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2002

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An Idempotent-Analytic ISS Small Gain Theorem with Applications to Complex Process Models

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An Idempotent-Analytic ISS Small Gain Theorem with Applications to Complex Process Models

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Dissertation

Presented to the Faculty of the Graduate School of

The University of Texas at Austin

in Partial Fulfillment

of the Requirements

for the Degree of

Doctor of Philosophy

The University of Texas at Austin

December 2002

To my brother and my sister. Family matters.

Acknowledgments

I would like to thank Drs. S. Joe Qin and Frank Allgöwer for their moral and financial support over these past years. To that, I would also like to thank Drs. Frank Doyle and Michael Hensen, whom I met while at the Institut für Systemtheorie Technischer Prozesse, for their help on finding interesting chemical engineering models to which I might apply my theory. I found discussions at the Institut für Systemtheorie Technischer Prozesse to be generally enlightening, especially those with Christian Ebenbauer and Rolf Findeisen. Back home, the discussions (debates?) with Jürgen Hahn and Sébastien Lextrait, if not always on the subject matter of this dissertation, were nonetheless thoroughly worthwhile. The same can be said of those with my roommate, J. Daniel Piper. Last, I would like to thank Riad Masri, my good friend in the Mathematics Department here at U.T., for keeping me interested in his field this whole long while.

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Austin, Texas

December, 2002

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December 2002

An Idempotent-Analytic ISS Small Gain Theorem with Applications to Complex Process Models

Publication No. ______________

Henry George Potrykus II, Ph.D.
The University of Texas at Austin, 2002

Supervisor: S. Joe Qin

In this dissertation a general nonlinear input-to-state stability small gain theory is developed using idempotent analytic techniques. The small gain theorem presented may be applied to system complexes, such as those arising in process modelling, and allows for the determination of a practical compact attractor in the system's state space. Thus, application of the theorem reduces the analysis of the system to one semi-local in nature. In particular, physically practical bounds on the region of operation of a complex system may be deduced. The theorem is proved within the context of the idempotent semiring $\mathcal{K} \subset \mathrm{End}_0^{\oplus}(\mathbb{R}_{\geq 0})$. We also show that particular to linear and power law input-to-state disturbance gain functions the deduction of the resulting sufficient condition for input-to-state stability may be performed efficiently, using any suitable dynamic programming algorithm. We indicate, through

examples, how an analysis of the (weighted, directed) graph of the system complex gives a computable means to delimit (in an easily understood form) robust input-to-state stability bounds. Applications of the theory to practical chemical engineering systems yielding novel results round out the work and conclude the main body of the dissertation.

Contents

List of Tables

List of Figures

Chapter 1

Introduction

1.1 Generalities and Motivation

Many questions concerning the qualitative behavior of systems of ordinary differential equations (ODE) deal with semi-local properties of the systems. For instance, one may study, purely locally, the stability of an equilibrium point; the behavior about a center manifold. Semi-local behavior, such as the topological nature of a flow around a (compact) attractor, might be considered the mainstay of the qualitative theory of ordinary differential equations. All this presupposes that the asymptotic (in the state space of the ODE), or better semi-global, behavior of the ODE has been previously characterized; only with the two may one rightly claim that the dynamics of the system is understood. In the case of a compact state space this question is already partially resolved. However, in physico-chemical models, whose state-spaces are *a priori* non-compact, the latter question, which ought from an epistemological view-point be the former of the two questions, has remained largely unaddressed.

The reason for this breach in knowledge is that Liapunov functions, the main tool in demonstrating the global stability of a system at hand, are notoriously difficult to construct for complex systems (here meaning systems whose state spaces contain more than a few dimensions and whose vector fields exhibit no apparent symmetry). Moreover, global stability is often not the proper property to be attempting to prove—chemical engineering systems exhibit a wide range of interesting semi-local behavior [2], [65], [36]. This latter issue is well dealt-with by a generalization of the Liapunov function technique known as La Salle's invariance principle [30]. This principle allows for regions in the state space in which the Liapunov function is not strictly decrescent along trajectories; the flow then simply stays in the appropriate connected component of the corresponding sub-level set.

A convenient simplification/expansion of La Salle's invariance principle is accomplished via Sontag's input-to-state stability (ISS) formalism [61]. This functionalization of the invariance principle not only reduces the complexity of the data incumbant on the investigator wanting to characterize stability-like behavior of the system (from a Liapunov function pair to that of an ISS gain and stability function pair), it also characterizes the behavior of the system when affected by input disturbances—that is, parameter variations in the ODE. The latter is facilitated by the *algebraic* behavior of ISS gain functions and the *sup*-norm (a so-called M-norm, [42]) with respect to the max operation.

This algebraic structure also enables one to produce a nonlinear small-gain theorem, [64]. This small gain theorem is instrumental in the so-called backstepping controller design, in which there has been much recent interest, [25]. As we shall see, this algebraic structure may be more fully exploited in the analysis of system complexes (giving rise to complex systems), i.e. interconnected systems of ODEs. Of course, any system of ODEs may be viewed in such a fashion. It is a point of this work that this view may be profitably exploited in the analysis of physical systems arising from the more or less detailed modelling of chemical engineering processes.

The mathematical formalism will follow below, but to be a bit more specific, the idempotent nature of the ISS gain functions of individual systems is exploited to show that a contraction condition (best expressed in terms of the loops in the weighted directed graph naturally associated to the system) allows one to conclude global stability of the entire system complex. Due to the nature of the vector fields arising in the modelling of physico-chemical systems, we must relax the desiderata of finding global attraction to a point to that of finding only a practical (compact) attractor. The formalism is easily modified to accomplish this. Thus, we have a means to determine a semi-global attractor of the system inside which the interesting, semi-local topological dynamics of the system may be further studied; the synthesis of the two being a complete picture of the dynamics.

There is no *a priori* reason that the method to be presented, being formal and idempotent algebraic in nature, need be applicable to any particular engineering system. However, in investigating practical examples we will learn to gain an intuition for what a contractive ('small-gain') interaction in a complex system is, for when the method will work, and for when and why it will fail outright.

1.2 Presentation of the Theory

The development herein proceeds as follows. We begin with an introduction to idempotent analysis. The presentation is meant to include the formal definitions necessary to carry the program through, while at the same time giving a vista into the world of idempotent analysis. We then move onto the ISS concept. We briefly review its meaning and at the same time we sketch Teel's form of the nonlinear small-gain theorem, the proof of which is subsumed by results given in the sequel: we subsequently introduce the idempotent analytic ISS formalism and then prove our small gain theorem.

For specific classes of gain functions a complete graph-theoretic characterization of the theory is possible. Examples of this phenomenon are hence given. That the theory does not allow for such a characterization in the general case, and thus, in that sense, is strictly stronger than Teel's small-gain theory, is shown in the next chapter. We give a few simple examples to indicate how the theory may be applied. The calculations are shown in detail. Moreover, the theory, being a mixture of analysis on weighted digraphs and systems theory, has an interesting phenomenology. We show an example of this by describing the construction of an elementary robustly stabilizing controller for a model system.

In approaching the conclusion of the work, we give applications of the theory to nontrivial chemical engineering examples: a CSTR, a continuous crystallizer, and a distillation column. To the knowledge of the author, the results in these sections have not been obtained heretofore by any other technique. Finally, we close with a broad view of the work and where it fits in the overall scheme of systems theory.

1.3 Note on Symbols Used

Especially as regards subscripting/superscripting, the notation can get a little dense. We stick to the following conventions for readability's sake. Typically time is taken as an argument or is superscripted (cf. 2.3); there is no other place for it. Then, indices, such as graph vertex labels or vector components, are subscripted. Time is rarely used, and in examples we will have need to take powers of numbers (e.g. $6^2 = 36$). Thus the indexing and powers will not occupy the same typographical place and hopefully confusion will be avoided. n and m will always refer to some state dimension; either of the system complex or of a simple (sub)system.

We include, at the end of each chapter, a "Symbols Used" section. If, in an example say, a symbol is used there, only there, and never alluded to again, we forego adding it to this section. Thus, these sections will serve as references for the reader on which nomenclature is *important* for the theory.

Chapter 2

Analytic Tools: Idempotent Analysis, Graph Theory, and ISS

2.1 Introduction and Background

In this work we exploit a formal, or algebraic, similarity between traditional transportation problems, e.g. the Bellman dynamic programming problem, formulated algebraically, and the standard formulation of input-to-state stability (ISS), as applied, in particular, to the nonlinear small gain theorem [38]. This similarity allows us to formulate a computationally efficient means of determining the resultant input/output stability properties of an interconnected set of i/o systems once the the ISS gain functions relating one basic system to the other are given. Since this methodology allows for an efficient way to determine the ISS of interconnected systems *algebraically*, it may be used, among other applications, to determine parametrically-demarked regions of robust stability.

That the Bellman problem, as well as other problems in discrete event systems theory such as Petri net modelling, can be given an effective, clear presentation within the algebraic formalism of *idempotent semirings*, and more specifically that of the max,+ semiring, is by now well documented; see [4], [1], [18] and references therein. In the standard formulation of ISS, a $\mathcal{K}$ and $\mathcal{KL}$ function are joined (in the mathematical sense, i.e. $\gamma(||\mathbf{d}||) \vee \beta(||\mathbf{x}^0||, t)$) in order to describe the (asymptotic) stability properties of the input/output system: the $\mathcal{K}$ function describes the increase in size of the attractor of practical stability ($= \gamma^{-1}(||\mathbf{d}||)$) as a function of the input magnitude, the $\mathcal{KL}$ function describes the rate of attraction ($\beta(||\mathbf{x}^0||, t) \to 0$ as $t \to \infty$) as a function of the initial condition of the state. The formal, algebraic similarity that is explioted here is precisely that the $\mathcal{K}$ functions comprise the (continuous) endomorphisms of the $(\mathbb{R}_{\geq 0}, \max)$ monoid, and that the Bellman max,+ algorithm may be applied to the endomorphism ring $\mathrm{End}_0^{\oplus}(\mathbb{R}^n_{\geq 0})$ over $(\mathbb{R}^n_{\geq 0}, \max)$ in order to check the 'small gain property' of the system complex.

The ISS paradigm was introduced by Sontag in [60]. Our work on ISS is based on the formalism in [64], and the style of that paper harkens back to the originally-introduced stability theory of Zames [69], [68] and Safonov [55]. In the latter papers 'functional' characterizations of certain notions of stability are given; the ISS framework is also a functional characterization, but in a norm-controlling sense: it gives a nonlinear L^∞-L^∞ norm bound of the output (state) signal by the input signal. We will remark later on further similarities between these two 'functional' approaches. It should be noted from the outset that ISS is basically a one-plus-time-dimensional characterization of the system being analyzed. Thereby, it is, in scope as well as formal appearance, much like control-Lyapunov function techniques wherein an energy function V, which is decresent along system trajectories, is used in systems analysis and control synthesis. There the system state space dimensions have been reduced to those of 'energy value by time,' and the ISS paradigm makes a similar reduction. We note that [62] shows the standard, functional description of ISS, to be used in this work, is equivalent to an ISS-Lyapunov description. Moreover, [28], by applying differential topological results including higher dimensional diffeomorphism (h-cobordism) theory, one shows that for state-space dimensions $\neq 4, 5$, via a diffeomorphism of the state-space and a regauging of the input signal strength, the canonical one dimensional picture of exponential asymptotic stability to a ball of radius = input signal strengh is rigorously equivalent to a generic ISS $\mathcal{K} \vee \mathcal{KL}$ functional pair.

The purpose of this work is twofold. First, as all examples known to the author of systems admitting a concrete ISS description either have one dimensional state spaces, or have sufficient symmetry (e.g. spherical) so that one may analyze it essentially as a one dimensional system, it is of considerable practical interest to detail a workable methodology allowing for the analysis of higher state-space dimension systems.[1] It will be shown here that the theory presented will allow for a practical ISS analysis of more complex systems.

Second, it is hoped that through the application of the methodology presented here we help to elucidate the types of systems ISS applies to. That is, we endeavor to make a phenomenological study of which systems nonlinear L^∞ norm bounds well characterize system input-output behavior. In our analysis,

[1]Cf. [40] for examples on how easily the ISS paradigm applies to systems with one state dimension.

given below, it is seen that certain chemical engineering process models admit just such a characterization—at least when viewing the problem semi-globally.

We begin by introducing the relevant terminology and symbols, both for idempotent analysis and nonlinear ISS theory. We then build up, from a ground-level review of Teel's work in [64], the abstract characterization of systems to which this generalized nonlinear small gain theory will apply. The next chapter proves the generalized nonlinear small gain theorem, entailed in inequality (3.2). After a discussion of this inequality we move onto the next chapter, where we give a complete characterization of the theory in terms of easily computable quantities for the linear and power-law gain function cases.

2.2 Idempotent Analysis, Graph Theory, and Signal Spaces

To begin with, the *usual, standard,* or *normal ring operations* on the reals, $\mathbb{R}$, will be multiplication and addition as induced from those on the natural numbers, $\mathbb{N}$. E. g. $4 \cdot 9 = 36$ and $5 + 11 = 16$. All other ring operation will be understood from context. Hopefully this will keep confusion to a minimum in the sequel.

A *partial order* on a set B is a subset L of $B \times B$ satisfying certain axioms. We write $a \leq b$ if $(a, b) \in L$. Then we must have $\forall a, b, c \in B$:

- $a \leq a$ (*reflexivity*);
- if $a \leq b$ and $b \leq a$ then $a = b$ (*symmetry*);
- finally, if $a \leq b$ and $b \leq c$ then $a \leq c$ (*transitivity*).

A partial order is a *(total) order* if $\forall a, b \in B$, either $a \leq b$ or $b \leq a$. $a < b$ means $a \leq b$,$a \neq b$. Given $C \subset B$, we write $c \leq C$ if $\forall a \in C : c \leq a$. The partially ordered set is *[sup-], (inf-)complete* if every bounded [from above] (from below) set $C \subset B$ has a [least upper bound] (greatest lower bound), written [$\sup C$] ($\inf C$). That is:

$$[\ (\exists d : C \leq d) \Rightarrow (\exists \text{“}\sup C\text{”}, C \leq \text{“}\sup C\text{”}\ \&\ \forall d, C \leq d \Rightarrow \sup C \leq d)\ ]$$

$$(\quad (\exists c : c \leq C) \Rightarrow (\exists \text{“}\inf C\text{”}, \text{“}\inf C\text{”} \leq C\ \&\ \forall c, c \leq C \Rightarrow c \leq \inf C) \quad)$$

A partially ordered set is *complete* if it is both *sup*- and *inf*-complete.

A function between partially ordered sets is *continuous* if $f(\sup C) = \sup f(C)$ and $f(\inf C) = \inf f(C)$; the element on one side of the equality existing whenever its opposite does. The *order interval* between two elements $c \leq d \in B$ is $[c, d] := \{b \in B : c \leq b \leq d\}$.

2.2.1 Idempotent Analysis

A *monoid* is a set A, a distinguished element $0 \in A$, and an operation $+ : A \times A \mapsto A$ such that $0 + a = a = a + 0$ for all $a \in A$ and $a + (b + c) = (a + b) + c$ for all $a, b, c \in A$, so that we may drop parenthesis when performing addition. An *idempotent monoid* is a monoid where $+$ (now written $\oplus$) satisfies $a \oplus a = a$ for all $a \in A$. We will deal exclusively with idempotent monoids, moreover only with those that are *abelian monoids*, i.e. $a + b = b + a$ for all $a, b \in A$.

A *semiring* R is an abelian monoid with another operation, multiplication, $\cdot : R \times R \mapsto R$ (juxtaposition often replacing the dot: $rs = r \cdot s$), such that $r(st) = (rs)t$, $r(s + t) = rs + rt$, and $(s + t)r = sr + tr$ for all $r, s, t \in R$. Multiplication will not normally be commutative, as our main idempotent semiring will be $\mathrm{End}_0^{\oplus}(A)$, the *endomorphisms* of $(A, 0, \oplus)$; maps from A to itself preserving 0, and commuting with $\oplus$ (= preserving addition): $\gamma \in \mathrm{End}_0^{\oplus}(A) \Leftrightarrow \gamma(0) = 0$ and $\gamma(a \oplus b) = \gamma(a) \oplus \gamma(b)$. General *morphisms* are defined similarly; they are maps between two (perhaps) different semigroups that preserve 0 and addition. $\mathrm{End}_0^{\oplus}(A)$ is a semiring under the additive operation 'point-wise addition' and the multiplicative operations 'functional composition:' $(\gamma \oplus \delta)(a) := \gamma(a) \oplus \delta(a)$ and $\gamma \cdot \mu := \gamma \circ \mu$, as the reader will easily confirm (the zero is the zero map which sends any element of A to 0). If two semirings are the same, i.e. there is a morphism between them (morphisms of semirings also preserve multiplication) that is 1-1 and onto (which is then called an *automorphism* if the underlying sets are identical, or more generally an *isomorphism*) so that the entire sets, the 0 elements, and the operations of multiplication and addition may be identified, we say the semirings are *isomorphic*, and denote the map either by $\xrightarrow{\sim}$ or $\cong$.

Our main idempotent monoid will be $(\mathbb{R}_{\geq 0}, 0, \oplus)$, where $\sqrt{5} \oplus 37.5 := \sqrt{5} \vee 37.5 := \max\{\sqrt{5}, 37.5\} = 37.5$, as $37.5 \geq \sqrt{5}$; $\sqrt{5}$ and 37.5 being here representative members of the non-negative reals.

We say a function $\gamma : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}_{\geq 0}$ is of class $\mathcal{K}$ ($\gamma \in \mathcal{K}$) when it is continuous, 0 at 0, and monotone nondecreasing (= monotone increasing, according to the French convention, otherwise we would write 'strictly monotone increasing'). Equivalently, the function is a continuous (in the lattice theoretic sense: it commutes with a transfinite number of the lattice operations $\vee$, $\wedge := -(-\cdot \vee - \cdot) = \min\{\cdot, \cdot\}$—this is, of course, intimately related to order theoretic continuity) endomorphism of the $\mathbb{R}_{\geq 0}$ monoid. Since our analysis is not so delicate as to require hen-pecking on whether the functions involved are somewhere discontinuous, we content ourselves to work with $\mathcal{K}$

instead of the larger $\mathrm{End}_0^{\oplus}(\mathbb{R}_{\geq 0})$ or the intermediate semi-continuous classes of functions.

Similarly, a function $\gamma : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}_{\geq 0}$ is of class $\mathcal{K}_{\emptyset}$ when it is continuous and monotone increasing, but not necessarily 0 at 0. $\mathcal{K}_{\emptyset} = \mathrm{End}^{\aleph(\oplus\wedge)}(\mathbb{R}_{\geq 0})$; where the crowded notation, which will be avoided in the future, should be clear. Also, a function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \mapsto \mathbb{R}_{\geq 0}$ is a $\mathcal{KL}$ function if it is separately continuous in each variable (joint continuity would restrict its applicability, but this is a complicated matter), a '$\mathcal{K}$ function in the first variable' (holding the second argument fixed), and for each $a \in \mathbb{R}_{\geq 0}$, $\beta(a,t) \to 0$ as $t \to \infty$. Lastly, corresponding to the above-discussed generalities on semirings, a function ρ is an *automorphism* of or *isotopic* on $(\mathbb{R}_{\geq 0}, 0, \oplus)$ if it is in $\mathrm{End}_0^{\oplus}(\mathbb{R}_{\geq 0})$ and is 1-1 and onto. This is denoted symbolically by $\rho \in \mathcal{K}_{\infty} = \mathrm{Aut}_0^{\oplus}(\mathbb{R}_{\geq 0})$. Note that ρ is then automatically continuous, that the same is true for its inverse, and that these are the only homeomorphisms (1-1, onto, continuous maps with continuous inverses) of $\mathbb{R}_{\geq 0}$.

We note the following relationships between the usual semiring operations on $\mathbb{R}_{\geq 0}$, the idempotent semiring operation on $\mathbb{R}_{\geq 0}$, and the functions in the endomorphism semiring associated to the semigroup $\mathbb{R}_{\geq 0}$. These relationships will be important later. In particular the monotonic behavior of the standard semiring operations on $\mathbb{R}_{\geq 0}$ will be used critically in the sections to follow.

$$(a+c) \oplus (b+c) = a \oplus b + c, \qquad (ca) \oplus (cb) = c(a \oplus b)$$

$$a_1 + \cdots + a_m \leq m \cdot (a_1 \oplus \cdots \oplus a_m), \qquad \gamma(a+b) \leq \gamma(a) + \gamma(b)$$

also

$$\gamma(a+b) \leq \gamma(a+\rho(a)) \oplus \gamma(b+\rho^{-1}(b)) \text{ for any } \rho \in \mathrm{Aut}_0^{\oplus}(\mathbb{R}_{\geq 0}). \qquad (2.1)$$

The third relation is clear for $m = 2$. The last relation is proved by considering the two cases $b \le \rho(a)$ and $b \ge \rho(a)$, and is actually a generalization of the third.

Now, $\mathbb{R}^n_{\ge 0}$ is also an idempotent monoid when we employ componentwise addition, we note that the associated endomorphism semiring is simply $\mathrm{End}^{\oplus}_0(\mathbb{R}^n_{\ge 0}) \supset \mathcal{K}^{n\times n}$. The order relation on $\mathbb{R}_{\ge 0}$ induces a partial order on $\mathbb{R}^n_{\ge 0}$: $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n_{\ge 0}, \mathbf{x} \le \mathbf{y} \Leftrightarrow x_i \le y_i \ \forall i$. $\mathbf{x} < \mathbf{y}$ means of course $\mathbf{x} \le \mathbf{y}$, $\mathbf{x} \ne \mathbf{y}$. $\mathbb{R}^n_{\ge 0}$ is what we will call the *positive cone* of $\mathbb{R}^n$. $\mathbb{R}^n_{\ge 0} = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \ge 0\}$. We will write $\mathbf{x} \ll \mathbf{y} \Leftrightarrow x_i < y_i \ \forall i$. Similarly for functions: $\gamma \le \delta \Leftrightarrow \forall\, a \in \mathbb{R}_{\ge 0} \gamma(a) \le \delta(a), \gamma, \delta \in \mathcal{K}$. And, most importantly, $\mathcal{E} \ll \Gamma \Leftrightarrow \forall \mathbf{x} \in \mathbb{R}^n_{\ge 0} \backslash \{0\} \quad \mathcal{E}\mathbf{x} \ll \Gamma \mathbf{x}$ where $\mathcal{E}, \Gamma \in \mathcal{K}^{n \times n}$.[2]

We take a short digression to give an idea of where idempotent semirings are relevant. The canonical example, the Bellman problem, will be discussed in a later section. Analogous to standard discrete time linear systems, one may study (max,+) discrete linear systems. These, as might be expected, take the form

$$x(k+1) = E(k+1)x(k+1) \oplus A(k)x(k) \oplus B(k)u(k).$$

Here matrix multiplication is induced from the semiring addition $\oplus = \vee$ and the semiring multiplication $\odot = +$ (the usual addition on the reals). The underlying idempotent semiring is $\overline{\mathbb{R}} := (\mathbb{R} \cup \{\pm\infty\}, -\infty, \oplus, \odot)$ $(-\infty \odot \infty = -\infty$ is forced by the semigroup axioms). The state component $x_i(k)$ is interpreted as the (discrete) instant of time at which the k^{th} parcel (of information, widgets, etc.) arrives at station i. Thus, the state is a monotone increasing function. This imposes an important causality constraint, expressible in terms of linear inequalities, on the above linear system. The first chapter of [41] gives

[2]For endomorphisms, when employing '$\ll$' we always ignore the trivial fixed point 0.

an overview of discrete systems, typically from operations research, modelled by such formulæ. Such systems include network queueing systems and petri nets. It is interesting to note that model predictive control has been formulated for such systems in [57] and for discrete event systems with noise in [58]. It will be interesting to see what affect these techniques will have on queuing network theory; already the model predictive control solution of some optimal allocation problems based on such discrete event systems representations has been shown in the above references to be more efficient than other existing algorithms.

In order to guarantee the stability of model predictive control one often imposes terminal constraints. In the discrete event systems case these terminal constraints take the form of asymptotic (linear) growth rates on the state vectors. That asymptotic linear growth rates for discrete event linear systems *always exist and obtain in finite time* is an interesting feature of these systems. This fact follows from a positive answer to the so-called Burnside problem for subsemigroups of $\mathrm{End}_{-\infty}^{(\oplus,\odot)}(\overline{\mathbb{R}}^n) \cong \mathcal{M}_{n\times n}(\overline{\mathbb{R}})$ (the matrices given above), [26]. The indicated isomorphism is analogous to the standard linear algebra fact that $\mathrm{End}(\mathbb{R}^n) \cong \mathcal{M}_{n\times n}(\mathbb{R})$, this latter fact being irrelevant here. The Burnside problem, one of the most famous problems in group theory, may be stated as follows. A semigroup A is *torsion* if for all $a \in A$ there exists an $n \in \mathbb{N}_0$ and a $c \in \mathbb{N}$ such that $a^{n+c} = a^n$, and *finitely generated* if there are a finite number of elements of A, (finite) products of which comprise all of A. Consider the following two propositions:

1. A is finite.

2. A is finitely generated torsion.

(1) $\Rightarrow$ (2) is trivial. The Burnside problem consists in finding classes of semi-groups such that (1) $\Leftarrow$ (2). The cited paper proves, using only elementary techniques, that if $A \subset \mathcal{M}_{n\times n}(\overline{\mathbb{R}})$ satisfies (2), then it satisfies (1). And with this remark we end the digression.

2.2.2 Graph Theory

We now move on to some graph theory concepts. All of our graphs will be directed graphs, so we will refer to them as graphs or digraphs interchangeably. A *directed graph* is a pair (V, E) where V is a finite set, the *vertices*, which, when the graph is drawn, are represented by dots: $\bullet_i \equiv x_i \in V$. $E \subseteq V \times V$ are the *arcs* (or edges), and are represented in the pictorial representation of the graph by arrows: $\bullet_i \to \bullet_j \equiv (ij) \in E$. The *dual* (or *opposite*) graph of a given graph is just the digraph with all the arrows reversed. Thus, when we are 'stepping back' somewhere in a digraph, we are 'stepping forwards' in the dual graph. A *weighted* directed graph is a digraph together with a function $\gamma : E \to \mathcal{S}$ (we are not actually confusing the matter or the reader by reusing the symbol γ here, we are doing him a favor!) on its arcs to a spaces of weights, $\mathcal{S}$. We will typically represent the weighting function, whose role in this work is substantial, by $(ij) \mapsto \gamma_{ji}$. In this way $\gamma_{\cdot\cdot}$ is a partial function on $(V \times V)^\top$.

One typical space of weights are the positive reals, $\mathcal{S} = \mathbb{R}_{\geq 0}$. Concretely, $\gamma_{\lightning\checkmark}$ might represent how long it takes a graduate student to run his thesis from his desk ($\checkmark$) over to his advisor's desk ($\lightning$). Note that, in general, $\gamma_{\checkmark\lightning} \neq \gamma_{\lightning\checkmark}$, and the return path, $(\lightning\checkmark)$, may not even exist! The story for this thesis is not so simple. Indeed, we will take the weights to be in $\mathcal{S} = \mathcal{K}$, or some space that includes it.

Associated to a weighted digraph is its adjacency matrix. Let $n = |V|$, the cardinality of the set of vertices. The *adjacency matrix*, $\tilde{\Gamma}$, is the $n \times n$ matrix with coefficients in $\mathcal{S} \cup \{0\}$ given by $\tilde{\Gamma}_{ji} = \gamma_{ji}$ if $(ij) \in E$, and $\tilde{\Gamma}_{ji} = 0$ otherwise. Note that if $\mathcal{S}$ is *a priori* a semiring, we may add and multiply different adjacency matrices defined on the same set of vertices using the rules for matrix addition and multiplication learned in high school, just as is indicated above in the digression on discrete event systems. Thus, the set of adjacency matrices over a fixed vertex set is a semiring in its own. In the special case $\mathcal{S} = \mathrm{End}_0^{\oplus}(\mathbb{R}_{\geq 0})$, we note that this semiring is simply $\mathrm{End}_0^{\oplus}(\mathbb{R}_{\geq 0}^n) \supset \mathcal{K}^{n \times n}$.

A digraph is said to be *strongly connected* if for every pair of vertices, i, k say, there exists a *path* from i to k. A path, l, from i to k is a finite sequence of arcs $(j_0 j_1), (j_1 j_2), \ldots, (j_{|l|-1} j_{|l|})$ such that $i = j_0$ and $j_{|l|} = k$. $|l|$ here is the number of arcs in the path. This number is sometimes called the *length* of the path. We will similarly employ the term *cost* for the value $\sum_{m=0}^{|l|-1} \gamma_{j_{m+1} j_m} = \gamma_{j_{|l|} j_{|l|-1}} + \cdots + \gamma_{j_1 j_0}$ when $\mathcal{S}$ is some semigroup.

It is easy to see that a digraph is strongly connected if and only if the associated adjacency matrix is *irreducible*, [11]. That is, it is not reducible: there is no simultaneous permutation of its rows and columns (corresponding to a renaming of the vertices) such that the adjacency matrix takes the (block matrix) form

$$\begin{pmatrix} R_{11} & 0 \\ R_{21} & R_{22} \end{pmatrix}.$$

An analog to the cost of a path weighted by a semigroup is $\prod_{m=0}^{|l|-1} \gamma_{j_{m+1} j_m} = \gamma_{j_{|l|} j_{|l|-1}} \cdots \gamma_{j_1 j_0}$. This is the *evaluated product* or *composition* of the weights along the path. These latter terms being defined in the case $\mathcal{S}$ is a semiring.

There is a correspondence between the case where $\mathcal{S}$ is first the semigroup $(\mathbb{R} \cup \{-\infty\}, +)$ and we consider *costs*, that is, *evaluated sums* along a path, and the case where $\mathcal{S}$ the semiring is $(\mathbb{R}_{\geq 0}, 0, \vee, \cdot)$ and we consider *evaluated products*. This correspondence is given by the exponential function; $\exp : (\mathbb{R} \cup \{-\infty\}, +) \to (\mathbb{R}_{\geq 0}, \cdot)$. The inverse to this correspondence is given by the logarithm function.

2.2.3 Signal Spaces

Lastly, we indicate to the reader the signal spaces we will be working (implicitly) with. We will either work with the space $[x(\cdot)] = L^{\infty}(\mathbb{R}_{\geq 0}; \mathbb{R}^m)$, the set of $\mathbb{R}^m$-valued essentially bounded functions on $\mathbb{R}_{\geq 0}$, which is the Banach space of measurable functions $x : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ with finite *sup*-norm:

$$||x||_{\infty} := \max_{i=1,\dots,m} \inf\{a_i : \mathrm{meas}(|x_i|^{-1}(a_i, \infty)) = 0\},$$

or we will work with the space of *eventually bounded signals* from $\mathbb{R}_{\geq 0}$ to $\mathbb{R}^m$, $[x(\cdot)] = L^{\to\infty}(\mathbb{R}_{\geq 0}; \mathbb{R}^m)$. This is the pre-Banach space of measurable functions $x : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ with finite semi-norm (*eventual bound*):

$$||x||_{\to\infty} := \max_{i=1,\dots,m} \inf_{t \to \infty} \inf\{a_i : \mathrm{meas}((t, \infty) \cap |x_i|^{-1}(a_i, \infty)) = 0\}.$$

$L^{\to\infty}(\mathbb{R}_{\geq 0}; \mathbb{R}^m)$ may not appear that interesting since its Banach space completion is isometric to $(\mathbb{R}^m, \max_{i=1,\dots,m} |\cdot_i|) =: l_{\infty}^m$,[3] but in systems theory it is

[3] This notation is not perfectly standard. Typically

$$l_{\infty} = (\{x^{\cdot} : \mathbb{N} \to \mathbb{R} : \sup_i |x^i| < \infty\},\ \sup_i |x^i|)$$

and sometimes

$$l_{\infty}^m = (l_{\infty})^m = (\{\mathbf{x}^{\cdot} : \mathbb{N} \to \mathbb{R}^m : \sup_i |\oplus \mathbf{x}^i| < \infty\},\ \sup_i |\oplus \mathbf{x}^i|).$$

These latter spaces would be critical for the analysis of discrete-time systems (cf. 6.3). We stick to the above notation throughout the thesis.

the original signal space that counts.

Since we shall make so much mention of them, we formally introduce l_∞^m balls. Figure 2.1 depicts the radius 1 l_∞^2 ball. It is simply $\{(x_1, x_2) \in \mathbb{R}^2 : x_1 \vee x_2 \leq 1\}$.

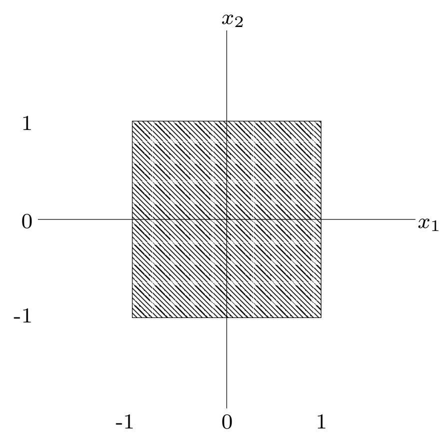

Figure 2.1: The Unit Ball in l_∞^2

In [55] there is a lengthy discussion about *extended signal spaces.* These are signal spaces which contain one of the above two signal spaces, but also contain elements whose (semi)norm is potentially infinite. That is, the seminorm on these vector spaces is *numerically valued.* Thus these extended signal spaces contain elements that 'blow up.' Extending the methodology to these spaces is one of reformulating definitions, and since our theory is a 'bounded-input-bounded-output' theory in its conceptual foundation, we do not see this extension as crucial; we will therefore not undertake it.

In order to reduce symbolic clutter, we generally dispense with, and leave ambiguous, the norming of a $L^\infty(\mathbb{R}_{\geq 0}; \mathbb{R}^n)$, say, signal. That is, x represents both $x : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}^n$ and $||x||$. We hope that the norm dropping convention will cause no confusion in the sequel. As the actual signals are not used in any of the mathematics of the methodology, the convention is sensible.

All the signals we shall be analyzing will be trajectories of differential equations of the type:

$$\dot{x}(t) = f(x(t), d(t))$$

with $x(0) = x^0$. f will be smooth, $d \in L^\infty$, so Peano's theorem (cf. [3]) guarantees existence of solutions, at least for some time. When we show ISS boundedness *a forteriori* we will have existence for all times.

The one ODE property we investigate in detail is that of *semi-global attraction to a compact set*. The set will be called a *semi-global compact attractor*,[4] and we, somewhat loosely, term it *practical* when the information it lends is, from a physical standpoint, useful. That is, when it gives us bounds on behavior we otherwise would not have known, but have use for in an application having perhaps nothing to do with the ISS concept. A subset, K, of a finite dimensional (normed) vector space (the latter is, or contains, the state space of the system) is compact when it is closed and bounded. It is an attractor when there exists an open set containing it such that all trajectories starting in that open set get asymptotically close to that set:

$$\exists U^{\text{open}} \supset K : \forall x^0 \in U : d_K(x(t)) \to 0 \text{ as } t \to \infty$$

where $d_K(x) := \inf\limits_{y \in K} ||x - y||$. It is a global attractor when U may be taken to be $\mathbb{R}^n$. It is a semi-global attractor, again, somewhat loosely, when a noncompact subset of the entire vector space may be taken to be the state space of the system, and in that state space the attractor is a 'global attractor.' Typically this noncompact subset will be the positive cone of $\mathbb{R}^n$.

[4]This terminology is not uniformly held to in the mathematical literature. Sometimes one requires that the attractor actually have a dense orbit within it. We forego this requirement.

We note that the two sup-norms given above are well suited for the characterization of signals attracted to l_∞^m balls (in $\mathbb{R}^m$). The rate of attraction to a compact attractor, as well as its radius, will be characterized via these sup-norms by a $\mathcal{K}_\emptyset \vee \mathcal{KL}$ pair, as will be shown in Section 2.4. This will be termed *ISS boundedness*.

2.3 The Bellman Algorithm: finding loops of positive cost

Our final foray into basic idempotent analysis will be to describe Bellman's dynamic programming algorithm in that framework. To wit, the Bellman algorithm, in the max,+ formalism,[5] is simply [17]

$$y^0 = \begin{bmatrix} -\infty \\ \vdots \\ -\infty \end{bmatrix}, y^k = Ay^{k-1} \oplus b$$

where b_i is the cost upon entry into vertex i, A_{ij} is the cost allowing for travel from vertex j to vertex i and it is immediate that

$$y^{k+1} = (I \oplus A \oplus \cdots \oplus A^k)b$$

is the vector of maximum cost for entrance into vertex (=row) i allowing paths of length $\leq k-1$. If there is a loop with positive cost then $y^{n+1} > y^n$, as may be seen by visualizing what paths the algorithm traces out. Otherwise $y^n = y^{n+1}$ and $y^n = Ay^n \oplus b$ is the minimum fixed point of $y \mapsto Ay \oplus B$. As is well known, this algorithm works in $O(n^3)$ time. We note that Howard's policy

[5]That is, all matrix operations are over the max,+ semiring.

iteration algorithm has also been algebraized within the max,+ framework, [18]. This algorithm, which gives the same information as the standard Bellman algorithm, generally outperforms the latter. Being able to find loops of positive cost will be important in the sequel, and so in practical applications these algorithms will be of use.

2.4 The ISS Stability Concept

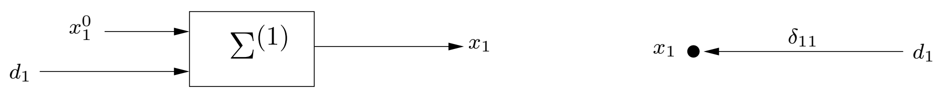

Figure 2.2: Representation of an elementary system

Figure (2.2) is the schematic presentation of the basic ISS relation $x_1 \leq \beta(x_1^0, t) \vee \delta_{11}(d_1)$. In words this relation states that the state signal's norm, x_1, is bounded by the maximum of the inputs signal's norm d_1, with nonlinear gain coefficient $\delta_{11} \in \mathcal{K}_\emptyset$ and the '$\beta-$decay rate' $\beta(x_1^0, t) \xrightarrow[t\to\infty]{} 0$, $\beta \in \mathcal{KL}$. The decay rate depends on the norm of x_1^0, of course. Such a system is said to be ISS if $\delta_{11} \in \mathcal{K}$, otherwise it is simply ISS bounded, or 'bounded-input-bounded-state stable.' Figure (2.3) shows the basic set-up to which the classical nonlinear small gain theorem is applied. The conditions used in the derivation are simply that $\gamma_{12} \circ \gamma_{21} \ll Id$ and $\gamma_{21} \circ \gamma_{12} \ll Id$.

In Figure (2.4) we depart from our conventional notation to conform to that in [64]. This figure is an exact graphical representation of the second small-gain theorem proved in the aforementioned paper. Thus, once our extended small gain theorem is proved for arbitrary systems graphs, it is im-

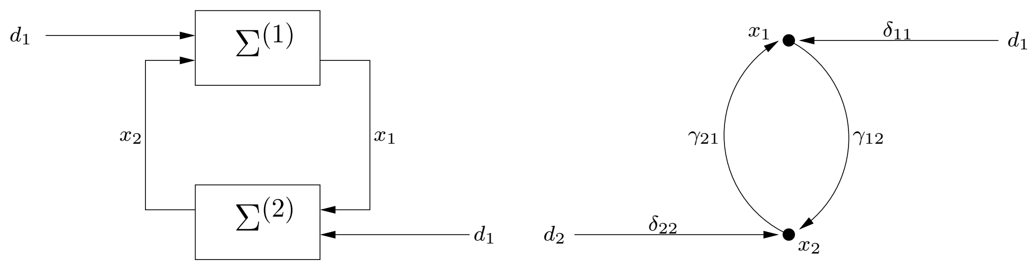

Figure 2.3: An elementary feedback system loop and the elementary Teel diagram

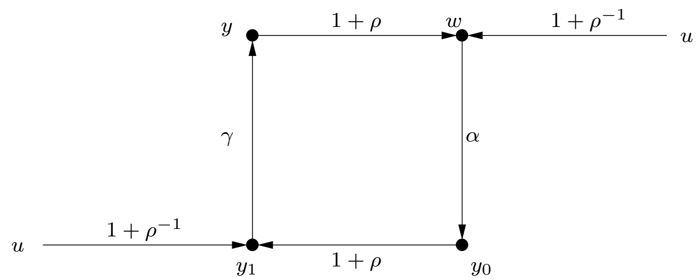

Figure 2.4: The other Teel diagram

mediate that both of Teel's small gain theorems are subsumed by the more general result given here. Notes on adapting this small gain theorem's formalism to our problem of minimizing the radius of the practical compact attractor to be computed will be given in the last chapter of this work. The signals in the figure are:

$$y = (y_1, y_2), \quad u = (u_1, u_2),$$

$$w = y + u, \quad y_1 = y_0 + u,$$

$$y_0 \le \alpha(w), \quad y \le \gamma(y_1).$$

and we change the pluses (in the second line above) to max's by using the gauge function $\rho \in \mathrm{Aut}_0^{\oplus}(\mathbb{R}_{\geq 0})$ and the inequality (2.1). We now return to our standard notation.

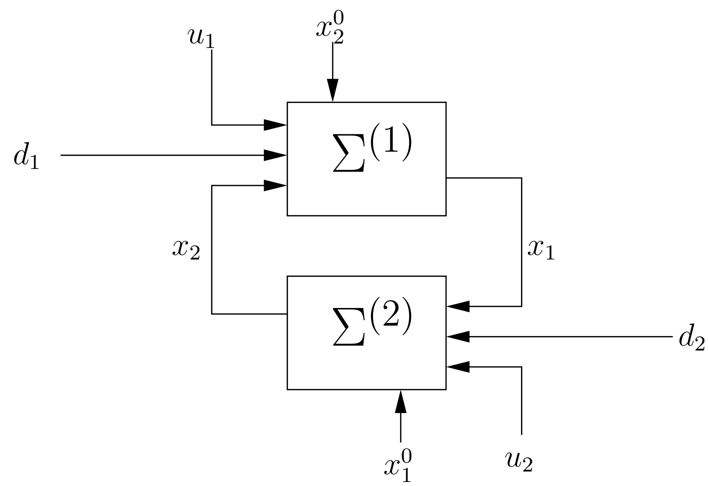

Figure 2.5: A feedback system loop with all possible inputs shown

Figure (2.5) shows, for an elementary loop, all input signals we can possibly handle in the theory. It must be mentioned that the control signals play a passive role throughout the development. Lastly, in Figure (2.6) we indicate a system which is more complex. Borrowing a convention from high energy physics diagrammatics, we do not demark the input nodes (which could just as easily be represented by an 'x,' say), and in this way distinguish them. In our case we ought eventually label them, $d_1, \ldots, d_m$, and the simple state spaces (the vertices, $\bullet$) $x_1, \ldots, x_n$, however.

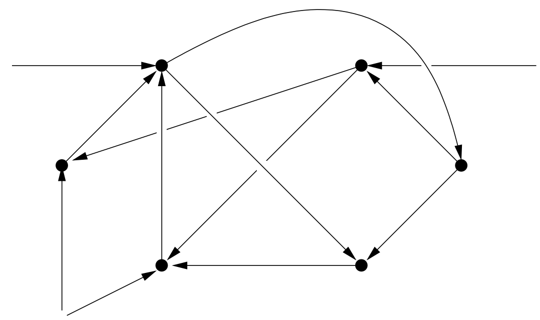

Figure 2.6: A more complicated system

2.5 Summary

The preceding exposition gave the necessary mathematical background needed to prove and interpret the extended nonlinear small gain theorem which will be proved in the next chapter. In particular, we gave the necessary idempotent-analytic and graph-theoretic concepts which will be used to extend the already standard ISS concept to complex systems; systems represented by a graph whose arcs relay ISS gain inter-dependance.

2.6 Symbols Used

Time is taken as an argument or is superscripted (cf. 3). Indices, such as graph vertex labels or vector components, are subscripted.

Table 2.1: Nomenclature

$\mathbb{N}$	the natural numbers
$\mathbb{R}_{\geq 0}$	the nonnegative reals
$\mathbb{R}^n_{\geq 0}$	the positive cone of the vector space $\mathbb{R}^n$
$\mathrm{End}_0^{\oplus}(A)$	the endomorphism semiring of the semigroup A
$\mathcal{K}$	continuous, monotone increasing, 0 at 0 functions on $\mathbb{R}_{\geq 0}$
	$\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$
$\mathcal{K}_{\emptyset}$	as above, but not necessarily 0 at 0
$\mathcal{K}_{\infty}$	as $\mathcal{K}$ but also onto
$\mathcal{KL}$	function $\mathbb{R}^2_{\geq 0} \to \mathbb{R}_{\geq 0}$, $\mathcal{K}$ in first coordinate,
	0 limit in second
$\mathcal{K}^{n\times n}$	$n \times n$ matrices whose entries are $\mathcal{K}$ functions
$L^{(\to)\infty}(\mathbb{R}_{\geq 0};\mathbb{R}^m)$	$\mathbb{R}^m$-valued (eventually) essentially bounded
	functions on $\mathbb{R}_{\geq 0}$
l^m_∞	the Banach Space $(\mathbb{R}^m, \max_{i=1,\ldots,m} \lvert\cdot\rvert)$
$\mathcal{M}_{n\times n}(\cdot)$	$n \times n$ matrices over some (semi)ring

Table 2.2: Mathematical Symbols

$\leq$	generic partial order
$\cdot \oplus \cdot,\ \cdot \vee \cdot$	the maximum of a pair
$\odot$	$+$
sup	least upper bound
inf	greatest lower bound
$\ll$	strictly less than at all points of evaluation
$\tilde{\Gamma}$	adjacency matrix of a weighted digraph

$\|\| \cdot \|\|_{(\to)\infty}$	(asymptotic) sup norm on $L^{(\to)\infty}(\mathbb{R}_{\geq 0}; \mathbb{R}^m)$
$[\cdot]$	space on which a certain variable is defined
	e.g. signal or state space of a variable
Id	the identity function $x \mapsto x$
γ	typically a $\mathcal{K}$ function
β	typically a $\mathcal{KL}$ function
ρ	typically a $\mathcal{K}_\infty$ function
t	time
x, y	(state) signals
w, u, d	(disturbance) signals
$\oplus\mathbf{x}$	max over components of $\mathbf{x}$
$d_K(x)$	$:= \inf\limits_{y \in K} \|\|x - y\|\|$, the distance function (to K)

Table 2.3: Abbreviations

ODE	ordinary differential equation
ISS	input to state stability

Chapter 3

The Extended Nonlinear Small Gain Theorem

3.1 Setup and Proof of Theorem

In this short chapter we set up and prove our extended nonlinear small gain theorem, which will be used to show the existence of practical compact attractors of complex systems.

To wit, take as given a system link-up graph with n vertices, where each weighted (by a monotone function) arc $(ji) \mapsto \gamma_{ij}$ represents that $x_i \leq \gamma_{ij}(x_j) \vee \cdots$, and, if a vertex i has no immediate predecessor, employ the trivial arc $(ii) \mapsto Id$, i.e. employ the inequality $x_i \leq x_i$ (this is also the way in which all input disturbances, d_i, are to be represented). We may then represent the entire dependance of the output signal norm on the input norms, as well as the output norms themselves, by a certain collection of gain query matrices. Before giving the most general expression, we give one which is simpler, more graph-theoretic, but has less freedom (and so is not as widely applicable). It will nonetheless prove useful in the next chapter. Thus, we have

$$\begin{aligned}\oplus_i x_i &\leq \left(\bigoplus_i \bigwedge_{m=0}^{\infty} \tilde{\Gamma}_i(\tilde{\Gamma})^m \mathbf{x}\right) \oplus \oplus_i \mathcal{G}\mathbf{d} \oplus \mathcal{B}(\mathbf{x}^0, t) \\ &=: \oplus_i \big(\tilde{\mathcal{E}}\mathbf{x} \oplus \mathcal{G}\mathbf{d}\big) \oplus \mathcal{B}(\mathbf{x}^0, t).\end{aligned} \tag{3.1}$$

Here $\tilde{\Gamma}_i$ is the row vector of non-linear gain functions which bound x_i and $\tilde{\Gamma}$ is obtained by concatenation (it is the adjacency matrix of the graph). This equation reads rather, in words, "to bound $\oplus_i x_i$ we may take the minimum of costs of stepping back along the arcs, where we step back uniformly, that is, we have gone, after the n^{th} power of $\tilde{\Gamma}$, $n+1$ arcs back from the starting vertices."

The inequality, to which we wish to apply small gain type reasoning, in its fullest generality is

$$\begin{aligned}\oplus_i x_i &\leq \left(\bigoplus_i \bigwedge_{|v|=1}^{\infty} (\Gamma)^v \mathbf{x}\right) \oplus (\oplus_i \mathcal{G}\mathbf{d}) \oplus \mathcal{B}(\mathbf{x}^0, t) \\ &=: \oplus_i \big(\mathcal{E}\mathbf{x} \oplus \mathcal{G}\mathbf{d}\big) \oplus \mathcal{B}(\mathbf{x}^0, t).\end{aligned} \tag{3.2}$$

We have used the notation $\Gamma^v = \Gamma^{v(1)}\Gamma^{v(2)} \cdots \Gamma^{v(N)}$ where v is a finite sequence of subsets of vertices, and the meet (="minimize") runs over all such finite sequences. Here $\Gamma^{\{i,\dots,k\}}$ is the matrix with rows $i, \dots, k$ being the vector of non-linear gain functions which bound $x_i, \dots, x_k$, identities on the diagonal excepting these rows, and zeros elsewhere. $\Gamma^{\{i,\dots,k\}}$ is to be regarded as the elementary 'simultaneous step back from vertices $i, \dots, k$' matrix—it corresponds to that particular ISS stability dependance query. Note that $\Gamma^{\{1,\dots,n\}} = \tilde{\Gamma}$. $\mathcal{G}$

are the "residual gain functions;" the collection of all dependence on $\mathbf{d}$ gained by "stepping back along the graph" to form $\mathcal{E}$. $\mathcal{G}$ is not, a priori, defined independent of the infinitary 'meet' operation performed above. It is found, e.g., only once a concrete cut-off, $|v| \leq N$ has been chosen (this is not the only way to cut-off the meet; cf. Chapter 4).[1] $\mathcal{B}(\mathbf{x}^0, t)$ is, similarly, the compound ISS decay function for the system complex. The equation reads, in words, "to bound $\oplus_i x_i$ we may take the minimum of costs of stepping back along the arcs, so long as we always include *all* contributing inputs input to a given system block = vertex." It will be noted that in the formation of both $\tilde{\mathcal{E}}$ and $\mathcal{E}$ it is the right action of $\mathcal{K}^{n\times n}$ on itself which effects the "stepping back" procedure.

This inequality is central to the theory. We therefore give an example of Γ matrices coming from a system graph given in figure (3.1).

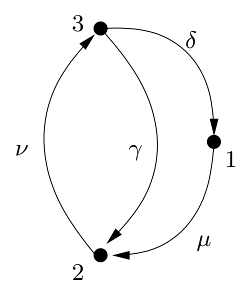

Figure 3.1: Stepping Back along a System Graph

[1]For a proof that $\mathcal{G}$ is well defined in the case where the theory works, i.e. $\mathcal{E} \ll Id$, also see the next chapter.

$$\Gamma^{\{2\}} = \begin{pmatrix} Id & 0 & 0 \\ \mu & 0 & \gamma \\ 0 & 0 & Id \end{pmatrix} \quad \Gamma^{\{3\}} = \begin{pmatrix} Id & 0 & 0 \\ 0 & Id & 0 \\ 0 & \nu & 0 \end{pmatrix}$$

$$\Gamma^{\{2,3\}} = \begin{pmatrix} Id & 0 & 0 \\ \mu & 0 & \gamma \\ 0 & \nu & 0 \end{pmatrix} \quad \Gamma^{\{1,2,3\}} = \begin{pmatrix} 0 & 0 & \delta \\ \mu & 0 & \gamma \\ 0 & \nu & 0 \end{pmatrix}$$

If $\mathcal{E}$ is a contraction (i.e. $\oplus_i \mathcal{E} \ll Id$, or to be completely explicit: $\oplus_i \mathcal{E}(\mathbf{x}) < \oplus_i \mathbf{x} \; \forall \mathbf{x} \in \mathbb{R}^n_{>0}$)[2] then because $\forall \epsilon, a, b \in \mathbb{R}_{\geq 0}, \epsilon < 1 : \big(a \leq \epsilon a \vee b \implies a \leq b\big)$, we may conclude that $\oplus_i x_i \leq \oplus_i \mathcal{G}\mathbf{d}$. We state this formally:

Theorem 3.1.1. *If $\mathcal{E}$ is a contraction then $\oplus_i x_i \leq \oplus_i \mathcal{G}\mathbf{d} \oplus \mathcal{B}(\mathbf{x}^0, t)$.*

[2] For endomorphisms, when employing '$\ll$' we always ignore the trivial fixed point 0.

3.2 Symbols Used

Table 3.1: Mathematical Symbols

$\leq$	partial order specific to $\mathbb{R}_{\geq 0}$
Id	the identity function $x \mapsto x$
$\cdot \oplus \cdot, \ \cdot \vee \cdot$	the maximum of a pair
$\wedge$	inf, greatest lower bound over choices
$\ll$	strictly less than at all points of evaluation
γ	gain function; typically a $\mathcal{K}$ function
β	typically a $\mathcal{KL}$ function
t	time
x	(state) signal norms
d	(disturbance) signal norms
Γ^v	gain function matrix of $\mathcal{K}$ functions
$\tilde{\Gamma}$	adjacency matrix of a weighted digraph
$\mathcal{E}$	collected gain function dependance of system complex
$\mathcal{G}$	collected disturbance gain function dependance of system complex
$\mathcal{B}$	collected $\mathcal{KL}$ decay rates of system complex
$\oplus_i \mathbf{x} = \oplus_i x_i$	max over components of $\mathbf{x}$

Chapter 4

Characterizations of the Theory

4.1 On Linear Gain Functions

4.1.1 Graph-Theoretic Characterization

In this section we give, for complex systems with linear gain functions (i.e. where the γ_{ji} are all of the form $\gamma_{ji}(x) = g_{ji} \cdot x$), an equivalent condition that $\mathcal{E}$ above be a contraction. Furthermore, combining the results of the two previous sections with that of this section, one has an effective means for concluding stability of the system complex. Thus let us state our first abutting proposition:

Proposition 4.1.1. *Given a system complex whose associated weighted system graph has* linear gain functions *as weights, assume that the graph is* strongly connected*. Then for $\mathcal{E}$ to be a contraction it is necessary and sufficient that all loops have gain value (when the product of the word of g_{ji}'s corresponding to the loops is evaluated)* strictly less than *1.*

Remark 1. Taking logarithms, we see that the development of Section 2.3 is directly applicable.

Remark 2. The condition of strong connectedness may be removed. In evaluating $\mathcal{E}_{ji}$ in this case, we must also consider paths $(j \cdots i)$ such that there is no return path. Given such a path, if j lies in a loop, we may ignore its cost. Otherwise, the cost of this path itself must be taken into consideration. It is easy to see that the Bellman scheme adapts well to this more complicated situation.

Remark 3. As with many proofs involving asymptotic (large iteration) behavior in discrete media, the proof essentially depends on 'pigeon hole' techniques.

Proof. Sufficiency. Suppose that $\forall$ *loop*s $\prod(loop) < 1$. Now, by 'tracing out $\mathcal{E}$' we mean that on starting at an arbitrary state vertex i we keep track of the words formed by all 'backwards steps' through the system graph. I.e. the unevaluated symbols going into row i of $\Gamma_i(\Gamma^v)^n$ for successively higher values of n. The picture-list thus formed will be of pyramidal shape and any word in the trace is of the form $[g_{i*} \cdots g_{**}] = [\{prefix\}\{loops\}\{suffix\}]$. The prefix and the suffix, not being loops, are of length at most $n-1$. Thus, arbitrarily long words, i.e., in the trace, stepping back enough times, forces a situation where *all* words appear with arbitrary large numbers of loops. As there are only a finite number of paths of length $\leq n-1$, we may bound the evaluated product of the prefix and suffix. It remains only to make the words starting at i, for each i, long enough so that $\prod loops$ is sufficiently small to force the entire product less than 1.

Necessity. By taking logs as mentioned in the first remark, we assume we are given a loop, with m arcs say, with each arc labelled by l_i where i is the vertex into which arc l_i points, such that $\sum_i l_i \geq 0$. We now answer the following question negatively:

Question. Can we pick a set of paths (of varying lengths) in the loop, one from each vertex, such that all paths have length < 0?

Suppose there exists a set of m words w_i (all with $1 \leq \#w_i < m$), word w_i beginning with l_i and ending with $l_{i(1)-1}$ (this defines the function $i : \mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$), such that the evaluated length, $\sum w_i < 0 \, \forall i$. We will make some multiple of the loop, every multiple of which has length ≥ 0, from these words w_i, each of which has length < 0, $\rightarrow\mid\leftarrow$.

Since $i : \mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$, one has $\exists$ (infinitely often) $p,\ l,\ l > p$ such that $i^l(1) = i^p(1)$. Let $k = l - p$ and $i^p(1) =: 1^*$ then $i^k(1^*) = 1^*$. Thus, starting at 1^* we have $k-1$ words w_{1^*}, etc. contributing to form some multiple of the loop-word. $\square$ [1]

Now that we have the proposition, let us give its most natural data: a linear system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{d}$$

such that the diagonals are all strictly negative. (We see presently that this condition is necessary to even *define* the *sup*-norm.) We then, from an elementary computation using the variation of parameters formula (assuming zero initial conditions): $x_i(t) = \int_0^t e^{a_{ii}(t-s)} a_{ij} x_j(s) + b_i d(s) ds$, find that $||x_i||_\infty \leq \left(\frac{|a_{ij}|}{-a_{ii}}\right)||x_j||_\infty + \left(\frac{|b_i|}{-a_{ii}}\right)||d||_\infty$. Defining $g_{ij} := l \cdot (|a_{ij}|/|a_{ii}|)$, etc., where l is the

[1] The following proof does not work because it does not handle the $\sum loop = 0$ case. It is so similar to the sufficiency proof, that it is included to show the ubiquity of the method. We will "make the loop infinitely often." Once again note that there are only a finite number of words of length $< m-1$, so we may bound from below their total length. And, as before, on going around the loop enough times (forming longer and longer concatenations of the w^is, we may dominate the lengths by the *positive length* loop, eventually making the total length positive. But $\sum w^i w^{i(1)} \cdots w^{i^p(1)}$ is less than 0 because each w^i is less than 0. $\rightarrow\mid\leftarrow$

number of entries of row $[\mathbf{A}\,\mathbf{B}]_i$ with non-zero value, we see that we are in the situation where the development of the previous section applies (each vertex i of the link-up graph corresponds to state x_i).

We shall see that linear gain functions may also arise when we analyze certain nonlinear process models some of whose state variable can be *a priori* bounded.

4.1.2 Example of the Computation of stability ranges for a Linear System Link-up

In the linear gain case ISS stability and boundedness are equivalent. Obviously the coefficients here then play the critical role. To fix the idea, we give the following example, which is perhaps the most elementary example for the theory that is possible.

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ a & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} * \\ * \end{pmatrix} d$$

an elementary calculation of the eigenvalues of $\mathbf{A}$ (which are $-1 \pm \sqrt{a}$) shows that the system is ISS (meaning $\exists L^\infty$ norm bounds) as long as $a \in (-\infty, 1)$. Now we perform the analysis allowed for above.

We may read off (assuming that $* \neq 0$) that $g_{12} = 2 \cdot 1$ and $g_{21} = 2 \cdot |a|$. Thus, as long as $|a| < 1/4$, we know our system is stable.

For a less trivial example consider

$$\mathbf{A} = \begin{pmatrix} -1 & 0 & 0 & a_{14} \\ a_{21} & -2 & 0 & a_{24} \\ 0 & a_{32} & -3 & 0 \\ 0 & a_{42} & a_{43} & -4 \end{pmatrix} , \quad \mathbf{B} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Then (all a_{ij} are below taken to be the absolute values of the a_{ij} above)

$$(g_{ij}) = \begin{pmatrix} 1 & 0 & 0 & 2a_{14} \\ a_{21} & 1 & 0 & a_{24} \\ 0 & a_{32}/3 & 1 & 0 \\ 0 & a_{42}/2 & a_{43}/2 & 1 \end{pmatrix}, \quad \delta_i(d) = \begin{pmatrix} 2d \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

whose associated digraph is shown in Figure 4.1. And, evaluating the value of

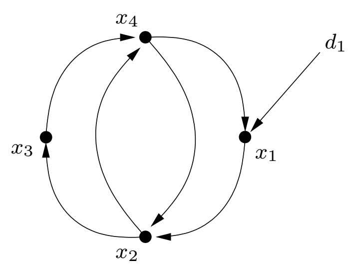

Figure 4.1: The Linear System Graph

the gains around all loops, Corollary 1 reads if

$$a_{14}\, a_{43}\, a_{32}\, a_{21} < 3 \qquad a_{24}\, a_{43}\, a_{32} < 6$$

$$a_{14}\, a_{42}\, a_{21} < 1 \qquad a_{42}\, a_{24} < 2$$

then the system is ISS stable. The ISS disturbance gain coefficient may be read off by maxing over a forward trace through the system graph.

As is clear, we have not used the technique of Section 2.3 to determine the parametrically demarked region of stability. Instead, we have used the list of all loops in the graph of the system complex. Generating the list of all loops of a digraph is an **NP**-hard problem, as it, e.g., gives an answer to the Hamiltonian cycle problem. Nonetheless, for graphs with low edge-to-vertex ratio (which is the case for most process models of interest, and is the case where our theory best applies, cf. Footnote 2.1), one may implement an algorithm which works quite well. We have written a Haskell function which automates the above procedure; in particular it uses a breadth-first search to determine the list of loops of the graph of the system complex.

Applying the classical Geršgorin criterion, [34], yields stability so long as $|a_{14}| < 1$, $|a_{21}| + |a_{24}| < 2$, $|a_{32}| < 3$, and $|a_{42}| + |a_{43}| < 4$. Comparing terms, our method seems conservative.[2]

4.2 On Power-Law Gain Functions

4.2.1 Graph-Theoretic Characterization

We say a function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a *power law* function if it is of the form $\gamma(x) = r \cdot x^a$ where $r, a \in \mathbb{R}_{\geq 0}$, $r > 0$, $a \geq 0$. We term r the *coefficient* and a the *power* or *exponent* of the power law function.

Proposition 4.2.1. *Given a complex system whose system link-up graph is strongly connected and weighted by power law ISS gain functions, and letting Λ be the set of all elementary loops of the link-up graph, consider the two following tuples:*

- $(\lambda_\mu)_{\mu \in \Lambda}$, *the list of* evaluated products *of the coefficients of the gain functions on each loop μ,*
- $(l_\mu)_{\mu \in \Lambda}$, *the list of* evaluated products *of the powers of the gain functions on each loop μ,*

then we have the following cases:

[2]but we will come back to this point in Section 6.1.1

- $\exists\ \nu \in \Lambda$ *s.t.* $l_\nu > 1$, *in which case ISS stability may not, by any means corresponding to this theory, be concluded.*

- $\forall\ \mu \in \Lambda\ l_\mu < 1$, *but* $\exists\ \nu \in \Lambda$ *s.t.* $\lambda_\nu > 1$, *in which case the system complex is ISS bounded.*

- $\forall\ \mu \in \Lambda\ l_\mu < 1\ \&\ \lambda_\mu \leq 1$, *in which case the system complex is ISS bounded.*

Remark 1. An analysis of the 'boundary' cases is straightforward. We list these cases here in a remark so that the proposition is itself not so cluttered:

- $\exists\ \nu \in \Lambda$ s.t. $l_\nu = 1$ and $\lambda_\nu < 1$. The sufficiency proof of the linear gain theory shows that we may infer ISS for the composite system, contingent upon the other loops satisfying 2^{nd} or 3^{rd} cases in the proposition.

- $\exists\ \nu \in \Lambda$ s.t. $l_\nu = 1$ and $\lambda_\nu \geq 1$. Stabiliy may not be inferred. A proof, analogous to that of 'necessity' for the characterization of the linear gain theory, is required.

Remark 2. Calculating the resulting $\mathcal{E}, \mathcal{G}$ requires some work. However, implicit in the proof of the proposition is monotonicity behavior (with respect to further compositions), which together with fixed point information, allows us to deduce practical compact attractors for a given problem. See also Remark 5 and Section 4.3.2.

Remark 3. As with the linear gain function case, one may relax the condition of strong connectivity. The statement to be found here will be analogous to that found in the linear gain function case.

Remark 4. It will be clear that only the gain functions between *state nodes* in every strongly connected component need be power law functions.

Remark 5. Even in the first case of the proposition, all is not lost; there will be an interval $(0, x^*)$, depending upon the coefficients of the gain functions, and possibly quite small, in which the contraction condition is met (due to the strict convexity of $x^{1+\epsilon}$ and that $0^{1+\epsilon} = 0, 1^{1+\epsilon} = 1$). If we may *a priori* bound the signal strength ($x < x^*$), because of small enough initial conditions and weak enough input disturbances, we may then conclude ISS for the system complex—*but only in this l_∞^n ball.* A fixed point theorem may be used to help find the point at which the loop gain functions become ≥ 1; cf. Section 4.3.2.

Proof. By successive compositions along a given loop we form gains of the form:

$$k \cdot \lambda^{\sum l^i} \cdot x^{l^n}$$

where k represents gain coefficients not part of a complete loop.

$(l < 1, \lambda \leq 1)$: To show the contraction property we must answer the following question affirmatively:

$$\forall x \in \mathbb{R}_{\geq 0}, \lambda \leq 1, l < 1, \exists ?N \text{ s.t. } \forall n \geq N : (\lambda \cdot \lambda^l \cdot \lambda^{l^2} \cdots \lambda^{l^n}) x^{l^n} < x/k$$

"$x \ggg 1$" is easy. The other case requires a bit of analysis. Let $x = 1$ Using that if $\forall i : 0 \geq u_i < 1$ then $(\prod_{i=1}^{\infty}(1 - u_i) > 0) \Leftrightarrow (\sum_{i=1}^{\infty} u_i$ converges) (cf. §15 in [54]), our question is reduced to: $\sum_{i=1}^{\infty}(1 - \lambda^{l^i}) < \infty$? We may use the ratio test to see that this series converges:

$$\lim_{i \to \infty} \frac{(1 - \lambda^{l^{i+1}})}{(1 - \lambda^{l^i})} < 1 \Leftrightarrow \lim_{\epsilon \to 0} \frac{\lambda^{l^{1/\epsilon+1}} l^{1/\epsilon+1}}{\lambda^{l^{1/\epsilon}} l^{1/\epsilon}} < 1 \Leftrightarrow l \frac{\lambda^{l^{i+1}}}{\lambda^{l^i}} < 1$$

In the first equivalence we have set $i = 1/\epsilon$ and used L'Hôpital's rule (which is easily justified). The last inequality is obviously true, and so at $x = 1$ there is an upper k beyond which we get no contraction. Because $x^{l^i} \to 1$ as $i \to \infty$, an analogous story holds for any k as long as x is small enough. We thus have the picture as given schematically in Figure 4.2 (showing continuity and monotonicity in x is a small matter). Since $l < 1$ it is easy to once

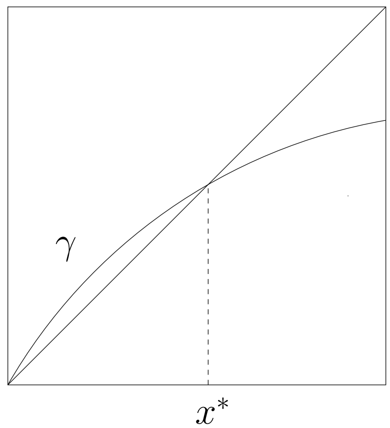

Figure 4.2: Limiting Behavior of the Power Law Gain Function

again employ the sufficiency arguments as given in the linear gain case (this time to the exponents in the composed gain functions) to conclude that for small x, this picture *must* occur in one of the gain queries, even for truncated (non-limit) compositions.

($l < 1, \lambda > 1$): The proof for this case is perfectly analogous to the case above. Now, however, $u_i = \lambda^{l^i} - 1$. Just as before $\lambda \cdot \lambda^l \cdot \lambda^{l^2} \cdots \lambda^{l^n}$ converges, and schematically Figure 4.2 is still correct.

($l > 1$): We view our successive loop compositions in the form

$$P := \prod_{i=0}^{\infty} \frac{x^{l^{i+1}}}{x^{l^i}} \lambda^{l^{i+1}}.$$

Is $P > x/k \; \forall k$? Taking logs:

$$\log P = \sum l^i((l-1)\log x \; + l \log \lambda).$$

It is thus clear that, so long as x is (fixed) large enough, P gets big as $i \to \infty$. The proof is not yet complete. We must also mention that for any fixed number of chosen compositions, we may find an x large enough such that one of the gains must be $> x$. Again, this statement follows from a pigeon hole type argument. Patching these two arguments together appropriately (i.e. choosing x uniformly large enough), we may conclude.

Nonetheless, for $x < 1/\lambda^{l/(l-1)}$ it should be clear from the expression for $\log P$ that, then, $P \to 0$. $\square$

4.2.2 Nonlinear Small-Gain for a 3-Vertex System

In this section we give an (admittedly artificial) example of the calculation of the ISS gain function for a complex system. To wit, consider

$$\begin{aligned}
&\text{S1:} \qquad \dot{x}_1 = -x_1^3 + x_3^2 \vee d_1 \\
&\text{S2:} \qquad \dot{x}_2 = -x_2 - 3x_2^3 + (1 + x_2^2)x_1 + x_2^2 x_3^{1/4} \\
&\text{S3:} \qquad \dot{x}_3 = -x_3 + x_2
\end{aligned}$$

where a '$\vee$' in the first elementary *system* has been employed mainly to simplify the calculation. The digraph associated to this system complex is given in Figure 4.3. Following Theorem 5.2 and the subsequent examples given in [40], we take a quadratic ISS-Liapunov function for all three elementary systems

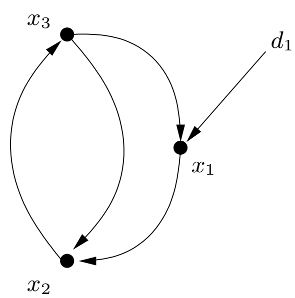

Figure 4.3: The 4 Vertex Graph

S_i: $V_i(x_i) = x_i^2$. Then

$$\begin{aligned}
\dot{V}_1 &= -x_1^4 + (x_3^2 \vee d_1)x_1 \\
&= -(1-\theta)x_1^4 - \theta x_1^4 + (x_3^2 \vee d_1)x_1 \\
&\leq -(1-\theta)x_1^4,\ \forall\, |x_1| \geq (|x_3^2 \vee d_1|/\theta)^{1/3} \\
\dot{V}_2 &= -x_2^2 - 3x_2^4 + x_2(1+x_2^2)x_1 + x_2^3 x_3^{1/4} \\
&\leq -x_2^4,\ \forall\, |x_2| \geq x_1 + x_3^{1/4} \\
&\qquad\qquad \Leftarrow (|x_2| \geq 2|x_1| \vee 2|x_3|^{1/4}) \\
\dot{V}_3 &= -x_3^2 + x_2 x_3 \\
&= -(1-\tau)x_3^2 - \tau x_3^2 + x_2 x_3 \\
&\leq -(1-\tau)x_3^2,\ \forall\, |x_3| \geq |x_2|/\tau
\end{aligned}$$

Thus

$$\begin{aligned}
\delta_{11}(d) &= (|d|/\theta)^{1/3} & \gamma_{13}(x_3) &= (|x_3|)/\sqrt{\theta})^{2/3} \\
\gamma_{21}(x_1) &= 2|x_1| & \gamma_{23}(x_3) &= 2|x_3|^{1/4} \\
\gamma_{32}(x_2) &= |x_2|/\tau & &
\end{aligned}$$

Collecting exponents and coefficients we find, for the outer loop, $\lambda_{\text{out}} = 2 \cdot (1/\tau) \cdot (1/\theta^{2/3}) > 1$, $l_{\text{out}} = 1 \cdot 1 \cdot 2/3 < 1$. Similarly, for the inner loop, $\lambda_{\text{in}} = 2/\tau > 1$, $l_{\text{in}} = 1/4 \cdot 1 < 1$. This example contains much of the basic structure implicit in Corollary 2. We content ourselves with the following remarks: we may immediately apply Corollary 2 (case 2) to conclude ISS boundedness with the disturbance gain's exponent being 1/3. Thus, starting with the system complex depicted in Figure 4.3, we have come to the point where Corollary 2 (case 2) may be applied to conclude ISS boundedness. It is not difficult to work out (again, basically through dynamical programming) that the ISS bound for the compound system is $x_1 \oplus x_2 \oplus x_3 \leq \frac{8}{\theta\tau^3} \oplus \left(\frac{8|d|}{\theta}\right)^{1/3}$.

4.3 General Sufficiency Results, Fixed Points, Characterization of the Theory

4.3.1 Computable Sufficiency Result

We also have the weaker but more general

Proposition 4.3.1. *To conclude ISS via the Small Gain Theorem it is sufficient that each loop-function be a contraction.*

Remark 1. The 'necessity' statement, while true for length 2 loops, is false in general.

Remark 2. It is this proposition that will be most useful in the sequel.

The proof of the proposition is identical to the proof of sufficiency of the hypotheses in Proposition 4.1.1. Now of course we argue point-wise; with each additional composition of (all) the loops of gain functions *every* value

$\tilde{\Gamma}^v(\mathbf{x})$ must decrease. Cf. Footnote 3.1. □

4.3.2 The Tarski Fixed Point Theorem

We now undertake the proof of the first remark above. We assume that all gain functions are 1-1. This poses no practical difficulty: the theory is stable under small perturbations, and a particular perturbing function of $+\epsilon \arctan$, say, will do the trick.

Under these assumptions then, the statement follows from fixed point considerations about the loop gain functions. The mean value theorem [10] may be applied directly, cf. [48]. The Tarski fixed point theorem may also be used to find whether or not such fixed points exist, and it is a practical as well as theoretical analytic tool. We choose to elaborate on and exploit this latter fixed point theorem as its formalism is, from a categorical point-of-view, more in-line with the mathematical presentation of this thesis. We indicate graphically these fixed point theorems in Figure 4.4. We note that the Tarski fixed point theorem for monotone functions on partially ordered sets, though elementary in nature, is one of the great unifying theorems in mathematics. As hinted to in Section 2.3, this theorem may be taken as the basis for dynamical programming. But this same statement ought to extend to the continuous time dynamic programming principle as well; the viscosity solution theory of Hamilton-Jacobi-Bellman equations might also be seen to be an application of the Tarski fixed point theorem on the space of sub- and supersolution candidates. The reader is referred to [29] for further applications of the Tarski fixed point theorem, especially for applications to recursive function theory.

Theorem 4.3.2. *Let B be a complete partially order set, and $f : B \to B$ be monotone ($a \leq b \Rightarrow f(a) \leq f(b)$), continuous. If $\exists c, d$ such that $c \leq f(c)$, $f(d) \leq d$, and either $c \gtreqless d$, then $\exists e : f(e) = e$.*

Proof. Restrict to $[c, d]$ (or $[d, c]$, when appropriate). Examine the bounded (from above or below) sets $\{b \in B : b \gtreqless f(b)\}$. A simple indirect argument shows that now the sup, now the inf of one of these sets gives us the least or greatest fixed points in the order interval $[c, d]$ (or $[d, c]$).

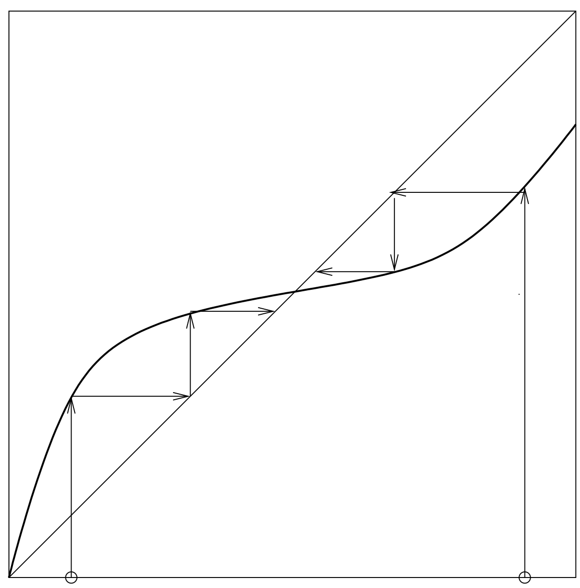

Figure 4.4: How to Find a Fixed Point of a Monotone Function Using the Tarski Fixed-Point Theorem

Remark 1. As the picture indicates, more may actually be concluded from this fixed point theorem. To wit, we remark on the utility of the Tarski fixed point theorem. The contraction condition used in Theorem 3.1.1 may be applied in any subinterval of $\mathbb{R}_{\geq 0}$. The subintervals wherein this condition is met may be determined precisely through the application of the Tarski fixed point theorem, as is indicated by its graphical representation in Figure 4.4.

Remark 2. We may relax the continuity condition when $c \leq d$. The proof basically goes through. Other extensions are also possible. We do not elaborate on details.

We now proceed with the proof of Remark 1 of Proposition 4.3.1: First,

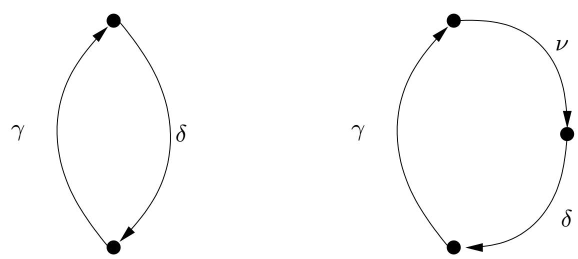

Figure 4.5: Length 2 Loop, Length 3 Loop

note that if $\delta, \gamma \in \mathcal{K}$ s.t. $\delta \circ \gamma \gg Id$, then $(\gamma \circ \delta) \circ (\gamma \circ \delta) \gg \gamma \circ \delta$ So that $\gamma \circ \delta$ cannot have any fixed point. (Likewise for $\ll$.) Furthermore, the Tarski fixed point theorem shows that we have only three cases: $\delta \circ \gamma \gg Id$, $\delta \circ \gamma \ll Id$, or there exists a fixed point. Thus we have characterized geometrically the fixed point behavior of our loop gain functions.[3]

Suppose the loop gain function $\delta \circ \gamma$ is not a contraction, then it must "bump up" for some time. The two possibilities after this "bump up" are shown in Figure 4.6. In the first case (it is easy to show that γ may also have no finite valued asymptote—thus the term 'asymptotic' below is fully justified), $\gamma \circ \delta$ must also be $\gg Id$ in this asymptotic region. But as $\gamma \ll Id$,

[3]In particular, we see that the classical hypotheses for the nonlinear small gain theorem are *redundant*. That is, if either $\gamma \circ \delta \ll Id$ or $\delta \circ \gamma \ll Id$ holds, the other is guaranteed to hold. Cf. [64]

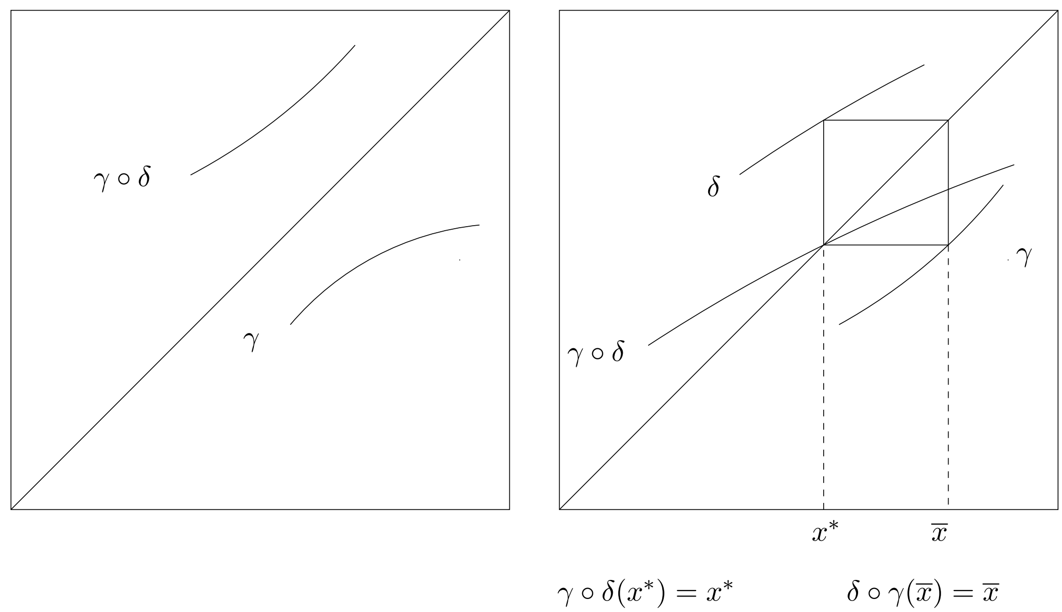

Figure 4.6: Why there is no possibility of "getting around the bump"

we must have $\delta \gg Id$. Hence there is no way that the gain functions starting with δ (and possibly stepping back) can be a contraction in this asymptotic region.

This leaves us with the second diagram. The fixed point $\gamma \circ \delta(x^*) = x^*$ is given by hypothesis. The sequence of evaluating this pair of functions on x^* is given graphically by tracing clockwise around the square, starting at (x^*, x^*). Likewise, we discover a fixed point $\overline{x}$ for $\delta \circ \gamma$ by tracing clockwise around the square starting at the other corner which intersects the diagonal $=: (\overline{x}, \overline{x})$. We may now conclude: by the monotonicity of δ, neither δ nor $\delta \circ \gamma$ may be a contraction at $\overline{x}$.

It is not difficult to describe a counter-example for the length 3 loop case. As an analogue, we briefly indicate how the obstruction occurring in the second diagram above may be removed through the intervention of a third gain function, ν: Assume that this is the only 'bump,' that the bump up begins at some point $x' > 0$, that $\nu \ll Id$, and that $\nu \in \mathcal{K}_\infty$. We may, first, use ν as the only ISS query for the vertex into which it is pointing. Then, if the interval $(0, x')$ is large enough, and (x', x^*) small enough, we may choosing $\delta \circ \nu \circ \gamma$, then $\delta \circ \nu$, then δ (the latter over the bump—"large enough" means here that $\nu^{-1}(0, x') \supset (x', x^*)$) so that there is no region where we cannot discover the contraction condition for each vertex. $\square$

4.3.3 The Well-Definedness of $\mathcal{G}$, $\mathcal{B}$ and the Value of $\mathcal{E}$

We give here a proof that $\mathcal{G}$ is well-defined if the theory would produce a useable result, $\oplus_i x_i \leq \oplus_i \mathcal{G}\mathbf{d} \oplus \mathcal{B}(\mathbf{x}^0, t)$, e.g. that all loops are contractions. Of course, if we terminate the infimum at some finite cutoff $|v| \leq N$, the values of $\mathcal{G}$ and $\mathcal{B}$ are what they are then given to be.

Proposition 4.3.3. *If all loops are contractions then the coefficients of $\mathcal{G}$ are bounded.*

Proof. Suppose for some $\mathbf{d}$ there exist paths of arbitrarily long lengths from *a priori* different inputs d_i so that the maximum of the coefficients of $\mathcal{G}$ becomes arbitrarily large (and $\mathcal{G}$ is not well defined). Then, by two applications of the pigeon-hole principle, first to the set $\{d_i\}$ (at least one of the d_i's, d_{i*}, must occur infinitely often), and second to the set of paths emanating out of d_{i*}, at least one path must be made arbitrarily long. This path eventually contains loops, at least one of which must have evaluated product gain (at $\mathcal{G}(\mathbf{d})) > 1$. By Proposition 4.3.1 $\mathcal{E}$ is not a contraction. $\square$

Lastly, we comment that if $\mathcal{E}$ is a contraction (after some finite cutoff: $|v| \leq N$) and the link-up graph is strongly connected, then we may "repeat" the application of $\mathcal{E}$ to drive the value of $\mathcal{E}$ arbitrarily close to zero. Thus, we have a sort of '0 or ≥ 1' law.

4.4 Summary

The extended nonlinear small gain theorem proved in the last chapter was here characterized for system complexes with linear and power law gain functions. It was seen that loop conditions in both cases were necessary and sufficient to characterize the theory. For gain functions not in this class, a general sufficient condition was proved, and an adjoining proposition on the well-definedness of $\mathcal{G}$ and $\mathcal{B}$ was given. In Subsection 4.3.2 a practical tool, the Tarski fixed point theorem, was also developed with an eye towards its application to the ISS extended nonlinear small gain theorem.

The last demonstration in Subsection 4.3.2 shows that the infimum in Inequality 3.2 is strictly stronger than any loop condition. Thus the theory, though characterized by loops in the particular cases of linear or power law gain functions, is in general not computable by such considerations. We readily conjecture the non-computability of the theory, even for smooth gain functions. The reason this will be true is that we must *a priori* check for a possibly infinite number of 'bumps up' along the positive real axis, once given a concrete graph, and there is no loop-theoretic technique that can, again *a priori*, aid us.

Nonetheless, in particular cases one may perform the graphical analysis of the gain functions by hand, as we shall see. Also, if all gain functions involved exhibit some certain regular behavior we can expect the theory to admit a computable resolution.

4.5 Symbols Used

Table 4.1: Mathematical Symbols

$\leq$	partial order specific to $\mathbb{R}_{\geq 0}$
Id	the identity function $x \mapsto x$
$\cdot \oplus \cdot,\ \cdot \vee \cdot$	the maximum of a pair
$\wedge$	inf, greatest lower bound over choices
$\ll$	strictly less than at all points of evaluation
γ, δ, ν	gain function; typically a $\mathcal{K}$ function
β	typically a $\mathcal{KL}$ function
t	time
x	(state) signal norms
d	(disturbance) signal norms
Γ^v	gain function matrix of $\mathcal{K}$ functions
$\tilde{\Gamma}$	adjacency matrix of a weighted digraph
$\mathcal{E}$	collected gain function dependance of system complex
$\mathcal{G}$	collected disturbance gain function dependance of system complex
$\mathcal{B}$	collected $\mathcal{KL}$ decay rates of system complex
$\oplus_i \mathbf{x} = \oplus_i x_i$	max over components of $\mathbf{x}$
ν, μ	index for a loop
λ, l	evaluated product along a loop of coefficients, powers
a, g, r	gain power, coefficient, coefficient

$\ggg$	much greater than

Table 4.2: Nomenclature

$\mathbb{Z}/m\mathbb{Z}$	$= \mathbb{Z} \mod m = \{0, 1, \ldots, m = 0, m + 1 = 1, \ldots\}$

Chapter 5

Applications to Chemical Engineering Systems

5.1 Introduction

Here we present three chemical engineering models which admit ISS boundedness characterization via the extended nonlinear small gain theorem. The rate of attraction to and radius of a compact attractor will be characterized by a $\mathcal{K}_\emptyset \vee \mathcal{KL}$ pair: what is actually calculated are $\mathcal{K}$ functions which in some region of signal strength are contractions. Then, in order to be able to apply the contraction condition we must either allow for ISS boundedness (by appending to the right hand side of 3.2 a positive constant—thereby changing the $\mathcal{K}$ functions into $\mathcal{K}_\emptyset$ function), or we must be able to consistently restrict all dynamics to a small region of the state space where the contraction condition is satisfied.

The systems, being of a practical nature, do not immediately admit ISS characterization via the theory developed above; other facts are often necessary for the program to go through. We use the follow, with due apology, in the sequel:

- invariance of (a physical) positive cone
- provable *a priori, uniform* physical bounds on certain state variables
- limits on physical parameters

Dependance on parameters, disturbance gain functions, and decay rate functions are also easily determined, and adjoining local stability results, one can obtain the entire dynamical picture of the system. We do not go this far with the analysis. In fact, the results presented in this chapter are highly preliminary. In order to optimally apply the theory, closer attention must be paid to the simple system dynamics, and state signal norm re-gauging functions ought to be employed. Cf. the end of this chapter and Section 6.1.

5.2 CSTR

We first examine a model in which the small gain theorem plays a background role; it is appealed to only at the end, once a significant amount of analysis of the component vector fields has been done. Thus, here, it is seen as a sort of glue which puts all necessary components together, in order to come up with a salient conclusion. The model, a CSTR with cooling jacket dynamics, has been extensively studied, [65], [31], [5]:

product

$$\dot{x}_1 = -x_1 + Da(1 - x_1) \exp \frac{x_2}{1 + x_2/\gamma}$$

tank temperature

$$\dot{x}_2 = -x_2 + \beta(x_2 - x_J) + Da(1 - x_1) \exp \frac{x_2}{1 + x_2/\gamma}$$

jacket temperature

$$\dot{x}_J = \delta_1 \delta_2 (x_J^0 - x_J) + \beta_J \delta_1 (x_2 - x_J)$$

The values for the constants, taken from [31], are $\gamma = 20.0$, $\beta = 0.3$, $Da = 0.072$, $\beta_J = 0.01$, $\delta_2 = 500$, and $B = 1.0$. This last value differs from [31], as discussed below.

We note first that if x_1 is negative, its vector field is strictly positive. Thus we need only find a gain function for it for positive values. This is then obtained from

$$(Da + \exp \frac{x_2}{1 + x_2/\gamma}) x_1 \geq Da \exp \frac{x_2}{1 + x_2/\gamma}.$$

The second vector field admits a gain function characterization through

$$(1 - \beta) x_2 \geq 2BDa \exp \frac{x_2}{1 + x_2/\gamma} \vee 2\beta |x_J| \tag{5.1}$$

where there is a bound on $(1 - x_1)$, it being less than or equal to 1 by the above discussion. The gain functions are obtained from these inequalities in the same manner as in Section 4.2.2. Figure 5.2 shows that there is a bounded region (which will be one of the two we consider here) in which this inequality allows for a gain function to be extracted. The dashed line indicates the diagonal $y = x$.

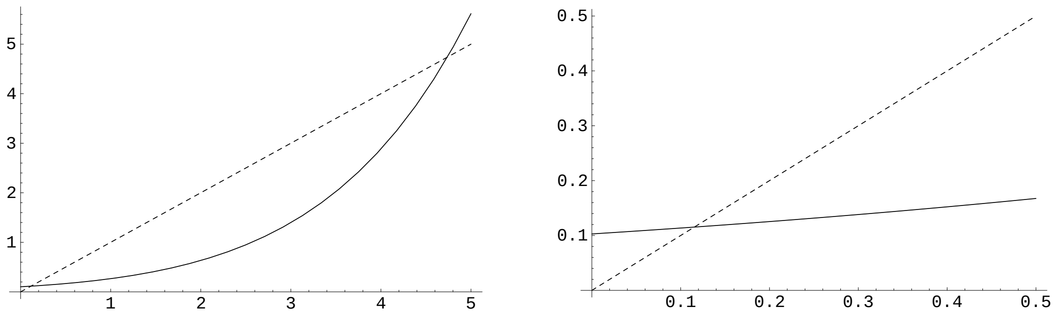

Figure 5.1: Region in Which Inequality 5.1 is Satisfied

Note that a (very rough) lower bound on x_2 is x_J^0; the second and third vector fields shows that x_2 cannot possibly get smaller than this. Now, in dimensionless units x_J is (much) smaller than γ, therefore we never reach the region in which the denominators of the exponentials become small. Finally, the jacket dynamics admit a linear-plus-offset gain characterization.

$$\delta_1(\delta_2 + \beta_J)|x_3| \geq \beta_J \delta_1 |x_2| + \delta_1 \delta_2 x_{30}$$

For simplicity we will let $x_J^0 = 0$ in the sequel.

The graph topology for the system, given in Figure 5.2, is particularly simple. We must only evaluate two simple (length 2) loop gain functions. Now, the top loop is automatically a contraction in the region depicted by Figure 5.2; all dependence of x_2 on x_1 is absorbed in the constants. The bottom loop is a linear contraction, also as long as $x_2 > 0.1$. Thus, we conclude ISS boundedness for this system. We remark that this l_∞^3 radius depends most sensitively on the Damköhler number, Da, (which is most usually the control parameter), the dimensionless adiabatic temperature rise, B, and the dimensionless activation energy, γ.

For larger (B, γ), greater than $(3, 5)$ say (and these values are physical), the theory does not apply. Though the required inequality does eventually

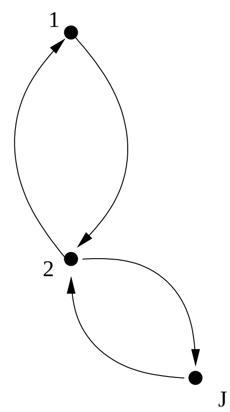

Figure 5.2: CSTR System Graph

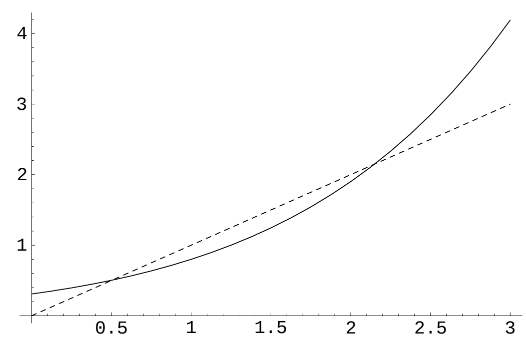

Figure 5.3: Region in Which Inequality 5.1 is Satisfied

occur (when the exponential is saturated, $x \ggg 5$), ISS boundedness radius is of no practical use; it is of the order $BDa\, e^{\gamma}$. Thus, for the theory to apply, we require either moderate B, $B \lesssim 3$, or small activation energy $\gamma \approx 1$. In the latter case, the saturation of the exponential term is achieved at a reasonable l^3_∞ radius. We graph in Figure 5.2 the region in which Inequality 5.1 is satisfied for $\gamma = 5.0$, $B = 3.0$. In lieu of discussing any further particulars of this system, we move onto a system with a more interesting graph topology.

5.3 Continuous Crystallizer

The following model, a continuous crystallizer moments model [36], is one of several that have been proposed to study the dynamic behavior of continuous crystallizers, [52], [51], [50]. It has been well documented that such systems have interesting dynamic behavior as exhibited in industrial crystallizers, laboratory crystallizers, and in computer simulation/theoretical analysis. We do not delve into the details of deriving the model(s), nor do lend more than short shrift to the local phenomena presented by these models; we simply apply our analysis technique to the model given below. With our technique, we calculate a semilocal attractor whose radius depends sensitively on the Damköhler number, Da. Functionally this attractor serves as a physical operating regime. Now, the problem of finding a Liapunov function for such (crystallizer) models has been open for some time, [52]. Our result can be viewed as a positive result in this direction, and, for this model, we explain why too much more may not be asked for.

Our system is, then

$$
\begin{aligned}
\dot{x}_0 &= -x_0 + (1 - x_3) Da\, e^{-F/y^2} \\
\dot{x}_1 &= -x_1 + y x_0 \\
\dot{x}_2 &= -x_2 + y x_1 \\
\dot{x}_3 &= -x_3 + y x_2 \\
\dot{y} &= \frac{1 - y - (\alpha - y) y x_2}{1 - x_3}
\end{aligned}
$$

The system graph is shown in Figure 5.4. It contains 8 loops, as found by running the Haskell program alluded to in Section 4.1.2. All loops save one run through the vertex corresponding to y, the dimensionless concentration;

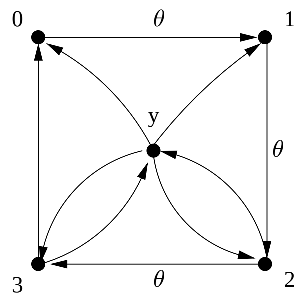

Figure 5.4: Crystallizer System Graph

this other loop is the outer loop as depicted in the figure. Now, applying the variation of parameters formula to the first four equations, and assuming a uniform bound on y (to be justified presently), we may calculate gain functions relating the various x_i's, exactly as is done at the end of Section 4.1 for linear systems. Also, if x_3 can be shown to be uniformly less than or equal to 1, and y to be uniformly positive, by evaluating the vector field at its boundary the positive cone, $\mathbb{R}^5_{\geq 0}$, of $[(\mathbf{x}(t), y(t))] = \mathbb{R}^5$ is easily checked to be invariant; we shall thus limit our considerations to this truncated state space.[1]

We thus must consider y's vector field. This is shown in Figure 5.5. As we always have that the dimensionless change in volume, α, is between 10 and 100, we conclude that uniformly in (positive) x_2, and $x_3 < 1$, y is bounded by the inequalities $0 < y < 1$. Due to this bound, all loops involving y are contractions for $y \geq 1$, so long as x_2 is positive and $x_3 < 1$.

These last two conditions will be consistently verified by evaluating the contraction condition for the outer loop. Three of the four gain functions on the loop are linear with gain coefficient 1. The last, obtained from the vector

[1]See the last paragraph in this section for further discussion on this point.

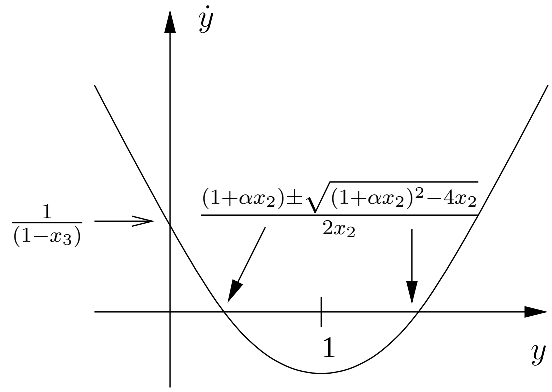

Figure 5.5: y's Vector Field

field for x_0 is a constant with value $Da\, e^{-F}$, which must be less than 1, for the argument to go though. The ranges of these variables are $0.1 \leq Da \leq 10^4$ and $10^{-4} \leq F \leq 10$. Thus we conclude physical operation of the plant, via this theory, only for $Da \leq 1$.

The first reason that a global Liapunov function for this system will never be found is that this moment model, and similarly other crystallizer models, exhibit the spontaneous formation of sustained oscillations in, e.g., x_0, x_1 and y for large $Da\,\alpha$. The mechanism for the appearance of such oscillations is through a Hopf bifurcation, [36]. Second, the sign of y's vector field, and hence the normal, physical value of y, is dependent on x_3 being less than one. If we do not *a priori* and consistently bound the value of x_3 below 1, the complete system's vector field will no longer satisfy any Lipshitz condition. In this case it would be necessary to consider generalized solution of the ODE. Moreover, if once the sign of y's vector field had flipped, it is no longer possible to bound y from below—it is possible that the entire system become instable.

5.4 Distillation Column

We shall study the following simple dynamic model for binary distillation, [6], [35]. The column is conceived to have 30 trays, and constant molar overflow as well as constant relative volatility is assumed. The feed is introduced at tray 17 as a saturated liquid.

condenser

$$A_c \dot{x}_1 = V(y_2 - x_1)$$

rectifier trays

$$A\dot{x}_i = L_r(x_{i-1} - x_i) - V(y_i - y_{i+1})$$

feed tray

$$A\dot{x}_{17} = Fx_{feed} + L_r x_{16} - L_s x_{17} - V(y_{17} - y_{18})$$

stripping trays

$$A\dot{x}_j = L_s(x_{j-1} - x_j) - V(y_j - y_{j+1})$$

reboiler

$$A_r \dot{x}_{32} = L_s x_{31} - (F - V + L_1)x_{32} - V y_{32}$$

where

$$L_s = F + L_r$$

$$V = D + L_r$$

$$RR = L_r/D$$

$$\frac{y.}{x.} = \alpha \frac{(1 - y.)}{(1 - x.)}$$

Using the last relationship, that expressing the relative volatility of the two components, we eliminate the vapor mole fractions from the model to obtain a standard set of ODEs.

By restricting attention to the vector field on each appropriate hyperplane, it is easy to see that the positive cone, $\mathbb{R}^{32}_{\geq 0}$ is invariant under the flow. We will take it to be our state space. We will show attraction to the unit ball, intersected with it, for all $G = (L_r + D)/L_r = 1 + 1/R \geq 1$. For larger G we will be able to show ISS to the origin. The system graph is given in Figure 5.6, and for the purposes of this exposition the (only) disturbance will be taken to be the feed input Fx_{feed}.

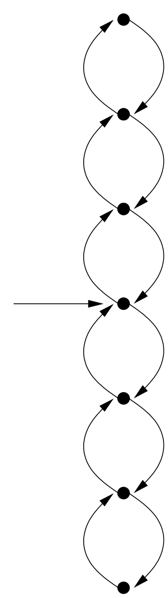

Figure 5.6: Distillation Column System Graph

Excepting the feed tray, the gain functions are obtained (by requiring the decrease of any $x.$ along the vector field component determining it) from $L_{r,s}x + Vy \geq 2L_{r,s}x_- \vee 2Vy_+$. The relative volatility expression being used to eliminate y, the left hand side is easily seen to be monotonic. Inverting the function, we obtain a complicated expression for the gain.[2] Given the gain functions and the graph topology (only simple, length 2 loops), we are ready to apply the small gain theorem. We plot, for chosen values $\alpha = 1.2,\ G = 1.5,\ F/L_r = 0.5$, the *loop* gain functions in Figure 5.4. Here the dashed line is the diagonal $y = x$. In plotting only 1/2 of the relevant loop gain functions, we have made use of the characterization of gain function behavior developed in Section 4.3.2; evaluating the gain functions along a loop in one particular order is sufficient—if that order shows the contraction property the other necessarily does as well.

For higher G, the loop gain functions become ever stricter contractions. Thus, we may conclude physical operation of the distillation plant; i.e. $0 \leq x \leq 1$ for all stages. Moreover, for reasonable reflux ratios, $G \gtrsim 1.4$, we may put a lower bound on the performance of the column: for small Fx_{feed} the system necessarily goes to a conversion as least as low as that indicated from the disturbance gain function—cf. Footnote 2. Reflux ratios of 3 and lower are quite commonly found.

Now, exploiting the only symmetry we have in this system, this same procedure may be repeated for the other component balance—it is easy to see that one must only send α to $1/\alpha$. The loop gain functions for this model, $\alpha = 1.2, G = 1.5, F/L_r = 0.5$, are shown in Figure 5.4.

Combining the information from both disturbance gain functions and all $\mathcal{KL}$ functions (the "β decay rates") one obtains a dynamical picture of

[2] In the rectification section we have

$$x \leq \frac{-(1+\alpha G - (\alpha-1)g) + \sqrt{(1+\alpha G - (\alpha-1)g)^2 + 4g(\alpha-1)}}{2(\alpha-1)}$$

where $g = 2x_- \vee 2Gy_+$ or on the feed tray $g = 3x_- \vee 3Gry_+ \vee 3Fx_{feed}/L_s$, $r = L_s/L_r < 1$. An analogue holds for the stripping section.

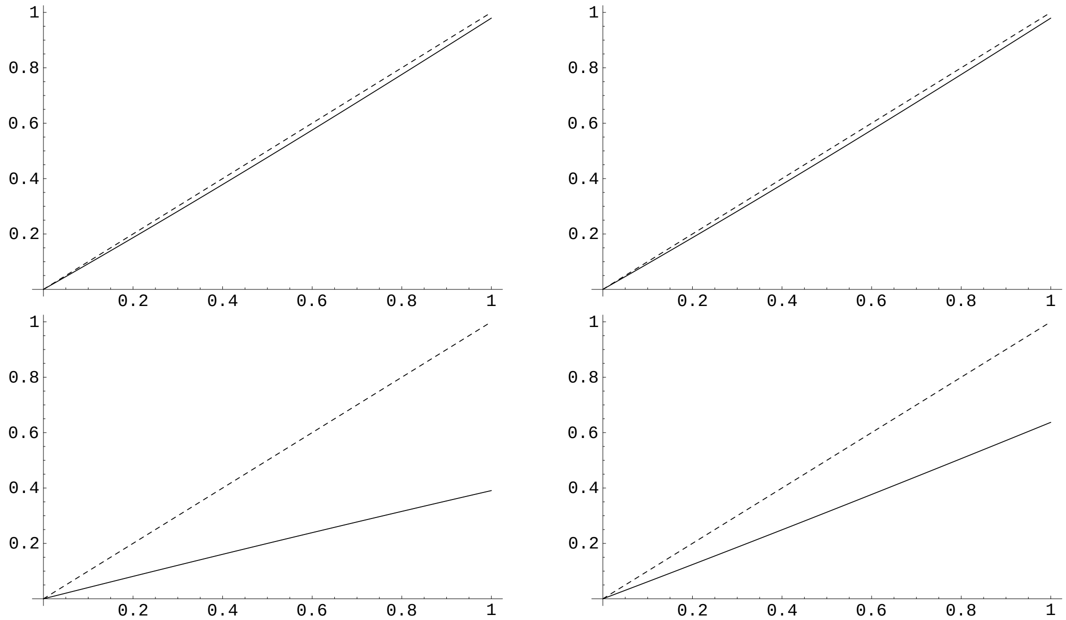

Figure 5.7: Rectifier, Rectifier-Feed, Stripper, Stripper-Feed Loops, Volatile Component

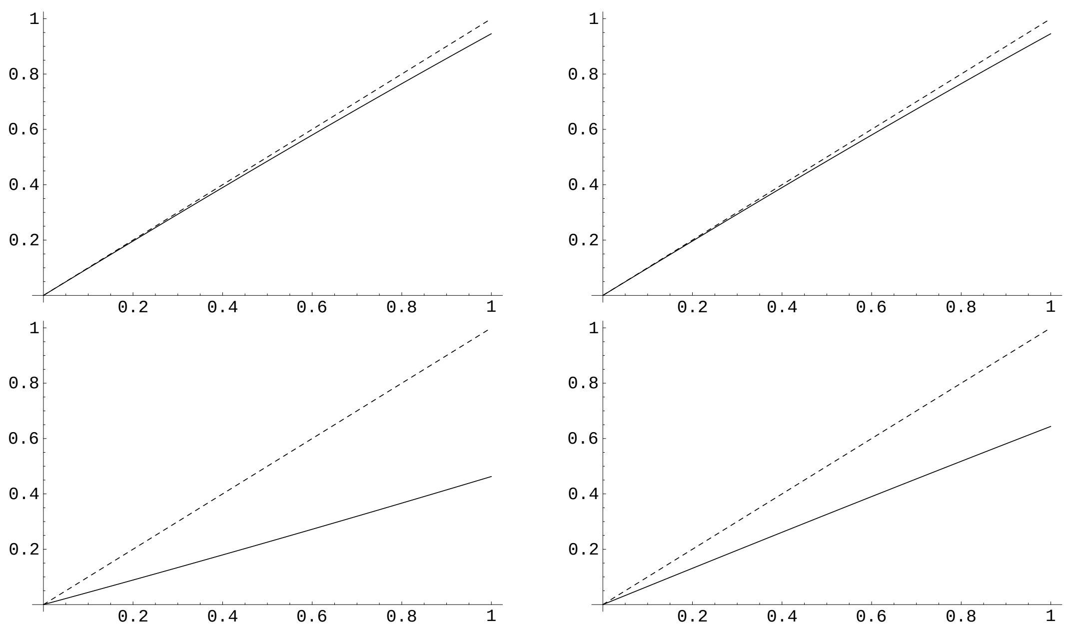

Figure 5.8: Rectifier, Rectifier-Feed, Stripper, Stripper-Feed Loops, Bottoms Component

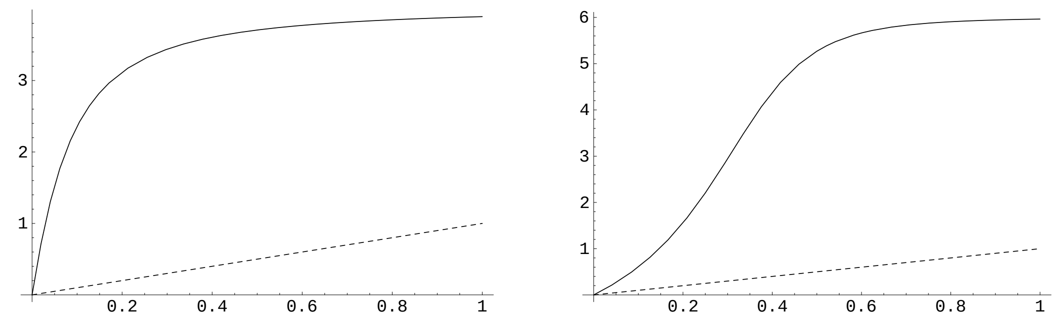

Figure 5.9: Disturbance-to-Condenser Gain Functions; 30 Trays; Volatile, then Involatile Component

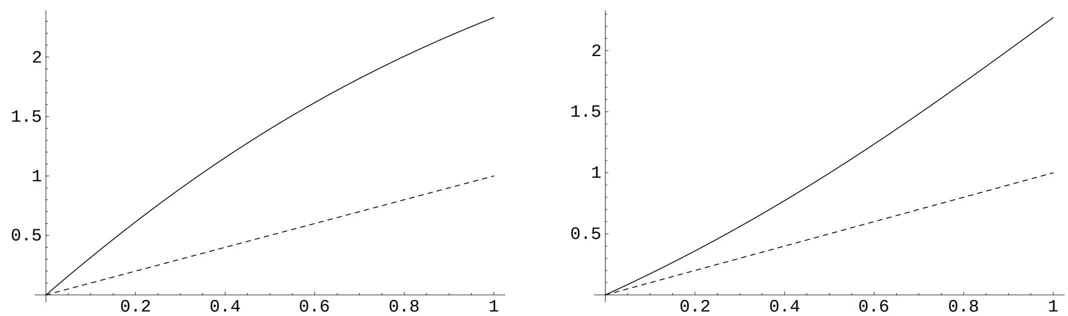

Figure 5.10: Disturbance-to-Condenser Gain Functions; 11 Trays; Volatile, then Involatile Component

the behavior of the system, though not so precise a picture. This is basically a one-dimensional description of the dynamics; its settling to local behavior. Without giving the precise functional description (as this is unilluminating), we elaborate. The disturbance gain functions from the *feed* to the condenser mole fractions are given in Figures 5.9 and 5.10.

The first being for the entire 30 tray model, the latter for a 11 tray model. All other disturbance gain functions are either contractive in nature or grow less slowly than these—it is thus these that determine the l_∞^n-ball's radius, this ball being the compact attractor for the system. Given nominal operating values $F/L_r = 0.5$, $x_{feed} = 0.5$, so that d, the disturbance input for the volatile component's mole fraction, respectively the the involatile component's fraction, is 0.25, then again 0.25, we see that for the 30 tray model our method reveals no useful information. For the 11 tray model, however, we find that $x_1 \leq 0.75$, then $(1 - x_1) \leq 0.4 \Leftrightarrow x_1 \geq 0.6$. A pictorial representation of this conclusion is given in Figure 5.11. All this we have concluded without

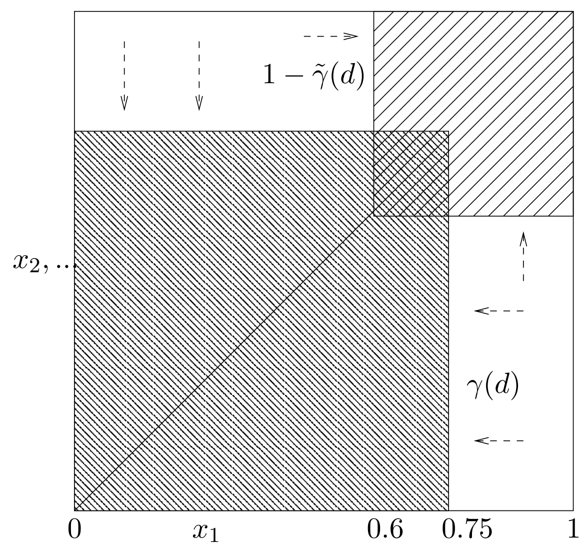

Figure 5.11: Determination of the operating region

resorting to an iterative calculation of a steady state, and we also know that the system is attracted to this region at a rate easily determined from $\mathcal{KL}$ functions; we forego giving the actual expression for these functions.

It should be noted that, due to the *bipartite* (= 2-partitionable, the meaning of which will be intuitively clear below) nature of this model's system graph (partition the vertices into even and odd tray numbers), Teel's nonlinear small gain theorem may also be applicable. However, one must then calculate the interaction gains as depicted in Figure 2.3. This may, of course, be done:

one takes the (point-wise) maximum of all gain functions out of one partition to determine one of the gain functions, then repeats this procedure with respect to the other partition to find the other gain function. This is of course also an idempotent analytic calculation. One may then check Teel's classical conditions. One sees immediately that this method for bipartite graphs is weaker, in general (and in this distillation column example in particular), than the small gain theory developed in this work. In the latter it is sufficient to check *elementary* loops.

5.5 Summary and Discussion

We recapitulate as to why the small gain theorem worked for the systems above. We do so in order to give a feel for when the theory works, and why it can often fail. First, for small γ, the destabilizing terms in two of the vector fields in the CSTR model exhibited saturation. This allowed for an asymptotic small gain condition to be immediately concluded. Otherwise stated, as long as a gain function for the second vector field was definable, the other gain functions were so strongly contractive that the small gain theory was easily applicable.

The gain functions for the crystallizer were found to be contractive, inside the physical operating region, because y had a quantifiable uniform behavior and the loop not involving y exhibited a cut-off suitable for the conclusion of the small gain condition for small enough Da.

Last, the gain functions for the distillation column were seen to be contractions for the very physical reason that in the inequality required to give decrease along any x_i, the vapor and liquid *decrescant* contributions on

System	Ranges of Validity
CSTR	$B \lesssim 3$ or $\gamma \lesssim 2$
Crystallizer	$Da \lesssim 1.0$
Dist. Column	$G \gtrsim 1.4$ and #trays $\lesssim 15$

Table 5.1: Applicability of the Small Gain Theorem

the stage were slightly greater than those *increscant* contributions coming from the other stages. These 'mass balance' facts allow for an (asymptotic) small-gain separation of the stages' vector fields.

We have seen that, in these practical examples, this theory works only over certain parameter ranges, though, nonetheless, over useful parameter ranges. This is perhaps inevitable; the theory is, at its root, 'small gain' in nature, and there will then be preordained limits wherein that condition is satisfied. This phenomenon might be likened to (somehow) determining a neighborhood around an equilibrium point wherein the linearization is really a good approximation. In some absolute scale this neighborhood will grow or shrink dependant on the parameters of the system. Further, it is one of the beauties of the qualitative theory of dynamical systems that parameter variations *produce truly radical changes in system performance.* We have, in this work, characterized (quantitatively) a certain behavior of systems; a semi-global elementary attractivity, or invariance, if you will. This behavior can be proven through this theory in particular parameter regimes only.

In Table 5.1 we summarize the regions of applicability of the nonlinear small gain theorem to the systems investigated in this chapter.

It should also be clear that in the preceding examples we have used only the most rough of bounds on variables; bounds whose values are nonetheless critical in concluding the contraction condition. A detailed analysis of such bounds would improve the limits given in Table 5.1. This and the proper "gauging" of the state variables (through the use of a $\mathcal{K}_\infty$ function—to be explained in the next chapter) are crucial to the optimal application of this small gain technique to practical systems.

5.6 Symbols Used

Table 5.2: Nomenclature

$\mathbb{R}^n_{\geq 0}$	the positive cone of the vector space $\mathbb{R}^n$
$\mathcal{K}$	continuous, monotone increasing, 0 at 0 functions $\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$
$\mathcal{K}_\emptyset$	as above, but not necessarily 0 at 0
$\mathcal{K}_\infty$	as $\mathcal{K}$ but also onto
$\mathcal{KL}$	function $\mathbb{R}^2_{\geq 0} \to \mathbb{R}_{\geq 0}$, $\mathcal{K}$ in first coordinate, 0 limit in second
l^m_∞	the Banach Space $(\mathbb{R}^m, \max_{i=1,\dots,m} \lvert \cdot \rvert)$

Table 5.3: Mathematical Symbols

$\leq$	partial order specific to $\mathbb{R}_{\geq 0}$
$\cdot \oplus \cdot,\ \cdot \vee \cdot$	the maximum of a pair
$\ggg$	much greater than
Id	the identity function $x \mapsto x$
$[\cdot]$	space on which a certain variable is defined e.g. signal or state space of a variable
t	time

Table 5.4: Abbreviations

CSTR	continuous stirred tank reactor
ISS	input to state stability

Table 5.5: CSTR Parameter Names

x_1	dimensionless reactant concentration $= (c_f - c)/c_f$
x_2	dimensionless temperature $= (E/RT_f)(T - T_f)/T_f$
x_J	dimensionless jacket temperature (as x_2, but $T = T_J$
Da	Damköhler Number $= k_0\, e^{-\gamma}\tau$
γ	Dimensionless activation energy $= E/RT_f$
τ	residence time in reactor $= V/F\lambda$
B	adiabatic temperature rise $= (E/RT_f)(-\Delta H c_f/\rho C_p T_f)$
β	dimensionless heat transfer coefficient $= hA\tau/V\rho C_p$
c	concentration
k_0	rate constant for 1^{st} order reaction
E	activation energy
R	universal gas constant
T	temperature
V	volume of reactor
λ	coefficient of recirculation
F	volumetric feed rate
ΔH	heat of reaction
ρ	density
C_p	specific heat
hA	heat transfer coefficient
$\cdot_f$	"feed"
$\cdot_J$	"jacket"

Table 5.6: Crystallizer Parameter Names

t	dimensionless time $= t/\tau$
x_0	dimensionless zeroth moment $= 8\pi\sigma^3 m_0$
x_1	dimensionless first moment $= 8\pi\sigma^2 m_1$
x_2	dimensionless second moment $= 4\pi\sigma m_2$
x_3	dimensionless third moment $= 4\pi m_3/3$
y	dimensionless concentration $= (c - c_s)/(c_0 - c_s)$
α	dimensionless volume change $= (\rho - c_s)/(c_0 - c_s)$
F	$= k_3 c_s^2/(c_0 - c_s)^2$
Da	Damköhler Number $= 8\pi\sigma^3 k_2 \tau$
σ	$= k_1 \tau (c_0 - c_s)$
k_1	constant for McCabe's growth law
k_2, k_3	constants for Volmer's nucleation law
ρ	density
c	concentration
$\cdot_s$	"saturation"
$\cdot_0$	"feed"

Table 5.7: Distillation Column Parameter Names

A	total molar holdup on each tray
F	feed flowrate
D	distillate flowrate
L_r	flowrate of the liquid in the rectification section
L_s	flowrate of the liquid in the stripping section
V	vapor flowrate in the column
RR	reflux ratio
$x.$	liquid composition of volatile component
x_{feed}	feed composition of volatile component
$y.$	vapor composition of volatile component
α	relative volatility
$\cdot_c$	condenser
$\cdot_r$	reboiler
$\cdot_i$	"i^{th} stage"

Chapter 6

Scope of the Theory

This chapter serves as a place where we collect all connections between this theory and those in the broader domain of systems theory as such. Hence, we give directions into which the theory may be extended: the general gauging of state signals to achieve optimum gain function bounds and controller synthesis. The former is necessary to give more satisfactory results upon application of the system to practical examples; the former demonstrates the interplay between the graph-theoretic and systems-theoretical aspects of the theory.

We also point to other classes of systems to which the theory ought to be fruitfully applied: complexes of discrete dynamical systems, Volterra series models, and distributed parameter models.

Lastly, we indicate how the work presented here is quite unique: We point to previous work on the stability of large-scale dynamical systems, comparing our work to that, and we close with a summary of what has been the content of this thesis.

6.1 Gauge Functions

A gauge function, $\rho \in \mathcal{K}_\infty = \mathrm{Aut}_0^{\oplus}(\mathbb{R}_{\geq 0})$, is any such function used to rescale interaction junctions like in Figure 2.4 ($\begin{smallmatrix}\searrow & \\ & \bullet \\ \nearrow & \end{smallmatrix}$) so that the resulting basic inequality 3.2 is modified. Gauge functions may be employed either to improve the size of robust stability regions, to reduce the radius of a compact attractor, or to calculate the smallest possible disturbance gain.[1]

6.1.1 Linear Systems; Robustness Analysis

We go back to our original example, the first system in Section 4.1.2, to make explicit the former construction just mentioned. There, by idempotent algebraic methods we were able to prove stability of the system for $|a| < 1/4$. Let us now consider linear gauge functions, $\rho(x) = \alpha \cdot x$, $\alpha \in \mathbb{R}_{>0}$. We employ Inequality 2.1; the coefficients g_{12} an g_{21} become $(1+\alpha)\cdot 1$ and $(1+\alpha')\cdot|a|$, respectively. The loop condition now gives (picking $\alpha \,\&\, \alpha'$ small) that the system is ISS for $|a| < 1$! Of course the disturbance gain deteriorates (with power -1) as we send α, $\alpha' \to 0$: it goes as $(1+\alpha^{-1})$. This deterioration, however, is to be expected: the eigenvalue $-1+\sqrt{a}$, used in the matrix exponential (and this in the variation of parameters formula) to deduce an L^∞ disturbance gain show the same qualitative behavior. The deterioration this time is only with power law exponent -1/2, though.

Now that we have this basic example behind us, let us move on to the

[1]These optimization problems have a long and distinguished lineage: ODE stability $\to$ Liapunov's direct method $\to$ La Salle's invariance principle $\to$ Sontag's ISS $\to$ Teel's nonlinear small gain theorem $\to$ the extended nonlinear small gain theorem $\to$ optimization over parameters.

other linear gain example, from which we can derive a basic theorem concerning linear gain complexes. Now, at each interconnection juncture ($\overset{\searrow}{\underset{\nearrow}{\bullet}}$) we may apply Inequality 2.1. This transformation sends every a_{ij} involved in such a juncture to $\frac{(1+\alpha)}{2}a_{ij}$. Thus, the loop gain conditions now read

$$\underbrace{\frac{(1+\epsilon)}{2}}_{a_{14}}\underbrace{\frac{(1+\mu^{-1})}{2}}_{a_{43}}\underbrace{\frac{(1+\alpha^{-1})}{2}}_{a_{21}}a_{14}\,a_{43}\,a_{32}\,a_{21} < 3$$

$$\underbrace{\frac{(1+\alpha)}{2}}_{a_{24}}\underbrace{\frac{(1+\mu^{-1})}{2}}_{a_{43}}a_{24}\,a_{43}\,a_{32} < 6$$

$$\underbrace{\frac{(1+\epsilon)}{2}}_{a_{14}}\underbrace{\frac{(1+\mu)}{2}}_{a_{42}}\underbrace{\frac{(1+\alpha^{-1})}{2}}_{a_{21}}a_{14}\,a_{42}\,a_{21} < 1$$

$$\underbrace{\frac{(1+\mu)}{2}}_{a_{42}}\underbrace{\frac{(1+\alpha)}{2}}_{a_{24}}a_{42}\,a_{24} < 2.$$

There are multiple parameters which we are to optimize: the a_{ij}. One way to resolve this dilemma is to admit scale factors: $a_{ij} =: \nu_{ijkl}a_{kl}$, the a_{ij} before tacitly taken to be positive, so likewise for the $\nu_{....}$. We then have a constrained nonlinear optimization problem; the reader will note that we have left the realm of (formally) linear problems, a hallmark heretofore of this thesis. However, we know we are working with positive polynomial inequalities and equalities (by relaxation—see just below). We could thus submit our problem to a computer algebra package for solution, as a (different) theorem of Tarski on the elimination of quantifiers in formal algebraic systems of inequalities, equalities, and nonequalities proves that such problems are computable (albeit generally **NP**-Hard), [49], [44]. Other, more traditional, optimization techniques allow us to go further, however, and we take this route. So, for this particular example all ν's will be set to 1. Then, taking logarithms and defining (relaxing)

$$\begin{aligned} A &= \log(1+\alpha) \\ A' &= \log(1+\alpha^{-1}) \\ M &= \log(1+\mu) \\ M' &= \log(1+\mu^{-1}) \\ F &= \log a \end{aligned}$$

we arrive at the following optimization problem with a linear objective function, linear inequality constraints, but nonlinear equality constraints (ϵ is, of course, set to 0):[2]

$$\begin{cases} \text{maximize } F = \pi_1 X \text{ subject to} \\ 4F + M' + A' \leq \log 24 \\ 3F + M' + A \leq \log 24 \\ 3F + M + A' \leq \log 8 \\ 2F + M + A \leq \log 8 \text{ and} \\ (e^A - 1)(e^{A'} - 1) = 1 \quad (e^M - 1)(e^{M'} - 1) = 1 \\ \text{on } [X] = [F, A, A', M, M'] = \mathbb{R} \times \mathbb{R}^4_{\geq 0}. \end{cases} \tag{6.1}$$

(Different ν's shift the values on the right hand sides of the inequalities.) Picking F small enough and $A = A' = M = M' = 0$, we see that there exists a feasible point. Working only with vectors larger than (partial order!) this feasible point compactifies the problem, and since it is hard to find a function smoother than π_1, we know there exists a solution.

To find the solution without recourse to the theory of positive polynomials, we may either employ exact penalization or the Karush-Kuhn-Tucker

[2] π_1 is the projection to the first factor.

conditions. Letting $\tilde{A} = [(e^A - 1)(e^{A'} - 1) = 1]$, etc., and $[\leq]$ be the set on which the linear inequalities are satisfied, we are ready to employ the former; we need only note that π_1 has Lipshitz constant 1, and so, [16], the function $\pi_1 + (1 + \epsilon')d_{\tilde{A}\cap\tilde{M}\cap[\leq]}$ has the same maxima as that in Problem 6.1, for all $\epsilon' > 0$. A gradient-based technique should be applicable to this problem.

The Karush-Kuhn-Tucker conditions also apply; as the inequality constraints at any optimization point form a linearly independent system of equations, the linear independence constraint qualification holds, [45]. The Karush-Kuhn-Tucker conditions then read, [8], that at any local maximizer, $\hat{X}$, of Problem 6.1 there exist $\zeta \in \mathbb{R}^4_{\geq 0}$, $(\xi_A, \xi_M) \in \mathbb{R}^2$ such that

$$\hat{F} + \sum \zeta_\nu p_\nu = 0$$
$$\sum\nolimits_\nu \zeta_\nu \delta_{\nu A} + \xi_A(\partial h_A/\partial A)(\hat{X}) = 0$$
$$\sum\nolimits_\nu \zeta_\nu \delta_{\nu A'} + \xi_A(\partial h_A/\partial A')(\hat{X}) = 0$$
$$\sum\nolimits_\nu \zeta_\nu \delta_{\nu M} + \xi_M(\partial h_M/\partial M)(\hat{X}) = 0$$
$$\sum\nolimits_\nu \zeta_\nu \delta_{\nu M'} + \xi_M(\partial h_M/\partial M')(\hat{X}) = 0$$

$h_A = (e^A - 1)(e^{A'} - 1) - 1 = 0$ are the equality constraints, p_ν is 4, 3, 3, or 2, $\delta_{\nu A}$ is 1 if A appears in the ν^{th} inequality and 0 otherwise. ζ_ν is nonzero if and only if that constraint is active at $\hat{X}$.

Working out the problem in some more detail, we find that there are no local optima whereby no or one constraint is active. For two active constraints, all pairs yield contradictions upon application of the Karush-Kuhn-Tucker conditions. All three constraint cases must be found by numerically solving transcendental equations. These local-cum-global maxima all yield $a \lesssim 1.47$ as the robust stability criterion. The four constraint case, after linearization and counting of independent directions, is easily seen generically not to occur. However, the optimization problem naturally seeks out this "corner," and, to numerical accuracy, we could not resolve it. Lastly, the criterion gives a more lenient bound than the Geršgorin sufficient condition given in Section 4.1.2.

The general theorem may be deduced from the above example. We need only a generalization of Inequality 2.1. The junction to be considered is shown

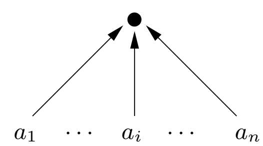

Figure 6.1: n Input Junction

in Figure 6.1.

For n incomming signals we are allowed $\binom{n}{2}$ gauging functions. We construct the generalization of Inequality 2.1 as follows. For each a_i designate one vertex of K_n, the complete undirected graph on n vertices. Now arbitrarily assign a direction and a gauge to each of the $N = \binom{n}{2}$ edges of K_n. $a_i \xrightarrow{\rho_{ji}} a_j$ then designates $\rho_{ji}(a_i) \lesseqgtr a_j$ (it is best to think of the subscripts ij of α as unordered). Each $\lesseqgtr$ is to be given one of the binary designations $\leq$ or $\geq$. This results in 2^N lists; each list containing N such relations. We now impose on every undirected loop of this graph the constraint that the composition of the gauges around that loop be the identity. This reduces the amount of information in the graph to the point where we may deduce our generalized gauging inequality,

$$a_1 + \cdots + a_n \leq (Id + \alpha_{21}^{\pm 1} + \cdots + \alpha_{n1}^{\pm 1})a_1 \vee (\alpha_{21}^{\mp 1} + 1 + \cdots)a_2 \vee \cdots$$

The parity of the exponents being determined by the arbitrary direction chosen for each edge. The way to prove this fact is by means of a finite induction, building up thereby, for each list separately, a graph of the relations imposed by the list. As each list has $\binom{n}{2}$ relations and n a_i's, this is a "total order;" every pair of elements is compared. By correctly posing the finite induction construction, one shows that there is always a maximal element, which is bare (that is, without a function operating on this signal). This bareness is achieved through possible swapping: $a_i \leq \rho_{ij}(a_j) \leftrightarrow \rho_{ij}^{-1}(a_i) \leq a_j$. If there are loops, because of the loop equals identity constraint, it does not matter which element is chosen to be the bare maximum. Lastly, using this graph of relations, we read off the inequality relations between each of the a_i; all relations reduce to a single $\rho_{ij}^{\pm 1}$, since by the complete connectivity of K_n and the loop equals identity constraint any sequence $\rho_{i\cdot}^{\pm 1} \circ \cdots \circ \rho_{\cdot l}^{\pm 1}$ may be reduced to a single $\rho_{il}^{\mp 1}$.[3] We may then deduce the inequality given above as a result of collecting the information in the 2^N lists, since, again because of the loop equals identity condition, the list of lists carries much redundant information; the inequality is the content of the remainder.

For the linear case, through the proper use of the tensors $\nu_{\cdots\cdot}$ and relaxation techniques, we arrive at a nonlinear programming problem much as before:

$$\begin{cases} \text{minimize } F = \pi_1 X \text{ subject to} \\ \textsc{Lin\_Ineq}(X) \text{ and} \\ \textsc{Transcend},\textsc{Poly\_Eq}(X, \alpha) \\ \text{on } [X] = [F, A, A', M, M', \alpha_{\cdot\cdot}, \ldots] = \mathbb{R} \times \mathbb{R}_{\geq 0}^{n+\binom{n}{2}+\cdots}. \end{cases} \tag{6.2}$$

[3]It is not necessary to impose the loop equals identity condition, but then we cannot reduce the programming problem to one analogous to that given above.

Now, there are also polynomial relations that cannot be easily eliminated. These come from the relaxations $e^A = (1 + \alpha_{21}^{\pm 1} + \cdots + \alpha_{n1}^{\pm 1}), \ldots$. The (bilinear) polynomial relations between these terms being difficult to eliminate algebraically. Using the fact that for a strongly connected graph each junction as in Figure 6.1 forces the existence of at least n cycles we come to a variable/inequality ratio less than 1. So, much like before, we will have enough inequalities to compactify our optimization domain (once the asymptotics connecting the α's and the A's is understood—but this is elementary); we need only find a feasible point. The existence of a feasible point is equivalent to there being some combination of coefficients such that ISS may be concluded. Thus, the general situation is analogous to that seen in the example, the applicability of the Karush-Kuhn-Tucker conditions, or exact penalization, being formally no more difficult.

6.1.2 Nonlinear Systems; Radius of Attractor Minimization

In the linear system case it is basic that nothing is gained (or little gain is lost...) by considering nonlinear gauges. The general, nonlinear systems case (again, perhaps restricting attention to linear gauges), is a difficult problem worthy of investigation. As an exposition, we show the how the theory may be developed for system complexes with power law gain functions. It is quickly recognized that to stay in the power law category, one must limit one's attention to linear gauge functions; else the gauging inequalities (Inequality 2.1, or that deduced above) net gain functions not of power law type. We also limit the discussion to junctions with two or less input signals; the more general case being analogous. Once again, then, the gain function before represented by $s \cdot x^a$ is transformed into $\frac{(1+r^{\pm 1})}{2} s \cdot x^a$. Now, as in Proposition 4.2.1, successive compositions along a given loop yield gains of the form:

$$k \cdot \lambda^{\sum l^i} \cdot x^{l^n}$$

Here, we ignore k and focus on the "topological" numbers λ and l; λ giving a fair averaged value of the k's along a loop anyhow.[4] Then, the condition for the fixed point which determines the attractor's radius is $\overline{x} = \lambda \overline{x}^l \Leftrightarrow \overline{x} = 1/(\lambda^{1/(l-1)})$. Our mathematical programming problem becomes

$$\begin{cases} \min\limits_{r_{\nu j}} \max\limits_{\nu \in \Lambda} \left(\frac{1}{\lambda_\nu}\right)^{1/(l-1)} \text{ where} \\ \lambda_1 = \frac{(1+r_{11})}{2} s_{11} \cdots \frac{(1+r_{1m_1})}{2} s_{1m_1}, \ldots. \end{cases} \tag{6.3}$$

Taking logs, collecting constants $e^{S_\nu} = s_{\nu 1} \cdots s_{\nu m_\nu}/2^{m_\nu}, \ldots$, and relaxing $e^{A_{\nu i}} = (1 + r_{\nu i})$ we come, finally, to the following programming problem

$$\begin{cases} \min \max\limits_{\nu \in \Lambda} \frac{-1}{l_\nu - 1} \big(S_\nu + A_{\nu 1} + \cdots + A_{\nu m_\nu}\big) \text{ subject to} \\ (e^{A_{\nu i}} - 1)(e^{A_{\nu' i'}} - 1) = 1, \ldots \\ \text{on } [A_{\nu j}] = \mathbb{R}_{\geq 0}^{\#\text{junctions}}. \end{cases} \tag{6.4}$$

Here we have allowed considerable redundancy in the labelling of the $r_{\nu j}$ and $A_{\nu j}$.

Existence of solutions follows from growth due to all slopes in the objective function being positive; the problem is posed on the positive cone $\mathbb{R}_{\geq 0}^{\#\text{junctions}}$. This problem may also be solved using exact penalization, a

[4] Or, we may view this restriction as a specialization to the case where we only work with the graph adjacency matrices $\tilde{\Gamma}$.

gradiant-based algorithm being appropriate: the Lipshitz constant of the objective function is easily found from $\bigvee_{\nu \in \Lambda} \left(\frac{-1}{l_\nu - 1}\right)$. A nonsmooth Lagrange multiplier rule may also be applied, we do not go into details.[5]

If we are trying to maximize a compact domain wherein the loop gain functions are contractions, i.e. when the l_ν's are greater than 1, the min-max problem changes to a max-min problem.

6.1.3 Gauging for Disturbance Attenuation

Another application of gauging, which is a sort of dual problem, and which we mention only in passing, would be to minimize the calculated disturbance-to-state gain, while all system parameters are held fixed. Assuming one disturbance input (the general case is just as easy), the problem is to minimize the gains in $\oplus_i x_i \leq (\gamma_1 \oplus \cdots \oplus \gamma_n)d$. This problem, for the linear gain case, is a sort of combination of the previous two problems: we have this minimax problem subject to the contraction constraints. Relaxing and taking logarithms, the problem may be analyzed much as before. The power law gain case is however more difficult, mainly because we are always working with practical stability and large gauging variances may result in a huge radius for the attractor. The junction at x_2 in Figure 4.3 is an example of a place at which the problem becomes explicit. The example associated with that graph is *ill-posed as regards*

[5] but the problem is solved by combining the results of Excersize 3.1.8 (h is smooth so it may be pulled out of the $\partial_L\{\cdots\}$) and Problem 1.11.14 (the functions inside the 'max' are linear, so the chain rule may be used without ado—otherwise a more extensive formula is given in Problem 1.11.17, and $\partial_L\{\max\}$ coincides with $\partial_P\{\max\}$, by geometric considerations) of [16]; the basic problem is considered in depth, using value function techniques, in Chapter 3 of this book

this optimization procedure.[6] Now the practical stability radius should be simultaneously tracked with the disturbance gain; once again we see a problem which is analogous—as an amalgam—to the other problems. There will, however, be special graphs (with enough disturbance signals) in which this latter problem is mitigated; we will again come to a minimax problem with equality constraints once the optimization domain is relaxed.

Finally, we note that gauge functions have nothing to do with coordinate changes of the state space. The relationship between state space coordinate changes and ISS for system complexes is another matter entirely.

6.2 Controller Design

We have mentioned that the small gain theory presented above allows control variables to be present, albeit in a passive fashion. But passive here may have two meanings. First, we may ignore their presence altogether (setting them at some nominal value), only to later come back and examine their affect on the system (within the framework of the theory). Second, we may follow [39] and redefine the ISS concept *assuming, and working implicitly the whole time* with a controller stabilized plant. It is the first viewpoint we take here; we do not elaborate on the second, which is essentially a partial ISS-functionalization of the control technique of proximal aiming. Proximal aiming, from a theoretical perspective, is now a mature concept and is closely related to optimal control,

[6] The ISS gain for the system is

$$\tau(x_1 \oplus x_2 \oplus x_3) \leq (1+\alpha)\delta_{11}(d) \oplus (1+\alpha)\Big(\frac{x_3}{\sqrt{\theta}}\Big)^{2/3} \oplus (1+\alpha^{-1})x_3^{1/4}.$$

MPC, and Hamilton-Jacobi-Bellman "value function" techniques; see [16], [13], [15], [14].

In order to carry our program through, we would need to be able to perform the operations

$$\text{system complex} \quad \rightarrow \quad \text{ISS gains on graph} \quad \rightarrow \quad \begin{cases} \text{disturbance gain} \\ \quad \text{attenuation} \\ \text{robust stability} \\ \text{radius of attractor} \end{cases}$$

automatically. The second operation has been the focus of the previous section. For the first, we need a formalized method (concerning a useful class of vector fields) for algebraically generating ISS gain functions. For linear systems, with output feedback, we sketch the program here. For the system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$
$$\mathbf{y} = \mathbf{C}\mathbf{x}$$

we

1. Express all u_i as $L_i\mathbf{C}\mathbf{x}$ (L_i indeterminant covectors)

2. Find L_i values and a coordinate transformation such that each x_i is stable when not perturbed by the other x_i[7]

3. Choose desired optimization problem, or an appropriate weighting of them

4. Pose and solve the optimization problem

[7]If the system is controllable and $C = I$ this can always be done. However, for which systems can this be done without the coordinate transformation? This is a question concerning setting all diagonals simultaneously negative.

The optimization problem thus posed will be nonlinear with inequality constraints. Now, as absolute values must be taken into account, one must be careful of sign changes. One must also maintain Condition 2 at all times. The problem may be relaxed to one with only polynomial constraints. It would then be interesting to have a computer algebra package symbolically solve an elementary control synthesis problem of this type. If logarithms are again used, the optimization problem maybe be formally simplified; we do not delve into the well-posedness of such a scheme.

6.3 Volterra Integral Equations, Discrete Time Systems, and PDE

In the next two sections, we connect up with that vast body of literature already present on nonlinear systems theory; in particular we touch on Volterra series models, PDE describing chemical engineering systems, discrete time systems, and work done on so-called large scale systems. The purpose of these sections is only to give a cursory view of these realms; the discussions are brief and unsatisfactory from a research point of view. That is, they present arenas into which the method here presented should be extended.

Nonlinear discrete dynamical systems have traditionally been studied in Wiener or Hammerstein form. The theory presented here may be useful as a guide in which to study more complicated discrete time models, in which the nonlinearity appears in specified sectors—like a neural-net type model. The formalism given here might be used as a superstructure for such investigations; all the while one would be able to carry along ISS information as one modified a particular empirical system complex.

The present interest in Wiener and Hammerstein discrete-time models is plain, as many practical, nonlinear processes may be modelled via one of these two classes; see [46] and references therein. The extension of ISS to discrete time systems is nearly immediate, cf. [37].

Of course, the simplest systems to apply a theory concerning the idempotent monoid $(\mathbb{R}_{\geq 0}, 0, \max)$ are those built from exactly its semiring of endomorphisms (and 'small' perturbations thereof). We have seen, in Section 2.2.1, that some processes, especially those coming from operations research, do admit such models. More generally, we may take as our (discrete time) plant:

$$x_{k+1} = f(x_k) \vee g(z_k) \tag{6.5}$$

The gain functions for the system being then any continuous upper envelope of f, g; the idempotent small gain analysis is then straightforward. However, two questions arise: what sort of perturbations are admissible, such that the theory is still applicable, and what (complex) systems, if any, admit models of this type?

Also, little has been said of asymptotic sup seminorms. Now, there is a (formally) well-developed theory of all generic asymptotic behavior for discrete time systems. This theory is based on the study of a *universal minimal discrete time dynamical system*, which may be taken as any pair (properly interpreted) $(L,\ 1+\cdot)$ where L is a minimal left ideal of $\beta\mathbb{N}$. Here $\beta\mathbb{N}$ is the Stone-Čech compactification of $\mathbb{N}$, and as such is also a (semi-) topological semigroup, [32]. $1+\cdot$ is the left semigroup action of $\mathbb{N}$ on $\beta\mathbb{N}$ "step forward by one unit of time." Any minimal left ideal of $\beta\mathbb{N}$ is uniquely determined by an idempotent $e \in \beta\mathbb{N}\backslash\mathbb{N}$ ($L = \beta\mathbb{N}e$), and so to give such a universal minimal dynamical system it is enough to choose such an e.[8] For systems such as those given by Equation 6.5, one may be able to profit from the study of the ISS small gain theorem (or the ISS concept in general) in light of this universal dynamical system and its relationship to the asymptotic sup seminorm.

Nonlinear equations of Volterra type, i.e. systems characterized by integral equations with memory that behave in a particular nonlinear fashion, seem also well suited to ISS analysis. Sandberg and others have detailed the application of integral equations of Volterra type to systems theory, [20]. In particular [56] uses (now) standard arguments in the theory of Volterra integral equations to establish a type of L^2 stability of a class of nonlinear systems. See also [19] and [21] for general information on nonlinear functional equations, and [67] for the adaptation of certain systems and control theoretic concepts, in particular the maximum principle, to functional equations of this class.

Lastly, we mention the possible application of the theory to systems described by partial differential equations. This extension would be particularly relevant to chemical engineering systems, as most models for such systems are naturally partial differential equations, [7]. The method in which the present theory would be applied to such distributed parameter systems would be by proving uniform l_∞^n $(n \to \infty)$ estimates for discretized versions of the PDEs. The program would depend on the method of discretization, of course, and it is unclear under which method (if any) and for which classes of PDEs the scheme would function. We mention that finding L^∞ bounds for the solutions of certain PDEs is not common—the usual techniques used in the regularity theory of (nonlinear, elliptic) PDE involve estimates for L^p, $1 < p < \infty$, [27], [12]. However, physical flows derived from PDEs, such as diffusions, say,

[8] $\beta\mathbb{N}\backslash\mathbb{N}$ is the space of all linear functionals in $(l_\infty)^*$ without support on $\mathbb{N}$, i.e. $(l_\infty/c_0)^*$.

when investigated analytically and numerically are often well controlled (uniformly over time) by the L^∞ norm, [22]; it is possible that our small gain technique may function as a new method to prove such bounds. The program may be rigorously carried through using standard discretization-convergence results on Sobolev spaces [9], [23], [12], or other convergence results such as those pertaining to the convergence of viscosity sub- and super-solutions of approximating PDEs to the (unique) viscosity solution of the PDE of interest [24].

6.4 Large Scale Systems Theory

There is an extensive literature on the properties, including stability, of system complexes (often called large scale dynamical systems). Our results, being ISS and idempotent-algebraic in nature, do not conform to previous studies. We refer the reader to [43], [59], [66], the more recent [63], and references therein.

More specifically, the method of Šiljak, in [59], is Liapunov function based, but it also uses a "diagonal dominant" matrix theory. The method requires that the perturbing off-diagonal vector fields are dominated (in the diagonal dominant matrix-theoretic sense) by $\mathcal{K}$ functions which are beforehand Liapunov functions for the unperturbed diagonal vector fields. Once the diagonal $\mathcal{K}$ functions have been found, these being the only nonlinear "scales" in the theory, the method reduces to one of linear systems analysis with the theory of diagonal dominant matrices standing at the fore. That is, the theory of quasi-diagonally dominant matrices is used to deduce asymptotic stability of the system via a differential inequality (i.e. Gronwall inequalities). It is believed that Šiljak's theory is strictly complementary to the one presented here, though no nonlinear example has as of yet been constructed to prove this last statement (mainly because constructing nontrivial examples for Šiljak's theory is no easy task). The example given in Subsection 4.1.2, and remarked on above, does indicate this trend: Šiljak's theory/the Geršgorin analysis is "additive" insofar as robustness analysis is concerned; the theory presented here is "functional compositive" (="multiplicative" for the linear gain case).

One of the methods of analysis in [43] is very similar to that just described. As above, so-called Minkowski matrices again play a critical role. Such matrices are also used in the analysis of abstract system complexes. There they are used to collect linear gain information of system complexes, and an extended small gain theorem is proved. This small gain theorem is substantially different even from our extended *linear* small gain, as it involves conditions on the principal minors of a matrix coming from an elementary feedback loop. Generally, and, as we have seen in all our nonlinear examples, such linear gain conditions are rarely applicable. Thus, for applicability's sake, the ISS small gain theorems constitute an important advance.

Also, from our perspective, if subplants of a system complex have been previously endowed with ISS gain functions, we may naturally proceed with a small gain analysis. We were not able to exhibit such an undertaking as, for chemical engineering process models, no such gain function are generally available.

6.5 Summary

In this chapter we have presented directions in which future work may be done. We have given a more or less complete picture of the easiest aspects of improving the information output from the extended nonlinear small gain theorem via gauging of the signals at junctions. In particular, we have generalized Inequality 2.1 from [64]. For general gain and gauge function the problem is quite challenging, but deserves to be looked at based upon the results of this chapter. Controller design via the methodology presented here has also been suggested. Last, we have merely listed systems other than ODEs to which the theory may be profitably applied. These classes of systems, Volterra integral equations, discrete time systems, and PDEs, were mentioned due to their importance in the modelling of engineering systems; especially in the modelling of chemical engineering systems.

6.6 Symbols Used

Time is taken as an argument or is superscripted (cf. 3). Indices, such as graph vertex labels or vector components, are subscripted.

Table 6.1: Nomenclature

$\beta\mathbb{N}$	the Stone-Čech compactification of $\mathbb{N}$
$\mathbb{R}^n_{\geq 0}$	the positive cone of the vector space $\mathbb{R}^n$
$\mathcal{K}$	continuous, monotone increasing, 0 at 0 functions $\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$
$\mathcal{K}_\infty$	as $\mathcal{K}$ but also onto
$L^{(\to)\infty}(\mathbb{R}_{\geq 0}; \mathbb{R}^m)$	$\mathbb{R}^m$-valued (eventually) essentially bounded functions on $\mathbb{R}_{\geq 0}$
l_∞	the Banach space $(\{x^\cdot : \mathbb{N} \to \mathbb{R} : \sup_i \lvert x^i \rvert < \infty\},\ \sup_i \lvert x^i \rvert)$

c_0	sequences converging to zero

Table 6.2: Mathematical Symbols

$\leq$	generic partial order
$\cdot \oplus \cdot,\ \cdot \vee \cdot$	the maximum of a pair
π_i	projection to the i^{th} factor
$\tilde{\Gamma}$	adjacency matrix of a weighted digraph
$\lvert\lvert \cdot \rvert\rvert_{(\rightarrow)\infty}$	(asymptotic) sup norm on $L^{(\rightarrow)\infty}(\mathbb{R}_{\geq 0}; \mathbb{R}^m)$
$[\cdot]$	space on which a certain variable is defined
	e.g. signal or state space of a variable
Id	the identity function $x \mapsto x$
γ	typically a $\mathcal{K}$ function
$\rho,\ \alpha,\ \mu,\ \epsilon$	typically a $\mathcal{K}_\infty$ function
$A, A', M, M', \ldots$	slack variables
a	parameter in system's vector field
$x_i,\ a_i$	(state) signals
d	(disturbance) signal
$\oplus_i x_i$	max over components of $\mathbf{x}$

Table 6.3: Abbreviations

ISS	input to state stability
ODE	ordinary differential equation
PDE	partial differential equation

Chapter 7

Conclusions

We have, in the preceding pages, developed an extended nonlinear ISS small gain theorem via an idempotent analytic presentation of the ISS paradigm. This theorem allows one, under hypotheses of a contractive condition, to characterize complex, many state dimension systems according to ISS. Since the analysis is (idempotent) algebraic, one can see simply the dependance of the resulting ISS gain functions on all parameters of the model. Further, the algebraic nature of the theory interacts harmoniously with the graph-theoretic representation of a system complex; for favorable classes of gain functions a complete, graph-theoretic characterization of the theory is thus possible. We have also been able to show that the theory applies, over certain parameter regions at least, to interesting chemical engineering process models.

The material here presented is most certainly of a theoretical nature. As the theory was here exposited for the first time, such a slant in presentation is unavoidable. Much work remains in order to see how far the theory may be applied with success; but its development, though mathematically rigorous and self-contained, was undertaken with application in mind. There are simply too many complex systems which completely elude any sort of nonlocal, but rigorous, stability analysis; it was our intention to “chip away” at this monolithic collection of models. The main limitation of the theory is of course its small gain nature, and we simply cannot relax this constraint. The best we can do is (if possible) find optimal state-variable regauging functions so that the small gain condition is more often satisfied. In the end, it will be this constraint that limits a wider application of the ISS theory presented in these pages.

We summarize the major contributions of this thesis in the following list:

- We proposed a new algebraic formalism for the stability analysis of complex nonlinear systems.
- We characterized the theory mathematically.
- We applied the theory to three chemical engineering systems to deduce new facts about their dynamics.
- We extended the theorys applicability by defining and applying a nonlinear programming scheme based on a graph-theoretic/algebraic analysis of the formalism.

It will be noted that our theory touches on both purely systems theoretic concepts, i.e. input-output behavior characterized by the ISS framework, and on purely topological information related to the qualitative theory of ODEs. We have sketched this correspondence in Section 2.2.3, but we may also access it in a graph theoretic fashion. That is, our ISS small gain theorem on system complex graphs is rather blind to whether there exist inputs. The theory relies on an input-output view of the system to be studied, but, as we have seen in the examples, there may in the end be no exogenous inputs; the system's graph may well be strongly connected. This is another interesting feature of the theory—that it skirts the boundary between the qualitative theory of ODEs and the system-theoretic view as exposited in [64] and [55]. The examples, being of the most preliminary type, were necessarily oriented towards the qualitative theory of ODEs—this direction being easier to develop quickly. They, however, prove that this boundary is nonempty and nontrivial.

Appendix

As per guidelines enforced by the dissertation committee, we present here a road map leading the interested reading into the present thesis.

To begin, one must be able to integrate ordinary differential equations. Abstract results on the feasibility of this task are necessarily to be understood because it is rare that we have concrete formulæ. See [47] and [33]. [16] has a thoroughly modern systems & control-theoretic treatment of the problem, with a view on numerical solution, as well. In the first two books above one will also found a treatment of Liapunov's $\epsilon - \delta$ definition of stability about an equilibrium trajectory $x(\cdot) \equiv x_{\text{eq}}$: $\forall \epsilon > 0 \exists \delta > 0 \forall x_0 : |x(0) - x_{\text{eq}}| < \delta \Rightarrow ||x(\cdot) - x_{\text{eq}}||_\infty < \epsilon$. The reader should understand that this appears to be a type of continuity because it *is* a type of continuity. In [30], one will learn about the stronger notion of *asymptotic stability*, which requires additionally that $\forall \epsilon > 0 \exists T \geq 0 \forall t \geq T : ||x(\cdot) - x_{\text{eq}}||_{\to\infty} < \epsilon$. One has to work harder to make this formally into a notion of continuity. These books contain information about the use of Liapunov's direct and indirect methods for the proof of local and global asymptotic stability; that is, linearization and Liapunov function techniques in the stability analysis of systems. Additionally, one will find in Hahn's book a exposition of LaSalle's invariance principle, something which was functionalized and generalized by E. Sontag's input-to-state stability concept. One can learn about this concept from Khalil's book [40], or Sontag's notes on the matter, [61]. Both give information on computing ISS gain functions for elementary (scalar or linear) systems.

Idempotent analysis, before now having nothing to do with systems analysis *per se*, is nicely described in [41].

Glossary

For purposes of readability the dissertation committee deemed it necessary that a glossary be included in the thesis.

Banach Space A vector space together with a norm; this norm inducing a metric topology on the space; the space being complete in this metric topology. Drop 'completeness' and one has only a pre-Banach space. Pre-M-spaces, a special such pre-Banach space, is a very useful environment in which to analyze signals from a functional point-of-view, as shown in this thesis.

Complex System Intentionally meant to be confused with system complex. A system complex is a well-defined entity, as per this thesis. Complex systems are those systems which exhibit "complex," or difficult-to-understand, behavior.

Crystallizer A unit operation in chemical engineering whose function is to facilitate the nucleation ("dropping out") of solid crystals from a liquor containing the chemical to be nucleated in liquidous (solvated) form.

CSTR A continuous stirred tank reactor is a reactor in which continuous flow through the unit (usually a drum) occurs concomitantly with a reaction to produce some chemical species.

Distillation An operation whereby differing thermodynamic properties (partial molar free energies in liquid and vapor) of two or more (miscible) chemical species are exploited to separate said species and thus 'purify' or 'extract' one species from the solution fed into the unit.

Epistemology The study of knowledge as such.

Gain querey The act of specifying how one state of a system depends in an ISS-defined fashion on inputs input to the system.

Haskell A functional programming language, similar in conceptual design to LISP.

Idempotent A binary operation on a set A, $\cdot \oplus \cdot : A \times A \to A$, is termed idempotent if it and the diagonal map $\Delta : A \hookrightarrow A \times A$ together factor the identity map. That is $Id = \oplus \circ \Delta$. Element-wise: $a \oplus a = a \; \forall a \in A$.

ISS Input-to-state stability is a simultaneous functionalization and generalization of LaSalle's invariance principle. It allows for the system's attraction to a compact attractor (this determined by the magnitude of the disturbance) to be characterized in terms of a pair of intuitively comprehensible functions.

Karush-Kuhn-Tucker conditions (Nonsmooth) differential necessary condition which optima must satisfy.

Liapunov Aleksandr Mikhailovich Liapunov (1857$*$-1918†) did important work on differential equations, potential theory, stability of systems and probability theory. His work, as background, is basic to the understanding this thesis, and the analysis of systems in general. Cf. the Appendix.

Local True in some neighborhood of some point. The 'diameter' of this neighborhood need not be specified in advance.

Moment Model In a distributed parameter—i.e. infinite dimensional—system moments may be taken with respect to the distributed parameter.

These moments normally correspond to definite and even measurable physical quantities and formally obey systems of ordinary differential equations. Under very weak hypotheses on the uniform (over time) boundedness of the moments, cf. [53] esp.§X.1, these moments reconstruct the dynamical distribution from which they came.

Process A mathematical representation or model of some collective of phenomena. There must be some notion of change or dynamics associated with these phenomena; they are connected temporally at least.

Robust Not destroyed when some (particular, well-defined) change in the system occurs.

Semi-global implies that there is some extended or decidable diameter to the property under investigation.

Closed Unit Ball The set of all points in a Banach space no farther than unity from the zero vector.

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This dissertation was typeset with LaTeX 2_ε[1] by the author.

[1]using macros by Dinesh Das, Bert Kay, and James A. Bednar.