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The functorial interpretation of the naive compactification of regular morphism from $\mathbb{P}^1$ to $\mathbb{P}^1$

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The functorial interpretation of the naive compactification of regular morphism from $\mathbb{P}^1$ to $\mathbb{P}^1$

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This thesis gives a functorial interpretation of the Naive Space of Maps $\mathbf{N_d} := \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^1}(d)) \oplus H^0(\mathcal{O}_{\mathbb{P}^1}(d)))$ as a parametrizing space for a family of maps from certain rational curves to $\mathbb{P}^1$.

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Chapter 1

Introduction

The space of maps from a curve C to a space X attracted considerable interest from both mathematicians and physicists. In string theory: If X is space time, a particle is indicated by a loop (a *string*), and as it moves through time it sweeps out a Riemann surface on X. A basic topological invariant of a topological space is π_2, homotopy classes of maps of S^2 into X. (Algebraic) maps of $\mathbb{P}^1$ into X is the obvious analog in algebraic geomety. The quantum cohomology of a symplectic manifold is a deformation of the ordinary singular cohomology, defined using counts of (pseudo) holomorphic curves. A striking example of the importance of maps of $\mathbb{P}^1$ into X is given in [Gross-Hacking-Keel] ([GHK11]), where a conjectural construction of the mirror to an affine Calabi-Yau manifold (with maximal boundary) is given using just the data of Gromov-Witten invariants, counting rational curves in X (and prove this in many cases).

Various studies are imposed on the compactification of both the space of regular maps from C to X and the space of regular maps from $\mathbf{P}^1$ to $\mathbf{P}^m$, which is the most simple case in the Algebraic Geomtric version of this problem. One of the most prominent results concerning the space of maps from

C to X is given by Kontsevich, the moduli space of stable maps. Kontsevich compactifies the space of regular maps from a smooth curve C to a scheme X by adding regular morphisms from stable curves to X as the boundary.

The space of degree d regular maps from $\mathbf{P}^1$ to $\mathbf{P}^m$ is an open subscheme in the projective space $\mathbb{P}(\oplus_{i=0}^m H^0(\mathcal{O}_{\mathbb{P}^1}, d)) \cong \mathbf{P}^{(m+1)(d+1)-1}$, with complicated boundary divisors. We refer to this projective space as the Naive Space of Maps. The closed points on the boundary in the Naive Space of Maps define rational maps from $\mathbb{P}^1$ to $\mathbb{P}^m$, and they fail to be regular on the points in $\mathbb{P}^1$ where all the $(m+1)$ sections vanish.

Alexander Givental studies the space of stable maps from $\mathbb{P}^1$ to $\mathbb{P}^1 \times \mathbb{P}^m$. He showes that the stable map space $\mathbf{Stab}(\mathbb{P}^1, \mathbb{P}^1 \times \mathbb{P}^m)$ has a birational map to the Naive Space of Maps. Adam Parker and the Mustata couple study this birational map in detail, and factor it into simple intermediate steps([MST]). The Mustata couple utilize their factorization and compute the cohomology groups of the stable map space $\mathbf{Stab}(\mathbb{P}^1, \mathbb{P}^1 \times \mathbb{P}^m)$. Yi Hu([YH]) starts from a totally different direction. He blows up the Naive Space of Maps successively along a filtration of the the boundary divisors and obtains a smooth scheme with normal crossing boundary. However, he was unable to realize the blowing-up as a parametrizing space of a nicely-described geometric family.

The author picked up where Yi Hu left off and tried to find a geometric interpretation for Yi Hu's blowing-up. She started with the very basic case where $m = 1$. Surprisingly, during her observation, she discovered that although many people researched into the Naive Space of Maps in various perspectives, no one realized that the Naive Space of Maps,*without any type of blowing-up's or more complicated constructions*, is, itself, a moduli space, where the boundary parameterizes maps from certain trees of rational curves, quite like the stable rational curves of Giventhal-Kontsevich. These rational curves are named "m-stable" curves in this thesis.

An m-stable curve is a rational curve obtained by gluing trivially thickened $\mathbb{P}^1$'s onto a fixed copy of $\mathbb{P}^1$ along finitely many discrete points. It comes naturally with two regular maps to $\mathbb{P}^1$.

A flat family over a base S is called an m-stable family if the fibers are all m-stable curves (with the two regular maps to $\mathbb{P}^1$).

The Naive Space of Maps $\mathbf{N_d}$ parametrizes the univerrsal family of m-stable curves. This is proved in the thesis by constructing the universal family explicitly from blowing up the space $\mathbf{N_d} \times \mathbb{P}^1$ along the base locus of the evaluation morphism:

$$ev : \mathbf{N_d} \times \mathbb{P}(D) \dashrightarrow \mathbb{P}(R)$$

$$ev(([f_0, f_1], [x_0, x_1])) \mapsto [f_0([x_0, x_1]), f_1([x_0, x_1])]$$

This construction can not be extended to the case when $m > 1$ and the obstruction is that the family is no longer flat in higher dimensions. However, it works in other direction.

Consider the space of rational maps from $\mathbb{P}^r$ to $\mathbb{P}^1$. Similar construction gives a flat family over the higher dimensional Naive Space of Maps

$\mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^r}(d)) \oplus H^0(\mathcal{O}_{\mathbb{P}^r}(d)))$ and this family is universal. Most facts that hold for the case $r = 1$ hold for $r > 1$. The only piece that is missing with the higher dimensional case is that the fibers are too complicated to describe explicitly. The author gives a brief and coarse description of the fibers in this thesis and keeps hope that she can find the explicit description in her future study.

Chapter 2

$\mathbf{N_d}$ parametrizes a flat family of morphisms

Let $\mathbf{k}$ be an algebraicly closed field and D and R be two vector spaces over $\mathbf{k}$ of dimension 2, then $\mathbb{P}(D) \cong \mathbb{P}(R) \cong \mathbb{P}^1$. Let $\{v_0, v_1\}$ and $\{w_0, w_1\}$ be a basis of D and R respectively. $\mathbb{P}(D)$ and $\mathbb{P}(R)$ have homogeneous coordinates $[x_0, x_1]$ and $[y_0, y_1]$ under the given bases.

Definition 2.1. Define *the Space of Maps* $\mathbf{M_d}(\mathbb{P}^r, \mathbb{P}^m)$ as the space of degree d morphisms from $\mathbb{P}^r$ to $\mathbb{P}^m$.

The Naive Space of Maps from $\mathbb{P}^r$ *to* $\mathbb{P}^m$ is defined to be $\mathbf{N_d}(\mathbb{P}^r, \mathbb{P}^m) := \mathbb{P}((\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^r}, d)) \otimes \mathbf{k}^{m+1}) \cong \mathbb{P}^{(m+1)\binom{r+1}{d}-1}$.

Closed points in $\mathbf{M_d}(\mathbb{P}^r, \mathbb{P}^m)$ are $(m+1)$ forms in $\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^r}, d)$ up to a scale factor without common zeros on $\mathbb{P}^r$. It is obvious that $\mathbf{M_d}(\mathbb{P}^r, \mathbb{P}^m)$ is an open subscheme in $\mathbf{N_d}(\mathbb{P}^r, \mathbb{P}^m)$ and $\mathbf{N_d}(\mathbb{P}^r, \mathbb{P}^m)$ is a compactification of $\mathbf{M_d}(\mathbb{P}^r, \mathbb{P}^m)$.

The notions of $\mathbf{M_d}$ and $\mathbf{N_d}$ will be used without clarifying the dimensions of domain and range, as long as it is clear from the context.

In Chapter 2 and Chapter 3, we discuss about maps from $\mathbb{P}(D)$ to $\mathbb{P}(R)$ ($\mathbf{M_d}(\mathbb{P}^1, \mathbb{P}^1)$) and the Naive Space of Maps we are talking about is $\mathbf{N_d} := \mathbb{P}(\mathrm{H}^0(\mathcal{O}_{\mathbb{P}(D)}, d) \otimes R) \cong \mathbb{P}((Sym_d(D^*) \otimes R) \cong \mathbf{P}^{2(d+1)-1}$.

The rational morphism:

$$ev : \mathbf{N_d} \times \mathbb{P}(D) \dashrightarrow \mathbb{P}(R)$$

$$ev(([f_0, f_1], [x_0, x_1])) \mapsto [f_0([x_0, x_1]), f_1([x_0, x_1])]$$

called *the evaluation morphism* is well defined as long as the point $[x_0, x_1]$ is not a common zero of the two forms $\{f_0, f_1\}$.

Let $\{V_0, V_1\}$ be a basis for D^*, $\{V_0^i V_1^j\}_{i+j=d}$ is a basis for $Sym_d(D^*)$, and $\{V_0^i V_1^j w_0, V_0^i V_1^j w_1\}_{i+j=d}$ is a basis for $\mathbf{N_d}$. Under this basis, $\mathbf{N_d}$ has homogeneous coordinates $[a_{ij}, b_{ij}]_{i+j=d}$ in a sense that any closed point in $\mathbf{N_d}$ with coordinates $[a_{ij}, b_{ij}]_{i+j=d}$ gives a concrete rational morphism from $\mathbb{P}(D)$ to $\mathbb{P}(R)$ by sending $[x_0, x_1]$ to $[y_0, y_1] = [\sum_{i+j=d} a_{i,j} x_0^i x_1^j, \sum_{i+j=d} b_{i,j} x_0^i x_1^j]$, so the evaluation morphism, expressed in coordinates, will be

$$ev([a_{ij}, b_{ij}]_{i+j=d}, [x_0, x_1]) = [\sum_{i+j=d} a_{i,j} x_0^i x_1^j, \sum_{i+j=d} b_{i,j} x_0^i x_1]$$

.

The evaluation morphism is defined by the two sections of the line bundle $\mathcal{O}_{\mathbf{N_d}}(1) \boxtimes \mathcal{O}_{\mathbb{P}^1}(d)$ over $\mathbf{N_d} \times \mathbb{P}(D)$, which in coordinates are $\sum_{i+j=d} a_{i,j} x_0^i x_1^j$ and $\sum_{i+j=d} b_{i,j} x_0^i x_1$. *The base locus of the evaluation morphism* Z_{ev} is the subscheme of $\mathbf{N_d} \times \mathbb{P}(D)$ cut out by these two sections.

Now take the blowing up of $\mathbf{N_d} \times \mathbb{P}(D)$ over the base locus Z_{ev} of the rational morphism ev.

Theorem 1 (Z_{ev} smooth)**.** *Z_{ev} is smooth and isomorphic to* $\mathbf{N_{d-1}} \times \mathbb{P}(D)$.

Proof. $\mathbf{N_d} \times \mathbb{P}(D)$ is non-singular, if $Z_{ev} \in \mathbf{N_d} \times \mathbb{P}(D)$ is a *local complete intersection* in , then Z_{ev} is Cohen-Macaulay ([AG], p.186, Proposition 8.23).

To show that Z_{ev} is a local complete intersection, we need to show that the ideal sheaf of Z_{ev} can be locally generated by $codim(Z_{ev}, \mathbf{N_d} \times \mathbb{P}(D))$ elements at every point ([AG], p185 Definition of local complete intersection).

Z_{ev} is cut out by two sections of the line bundle $\mathcal{O}_{\mathbf{N_d}}(1) \boxtimes \mathcal{O}_{\mathbb{P}(D)}(d)$ over $\mathbf{N_d} \times \mathbb{P}(D)$, so the ideal sheaf of Z_{ev} is locally generated by two elements (the two sections).

We want to show that Z_{ev} is codimensional two in $\mathbf{N_d} \times \mathbb{P}(D)$ by showing that each fiber of Z_{ev} over $\mathbb{P}(D)$ has codimension two.

Consider the following diagram:

$$\begin{tikzcd} Z_{ev} \arrow[r, "i"] \arrow[dr, "\phi"'] & \mathbf{N_d} \times \mathbb{P}(D) \arrow[d, "p_2"] \\ & \mathbb{P}(D) \end{tikzcd}$$

Z_{ev} is considered a closed subscheme of $\mathbf{N_d} \times \mathbb{P}(D)$. p_2 is the projection of $\mathbf{N_d} \times \mathbb{P}(D)$ onto its second factor.

The two sections that cut out Z_{ev}, expressed in coordinates, are $\sum_{i+j=d} a_{i,j} x_0^i x_1^j$ and $\sum_{i+j=d} b_{i,j} x_0^i x_1$.

Fix $[x_0, x_1] \in \mathbb{P}^(D)$, the fiber $\phi^{-1}([x_0, x_1])$ is cut out by the two forms

$\{\sum_{i=0}^{d} a_i x_0^i x_1^{d-i}, \sum_{i=0}^{d} b_i x_0^i x_1^{d-i}\}$. These two forms are linear forms with separate cooridinates in $\mathbf{N_d}$, so the fibers they cut out are irreducible and codimensional two in $\mathbf{N_d}$ (They are codimensional two hyperplanes in $\mathbb{N}$).

Now that the fibers of ϕ are all irreducible with the same dimension, and that $\mathbb{P}(D)$ is irreducible, Z_{ev} is irreducible and has dimension $(dim\phi^{-1}([x_0, x_1]) + dim\mathbb{P}(D)$, so $dimZ_{ev} = dim\mathbf{N_d} - 2 + dim\mathbb{P}(D)$ has codimension 2 in $\mathbf{N_d} \times \mathbb{P}(D)$. Z_{ev} is then a local complete intersection, therefore is Cohen-Macaulay.

Z_{ev} is Cohen-Macaulay, $\mathbb{P}(D)$ is smooth and $dim\phi^{-1}([x_0, x_1]) = dimZ_{ev} - dim\mathbb{P}(D)$, $\forall [x_0, x_1] \in \mathbb{P}(D)$. By Theorem 14 of flatness condition in Chapter 5, Z_{ev} is flat over $\mathbb{P}(D)$ under the map ϕ.

The fibers $\phi^{-1}([x_0, x_1])$ are non-singular (hyperplanes) $\forall [x_0, x_1] \in \mathbb{P}(D)$, ϕ is flat, and $\mathbb{P}(D)$ is smooth, so we have that Z_{ev} is non-singular.

The next thing we want to prove is that Z_{ev} is isomorphic to $\mathbf{N_{d-1}} \times \mathbb{P}(D^*)$. To see this, we want to prove that the following morphism is a closed embedding and its image is Z_{ev}.

$$\iota \colon \mathbf{N_{d-1}} \times \mathbb{P}(D^*) \longrightarrow \mathbf{N_d} \times \mathbb{P}(D)$$
$$\iota([g_0, g_1], [l]) \mapsto ([g_0 \cdot l, g_1 \cdot l], [x_0, x_1])$$

where l is a linear form in $\mathbb{P}(D^*)$ and $l([x_0, x_1]) = 0$ (l and $[x_0, x_1]$ determine each other uniquely up to a scalor factor).

It is obvious that $\mathbf{Im}\iota \subseteq Z_{ev}$ by definition. On the set-theoretic level, $\forall([f_0, f_1], [x_0, x_1]) \in Z_{ev}, \exists [x_0, x_1] \in \mathbb{P}(D)$ where f_0 and f_1 vanish simultaneously. $[x_0, x_1]$ determines a linear form $l \in \mathbb{P}(D^*)$ up to a scalor factor, which will be a linear common factor of f_0 and f_1, so we can factor $[f_0, f_1]$ into $[l \cdot g_0, l \cdot g_1]$ with $[g_0, g_1] \in \mathbf{N_{d-1}}$. Therefore, at the set-theoretic level, $\mathbf{Im}\iota = Z_{ev}$.

Consider the following diagram:

$$\begin{tikzcd} \mathbf{N_{d-1}} \times \mathbb{P}(D^*) \arrow[r, "\iota"] \arrow[dr, "\eta"'] & \mathbf{N_d} \times \mathbb{P}(D) \arrow[d, "p_2"'] \\ & \mathbb{P}(D) \end{tikzcd}$$

By Theorem 5.3 of closed embedding, ι is a closed embedding, if its restriction on each fiber over $\mathbb{P}(D)$ is a closed embedding.

Take a closed point $[x_0, x_1] \in \mathbb{P}(D)$.

$\eta^{-1}([x_0, x_1]) = \mathbf{N_{d-1}}$. $p_2^{-1}([x_0, x_1]) = \mathbf{N_d}$.

$\iota|_{[x_0,x_1]} : \mathbf{N_{d-1}} \to \mathbf{N_d}$

is defined by $\iota|_{[x_0,x_1]}([a'_{ij}, b'_{ij}]_{i+j=d-1}) = [a_{ij}, b_{ij}]_{i+j=d}$,
where $a_{ij} = -x_1 a'_{(i-1)j} + x_0 a'_{i(j-1)}, b_{ij} = -x_1 b'_{(i-1)j} + x_0 b'_{i(j-1)}$
(let $a'_{(-1)d} = a'_{d(-1)} = b'_{(-1)d} = b'_{d(-1)} = 0$).

This map is obviously linear, so it induces an isomorphism from $\mathbf{N_{d-1}}$ to its image, and therefore $\iota|_{[x_0,x_1]}$ is a closed embedding on the closed point $[x_0, x_1] \in \mathbb{P}(D)$. In fact, any regular linear morphism from $\mathbb{P}^m$ to $\mathbb{P}^n$ is a closed embedding. By Theorem 5.3, ι is a closed embedding, so $\mathbf{N_{d-1}} \times \mathbb{P}(D^*)$ is isomorphic to its image under ι. We have already shown that set-theoretically, the image is Z_{ev}. But Z_{ev} is non-singular, in particular reduced, therefore $\iota : \mathbf{N_{d-1}} \times \mathbb{P}(D) \hookrightarrow Z_{ev}$ is an isomorphism.

□

Theorem 2 ($\mathcal{C}$ smooth)**.** *The universal family* $\mathcal{C} = \mathbf{Bl}_{Z_{ev}}(\mathbf{N_d} \times \mathbb{P}(D))$ *is non-singular.*

Proof. $\mathbf{N_d} \times \mathbb{P}(D)$ and Z_{ev} are both non-singular, therefore the blowup $\mathcal{C}$ is non-singular ([AG], p186., Theorem 8.24(a)). □

The blowing-up scheme $\mathcal{C} := \mathbf{Bl}_{Z_{ev}}(\mathbf{N_d} \times \mathbb{P}(D))$ is isomorphic to the closure of the graph of the rational morphism ev in $\mathbf{N_d} \times \mathbb{P}(D) \times \mathbb{P}(R)$ following Proposition 5.6 which we will prove later.

Theorem 3 ($\mathcal{C}$ is flat over $\mathbf{N_d}$)**.** *The blowing-up* $\mathcal{C} = \mathbf{Bl}_{Z_{ev}}(\mathbf{N_d} \times \mathbb{P}(D))$ *is flat over the base* $\mathbf{N_d}$.

Proof. By Theorem 5.1 of flatness, $\mathcal{C}$ is Cohen-Macaulay (since it is non-singular) $\mathbf{N_d}$ is non-singular, the morphism $\pi : \mathcal{C} \to \mathbf{N_d} \times \mathbb{P}(D) \to \mathbf{N_d}$ is flat, if the fibers $\pi^{-1}([f_0, f_1])$, $\forall [f_0, f_1] \in \mathbf{N_d}$ has dimension $= dim\mathcal{C} - dim\mathbf{N_d}$ $= 1$.

The fibers of the projection $p_2 : \mathbf{N_d} \times \mathbb{P}(D) \to \mathbf{N_d}$ are copies of $\mathbb{P}(D)$. In another word, $p_2^{-1}([f_0, f_1]) = \{[f_0, f_1]\} \times \mathbb{P}(D)$.

The morphism $p : \mathcal{C} \to \mathbf{N_d} \times \mathbb{P}(D)$ is a blowing-up, so p is an isomorphism on $\mathbf{N_d} \times \mathbb{P}(D) \setminus Z_{ev}$, and the fibers of p on Z_{ev} are of dimension $(dim\mathbf{N_d} \times \mathbb{P}(D) - 1) - dim Z_{ev}$ (Fulton intersetion theory), since Z_{ev} is regular embedded in $\mathbf{N_d} \times \mathbb{P}(D)$. Z_{ev} is codimenional two in $\mathbf{N_d} \times \mathbb{P}(D)$ (see proof of Theorem 2.1), so the fibers of p on Z_{ev} are of dimension 1.

$\forall [f_0, f_1] \in \mathbf{N_d}$, $\{[f_0, f_1]\} \times \mathbb{P}(D) \cap Z_{ev}$ is either empty or a set of finitely many isolated points, because both f_0 and f_1 vanish only on finitely many points in $\mathbb{P}(D)$. Therefore, $\pi^{-1}([f_0, f_1] = p^{-1}(\{[f_0, f_1]\} \times \mathbb{P}(D))$ is either a copy of $\mathbb{P}(D)$ or $\mathbb{P}(D)$ with dimenion 1 branches, which means that the fibers $\pi^{-1}([f_0, f_1])$ are of dimension 1, so $\mathcal{C}$ is flat over $\mathbf{N_d}$.

□

Chapter 3

m-stable and (1,d) curves

Consider the following data:

1) $\mathbb{P}(D)$ with finitely many marked points $\{p_i\}$ and weights $\{w_i\}$;

2) A regular morphism $g : \mathbb{P}(D) \to \mathbb{P}(R)$ of degree k.

The thickened points at p_i with weights w_i are subschemes of $\mathbb{P}(D)$ with ideal sheaf $\mathcal{I}_{p_{w_i}} = (\mathbf{m}_{p_i})^{w_i+1}$, where $\mathbf{m}_{p_i}$ is sheaf of the maximal ideal at p_i. $p_{w_i} =: Spec(\mathbf{k}[\varepsilon]/\varepsilon^{w_i+1})$ abstractly as a scheme.

Glue the trivial ribbons $p_{w_i} \times \mathbb{P}(R)$ onto $\mathbb{P}(D)$ along p_{w_i} with the following morphisms:

$i_1 : p_{w_i} \hookrightarrow \mathbb{P}(D)$ the closed immersion of p_{w_i} as a subscheme,

and $i_2 : p_{w_i} \hookrightarrow p_{w_i} \times \mathbb{P}(R)$ which is the identity morphism on the first factor and the restriction of the given morphism g onto $p_{w_i} : g|_{p_{w_i}}$, or i_2 is the graph of g on p_{w_i}.

The gluing result is a scheme, which will be studied in details in Chapter 5 section 2, and we name it $C = \mathbb{P}(D) \coprod_{p_{w_i}} (p_{w_i} \times \mathbb{P}(R))$.

C has a natural regular morphism to $\mathbb{P}(D)$ and $\mathbb{P}(R)$ respectively from its universal property:

$\phi_1 : C \to \mathbb{P}(D)$ is the identity morphism on the component $\mathbb{P}(D)$ and a projection onto p_{w_i} on the components of $p_{w_i} \times \mathbb{P}(R)$. ϕ_1 is regular of degree 1.

$\phi_2 : C \to \mathbb{P}(R)$ is the morphism g on the component $\mathbb{P}(D)$ and a projetion onto $\mathbb{P}(R)$ composed with the linear isomorphism l_i on the components of $p_{w_i} \times \mathbb{P}(R)$. ϕ_2 is regular of degree $d = k + \sum_i w_i$.

Definition 3.1 (m-stable curve)**.** A rational curve is called an *m-stable curve* if it is the gluing scheme with the two natural regular morphisms to $\mathbb{P}(D)$ and $\mathbb{P}(R)$ uniquely determined by the given data above.

$d = k + \sum_i w_i$ is called *the degree of the m-stable curve.*

Lemma 1 ((d,1)-curves)**.** *An m-stable curve embeds into $\mathbb{P}(D) \times \mathbb{P}(R)$ as a (d,1)-curve. Conversely, any (d,1)-curve in $\mathbb{P}(D) \times \mathbb{P}(R)$ is an m-stable curve.*

Proof. Let C be an m-stable curve, then it has two regular maps ϕ_1 and ϕ_2 to $\mathbb{P}(D)$ and $\mathbb{P}(R)$ respectively, construct a morphism $\phi_1 \times \phi_2 : C \to \mathbb{P}(D) \times \mathbb{P}(R)$ of bi-degree (1,d). It is a closed embedding on each of the components of C, following the universal property of the gluing scheme proven in Chapter 5, $\phi_1 \times \phi_2$ is then a closed embedding of C into $\mathbb{P}(D) \times \mathbb{P}(R)$, so the image of $\phi_1 \times \phi_2$ is a (d,1)-curve in $\mathbb{P}(D) \times \mathbb{P}(R)$.

Conversly, let C be a (d,1)-curve in $\mathbb{P}(D) \times \mathbb{P}(R)$. It has two natural regular morphism of the degree 1 and d respectively to $\mathbb{P}(D)$ and $\mathbb{P}(R)$. Any (d,1)-curve is cut out by a section $y_0 f_0([x_0, x_1]) + y_1 f_1([x_0, x_1])$, where $f_0, f_1 \in$

$\mathrm{H}^0(\mathcal{O}_{\mathbb{P}(D)}, d)$. $(f_0, f_1) = \prod_i l_i^{w_i}(g_0, g_1)$, where l_i are linear forms and g_0 and g_1 are relatively prime. $[g_0, g_1]$ defines a regular morphism of degree $k = d - (\sum_i w_i)$ from $\mathbb{P}(D)$ to $\mathbb{P}(R)$. l_i have a unique zero on $\mathbb{P}(D)$ and free on $\mathbb{P}(D)$. C can be decomposed into isomorphic copies of $\mathbb{P}(R)$ and a curve defined by $y_0 g_0([x_0, x_1]) + y_1 g_1([x_0, x_1])$. The later curve is ismorphic to $\mathbb{P}(D)$ and has a degree k regular morphism to $\mathbb{P}(R)$.

$\square$

Definition 3.2 (family of m-stable curves)**.** *A family $\mathcal{M}$ of m-stable curves* over a base scheme S is a flat scheme over S with two regular morphisms:

$\phi : \mathcal{M} \to S \times \mathbb{P}(D)$, and

$\psi : \mathcal{M} \to S \times \mathbb{P}(R)$,

such that the fibers $\mathcal{M}_S$ of $\mathcal{M}$ over the closed points in S are m-stable curves of the same degree.

Definition 3.3 (family of (d,1) curves)**.** *A family $\mathcal{M}$ of (d,1)-curves* in $\mathbb{P}(D) \times \mathbb{P}(R)$ over a base scheme S is a flat scheme over S such that every fiber is a (d,1)-curve in $\mathbb{P}(D) \times \mathbb{P}(R)$.

Definition 3.4 (family of Cartier divisors)**.** Let X be a scheme over a base S with a fixed line bundle L on X. A flat family of Cartier divisors parametrized by S in the fixed linear system $|L|$ over X is a Cartier divisor $\mathbf{D}$ over $X \times S$ such that for any $s \in S$, $\mathbf{D}_s \in L$.

Theorem 4 (equivalence)**.** *The following families are equivalent to one another:*

1) family of (d,1)-curves

2) family of m-stable curves

3) flat family of Cartier divisors in the fixed linear system $|\mathcal{O}_{\mathbb{P}(D)}(d) \boxtimes \mathcal{O}_{\mathbb{P}(R)}(1)|$ *over* $X = \mathbb{P}(D) \times \mathbb{P}(R)$ *parametrized by a base scheme* S*.*

Proof. The equivalence of the first two is directly from the Lemma proven above.

1) $\Rightarrow$ 3)

A family of (d,1)-curves in $\mathbb{P}(D) \times \mathbb{P}(R)$ is a flat family of Cartier divisors over $\mathbb{P}(D) \times \mathbb{P}(R)$ parametrized by the base S. The picard group $\mathbf{Pic}(\mathbb{P}(D) \times \mathbb{P}(R)) = \mathbb{Z} \times \mathbb{Z}$ is discrete, so algebraically equivalent divisors are linearly equivalent. Any flat family of Cartier divisors are automatically in the same linear system. Here, the linear system is obviously $|\mathcal{O}_{\mathbb{P}(D)}(d) \boxtimes \mathcal{O}_{\mathbb{P}(R)}(1)|$ over $\mathbb{P}(E) \times \mathbb{P}(R)$.

3) $\Rightarrow$ 1)

A flat family of Cartier divisors $\mathbf{D}$ in the fixed linear system $|\mathcal{O}_{\mathbb{P}(D)}(d) \boxtimes \mathcal{O}_{\mathbb{P}(R)}(1)|$ over $X = \mathbb{P}(D) \times \mathbb{P}(R)$ parametrized by S is a flat scheme over S, with fibers $\mathbf{D}|_s \in L = \mathcal{O}_{\mathbb{P}(D)}(d) \boxtimes \mathcal{O}_{\mathbb{P}(R)}(1)$, which are (d,1)-curves in $\mathbf{D} \times \mathbf{P}(R)$. □

Theorem 5 (universal property for divisors)**.** *The universal family* $\mathcal{D}$ *of the flat family of Cartier divisors parametrized by* S *in the linear system* $|L|$ *over* X *is a flat family of Cartier divisors parametrized by* $\mathbb{P}(H^0(X, \mathcal{L}))$*, with fibers* $\{x \in X : \sigma(x) = 0\}$*,* $\forall \sigma \in H^0(X, \mathcal{L})$*,*

Theorem 6. $\phi : \mathcal{C} = \mathbf{Bl}_{Z_{ev}}(\mathbf{N_d} \times \mathbb{P}(D)) \to \mathbf{N_d}$ *is the universal family of m-stable curves.*

Proof. Let $\mathcal{M}$ over a base S be any family of m-stable curves. By Theorem 4, $\mathcal{M}$ is a family of Cartier divisors on $X = \mathbb{P}(D) \times \mathbb{P}(R)$ in a fixed linear system $|L| = |\mathcal{O}(d,1)|$. By Theorem 4, for any scheme X with a line bundle L over X,there exists a universal family $\mathcal{D}$ of Cartier divisors on X in the fixed linear system $|L|$, parametrized by the projective space $\mathbb{P}(H^0(X,L))$, or $\mathbb{P}(H^0(\mathbb{P}(D) \times \mathbb{P}(R), (d,1)))$ in our particular case.

By Theorem 4, $\mathcal{C} = \mathbf{Bl}_{Z_{ev}}(\mathbf{N_d} \times \mathbb{P}(D))$ is a flat family of m-stable curves if and only if it is a flat family of (d,1)-curves in $\mathbb{P}(D) \times \mathbb{P}(R)$. By Theorem 3, $\mathcal{C}$ is flat over $\mathbf{N_d}$.

The picard group $\mathbf{Pic}(\mathbb{P}(D) \times \mathbb{P}(R)) = \mathbb{Z} \times \mathbb{Z}$ is discrete, so algebraically equivalent divisors are linearly equivalent. Any flat family of Cartier divisors are automatically in the same linear system. In order to show that $\mathcal{C}$ is a family of (d,1)-curves curves, we only need to show that at least one fiber of ϕ is a (d,1)-curve in $\mathbb{P}(D) \times \mathbb{P}(R)$. The generic fibers of $\mathcal{C}$ over $\mathbf{N_d}$ are graphs of regular maps of degree d from $\mathbb{P}(D)$ to $\mathbb{P}(R)$. They are clearly degree $(d,1)$ curves in $\mathbb{P}(D) \times \mathbb{P}(R)$. Therefore, $\mathcal{C}$ is a family of (d,1)-curves.

The family $\mathcal{C}$ is the pull back family onto $\mathbf{N_d}$ of the universal family $\mathcal{D}$ over $\mathbb{P}(H^0(\mathbb{P}(D) \times \mathbb{P}(R), (d,1)))$ under a regular morphism $\psi : \mathbf{N_d} \to \mathbb{P}(H^0(\mathbb{P}(D) \times \mathbb{P}(R), (d,1)))$, where $\psi(s)$, $s \in \mathbf{N_d}$, is the line spanned by a section of degree $(d,1)$ which cuts out the fiber $\phi^{-1}(s) \subseteq \mathcal{C}$. We will show that ψ

is an isomorphism.

Inheriting the notations from Chapter 2, $\{V_0, V_1\}$ is a basis of $H^0(\mathcal{O}_{\mathbb{P}(D)}(1)) = D^*$, $\{W_0, W_1\}$ be a basis of $H^0(\mathcal{O}_{\mathbb{P}(R)}(1)) = R^*$ and and $\{V_0^i V_1^j w_0, V_0^i V_1^j w_1\}_{i+j=d}$ is a basis for $\mathbf{N_d} := \mathbb{P}(\mathrm{H}^0(\mathcal{O}_{\mathbb{P}(D)}, d) \otimes R)$. Under this basis, $\mathbf{N_d}$ has homogeneous coordinates $[a_{ij}, b_{ij}]_{i+j=d}$, which means that each point $s \in \mathbf{N_d}$ with coordinates $[a_{ij}, b_{ij}]_{i+j=d}$, we have a rational map defined by $[\sum_{i+j=d}(a_{ij}x_0^i x_1^j), \sum_{i+j=d}(b_{ij}x_0^i x_1^j)]$ from $\mathbb{P}(D)$ to $\mathbb{P}(R)$.

$\{W_1 V_0^i V_1 j, -W_0 V_0^i V_1^j\}_{i+j=d}$ is basis of $\mathbb{P}(H^0(\mathbb{P}(D) \times \mathbb{P}(R), (d, 1)))$. Let $s \in \mathbf{N_d}$ be a generic closed point with coordinates $[a_{ij}, b_{ij}]_{i+j=d}$. s is a regular morphism from $\mathbb{P}(D)$ to $\mathbb{P}(R)$ defined in coordinates as $[\sum_{i+j=d}(a_{ij}x_0^i x_1^j), \sum_{i+j=d}(b_{ij}x_0^i x_1^j)]$. The fiber of s $\phi^{-1}(s) \subseteq \mathcal{C}$ is the graph of the regular morphism s. Obviously, in coordinates, the section that cuts out the fiber $\phi^{-1}(s) \subseteq \mathcal{C}$ is given by $y_1(\sum_{i+j=d}(a_{ij}x_0^i x_1^j)) - y_0(\sum_{i+j=d}(b_{ij}x_0^i x_1^j))$. In $\mathbb{P}(H^0(\mathbb{P}(D) \times \mathbb{P}(R), (d, 1)))$ under the given basis, $\phi^{-1}(s)$ has coordinates exactly $[a_{ij}, b_{ij}]_{i+j=d}$. Now identify $\mathbb{P}(H^0(\mathbb{P}(D) \times \mathbb{P}(R), (d, 1)))$ with $\mathbf{N_d}$ using the given bases. The regular morphism ψ is an identity on generic points of $\mathbf{N_d}$. $\mathbf{N_d}$ is separated, ψ is continuous and is identity on generic points, so ψ is an identity all over $\mathbf{N_d}$.

Now we have that ψ is an isomorphism from $\mathbf{N_d}$ to $\mathbb{P}(H^0(\mathbb{P}(D) \times \mathbb{P}(R), (d, 1)))$, so $\mathcal{C}$ is isomorphic to the universal family $\mathcal{D}$.

$\square$

Chapter 4

Higher dimensional case

Let $\mathbf{k}$ be an algebraicly closed field and W and R be two vector spaces over $\mathbf{k}$ with dimension (r+1) and 2 respectively, then $\mathbb{P}(E) \cong \mathbb{P}^r$ and $\mathbb{P}(R) \cong \mathbb{P}^1$. Let $\{v_0, v_1, \cdots, v_r\}$ be a basis of E and $\{w_0, w_1\}$ be a basis of R. $\mathbb{P}(E)$ and $\mathbb{P}(R)$ have homogeneous coordinates $[x_0, x_1, \cdots, x_r]$ and $[y_0, y_1]$ under the given bases.

As is defined and remarked in Chapter 2, we take $\mathbf{N_d}$ as an abuse of notation to be the Naive Space of Maps $\mathbf{N_d}(\mathbb{P}^r, \mathbb{P}^1)$ in this Chapter, and $\mathbf{M_d}$ as the space of degree d rational (for $r > 1$) morphisms from $\mathbb{P}(E)$ to $\mathbb{P}(R)$. $\mathbf{M_d}$ is open in $\mathbf{N_d}$ and $\mathbf{N_d}$ is a compactication of $\mathbf{M_d}$. $\mathbf{N_d} =: \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}(E)}, d) \otimes R) \cong \mathbb{P}((Sym_d(E^*) \otimes R) \cong \mathbf{P}^{2\binom{r+1}{d}-1}$.

Similar to the case where $r = 1$, we have the *evaluation morphism*:

$$ev : \mathbf{N_d} \times \mathbb{P}(E) \dashrightarrow \mathbb{P}(R)$$

$$ev(([f_0, f_1], [x_0, x_1, \cdots, x_r])) \mapsto [f_0([x_0, x_1, \cdots, x_r]), f_1([x_0, x_1, \cdots, x_r])].$$

The evaluation morphism is a rational morphism and its base locus is

$$Z_{ev} = \{([f_0, f_1], [x_0, x_1, \cdots, x_r]) \in \mathbf{N_d} \times \mathbb{P}(E)\}$$

where $f_0([x_0, x_1, \cdots, x_r]) = f_1([x_0, x_1, \cdots, x_r] = 0$.

Let $\{V_0, V_1, \cdots, V_r\}$ be a basis for E^*, $\{V_0^{i_0} V_1^{i_1} \cdots V_r^{i_r}\}_{i_0+\cdots+i_r=d}$ be a basis for $Sym_d(E^*)$, and $\{V_0^{i_0} V_1^{i_1} \cdots V_r^{i_r} w_0, V_0^{i_0} V_1^{i_1} \cdots V_r^{i_r} w_1\}_{i_0+\cdots+i_r=d}$ be basis for $\mathbf{N_d}$. Under these bases, $\mathbf{N_d}$ has homogeneous coordinates $[a_{i_0,\cdots,i_r}, b_{j_0,\cdots,j_r}]_{i_0+\cdots+i_r=d}$, in a sense that any closed point in $\mathbf{N_d}$ with coordinates $[a_{i_0,\cdots,i_r}, b_{j_0,\cdots,j_r}]_{i_0+\cdots+i_r=d}$ gives a concrete rational morphism from $\mathbb{P}(E)$ to $\mathbb{P}(R)$ by sending $[x_0, x_1, \cdots, x_r]$ to

$$[y_0, y_1] = [\sum_{i_0+\cdots+i_r=d} a_{i_0,\cdots,i_r} x_0^{i_0} \cdots x_r^{i_r}, \sum_{j_0+\cdots+j_r=d} b_{j_0,\cdots,j_r} x_0^{j_0} \cdots x_r^{j_r}]$$

,

The evaluation morphism, expressed in coordinates, will be

$$ev([a_{i_0,\cdots,i_r}, b_{j_0,\cdots,j_r}]_{i_0+\cdots+i_r=d}, [x_0, x_1, \cdots, x_r])$$

$$= [\sum_{i_0+\cdots+i_r=d} a_{i_0,\cdots,i_r} x_0^{i_0} \cdots x_r^{i_r}, \sum_{j_0+\cdots+j_r=d} b_{j_0,\cdots,j_r} x_0^{j_0} \cdots x_r^{j_r}]$$

where $[a_{i_0,\cdots,i_r}, b_{j_0,\cdots,j_r}]$ are coordinates of closed points in $\mathbf{N_d}$.

Blow up $\mathbf{N_d} \times \mathbb{P}(E)$ over the base locus Z_{ev} of the rational morphism ev. We have, in the following, a few results and the proofs exactly parallel to the case where $r = 1$, which we discussed in details in Chapter 2 and Chapter 3.

Theorem 7. *Z_{ev} is smooth.*

Proof. $\mathbf{N_d} \times \mathbb{P}(E)$ is non-singular, if $Z_{ev} \subseteq \mathbf{N_d} \times \mathbb{P}(E)$ is a *local complete intersection*, then Z_{ev} is Cohen-Macaulay ([AG], p.186, Proposition 8.23).

To show that Z_{ev} is a local complete intersection, we need to show that the ideal sheaf of Z_{ev} can be locally generated by $codim(Z_{ev}, \mathbf{N_d} \times \mathbb{P}(E))$ elements at every point ([AG], p185 Definition of local complete intersection).

Z_{ev} is cut out by two sections of the line bundle $\mathcal{O}_{\mathbf{N_d}}(1) \boxtimes \mathcal{O}_{\mathbb{P}(E)}(d)$ over $\mathbf{N_d} \times \mathbb{P}(E)$, so the ideal sheaf of Z_{ev} is locally generated by two elements (the two sections).

We want to show that Z_{ev} is codimensional two in $\mathbf{N_d} \times \mathbb{P}(E)$.

Consider the following diagram:

$$\begin{tikzcd} Z_{ev} \arrow[r, "i"] \arrow[dr, "\phi"'] & \mathbf{N_d} \times \mathbb{P}(E) \arrow[d, "p_2"] \\ & \mathbb{P}(E) \end{tikzcd}$$

Z_{ev} is considered a closed subscheme of $\mathbf{N_d} \times \mathbb{P}(E)$. p_2 is the projection of $\mathbf{N_d} \times \mathbb{P}(E)$ onto its second factor.

Fix a closed point $[x_0, x_1, \cdots, x_r] \in \mathbb{P}(E)$, the fiber $\phi^{-1}([x_0, x_1, \cdots, x_r])$ is cut out by the two linear forms

$$\{ \sum_{i_0+\cdots+i_r=d} a_{i_0,\cdots,i_r} x_0^{i_0} \cdots x_r^{i_r}, \sum_{j_0+\cdots+j_r=d} b_{j_0,\cdots,j_r} x_0^{j_0} \cdots x_r^{j_r} \}$$

with separate cooridinates in $\mathbf{N_d}$ and thereofore these two linear forms are linearly independent. The fibers they cut out are codimensional two linear subspaces in $\mathbf{N_d}$. It follows that the fibers are non-singular, and in particularly, reduced and irreducible.

Now that the fibers of ϕ are all irreducible with the same dimension, ϕ is surjective, and $\mathbb{P}(E)$ is irreducible, it follows that Z_{ev} is irreducible and has dimension $(dim\phi^{-1}([x_0, x_1, \cdots, x_r]) + dim\mathbb{P}(E)$, so $dimZ_{ev} = dim\mathbf{N_d} - 2 + dim\mathbb{P}(E)$ has codimension 2 in $\mathbf{N_d} \times \mathbb{P}(E)$. Z_{ev} is then a local complete intersection, therefore is Cohen-Macaulay.

Z_{ev} is Cohen-Macaulay, $\mathbb{P}(E)$ is smooth and $\forall [x_0, x_1, \cdots, x_r] \in \mathbb{P}(E)$, $dim\phi^{-1}([x_0, x_1, \cdots, x_r]) = dimZ_{ev} - dim\mathbb{P}(E)$. By Theorem 5.1 of flatness condition, Z_{ev} is flat over $\mathbb{P}(E)$ under the map ϕ.

$\forall [x_0, x_1, \cdots, x_r] \in \mathbb{P}(E)$, the fibers $\phi^{-1}([x_0, x_1, \cdots, x_r])$ are non-singular and equidimensional, and ϕ is flat. It follows that ϕ is smooth ([AG], p269, Theorem 10.2). Since $\mathbb{P}(E)$ is non-singular (smooth over $Spec(\mathbf{k})$), Z_{ev} is non-singular by composition.

□

Theorem 8 ($\mathcal{C}$ smooth)**.** *The universal family* $\mathcal{C} = \mathbf{Bl}_{Z_{ev}}(\mathbf{N_d} \times \mathbb{P}(E))$ *is non-singular.*

Proof. $\mathbf{N_d} \times \mathbb{P}(E)$ and Z_{ev} are both non-singular, therefore the blowup $\mathcal{C}$ is non-singular ([AG], p186., Theorem 8.24(a)). □

The blowing-up scheme $\mathcal{C} := \mathbf{Bl}_{Z_{ev}}(\mathbf{N_d} \times \mathbb{P}(E))$ is isomorphic to the closure of the graph of the rational morphism ev in $\mathbf{N_d} \times \mathbb{P}(E) \times \mathbb{P}(R)$ following

Proposition 5.6 which we will prove later.

Theorem 9 ($\mathcal{C}$ is flat over $\mathbf{N_d}$)**.** *The blowing-up* $\mathcal{C} = \mathbf{Bl}_{Z_{ev}}(\mathbf{N_d} \times \mathbb{P}(E))$ *is flat over the base* $\mathbf{N_d}$*.*

$\pi : \mathcal{C} \to \mathbf{N_d}$ *is defined by* $\pi = \pi_{bl} \circ p_2$*, where* π_{bl} *is the blowing-up and* p_2 *is the projection onto the second factor.*

$$\begin{tikzcd} \mathcal{C} \arrow[d, "\pi_{bl}"] \\ \mathbf{N_d} \times \mathbb{P}(E) \arrow[d, "p_2"] \\ \mathbb{P}(E) \end{tikzcd}$$

Proof. $\mathcal{C}$ is Cohen-Macaulay (since it is non-singular) and $\mathbf{N_d}$ is non-singular, by Theorem 5.1 of flatness, the composition morphism π is flat, if the fibers $\pi^{-1}([f_0, f_1])$, $\forall [f_0, f_1] \in \mathbf{N_d}$ has dimension $= dim\mathcal{C} - dim\mathbf{N_d} =$ r.

The fibers of the projection $p_2 : \mathbf{N_d} \times \mathbb{P}(E) \to \mathbf{N_d}$ are copies of $\mathbb{P}(E)$. In another word, $p_2^{-1}([f_0, f_1]) = \{[f_0, f_1]\} \times \mathbb{P}(E)$.

The morphism $p : \mathcal{C} \to \mathbf{N_d} \times \mathbb{P}(E)$ is a blowing-up, so p is an isomorphism on $\mathbf{N_d} \times \mathbb{P}(E) \setminus Z_{ev}$, and the fibers of p on Z_{ev} are of dimension

$(dim(\mathbf{N_d} \times \mathbb{P}(E)) - 1) - dim Z_{ev}$ ([IT],p439, B.8.2), since Z_{ev} is regularly embedded in $\mathbf{N_d} \times \mathbb{P}(E)$. Z_{ev} is codimenional two in $\mathbf{N_d} \times \mathbb{P}(E)$ (see proof of Theorem 2.1), so the fibers of p on Z_{ev} are of dimension 1.

$\forall [f_0, f_1] \in \mathbf{N_d}$, $(\{[f_0, f_1]\} \times \mathbb{P}(E)) \cap Z_{ev}$ is a proper subscheme of $\{[f_0, f_1]\} \times \mathbb{P}(E)$. This is obvious since the two forms that cut out the intersection:

$$\{\sum_{i_0+\cdots+i_r=d} a_{i_0,\cdots,i_r} x_0^{i_0} \cdots x_r^{i_r}, \sum_{j_0+\cdots+j_r=d} b_{j_0,\cdots,j_r} x_0^{j_0} \cdots x_r^{j_r}\}$$

can not both be zero unless $(a_{i_0,\cdots,i_r}, b_{i_0,\cdots,i_r}) = 0$ or $f_0 = f_1 \equiv 0$. The fibers of $\mathcal{C}$ over $\mathbf{N_d}$ will be $\mathbb{P}(E)$ with dimension 1 branches along a proper subscheme, and thus have dimension $r = dim\mathbb{P}(E)$, which completes the proof. $\square$

Theorem 10. *1)* $\mathcal{C} := \mathbf{Bl}_{Z_{ev}}(\mathbf{N_d} \times \mathbb{P}(E))$ *is a flat family of* $(d, 1)-$*hypersufaces* $\subset \mathbb{P}(E) \times \mathbb{P}(R)$ *over the base* $\mathbf{N_d}$.

2) $\mathcal{C} := \mathbf{Bl}_{Z_{ev}}(\mathbf{N_d} \times \mathbb{P}(E))$ *is a flat family of of Cartier divisors parametrized by* $\mathbf{N_d}$ *in the fixed linear system* $|\mathcal{O}_{\mathbb{P}(E)}(d) \boxtimes \mathcal{O}_{\mathbb{P}(R)}(1)|$ *over* $\mathbb{P}(E) \times \mathbb{P}(R)$.

Proof. The flatness of $\mathcal{C}$ over $\mathbf{N_d}$ is given by Theorem 4.3.

For 1), in order to show that $\mathcal{C}$ is a family of (d,1)-hypersufaces, we only need to show that at least one fiber of ϕ is a (d,1)-hypersufaces in $\mathbb{P}(E) \times \mathbb{P}(R)$.

$$\begin{tikzcd}
\mathcal{C} \subset \mathbf{N_d} \times \mathbb{P}(E) \times \mathbb{P}(R) \arrow[d, "\pi_{bl}"] \\
\mathbf{N_d} \times \mathbb{P}(E) \arrow[d, "p_2"] \\
\mathbb{P}(E)
\end{tikzcd}$$

Fixing $[f_0, f_1] \in \mathbf{N_d}$ and a generic point $[x_0, x_1, \cdots, x_r] \in \mathbb{P}(E)$, π_b is a blowing-up and thus birational on $\mathbf{N_d} \times \mathbb{P}(E)$, so $\pi^{-1}([f_0, f_1])$ has degree 1 on $\mathbb{P}(R)$.

The generic fibers of $\mathcal{C}$ over $\mathbf{N_d}$ are graphs of regular maps of degree d from $\mathbb{P}(E)$ to $\mathbb{P}(R)$. They are clearly degree $(d, 1)$ curves in $\mathbb{P}(E) \times \mathbb{P}(R)$. Therefore, $\mathcal{C}$ is a family of (d,1)-hypersufaces.

1) $\Rightarrow$ 2)

A family of (d,1)-hypersufaces in $\mathbb{P}(E) \times \mathbb{P}(R)$ is a flat family of Cartier divisors over $\mathbb{P}(E) \times \mathbb{P}(R)$ parametrized by the base S. The picard group $\mathbf{Pic}(\mathbb{P}(E) \times \mathbb{P}(R)) = \mathbb{Z} \times \mathbb{Z}$ is discrete, so algebraically equivalent divisors are linearly equivalent. Any flat family of Cartier divisors are automatically in the same linear system. Here, the linear system is obviously $|\mathcal{O}_{\mathbb{P}(E)}(d) \boxtimes \mathcal{O}_{\mathbb{P}(R)}(1)|$ over $\mathbb{P}(E) \times \mathbb{P}(R)$.

2) $\Rightarrow$ 1)

A flat family of Cartier divisors $\mathbf{D}$ in the fixed linear system $|\mathcal{O}_{\mathbb{P}(E)}(d)\boxtimes \mathcal{O}_{\mathbb{P}(R)}(1)|$ over $X = \mathbb{P}(E)\times\mathbb{P}(R)$ parametrized by S is a flat scheme over S, with fibers $\mathbf{D}|_s \in L = \mathcal{O}_{\mathbb{P}(E)}(d)\boxtimes\mathcal{O}_{\mathbb{P}(R)}(1)$, which are (d,1)-hypersufaces in $\mathbf{E}\times\mathbf{P}(R)$.

$\square$

Theorem 11. $\phi : \mathcal{C} = \mathbf{bl}_{Z_{ev}}(\mathbf{N_d}\times\mathbb{P}(E)) \to \mathbf{N_d}$ *is the universal family of the flat family of Cartier divisors in the fixed linear system* $|\mathcal{O}_{\mathbb{P}(E)}(d)\boxtimes\mathcal{O}_{\mathbb{P}(R)}(1)|$ *over* $\mathbb{P}(E)\times\mathbb{P}(R)$.

Proof. The family $\mathcal{C}$ is the pull back family onto $\mathbf{N_d}$ of the universal family $\mathcal{D}$ over $\mathbb{P}(H^0(\mathbb{P}(E)\times\mathbb{P}(R),(d,1)))$ under a regular morphism $\psi : \mathbf{N_d} \to \mathbb{P}(H^0(\mathbb{P}(E)\times\mathbb{P}(R),(d,1)))$, where $\psi(s)$, $s\in\mathbf{N_d}$, is the line spanned by a section of degree $(d,1)$ which cuts out the fiber $\phi^{-1}(s)\subseteq\mathcal{C}$. We will show that ψ is an isomorphism.

$H^0(\mathcal{O}_{\mathbb{P}(E)}(1)) = E^*$ and $H^0(\mathcal{O}_{\mathbb{P}(R)}(1)) = R^*$. Inherit the previous notations and let $\{V_0, V_1,\cdots,V_r\}$ be a basis for E^* and $\{W_0, W_1\}$ be a basis for R^*.

Under the basis $\{V_0^{i_0}V_1^{i_1}\cdots V_r^{i_r}w_0, V_0^{i_0}V_1^{i_1}\cdots V_r^{i_r}w_1\}_{i_0+\cdots+i_r=d}$, $\mathbf{N_d}$ has homogeneous coordinates $[a_{i_0,\cdots,i_r}, b_{j_0,\cdots,j_r}]_{i_0+\cdots+i_r=d}$.

Let $\{V_0^{i_0}V_1^{i_1}\cdots V_r^{i_r}W_1, -V_0^{i_0}V_1^{i_1}\cdots V_r^{i_r}W_0\}_{i_0+\cdots+i_r=d}$ be a basis of $\mathbb{P}(H^0(\mathbb{P}(D)\times\mathbb{P}(R),(d,1)))$.

$\mathbf{N_d} \cong \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}(D)}(d)) \oplus H^0(\mathcal{O}_{\mathbb{P}(D)}(d)))$. $\{x_0^i x_1^j\}_{i+j=d}$ is a basis of $H^0(\mathcal{O}_{\mathbb{P}(D)}(d))$. $\mathbf{N_d}$ then has homogeneous coordinates $[a_{ij}, b_{ij}]_{i+j=d}$, which means that each point $s \in \mathbf{N_d}$ with coordinates $[a_{ij}, b_{ij}]_{i+j=d}$, we have a rational map defined by $[\sum_{i+j=d}(a_{ij}x_0^i x_1^j), \sum_{i+j=d}(b_{ij}x_0^i x_1^j)]$ from $\mathbb{P}(D)$ to $\mathbb{P}(R)$.

$\{y_1 x_0^i x_1 j, -y_0 x_0^i x_1^j\}_{i+j=d}$ is basis of $\mathbb{P}(H^0(\mathbb{P}(D) \times \mathbb{P}(R), (d,1)))$. Let $s \in \mathbf{N_d}$ be a generic closed point with coordinates $[a_{ij}, b_{ij}]_{i+j=d}$. s is a regular morphism from $\mathbb{P}(D)$ to $\mathbb{P}(R)$ defined in coordinates as $[\sum_{i+j=d}(a_{ij}x_0^i x_1^j), \sum_{i+j=d}(b_{ij}x_0^i x_1^j)]$. The fiber of s $\phi^{-1}(s) \subseteq \mathcal{C}$ is the graph of the regular morphism s. Obviously, in coordinates, the section that cuts out the fiber $\phi^{-1}(s) \subseteq \mathcal{C}$ is given by $y_1(\sum_{i+j=d}(a_{ij}x_0^i x_1^j)) - y_0(\sum_{i+j=d}(b_{ij}x_0^i x_1^j))$. In $\mathbb{P}(H^0(\mathbb{P}(D) \times \mathbb{P}(R), (d,1)))$ under the given basis, $\phi^{-1}(s)$ has coordinates exactly $[a_{ij}, b_{ij}]_{i+j=d}$. Now identify $\mathbb{P}(H^0(\mathbb{P}(D) \times \mathbb{P}(R), (d,1)))$ with $\mathbf{N_d}$ using the given bases. The regular morphism ψ is an identity on generic points of $\mathbf{N_d}$. $\mathbf{N_d}$ is separated, ψ is continuous and is identity on generic points, so ψ is an identity all over $\mathbf{N_d}$.

Now we have that ψ is an isomorphism from $\mathbf{N_d}$ to $\mathbb{P}(H^0(\mathbb{P}(E) \times \mathbb{P}(R), (d,1)))$, so $\mathcal{C}$ is isomorphic to the universal family $\mathcal{E}$.

$\square$

Chapter 5

Fundamentals and backgrounds

5.1 Closed embedding

Theorem 12 (Closed immersion)**.** *Consider the following commutative diagram:*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\phi"'] & Y \arrow[dl, "\psi"'] \\ Z & \end{tikzcd}$$

where X,Y and Z are schemes finite type over a field k and f is a projective morphism.

f is a closed immersion, if and only if $\forall z \in Z$ a closed point, the morphism restricted on the fiber $f_z \colon X_z \to Y_z$ is a closed immersion.

Proof. It is obvious that if f is a closed immersion, then f restricted on the fibers over closed points of Z is a closed immersion.

The assumption that f is a closed immersion on each closed point in Z implies that f restricted on the fiber over each closed point in Z f is injective, inparticular, f is quasi finite. Assuming that f is a projective morphism, in particular f is proper, it follows that f is a finite morphism ([AG] p.280 Ex 11.2).

Closed immersion is a local condition in Y, so we take any closed point $y \in Y$ and its affine neighbourhood B, $y \in SpecB \subset Y$. B is a local ring with $\mathbf{m}_y$ being the maximal ideal. f is finte, so any affine open subset in Y pull back to an affine open subset in X. Name it $SpecA$. A is a finite B module. Let $SpecC$ be the affine open subset in Z where $z = \psi(y)$ lives in. We have the following diagram:

$$\begin{tikzcd} A & B \arrow[l, "f^\sharp"'] \\ C \arrow[u, "\phi^\sharp"] \arrow[ur, "\psi^\sharp"'] & \end{tikzcd}$$

$f^\sharp$ is finite.

Qutient out (the image of) the maximal ideal $\mathbf{m}_z$ of the closed point $z \in Z$ and the maximal ideal $\mathbf{m}_y$ of $y \in Y$. We get the following commutative diagram:

$$
\begin{tikzcd}
A/\mathbf{m}_y A & B/\mathbf{m}_y \arrow[l, "f_x^\sharp"'] \\
A/\mathbf{m}_z A \arrow[u, "\gamma_B"'] & B/\mathbf{m}_z B \arrow[l, "f_z^\sharp"] \arrow[u, "\gamma_A"']
\end{tikzcd}
$$

$\mathbf{m}_z$ lives in $\mathbf{m}_y$ under the morphism $\psi^\sharp$, since $\mathbf{m}_x$ is the only maximal ideal in B, so both γ_A and γ_B are surjective. $f_z^\sharp$ is surjective follows from the assumption that f_z is a closed immersion. $f_x^\sharp$ is forced to be surjective in the previous diagram.

We can choose C to be a local ring with maximal ideal $\mathbf{m}_z$, and $f^\sharp$ is a C-linear morphism is the same as saying that it is a module homomorphism over C. Now we can apply Nakayama's Lemma's corollary about module endormorphism, which we will state after the proof of this theorem, and conlude that $f^\sharp$ is surjective.

f to be injective on the topological space of X follows directly from the assumption that f is an immersion fiber-wise. $f^\sharp$ is surjective locally, so f is a closed immersion from X to Y.

$\square$

Proposition 2 (Corollary of Nakayama's Lemma)**.** *Suppose that R is a local ring with maximal ideal $\mathbf{m}$, and M, N are finitely generated R-modules. If $\phi : M \to N$ is an R-linear map such that the quotient $\phi_{\mathbf{m}} : M/\mathbf{m}M \to N/\mathbf{m}N$ is surjective, then ϕ is surjective.*

Please refer to [CA] p184 Corollary 4.8

5.2 Cohen-Macaulay scheme

Theorem 13. *If a scheme X is non-singular, then X is Cohen-Macaulay.*

Please refer to [AG] p184 Theorem 8.21A (regular implies Cohen-Macaulay).

Theorem 14 (Flatness)**.** *Let $f\colon X \to Y$ be a morphism of finite type of schemes, where X is Cohen-Macaulay and Y is smooth. Then X is flat over Y if and only if $\forall x \in X, y = f(x), dim_x X = dim_y Y + dim_x X_y$, where $X_y = f^{-1}(y)$ is the fiber over y in X under the map f.*

Proof. Equidimensional fibers $\Rightarrow$ flatness:

Let $\{y_1, y_2, \cdots, y_m\}$ be a regular sequence in the ring $\mathcal{O}_{Y,y}$ for some integer m , and $\{x_1, x_2, \cdots, x_m\}$ be the pull back of the regular sequence in $\mathcal{O}_{X,x}$, i.e. $x_i = f_x^\sharp(y_i)$, for $i = 1, 2, \cdots, m$. Let Y' be the subscheme in Y defined by the sequence $\{y_1, y_2, \cdots, y_m\}$. Because the sequence is regular, $dim_y Y' = dim_y Y - m$.

X is a Cohen-Macaulay scheme. $\forall x \in X$, the sequence $\{x_1, x_2, \cdots ,_m\} \subseteq \mathcal{O}_{X,x}$ is a regular sequence if and only if $dim_x X' = dim_x X - m$, where X' is the subscheme in X defined by the given sequence. ([IT] p.418 Proposition 6.3)

Now consider the case where Y is a smooth scheme.

A scheme Y is smooth if and only if for every point in Y, $\exists$ a regular sequence defining the maximal ideal of this single point. Let$\{y_1, y_2, \cdots, y_m\} \subseteq \mathcal{O}_{Y,y}$ be the regular sequence defining $\{y\}$, then $m = dim_y Y$. Inheritting the notation from the beginning of the proof, the pullback sequence $\{x_1, x_2, \cdots ,_m\} \subseteq \mathcal{O}_{X,x}$ cuts out the fiber X_y of X over y under the morphism f.

By assumption, the dimension of the fiber $dim_x X_y = dim_x X - dim_y Y = dim_x X - m$. The assumption that X is Cohen-Macaulay, guarantees that $\{x_1, x_2, \cdots ,_m\} \subseteq \mathcal{O}_{X,x}$ is a regular sequence. ([IT]p.397 Theorem 5.17)

A corollary of the local criterion of flatness states that as long as the sequence $\{x_1, x_2, \cdots ,_m\} \subseteq \mathcal{O}_{X,x}$ is a regular sequence, X is flat over Y on an open neighbourhood of x if and only if X' is flat over Y'. With the corollary, the fact that any scheme is flat over a point puts an end to the proof by taking Y' to be $\{y\}$ the scheme with the single point y.

Therefore, $\forall x \in X$, X is flat on a neighbourhood of x over Y if $dim_x X = dim_y Y + dim_x X_y$ giving the result that X is flat over Y.

Flatness $\Rightarrow$ equidimensional fiber:

One direction is obvious that $dim_x X \leqq dim_y Y + dim_x X_y$.

To prove that $dim_x X \geqq dim_y Y + dim_x X_y$, we need the fact that the going-down property holds for flat extension of rings([IT] p.436). Consider the local rings $f^\sharp : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$. Let $\mathcal{I}_{X_y}$ be the sheaf of ideal of X_y. From the gowing-down property, given any chain in $\mathcal{O}_{Y,y}$ starting from $\Updownarrow_y$ the sheaf of ideal of $\{y\}$, there exists a chain in $\mathcal{O}_{X,x}$ starting from $\mathcal{I}_{X_y}$ lying over the given chain, with the observation that $\mathcal{I}_{X_y}$ lies over $\Updownarrow_y$ under the morphism $f^\sharp$. Therefore, $codim_x X_f \geqq dim_y Y$, which is equivalent to $dim_x X \leqq dim_x X - dim_y Y$.

$\square$

5.3 Glueing scheme

Definition 5.1. Let X, Y, Z be schemes of finite type over a commutative ring A. Z is closed embedded into both X and Y with $i_1 : Z \hookrightarrow X$ and $i_2 : Z \hookrightarrow Y$. *The gluing scheme* C of X and Y over Z, denoted as $C = X \coprod_Z Y$, is defined in the following way:

1) The topological space of C is the gluing of the topological spaces of X and Y under the equivalence relation : $\forall z \in Z$, $i_1(z) \sim i_2(z)$. (Take the disjoint union of X and Y and identify $i_1(z)$ with $i_2(z)$ for all $z \in Z$.)

2)If locally, X is $SpecA$, Y is $SpecB$, and Z is $SpecD$ for some commutative rings A, B, D, then C is $SpecK$, where K is the kernel of the sujective homomorphism from $A \times B$ to D given by $i_1^\sharp - i_2^\sharp$:

$$0 \to K \to A \times B \to D \to 0$$

. The gluing scheme is universal, in a sense that every regular function over X and Y that agrees on Z, pushes out to a regular function on C. This is obvious since locally their pullback onto A and B coincides in D, and thus is in the kernel of the exact sequence which is a regular function on the gluing scheme we defined.

Lemma 3. *Let A be a commutative ring, and I,J be two ideals in A. The following sequence is exact:*

$$0 \to A/(I \cap J) \to A/I \oplus A/J \to A/(I + J) \to 0.$$

Please refer to ([CA] p.187 Proposion 2.4) for proof details.

Example: If X is cut out by one single equation f in a UFD and Y by another equation g, and $\{f, g\}$form a regular sequence, then $C = X \coprod_Z Y$ is cut out by fg.

The reason is simply based on the fact that $(f) \cap (g) = (fg)$ if $\{f, g\}$ form a regular sequence in a UFD.

5.4 Blowing up

Theorem 15. *Let $\phi : X \dashrightarrow \mathbb{P}^m$ be a rational morphism from an integral scheme X to the projective space of dimension m, and Z be the base locus of ϕ. The blowing-up $\mathbf{Bl}_Z X$ is isomorphic to $\overline{\Gamma_\phi}$, the closure of the graph of ϕ in $X \times \mathbb{P}^m$.*

Proof. Rational morphisms to the projective space $\mathbb{P}^m$ are given by $(m+1)$ global sections $\{s_i \in \mathrm{H}^0(X, \mathcal{L})\}_{i=0,1,\cdots,m}$, where $\mathcal{L}$ is an invertable sheaf over X globally generated by $\{s_i\}_{i=0,1,\cdots,m}$.

$\phi = [s_0, s_1, \cdots, s_m]$, Z is the subscheme in X cut out by the $(m+1)$ sections, so the invertable sheaf $\mathcal{L}$ defining ϕ is infact the ideal sheaf of Z over X.

$\mathbf{Bl}_Z X$ by definition is $\mathbf{Proj}(\oplus_{n \geqq 0} \mathcal{L}^n)$. $Proj \mathcal{O}_X(s_0, \cdots, s_m) = X \times \mathbb{P}^m$, so $\mathbf{Bl}_Z X$ can be considered as a closed subscheme of $X \times \mathbb{P}^m$. $\overline{\Gamma_\phi}$ is obviously a closed subscheme of $X \times \mathbb{P}^m$. Since X is integral, so is $X \times \mathbb{P}^m$, if either of the closed subscheme contains the other and they have the same dimension, they should be equivalent.

Let $\mathcal{I}$ be the sheaf of ideal of $\overline{\Gamma_\phi}$.

$$\mathrm{H}^0(X, \mathcal{L}) \otimes \mathcal{O}_X \twoheadrightarrow \mathcal{I} \otimes \mathcal{L}$$

$$\mathcal{O}_X[s_0, \cdots, s_m] \cong Sym_{\mathcal{O}_X}(\mathrm{H}^0(X, \mathcal{L})) \twoheadrightarrow \mathcal{O}_X \oplus \mathcal{I} \otimes \mathcal{L} \oplus \cdots$$

$$\mathcal{O}_X[s_0, \cdots, s_m] \otimes \mathcal{O}_X[s_0, \cdots, s_m]/\mathcal{I} \cong Sym_{\mathcal{O}_X}(\mathrm{H}^0(X, \mathcal{L})) \otimes \mathcal{O}_X[s_0, \cdots, s_m]/\mathcal{I}$$

$$\twoheadrightarrow \mathcal{O}_X \oplus \mathcal{L} \oplus \mathcal{L}^2 \oplus \cdots$$

Take Proj on both sides. We have an injection from $\mathbf{Bl}_Z X$ to the closure of the graph. They have the same dimension and in the same integral ambient space, so they are isomorphic to each other.

$\square$

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