative Durable Product

2.1 Introduction

2.1.1 Motivation

This research addresses the strategic interactions between manufacturers with innovative durable products (IDPs) and the specialized retailers to sell the IDPs to final users; it was motivated by the distribution of Computer Aided Design (CAD) hardware and software; however, the models developed and the results obtained are applicable to the distribution of multiple innovative industrial products. These products are technically very sophisticated, and buyers require extensive technical information and attention before they commit to purchasing a unit. In this context, the distributor/retailer needs to devote important resources to the distribution of the product. The motivation for a manufacturer to use this Value Adding Resellers (VARs) channel is to reach the VARs' current customer base faster and more efficiently. Since these VARs are already experienced and knowledgeable in dealing with potential product adopters as they are already providing them with other related products and services, these VARs can reach potential customers faster and more economically; moreover, since they are typically located in the same geographical region as their customer base, they are more efficient in providing field support and technical assistance regarding the product's utilization. From the perspective of the customer, the cost of buying the IDP include not only the price of the physical product, but also the price of the service component provided by the VAR. In this research we aim to provide insights that will help IDP manufacturers to understand better the strategic interactions and challenges specific to this type of distribution channel, and provide guidelines to help them improve the efficiency and profitability of their distribution systems. To the best of our knowledge this is the first paper to address supply chain coordination issues in a multi-period dynamic environment for a durable product.

To analyze the strategic interactions between the manufacturer and the VARs, we consider a stylized model of a supply chain in which a monopolist manufacturer produces an IDP, and sells it through an independent retailer (a VAR operating in a geographical region) who serves the final market. We assume the retailer buys the product from the manufacturer at a wholesale price, she adds a profit margin on the IDP, and then she may charge a premium for the value adding services she provides; for simplicity in the rest of the paper, we will refer by retailer price to the total cost of acquisition and deployment for the final customer including the price paid by the customer for additional necessary services provided by the VAR such as technical assistance, training, field service etc. However, we would like to emphasize that this view of life-cylce differs from the conventional definition; in particular, if the window of opportunity to sell the product is small enough, the IDP will never reach stages of maturity and decline. We assume there is an exogenously determined (by competition or other factors) window of opportunity to sell the IDP, and we will refer to this window of opportunity as the planning horizon or product life-cycle. We assume there is a finite number of potential adopters for the IDP and each potential adopter would purchase at most one unit (no repeat purchases). The advantages of distributing an IDP through VARs are significant, however, as we shall discuss below, introducing an intermediary in this distribution channel creates a host of coordination problems for the channel. In our analysis we concentrate on the following three challenges: (a) conflict of inter-temporal optimization objectives (“myopic”, i.e., short-term, versus “far-sighted”, i.e., long-term) between the manufacturer and the VAR, (b) competition for resources between the IDP in question and other products carried by the VAR and (c) double marginalization problems in a dynamic multi-period environment. Below we elaborate each of these challenges.

(a) Conflict of Inter-Temporal Optimization Objectives. The life-cycle sales of the IDP are influenced by retail price as well as a variety of factors including word-of-mouth or network effects, which work through the interaction between the current and future potential adopters. This word-of-mouth interaction suggests that future product sales are influenced by cumulative past sales. Correspondingly, there is a trade-off between current and future profits when we aim to maximize the IDP’s life-cycle profits. Specifically, lowering current period prices may stimulate immediate sales possibly at the expense of immediate profits, while an increase in the number of current adopters may increase future demand through word-of-mouth or network influences and possibly leading to larger future profits. Furthermore, if both the manufacturer and the retailer make independent pricing decisions, neither of them has full control of their profitability or the profitability of the supply chain.

By setting the retail prices, the retailer can affect retail demand for the IDP, but her profitability is also affected by the wholesale prices charged by the manufacturer. The manufacturer on the other hand can affect retail prices and sales only indirectly. Even in cases in which the manufacturer has the ability to set the customer price for the physical IDP, the VAR can affect the final deployed cost for the customer by varying the prices charged by her services and hence affect the customer demand. Moreover, both of these supply chain partners might place significantly different values to the trade-off between immediate and future profits.

(b) Competition for Critical Selling Resources. An independent VAR would carry multiple products thus creating a competition for limited critical selling resources. We will refer to this case as the *shared resource setting.* If alternative products provide the VAR with a high profit margin per unit of resource utilized, the VAR will increase the IDP's sale price to increase her profit margin; the justification of this price increase is to make it profitable for her to allocate selling resources to the IDP, but it will also have the effect of reducing the IDP's sales volume. This competition for resources will affect negatively the manufacturer's profits and it will hinder the IDP's diffusion as well.

(c) Multi-Period Double Marginalization. Since the VAR is an independent decision maker she will formulate her pricing strategies to maximize her own profits disregarding the profitability of the manufacturer's as well as the distribution channel's. In our context, this will consit of a series of myopic local optimizations (on a rolling horizon) or of a multi-period local optimization by the VAR, leading the manufacturer to select the wholesale pricing strategy that maximizes his own profits over the IDP's life-cycle.

The maximization of life-cycle profits derived from an IDP presents us with a multi-period, dynamic supply chain coordination problem, and we address the following research questions: Is it desirable for the manufacturer, through an up-front fee, to induce the VAR to dedicate a number of her selling resources to the IDP? This type of agreement is analogous to the "store within a store" practice utilized in the retailing of consumer products. What is the benefit of such an agreement for the manufacturer, for the retailer, and for the supply chain? Finally, we ask if it is possible to fully coordinate the supply chain throughout the entire life-cycle of the IDP? If so, what are the terms of such contract? How will the profits be split between the manufacturer and the retailer under a coordinating contract?

The complexities of the supply chain coordination issues described above cannot be captured by a static or a single period model. In this research we develop and analyze a dynamic multi-period model to address and provide insights into the above questions. Further we emphasize that the scope of application of this research is not limited to the distribution of CAD hardware. The tradeoffs and conflicts described above are present in the distribution of IDPs such as complex industrial products, high-end audio products, as well as hardware-software systems for commercial applications; in particular, the VARs distribution channels is also intensively used by both IBM and HP to distribute computer hardware and specialized software to commercial customers. In this latter case the services added by the intermediaries include installation, training and technical support.

2.1.2 Overview of the Model

Following the marketing literature, we assume that the underlying retail demand of an IDP follows a Bass-type diffusion process with market dynamics modeled with a nonlinear differential equation. According to the Bass (1969) model, the purchase decisions of potential adopters of an IDP are affected by two market influences: external and internal market influence. Examples of external market influence include advertising, such as the manufacturer's national advertising in the mass media and specialized trade magazines, shows and conventions, as well the retailer's (in the rest of the paper, we will refer to the VAR as the "retailer") local advertising and promotional activities and sales efforts. The internal market influence works through the word-of-mouth and network effects spreading from the previous adopters to the potential adopters. We incorporate the impact of retail price on retail demand, enabling us to capture the role of an independent retailer on the dynamics of the supply chain.

We formulate the problem in an optimal control framework. We assume that the manufacturer takes the leader role in his relationship with the retailer. Specifically, the manufacturer and retailer play a Stackelberg (sequential) differential game: the manufacturer announces his wholesale price to the retailer, and the retailer sets the retail price that maximizes her profits taking the manufacturer's contractual wholesale price as given. The manufacturer takes the retailer's optimal reaction into consideration when he makes his wholesale price decision. The solution concept for the Stackelberg differential game we identify is an open-loop equilibrium which means, at the start of the game, the manufacturer and retailer decide on a strategy that depends on time. In this study, we assume that the manufacturer is able to credibly commit to his wholesale price strategy.

2.1.3 Key Results

We have identified a conflict of preferences between the manufacturer and the retailer. First, a manufacturer will not always prefer the retailer to take a long-term (i.e, far-sighted) optimization strategy; that is, the manufacturer sometimes is better off if the retailer has a short-term (i.e., myopic) optimization focus. That is, in some instances the manufacturer will prefer the retailer to react to the wholesale prices by setting retail prices that maximizes her immediate (instantaneous) profits at any instant rather than her long-term profits over the entire horizon. On the other hand, the retailer's preferences over her optimization focus (short-term versus long-term) change with the market characteristics of the IDP and they do not always agree with the manufacturer's preferences. It is not immediately obvious that a seemingly myopic retailer behavior may enhance the performance of the supply chain.

In the shared resources setting (the VAR distributes other products in addition to the IDP), the manufacturer and the entire channel make lower profits than with a dedicated resources setting in which the retailer dedicates a share of her resources to sell exclusively the manufacturer's product. We explore the possibility of a two-part tariff, wholesale price and an up-front fee (a fee paid by the manufacturer to the retailer for the exclusive use of a given quantity of resources), to partially improve the channel performance.

Finally, we demonstrate that revenue sharing contracts are in principle capable of coordinating a durable product supply chain with a long-term as well as a short-term retailer profitability focus and arbitrarily allocate the channel profit. More specifically, we show that the coordinating wholesale price for a retailer with a long term focus is constant over the IDP's life cycle while the coordinating wholesale price for a retailer with a short-term focus varies over time.

2.1.4 Organization

The rest of the paper is organized as follows. In the next section, we review the related literature. In Section 2.3, we introduce the demand model. In Section 2.4, we study the case of a retailer with a long-term profitability focus. In Section 2.5, we study the case of a retailer with a short-term focus. In Section 2.6, we present a numerical study that compares the case of long-term and short-term focus. In Section 2.7, we propose an up-front fee to improve the channel performance. We use revenue sharing contracts to fully coordinate the channel in Section 2.8. We conclude the paper by summarizing the results and summarizing the managerial implications and pointing out future research avenues in Section 2.9.

2.2 Literature Review

This work is related to multiple streams of literature, but the three most closely related literatures are optimal dynamic pricing for new products, revenue sharing contracts in supply chain management literature, and differential games with applications in management science.

In the marketing literature, Bass (1969) and its variants have been widely used to forecast the demand of a new durable product. We refer readers to Mahajan et al. (1990) and Mahajan et al. (2000) for comprehensive reviews on diffusion models. The original Bass (1969) model does not include the pricing variables. A number of later papers extended the Bass model by incorporating the (competitive) price impact on retail demand of an IDP, including Robinson and Lakhani (1975), Bass (1980), Dolan and Jeuland (1981), Bass and Bultez (1982), Kalish (1983), Kalish and Lilien (1983), Clarke and Dolan (1984), Thompson and Teng (1984), Rao and

Bass (1985), Eliashberg and Jeuland (1986), Raman and Charterjee (1995), and Krishnan at el. (1999). Regarding the market conditions, Eliashberg and Jeuland (1986), Thompson and Teng (1984) analyze oligopoly pricing strategies while the rest analyze the optimal monopolist pricing strategies.

In order to derive the dynamic pricing strategies, a researcher needs to make a key assumption about the firm's profit-maximizing strategy, i.e., the firm maximizes the short-term or long term profits? Bass (1980) and Bass and Bultez (1982) assume the firm maximizes the current period (instantaneous) profits. The corresponding pricing strategies in these two papers are called myopic pricing strategies as compared to (global) optimal pricing strategies which maximize the firm's aggregated profits over the product's life cycle. Robinson and Lakhani compared the results from the optimal pricing with the myopic pricing strategies. Their numerical results show that the differences are significant while Bass and Bultez (1982) reported small difference.

As noted by Dolan and Jeuland (1981), it is very critical to properly incorporate the pricing impact into the demand model. Several papers, including Robinson and Lakhani (1975), Dolan and Jeuland (1981), and Thompson and Teng (1984), assume the demand is an exponential function of price. In contrast, like Eliashberg and Jeuland (1986) and Raman and Chatterjee (1995), we assume that the demand is a linearly decreasing function of retail price. We selected this demand model to be able to extend our analysis to explore contracting and coordination issues with two independent echelons supply chain.

All of above papers assume a centralized decision maker will decide the pricing strategy and by implication, production quantities. Since the above models assume centralized decision making they are unable to examine the role that an independent retailer may play in distributing the IDP.

In the supply chain management literature, various types of supply contracts have been designed to mitigate or eliminate the double marginalization and incentive misalignment problems due to the independent decisions of a retailer in a decentralized channel. We refer the readers to Krishnan et al. (2004) and Cachon (2003) for excellent reviews on the supply contracting literature. The most relevant papers are those that study revenue sharing contracts. For example, Gerchak and Wang (2004) study the revenue-sharing contracts between an assembler/ retailer and its component suppliers. In their paper, the assembler sets the shares of the revenue then the suppliers decide delivery quantities. They show that revenue share alone cannot coordinate the assembly system. However, a revenue sharing scheme coupled with a subsidy paid by the assembler to component suppliers can coordinate the assembly supply chain. Cachon and Lariviere (2005), in a newsvendor setting, study the revenue sharing contracts between a retailer and manufacturer who sets the wholesale price. Gerchak et al. (2006) study the revenue sharing contracts in a decentralized Stackelberg setting in which the video rental channel and the studio make independent decisions. However, all the above papers focus the one-shot interaction between the supplier (manufacturer) and the retailer. In contrast, we study the channel coordination between a manufacturer and a retailer over the life-cycle of the IDP in a dynamic environment, i.e., both channel members make dynamic retail and wholesale pricing decisions rather than static decisions.

We assume that the manufacturer and the retailer play a Stackelberg differential game. The differential game approach is very popular to study the problems involving dynamic environments. Mathematically, in close spirit to our approach, Jørgensen et al. (2003) study the dynamic advertising strategies of a manufacturer and a retailer in a decentralized setup in which a retailer can be myopic (maximizes the instantaneous payoff) or far-sighted (maximizes the long term payoff). Eliashberg and Steinberg (1987) formulated a Stackelberg differential game to study the interactions between a manufacturer and a downstream distributor. However, their focus is on the joint inventory and pricing strategies of the manufacturer and the distributor. Additionally, since they assume a constant (not varying over time) transfer price, they do not allow the manufacturer to dynamically set the wholesale price. By contrast, we study the optimal dynamic wholesale prices as well as dynamic retail prices. Additionally, their specific demand model captures the seasonal sales fluctuation while there is no diffusion process involved.

2.3 The Demand Model

A manufacturer produces an innovative durable product whose retail demand follows Bass type diffusion process. Let $x(t)$ be the instantaneous sales rate at time t. The demand dynamics are described by the following differential equation:

$$x(t) = \dot{X}(t) = \frac{dX(t)}{dt} = (M - X(t))(\alpha + \beta X(t))(1 - \gamma r(t)),$$

where $X(t)$ is the cumulative sales up to time t, M is the potential market size, the term $(M - X(t))$ is the unsaturated market size, α and β are positive coefficients of external and internal market influences, respectively, and γ is a positive parameter that measures the customers' sensitivity to the retail price $r(t)$. According to our formulation, $x(t)$ is determined by three factors: the external market influence, the internal market influence and price sensitivity.

A few additional comments are in order now. First, we use a multiplicatively separable function to model the impact of price and cumulative sales (past sales) on the instantaneous demand rate. Second, the instantaneous sales rate $x(t)$ is a linearly decreasing function of retail price. Linear demand functions have been used by researchers in the stream of dynamic pricing that particularly used Bass model, including Eliashberg and Jeuland (1988), Raman and Chartejee (1995), and Kalish (1983). Third, we observe that the main drivers of sales change during the entire selling horizon an IDP. Initially, the market saturation level is low, diffusion effect outweighs saturation effect (shrinking potential market size). However, if the selling horizon is very long, after a certain period of time, the market gets highly saturated and every additional sale is more difficult thus we can say that the saturation effect dominates the diffusion effect.

The retailer needs certain critical resources to sell the IDP. Examples of such resources include specialized salespeople, and in some cases equipment and facilities. We assume that the amount of resources required are proportional to the sales volume. Let K be the capacity of retailer's critical resources. The retailer may have alternative profitable uses other than for the manufacturer's product. It is in the retailer's best interest to allocate the K units of resources flexibly among the products she sells. Let O be the profit margin of the alternative use; we will refer to O as the "outside" profit margin for the retailer as it refers to products that are external to the supply chain of the IDP in consideration. By implication, O will affect the retailer's pricing decisions hence the sales volume for the IDP. By defining the resource units appropriately, we assume that each unit of sales requires a unit of resource. We refer to these resources as *dedicated resources* if the retailer only uses them to sell the manufacturer's product and *flexible resources* if she can flexibly allocates them among alternative products.

We use the superscripts "L" and "S" denote the long-term and short-term retailer profitability strategy (focus), respectively. Subscripts "M", "R", and "C" denote the manufacturer, the retailer and the channel, respectively.

2.4 A Retailer with a Long-term Focus

Consider the case of a retailer with a long-term profitability strategy who maximizes her profits over the entire life cycle T of the IDP. We assume that she has alternative uses for her critical sales resources and flexibly allocates them among the different products she sells. We assume that the manufacturer and retailer play a Stackelberg differential game with the manufacturer acting as the leader. That is, the manufacturer announces the wholesale price path $\{w^L(t) : t \in [0, T]\}$ at time 0. Then the retailer decides a retail price path $\{r^L(t) : t \in [0, T]\}$. This retail price includes the price the consumer pays for the physical IDP as well as the price charged by the retailer for the service component required by the product. The retailer's instantaneous profit function is given by $[r^L(t) - w^L(t) - s]\, \dot{X}^L(t) + \left[K - \dot{X}^L(t)\right] O$, where s is the retailer's cost associating with selling the product. This cost should include not only the variable costs associated with closing the sale of the physical IDP, but it should also include the variable cost of the additional services provided. As stated previously, the retailer with a long-term strategy maximizes the life cycle profits $\Pi_R^L(T)$:

$$\Pi_R^{L*}(T) = \max_{r^L(t)} \int_0^T \left\{ [r^L(t) - w^L(t) - s]\, x^L(t) + \left(K - \dot{X}^L(t)\right) O \right\} dt \tag{2.1}$$

$$s.t. \quad x^L(t) = (M - X^L(t))\,(\alpha + \beta X^L(t))\,(1 - \gamma r^L(t)) \tag{2.2}$$

$$X^L(0) = X_0^L \tag{2.3}$$

where $X^L(0)$ is the initial sales condition. Note that (2.1)-(2.3) is an optimal control problem with $r^L(t)$ and $X^L(t)$ as control and state variables, respectively. The differential equation (2.2) along with the initial condition (2.3) explicitly describes how the the cumulative sales $X^L(t)$ and retail price $r^L(t)$ jointly determine the immediate sales (demand) rate $x^L(t)$. We shall assume that the retailer has enough capacity to sell the manufacturer's product, i.e., $K \geq x^L(t)$ holds for $\forall t \in [0, T]$.

We first solve the retailer's problem and use her best response to formulate the manufacturer's problem. We formulate the retailer's problem in the optimal control framework with a control variable $r^L(t)$ and a state variable $X^L(t)$. From now on, for notational simplicity, we may omit the time argument in some equations. The retailer's Hamiltonian H_R^L is given by:

$$H_R^L = F^L\left(1 - \gamma r^L\right)\left[r^L - w^L - s - O + \lambda_R^L\right], \tag{2.4}$$

where $F^L = \left(M - X^L\right)\left(\alpha + \beta X^L\right)$ and λ_R^L is the shadow price associated with the state variable X^L. Note that we ignore the constant term KO when formulating the retailer's Hamiltonian. Define $f^L = \frac{dF^L}{dX^L} = -\alpha + M\beta - 2\beta X^L$. The shadow price λ_R^L satisfies the following condition:

$$\dot{\lambda}_R^L = -\frac{\partial H_R^L}{\partial X^L} = -f^L\left(1 - \gamma r^L\right)\left[r^L - w^L - s - O + \lambda_R^L\right], \tag{2.5}$$

with the terminal value $\lambda_R^L(T) = 0$ (because $X^L(T)$ is free to move). Let r^{L*} be the retailer's best response retail price. The necessary first order condition to maximize H_R^L is given by:

$$\frac{\partial H_R^L}{\partial r^L} = 0 \Longrightarrow \quad r^{L*} = \frac{1 + k\left(w^L + s + O - \lambda_R^L\right)}{2\gamma}. \tag{2.6}$$

The economic interpretation of $\lambda_R^L(t)$ is the value of additional unit of sales. $\lambda_R^L(t) > 0$ implies that the retailer benefits from current sales (see Sethi and Thompson 2000 for detailed discussion of the economic interpretation of the shadow price); accordingly, the retailer sets $r^L(t)$ below the myopic retail response which is defined as the price that would result if we set $\lambda_R^L(t) = 0$. With the myopic retail response, the retailer does not take into account the impact of current sales on future sales. On the other hand, when $\lambda_R^L(t) < 0$, the retailer has no incentive to sacrifice current profits for future profits, and she will increase $r^L(t)$ above the myopic price level.

Let H_R^{L*} be the maximized Hamiltonian. H_R^{L*} is given by:

$$H_R^{L*} = \frac{F^L\left[1 - k\left(w^L + s + O - \lambda_R^L\right)\right]^2}{4\gamma}.$$

It is easy to verify that H_R^{L*} is concave and continuously differentiable with respect to X^L for all $t \in [0, T]$. Therefore r^{L*} is an optimal path.

Note that for each wholesale price path $\{w^L(t) : t \in [0, T]\}$ the manufacturer announces, there is a corresponding optimal retail price path $\{r^{L*}(t) : t \in [0, T]\}$. The manufacturer takes the retailer's best response into consideration when solving his optimization problem. Assume that the manufacturer incurs a constant per unit production cost c_0. The manufacturer's optimization problem is given by:

$$\Pi_M^{L*}(T) = \max_{w^L} \int_0^T \left[w^L - c_0\right] x^{L*} dt$$

$$s.t. \quad x^{L*} = \frac{F^{L*}\left\{1 - \gamma\left[w^L + s + O - \lambda_R^L\right]\right\}}{2} \tag{2.7}$$

$$\dot{\lambda}_R^L = -\frac{f^{L*}\left\{1 - \gamma\left[w^L + s + O - \lambda_R^L\right]\right\}^2}{4k} \tag{2.8}$$

$$X^L(0) = X_0^L, \lambda_R^L(T) = 0, \tag{2.9}$$

where the expressions for x^{L*} and $\dot{\lambda}^L_R$ are obtained by substituting (2.6) into (2.2) and (2.5), respectively. Note that the manufacturer has two state variables: X^{L*} and $\dot{\lambda}^L_R$. Dockner et. al. (2000) used a similar approach. By now, we have defined a two-player differential game with two control variables w^L and r^L. The manufacturer's Hamiltonian equation H^L_M is given by:

$$H^L_M = \left(w^L - c_0 + \lambda^L_M\right) x^L + \mu \dot{\lambda}^L_R \tag{2.10}$$

where λ^L_M and μ are the shadow prices associated with x^{L*} and $\dot{\lambda}^L_R$, respectively. λ^L_M and μ satisfy the following conditions:

$$\dot{\lambda}^L_M = -\frac{\partial H^L_M}{\partial X^L}, \dot{\mu} = -\frac{\partial H^L_M}{\partial \lambda^L_R}.$$

Specifically, we have:

$$\dot{\lambda}^L_M = -\frac{f^L\left(w^L - c_0 + \lambda^L_M\right)\left[1 - \gamma\left(w^L + s + O - \lambda^L_R\right)\right]}{2} \tag{2.11}$$

$$\dot{\mu} = -\frac{\gamma F^L\left(w^L - c_0 + \lambda^L_M\right)}{2} + \frac{\mu f^L\left[1 - \gamma\left(w^L + s + O - \lambda^L_R\right)\right]}{2}, \tag{2.12}$$

with boundary conditions $\lambda^L_M(T) = 0$ and $\mu(0) = 0$. We impose $\lambda^L_R(T) = 0$ because $X^L(T)$ is free to move and impose $\mu(0) = 0$ because our problem is controllable, i. e., the associated initial state $\lambda^L_R(0)$ is dependent on w^L. Substituting 2.7 into 2.10, we get:

$$\begin{aligned} H^L_M &= \frac{F^L\left(w^L - c_0 + \lambda^L_M\right)\left\{1 - \gamma\left[w^L + s + O - \lambda^L_R\right]\right\}}{2} \\ &\quad -\frac{\mu f^L\left\{1 - \gamma\left[w^L + s + O - \lambda^L_R\right]\right\}^2}{4\gamma}. \end{aligned} \tag{2.13}$$

The necessary first order condition to maximize H_{M}^{L} is given by:

$$\frac{\partial H_M^L}{\partial w^L} = 0 \Longrightarrow \quad w^{L*} = \frac{1+\gamma\left(\lambda_R^L - s - O\right)}{\gamma} - \frac{F^{L*}\psi^L}{k\left(2F^{L*}+\mu f^{L*}\right)}, \tag{2.14}$$

where $\psi^L = 1 + k\left(\lambda_M^L + \lambda_R^L - s - O - c_0\right)$. Substituting (2.14) into (2.16), we have

$$r^{L*} = \frac{1}{\gamma} - \frac{F^{L*}\psi^L}{2\gamma\left(2F^{L*}+\mu f^{L*}\right)} \tag{2.15}$$

Substituting (2.14) into (2.13), after simplifying the expression, the maximized Hamiltonian equation H_M^{L*} is given by:

$$H_M^{L*} \quad = \frac{\left(\psi^L F^{L*}\right)^2}{4\gamma\left(2F^{L*}+\mu f^{L*}\right)}.$$

Substituting (2.14) into (2.7), (2.8), (2.11), and (2.12), respectively, we have:

$$x^{L*} = \frac{\psi^L\left[F^{L*}\right]^2}{2\left(2F^{L*}+\mu f^{L*}\right)} \tag{2.16}$$

$$\dot{\lambda}_R^L = \frac{-f^{L*}\left(\psi^L F^{L*}\right)^2}{4\gamma\left(2F^{L*}+\mu f^{L*}\right)^2} \tag{2.17}$$

$$\dot{\lambda}_M^L = -\frac{F^{L*}\left[\psi^L\right]^2\left\{2f^{L*}F^{L*}+\mu\left[2\left(f^{L*}\right)^2+2\beta F^{L*}\right]\right\}}{4\gamma\left(2F^{L*}+\mu f^{L*}\right)^2} \tag{2.18}$$

$$\dot{\mu} = -\frac{\psi^L\left(F^{L*}\right)^2}{2\left(2F^{L*}+\mu f^{L*}\right)}. \tag{2.19}$$

with boundary conditions

$$X^L(0) = X0^L, \mu(0) = 0, \lambda_R^L(T) = \lambda_M^L(T) = 0. \tag{2.20}$$

We assume that $Sign\left(\psi^L\right) = Sign\left(2F^{L*}+\mu f^{L*}\right)$ to guarantee that the right-hand side of Equation (2.16) is positive. Comparing Equation (2.9) to Equation (2.16), we find that $\dot{\mu} = -\dot{X}^{L*} < 0$, for $\forall t \in [0, T]$. This implies that $\mu(t) < 0, \forall t \in [0, T]$, because we have $\mu(0) = 0$. Equations (2.16)-(2.19) consist a system of four differential equations with four unknowns, which, along with the boundary conditions (2.10), imply a solution; however, it is very difficult to derive analytical solutions for all the variables as functions of time and system parameters (See Eliashberg and Jeuland (1986) for a discussion of the complexity of the solutions to a similar system of non-linear differential equations.).

We observe, from Equation (2.17), that the sign of $\dot{\lambda}_R^L$ depends on the sign of f^{L*}: if $f^{L*} = -\alpha + M\beta - 2\beta X^{L*} > 0$, $\dot{\lambda}_R^L < 0$, i.e., λ_R^L is decreasing; otherwise, $\dot{\lambda}_R^L \geq 0$.

2.5 A Retailer with a Short-term Focus

For a given wholesale price contract, it is always to the retailer's advantage to set the retail prices with a life-cycle (global) optimization objective. However, the manufacturer adjusts the wholesale prices that he will offer the retailer taking into account the retailer's optimization objective; in this situation, it is not clear that the life-cycle optimization strategy will be in the retailer's best interest. Specifically, if the manufacturer knows that the retailer sets retail prices with a short-term profitability strategy, and then offers the retailer the wholesale prices under this assumption, the retailer may be better off than if the manufacturer assumes that she sets the retail prices with a long-term profitability strategy and offers her wholesale prices reflecting this long-term strategy. To study this hypothesis, we model the optimal pricing strategy of the manufacturer under the assumption that the retailer sets the retail prices with a short-term optimization objective. At the end of section, we discuss alternative incentives that may lead to the short-term retailer profitability strategy.

With a short-term profitability strategy, at any time t, the retailer maximizes her instantaneous profit rate, taking the wholesale price $w^s(t)$ as given:

$$\begin{aligned} \pi_R^{S*}(t) &= \max_{r^S(t)} \{[r^s(t) - w^s(t) - s]\, x^s(t) + O[K - x^s(t)]\} \\ s.t. \quad x^s(t) &= F^s(X^s(t))[1 - \gamma r^s(t)], X^s(0) = X_0^s, \forall t \in [0, T] \end{aligned} \tag{2.21}$$

where $F^s(X^s(t)) = (M - X^s(t))(\alpha + \beta X^s(t))$. Similar to the case of a long-term profitability strategy, we shall assume that $K \geq x^s(t)$ holds for $\forall t \in [0, T]$. From the first order condition, we obtain the retailer's best response:

$$r^{S*} = \frac{1}{2\gamma}[1 + \gamma(w^s + s + O)]. \tag{2.22}$$

The manufacturer takes the retailer's best response into the consideration when solving his own optimization problem. He maximizes the life cycle profits and his optimization problem is given by:

$$\begin{aligned} \Pi_M^{S*}(T) &= \max_{w^S(t)} \int_0^T [w^s(t) - c_0]\, x^s(t)\, dt \\ s.t. \quad x^s &= \frac{1}{2} F^s[1 - \gamma(w^s + s + O)], X^s(0) = X_0^s. \end{aligned} \tag{2.23}$$

where the expression for x^s is obtained by substituting (2.22) into (2.21). The manufacturer has a control variable w^s and a state variable x^s. His Hamiltonian equation H^s is given by

$$H^s = \frac{1}{2} F^s[1 - k(w^s + s + O)][w^s - c_0 + \lambda^s],$$

where λ^s is the shadow price associated with the state variable X^s. Let $f^s = \frac{dF^s}{dX^s} = -\alpha + M\beta - 2\beta X^s$. The shadow price λ^s satisfies the following equation:

$$\dot{\lambda}^s = -\frac{\partial H^s}{\partial X^s} = -\frac{1}{2} f^s \left[1 - \gamma\left(w^s + s + O\right)\right]\left[w^s - c_0 + \lambda^s\right] \tag{2.24}$$

with the boundary condition $\lambda^s(T) = 0$. We can determine the optimal control w^s from the first order condition of $\frac{\partial H^s}{\partial w^s} = 0$:

$$w^{s*} = \frac{1}{2k}\left[1 + \gamma\left(c_0 - s - O - \lambda^s\right)\right]. \tag{2.25}$$

The maximized Hamiltonian $H^{s*}$is given by:

$$H^{s*} = \frac{F^s\left[1 + \gamma\left(\lambda^s - c_0 - s - O\right)\right]^2}{8k}.$$

We can verify that H^{s*} is concave in X^s and continuously differentiable with respect to X^s for all $t \in [0, T]$. Therefore w^{s*} is an optimal path.

Substituting w^{s*} into (2.21)-(2.23), we derive the optimal retail price, instantaneous sales rate, and shadow price as follows:

$$r^{s*} = \frac{1}{4\gamma}\left[3 - \gamma\left(\lambda^s - s - O - c_0\right)\right], \tag{2.26}$$

$$x^{s*} = \frac{F^{s*}}{4}\left[1 + \gamma\left(\lambda^s - s - O - c_0\right)\right], \tag{2.27}$$

$$\dot{\lambda}^s = -\frac{f^{s*}}{8\gamma}\left[1 + \gamma\left(\lambda^s - s - O - c_0\right)\right]^2. \tag{2.28}$$

The solution to the problem is determined by (2.27) and (2.28) together with boundary conditions: $X^s(0) = X_0^s$ and $\lambda^s(T) = 0$. We will characterize the equilibrium in terms of cumulative sales in the following several lemmas.

The following lemma derives the relationship between λ^s and X^{s*}. Define $\bar{\phi} = 1 - \gamma(c_0 + s + O)$, a function of the system parameters, and assume the values of the parameters are such that $\bar{\phi} > 0$.

Lemma 2.1. *The shadow price trajectory $\lambda^s(t)$ is given by*

$$\lambda^s(t) = \frac{\bar{\phi}}{\gamma}\left[\sqrt{\frac{F^{s*}(X^{s*}(T))}{F^{s*}(X^{s*}(t))}} - 1\right], t \in [0, T]. \tag{2.29}$$

Proof: Combining Equations (2.27) and (2.28), we have

$$\begin{aligned}
&\frac{\dot{\lambda}^s(\tau)}{x^{s*}(\tau)} = -\frac{f^{s*}(x^{s*}(\tau))\left[1 + \gamma(\lambda^s(\tau) - s - O - c_0)\right]}{2\gamma F^{s*}(x^{s*}(\tau))} \\
\Longrightarrow\ &\frac{\gamma d\lambda^s(\tau)}{\left[1 + \gamma(\lambda^s(\tau) - s - O - c_0)\right]} = -\frac{f^{s*}(x^{s*}(\tau))\, dX^{s*}(\tau)}{2F^{s*}(x^{s*}(\tau))} \\
\Longrightarrow\ &\int_{\lambda^S(t)}^{\lambda^S(T)} \frac{\gamma d\lambda^s(\tau)}{\left[1 + \gamma(\lambda^s(\tau) - s - O - c_0)\right]} = -\int_{X^{S*}(t)}^{X^{S*}(T)} \frac{f^{s*}(x^{s*}(\tau))\, dX^{s*}(\tau)}{2F^{s*}(x^{s*}(\tau))} \\
\Longrightarrow\ &\ln\left[1 + \gamma(\lambda^s(\tau) - s - O - c_0)\right] \Big|_{\lambda^S(t)}^{\lambda^S(T)} = -\ln\sqrt{F^{s*}(x^{s*}(\tau))} \Big|_{X^{S*}(t)}^{X^{S*}(T)} \\
\Longrightarrow\ &\frac{1 - \gamma(s + O + c_0)}{1 + \gamma(\lambda^s(t) - s - O - c_0)} = \sqrt{\frac{F^{s*}(x^{s*}(t))}{F^{s*}(x^{s*}(T))}} \\
\Longrightarrow\ &\lambda^s(t) = \frac{1 - \gamma(c_0 + s + O)}{\gamma}\left[\sqrt{\frac{F^{s*}(x^{s*}(T))}{F^{s*}(x^{s*}(t))}} - 1\right],
\end{aligned}$$

with $\lambda^s(T) = 0$. We obtain the expression in Lemma 2.1.□

REMARKS. We have $Sign[\lambda^s(t)] = Sign\left[\sqrt{\frac{F^{S*}(X^{S*}(T))}{F^{S*}(X^{S*}(t))}} - 1\right]$. When $\sqrt{\frac{F^{S*}(X^{S*}(T))}{F^{S*}(X^{S*}(t))}} > 1$, i.e., $F^{s*}(X^{s*}(T)) > F^{s*}(X^{s*}(t))$, $\lambda^s(t) > 0$; if $\sqrt{\frac{F^{S*}(X^{S*}(T))}{F^{S*}(X^{S*}(t))}} < 1$, i.e., $F^{s*}(X^{s*}(T)) < F^{s*}(X^{s*}(t))$, $\lambda^s(t) < 0$. Lemma 2.1 enables us to eliminate λ^s from the optimality conditions and characterize the variables in terms of the cumulative sales $X^{s*}(t)$.

Lemma 2.2. *With a short-term retailer profitability strategy, for $t \in [0, T]$,* *(i) The instantaneous shadow price $\dot{\lambda}^{S}(t)$ is given by*

$$\dot{\lambda}^{S}(t) = -\frac{\bar{\phi}^{2} f^{S*}(X^{S*}(t))}{8\gamma} \frac{F^{S*}(X^{S*}(T))}{F^{S*}(X^{S*}(t))}.$$

(ii) The equilibrium retail price trajectory $r^{S*}(t)$ and wholesale price trajectory $w^{S*}(t)$ are:

$$\begin{aligned} r^{S*}(t) &= \frac{1}{4\gamma}\left[4 - \bar{\phi}\sqrt{\frac{F^{S*}(X^{S*}(T))}{F^{S*}(X^{S*}(t))}}\right] \\ w^{S*}(t) &= \frac{1}{2\gamma}\left[2(1-\gamma(s+O)) - \bar{\phi}\sqrt{\frac{F^{S*}(X^{S*}(T))}{F^{S*}(X^{S*}(t))}}\right]. \end{aligned}$$

(iii) The equilibrium instantaneous sales rate $x^{S*}(t)$ is given by

$$x^{S*}(t) = \frac{\bar{\phi}}{4}\sqrt{F^{S*}(X^{S*}(t))\,F^{S*}(X^{S*}(T))}.$$

(iv) $R^{S*}(t) = \sqrt{\frac{F^{S*}(X^{S*}(T))}{F^{S*}(X^{S*}(t))}} = \frac{x^{S*}(T)}{x^{S*}(t)}$.

(v) The retailer's and manufacturer's equilibrium (optimal) instantaneous profits are given by

$$\begin{aligned} \pi_{R}^{S*}(t) &= \frac{\bar{\phi}^{2} F^{S*}(X^{S*}(T))}{16\gamma} + KO \\ \pi_{M}^{S*}(t) &= \frac{\bar{\phi}^{2}}{8\gamma}\left(2\sqrt{F^{S*}(X^{S*}(t))\,F^{S*}(X^{S*}(T))} - F^{S*}(X^{S*}(T))\right). \end{aligned}$$

Proof: (i) Substituting the result from Lemma 2.1 into Equation (2.28), we have

$$\begin{aligned}\dot{\lambda}^s(t) &= -\frac{f^{s*}(x^{s*}(t))\left[1+\gamma\left(\lambda^s(t)-s-O-c_0\right)\right]^2}{8\gamma} \\ &= -\frac{\bar{\phi}^2 f^{s*}(x^{s*}(t))}{8\gamma}\frac{F^{s*}(x^{s*}(T))}{F^{s*}(x^{s*}(t))}.\end{aligned}$$

(ii) Substituting (2.29) into (2.25) and (2.26), we have

$$r^{s*}(t) = \frac{1}{4\gamma}\left[4-\bar{\phi}\sqrt{\frac{F^{s*}(x^{s*}(T))}{F^{s*}(x^{s*}(t))}}\right]$$

and

$$w^{s*}(t) = \frac{1}{2\gamma}\left[2\left(1-\gamma\left(s+O\right)\right)-\bar{\phi}\sqrt{\frac{F^{s*}(x^{s*}(T))}{F^{s*}(x^{s*}(t))}}\right].$$

(iii) Substituting (2.29) into (2.27), we have

$$x^{s*}(t) = \frac{\bar{\phi}}{4}\sqrt{F^{s*}(x^{s*}(t))\,F^{s*}(x^{s*}(T))}.$$

(iv) From (iii), we have the results.

(v) We have

$$\pi_R^{s*}(t) = \left(r^{s*}-w^{s*}-s\right)x^{s*}+O\left(K-x^{s*}\right)$$

and

$$\pi_M^{s*}(t) = \left(w^{s*}-c_0\right)x^{s*}.$$

Substituting the results in (ii) and (iii) into π_R^{S*}, we have

$$\pi_R^{S*}(t) = \frac{\bar{\phi}^2 F^{S*}\left(x^{S*}(T)\right)}{16\gamma} + KO.$$

We have $\pi_M^{S*}(t) = (w^{S*} - c_0)\, x^{S*}$. Substituting the results in (ii) and (iii) into $\pi_M^{S*}(t)$, we have

$$\pi_M^{S*}(t) = \frac{\bar{\phi}^2}{8\gamma}\left[2\sqrt{F^{S*}\left(x^{S*}(t)\right) F^{S*}\left(x^{S*}(T)\right)} - F^{S*}\left(x^{S*}(T)\right)\right].$$

REMARKS. According to part (i), the sign of $\dot{\lambda}^S(t)$ is determined by the sign of $f^{S*}\left(X^{S*}(t)\right)$: we have $\dot{\lambda}^S(t) > 0$ when

$$f^S\left(X^{S*}(t)\right) = -\alpha + M\beta - 2\beta X^{S*}(t) < 0$$

, *i.e.*, $X^{S*}(t) > \frac{-\alpha+M\beta}{2\beta}$, and $\dot{\lambda}^S(t) < 0$, when $X^{S*}(t) < \frac{-\alpha+M\beta}{2\beta}$. This result is consistent with that in the case of a retailer with a long-term profitability strategy. According to part (iv), $R^{S*}(t)$ is equal to the ratio of the end-of-horizon instantaneous sales rate to the instantaneous sales rate at any instant t.

According to part (v), the retailer achieves a constant instantaneous profit rate over time. This result is surprising because both the instantaneous sales volume $x^{S*}(t)$ and the retail margin vary over time. Our assumption of multiplicativly separable demand function partially contributes to this result. On the other hand, the manufacturer's instantaneous profit rate varies over time.

PROPOSITION 2.1. *The optimal retail price r^{S*}, wholesale price w^{S*} and instantaneous sales rate $\dot{X}^{S*}$peak at the same time. We can observe three retail/ wholesale pricing patterns, depending on the system parameters.*

(i) When $X^{S*}(T) < \frac{M}{2} - \frac{\alpha}{2\beta}$, $r^{S*}(t)$ and $w^{S*}(t)$ are both monotonically increasing over the entire selling horizon.

(ii) When $X^{S*}(T) > \frac{M}{2} - \frac{\alpha}{2\beta} > X^{S}(0)$, $r^{S*}(t)$ and $w^{S*}(t)$ are increasing up to the peak sales and decreasing thereon.

(iii) When $X^{S}(0) > \frac{M}{2} - \frac{\alpha}{2\beta}$, $r^{S*}(t)$ and $w^{S*}(t)$ are monotonically decreasing over the entire selling horizon.

Proof: Examing the results in Lemma 2.2(ii) and 2.2(iii), r^{S*}, w^{S*} and x^{S*}peak when $\sqrt{\frac{F^{S*}(X^{S*}(T))}{F^{S*}(X^{S*}(t))}}$ achieves its minimum.

(i) When $X^{S*}(T) \leq \frac{M}{2} - \frac{\alpha}{2\beta}$, $F^{S*}(X^{S*}(t))$ is an increasing function of $X^{S*}(t)$ throughout the horizon with $t \in [0, T]$. $\sqrt{\frac{F^{S*}(X^{S*}(T))}{F^{S*}(X^{S*}(t))}}$ decreases over time t. From Lemma 2.2(ii),$r^{S*}$and $w^{S*}$are both increasing over the entire selling horizon.

(ii) When $X^{S*}(T) \geq \frac{M}{2} - \frac{\alpha}{2\beta}$, $F^{S*}(X^{S*}(t))$ is an increasing function of $X^{S*}(t)$ for $t \in [0, t_1]$, where t_1 is such that $X(t_1) = \frac{M}{2} - \frac{\alpha}{2\beta}$, while $F^{S*}(X^{S*}(t))$ is a decreasing function of $X^{S*}(t)$ throughout the horizon for $t \in [t_1, T]$. Therefore r^{S*} and w^{S*} increase up to $t_1$then decrease.

(iii) When $X_0 > \frac{M}{2} - \frac{\alpha}{2\beta}$, $F^{S*}(X^{S*}(t))$ is an decreasing function of $X^{S*}(t)$ throughout the entire horizon. Therefore r^{S*} and w^{S*} are monotonically decreasing over the entire horizon.

Proposition 2.1 states that we may observe three different patterns of retail and wholesale prices: monotonically increasing, increasing then declining, and monotonically declining. The ultimate determinant of pricing patterns is the interaction between the demand dynamics: the diffusion effect (word-of-mouth) and the saturation effect. When the market saturation level is low, the word-of-mouth effect stimulates sales, i.e., the diffusion effect outweighs the saturation effect, the retailer (manufacturer) will start with relatively low price to stimulate early sales. As the sales grows, the saturation effect takes its turn, the retailer (manufacturer) will price high to capture the immediate profit rather than sacrificing current profits for future profits. This result is a generalization of Kalish (1983). Kalish examined the optimal dynamic pricing strategy within a centralized channel setting. Here we show that with a short-term retailer profitability strategy, the optimal retail price and wholesale price patterns should follow the sales curves. In contrast, in the decentralized channel with a long-term retailer profitability strategy, neither the retail price nor the wholesale price pattern mimic the sales curves as illustrated by Figures 2.1 and 2.2.

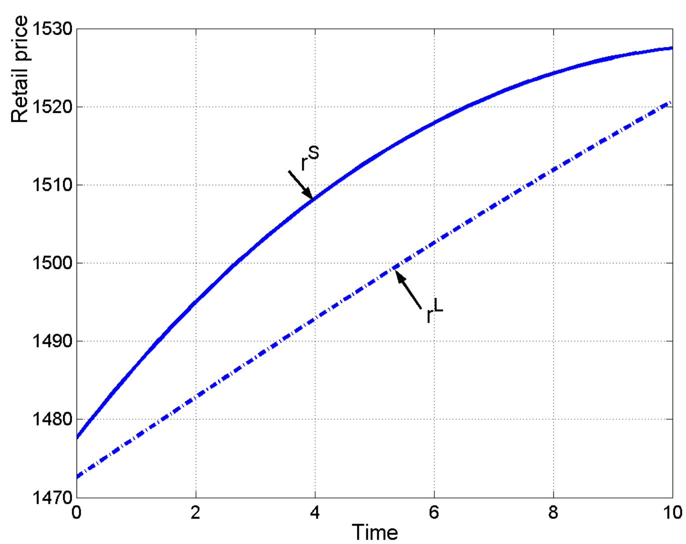

Figure 2.1: Retail price under long-term and short-term strategies

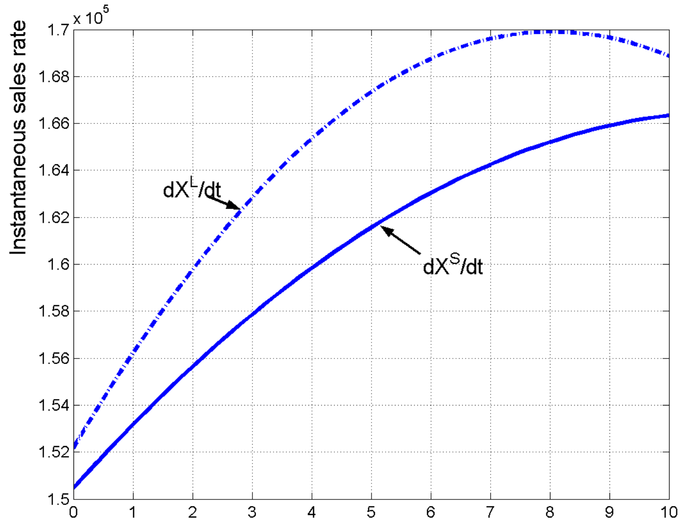

Figure 2.2: Instantaneous sales rate under long-term and short-term strategies

Note: the parameters are $M = 4 \times 10^7, X_0 = 1 \times 10^7, \alpha = 0.016, \beta = 8 \times 10^{-9}, \gamma = 5 \times 10^{-4}, c_0 = \$100, s = \$10, O = 0, T = 10$.

From Lemma 2.2, we have demonstrated the importance of obtaining the cumulative sales trajectory $X^{s*}(t)$. The following provides a method to calculate the equilibrium cumulative sales up to time t, $X^{s*}(t)$.

Lemma 2.3. *The equilibrium (optimal) cumulative sales trajectory $X^{s*}(t)$ can be determined by the unique solution to the following equation:*

$$\tan^{-1}\left[\frac{f^{s*}(X^{s*}(t))}{2\sqrt{\beta F^{s*}(X^{s*}(t))}}\right] = \tan^{-1}\left[\frac{f^{s}(X^{s}(0))}{2\sqrt{\beta F^{s}(X^{s}(0))}}\right] - \frac{\bar{\phi}t\sqrt{\beta F^{s*}(X^{s*}(T))}}{4} \tag{2.30}$$

Proof: Define a function $G = -\frac{f^S}{\sqrt{F^S}}$. We can show that $G(t)$ is an increasing function of $X^s(t)$ for $t \in [0, T]$ by taking the first derivative of $G(t)$ with regard to

$X^s(t)$:

$$\frac{dG}{dX^s} = \frac{\beta(\alpha+\beta X^s)^2}{2(F^s)^{3/2}} > 0.$$

The proof of Lemma 2.3 will proceed in two steps. First we show that Equation (2.30) holds. Second, we show the uniqueness of solution. The second step itself includes 3 substeps as we will see shortly. From Lemma 2.2, we have

$$\begin{aligned} x^{s*}(t) &= \frac{\bar{\phi}}{4}\sqrt{F^{s*}(X^{s*}(t))\,F^{s*}(X^{s*}(T))} \\ \Longrightarrow \frac{dX^{s*}(t)}{\sqrt{F^{s*}(X^{s*}(t))}} &= \frac{\bar{\phi}\sqrt{F^{s*}(X^{s*}(T))}dt}{4}. \end{aligned}$$

Integrating the left hand side of equation from $X^s(0)$ to $X^{s*}(t)$ and right hand side from 0 to t, we have

$$\int_{X^S(0)}^{X^{S*}(t)} \frac{dX^{s*}(\tau)}{\sqrt{F^{s*}(X^{s*}(\tau))}} = \frac{\bar{\phi}\sqrt{F^{s*}(X^{s*}(T))}}{4}\int_0^t d\tau$$

$$\frac{1}{\sqrt{\beta}}\tan^{-1}\left[\frac{\alpha-\beta M+2\beta X^{s*}(\tau)}{2\sqrt{\beta F^{s*}(X^{s*}(\tau))}}\right]_{\tau=0}^{\tau=t} = \frac{\bar{\phi}t\sqrt{F^{s*}(X^{s*}(T))}}{4}$$

Rearranging the terms, we will get the result in (2.30).

Next we shall show the uniqueness of solution to (2.30). In the first substep, we show that the following equation provides a unique solution for $X^{s*}(T)$:

$$\tan^{-1}\frac{f^{s*}(X^{s*}(T))}{2\sqrt{\beta F^{s*}(X^{s*}(T))}} = \tan^{-1}\frac{f^{s}(X^{s}(0))}{2\sqrt{\beta F^{s}(X^{s}(0))}} - \frac{\bar{\phi}T\sqrt{\beta F^{s*}(X^{s*}(T))}}{4}.$$

Define the following functions:

$$
\begin{aligned}
F_1(X) &= -\tan^{-1}\frac{f^{s*}(X)}{2\sqrt{\beta F^{s*}(X)}} + \tan^{-1}\frac{f^{s}(X^{s}(0))}{2\sqrt{\beta F^{s}(X^{s}(0))}};\\
F_2(X) &= \frac{T\bar{\phi}\sqrt{\beta F^{s*}(X)}}{4};\\
\triangle F &= F_1(X) - F_2(X).
\end{aligned}
$$

where $F^{s*}(X) = (M - X)(\alpha + \beta X)$ and $f^{s*}(X) = -\alpha + \beta M - 2\beta X$. We have shown that have G is a strictly increasing function of X, hence F_1 is strictly increasing in X as well. So the uniqueness of solution can be easily extended to any instant t. We will focus on proving the uniqueness of the solution to (2.30). We will proceed the proof in three steps listed as follows:

Step One: Prove that there is at least one solution to the equation;

Step Two: Prove that there is at most one extreme point for $\triangle F$;

Step Three: Prove F_1 and F_2 intersect with each other either within $\left(X^{s}(0), \frac{M}{2} - \frac{\alpha}{2\beta}\right)$ or within $\left(\frac{M}{2} - \frac{\alpha}{2\beta}, M\right)$, but not both.

Next we will prove this part according to the steps specified above.

Step One: Prove that there is at least one solution to the equation.

Checking the sign of $\triangle F$ at $X^{s}(0)$ and $X \to M$. We have $\lim_{X\to M} F_2(X) = 0$. Since F_1 is increasing in X, we have $\triangle F(X \to M) = \lim_{X\to M} \triangle F = \lim_{X\to M}(F_1 - F_2) = \lim_{X\to M} F_1 > 0$. At $X = X_0$,we have

$$
\triangle F(X = X_0) = -\frac{T\bar{\phi}\sqrt{\beta F^{s*}(X^{s}(0))}}{4} < 0.
$$

Therefore we have $\triangle F$ different signs at $X = X_0$ and $X \to M$. We conclude that there is at least one solution to equation (2.30) within $(X^{s}(0), M)$.

Step Two: Prove that there is at most one extreme point for $\triangle F$.

Taking the derivative of $\triangle F$ with regard to X, we have

$$\frac{d\triangle F}{dX} = \frac{\sqrt{\beta}}{\sqrt{F^{s*}(X)}} \frac{8 - T\bar{\phi} f^{s*}(X)}{8}.$$

The unique solution to $\frac{d\triangle F}{dX)} = 0$ is $X = \frac{M}{2} - \frac{\alpha}{2\beta} - \frac{4}{T\beta\bar{\phi}}$. Hence we conclude that there is at least one extreme point over $(X^s(0), M)$, depending on whether $\frac{M}{2} - \frac{\alpha}{2\beta} - \frac{4}{T\beta\bar{\phi}}$ is within or out of $(X^s(0), M)$.

Step Three: Prove that there is a unique solution.

It is easy to show that F_2 is an increasing function of X for an arbitrary $X \in \left(X^s(0), \frac{M}{2} - \frac{\alpha}{2\beta}\right)$ and an decreasing function of X for $X \in \left(\frac{M}{2} - \frac{\alpha}{2\beta}, M\right)$. We will show that F_1 and F_2 intersect with each other only once, i.e, they either intersect within the region $\left(X^s(0), \frac{M}{2} - \frac{\alpha}{2\beta}\right)$ or within $\left(\frac{M}{2} - \frac{\alpha}{2\beta}, M\right)$, but not both.

We will check the sign of the function $\triangle F$ at three points, i.e., $X^s(0)$, $\frac{M}{2} - \frac{\alpha}{2\beta}$ and M:

$$\triangle F\left(\frac{M}{2} - \frac{\alpha}{2\beta}\right) = \tan^{-1} \frac{f^s(X^s(0))}{\sqrt{2\beta F^s(X^s(0))}} - \frac{T\bar{\phi}(\alpha + \beta M)}{2}$$

Note that for $\triangle F\left(\frac{M}{2} - \frac{\alpha}{2\beta}\right) > 0$ to hold, we need

$$\begin{aligned} & \tan^{-1} \frac{f^s(X^s(0))}{\sqrt{\beta F^s(X^s(0))}} - \frac{T\bar{\phi}(\alpha + \beta M)}{2} > 0 \\ \Longleftrightarrow \quad & \frac{f^s(X^s(0))}{\sqrt{\beta F^s(X^s(0))}} > \tan\left[\frac{T\bar{\phi}(\alpha + \beta M)}{2}\right] \end{aligned}$$

When $\frac{f^s(X^s(0))}{\sqrt{\beta F^s(X^s(0))}} < \tan\left[\frac{T\bar{\phi}(\alpha+\beta M)}{2}\right]$ holds, $\triangle F(X = M)$ and $\triangle F\left(X = \frac{M}{2} - \frac{\alpha}{2\beta}\right)$ have different signs. Because there is no extreme point, there is only one intersection only.

When $\frac{f^S(X^S(0))}{\sqrt{\beta F^S(X^S(0))}} > \tan\left[\frac{T\bar{\phi}(\alpha+\beta M)}{2}\right]$ holds, $\triangle F\left(X = X_0\right)$ and $\triangle F\left(X = \frac{M}{2} - \frac{\alpha}{2\beta}\right)$ have different signs. We know that the extreme point occurs at $X^S = \frac{M}{2} - \frac{\alpha}{2\beta} - \frac{4}{T\beta\bar{\phi}}$. If $X^S(0) < \frac{M}{2} - \frac{\alpha}{2\beta} - \frac{4}{T\beta\bar{\phi}} < \frac{M}{2} - \frac{\alpha}{2\beta}$, then there is an extreme point within $(X^S(0), M)$, otherwise there is no extreme point. Furthermore, if

$$Sign\left[\triangle F\left(X = \frac{M}{2} - \frac{\alpha}{2\beta} - \frac{4}{T\beta\bar{\phi}}\right)\right] = Sign\left[\triangle F\left(X = X_0\right)\right],$$

the intersection must fall in $\left(X(0), \frac{M}{2} - \frac{\alpha}{2\beta} - \frac{4}{T\beta\bar{\phi}}\right)$; otherwise, if

$$Sign\left[\triangle F\left(X = \frac{M}{2} - \frac{\alpha}{2\beta} - \frac{4}{T\beta\bar{\phi}}\right)\right] = Sign\left[\triangle F\left(X = \frac{M}{2} - \frac{\alpha}{2\beta}\right)\right],$$

then the intersection must fall in $\left(\frac{M}{2} - \frac{\alpha}{2\beta} - \frac{4}{T\beta\bar{\phi}}, M\right)$.

Now we have proved that there is unique solution of $X^{S*}(T)$. We can show that there is unique solution of $X^{S*}(t)$. $\square$.

Managerially, Lemma 2.3 is very useful for two major reasons. First, it is useful for forecasting purposes. The manufacturer can assess the life time cumulative sales $X^{S*}(T)$ using (30) to solve for the unique solution. Note that $X^{S*}(T)$ is a function of initial sales $X^S(0)$ and parameters of the problem. Once the life time cumulative sales is obtained, the cumulative sales trajectory $X^{S*}(t)$ for any given time t is determined as well as the wholesale and retail price trajectories. Second, the managers can plan for operations decisions such as production rate, production capacity, and distribution channel capacity installation at any instant t.

Let $\Pi^{S*}_{SC}(T)$ denote the channel profit up to time t, which is the sum of the manufacturer's and retailer's profits from selling the manufacturer's product. The following result characterizes the life cycle equilibrium (optimal) profits up to time

t, $\Pi_{R}^{S*}(t)$ and $\Pi_{M}^{S*}(t)$ and $\pi_{SC}^{S*}(T)$, in terms of the cumulative sales $X^{S*}(t)$.

Lemma 2.4. *Let $\triangle X^{S*}(t) = X^{S*}(t) - X^{S}(0)$.*

(i) The manufacturer's optimal cumulative profit up to time t, $\Pi_{M}^{S*}(t)$, and retailer's optimal cumulative profit up to time t, $\Pi_{R}^{S*}(t)$, are are given by:

$$\Pi_{M}^{S*}(t) = \frac{\bar{\phi}\triangle X^{S*}(t)}{\gamma} - \frac{\bar{\phi}^2 t F^{S*}(T)}{8\gamma},$$
$$\Pi_{R}^{S*}(t) = \frac{\bar{\phi}^2 t F^{S*}(T)}{16\gamma} + tOK;$$

$\Pi_{M}^{S*}(t)$ is a decreasing function of O, i.e., $\frac{\partial \Pi_{M}^{S*}(t)}{\partial O} < 0$.

(ii) The life-cycle supply chain profits $\Pi_{SC}^{S*}(T)$ is given by

$$\Pi_{SC}^{S*}(T) = \frac{[1-\gamma(c_0+s)]\triangle X^{S*}(T)}{\gamma} - \frac{\bar{\phi}^2 T F^{S*}(X^{S*}(T))}{16\gamma}.$$

Proof: 2.4(i) From Lemma 2.2, we have $\pi_{R}^{S*}(t) = \frac{\bar{\phi}^2 F^{S*}(x^{S*}(T))}{16\gamma} + KO$. We integrate $\pi_{R}^{S*}(t)$ from time 0 to t to obtain the retailer's cumulative profit up to time t as $\Pi_{R}^{S*}(t) = \frac{t\bar{\phi}^2 F^{S*}(x^{S*}(T))}{16\gamma} + tKO$. From Lemma 2.2(iii), we have $x^{S*}(t) = \frac{\bar{\phi}}{4}\sqrt{F^{S*}(x^{S*}(t))\,F^{S*}(x^{S*}(T))}$. Integrating from 0 to t, we have

$$\int_0^t x^{S*}(\tau)\,d\tau = \frac{\bar{\phi}}{4}\sqrt{F^{S*}(x^{S*}(T))}\int_0^t \sqrt{F^{S*}(x^{S*}(\tau))}d\tau$$
$$\Longrightarrow \int_0^t \sqrt{F^{S*}(x^{S*}(\tau))}d\tau = \frac{4\left[X^{S*}(t) - X^{S}(0)\right]}{\bar{\phi}\sqrt{F^{S*}(x^{S*}(T))}}$$

From Lemma 2.2(v), we have

$$\pi_{M}^{S*}(t) = \frac{\bar{\phi}^2}{8\gamma}\left[2\sqrt{F^{S*}(x^{S*}(t))\,F^{S*}(x^{S*}(T))} - F^{S*}(x^{S*}(T))\right].$$

Integrating from 0 to t, we have:

$$\begin{aligned}
\Pi_{M}^{s*}(T) = \int_0^t \pi_{M}^{s*}(\tau)\,d\tau \quad &= \frac{\bar{\phi}^2\sqrt{F^{s*}(x^{s*}(T))}}{4\gamma}\int_0^t \sqrt{F^{s*}(x^{s*}(\tau))}d\tau - \frac{t\bar{\phi}^2 F^{s*}(x^{s*}(T))}{8\gamma} \\
&= \frac{\bar{\phi}^2\sqrt{F^{s*}(x^{s*}(T))}}{4\gamma}\frac{4\left[X^{s*}(t) - X^{s}(0)\right]}{\bar{\phi}\sqrt{F^{s*}(x^{s*}(T))}} - \frac{t\bar{\phi}^2 F^{s*}(x^{s*}(T))}{8\gamma} \\
&= \frac{\bar{\phi}\left[X^{s*}(t) - X^{s}(0)\right]}{\gamma} - \frac{t\bar{\phi}^2 F^{s*}(X^{s*}(T))}{8\gamma}
\end{aligned}$$

From (2.30), for $t = T$, we have:

$$\tan^{-1}\frac{f^{s*}(X^{s*}(T))}{2\sqrt{\beta F^{s*}(X^{s*}(T))}} = \tan^{-1}\frac{f^{s}(X^{s}(0))}{2\sqrt{\beta F^{s}(X^{s}(0))}} - \frac{\bar{\phi}T\sqrt{\beta F^{s*}(X^{s*}(T))}}{4}.$$

Taking the derivative of both sides of (2.30) with regard to O, we have:

$$\begin{aligned}
& \frac{1}{\sqrt{F^{s*}(X^{s*}(T))}}\frac{\partial X^{s*}(T)}{\partial O} = -\frac{\gamma T\sqrt{F^{s*}(X^{s*}(T))}}{4} + \frac{\bar{\phi}T f^{s*}(X^{s*}(T))}{8\sqrt{F^{s*}(X^{s*}(T))}}\frac{\partial X^{s*}(T)}{\partial O} \\
\Longrightarrow \quad & \frac{\partial X^{s*}(T)}{\partial O} = -\frac{\gamma T F^{s*}(X^{s*}(T))}{4} + \frac{\bar{\phi}T f^{s*}(X^{s*}(T))}{8}\frac{\partial X^{s*}(T)}{\partial O} \\
\Longrightarrow \quad & \frac{\partial X^{s*}(T)}{\partial O}\left[1 - \frac{\bar{\phi}T f^{s*}(X^{s*}(T))}{8}\right] = -\frac{\gamma T F^{s*}(X^{s*}(T))}{4} \\
\Longrightarrow \quad & \frac{\partial X^{s*}(T)}{\partial O} = -\frac{\gamma T F^{s*}(X^{s*}(T))}{8 - \bar{\phi}T f^{s*}(X^{s*}(T))}
\end{aligned}$$

Taking the derivative of $\Pi_{M}^{s*}(T)$ with regard to O, we have:

$$\begin{aligned}
\frac{\partial \Pi_{M}^{s*}(T)}{\partial O} \quad &= -\triangle X^{s*}(T) + \frac{\bar{\phi}}{\gamma}\frac{\partial X^{s*}(T)}{\partial O} + \frac{T\bar{\phi}F^{s*}(X^{s*}(T))}{4} - \frac{T\bar{\phi}^2 f^{s*}(X^{s*}(T))}{8\gamma}\frac{\partial X^{s*}(T)}{\partial O} \\
&= -\triangle X^{s*}(T) + \frac{\bar{\phi}}{\gamma}\frac{\partial X^{s*}(T)}{\partial O}\left(1 - \frac{T\bar{\phi}^2 f^{s*}(X^{s*}(T))}{8\gamma}\right) + \frac{T\bar{\phi}F^{s*}(X^{s*}(T))}{4} \\
&= -\triangle X^{s*}(T) < 0
\end{aligned}$$

2.4(ii) From the definition, we have

$$\begin{aligned}
\Pi_{SC}^{s*}(T) &= \Pi_{M}^{s*}(T) + \Pi_{R}^{s*}(T) - \int_0^T (K - x^{s*})\,Odt \\
&= \Pi_{M}^{s*}(T) + \Pi_{R}^{s*}(T) - KOT + O\triangle X^{s*}(T) \\
&= \frac{\bar{\phi}\triangle X^{s*}(T)}{\gamma} - \frac{T\bar{\phi}^2 F^{s*}(X^{s*}(T))}{8\gamma} + \frac{T\bar{\phi}^2 F^{s*}(X^{s*}(T))}{16\gamma} \\
&+KOT - KOT + O\triangle X^{s*}(T) \\
&= \frac{[1-\gamma(c_0+s)]\triangle X^{s*}(T)}{\gamma} - \frac{T\bar{\phi}^2 F^{s*}(X^{s*}(T))}{16\gamma}
\end{aligned}$$

We conclude this section by pointing out that the retailer's strategy of having a short-term strategy may or may not be optimal from the retailer's perspective, as will be discussed in the following section.

2.6 Numerical Analysis: Long-term Focus versus Short-term Focus

We have analyzed the models with both the long-term and short-term retailer profitability strategies. Our analysis so far leaves open the following questions: Will the retailer and manufacturer have conflicts over the preferred retailer profitability strategy? If so, under what conditions will they have conflicts? A priori, it is not clear whether the long-term profitability objective will be preferred when both the retailer and manufacturer solve their global optimization problem. We must recognize that this sequential local maximization will not necessarily lead to a better global solution than if the retailer has a short term optimization focus. Let i and j denote the manufacturer's and retailer's preferences, respectively. Accordingly, there are four possible combinations of preferences, i.e., $\{i, j\} = \{L, L\}, \{L, S\}, \{S, L\}, \{S, S\}$.

Preferences Over the Retailer's Profitability Strageties. We conducted a numerical study with different values of T and β. Table 2.1 reports the preferred retailer strategy, market saturation level, and profit gain with different combinations of T and β. We have a few observations. First, we can observe all four combinations of preferences. In some cases, the manufacturer and the retailer have aligned preferences over the retailer profitability strategy while in other cases, they may have conflicts. Second, for a fixed value of β, preferences shift in the following order: $\{L, L\}, \{L, S\}, \{S, S\}$ and $\{S, L\}$ as T increases. Moreover, for a fixed value of T, preferences shifts in the same order as β increases. Third, the preferences of both parties depend on the level of market saturation at the end of the selling horizon. Finally, the conflicts between the manufacturer and the retailer can be significant when the market is highly saturated at the end of the selling horizon, i.e., when either T or β or both are very large. For example, when the market penetration level is over 80%, the manufacturer's profit gain is between 10% and 18% with his preferred profitability strategy while the retailer's profit gain is more significant: it can be from 13% to 115%. However, when the market is not highly saturated, the conflict is far less significant.

In a decentralized channel, due to the differences in the cost structure, the manufacturer and the retailer see different future benefits of current sales therefore they make a different adjustment of their pricing strategies when they are trading off current with future profits. For both the manufacturer and the retailer, shadow prices represent the future value of an additional sale. The determination of the magnitude of the shadow prices and therefore the trade-offs depends on the market saturation level which itself depends on the particular parameter settings such as T,

α and β. When the combination of these parameters leads to high levels of market penetration, the saturation effect will reduce the retailer's shadow price eventually reaching negative levels; this saturation effect will lead the retailer to increase her profit margins. A higher retail price is not in the manufacturer's interest therefore the manufacturer will switch his preference to a "myopic" retailer profitability strategy when the market reaches high enough saturation level at the end of the horizon.

The retailer's preferences are more complex to explain because she needs to balance (a) the cost due to the myopic pricing decision and (b) the gain, in terms of favorable wholesale price terms received from the manufacturer under this myopic retailer strategy. When the window of opportunity to sell the IDP leads to a low level of market saturation, the manufacturer is interested in stimulating the diffusion of the IDP and he offers favorable wholesale price terms when the retailer cooperates with his diffusion goal by lowering the retail price (a long-term profitability strategy). When the window of opportunity is long enough, the manufacturer's wholesale price terms are not attractive to the retailer and she switches her preference to a short-term profitability strategy. When the market saturation level is extremely high, the retailer knows additional sales will be very low so she is interested in "milking" the market and switches her preference to a long-term profitability strategy (in this case the shadow price is negative so she will increase her profit margins) regardless of the manufacturer's wholesale price increase as a reaction to her long-term optimization preference.

Table 2.1: Prefered retailer focus, market saturation, and profit gain

T	$\beta = 2.5\times10^{-9}$			$\beta = 8\times10^{-9}$			$\beta = 2\times10^{-8}$			$\beta = 5\times10^{-8}$		
	P	SL	G	P	S	G	P	SL	G	P	SL	G
1	–	–	–	$\{L,L\}$	$(27,27)$	<1	$\{L,L\}$	$(29,29)$	$(2,2)$	$\{L,L\}$	$(36,36)$	$(5,3)$
2	–	–	–	$\{L,L\}$	$(29,29)$	$(2,1)$	$\{L,L\}$	$(34,33)$	$(4,3)$	$\{L,S\}$	$(49,48)$	$(5,4)$
5	$\{L,L\}$	$(28,28)$	$(3,1)$	$\{L,L\}$	$(35,34)$	$(3,2)$	$\{L,S\}$	$(49,48)$	$(4,4)$	$\{S,S\}$	$(74,74)$	$(5,19)$
10	$\{L,L\}$	$(32,32)$	$(5,1)$	$\{L,S\}$	$(45,45)$	$(4,2)$	$\{S,S\}$	$(68,68)$	$(2,17)$	$\{S,S\}$	$(90,90)$	$(12,10)$
25	$\{L,S\}$	$(44,45)$	$(1,3)$	$\{S,S\}$	$(70,70)$	$(4,17)$	$\{S,S\}$	$(88,88)$	$(12,9)$	$\{S,L\}$	$(91,98)$	$(13,23)$
100	$\{S,S\}$	$(80,80)$	$(10,13)$	$\{S,L\}$	$(87,95)$	$(14,8)$	$\{S,L\}$	$(94,99)$	$(11,50)$	$\{S,L\}$	$(97,99)$	$(18,115)$
200	$\{S,L\}$	$(83,92)$	$(14,1)$	–	–	–	–	–	–	–	–	–

Notes. Parameters are $M = 4\times10^7$, $X_0 = 1\times10^7$, $\alpha = 0.016$, $k = 5\times10^{-4}$, $c_0 = \$100$, *and* $O = 0$. Let $P = \{P_M, P_R\}$ denotes the manufacturer and retailer's preferred profitability foci and $SL = \{SL_M, Sl_R\}$ denote the market saturation level at the end of the selling horizon, i.e., $SL_{M(R)} = \frac{X_{M(R)}(T)}{M}\times 100\%$. Let $\bar{P} = \left(\bar{P}_M, \bar{P}_R\right)$ denote the combination of the manufacturer's and retailer's unpreferred profitability foci. Let $G = \{G_M, G_R\}$ and $G_{M(R)} = \left(\frac{\pi^P_{M(R)}(T)}{\pi^{\bar{P}}_{M(R)}(T)} - 1\right)\times$ 100%, i.e., the percentage of the profit gain to the manufacturer (retailer) with a preferred retailer profitability strategy, where $\pi^P_{M(R)}(T)$ denote the manufacturer (retailer) 's life-cycle profit under his (her) preferred retailer profitability strategy.

Wholesale Price Contracts Implementation Issues. Clearly, it is important to understand what the resulting wholesale price contracts will be under different scenarios. For scenarios $\{L,L\}$ and $\{L,S\}$, the manufacturer offers the wholesale price contract assuming that the retailer will be farsighted and he may safely do so because for a given wholesale price strategy, the retailer is better off being far-sighted no matter her preferences are. For the scenarios $\{S,S\}$ and $\{S,L\}$, the manufacturer offers the wholesale price contract assuming that the retailer will be myopic. However, in order to implement these wholesale price contracts, the manufacturer needs to monitor the retailer's sales volume or retail price at any instant of time. Without monitoring, for a given wholesale price, the retailer is better off setting the retail price with a long-term perspective. Note that for scenarios $\{L, S\}$ and $\{S, L\}$, the resulting wholesale price contracts are consistent with the manufacturer's preferences, but not in the retailer's first preference, so there may be some implementation resistance from the retailer.

Although larger $\alpha$and β lead to increased life-cycle supply chain profits, the effect of larger values of $\alpha$and β on the split of this profit between the manufacturer and the retailer is not straightforward. When the selling window leads to a low level of market saturation, the retailer will tend to capture a higher share of the profit than when the selling window allow high levels of market saturation.

2.7 Two-part Contracts

Without additional transfers of profits from the manufacturer to the VAR, the VAR will consider the opportunity cost O of diverting sale resources from alternative products to the IDP when making pricing decisions (see (2.1) and (2.21)). We shall demonstrate in this section that the existence of alternative products in the retailer's portfolio competing with the IDP for sales resources, creates for the retailer a flexibility in allocation of resources, which hurts the manufacturer as well as the IDP's supply chain. As a possibility to overcome this problem, we consider a lump-sum up-front payment A paid by the manufacturer to the retailer for the exclusive use of sale resources to the IDP. Once these resources are contractually allocated to the IDP, the retailer foregoes the possibility of using these resources for an alternative product, making the opportunity cost of dedicating these resources to the IDP effectively equal to zero. We assume that this contractual agreement is initiated by the manufacturer. If the retailer accepts the terms of trade with an upfront fee, she entirely dedicates the K units of resources to the manufacturer.

We assume that the manufacturer knows the profitability objective of the retailer, i.e., either a long-term or a short-term strategy. The sequence of events is as follows: At date 1, the manufacturer launches a new product and proposes two terms of trade (contracts), a two-part contract, i.e., a wholesale price coupled with an up-front fee, $\{A, w(t)\}$, and a wholesale-price-only contract, $\{w(t)\}$. At date 2, the retailer decides which contract to choose. At date 3, she decides the retail price path and thereby her instantaneous sales rate, depending on the contract she chose at date 2. At date 4, production and trade occur. We assume that when the retailer is indifferent between the two contracts, she will choose the one with an up-front fee.

2.7.1 Two-part Contracts with a Long-term Focus

Let $\Pi_R^{LD*}(T)$ be the life cycle profits obtained by the retailer who accepts the two-part contract $\{A, w(t)\}$ to dedicate her selling resources to the manufacturer's IDP. Note that $\Pi_R^{LD*}(T)$ can be obtained by solving (1) with $O = 0$. A rational retailer with a long-term strategy accepts a two-part contract with an upfront payment A only if she earns no less than she does under the wholesale-price-only contract (Incentive Compatibility Constraint), mathematically,

$$\Pi_R^{LD*}(T) + A \geq \Pi_R^{L*}(T). \tag{2.31}$$

Given her acceptance of the two-part contract, the retailer regards the outside profit margin O and the allowance A as sunk costs and foregoes them when setting her dynamical retail price strategy.

The manufacturer decides whether to induce the dedicated retail resources. He needs to compare the maximum profit he earns with a two-part contract to that with a wholesale price contract. If it is profitable for him to induce the retailer to accept the up-front payment, the manufacturer would extract the entire gain in channel profit, i.e., the manufacturer will make the retailer break-even between the two-part contract and wholesale-price-only contract. The manufacturer will offer the retailer an up-front fee $A^{BEL} = \Pi_R^{L*}(T) - \Pi_M^{LD*}(T)$ coupled with the corresponding transfer price $w^{LD*}(t)$ if it is to his advantage or just a transfer price contract $w^{L*}(t)$. Let $\pi_M^{LD*}(T)$ and $\pi_M^{L*}(T)$ be the manufacturer's profits with dedicated and flexible resources, respectively. The manufacturer offers an up-front fee only if he is not worse off than with a wholesale-price alone contract, i.e., $\Pi_M^{LD*}(T) - A^{BEL} \geq \Pi_M^{L*}(T)$.

2.7.2 Two-part Contracts with a Short-term Focus

Let $\Pi_R^{SD*}(T)$ be the retailer's life cycle profit with a short-term profitability strategy who accepts the two-part contract $\{A, w(t)\}$. The analysis in the section parallels that of a retailer with a long-term profitability strategy. Her participation constraint for a two-part contract is:

$$\Pi_R^{SD*}(T) + A \geq \Pi_R^{S*}(T), \tag{2.32}$$

Let $\pi_R^{SD*}(T)$ be the manufacturer's profit from the sales of the product (with the dedicated resources). The two-part contract offered to the retailer will be $\{A^S, w^{SD*}(t)\}$ where $A^{BES} = \Pi_R^{S*}(T) - \Pi_R^{SD*}(T)$. Alternatively, if $\Pi_M^{SD*}(T) - A^{BES} < \Pi_M^{S*}(T)$, the manufacturer will simply offer a transfer price contract $w^{S*}(t)$. Let $\phi =$

$1-\gamma\left(c_0+s\right)$, a function of parameters.

Proposition 2.2. *Given the acceptance of* $\left\{A^{BES}, w^{SD*}(t)\right\}$*, where* $w^{SD*}(t)$ *is given by*

$$w^{SD*}(t)=\frac{1}{2\gamma}\left[2\left(1-\gamma s\right)-\phi\sqrt{\frac{F^{SD*}\left(X^{SD*}(T)\right)}{F^{SD*}\left(X^{SD*}(t)\right)}}\right]$$

and A^{BES} *is given by:*

$$A^{BES}=KOT+\frac{\bar{\phi}^2}{16\gamma}TF^{S*}\left(X^{S*}(T)\right)-\frac{\phi^2}{16\gamma}TF^{SD*}\left(X^{SD*}(T)\right);$$

(i) The retailer's cumulative life-cycle profit $\Pi_R^{SD*}(T)$ *is given by* $\Pi_R^{SD*}(T)=\frac{\phi^2 TF^{SD*}\left(X^{SD*}(T)\right)}{16\gamma}$.

(ii) The manufacturer's life-cycle profit $\Pi_M^{SD*}(T)$ *is given by*

$$\Pi_M^{SD*}(T)=\frac{\phi\triangle X^{S*}(T)}{\gamma}-\frac{\phi^2 TF^{SD*}(T)}{8\gamma}.$$

$\Pi_M^{SD*}(T)$ *is a decreasing function of* O*:* $\frac{\partial\Pi_M^{SD*}(T)}{\partial O}=-\triangle X^{SD*}(T)<0$.

Two-part contracts may improve the manufacturer-retailer channel efficiency; however, for many situations (e.g., a large outside profit margin), such contracts may not be feasible, as the additional profit cannot recoup the retailer's loss. In other words, the manufacturer cannot afford to pay a high up-front fee to secure the retailer's resources. Even in the situations that this two-part tariff contract is implementable, such a mechanism cannot achieve full channel coordination because it cannot overcome the double marginalization problem. In the next section, we explore the utilization of revenue sharing contracts to increase the supply chain performance.

2.8 Revenue Sharing Contracts

In this section, we use revenue sharing contracts to coordinate the channel. Subsection 1.8.1 determines the benchmark solution, i.e., the integrated channel's retail price, sales and profit rate trajectories. Subsection 1.8.2 studies the coordination with a long-term retailer profitability strategy. Subsection 1.8.3 studies the revenue sharing contract with a short-term retailer profitability strategy. Subsection 1.8.4 discusses the channel coordination implications of retailer profitability strategy and combined sharing contract with an upfront fee.

2.8.1 Integrated Channel

We now consider an integrated channel in which the manufacturer is the central decision maker. The channel maximizes the life cycle profit obtained from selling the IDP. The channel incurs a constant per unit production cost c_0 and a selling cost s. This problem corresponds to a specialization of the demand function in Kalish (1983). For this special case, we obtain an implicit expression for the optimal sales trajectory (Lemma 2.7) and we are able to express the optimal retail price trajectory (Lemma 2.6) and the optimal profit trajectory (Lemma 2.8) as functions of the cumulative sales. The integrated channel's profit maximization problem is given by

$$\Pi^{I*}(T) = \max_{r^I(t)} \int_0^T \left[r^I(t) - c_0 - s\right] \dot{X}^I(t)\, dt$$
$$s.t. \quad x^I = \left(M - X^I\right)\left(\alpha + \beta X^I\right)\left(1 - \gamma r^I\right), X^I(0) = X_0^I$$

Define $f^I = \frac{dF_I}{dX^I} = -\alpha + \beta M - 2\beta X^I$. Let $\lambda^I(t)$ denote the shadow price associated with the state variable X^I. Using a similar approach to the one applied in the previous several sections, we establish the relationship between the optimal cumulative sales X^{I*} and shadow price λ^{I} trajectories in the following lemma. Let $\phi = 1 - \gamma (c_0 + s)$ and assume that $\phi > 0$. Let $R^{I*}(t) = \sqrt{\frac{F^{I*}(X^{I*}(tT))}{F^{I*}(X^{I*}(t))}}$, where $F^{I*}(t) = (M - X^{I*}(t))(\alpha + \beta X^{I*}(t))$, $\forall t \in [0, T]$.

Lemma 2.5. *$\lambda^{I}(t)$ is given by*

$$\lambda^{I}(t) \quad = \quad \frac{\phi}{\gamma}\left(\sqrt{\frac{F^{I*}(X^{I*}(T))}{F^{I*}(X^{I*}(t))}} - 1\right), t \in [0, T].$$

Proof: Similar to the proofs of lemma 2.2.1 and 2, respectively.

Lemma 2.6. *For the integrated channel, the optimal retail price r^{I*}, the instantaneous sales rate x^{I*}, and the instantaneous profit rate π^{I*} are given by:*

$$r^{I*}(t) = \frac{1}{2\gamma}\left[2 - \phi\sqrt{\frac{F^{I*}(X^{I*}(T))}{F^{I*}(X^{I*}(t))}}\right]$$
$$x^{I*}(t) = \frac{\phi}{2}\sqrt{F^{I*}(X^{I*}(T))\,F^{I*}(X^{I*}(t))}$$
$$\pi^{I*}(t) = \frac{\phi^2}{4\gamma}\left[2\sqrt{F^{I*}(X^{I*}(T))\,F^{I*}(X^{I*}(t))} - F^{I*}(X^{I*}(T))\right]$$

Proof: Similar to the proofs of Lemma 2.1 and 2.2, respectively.

Lemma 2.7. *The optimal cumulative sales trajectory $X^{I*}(t)$ is determined by the unique solution to the following equation:*

$$\tan^{-1}\left[\frac{f^{I*}(X^{I*}(t))}{2\sqrt{\beta F^{I*}(X^{I*}(t))}}\right] = \tan^{-1}\left[\frac{f^{I}(X^{I}(0))}{2\sqrt{\beta F^{I}(X^{I}(0))}}\right] - \frac{\phi t\sqrt{\beta F^{I*}(X^{I*}(T))}}{2}$$

Using the results from Lemma 2.6 and 7, we can now establish the profit trajectory to the integrated channel.

Proof: From Lemma 2.6, we have

$$x^{I*}(t) = \frac{\phi}{2}\sqrt{F^{I*}\left(X^{I*}(t)\right)F^{I*}\left(X^{I*}(T)\right)}$$
$$\Longrightarrow \quad \frac{dX^{I*}(t)}{\sqrt{F^{I*}\left(X^{I*}(t)\right)}} = \frac{\phi\sqrt{F^{I*}\left(X^{I*}(T)\right)}dt}{2}$$

Integrating the left hand side of equation from $X^{I}(0)$ to $X^{I*}(t)$ and right hand side from 0 to t, we have

$$\int_{X^{I}(0)}^{X^{I*}(t)} \frac{dX^{I*}(\tau)}{\sqrt{F^{I*}\left(X^{I*}(\tau)\right)}} = \frac{\phi\sqrt{F^{I*}\left(X^{I*}(T)\right)}}{2}\int_{0}^{t} d\tau$$
$$\Longrightarrow \quad \frac{1}{\sqrt{\beta}}\tan^{-1}\left[\frac{\alpha - \beta M + 2\beta X^{I*}(\tau)}{2\sqrt{\beta F^{I*}\left(X^{I*}(\tau)\right)}}\right]_{\tau=0}^{\tau=t} = \frac{t\phi\sqrt{F^{I*}\left(X^{I*}(T)\right)}}{2}$$

Rearrange the above equation and we will get the equation in Lemma. $\square$.

Lemma 2.8. *Define* $\triangle X^{I*}(t) = X^{I*}(t) - X^{I}(0)$. *The optimal integrated channel's cumulative profit up to time* t, $\Pi^{I*}(t)$, *is given by:*

$$\Pi^{I*}(t) = \frac{\phi\triangle X^{I*}(t)}{\gamma} - \frac{t\phi^{2}F^{I*}\left(X^{I*}(T)\right)}{4\gamma}.$$

Proof: From Lemma 2.6, we have

$$x^{I*}(t) = \frac{\phi}{2}\sqrt{F^{I*}\left(X^{I*}(t)\right)F^{I*}\left(X^{I*}(T)\right)}$$
$$\Longrightarrow \quad \int_{X^{I}(0)}^{X^{I*}(t)} dX^{I*}(t) = \frac{\phi}{2}\sqrt{F^{I*}\left(X^{I*}(T)\right)}\int_{0}^{t}\sqrt{F^{I*}\left(X^{I*}(t)\right)}dt$$
$$\Longrightarrow \quad \triangle X^{I*}(t) = \frac{\phi}{2}\sqrt{F^{I*}\left(X^{I*}(T)\right)}\int_{0}^{t}\sqrt{F^{I*}\left(X^{I*}(t)\right)}dt$$
$$\Longrightarrow \quad \int_{0}^{t}\sqrt{F^{I*}\left(X^{I*}(t)\right)} = \frac{2\triangle X^{I*}(t)}{\phi\sqrt{F^{I*}\left(X^{I*}(T)\right)}}$$

Integrating the left hand side equation from $\pi^{I}(0)=0$ to $\pi^{I*}(t)$ and the right hand side equation from $X^{I}(0)$ to $X^{I*}(t)$, we have

$$\begin{aligned}\Pi^{I*}(t)=\int_0^t \pi^{I*}(\tau)\,d\tau &= \frac{\phi^2\sqrt{F^{I*}(X^{I*}(T))}}{4\gamma}\int_0^t\left(2\sqrt{F^{I*}(X^{I*}(t))}-\sqrt{F^{I*}(X^{I*}(T))}\right)dt \\ &= \frac{\phi^2\sqrt{F^{I*}(X^{I*}(T))}}{4\gamma}\left[\frac{4\triangle X^{I*}(t)}{\phi\sqrt{F^{I*}(X^{I*}(T))}}-t\sqrt{F^{I*}(X^{I*}(T))}\right] \\ &= \frac{\phi\triangle X^{I*}(t)}{\gamma}-\frac{t\phi^2F^{I*}(X^{I*}(T))}{4\gamma}\end{aligned}$$

We will use these results to study the channel coordination with revenue sharing contracts in the next two subsections.

2.8.2 Revenue Sharing with a Long-term Focus

We consider a revenue sharing contract with two parameters $\{q^{L},\hat{w}^{L}(t)\}$, where $q^{L}\in[0,1]$ is the manufacturer's share of revenue per unit sold by the retailer and $\hat{w}^{L}(t)$ is the wholesale price that the manufacturer charges the retailer per unit at time t. q^{L} is assumed to be constant over time. Note that we use the hat-accent " $\hat{}$ " to indicate that the variable is associated with a revenue sharing contract. The manufacturer's objective is to set $\hat{w}^{L}(t)$ such that the supply chain profit (sales) is the same as that with achieved by an integrated channel. The sequence of events is as follows: On date 1, the manufacturer announces the $\{q^{L},\hat{w}^{L}(t)\}$; On date 2, the retailer decides the retail price trajectory thereby the instantaneous sales rate with an aim to maximize her life-cycle profit; On date 3, production starts and trade occurs. We assume that the revenue sharing contract is not accompanied by an up-front fee so the retailer takes the O into consideration when dynamically setting the retail price.

The problem for the retailer with a long-term strategy is formulated as follows:

$$\hat{\pi}_{R}^{L*}(T) = \max_{\hat{r}^{L}(t)} \int_{0}^{T} \left\{ \left[\left(1-q^{L}\right) \hat{r}^{L}(t) - \hat{w}^{L}(t) - s \right] \dot{\hat{X}}^{L}(t) + O\left[K - \dot{\hat{X}}^{L}(t) \right] \right\} dt$$

$$s.t. \quad \dot{\hat{X}}^{L}(t) = \hat{F}^{L}\left(\hat{X}^{L}(t)\right)\left[1-\gamma \hat{r}^{L}(t)\right], \hat{X}^{L}(0) = \hat{X}_{0}^{L}$$

Let $\hat{\lambda}_{R}^{L}(t)$ be the shadow price associated with the state variable $\dot{\hat{X}}^{L}(t)$. Define $\hat{\pi}_{SC}^{L*}(T)$ as the maximum channel profits obtained from selling the manufacturer's product.

Theorem 2.1. *Consider a revenue sharing contract with $q^{L} \in [0,1]$ and the wholesale price trajectory $\hat{w}^{L*}(t)$ set as follows $\hat{w}^{L*}(t) = \left(1-q^{L}\right) c_{0} - q^{L} s - O$.*

(i)The retailer's instantaneous profit rate is given by

$$\hat{\pi}_{R}^{L*}(t) = \left(1-q^{L}\right) \pi^{I*}(t) + KO$$

and her life-cycle profit is given by $\hat{\Pi}_{R}^{L*}(T) = \left(1-q^{L}\right) \Pi^{I*}(T) + KOT$.

(ii) The manufacturer's instantaneous profit rate is given by

$$\hat{\pi}_{M}^{L*}(t) = q^{L} \pi^{I*}(t) - O x^{I*}(t)$$

and his life-cycle profit is given by $\hat{\Pi}_{M}^{L*}(T) = q^{L} \Pi^{I*}(T) - O \triangle X^{I*}(T)$.

(iii) The above revenue sharing contract coordinates the channel, i.e., $\hat{\Pi}_{SC}^{L*}(T) = \Pi^{I*}(T)$. The retailer's instantaneous sales rate is $x^{I*}(t)$ and the retail price is set at $r^{I*}(t)$.

(iv) When $O = 0$, the revenue sharing contract leads to profit sharing between the manufacturer and retailer.

Proof: To get the optimal retail price, we take the first order condition of $\hat{H}^L_R$ with regard to $\hat{r}^L$ to obtain:

$$\frac{\partial \hat{H}^L_R}{\partial \hat{r}^L} = 0 \implies -\gamma \left[(1-q)\,\hat{r}^L(t) - \hat{w}^L(t) - s - O + \hat{\lambda}^L_R(t) \right] + (1-q^L)\left[1 - \gamma \hat{r}^L(t)\right] = 0$$
$$\implies \hat{r}^L(t) = \frac{1 - q^L + \gamma\left(\hat{w}^L(t) + s + O - \hat{\lambda}^L_R(t)\right)}{2\gamma(1-q^L)}$$

The shadow price $\hat{\lambda}^L_R(t)$ satisfies the following equation:

$$\begin{aligned}\dot{\hat{\lambda}}^L_R(t) &= -\frac{\partial \hat{H}^L_R}{\partial \hat{X}^L} \\ &= -\hat{f}^{L*}(1-\gamma\hat{r}^L)\left[(1-q^L)\,\hat{r}^L - \hat{w}^L - s - O + \hat{\lambda}^L_R\right]\end{aligned}$$

The manufacturer sets the wholesale price such that

$$\hat{r}^L(t) = r^{I*} = \frac{1}{2\gamma}\left[2 - \phi\sqrt{\frac{F^{I*}(X^{I*}(T))}{F^{I*}(X^{I*}(t))}}\right].$$

The corresponding $\hat{w}^{L*}(t)$ is obtained by;

$$\frac{1 - q^L + \gamma\left(\hat{w}^L(t) + s + O - \hat{\lambda}^L_R(t)\right)}{2\gamma(1-q^L)} = \frac{1}{2\gamma}\left[2 - \phi\sqrt{\frac{F^{I*}(X^{I*}(T))}{F^{I*}(X^{I*}(t))}}\right]$$
$$\implies \hat{w}^{L*}(t) = \frac{1-q^L}{\gamma}\left[1 - \phi\sqrt{\frac{F^{I*}(X^{I*}(T))}{F^{I*}(X^{I*}(t))}}\right] + \hat{\lambda}^L_R(t) - s - O$$

Substituting $\hat{w}^L$ and $\hat{r}^L(t)$ into $\dot{\hat{\lambda}}^L_R(t)$, after simplification, we have

$$\dot{\hat{\lambda}}^L_R(t) = -\frac{(1-q^L)\,\phi^2 f^{I*}(X^{I*}(t))}{4\gamma}\frac{F^{I*}(X^{I*}(T))}{F^{I*}(X^{I*}(t))}$$

Recall that we have $\hat{x}^L_R(t) = x^I(t) = \frac{\phi}{2}\sqrt{F^{I*}(X^{I*}(T))\,F^{I*}(X^{I*}(t))}$. We can establish the relationship between $\dot{\hat{\lambda}}^{L}_{R}(t)$ and $\hat{x}^{L}_{R}(t)$ as:

$$\begin{aligned}
\frac{\dot{\hat{\lambda}}^{L}_{R}(t)}{\hat{x}^{L}_{R}(t)} &= -\frac{\phi f^{I*}\left(X^{I*}(t)\right)\left(1-q^{L}\right)\sqrt{F^{I*}\left(X^{I*}(T)\right)}}{2\gamma\left[F^{I*}\left(X^{I*}(t)\right)\right]^{3/2}} \\
\Longrightarrow \quad \int_{\lambda^{L}_{R}(t)}^{\lambda^{L}_{R}(T)} d\hat{\lambda}^{L}_{R}(\tau) &= -\frac{\phi\left(1-q^{L}\right)\sqrt{F^{I*}\left(X^{I*}(T)\right)}}{2\gamma}\int_{F^{I}(t)}^{F^{I}(T)}\left[F^{I*}\left(X^{I*}(t)\right)\right]^{-3/2} dF^{I}(\tau) \\
\Longrightarrow \quad \hat{\lambda}^{L}_{R}(t) &= -\frac{\phi\left(1-q^{L}\right)\sqrt{F^{I*}\left(X^{I*}(t)\right)}}{2\gamma}\left[F^{I*}\left(X^{I*}(t)\right)\right]^{-3/2}\Big|_{F^{I}(t)}^{F^{I}(T)} \\
&= \frac{\phi\left(1-q^{L}\right)}{\gamma}\left(\sqrt{\frac{F^{I*}\left(X^{I*}(T)\right)}{F^{I*}\left(X^{I*}(t)\right)}}-1\right)
\end{aligned}$$

We note that $\hat{\lambda}^{L}_{R}(t) = \left(1-q^{L}\right)\lambda^{I}_{R}(t)$. Substituting $\hat{\lambda}^{L}_{R}(t)$ and $\lambda^{I}(t)$ into $\hat{w}^{L*}(t)$, we can simplify $\hat{w}^{L*}$ as:

$$\begin{aligned}
\hat{w}^{L*}(t) &= \frac{1-q^{L}}{\gamma}\left[1-\phi\sqrt{\frac{F^{I*}\left(X^{I*}(T)\right)}{F^{I*}\left(X^{I*}(t)\right)}}\right]+\hat{\lambda}^{L}_{R}(t)-s-O= \\
&= \frac{1-q^{L}}{\gamma}\left[1-\phi\sqrt{\frac{F^{I*}\left(X^{I*}(T)\right)}{F^{I*}\left(X^{I*}(t)\right)}}\right]+\frac{\phi\left(1-q^{L}\right)}{\gamma}\left(\sqrt{\frac{F^{I*}\left(X^{I*}(T)\right)}{F^{I*}\left(X^{I*}(t)\right)}}-1\right) \\
&\quad -s-O \\
&= \left(1-q^{L}\right)\left(c_{0}+s\right)-s-O
\end{aligned}$$

Obviously, $\hat{w}^{L}(t)$ is constant across time.

Theorem 2.1(i). Substituting $\hat{w}^{L*}(t)$ into $\hat{\pi}^{L*}_{R}(t)$, we have

$$\begin{aligned}
\hat{\pi}^{L*}_{R}(t) &= \left[\left(1-q^{L}\right)r^{L*}(t)-\hat{w}^{L*}(t)-s\right]\hat{x}^{L*}+O\left(K-\hat{x}^{L*}\right) \\
&= \left(1-q^{L}\right)\pi^{I*}(t)+KO.
\end{aligned}$$

Integrating $\hat{\pi}_R^L(t)$ from 0 to t, we have

$$\begin{aligned}
\hat{\Pi}_R^{L*}(T) &= \int_0^T \hat{\pi}_R^{L*}(t)\,dt \\
&= KOT + (1-q^L)\int_0^T \pi^{I*}(t)\,dt \\
&= KOT + (1-q^L)\,\Pi^{I*}(T)
\end{aligned}$$

Theorem 2.1 (ii). The manufacturer's instantaneous profit rate $\hat{\pi}_M^{L*}(t)$ is given by:

$$\begin{aligned}
\hat{\pi}_M^{L*}(t) &= \left[\hat{w}^{L*} - c_0 + q^L \hat{r}^{L*}\right]\hat{x}^{L*} \\
&= \left[(1-q^L)(c_0+s) - s - O - c_0 + q^L \hat{r}^{L*}\right]\hat{x}^{I*} \\
&= \left[-q^L(c_0+s) - O + q^L \hat{r}^{L*}\right]\hat{x}^{I*} \\
&= q^L \pi^{I*}(t) - O\hat{x}^{I*}
\end{aligned}$$

The manufacturer's life-cycle profit $\hat{\Pi}_M^{L*}(T)$ is given by

$$\begin{aligned}
\hat{\Pi}_M^{L*}(T) &= \int_0^T \left[\hat{w}^{L*} - c_0 + q^L \hat{r}^{L*}\right]\hat{x}^{L*}dt \\
&= q^L \Pi^{I*}(T) - O\triangle X^{I*}(T)
\end{aligned}$$

Theorem 2.1(iii) According to the definition,

$$\hat{\Pi}_{SC}^{L*}(T) = \hat{\Pi}_M^{L*}(T) + \hat{\Pi}_R^{L*}(T) - \int_0^T O\left(K - \hat{x}^{L*}\right)dt.$$

Substituting $\hat{\Pi}_M^{L*}(T)$ and $\hat{\Pi}_R^{L*}(T)$ into $\hat{\Pi}_{SC}^{L*}(T)$ we have

$$\begin{aligned}\hat{\Pi}_{SC}^{L*}(T) &= \hat{\Pi}_M^{L*}(T) + \hat{\Pi}_R^{L*}(T) - \int_0^T O\left(K - \hat{x}^{L*}\right) dt \\ &= q^L \Pi^{I*}(T) - O\triangle X^{I*}(T) + KOT + \left(1 - q^L\right)\Pi^{I*}(T) - KOT + O\triangle \hat{X}^{L*}(T) \\ &= \Pi^{I*}(T)\end{aligned}$$

Theorem 2.1(iv) When $O = 0$, we have

$$\hat{\Pi}_R^{L*}(T) = \left(1 - q^L\right)\Pi^{I*}(T)$$

and $\hat{\Pi}_M^{L*}(T) = q^L \Pi^I(T)$. $\square$

We have a few observations from Theorem 2.1. First, the wholesale is constant over time and below the manufacturer's per unit production cost. Therefore, the manufacturer loses money in his wholesale transaction, and only makes money by sharing the retailer's revenue. Second, when O is positive, the share of the profits earned by the manufacturer is smaller than his share of revenue q^L.

2.8.3 Revenue Sharing with a Short-term Retailer Strategy

In this section, we consider a revenue sharing contract $\{q^S, \hat{w}^S(t)\}$ signed by a manufacturer and a retailer with a short-term profitability strategy. Note that the revenue sharing contract is not accompanied with up-front fees hence the retailer flexibly allocates her resources between the products that she sells. Suppose that the manufacturer's objective is to set the wholesale price $\hat{w}^S$ such that the instantaneous sales rate $\hat{x}^S = x^I$. The retailer's problem is to maximize her instantaneous profit rate:

$$\hat{\pi}_{R}^{S*}(t) = \max_{\hat{r}^{S}(t)} \{[(1-q^{S})\hat{r}^{S}(t) - \hat{w}^{S}(t) - s]\hat{x}^{S}(t) + O[K - \hat{x}^{S}(t)]\}$$
$$s.t. \quad \hat{x}^{S}(t) = \hat{F}^{S}\left(\hat{X}^{s}(t)\right)[1-\gamma\hat{r}^{S}(t)], \hat{X}^{S}(0) = \hat{X}_{0}$$

Let $\pi_{SC}^{S*}(T)$ be the channel's optimal life cycle profits.

Theorem 2.2. *Consider a revenue sharing contract with $q^{S} \in [0,1]$ and the wholesale price trajectory $\hat{w}^{S*}(t)$ set as*

$$\hat{w}^{S*}(t) = \frac{1}{\gamma}(1-q^{S})\left[1-\phi\sqrt{\frac{F^{I*}(X^{I*}(t))}{F^{I*}(X^{I*}(t))}}\right] - s - O.$$

(i) The resulting retailer's instantaneous profit rate is given by

$$\hat{\pi}_{R}^{S*}(t) = \frac{\phi^{2}}{4\gamma}(1-q^{S})F^{I*}(X^{I*}(T)) + KO.$$

When $O=0$, the retailer's optimal instantaneous profit rate $\hat{\pi}_{R}^{S*}(t)$ is constant over time. The retailer's cumulative life-cycle profit is given by

$$\hat{\Pi}_{R}^{S*}(T) = \frac{(1-q^{S})\phi^{2}}{4\gamma}TF^{I*}(X^{I*}(T)) + KOT.$$

(ii) The manufacturer's instantaneous profit rate $\hat{\pi}_{M}^{S*}(t)$ is given by:

$$\hat{\pi}_{M}^{S*}(t) = \left[\frac{\bar{\phi} - \phi R^{I*}(X^{I*}(t))}{\gamma} + \frac{q^{S}\phi R^{I*}(X^{I*}(t))}{2\gamma}\right]x^{I*}(t)$$

and the cumulative life cycle profit $\hat{\Pi}_{M}^{S*}(T)$ *is given by*

$$\hat{\Pi}_{M}^{S*}(T) = \frac{\bar{\phi}}{\gamma}\triangle X^{I*}(T) + \frac{T\phi^2(q^S - 2)F^{I*}(X^{I*}(T))}{4\gamma}.$$

(iii) The above revenue sharing contract coordinates the the channel, i.e., $\hat{\Pi}_{SC}^{S*}(T) = \Pi^{I*}(T)$*. At any time* t*, the retailer's instantaneous sales rate is* $x^{I*}(t)$ *and the retail price* $r^{I*}(t)$*.*

Proof: For a given wholesale price trajectory $\hat{w}^S(t)$, the retailer's best response retail price trajectory $\hat{r}^S(t)$ is obtained by taking the first order condition of $\hat{\pi}_R^S$ with regard to $\hat{r}^S$:

$$\begin{aligned}
\tfrac{\partial \hat{\pi}_R^S}{\partial \hat{r}^S} = 0 \quad &\Longrightarrow (1-q^S)\left[1-\gamma\hat{r}^S\right] - \gamma\left[(1-q^S)\hat{r}^S - \hat{w}^S - s - O\right] = 0 \\
&\Longrightarrow \hat{r}^{S*} = \frac{1-q^S+\gamma(\hat{w}^S+s+O)}{2\gamma(1-q^S)}.
\end{aligned}$$

Suppose that manufacturer sets the wholesale price $\hat{w}^{S*}$ in a way such that $r^S = r^I$. Then the optimal wholesale price is obtained by solving the following equation:

$$\begin{aligned}
\frac{1-q^S+\gamma(\hat{w}^{S*}+s+O)}{2\gamma(1-q^S)} &= \frac{1}{2\gamma}\left[2-\phi\sqrt{\frac{F^{I*}(X^{I*}(T))}{F^{I*}(X^{I*}(t))}}\right] \\
\Longrightarrow \quad \hat{w}^{S*} &= \frac{(1-q^S)}{\gamma}\left[1-\phi\sqrt{\frac{F^{I*}(X^{I*}(T))}{F^{I*}(X^{I*}(t))}}\right] - s - O.
\end{aligned}$$

Let $R^{I*}\left(X^{I*}\left(t\right)\right)=\sqrt{\frac{F^{I*}\left(X^{I*}\left(T\right)\right)}{F^{I*}\left(X^{I*}\left(t\right)\right)}}$. Then, we have:

$$\begin{aligned}
\hat{w}^{S*} &= \frac{\left(1-q^{S}\right)\left[1-\phi R^{I*}\left(X^{I*}\left(t\right)\right)\right]}{\gamma}-s-O,\\
\hat{r}^{S*}\left(t\right) &= \frac{2-\phi R^{I*}\left(X^{I*}\left(t\right)\right)}{2\gamma},\\
\hat{x}^{S*}\left(t\right) &= \frac{\phi F^{I*}\left(X^{I*}\left(t\right)\right)R^{I*}\left(X^{I*}\left(t\right)\right)}{2}.
\end{aligned}$$

Theorem 2.2(i). Substituting $w^{S}\left(t\right)$ into the manufacturer's instantaneous profit rate $\hat{\pi}_{R}^{S*}\left(t\right)$, we have

$$\begin{aligned}
\hat{\pi}_{R}^{S*}\left(t\right) &= \left[\left(1-q^{S}\right)\hat{r}^{S*}\left(t\right)-\hat{w}^{S*}\left(t\right)-s-O\right]\hat{x}^{S*}\left(t\right)+KO\\
&= \left[\left(1-q^{S}\right)\frac{2-\phi R^{I*}\left(X^{I*}\left(t\right)\right)}{2\gamma}-\frac{\left(1-q^{S}\right)\left[1-\phi R^{I*}\left(X^{I*}\left(t\right)\right)\right]}{\gamma}\right]\hat{x}^{S*}\left(t\right)+KO\\
&= \left(1-q^{S}\right)\frac{\phi R^{I*}\left(X^{I*}\left(t\right)\right)}{2\gamma}\frac{\phi F^{I*}\left(X^{I*}\left(t\right)\right)R^{I*}\left(X^{I*}\left(t\right)\right)}{2}+KO\\
&= \frac{\left(1-q^{S}\right)\phi^{2}F^{I*}\left(X^{I*}\left(T\right)\right)}{4\gamma}+KO
\end{aligned}$$

To derive the retailer's life-cycle profit, we integrate $\hat{\pi}_{R}^{S*}\left(t\right)$ from 0 to t:

$$\hat{\Pi}_{R}^{S*}\left(T\right)=\int_{0}^{T}\hat{\pi}_{R}^{S*}\left(t\right)dt=\frac{T\left(1-q^{S}\right)\phi^{2}F^{I*}\left(X^{I*}\left(T\right)\right)}{4\gamma}+KOT$$

Theorem 2.2(ii). The manufacturer's instantaneous profit rate is given by

$$\begin{aligned}
\hat{\pi}_{M}^{S*}\left(t\right) &= \left[\hat{w}^{S*}\left(t\right)+q^{S}\hat{r}^{S*}\left(t\right)-c_{0}\right]\hat{x}^{S*}\left(t\right)\\
&= \left[\frac{\left(1-q^{S}\right)\left[1-\phi R^{I*}\left(X^{I*}\left(t\right)\right)\right]}{\gamma}+\frac{q^{S}\left[2-\phi R^{I*}\left(X^{I*}\left(t\right)\right)\right]}{2\gamma}-s-O-c_{0}\right]x^{I*}\left(t\right)\\
&= \left[\frac{\bar{\phi}-\phi R^{I*}\left(X^{I*}\left(t\right)\right)}{\gamma}+\frac{q^{S}\phi R^{I*}\left(X^{I*}\left(t\right)\right)}{2\gamma}\right]x^{I*}\left(t\right)
\end{aligned}$$

The cumulative life cycle profit to the manufacturer is given by

$$
\begin{aligned}
\hat{\Pi}_M^{S*}(T) = \int_0^T \hat{\pi}_M^{S*}(\tau)\, d\tau &= \int_0^T \left[\frac{\bar{\phi} - \phi R^{I*}(X^{I*}(\tau))}{\gamma} + \frac{q^S \phi R^{I*}(X^{I*}(\tau))}{2\gamma}\right] x^{I*}(\tau)\, d\tau \\
&= \frac{\bar{\phi}}{\gamma} \triangle X^{I*}(T) - \frac{T\phi^2 F^{I*}(X^{I*}(T))}{2\gamma} + \frac{Tq^S \phi^2 F^{I*}(X^{I*}(T))}{4\gamma} \\
&= \frac{\bar{\phi}}{\gamma} \triangle X^{I*}(T) + \frac{T\phi^2 (q^S - 2) F^{I*}(X^{I*}(T))}{4\gamma}
\end{aligned}
$$

2.2(iii) We can derive $\hat{\Pi}_{SC}^S(T)$ as follows:

$$
\begin{aligned}
\hat{\Pi}_{SC}^{S*}(T) &= \hat{\Pi}_M^{S*}(T) + \hat{\Pi}_R^{S*}(T) - \int_0^T O\left(K - \hat{x}^{L*}\right) dt \\
&= \frac{\bar{\phi}}{\gamma} \triangle X^{I*}(T) + \frac{T\phi^2 (q^S - 2) F^{I*}(X^{I*}(T))}{4\gamma} + \frac{T\phi^2 (1 - q^S) F^{I*}(X^{I*}(T))}{4\gamma} \\
&\quad + O \triangle X^{I*}(T) \\
&= \frac{\phi \triangle X^{I*}(T)}{\gamma} - \frac{T\phi^2 F^{I*}(X^{I*}(T))}{4} = \Pi^{I*}(T)
\end{aligned}
$$

Different from the revenue sharing with a long-term retailer profitability strategy, the wholesale price is no longer constant under a short-term retailer profitability strategy. The manufacturer will offer lower $\hat{w}^S(t)$ with a higher O.

A computational experiment over the ranges of parameters in Table 1, shows that the revenue sharing contract has the potential to increase the supply chain profit by up to 72% when market saturation levels reach 45%.

2.9 Conclusion

In this paper, we focus on the strategic interactions and decisions in a supply chain to launch an innovative durable production in the context of a decentralized distribution channel. We analyze the potential conflicts of interests between a manufacturer and an independent retailer in the retailer's choice of profitability strategy and her pricing decisions. Our results show that the length of selling horizon may greatly impact the retailer's preference over her profitability strategy, and thereby her pricing strategy. Additionally, we investigate the impact of retailer's resource allocation on the channel performance, and we propose a two-part tariff to partially improve the channel performance. Finally, we use the revenue sharing contracts to coordinate the channel with both types of retailer profitability strategy.

Our model allows the managers to improve the decisions in three ways. First, we highlight the impact of retailer's profit optimizing strategy on the manufacturer's as well as channel's profit. The manufacturer may need to monitor the retailer's sales volume and retail price in order to implement his most preferred profitability strategy. Second, we demonstrate that the manufacturer may propose the revenue sharing contracts to achieve full coordination. It is important to emphasize that a sale commission agreement will have the properties of a revenue sharing agreement, and we can point out that commission sales agreement are actually common in the VARs distribution channel. However, for a commission sales agreement to coordinate the channel we need the price of the product to be inclusive of the service component of the IDP paid by the customer. That is, a commission sales agreement between a manufacturer and a VAR that permits the VAR to charge additional fees for services to the customer will not coordinate the channel. What is required is that the manufacturer set a price for the IDP inclusive of all the required services and then negotiate a commission with the VAR over the inclusive price precluding them to charge additional fees to the customer. Finally, as a by-product, we characterize the equilibrium retail and wholesale pricing strategies as functions of cumulative sales rather than shadow prices as was done by in the literature. This leads to more convenient pricing guidelines.

Our diffusion model in the context of a decentralized supply chain opens up several avenues for future research. One direction is to model manufacturer level competition. It would be interesting to investigate how the manufacturer's pricing strategy and his preference over the retailer profitability strategy change in competitive environments. Furthermore, it would also be interesting to study how the channel will be coordinated with competing manufacturers. Future research may consider the different demand situations, such as products with repeat purchases, and incorporation the alternative demand functions. Our model assumes that the consumers do not postpone purchase on purpose in anticipation of future product price. Modeling such strategic consumer behavior is a natural extension to this work. Our model can also be extended to study dynamic two-part tariffs to secure different amounts of selling resources throughout the planning horizon.

Chapter 3

Dynamic Slotting and Pricing Decisions in a Durable Product Supply Chain

3.1 Introduction

Consider a supply chain in which a manufacturer (he) sells an innovative durable product (IDP) to an independent retailer (she) over its life-cycle of a fixed time horizon. During this period, the retailer makes decisions to influence the retail demand for the IDP in order to maximize her profit objective. We focus on four important factors that affect the retail demand of the IDP. (a) Diffusion effect (network effect) by which we mean that the customers who have purchased the IDP inform those who have not. (b) Saturation effect: Because the product is durable, the consumers purchase only once during the selling horizon of the IDP. Therefore, the more the cumulative adopters, the smaller is the remaining potential market. (c) Retail price: The consumer demand for the product is inversely related to the unit retail price charged. (d) Shelf space: While the shelf space has a positive impact on the demand for the IDP, it is a scare resource for the retailer.

We formulate the problem in a Stackelberg differential game framework. We assume that the manufacturer takes the role of the leader in his relationship with the retailer. Thus, the manufacturer announces his wholesale price to the retailer, and the retailer decides on the retail price and the shelf space allocation over time in order to maximize her profit objective, taking the wholesale price as given. When setting the wholesale price, the manufacturer takes the retailer's best response into consideration in order to maximize his life-cycle profit over the horizon. We consider the following two retailer's profit strategies: (i) far-sighted strategy and (ii) myopic strategy. By far-sighted we mean that the retailer maximizes her life-cycle profit over the selling horizon, whereas by myopic we mean that the retailer maximizes her instantaneous profit rates at each time instant in the selling horizon.

We address the following research questions. (1) For a given retailer's profit strategy, what are the optimal pricing and slotting policies for the retailer and the optimal wholesale pricing policy for the manufacturer? (2) Should the retailer be far-sighted or myopic? (3) Does the manufacturer prefer the retailer to be far-sighted or myopic? (4) Is there a conflict of preference between the manufacturer and the retailer?

The solution concept for the Stackelberg differential game that we use is the open-loop equilibrium. This means that the manufacturer and the retailer decide on the their respective policies at the start of the game. It is known that an open-loop equilibrium is time inconsistent if the leader cannot credibly commit to his policy, i.e., if the leader is given the opportunity to revise his policy, he would, at sometime during the selling horizon, switch to another policy different from the one he chose at the beginning of the game. Hence, open-loop policies make sense only when the leader can pre-commit to his policy (Jørgensen 2003). In this study, we assume that the manufacturer is able to commit to his wholesale pricing policy.

There is a vast literature on supply chain management dealing with stochastic demand and modeled as newsvendor problems (see De Kok and Graves 2003). A serious limitation of such analyses is that they address only single period models. Yet the real world problems are dynamic. Our work is an early attempt to formulate supply chain management problems as dynamic games of Stackelberg type.

Our contributions include characterization of optimal solutions and their numerical computations. Furthermore, based on these, we conclude that the following possibilities may arise in terms of the preferred profit strategies of the game players. First, the manufacturer may not always prefer the retailer to have a far-sighted strategy, i.e., the manufacturer sometimes is better off if the retailer is myopic. On the other hand, the retailer's preference (myopic /far-sighted) changes with market characteristics and they do not always agree with the manufacturer's preference. Our results show that both the manufacturer and the retailer are better off if the retailer is far-sighted when the final market saturation level is low. However, if the level is high at the end of the horizon, the manufacturer is better off with a myopic retailer, while the retailer prefers that the manufacturer sets his wholesale prices assuming that she is far-sighted. The conflict between the manufacturer and the retailer over the preferred retail profit focus raises some contract implementation issues that are worth exploring as a future research topic.

The rest of the paper is organized as follows. In the next section, we review the related literature. In Section 3.3 we introduce the demand model. In Section 3.4 we study the case of a myopic retailer. In Section 3.5 we study the case of a far-sighted retailer. In Section 3.6 we present a numerical study that compares the cases of far-sighted and myopic foci. In Section 3.7 we conclude by summarizing the results and pointing out future research directions.

3.2 Literature Review

This research is closely related to the new product diffusion and shelf space allocation models in the marketing literature and the differential game models in the optimal control literature.

The Bass (1969) diffusion model and its variants have been widely used to forecast demand of a durable new product. We refer the readers to Mahajan et al. (1990, 2000) for comprehensive reviews of this literature. The original Bass (1969) model does not include price as a variable. A number of later papers extended the Bass model by incorporating the (competitive) price impact on retail demand of an IDP. These include Robinson and Lakhani (1975), Bass (1980), Dolan and Jeuland (1981), Bass and Bultez (1982), Kalish (1983), Kalish and Lilien (1983), Clarke and Dolan (1984), Thompson and Teng (1984), Rao and Bass (1985), Eliashberg and Jeuland (1986), Raman and Charterjee (1995), and Krishnan at el (1999). Regarding the market conditions, Eliashberg and Jeuland (1986) and Thompson and Teng (1984) analyze equilibrium oligopoly pricing, whereas the others study the optimal monopolist pricing.

In the context of optimal dynamic pricing, an important consideration is whether a firm maximizes its short-term profit or its long-term profit. Bass (1980) and Bass and Bultez (1982) assume that the firm maximizes the current period (instantaneous) profit, and the pricing strategies that they obtain are myopic in nature, as compared to (far-sighted) optimal pricing policies which maximize the firm's aggregated profit over the product's life cycle. Robinson and Lakhani (1975) compared the total profits resulting from far-sighted optimal pricing and myopic optimal pricing. Their numerical results show that the differences in profits are significant, whereas Bass and Bultez (1982) report only small differences.

As noted by Dolan and Jeuland (1981), it is very critical to properly incorporate the impact of pricing into the demand model. Several papers, including Robinson and Lakhani (1975), Dolan and Jeuland (1981), and Thompson and Teng (1984), assume that the demand is an exponential function of price. In contrast, as in Eliashberg and Jeuland (1986) and Raman and Chatterjee (1995), we assume that the demand is a linearly decreasing function of retail price.

Over the past three decades, the marketing researchers have devoted much attention to study the impact of shelf space allocation on the sales. A number of theoretical and empirical studies have documented that sales increase with the amount of allocated shelf space (Cox 1970, Curhan 1971, Curhan 1973). Curhan (1971, Dreze et al. 1994) studies the relationship between allocated shelf space and sales in supermarkets. Curhan hypothesizes that the shelf space elasticity is a function of a product's physical properties, merchandising characteristics, and use characteristics. Curhan concludes that the impact of changes in shelf space on sales is small relative to the effects of the other variables such as retail price, brand name, and advertising. through a series of field experiments, Dereze et al. (1994) found modest gains (4%) in sales and profits from increased customization of shelf space.

Recently, researchers have integrated marketing research (studies on the impact of shelf space on sales) and operations management (inventory management) by developing models that incorporate the impact of displayed inventory on demand, including Brown and Tucker (1961), Corstjens and Doyle (1981, 1983), Bultez and Naert (1988), Borin et al. (1994), Urban (1998), Wang and Gerchak (2001), Lim et al. (2004). Urban (1998) and Wang and Gerchak (2001) assume that the demand rate at the retail level depends on the shelf space allocated to the product. Specifically, in their models, the demand is an increasing and concave function of the merchandise displayed on the shelf, and they addressed the optimal level of that inventory to be on the shelf. Recently, there are a few papers that jointly consider the product assortment and pricing (Green and Saviltz 1994, Mcintyre and Miller 1999).

Differential game models have been applied to analyze the strategic dynamic interactions between the players. In the supply chain management literature, Elishaberg and Steinberg (1986), Bykadorov (2007), and Gutierrez and He (2007) have applied these models in the context of a channel with a manufacturer and a retailer (distributor). Elishaberg and Steinberg (1986) focus on the inventory and pricing decisions of the manufacturer and the distributor who faces a stochastic demand. In a differential game framework, Bykadorov et al. (2007) derive the optimal control of the manufacturer's profit via discounts. They analyze the types of games played between the manufacturer and the retailer: a Stackelberg differential game with the manufacturer being the leader, and a Nash game. They compare the Stackelberg equilibrium solution with the Nash equilibrium solution. Our work is closest in spirit and structure to Gutierrez and He (2007), who analyze dynamic pricing decisions in a Stackelberg differential game framework in the context of an IDP. We extend Gutirrez and He (2007) by considering the impact of shelf space allocation on the retail demand. This enables us to study the dynamic slotting decisions of the retailer in addition to her pricing decisions. To our knowledge, this is the first paper that determines optimal pricing and shelf space allocation simultaneously in a dynamic game framework.

3.3 The Demand Model

The manufacturer produces an IDP whose retail demand follows a Bass type diffusion process. Let $x(t)$ and $X(t)$ be the instantaneous demand (rate of sales) and cumulative sales at time t, $0 \leq t \leq T$, where T denotes the selling horizon of the IDP. The demand dynamics are described by the differential equation

$$x(t) = \dot{X}(t) = \frac{dX(t)}{dt} = \sqrt{p(t)}\,(M - X(t))\,(\alpha + \beta X(t))\,(1 - \gamma r(t))\,, X(0) = X_0 \tag{3.1}$$

where $p(t)$ is the shelf space allocated to the product at time $t \in [0, T]$, M denotes the potential market size, $(M - X(t))$ is the unsaturated market size, α and β are positive coefficients of external and internal market influences, respectively, and the parameter $\gamma > 0$ measures the customer's sensitivity to the retail price $r(t)$. According to our formulation, the sales rate $x(t)$ is determined by four factors: the external market influence, the internal market influence, the slotting decision, and the retail price. We use a multiplicatively separable function in (3.1) to model the impact of price, shelf space, and cumulative sales on the instantaneous demand rate. The instantaneous demand rate is a linearly decreasing function of the retail price. Linear demand functions have been used by a number of papers in the stream of dynamic pricing that use the Bass model, such as Eliashberg and Jeuland (1988), Raman and Chartejee (1995), and Kalish (1983). The $\sqrt{p(t)}$ term in (3.1) signifies that the shelf space has marginal diminishing returns with respect to the instantaneous demand . The effect of the cumulative sales $X(t)$ on the instantaneous demand $x(t)$ is as follows. Initially, the market is not saturated and the diffusion effect outweighs the saturation effect. Over time, the market gets saturated, which makes additional sales more difficult; thus as time progresses, the saturation effect starts dominating the diffusion effect.

We use superscripts "M", "R", and "C" to denote the manufacturer, the retailer, and the channel variables, respectively. We let $\pi_R(t)$, $\Pi_R(t)$, $\pi_M(t)$ and $\Pi_M(t)$ denote the retailer's profit rate at time t, her total profit over the horizon T, the manufacturer's profit rate at time t, and his total profit

3.4 Myopic Retailer

We consider the case of a retailer with short-term profit focus. The manufacturer and the retailer play a Stackelberg differential game. The sequence of the events is as follows. The manufacturer announces the wholesale price trajectory $w(\cdot)$. Then the retailer simultaneously decides the retail price trajectory $r(\cdot)$ and the shelf space trajectory $p(\cdot)$. Rewriting $p(t)$ as $c^2(t)$, and calling $c(t)$ as the slotting decision, and with s_0 denoting the retailer's unit selling cost and s denoting the unit cost of the shelf space, we formulate the retailer's optimization problem at instant t for any given manufacturer's wholesale price trajectory $w(\cdot) = w(t), 0 \leq t \leq T$. The myopic retailer's profit:

$$\pi_R(t) = \pi_R(Xt(t), r(t), c(t); w(t)) = [r(t) - w(t) - s_0] \dot{X}(t) - sc^2(t) \quad (3.2)$$

The problem is to choose $r(t)$ and $c(t)$ to maximize $\pi_R(t)$,$0 \leq t \leq T$, subject to

$$x(t) = c(t)(M - X(t))(\alpha + \beta X(t))[1 - \gamma r(t)], X(0) = X_0. \quad (3.3)$$

It should be obvious in the myopic case that the retailer's best response at time t will be depend only on the past of the announced wholesale price trajectory, i.e., on $\{w(\tau), 0 \leq \tau \leq t\}$. Moreover, we can characterize the structure of these decisions by using the first order conditions for the maximum of π_R . Specifically, solving $\partial \pi_R(t) \diagup \partial r(t) = 0$ and $\partial \pi_R(t) \diagup \partial c(t) = 0$ simultaneously gives the best response

$$r(t) = r^*(X(t); w(\cdot)) = \frac{1 + \gamma(w(t) + s_0)}{2\gamma}, \tag{3.4}$$

$$c(t) = c^*(X(t); w(\cdot)) = \frac{F(X(t))(1 - \gamma r(t))[r(t) - w(t) - s_0]}{2s}, \tag{3.5}$$

where $F(X(t)) = (M - X(t))(\alpha + \beta X(t))$. Here we have suppressed the argument t as is in the control theory literature, and we shall do so from now on where the arises no confusion in doing so. Substituting r into x and c we have

$$x = \frac{F^2(X)[1 - \gamma(w + s_0)]^3}{16\gamma s}, \quad c = \frac{F(X)[1 - \gamma(w + s_0)]^2}{8\gamma s}, \tag{3.6}$$

where $F^2(X)$ denotes $(F(X))^2$. Note that the best retail price response $r(t)$ depends only on $w(t)$ and the best slotting response $c(t)$ depends on the cumulative sales $X(t)$ and the wholesale price $w(t)$at time t. The manufacturer takes the retailer's best response (3.6) into consideration when solving his problem over the selling horizon T. That is, he uses (3.6) in (3.3) to obtain his state equation, and his problem is

$$\max_{w(\cdot)} \int_0^T [w(t) - c_0]\dot{X}(t)\,dt, \tag{3.7}$$

$$\dot{X}(t) = \frac{F^2(X(t))[1 - \gamma(w(t) + s_0)]^3}{16\gamma s}, \quad X(0) = X_0, \tag{3.8}$$

where c_0 is the manufacturer's unit production cost. We shall use the Maximum principle to solve this optimal control problem (see Sethi and Thompson 2001). The manufacturer's Hamiltonian

$$\begin{aligned} H_M(t) &\equiv H_M(X(t), w(t), \lambda_M(t)) = (w(t) - c_0 + \lambda_M(t))\dot{X}(t) \\ &= \frac{F^2(X(t))(w(t) - c_0 + \lambda_M(t))[1 - \gamma(w(t) + s_0)]^3}{16\gamma s}, \end{aligned} \tag{3.9}$$

where λ_M, the shadow price associated with the state variable $X(t)$, satisfies the adjoint equation

$$\dot{\lambda}_M = -\frac{\partial H_M}{\partial \bar{X}} = -\frac{F'(X)F(X)(w - c_0 + \lambda_M)[1 - \gamma(w + s_0)]^3}{8\gamma s}, \tag{3.10}$$

with $\lambda_M(T) = 0$, which raises from the fact that $X(T)$ is free. Using the first order condition, we derive the optimal wholesale price as follows:

$$\begin{aligned} \tfrac{\partial H_M}{\partial w} = 0 &\implies [1 - \gamma(w + s_0)]^3 - 3\gamma[w - c_0 + \lambda_M][1 - \gamma(w + s_0)]^2 = 0 \\ &\implies w(t) = \frac{1 + \gamma[3c_0 - 3\lambda_M(t) - s_0]}{4\gamma}. \end{aligned} \tag{3.11}$$

We note that the higher the λ_M, the lower is the wholesale price w. This result is the demonstration of the economic interpretation of the shadow price: λ_M is the future value of an additional unit of sales. When $\lambda_M > 0$, the higher the λ_M, the larger is the future value of the additional sales. Thus, the manufacturer has an incentive to lower the wholesale price $w(t)$ to stimulate immediate sales. Substituting the optimal wholesale price $w(t)$ from (3.11) into $r(t)$, $x(t)$, $\dot{\lambda}_M(t)$, and $c(t)$, we have their values in equilibrium:

$$r(t) = \frac{5 + 3\gamma\left[c_0 + s_0 - \lambda_M(t)\right]}{8\gamma}, \tag{3.12}$$

$$x(t) = \frac{27F(X(t))^2\left[1 - \gamma\left(c_0 + s_0 - \lambda_M(t)\right)\right]^3}{1024\gamma s}, \tag{3.13}$$

$$c(t) = \frac{F(X(t))\left[1 - \gamma\left(c_0 + s_0 - \lambda_M(t)\right)\right]^2}{128\gamma s}, \tag{3.14}$$

$$\dot{\lambda}_M(t) = -\frac{27F'(X(t))F(X(t))\left[1 - \gamma\left(c_0 + s_0 - \lambda_M(t)\right)\right]^4}{2048 s\gamma^2}. \tag{3.15}$$

Note that the equilibrium values of $w(t)$, $r(t)$, $x(t)$, $\dot{\lambda}_M(t)$, and $c(t)$ are functions of the state variable $X(t)$ and shadow price $\lambda_M(t)$. Also, since the equilibrium values of $r(t)$ and $x(t)$ depend on $\lambda_M(t)$, the retailer's optimal decisions depend on the entire wholesale price trajectory $w(\cdot)$. Furthermore, one can easily see that there is no reason for either player to change their policy in the middle of the game, and therefore, the equilibrium in the myopic case is time consistent.

In the following lemma, we express $\lambda_M(t)$ in terms of $X(t)$.

Lemma 3.1. *The shadow price trajectory $\lambda_M(t)$ is given by*

$$\lambda_M(t) = \frac{1 - \gamma(c_0 + s_0)}{\gamma}\left[\sqrt{\frac{(M - X(T))(\alpha + \beta X(T))}{(M - X(t))(\alpha + \beta X(t))}} - 1\right], t \in [0, T]. \tag{3.16}$$

Proof: From Equations (3.13) and (3.15), we have:

$$\frac{\dot{X}(t)}{\dot{\lambda}_M(t)} = -\frac{2\gamma F(X(t))}{F'(X(t))\left[1 - \gamma\left(c_0 + s_0 - \lambda_M(t)\right)\right]},$$

which can be written as

$$\frac{F'(X(t))\,dX(t)}{F(X(t))} = d\ln F(X(t)) = -\frac{2\gamma d\lambda_{M}(t)}{[1-\gamma(c_0+s_0-\lambda_{M}(t))]}.$$

We can now integrate both sides of the equation from t to T and simplify to obtain (3.16). □

We observe that the sign of $\lambda_{M}(t)$ depends on the ratio of $F(X(T))$ to $F(X(t))$. Thus, when $F(X(T)) \geq F(X(t))$, $\lambda_{M}(t) > 0$ and when $F(X(T)) \leq F(X(t))$, $\lambda_{M}(t) < 0$.

Lemma 3.2. *In an optimal solution for the myopic retailer, the wholesale price, the retail price, the slotting decision, and the instantaneous sales rate, respectively,*

$$w(t) = \frac{4(1-\gamma s_0) - 3\sqrt{\frac{F(X(T))}{F(X(t))}}\,[1-\gamma(c_0+s_0)]}{4\gamma}, \tag{3.17}$$

$$r(t) = \frac{8 - 3\sqrt{\frac{F(X(T))}{F(X(t))}}\,[1-\gamma(c_0+s_0)]}{8\gamma}, \tag{3.18}$$

$$c(t) = \frac{9F(X(T))\,[1-\gamma(c_0+s_0)]^2}{128\gamma s}, \tag{3.19}$$

$$x(t) = \dot{X}(t) = \frac{27F^{\frac{1}{2}}(X(t))\,F^{\frac{3}{2}}(X(T))\,[1-\gamma(c_0+s_0)]^3}{1024\gamma s}. \tag{3.20}$$

Furthermore, the slotting decision is constant over time, and the retail price, the wholesale price, and the instantaneous sales rate all peak at the same time.

Proof: Substitute for $\lambda_{M}(t)$ from (3.16) to (3.10), (3.11), (3.12), (3.13) and (3.14) to obtain (3.17), (3.18), (3.19), and (3.20), respectively. The last part of the lemma is obvious from a comparison of (3.17), (3.18) and (3.19). □

Lemma 3. *The optimal cumulative sales $X(t)$ is the unique solution to the* *equation*

$$\tan^{-1}\left[\frac{\alpha-\beta M+2\beta X(t)}{2\sqrt{\beta F(X(t))}}\right] = \tan^{-1}\left[\frac{\alpha-\beta M+2\beta X(0)}{2\sqrt{\beta F(X(0))}}\right] + \\ + \frac{27\sqrt{\beta}tF^{3/2}(X(T))\left[1-\gamma\left(c_0+s_0\right)\right]^3}{1024\gamma s}. \qquad (3.21)$$

PROOF. Integrate (3.19) from 0 to t and rearrange terms to obtain (3.21). In order to show the uniqueness, one can show that for $t = T$, (3.21) has a unique solution for $X(T)$. In view of the fact that the left-hand side is increasing in $X(t)$, it follows that $X(t)$ is unique. □

LEMMA 3.4. *In the optimal solution with the myopic retailer,*

$$\pi_R(t) = \frac{81F^2(X(T))\left[1-\gamma\left(c_0+s_0\right)\right]^4}{s(128\gamma)^2} \qquad (3.22)$$

is constant over time, and her total profit over the horizon T is

$$\Pi_R(T) = \int_0^T \pi_R(\tau)\, d\tau\tau = \frac{81TF^2(X(T))\left[1-\gamma\left(c_0+s_0\right)\right]^4}{s(128\gamma)^2}. \qquad (3.23)$$

The manufacturer's instantaneous profit rate

$$\pi_M(t) = \frac{27}{s(64\gamma)^2}F^{\frac{1}{2}}(X(t))\,F^{\frac{3}{2}}(X(T))\left(4-3\sqrt{\frac{F(X(T))}{F(X(t))}}\right)\left[1-\gamma\left(c_0+s_0\right)\right]^4 \qquad (3.24)$$

and his total profit over the horizon T is

$$\begin{aligned}\Pi_M(T) &= \int_0^T \pi_M(\tau)\, dt\tau \\ &= \frac{[X(T) - X(0)][1-\gamma(c_0+s_0)]}{\gamma} - \frac{81TtF^2(X(T))[1-\gamma(c_0+s_0)]^4}{s(64\gamma)^2}.\end{aligned} \tag{3.25}$$

Proof: Substituting $r(t)$,$w(t)$, $x(t)$, and $c(t)$ obtained in Lemma 3.2 into (3.2) gives (3.22). Integrating (3.22) from 0 to t gives (3.23). Substituting $w(t)$ and $x(t)$ obtained in Lemma 3.2 in the integrand of (3.7) gives (3.24). Integrating (3.24) from 0 to t gives (3.25). □

Based on these results, we conduct a numerical study with the following parameter values $M = 1 \times 10^7, X_0 = 0$, $\alpha = 0.016$, $\beta = 8 \times 10^{-9}$, $c_0 = 100$, $s_0 = 20$, $s = 5 \times 10^7$, and $\gamma = 5 \times 10^{-4}$. We compute the decisions for various values of the horizon T. Figure 3.1 shows the wholesale price trajectories for different values of T. We observe two patterns of wholesale price trajectories: increasing over time (for $T = 25, 40, 45, 60$) and initially increasing then decreasing (for $T = 75, 80$). Also, a comparison of (3.17), (3.18), and (3.19), or Figures 3.1-3.3, reveals that these trajectories of $w(\cdot)$, $r(\cdot)$, and $x(\cdot)$ mimic one another. Furthermore, we observe that for lower values of T, the wholesale price curves move downward as T increases, and beyond a certain value of T, the wholesale price curves move upward as T increases.

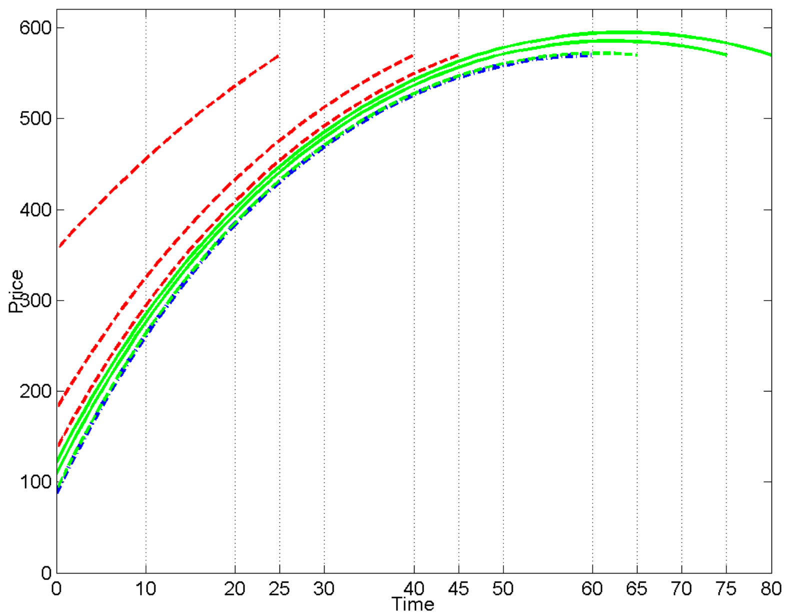

Figure 3.1: Wholesale price trajectories with different horizons: Myopic retailer

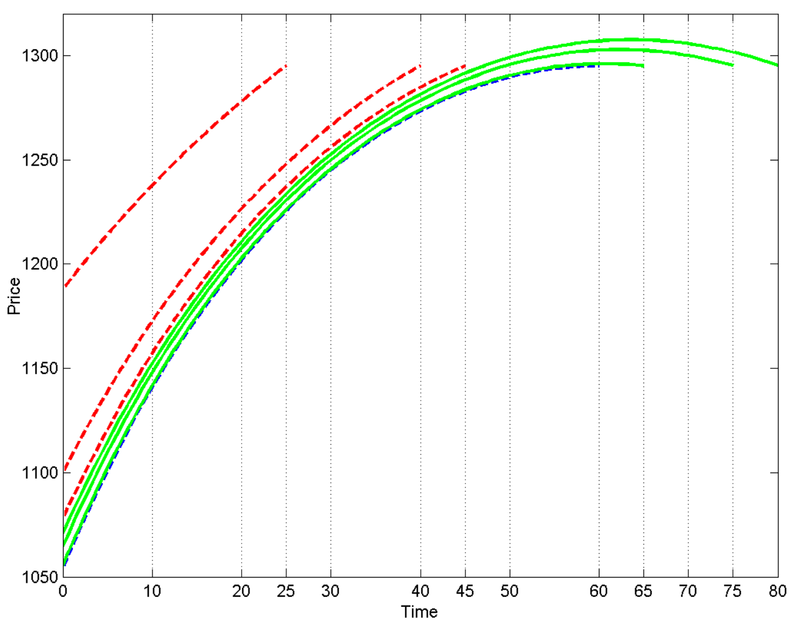

Figure 3.2: Retail price trajectories with different horizons: Myopic retailer

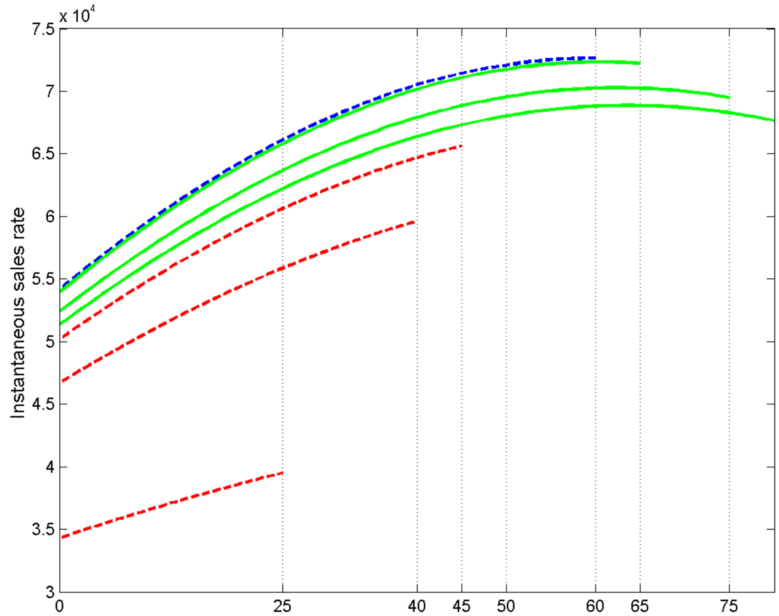

Figure 3.3: Instantaneous saes rate with $T = 75$: Myopic retailer

Using the same parameter values, we continue our numerical study to get further insights. Table 3.1 reports the manufacturer's profit $\Pi_M(T)$, the retailer's profit $\Pi_R(T)$, and the profit ratios $\Pi_M(T)/\Pi_C(T)$ and $\Pi_R(T)/\Pi_C(T)$, where $\Pi_C(T) = \Pi_M(T) + \Pi_R(T)$ denotes the channel's profit, and the slotting decisions c, which is constant throughout the horizon, and the market saturation level $X(T)/M$ for different values of T. Interestingly, we find that the impact of T on the manufacturer's (retailer's) share of the channel profit is not uniform in T. That is, his (her) share of the channel profit initially decreases (increases) as T increases, and beyond a certain value of T, his (her) share of profit increases (decreases) as T increases. As for the slotting decision c, it initially increases as T increases, and beyond a certain value of T, c decreases in T.

Table 3.1 Profits, profit ratios, shelf decisions, and market saturation for different T: myopic retailer

T	5	10	25	40	45	60	70	75	80	150	250
$\Pi_M(T)\,(\times 10^8)$	0.5521	1.1610	3.4700	6.9368	8.4111	13.377	16.766	18.419	20.031	37.930	53.689
$\Pi_R(T)\,(\times 10^8)$	0.431	0.9618	3.4819	8.4025	10.407	15.367	17.543	18.375	19.076	22.821	23.177
$\frac{\Pi_M(T)}{\Pi_C(T)} \times 100\%$	55.69	54.05	47.73	40.98	40.05	42.04	44.95	46.45	47.90	61.61	70.15
$\frac{\Pi_R(T)}{\Pi_C(T)} \times 100\%$	44.31	45.95	52.27	59.02	59.95	57.96	55.05	53.55	52.10	38.39	29.85
$c = \sqrt{p}$	0.4167	0.4386	0.5278	0.6482	0.6801	0.7157	0.708	0.70	0.69	0.56	0.4306
$\frac{X(T)}{M} \times 100\%$	1.22	2.66	9.25	21.57	26.62	39.81	46.24	48.89	51.24	68.73	77.87

3.5 Far-sighted Retailer

In this section we study the model of a retailer focused on the long-term profit. We use a bar over the variables to signify the retailer's far-sighted focus. The retailer's problem for a given wholesale price trajectory $\bar{w}(\cdot)$ is

$$\max_{\bar{r}(\cdot),\bar{c}(\cdot)} \int_0^T \left\{\left[\bar{r}(t) - \bar{w}(t) - s_0\right]\bar{x}(t) - s\bar{c}^2(t)\right\} dt, \tag{3.26}$$

$$\bar{x}(t) = \dot{\bar{X}}(t) = \bar{c}(t)\left(M - \bar{X}(t)\right)\left(\alpha + \beta\bar{X}(t)\right)\left[1 - \gamma\bar{r}(t)\right], \bar{X}(0) = \bar{X}_0. \tag{3.27}$$

The retailer's Hamiltonian $\bar{H}_R(t)$ at time t given the wholesale price trajectory $\bar{w}(t)$ is

$$\begin{aligned}\bar{H}_R(t) &= H_R\left(\bar{X}(t), \bar{r}(t), \bar{c}(t), \bar{\lambda}_R(t); \bar{w}(t)\right) \\ &= \bar{c}(t)\,F\left(\bar{X}(t)\right)\left(1 - \gamma\bar{r}(t)\right)\left(\bar{r}(t) - \bar{w}(t) - s_0 + \bar{\lambda}_R(t)\right) - s\bar{c}^2(t).\end{aligned} \tag{3.28}$$

where $\bar{\lambda}_R(t)$, the shadow price associated with $\bar{X}(t)$, satisfies the adjoint equation

$$\dot{\bar{\lambda}}_R(t) = -\frac{\partial \bar{H}_R(t)}{\partial \bar{X}(t)}, \ \bar{\lambda}_R(T) = 0. \tag{3.29}$$

The first order conditions for profit maximization are given by

$$\frac{\partial \bar{H}_R}{\partial \bar{r}} = -\gamma F\left(\bar{X}\right)\left[\bar{r} - \bar{w} - s_0 + \bar{\lambda}_R\right] + F\left(\bar{X}\right)(1 - \gamma\bar{r}) = 0, \tag{3.30}$$

$$\frac{\partial \bar{H}_R}{\partial \bar{c}} = F\left(\bar{X}\right)(1 - \gamma\bar{r})\left[\bar{r} - \bar{w} - s_0 + \bar{\lambda}_R\right] - 2sc = 0. \tag{3.31}$$

Their solution yields the best response retail price $\bar{r}(t)$ and the best response shelf space decision $\bar{c}(t)$ as follows:

$$\bar{r}(t) = \frac{1 + \gamma\left(\bar{w}(t) + s_0 - \bar{\lambda}_R(t)\right)}{2\gamma}, \tag{3.32}$$

$$\bar{c}(t) = \frac{F\left(\bar{X}(t)\right)(1 - \gamma\bar{r}(t))\left[\bar{r}(t) - \bar{w}(t) - s_0 + \bar{\lambda}_R(t)\right]}{2s}. \tag{3.33}$$

We make a number of observations from (3.32) and (3.33). First, for a given $\bar{w}$, the retail price $\bar{r}$ is a decreasing function of $\bar{\lambda}_R$. Intuitively, when $\bar{\lambda}_R > 0$, meaning that there is a positive future value of additional sales, the retailer is willing to lower the retail price below the myopic level ($\bar{\lambda}_R = 0$). On the other hand, for a given $\bar{w}$, the shelf space allocation $\bar{c}^2$ is an increasing function of $\bar{\lambda}_R$. Thus, when there is positive future value of additional sales, the retailer is willing to allocate more shelf space to the product to increase its sales when compared to the myopic case. Finally, since the retailer's best response also depends on $\bar{\lambda}_R$, which is affected by the future portion of the announced wholesale price trajectory, i.e„ on $\{\bar{w}(\tau), t \leq \tau \leq T\}$, we see that the best response depends indeed on the entire wholesale price trajectory $\bar{w}$. This provides an important contrast to the case of the myopic retailer. As we will see, this dependence on $\bar{\lambda}_R$ requires us to treat $\bar{\lambda}_R$ as a state variable in the formulation of the manufacturer's optimization problem. It is this requirement that causes time inconsistency in the far-sighted case. For further details on the theory of the Stackelberg differential games, see Dockner et al. (2000).

By substituting (3.32) into (3.33), we have the slotting decision

$$\bar{c}(t) = \frac{F\left(\bar{X}(t)\right)\left[1+\gamma\left(\bar{\lambda}_R(t) - \bar{w}(t) - s_0\right)\right]^2}{8\gamma s}. \tag{3.34}$$

Substitution of (3.32) and (3.34) into (3.27) gives the instantaneous sales rate as a function of $\bar{w}$, i.e.,

$$\bar{x}(t) = \dot{\bar{X}}(t) = \frac{F^2\left(\bar{X}(t)\right)\left[1+\gamma\left(\bar{\lambda}_R(t) - \bar{w}(t) - s_0\right)\right]^3}{16\gamma s}, \bar{X}(0) = \bar{X}_0. \tag{3.35}$$

By substituting (3.32), (3.33), and (3.35) into (3.28), we obtain the maximized Hamiltonian

$$\bar{H}_R^*\left(\bar{X}, \bar{\lambda}_R; \bar{w}\right) = \frac{F^2\left(\bar{X}\right)\left[1+\gamma\left(\bar{\lambda}_R - \bar{w} - s_0\right)\right]^4}{64 s\gamma^2}. \tag{3.36}$$

The adjoint equation (3.29) for the shadow price $\bar{\lambda}_R(t)$ can now be written as

$$\begin{aligned} \dot{\bar{\lambda}}_R(t) &= -\frac{\partial \bar{H}_R}{\partial \bar{X}} = -\frac{\partial \bar{H}_R^*}{\partial \bar{X}} = \\ &= -\frac{F'\left(\bar{X}(t)\right)F\left(\bar{X}(t)\right)\left[1+\gamma\left(\bar{\lambda}_R(t) - \bar{w}(t) - s_0\right)\right]^4}{32 s\gamma^2}, \ \bar{\lambda}_R(T) = 0. \end{aligned} \tag{3.37}$$

The manufacturer takes the retailer's best response into consideration. Thus, his problem is

$$\max_{w(t)} \int_0^T \left[\bar{w}(t) - c_0\right] \bar{x}(t)\, dt, \tag{3.38}$$

subject to (3.35) and (3.37).

Note interestingly that the instantaneous sales rate $\bar{x}(t)$ and the retailer's shadow price $\bar{\lambda}_R(t)$ are the manufacturer's state variables. For further details on the theory of the Stackelberg differential games, see Dockner et al. (2000) or Jorgensen and Zaccour (2001).

The manufacturer's Hamiltonian

$$\begin{aligned} \bar{H}_M &= H_M\left(\bar{X}, \bar{\lambda}_R, \bar{w}, \bar{\lambda}_M, \bar{\mu}\right) = \left[\bar{w} - c_0 + \bar{\lambda}_M\right] \dot{X} + \bar{\mu}\dot{\bar{\lambda}}_R \\ &= \frac{F^2\left(\bar{X}\right)\left(\bar{w} - c_0 + \bar{\lambda}_M\right)\left[1 + \gamma\left(\bar{\lambda}_R - \bar{w} - s_0\right)\right]^3}{16\gamma s} \\ &\quad - \frac{\bar{\mu} F'\left(\bar{X}\right) F\left(\bar{X}\right)\left[1 + \gamma\left(\bar{\lambda}_R - \bar{w} - s_0\right)\right]^4}{32\gamma^2 s}, \end{aligned} \tag{3.39}$$

where $\bar{\lambda}_M$ and $\bar{\mu}$ are the shadow prices associated with $\bar{X}$ and $\dot{\bar{\lambda}}_R$, and they satisfy the adjoint equations

$$\begin{aligned} \dot{\bar{\lambda}}_M = &-\frac{\partial \bar{H}_M}{\partial \bar{X}} = -\frac{F'\left(\bar{X}\right) F\left(\bar{X}\right)\left(\bar{w} - c_0 + \bar{\lambda}_M\right)\left[1 + \gamma\left(\bar{\lambda}_R - \bar{w} - s_0\right)\right]^3}{8\gamma s} + \\ &\frac{\bar{\mu}\left(\left(F'\left(\bar{X}\right)\right)^2 - 2\beta F\left(\bar{X}\right)\right)\left[1 + \gamma\left(\bar{\lambda}_R - \bar{w} - s_0\right)\right]^4}{32\gamma^2 s}, \bar{\lambda}_M(T) = 0, \end{aligned} \tag{3.40}$$

$$\begin{aligned} \dot{\bar{\mu}} = &-\frac{\partial \bar{H}_M}{\partial \bar{\lambda}_R} = -\frac{3F^2\left(\bar{X}\right)\left(\bar{w} - c_0 + \bar{\lambda}_M\right)\left[1 + \gamma\left(\bar{\lambda}_R - \bar{w} - s_0\right)\right]^2}{16 s} + \\ &\frac{\mu F'\left(\bar{X}\right) F\left(\bar{X}\right)\left[1 + \gamma\left(\bar{\lambda}_R - \bar{w} - s_0\right)\right]^3}{8\gamma s}, \bar{\mu}(0) = 0. \end{aligned} \tag{3.41}$$

The boundary conditions $\bar{\lambda}(T) = 0$ and $\bar{\mu}(0) = 0$ arise from the fact that $x(T)$ and $\bar{\lambda}_R(0)$ are free. Note that $\dot{\lambda}_R(t) > 0$(resp. < 0) when $F'\left(\bar{X}(t)\right) > 0$ (resp. < 0).

To derive the optimal control $\bar{w}$, we use the first order condition

$$\frac{\partial \bar{H}_M}{\partial \bar{w}} = \frac{F^2\left(\bar{X}\right)\left[1+\gamma\left(\bar{\lambda}_R-\bar{w}-s_0\right)\right]^3}{16\gamma s} - \frac{3\gamma F^2\left(\bar{X}\right)\left(\bar{w}-c_0+\bar{\lambda}_M\right)\left[1+\gamma\left(\bar{\lambda}_R-\bar{w}-s_0\right)\right]^2}{16\gamma s}$$
$$+\frac{\bar{\mu}F'\left(\bar{X}\right)F\left(\bar{X}\right)\left[1+\gamma\left(\bar{\lambda}_R-\bar{w}-s_0\right)\right]^3}{8\gamma s} = 0. \tag{3.42}$$

Note that $1+\gamma\left(\bar{\lambda}_R-\bar{w}-s_0\right)=0$ is ruled out as it would lead to $\bar{x}=0$ according to (3.35). Then, the other factor in (3.38) gives the equilibrium wholesale price as

$$\bar{w} = \frac{F\left(\bar{X}\right)\left[1+\gamma\left(3c_0-3\bar{\lambda}_M+\bar{\lambda}_R-s_0\right)\right]+2F'\left(\bar{X}\right)\bar{\mu}\left[1+\gamma\left(\bar{\lambda}_R-s_0\right)\right]}{2\gamma\left(2F\left(\bar{X}\right)+F'\left(\bar{X}\right)\bar{\mu}\right)} \tag{3.43}$$

By substituting (3.44) into (3.38), (3.39), (3.42), and (3.43), we obtain a two-boundary value problem consisting of four differential equations. We solve this problem numerically for the same set of the parameters values in Section 3.3 for the myopic retailer.

The open-loop equilibrium we obtain in this case is time inconsistent. This is because $\bar{u}(t)$ does not stay at its initial value of zero. So if $\bar{u}(t)\neq 0$ at some time $\tau>0$, then it is in the manufacturer interest to re-solve the problem at τ and choose a new wholesale price trajectory from τ on that satisfies $\bar{u}(t)=0$. The intuition behind this behavior is that the manufacturer announces a wholesale price trajectory at time zero that leads to the retailer's decisions that are favorable to him. But by time τ, the retailer has executed her decisions in the interval $[0,\tau]$, and the manufacturer has no incentive to keep his promise. It is for this reason, we have assumed that the outset of this paper that the manufacturer commits to his announced wholesale price policy.

We make a number of observations. The retail and wholesale price trajectories no longer mimic the instantaneous demand trajectory (Figures 3.4-3.6), as they did in the myopic case. Instead, we observe two patterns of wholesale price trajectories: decreasing over time (for $T = 5, 15, 20, 25$) and initially decreasing then increasing (for $T = 45, 75, 100$). As for the retail price, for all values of T in this study, it increases over time. We observe that for lower values of T, the retail price trajectories $\bar{r}(\cdot)$ move downward as T increases, and beyond certain value of T, the retail price curves move upward as T increases. A similar observation holds for the wholesale price curves. We observe that for all values of T, the instantaneous sales rate trajectory $\bar{x}(\cdot)$ rises upward in T (Figure 3.5). On the other hand, the behavior of the slotting decision $\bar{c}\,(\cdot)$ is not constant over time, and is not uniform in T (see Figure 3.6). We see that $\bar{c}(\cdot)$ initially rises in T, and beyond a certain value of T, it moves downward as T increases. Unlike the case of the myopic retailer, the manufacturer's (retailer's) share of the channel profit in the far-sighted case is not inform in T (Table 3.2). Instead, his/her share of the entire channel profit initially decreases as T increases, and beyond a certain value of T, his/her share of the channel profit increases/decreases as T increases.

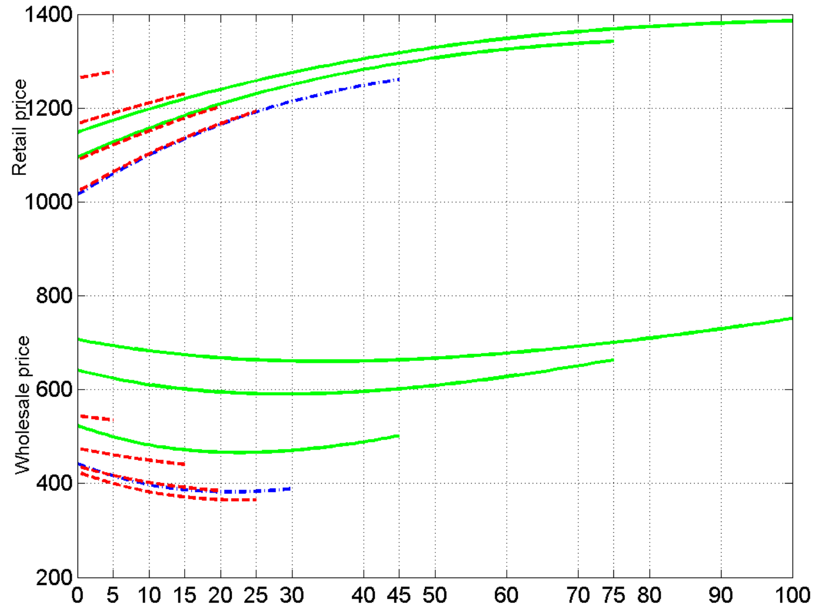

Figure 3.4: Wholesale price and retail price trajectories: Far-sighted Readier

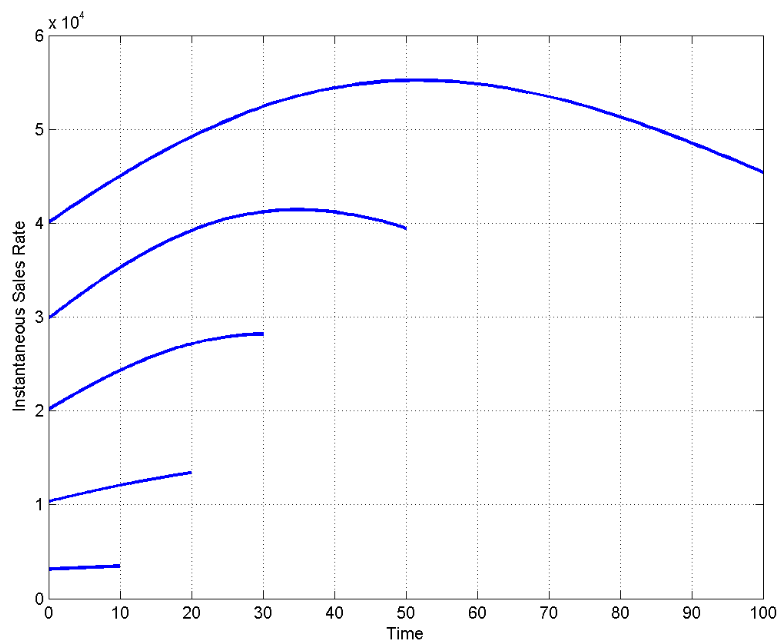

Figure 3.5: Instantaneous sales rate: Far-sighted Readier

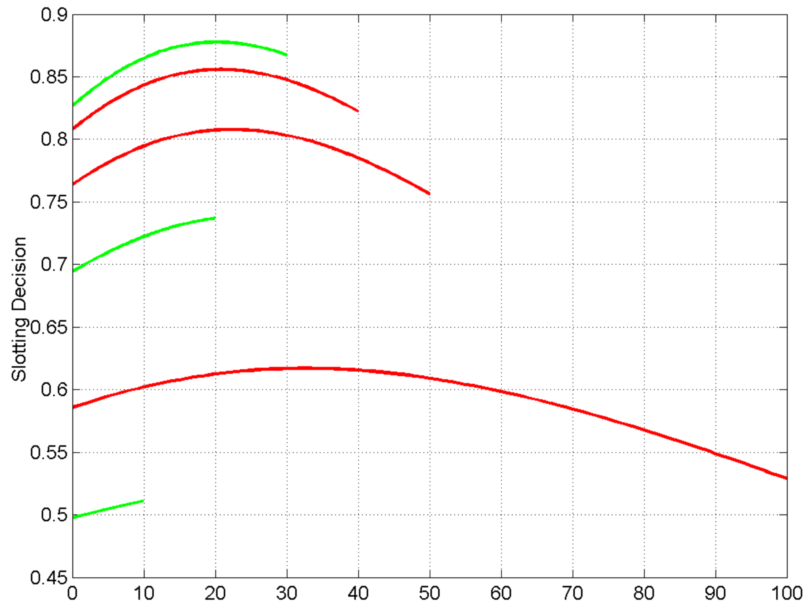

Figure 3.6: Shelf space allocation: Far-sighted Readier

Table 3.2: Profits, profit ratios, and market saturation levels: Far-sighted retaier

T	5	10	15	25	40	75	100
$\bar{\Pi}_M(T)\,(\times 10^8)$	0.6002	1.3343	2.2860	5.4804	12.085	22.827	30.481
$\bar{\Pi}_R(T)\,(\times 10^8)$	0.3801	1.0138	1.9584	5.1236	8.3747	11.563	13.316
$\frac{\bar{\Pi}_M(T)}{\bar{\Pi}_C(T)}\times 100\%$	64.18	59.00	56.47	56.50	63.47	69.06	71.40
$\frac{\bar{\Pi}_R(T)}{\bar{\Pi}_C(T)}\times 100\%$	35.82	41.00	43.53	43.50	36.53	30.94	28.60
$\frac{X(T)}{M}\times 100\%$	1.35	3.30	6.45	19.28	33.26	44.93	50.75

We use the same parameter values as ones used for in the myopic retailer to conduct a numerical study in the case of a far-sighted retailer (Table 3.2). Like the case of the myopic retailer, the manufacturer's (retailer's) share of the channel profit is not uniform in T. His (her) share of the entire channel profit initially decreases (increases) as T increases, and beyond a certain value of T, his (her) share of profit increases (decreases) as T increases.

3.6 Myopic Focus versus Far-sighted Focus

So far, we have derived the pricing and slotting decisions separately for the myopic and far-sighted retailers. We now address the following interesting questions: Will the retailer be better off with a far-sighted or myopic focus? If so, when? What is the manufacturer preference for the retailer's focus? Will the manufacturer and the retailer have conflict over the retailer's focus? If so, when? For a fixed transfer price charged by the manufacturer, the retailer is certainly better off having a far-sighted focus. However, the manufacturer adjusts his wholesale price accordingly. Therefore, it is not obvious that the retailer will always be better off with a far-sighted focus.

In Table 3.3, we report the results based on our numerical computations, and compare the player's life-cycle profits in the far-sighted retailer case to their profits in the myopic retailer case. We observe that the manufacturer and the retailer may have four different combinations of preferred retailer's profit foci, i.e., both prefer a far-sighted retailer, both prefer a myopic retailer, one prefers a far-sighted retailer and the other prefers a myopic retailer, and vice verse. Specifically, the manufacturer as well as the supply chain prefers a far-sighted retailer when the market saturation level is low (corresponding to the cases of $T = 30$ and $T = 50$), whereas he prefers a myopic retailer when the market saturation level is high (corresponding to the cases of $T = 250$ and $T = 1200$). When the market saturation level is low, the retailer's preference is aligned with the manufacturer. That is, the retailer prefers the manufacturer to offer wholesale prices assuming a far-sighted profit focus. However, as the market saturation level increases furthermore, the retailer switches her preference to myopic profit focus. If the market saturation level is extremely high (for example when $T = 1200$), the retailer again prefers to be far-sighted.

Table 3.3: Preferred Retailer foci

T	M's profit ($\times 10^8$)	R's profit ($\times 10^8$)	C's profit($\times 10^8$)	% Saturation Level
30	**7.664**, 4.467	**6.476**, 4.818	**14.140**, 9.286	25.29,12.63
50	**16.107**, 10.006	9.713, **12.295**	**25.822**, 22.301	38.39,31.48
250	51.021, **53.689**	16.791, **23.177**	67.812, **76.866**	62.23,77.87
1200	84.314, **98.274**	**19.457**, 18.707	103.777, **116.98**	75.44, 92.08

Note. Parameters are $M = 1\times 10^7$,$X_0 = 0, \alpha = 0.016$, $\beta = 8\times 10^{-9}$, $\gamma = 5\times 10^{-4}$,$s_0 = \$20, s = 5 \times 10^7$, and $c_0 = \$100$.

3.7 Conclusion

In this paper, we study the dynamic wholesale and retail pricing and shelf-space allocation in a decentralized durable product supply chain consisting of a manufacturer and a retailer. We formulate the problem as an open-loop Stackelberg differential game with the manufacturer as the leader and the retailer as the follower. Our demand model extends the Bass-type diffusion model by incorporating the impact of retail price and shelf space allocation on the retail demand. We study two retailer foci: myopic and far-sighted. We provide analytical results and numerical analysis to obtain the Stackelberg equilibria for different life-cycle lengths. Furthermore, we develop insights into conditions under which the both players prefer a far-sighted retailer, both prefer a myopic retailer, and one prefers a far-sighted and the other prefers a myopic retailer and vice verse.

Our analysis opens up several opportunities for future research. First, the open-loop equilibrium that we use is time inconsistent in the far-sighted retailer case. It would be interesting to look into the feedback Stackelberg equilibria and related time consistency issues. Second, further analysis of our model could be carried out the issue of channel coordination. Third, our model can be extended to allow for multiple competing retailers. This could combine our demand model with earlier research by Eliashberg and Jeuland (1986) and Savin and Terwiesch (2005). Finally, our model can be extended to allow for multiple products competing for a limited shelf space.

Chapter 4

A Review of Stackelberg Differential Game Models in Supply and Marketing Channels

4.1 Introduction

Stackelberg differential game (DG) models have been used to study issues such as inventory and production policies, outsourcing, capacity and shelf space allocation decisions, dynamic competitive advertising strategies and pricing for new products in the marketing literature (see Erickson 1995 for a review of Nash game models in competitive advertising strategies). Stackelberg DG models have also been used to the government's subsidy policy in new technology (Jørgensen and Zaccour 1999); R&D investment in the energy industry (Harris and Vickers 1995), and monetary and fiscal policies in economics (Xie 1997).

Most studies in the supply chain management have used the single-period newsvendor model as a means to study the strategic interactions between the channel members. For fashionable products, the one-period newsvendor model may be an appropriate approach. However, there are many market situations where this is not appropriate. There has been some work recently that investigates the dynamic interactions between the channel members. Recently, a number of papers have applied DG models to treat dynamic interactions between the channel members in decentralized supply and marketing channels. This review focuses on these applications. Specifically, we review papers that analyze retail and wholesale pricing and/or advertising strategies, slotting and pricing decisions to launch innovative durable products, pricing and production, and investment in supply chain infrastructure. We focus primarily on Stackelberg equilibria as the solution concept for the games under consideration. We shall begin our review with an introduction to the basics of the Stackelberg DGs. We then summarize the important managerial insights obtained in each of the studies being reviewed. Finally, we point out future research avenues for applications of DGs in supply chain management.

The review is organized as follows. In Section 4.2, we introduce the basic concepts of the Stackelberg differential games. In Section 4.3, we review the models that derive the Stackelberg equilibria in the area of supply chain management. Section 4.4 dicusses the applications to marketing channels. Miscellaneous applications are reviewed in Section 4.5. In Section 4.6, we conclude the paper and point out some future research directions.

4.2 Basics of the Stackelberg Differential Game

A differential game has the following structure (1) the state of the dynamic system at any time t is characterized by a set of variables called the state variables, (2) there are controls to be decided by the game players, (3) the evolution of the state variables over time is described by a set of differential equations involving both state and control variables, and (4) each player has an objective function that he/she wants to maximize by his/her choice of decisions.

We illustrate the basics of a Stackelberg differential game involving two players playing the game over a fixed finite horizon T: a leader and a follower. Let X denote the vector of state variables, w denote the control vector of the leader, and r denote the control vector of the follower. The sequence of the play is as follows. The leader announces the control path $w(\cdot)$. Then the follower decides on the control path $r(\cdot)$. Let π_R and π_M denote the follower's and the leader's instantaneous profit functions, respectively. The solution procedure for the Stackelberg differential game is the backward induction. That is, we first solve the follower's problem by deriving the follower's best response to the leader's announced policy. We then substitute the follower's response into the leader's problem to solve for the leader's optimal policy. This policy of the leader to together with the retailer's best response to this policy constitutes a Stackelberg equilibrium solution. Formally, the follower's problem is

$$\begin{aligned} \max_{r(\cdot)} \{J_R(X_0, r(\cdot); w(\cdot)) &= \int_0^T e^{-\rho t} \pi_R (X(t), w(t), r(t))\, dt \Big\}, \\ \dot{X}(t) &= F(X(t), w(t), r(t)), \quad X(0) = X_0, \end{aligned} \tag{4.1}$$

where the function F represents the rate of sales, ρ is the follower's discount rate, and X_0 is the initial condition. The follower's Hamiltonian

$$H_R(X, r, \lambda_R, w) = \pi_R(X, w, r) + \lambda_R F(X, w, r), \tag{4.2}$$

where λ_R is the vector of the shadow prices associated with the state variable X, and it satisfies the adjoint equation

$$\dot{\lambda}_R = \rho\lambda_R - \frac{\partial H_R(X, r, \lambda_R, w)}{\partial X}, \quad \lambda_R(T) = 0. \tag{4.3}$$

Here we have suppressed the argument t as is standard in the control theory literature, and we will do whenever convenient and when there arises no confusion in doing so.

The necessary optimality condition for the follower's problem satisfy

$$\frac{\partial H_R}{\partial r} = 0 \Longrightarrow \frac{\partial \pi_R(X, w, r)}{\partial r} + \lambda_R \frac{\partial F(X, w, r)}{\partial r} = 0. \tag{4.4}$$

We assume that the Hamiltonian H_R is jointly concave in the variables X and r for any given w. Then condition (4.4) is sufficient for the optimality of r for a given w. From the necessary condition (4.4), we derive the follower's best response $r(X, w, \lambda_R)$.

The leader's problem is

$$\begin{aligned}
\max_{w(\cdot)} \left\{ J_M(X_0, w(\cdot)) \right. &= \left. \int_0^T e^{-\mu t} \pi_M(X, w, r(X, w, \lambda_R))\, dt \right\}, \\
\dot{X} &= F(X, w, r(X, w, \lambda_R)), \quad X_0(0) = X_0 \qquad (4.5) \\
\dot{\lambda}_R &= \rho \lambda_R - \frac{\partial H_R(X, r(X, w, \lambda_R), \lambda_R, w)}{\partial X}, \quad \lambda_R(T) = 0, (4.6)
\end{aligned}$$

where μ is the leader's discount rate and the differential equations in (4.5) and (4.6) are obtained by substituting the follower's best response $r = r(X, w, \lambda_R)$ in the state equation (4.1) and the adjoint equation (4.3), respectively.. We formulate the leader's Hamiltonian

$$\begin{aligned}
H_M &= \pi_M(X, \lambda_R, w, r(X, w, \lambda_R), \lambda_M, \psi) + \lambda F(X, w, r(X, w, \lambda_R)) \\
&\quad - \mu \frac{\partial H_R(X, r(X, w, \lambda_R), \lambda_R, w)}{\partial X},
\end{aligned} \tag{4.7}$$

where λ_M and μ are the shadow prices associated with X and λ_R, respectively, and they satisfy the adjoint equations

$$\begin{aligned}
\dot{\lambda}_{M} &= \mu\lambda_{M} - \frac{\partial H_{M}\left(X, \lambda_{R}, w, r(X, w, \lambda_R), \lambda_{M}, \psi\right)}{\partial X} \\
&= \mu\lambda_{M} - \frac{\partial \pi_{M}\left(X, w, r(X, w, \lambda_R)\right)}{\partial X} - \lambda_{M}\frac{\partial F\left(X, w, r\left(X, w, \lambda_{R}\right)\right)}{\partial X} \\
&\quad -\mu\frac{\partial^2 H_{R}\left(X, r\left(X, w, \lambda_{R}\right), \lambda_{R}, w\right)}{\partial X^2},
\end{aligned} \tag{4.8}$$

$$\begin{aligned}
\dot{\psi} &= \mu\psi - \frac{\partial H_{M}\left(X, \lambda_{R}, w, r(X, w, \lambda_R), \lambda_{M}, \psi\right)}{\partial \lambda_{R}}, \\
&= \mu\psi - \lambda_{M}\frac{\partial F\left(X, w, r\left(X, w, \lambda_{R}\right)\right)}{\partial \lambda_{R}} - \mu\frac{\partial^2 H_{R}\left(X, r\left(X, w, \lambda_{R}\right), \lambda_{R}, w\right)}{\partial \lambda \partial X},
\end{aligned} \tag{4.9}$$

with the boundary conditions $\lambda_{M}(T) = 0$ and $\psi(0) = 0$.

Note that we use the open-loop Stackelberg solution concept to solve the leader and the follower's problem. There are two types of Stackelberg equilibria: *open-loop and closed-loop.* Open-loop solutions are said to be static in the sense that decisions can be derived at one point in time, independent of the state variable solutions obtained beyond that time. In contrast, closed-loop equilibrium strategies are functions of time as well as the state variable. They are subgame perfect, if they do not depend on initial conditions (closed-loop strategies that do not depend on initial conditions are called *feedback* strategies.) They are perfect state-space equilibria because the necessary conditions for optimality are required to hold for all values of the state variables, not just values that lie on the optimal state-space trajectories. Therefore, the solutions obtained should remain optimal even after the game has begun. It is known that most open-loop Stackelberg equilibria have an inherent instability of being time inconsistent. This means that given the opportunity to revise his strategy at any instant of time after the initial one, the leader would like to choose another strategy than the one he chose at the initial instant of time. Thus, an open-loop Stackelberg equilibrium only make sense if the leader can credibly pre-commit to his strategy.

4.3 Supply Chain Management Applications

In this section we shall review applications in the supply chain management area. the supplier or the manufacturer decides on the wholesale price and/ or his production rate, whereas the retailer's decision variables are retail price and/or shelf-space allocation.

4.3.1 Gutierrez and He (2007): Life-Cycle Channel Coordination

Gutierrez and He (2007) study a decentralized channel composed of a manufacturer and a retailer to launch an innovative durable product (IDP). The manufacturer, being the leader, announces the wholesale price trajectory first, and then the retailer follows by deciding on the retail price trajectory. While the manufacturer is assumed to maximize his life-cycle profit from selling the IDP, the authors consider two types of retailer's foci: (1) a far-sighted strategy of maximizing the life-cycle profit from the IDP, and (2) a myopic strategy of maximizing the instantaneous profit rate at any time t. They address the following research questions: Does the manufacturer prefer the retailer to be far-sighted or myopic? Does the retailer prefer the manufacturer to set the wholesale price assuming she is far-sighted or myopic?

Far-sighted Retailer. Consider the case when the retailer is far-sighted. For a given wholesale price path $w(\cdot)$, the retailer decides a retail price path $r(\cdot)$ by solving the problem:

$$\max_{r(\cdot)} \int_0^T \left\{ \left[r\left(t\right) - w\left(t\right) - s \right] \dot{X}\left(t\right) \right\} dt \tag{4.10}$$
$$\dot{X}\left(t\right) = \left(M - X\left(t\right)\right) \left(\alpha + \beta X\left(t\right)\right) \left(1 - \gamma r\left(t\right)\right), X\left(0\right) = X_0, \tag{4.11}$$

where s is the selling cost including any opportunity cost, α and β are internal and external influence parameters, γ is the price sensitivity parameter, and X is the initial value of the sold market. Let

$$F(X) = \left(M - X\right) \left(\alpha + \beta X\right).$$

The manufacturer's problem is

$$\max_{w(\cdot)} \int_0^T \left[w(t) - c_0 \right] \dot{X}(t) dt, \tag{4.12}$$
$$\dot{X}(t) = \frac{F(X(t)) \left\{ 1 - \gamma \left[w(t) + s - \lambda_R(t) \right] \right\}}{2}, X\left(0\right) = X_0, \tag{4.13}$$
$$\dot{\lambda}_R(t) = -\frac{F'(X(t)) \left\{ 1 - \gamma \left[w(t) + s - \lambda_R(t) \right] \right\}^2}{4\gamma}, \lambda_R\left(T\right) = 0, \tag{4.14}$$

where c_0 is the per unit production cost and λ_R is the retailer's shadow price. Note that X and λ_R are the manufacturer's two state variables, and their evolution incorporates the retailer's best response.

Myopic Retailer. The retailer selects $r(t)$ to maximize her instantaneous profit rate $\left[r\left(t\right) - w\left(t\right) - s \right] \dot{X}\left(t\right)$ subject to (4.11). The manufacturer maximizes his life cycle profits and his optimization problem is given by (4.12) and (4.13).

Gutierrez and He identified a conflict of preferences between the manufacturer and the retailer. First, a manufacturer does not always prefer the retailer to take a long-term optimization focus; that is, he sometimes is better off if the retailer is myopic. On the other hand, the retailer's preferred focus changes with the market conditions, and it does not always agree with the manufacturer's preference. Their results show that both the manufacturer and the retailer are better off if the retailer is far-sighted when the final market is insufficiently penetrated. However, if the market saturation level is high like at the end of the planning horizon, the manufacturer will shift his preference and will be better off with a myopic retailer, while the retailer prefers that the manufacturer sets the wholesale prices assuming that she is far-sighted. However, monitoring the retailer sales volume or retail price becomes an implementation necessity when the manufacturer offers a wholesale price contract assuming the retailer is myopic. It is not immediately obvious that a seemingly myopic retailer behavior may enhance the performance of the supply chain.

4.3.2 He and Sethi (2007): Pricing and Slotting Decisions

He and Sethi (2007) extend the work of Gutierrez and He (2007) by considering the impact of shelf space allocation on the retail demand. They assume the retail demand to be an increasing and concave function of the merchandise displayed on the shelf. They do this by introducing the multiplicative term $\sqrt{p(t)}$to the right-hand side of (4.11), where $p(t)$ is the shelf space allocated to the product at time t. They consider a linear cost of shelf space to be included in the retailer's objective function (4.10). They solve for equilibrium wholesale and retail pricing and slotting decisions, In connection with the strategic foci of the retailer, myopic and far-sighted, they obtain results similar to those in Gutierrez and He (2007).

4.3.3 Eliashberg and Steinberg (1987): Pricing and Production Decisions

Eliashberg and Steinberg (1987) consider a decentralized assembly system composed of a single manufacturer (he) and a single distributor (she). The distributor processes further the product whose demand has seasonal fluctuations. Pekelman (1974) uses a general time varying demand function $D_R(t) = a_R(t) - b_R(t) P_R(t)$, where $a_R(t)$ is the time varying total market potential, b_R is the coefficient of price sensitivity, and $P_R(t)$ is the distributor's price. Eliashberg and Steinberg (1987) assumes that $b_R(t)$ is constant. Specifically, they assume that the time varying market demand potential $a_R(t)$ has the following form

$$a_R(t) = -\alpha_1 t^2 + \alpha_2 t + \alpha_3, 0 \leq t \leq T$$

where $T = \frac{\alpha_2}{\alpha_1}$. α_1, α_2 and $\alpha_3$are positive parameters.

The manufacturer and the distributor play a Stackelberg game with the manufacturer acting as the leader and the distributor the follower. The distributor decides on the processing, pricing, and inventory policies. The manufacturer decides the inventory and pricing policies. The distributor's problem, given that P_M is the constant transfer price charged by the manufacturer, is

$$\begin{aligned} \max_{I_R(\cdot),P_R(\cdot)} \{J_R &= \int_0^T \left[P_R(t) D_R(t) - P_M Q_R(t) - f_R(Q_R(t)) - h_R I_R(t)\right] dt \Big\}, \\ \dot{I}_R(t) &= Q_R(t) - D_R(t), \ I_R(0) = I_R(T) = 0, \ I_R(t) \geq 0, \end{aligned}$$

where $Q_R(t)$ is her processing rate, $f_R(\cdot)$ is her processing cost function, h_R is her inventory holding cost per unit, and $I_R(t)$ is her inventory level. Similar to

Pekelman, Eliashberg and Steinberg (1987) use a linear holding cost function. They also assume that the processing cost function f_R is increasing and strictly convex.

The authors show that the distributor follows a two-part processing strategy. During the first part of her processing schedule, she processes at a constantly increasing rate. This policy builds up inventory initially and then draws down inventory until it reaches zero at a time t_R^*. During the second part, which begins at the stockless point t_R^*, she processes at precisely her market demand rate. She also follows a two-part pricing strategy. The price charged by the distributor is first increasing at a decreasing rate and then decreasing at an increasing rate. The inventory builds up for a while and then reaches zero; from then on, the distributor processes just enough to meet demand.

An intuitive interpretation is as follows. The distributor, facing a seasonal demand which increases and then decreases, can smooth out her processing operations. The reason that she may carry inventory initially is due to the assumption of convex processing cost. In other words, if she does not carry any inventory throughout the entire horizon, she could incur higher costs due to the convexity of her processing cost function.

Now we turn to the manufacturer's problem. Let Q_M, I_M, h_M, and $f_M(\cdot)$ denote his production rate, inventory level, inventory holding cost per unit, and production cost function, respectively. The manufacturer's problem is given by

$$\max_{Q_M(\cdot),P_M} \left\{J_M = \int_0^T \left[(P_M - C_M)\, Q_R(P_M,t) - f_M\left(Q_M(t)\right) - h_M I_M(t)\right] dt\right\}$$

$$\begin{aligned} s.t.\quad & \dot{I}_M(t) = Q_M(t) - Q_R(P_M,t),\ \ I_M(0) = I_M(T) = 0, \\ & Q_M(t) \geq Q_R(t) \geq 0,\ \ I_M(t) \geq 0, \end{aligned}$$

where C_M is his cost per unit transferred to the distributor and $Q_R(P_M, t)$ is the best response of the distributor given P_M. They assume that $P_M > C_M$. Different from Pekelman (1974) which uses a strictly convex increasing production cost function, Eliashberg and Steinberg (1987) use a quadratic production cost function.

The authors characterize the manufacturer's policies as follows. The manufacturer follows a two-part production policy. During the first part, he produces at a constantly increasing rate. During the second part, which begins at the manufacturer's stockless point t_M^*, he produces at exactly the distributor's processing rate. In general, if the manufacturer's inventory holding cost per unit is sufficiently low and the distributor's processing efficiency and inventory holding cost per unit are high, then the manufacturer can also smooth out his operations.

4.3.4 Desai (1992): Marketing-Production Channel under Independent and Integrated channel

Desai (1992) analyzes the production and pricing decisions in a marketing-production channel in which the retailer buys the good from the manufacturer and sells it to the final consumers. The retailer faces a price-dependent seasonal demand. Like Eliashberg and Steinberg (1987), Desai (1992) uses a general time varying demand function $D_R(t) = a_R(t) - bp_R(t)$, where $a_R(t) = \alpha_1 + \alpha_2 sin\left(\alpha_3 t\right)$; $\alpha_3 T = \pi$, where T is the duration of the season. Different from Pekelman (1974) and Eliashberg and Steinberg (1987) who use a linear holding cost function, Desai (1992) assumes quadratic total production and holding cost functions and does not allow the retailer to carry inventory.

Desai (1992) shows that once the production rate becomes positive, it does not become zero again, which implies production smoothing. However, none of the gains of production smoothing are passed on to the retailer. The optimal production rate and the inventory policy are a linear combination of the nominal demand rate, the peak demand factor, and the salvage value, and the initial inventory. In the scenario where the retailing operation does not require an effort, the pricing policies of the manufacturer and the retailer and the production policy of the manufacturer have a synergistic effect, i.e., an increase in the manufacturer's price or production rate or the retailer's price leads to an increase in the rate of change of inventory. However, in the scenario where the retailing operation does benefit from the effort, the retailer's pricing policy may not necessarily be synergistic with the other policies.

4.3.5 Desai (1996): Marketing-production channel under seasonal demand

Desai (1996) differs from Eliashberg and Steinberg (1987) in three ways. First, Eliashberg and Steinberg (1987) restrict themselves to a contract in which the manufacturer's wholesale price remains constant throughout the season, while Desai (1996) allows the manufacturer to change the wholesale price over time (i.e., more general arrangement). Second, he does not allow the retailer to carry inventory. Third, he assumes a quadratic holding cost function, while Eliashberg and Steinberg (1997) use a linear cost function.

The manufacturer makes the production and pricing decisions while the retailer decides on the processing rate and pricing policies. Desai (1996) considers three types of contracts: contracts under which the manufacturer charges a constant price throughout a season, contracts under which the retailer processes at a constant rate throughout the season, and contracts under which the manufacturer and retailer cooperate to make decisions jointly. He compares the optimal policies under three different contract types. He shows that the type of contract does not significantly impact the retailer's price. However, the type of contract has an impact on the manufacturer's price and the production rate as well as the retailer's processing rate. If the demand is not highly seasonal, a constant processing rate contract will lead to higher production and processing rates, and a lower manufacturer's price compared to a constant manufacturer's price contract.

4.3.6 Kogan and Tapiero (2007)a: Inventory Game with Endogenous Demand

Kogan and Tapiero (2007)a consider a supply chain consisting of a manufacturer (leader) and a retailer (follower) facing time-dependent endogenous demand depending on price set by the retailer. Furthermore, the retailer has a finite production capacity, which requires consideration of the effect of inventory. Thus, the retailer must decide on the price $p(t)$as well as the production rate $u(t)$. The manufacturer, on the other hand, has ample capacity and must decide on only the wholesale price. The authors assume that the game is played over a season of length T which includes a short production period $\left[t_s, t_f\right]$ such as the Christmas time, during which both the demand potential $a(t)$ and the customer price sensitivity $b(t)$ are high. Specifically, the demand $D_R = a(t) + b(t)\,p(t)$, where

$$a(t) = \begin{cases} a_1,\ t < t_S \ \mathit{and}\ \ t \geq t_f \\ a_2,\ t_S \leq t < t_f \end{cases}$$

$$b(t) = \begin{cases} b_1,\ t < t_S \ \mathit{and}\ \ t \geq t_f \\ b_2,\ t_S \leq t < t_f \end{cases}$$

with $a_2 > a_1$ and $b_2 > b_1$. Kogan and Tapiero (2007)a limit the manufacturer to set a constant wholesale price w_1 in the regular periods and $w_2 \leq w_1$ in the promotion period. The manufacturer aims to maximize his profits, and so his problem is

$$\max_{w} \int_0^T \left[w(t)\, u(t) - c_s u(t) \right] dt$$

subject to $w(t) \geq c_s$.

The retailer's problem is

$$\begin{aligned} & \max_{u,p} \int_0^T \left[w(t)\,(a(t) - b(t)\,p(t)) - c_r u(t) - w(t)\,u(t) - h(X(t)) \right] dt \\ s.t. \quad & \dot{X}(t) = u(t) - (a(t) - b(t)\,p(t)); \\ & 0 \leq u(t) \leq U; \\ & a(t) - b(t)\,p(t) \geq 0; p(t) \geq 0. \end{aligned}$$

Kogan and Tapiero (2007)a obtain the optimal solution to the centralized problem as well as the Stackelberg equilibrium. They require the system to begin in a steady state at time 0, go to a transient state in response to promotional decisions, and then revert back to the steady state by the end of the season at time T. Thus, their solution is meant to be implemented in a rolling horizon fashion.

Under reasonable conditions on the parameters, the authors are able to derive formulas for the equilibrium values of the regular and promotional wholesale prices for the manufacturer. Then they show that it is beneficial for the retailer to change pricing and processing policies in response to the reduced promotional wholesale price and the increased customer price sensitivity during the promotion. The change is characterized by instantaneous jump upward in quantities ordered and downward in retailer prices at the instant when the promotion period starts and vice versa just when the promotion ends. In fact, the retailer starts lowering her prices sometime before the promotion starts. This causes a greater demand when the promotion period begins, thereby taking advantage of the reduced wholesale price during the promotion. This is accomplished gradually to strike a trade-off between the surplus/ backlog cost and the wholesale price over time. Specifically, any reduction in the wholesale price results first in backlog and then in surplus. An opposite scenario takes place on the side when the promotion periods ends.

In the symmetric case when unit backlog and surplus costs are equal, the authors obtain the typical equilibrium as shown in Figure 4.1. As can be seen, due to symmetric costs, the transient solution is symmetric with respect to the midpoint of the promotion phase.

Finally, the authors show that due to inventory dynamics, the traditional two-part tariff does not coordinate the supply chain as it does in the static case. This is because the manufacturer when setting the promotional wholesale price ignores not only the retailer's profit margin from sales, but also the profit margin from handling inventories. Of course, in the very special case when the manufacturer fixes a wholesale price throughout the season, the retailer's problem becomes identical to the centralized problem and the supply chain is coordinated.

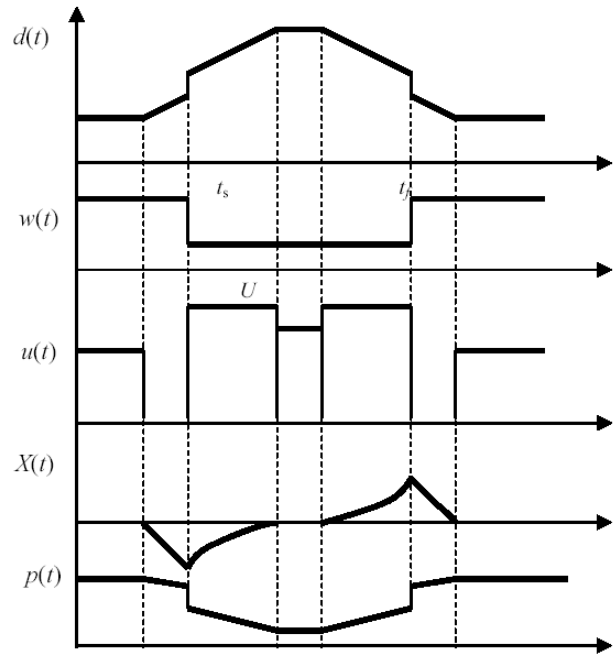

Fig 4.1: Optimal policies with promotion

4.3.7 Kogan and Tapiero (2007)b: Inventory Game with Exogenous Demand

This game differs from Kogan and Tapiero (2007)a in two ways. First, the demand is no longer price-dependent, and so pricing is not an issue. This simplifies the problem. Second, both the manufacturer and the retailer have limited capacities in contrast to the previous game in which only the retailer has a limited capacity. This complicates the ordering decisions which require multiple switching points induced by coordinating inventory decision and capacity limitation. While the demand is not price dependent, it varies with time. Thus, the optimal control problems faced by each of the players is a standard dynamic inventory problem as discussed, e.g., by Bensoussan et al. (1974).

Since production control appears linearly in these problem, the optimal production can have three possible regimes: maximum production, (singular) production on demand, and zero production. Sequencing of these regimes depends on the time-varying nature of the demand and inventory cost parameters.

Kogan and Tapiero (2007)b consider the retailer as the leader and the manufacturer as the follower. They solve the problem explicitly in a special case of demand.

As for coordination, the authors show that if the retailer pays the manufacturer for his inventory related cost, then the centralized solution is attained. They also show that a two-part tariff contract can also be obtained to coordinate the supply chain.

4.3.8 Kogan and Tapiero (2007)c: Production Balance with Subcontracting

Kogan and Tapiero (2007)c consider a supply chain consisting of one manufacturer (follower) and one supplier/ subcontracter (leader). The supplier has ample capacity and so the inventory dynamics is not an issue. The manufacturer, on the other hand, has a limited capacity, and his decisions depend on the available inventory. The product demand rate at time t is $a(t)D$, where D is a random variable and $a(t)$ is known as the demand shape parameter. This assumption is reasonable in the case of fashion goods.

The realization of D is observed only at the end of a short selling season, and as a result, the manufacturer can only place an advance order to obtain an initial end-product inventory, which is then used to balance production over time with the limited in-house capacity.

Thus, the problem is a dynamic version of the newsvendor problem which incorporates production control. The supplier's problem is to set a constant wholesale price to maximize his profit from the advance order from the manufacturer. The manufacturer must decide on the advance order as well as the production rate over time in order to minimize his total expected cost of production, inventory/ backlog, and advance order. The authors are able to transform this problem to a deterministic optimal control problem, which can be solved to obtain the manufacturer's best response to the supplier's announced wholesale price.

Kogan and Tapiero (2007)c consider unit in-house production cost to be greater than the supplier's cost. They show that if the supplier makes profit (i.e., has a positive margin), then the manufacturer produces more in-house and subcontracts less than the centralized solution. This is due to the double marginalization not unlike in the static newsvendor problem. Furthermore, if the manufacturer is myopic, he also orders less than the centralized solution even though he does not produce in-house since he does not take into account the inventory dynamics.

While the optimal production rate over time would depend on the nature of the demand, it is clear that optimal production will have intervals of zero production, maximum production, and a singular level of production. The authors also solve a numerical example and obtain the equilibrium wholesale price, the manufacturer's advance order quantity, and his production rate over time.

4.3.9 Kogan and Tapiero (2007)d: Outsourcing Game

Kogan and Tapiero (2007)d consider a supply chain consisting of one producer and multiple suppliers, all having limited production capacities. The suppliers are the leaders and set their wholesale prices over time that maximize their profits. In response to these, the producer decides on his in-house production rate and supplements this by ordering from a selection of suppliers over time in order to meet a random demand at the end of a planning period T. The producer incurs a penalty for any unmet demand. Unlike in the previous section, no assumption regarding the cost of in-house production and the supplier's production cost is made in this model. The producer's goal is to minimize his expected cost.

As in the previous section, the authors transform the producer's problem into a deterministic optimal control problem. Because the producer's problem is linear in his decisions, his production rate can have one of three regimes as in the previous section, and his odering rate from any chosen supplier will also have similar three regimes.

The authors show that the greater the wholesale price of a supplier, the longer the producer waits before he orders from that supplier. This is because the producer has an advantage over that supplier up to and until a breaking point in time for outsourcing to this supplier is reached. As in the previous model, here aslo when a supplier sets his wholesale price strictly above his cost over the entire horizon, then the oursourcing order quantity is less than that in the centralized solution.

Once again, the supply chain is coordinated if the suppliers set their wholesale price equal to their cost and get lump sum transfers from the producer.

4.3.10 Bykadorov et al. (2007): Trade Discount Policies

Bykadorov et al. studies the trade discount in the context of differential games framework. The manufacturer controls the dynamic discount rate and the retailer controls the dynamic shelf-price (pass-through). The paper considers two cases: the case with the manufacturer being the game leader and the case with the retailer being the game leader.

4.4 Marketing Channel Applications

4.4.1 Jørgensen et al. (2000): Dynamic Cooperative Advertising

Jørgensen et al. (2000) studies a two-member channel in which a manufacturer and a retailer can make advertising expenditures that have both long-term and short-term impacts on the retail demand.

The manufacturer controls his rate of short-term advertising effort and long-term advertising advertising effort. The manufacturer and the retailer can enter into a cooperative advertising program in which the manufacturer pays a certain share of the retailer's advertising expenditure. The manufacturer is a Stackelberg game leader in designing the program: He announces his advertising strategies and support rates for the retailer's long-term and short-term advertising efforts.

The results show that both the manufacturer and the retailer prefer full support to any of the two kinds of support, which is preferred to no support at all.

4.4.2 Jørgensen et al. (2001) : Impact of Stackelberg Leadership on Channel Efficiency

Jørgensen et al. (2001) studies the effects of strategic interactions in both pricing and advertising in a channel consisting of a manufacturer and a retailer. They consider three scenarios: each channel member simultaneously decides its margin and advertising rate, the scenaro with the retailer acting as the leader, and the scenario with the manufacturer acting as a leader. The manufacturer controls his margin $m_M(t)$ and rate of advertising $A_M(t)$. The retailer controls her margin $m_R(t)$ and her advertising rate $A_R(t)$. The retailer price

The demand rate $D_R(t)$ is given as

$$D_R(t) = A_R(t)\left(a_R - b_R P_R(t)\right)\gamma\sqrt{G(t)},$$

where $P_R(t)$ is the retail price; a_R and b_R are positive parameters; $G(t)$ is the stock of brand goodwill. Jørgensen et al. (2001) assumes that the retailer is myopic, meaning that she is only concerned with the short-term effects of her pricing and advertising decisions. The manufacturer is concerned with his brand image, as reflected in the goodwill stock which evolves according to the Nerlove-Arrow (1962) dynamics:

$$\dot{G}(t) = \alpha_M(t) - \delta G(t)\,,\ G(0) = G_0 \geq 0,$$

where $\delta > 0$ is a decay constant. The manufacturer's objective functional is

$$J_M = \int_0^\infty e^{-\rho t}\left[m_M(t)\,\alpha_R(t)\left(\alpha - \beta p_R(t)\right)\sqrt{G(t)} - \frac{1}{2}w_M\alpha_M^2(t)\right]dt$$

and the retailer's is

$$J_R = \int_0^\infty e^{-\rho t}\left[m_R(t)\,\alpha_R(t)\left(\alpha - \beta p_R(t)\right)\sqrt{G(t)} - \frac{1}{2}w_R\alpha_R^2(t)\right]dt,$$

The authors show that the manufacturer and retailer leadership in a channel are not symmetric as in pure pricing games. The manufacturer leadership improves channel efficiency and is desirable in terms of consumer welfare, but the retailer leadership is not desirable for channel efficiency and for consumer welfare.

4.4.3 Jørgensen et al. (2003): Retail Promotions with Negative Brand Image Effects

Jørgensen et al. (2003) consider a distribution channel with a manufacturer and a retailer. The manufacturer (he) advertises in the national media to build up the image for his brand. The retailer (she) promotes locally the brand (by such means as local store displays and advertising in local newspapers) to increase sales revenue, but these local promotional efforts are assumed to be harmful to the brand image. Jørgensen et al. (2003) analyze two firms in a cooperative program in which the manufacturer supports the retailer's promotional efforts by paying part of the cost incurred by the retailer when promoting the brand. The two firms play a Stackelberg game where the manufacturer is the leader. They address the question of whether the cooperative promotion program is possible and whether the retailer's decision on being a myopic or far-sighted will affect the implementation of a cooperative program.

Let $A(t)$, $B(t)$, and $G(t)$ denote the manufacturer's advertising rate, the retailer's promotional rate, and the brand image, respectively. The dynamics of $G(t)$ is described by the differential equation

$$\dot{G}(t) = aA(t) - bB(t) - \delta G(t), G(0) = G_0 > 0,$$

where a and b are positive parameters measuring the impact of the manufacturer's advertising and retailer's promotion, respectively, on the brand image and δ is the decay rate of the brand image. The sales revenue rate of the product is $Q(B(t), G(t)) = dB(t) + eG(t)$, where d and e are positive parameters that represent the effects of promotion and brand image on current sales revenue. With this formulation of the demand and the revenue, the retailer faces a trade-off between the sales volume and the negative impact of the local advertising on the brand image.

The manufacturer and the retailer incur quadratic advertising and promotional costs $C(A(t)) = \mu_A A^2/2$ and $C(B(t)) = \mu_B B^2/2$, respectively. Let $D(t)$ denote the amount the manufacturer contributes to the retailer's promotion cost. The manufacturer's objective functional is

$$J_M = \int_0^\infty \left\{ e^{-\rho t} \pi \left[dB(t) + eG(t) \right] - \frac{\mu_A A(t)^2}{2} - \frac{\mu_B D(t) B(t)^2}{2} \right\} dt,$$

and the retailer's is

$$J_R = \int_0^\infty \left\{ e^{-\rho t} (1-\pi) \left[dB(t) + eG(t) \right] - -\frac{\mu_B (1 - D(t)) B(t)^2}{2} \right\} dt.$$

Jørgensen et al. show that a cooperative program is implementable if the initial value of the brand image G_0 is sufficiently small, and if the initial brand image is "intermediate" but promotion is not "too damaging" (i.e., b is small) to the brand image.

4.5 Miscellaneous Applications

4.5.1 Jørgensen and Zaccour (1999): New Technology Subsidy

This paper studies the problem of new technology subsidy problem. Specifically, the government uses two instruments: price subsidies and guaranteed buys, to accelerate the adoption of new technology. The government acts as the leader and the firm is the follower.

4.5.2 Kogan and Tapiero (2006): Co-Investment in Supply Chain Infrastructure

This paper considers a supply chain with N firms. Let $K(t)$ denote the current level of the supply chain infrastructure capital, $L = (L_1, ...L_N)$ a vector of the labor force, $I = (I_1, ...I_N)$ a vector of investment policy, and $Q_j = f(K, L_j)$ an aggregate production function of the j^{th} firm, where $\frac{\partial f}{\partial K} \geq 0$, $\frac{\partial f}{\partial L_j} > 0$ for $L \neq 0$, $\frac{\partial f(K,0)}{\partial L_j} = 0$, and $\frac{\partial^2 f}{\partial L_j^2} < 0$. The process of capital accumulation is given by

$$\dot{K}(t) = -\delta K(t) + \sum_{j=1}^{N} I_j(t), \ K(0) = K_0, \ f(K(t), \ L_j(t)) \geq I_j(t) \geq 0, j = 1, ...N.$$

The jth firm's objective is to maximize its discounted total profit, i.e.,

$$\max_{L_j(t), I_j(t)} \left\{ \int_0^\infty e^{-\rho_j t} \left[p_j(t) f\left(K(t), L_j(t)\right) - c_j(t) L_j(t) - C_I\left((1-\theta) I_j(t)\right) \right] dt \right\}, j = 1, ...N,$$

where p_j is the price, c_j is the unit labor cost, $C_I(.)$ is the investment cost function, and θ is the portion of the cost that is subsidized.

The authors derive the Nash strategy as well as the Stackelberg strategy for the supply chain where firms are centralized and controlled by a "supply chain manager". Their results show that the Stackelberg strategy applied to consecutive subsets of firms will result in an equilibrium identical to that obtained in case of a Nash supply chain. The implication is that it does not matter who the leader is and who the followers are.

4.6 Conclusion

The supply chain and marketing channel management has attracted a great deal of attention in the operations management and marketing literature in the last decade. In the supply chain management literature, most of the models are based on the one-period newsvendor models and, therefore, are limited to examine the one-shot interactions between the channel members. In practice, the channel members may often interact with each other frequently. It is thus natural to explore how their decisions evolve over time. Unfortunately, the insights under the assumption of one-shot interaction cannot be extended into the dynamic situations. For such situations, the differential game modeling approach can be used. However, we have not observed many models that use this approach.

There are a number of reasons that have limited the applications of the Stackelberg differential games to the supply chain management area. Open-loop Stackelberg equilibria are used because of their mathematical tractability. But these equilibria are in general not time consistent. On the other hand, the closed-loop equilibria are hard to obtain. Even in the open-loop case, numerical analysis is needed to get insights into the impact of key parameters on the issues under examination. It may be possible to limit the class of the feedback policies for analysis of the closed-loop equilibria. Finally, researchers have adopted deterministic differential game models, even though the situations that are modeled are affected by uncertain factors. For these applications, it would be of interest to apply the stochastic differential game models.

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