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Collapsing Dimensions, Physical Limitation, and other Student Metaphors for Limit Concepts: An Instrumentalist Investigation into Calculus Students' Spontaneous Reasoning

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Collapsing Dimensions, Physical Limitation, and other Student Metaphors for Limit Concepts: An Instrumentalist Investigation into Calculus Students' Spontaneous Reasoning

by

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Dedication

To my wife, Beth, with whom I am fortunate to share my life. We discovered our love in a place of tragedy but redeemed through God's compassion. May that always serve as a model for our passion for one another and, in turn, orient us toward service to others.

To my parents, Robert and Anne Oehrtman, who are my greatest role models and more than any other people, shaped who I am. They always had abundant time for our family and still emerged as leaders and servants in our community.

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To my students, from whom I never cease to learn. They have truly inspired me to pursue a career as a teacher.

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Collapsing Dimensions, Physical Limitation, and other Student Metaphors for Limit Concepts: An Instrumentalist Investigation into Calculus Students' Spontaneous Reasoning

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This research is an investigation of first-year calculus students' spontaneous reasoning about limit concepts. The central theoretical perspective guiding the study design and the data analysis is based on an interaction theory of metaphorical reasoning first proposed by Ivor Richards and later developed by Max Black. From this perspective, strong metaphors are characterized as those that both support high degrees of elaborative implication and are ontologically creative. This study investigates students' spontaneous reasoning about limits that may be described as metaphorical in this sense and are thus implicative for the students' emerging understandings. John Dewey's instrumentalism, a theory of inquiry as the application of cognitive tools against problematic situations, provides a focus on functional as well as structural aspects of students' metaphors.

Descriptive answers to the following questions are sought: 1) What conceptual metaphors do students use to reason about limit concepts, and how are they applied in specific problem contexts? 2) How do these metaphors affect students' interpretations of content presented in class? 3) What are the implications of trying to directly influence students' metaphorical reasoning? The methodology is a micro-ethnographic study of students' problem solving through clinical interviews, writing assignments, and classroom observation.

The main result is the characterization of five metaphor clusters for limits that were used in a variety of problem contexts by several students. These metaphors involved reasoning about limits in terms of a collapse in dimension, approximation, closeness in a spatial domain, a physical limit for which nothing smaller could exist, and the treatment of infinity as a number. Contrary to the implications of much of the informal language associated with limits, students were not observed to use motion imagery in significant ways to reason about limit concepts. While many aspects of the students' metaphors generated mathematically incorrect entailments for limits, most students were still able to use them to great conceptual advantage. Two important factors were whether students critically reflected on their own reasoning to make refinements and whether they attempted to make connections to relevant mathematical structures.

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Chapter 1: Introduction

Limit concepts have proven to be notoriously difficult for students to learn (Cottrill, et. al., 1996; Cornu, 1991; Davis, 1986; Davis & Vinner, 1986; Sierpinska, 1987; Simonsen, 1995; Tucker, 1986; Williams, 1991). Even historically, the development of mathematically rigorous formulations for intuitive arguments such as the Eudoxus' method of exhaustion, Newton's fluents and fluxions, or Leibniz's infinitesimals represents a highly nontrivial intellectual feat (Cornu, 1991; Kaput, 1994). Consequently, as students struggle to understand and use limits concepts presented in introductory calculus courses, their conceptualizations are often highly influenced by informal notions and ontological commitments regarding infinity (Sierpinska, 1987; Tall, 1992; Tirosh, 1991), infinitesimals (Artigue, 1991; Tall, 1990), and the structure of the real numbers (Cornu, 1991; Tall & Schwarzenberger, 1978). Nonmathematical intuitions about things such as speed limits, physical barriers, and motion also play a role in their developing understandings (Thompson, 1994b; Williams, 1989, 1991) as do even their epistemological beliefs about mathematics in general (Sierpinska, 1987; Szydlick, 2000).

Previous research, for the most part, has focused on characterizing students' naïve conceptualizations and conceptual difficulties regarding limit concepts. The research presented in this dissertation is intended to investigate the role that such informal thinking plays in students' current reasoning and interpretation and in their ongoing conceptual development. As described briefly below and developed more fully in the following chapters, the theoretical perspective is based on Max Black's (1962a, 1977) interaction theory of metaphorical reasoning, where students' application of intuitive ideas are seen as their metaphors for limit concepts.

RESEARCH GOALS AND QUESTIONS

The primary goal of this research is to characterize students' spontaneous language and patterns of reasoning about limits as they emerge in the process of learning. Using a theoretical perspective based on conceptual metaphors and metaphorical reasoning, this goal is translated to the following five specific questions:

1. What intuitive concepts do students apply metaphorically to reason about limit concepts?
2. What are the structural elements, logical relations, and entailments of these metaphors?
3. How do students apply these metaphors as conceptual tools to understand new ideas or to work challenging problems?
4. How do students' spontaneous reasoning and metaphors affect their organization and interpretation of concepts presented in the classroom?
5. What are the implications for directly trying to influence students' metaphorical reasoning and use of language around specific intuitive ideas related to limits?

The secondary goal of this research is to contribute to the knowledge of the mathematics education research community on both the investigation of functional aspects of students' knowledge, and the use of metaphor in a theoretical perspective.

The remainder of this chapter develops the meaning of these questions and describes how they are answered in this study. The use of the construct of metaphorical reasoning is motivated by both theoretical considerations and the initial findings of an exploratory study. Aspects of Vygotsky's perspective on instruction and learning provide a general framework for the development of spontaneous and scientific concepts and their mediation by one another and by language and other signs. In addition, features of linguistic cues and conceptual schemas were observed in use by students during an exploratory study. The relation of these features to Vygotsky's perspective is briefly discussed here and developed more fully in Chapter 3.

Scientific and Spontaneous Concepts

Many aspects of this research were motivated by Lev Vygotsky's (1978, 1987) perspective characterizing conceptual development as a complex interplay between intuitive (spontaneous) and formally structured (scientific) thought. These two types of thought are distinguished by their relationship to the objects of reference and by the nature of thought available to them. Spontaneous concepts develop first through a direct encounter with the object, such as physical interaction, and form the basis of experiential knowledge developed informally over long periods of time. They are intuitive in nature and can be applied spontaneously, without conscious reflection on their meaning. On the other hand, they cannot solve abstract problems in non-concrete situations. Scientific concepts emerge later through a mediated relationship to the object, such as a verbal definition. They are expressed and initially applied only in abstract ways affording quick mastery of operations and relationships, but they are disconnected from personal experience or meaning.

Especially within a field as structurally rich as mathematics, scientific concepts are defined by their systematicity. Within a spontaneous concept system where the only relationships possible are relationships between objects (and not between concepts), verbal thinking is governed by the logic of graphic imagery and thus is highly dependent on perception. Corresponding concepts are presyncretic, that is, they are not tied to other concepts in meaningful ways. It is the appearance of higher order concepts that allows this to change; the unification of concepts within a single structure, under a single superordinate concept, allows for the comparison and analysis of subordinate concepts. To recognize contradictions or evaluate one conceptualization against another, the individual must understand two different concepts as relating to the same thing within the single overlying structure. Comprehending the structure of a scientific concept, therefore, requires the learner to develop higher levels of reasoning, to form new categories of relationships, and to generalize.

Thus, the strengths of the scientific concept are the weaknesses of the spontaneous concept, and vice-versa. By means of their complementarities, each one lays the foundation for the development of the other. The development of the scientific concept is mediated by the spontaneous concept as intuitive modes of analysis become available to it. The spontaneous concept is in turn transformed through this mediation much in the same way that one's native language is transformed when it mediates the learning of a foreign language. The structure provided by the scientific concept enables the spontaneous concept to grow and become more available to abstract functioning.

Moving ahead of development, one purpose of instruction is to encourage in the student conscious awareness and volitional use of their spontaneous knowledge. This occurs as thinking within a system that is just beyond the current comprehension of the student but within their ability to imitate is modeled. Vygotsky argues that imitation is not an act of thoughtless mimicry but rather requires a beginning grasp of the structure of the system, noting that animals cannot imitate except through training. As opposed to performing a trained behavior, a student can only spontaneously imitate if the task lies within the zone of his or her own intellectual potential, the so-called "zone of proximal development."

Apart from the particular examples Vygotsky provided (mostly concerning the learning of language, such as the impact of learning a foreign language on one's native language or the development of causal and adversative relations), he was not specific about the mechanisms by which scientific concepts become more fully developed and more fully integrated with the learner's spontaneous concepts. The purpose of this research is to illuminate the nature of the interaction and influence between students' scientific and spontaneous reasoning about limit concepts.

Framing the Research Questions in Terms of Metaphor

This research began with an exploratory study with the general goal of identifying, for further investigation, cognitive mechanisms involved as calculus students refine their understandings of core concepts throughout a year-long course. Two main themes emerged from the data indicating that students used linguistic cues to recall and make decisions about strategies and that they subsequently built elaborate conceptual schemas around intuitive, nonmathematical ideas to reason about the mathematics. The construct of conceptual metaphors used in this study combines these aspects of reasoning. It not only allows for, but focuses on, the generative mathematical activity of students even though they may develop nonstandard interpretations or reason based on typically nonmathematical contexts. The exploratory study and its impact on the theoretical perspective are described in Chapter 3.

Language and other signs, according to Vygotsky, are used as important conceptual tools in reasoning (Cole & Scribner, 1974; Vygotsky, 1978), reorganizing the experiences in which they are used. That is, the cognitive activities engaged with such a tool are qualitatively different than they would be otherwise. The construct of metaphorical reasoning, as developed for this research, characterizes the students' connections between their spontaneous and scientific concepts through linguistic aspects of thought, that is, as revealed in the language, symbols and images, and even entire domains of nonmathematical experience used as signs to point to mathematical ideas that are used by the students.

Conceptual schemas may be constructed from students' existing intuitive spontaneous concepts or from their system of scientific mathematical knowledge, yet, somehow new ways of understanding emerge. The generative dynamic created when these complementary domains are allowed to interact is similar in nature to the creation of new meaning through crossing categories in a metaphorical expression. Specifically, in metaphor, a term or concept (the metaphorical domain) is applied to some other term or concept in a way that does not reflect its literal meaning. For this study, the metaphorical domain will always be intuitively understood concepts. They will be applied to the students' emerging understanding of the mathematics. This application provides a concrete carrier for the abstract structures of the mathematics, while at the same time it systematizes the intuitive concepts being applied. In Vygotsky's terms, this dialectic allows each of the spontaneous and scientific concepts to blaze the path for further development of the other.

ANSWERING THE RESEARCH QUESTIONS

This research seeks descriptive answers to the questions posed above. That is, instead of testing specific hypotheses about students' understanding, basic characterizations of students' metaphors for limit concepts will be developed through data collected from various instruments asking students to explain their understanding of their problem solving activities in detail. These instruments are open-ended in nature to provide students with wide latitude of potential responses, and they prompt for detailed responses to capture fine-grained characterizations of specific students' reasoning in particular situations from rich collections of data. The methodology used to collect and analyze the data is described in Chapter 4.

A Functionalist Perspective

Vygotsky's (1987) characterization of spontaneous and scientific thought includes both the nature of the relevant conceptual structures and the ways in which those concepts are used. He argues that if only the content of thought is considered or only cognitive functions are considered (as in two main competing lines of thought in psychology at the time), then knowledge and thinking are incommensurable and it is impossible to understand the problem of conceptual development. "In contrast," he suggests, "if we attempt to unite the structural and functional aspects in the study of thinking, that is, if we begin with the idea that what functions influences the process of functioning, the problem not only becomes accessible but is solved."

Such functional aspects of knowledge are incorporated into this study in three parallel ways. First, the theory of metaphor that is adopted accounts for meaning in terms of interactions that occur between metaphorical and literal domains (Black, 1962a, 1977). Thus it is in the process of applying a metaphor in a specific situation that its meaning is developed. Black's interaction theory of metaphor is developed against the backdrop of several competing theories in

Chapter 2. Second, we treat metaphors as cognitive tools used in problem solving situations as in John Dewey's characterization of instrumentalism (Hickman, 1990; Prawat & Floden, 1994). Finally, we view even the structure of conscious knowledge as instantiated, emerging from a largely unstructured tacit knowledge base in response to particular situations (Alexander, Schallert, & Hare, 1991). The constructs of instrumentalism and instantiation of explicit knowledge are discussed in Chapter 3 as part of the development of the theoretical perspective for this research.

The Metaphors

There is no reason to expect that different students will construct and use the same metaphors as they reason about limit concepts. Nevertheless, we will look for commonalities as well as idiosyncrasies. Students' metaphors for limit concepts will be characterized both as amalgams of different students' metaphors and as applications of ideas by individual students in specific problem contexts. In addition to respecting both individual and general aspects of students' metaphorical reasoning, this type of characterization also treats as essential the specific ways in which these metaphors are used. That is, the functional as well as the structural aspects of though must both be included. These characterizations, called "metaphor clusters," are presented as the main part of the summary of data in Chapter 5.

Limitations of the Research

This research is designed to obtain and analyze detailed data on students' thought processes regarding limit concepts. As a result there are a number of limitations that must be acknowledged. As with any study based on qualitative data from a small number of students, the results do automatically not generalize to other students and other settings. In addition there are limitations specific to the nature of the particular methods of data collection and analysis used in the study.

The data was collected through a variety of methods such as clinical interviews and short writing assignments, thus placing students in somewhat artificial settings. For example, for most students, the natural process of thinking through a problem does not likely involve explicitly describing their thoughts in the process. The students were also placed in a situation where they may have felt pressure to appear knowledgeable about material over which they had little command. In parts of the study, students were intentionally given problems slightly beyond their capability in order to elicit dynamic problem solving activity. As a result of these conditions, students, though not intending to be disingenuous, may engage in thought processes in different ways than they normally do or respond with ad-hoc conceptualizations to meet the immediate needs of the interview or written questioning. Finally, in any cognitive research, there is always a disconnect between concepts that are actually in the subjects' mind and the observable data. This study does not attempt to control for these factors but acknowledges them forthright as a part of the background.

In addition to methodological issues, there are also several aspects of the data analysis that contribute to the limitations of this research. First, it is necessary in any theory on cognition to idealize or simplify thoughts and thought processes, which pertinent to this study, are not literally metaphors or conceptual tools. Second, in this study, students are not presented with preset categories of possible responses and are, instead, given open-ended tasks to which they may respond in an unlimited number of ways. An attempt is subsequently made to make decisions about analysis guided by the theoretical perspective and recurring themes in the data. Nevertheless, these decisions are subjective, thus any resulting categorization of students' responses is inherently problematic. These considerations point to the need to be critically reflective in this type of research. Since it is impossible to entirely remove oneself from influencing the data being collected or to remove one's personal perspective from the analysis of that data, it is necessary is to acknowledge what one's perspective is, and to reflect on the impact that has on the research.

Chapter 2: Review of the Relevant Literature

This study investigates first-year calculus students' emerging understandings of limit concepts through a theoretical lens of metaphor. Significant research has been conducted in each of the areas of limits and metaphor, with limits naturally falling under the domain of mathematics education researchers and metaphors treated by philosophers of language. This chapter presents a survey of the existing research literature in each of these areas.

LIMITS

Students' difficulties in learning limit concepts have been well documented. Evidence of the extreme difficulty for novice students with formal definitions, in particular, led the Content Workshop of the "Conference/Workshop to Develop Curriculum and Teaching Methods for Calculus at the College Level" held at Tulane University in January of 1986 to explicitly leave out references to epsilons and deltas in the recommended course syllabus for introductory collegiate level calculus (Tucker, 1986). Limit concepts, in general, have even been shown to be difficult for school teachers with decades of experience teaching calculus (Simonsen, 1995). In this section, we first discuss researchers' attempts to better understand the nature of students' difficulties and the resulting misconceptions about limits that students develop. We then examine some speculations on the cognitive obstacles involved and the constructions that are necessary for students to make in developing a mature limit concept. Next, we explore the nature of spontaneous models about limits that students use in working with limits. Finally, we give a brief review of some general linguistic considerations in the mathematics education literature, before turning to the section on metaphor.

Students' Misconceptions

Limit concepts are notoriously difficult for students of calculus, and anyone who has taught the course has probably noticed a number of their students' misinterpretations. The most obvious line of research related to the learning of limit concepts (and perhaps necessarily the first) is thus to systematically catalog students' errors in dealing with limits and to identify the conceptual difficulties. Several researchers have shown that students often conceive of limits as a boundary beyond which a function or sequence cannot pass (Williams, 1990, 1991; Davis & Vinner, 1986; Tall & Vinner, 1981), as a value that is actualized at "the end" of the limit process (Orton, 1983; Davis & Vinner, 1986; Thompson, 1994b), as an infinite process that can never be completed (Orton, 1983; Williams, 1991), and as an approximation (Williams, 1990, 1991). Students also confuse average rate and instantaneous rate (Orton, 1983), interpret technical language about limits in terms of everyday rather than mathematical meanings (Frid, 1994; Davis & Vinner, 1986; Tall, 1992), and focus on incidental and misleading aspects of graphs presented about limits (Orton, 1983; Monk, 1987).

Concept Image and Concept Definition

Several of the research studies on students' understandings of limits have been based on the distinction drawn by Tall & Vinner (1981) between a student's *concept image* and their *concept definition*. Defining the former, they say,

> We shall use the term *concept image* to describe the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes. It is built up over the years through experiences of all kinds, changing as the individual meets new stimuli and matures.

Students' concept images are taught and are often influenced by "naïve conceptualizations" assembled from previous non-mathematical, as well as mathematical, experiences (Davis & Vinner, 1986). Part of one's concept image is a *concept definition,* that is, one's personal "form of words used to specify that concept," which may differ from the accepted mathematical definition (Tall & Vinner, 1981). Tall and Vinner suggest that students are likely to invoke different aspects of their concept image when presented with varying problems. One's concept image may contain several potentially conflicting factors. Since different aspects of the concept image will be activated at different times, however, contradictions may go unnoticed until they are evoked simultaneously, causing actual cognitive conflict. Learning new ideas does not necessarily obliterate old ones, so students may often retain early misconceptions alongside more acceptable, subsequently developed interpretations. Thus, when students present incorrect conceptualizations, they do not necessarily lack the correct ones; rather, the issue is often the selection of which idea (or combination of ideas) to retrieve.

Davis and Vinner (1986) conducted a teaching experiment in which they sought to develop early understanding of limit concepts with 15 eleventh-grade students by explicitly addressing their naïve conceptualizations. The students were given a two-year calculus course based on the notion that "mathematics should be seen to be based on reasonable responses to reasonable challenges." In this course, the students were presented with various types of sequences (e.g. to find $\sqrt{2}$ or the area of a circle), asked to generate their own sequences with various properties for the consideration of a general treatment, then asked to help develop a definition of a limit that contains all of the examples. Several misconceptions of limits were revealed, and dealt with in this process. For example, suggestions included "the number that the terms are approaching" (such a number is not unique and this statement is only true for monotonic sequences) and "the number that you get to after infinitely many refinements" (which fails to realize that one wants to restrict criteria to actions which, at least in principle, can be carried out). Eventually they arrived at three versions of a definition, the standard ε-N definition and two informal but equivalent versions based on approximation language and on a graphical representation. Students were then expected to construct basic proofs about properties of limits using these definitions.

At the end of the first year, the students "were able to prove typical theorems, state correct definitions, produce exemplar sequences to demonstrate weaknesses in incorrect definitions, and so on." At the beginning of the following year, after a summer vacation, the students were given a surprise quiz asking for both a "precise formal definition" and a "description of a limit of a sequence in intuitive or informal terms." Davis and Vinner analyzed the responses to determine how the standard interpretations (learned by the students in the previous year) fared in competition with their naïve conceptualizations and then classified the students' errors into nine categories:

(1) A sequence "must not reach its limit."
(2) Implicit monotonicity for a_n - regarding the phrase "going *toward* a limit" as having its everyday literal meaning.
(3) Confusing *limit* with *bound*, requiring that a limit be an upper or lower bound for all a_n in the sequence.
(4) Assuming that the sequence has a "last" term, a sort of a_∞.
(5) Assuming that you can somehow "go through infinitely many terms" of the sequence.
(6) Confusing $f(x_0)$ with $\lim_{x \to x_0} f(x)$.
(7) Assuming that sequences must have some obvious, consistent pattern (or even a simple algebraic formula for a_n).
(8) Neglect of the important role of *temporal order* (first selecting an N and expecting that for $n > N$, $L - \varepsilon \le a_n \le L + \varepsilon$ will be true for any positive ε).
(9) Confusion between the fact that *n does not reach infinity* and the question of whether a_n may possibly "reach" the number L.

Davis and Vinner labeled the first 7 of these as "naïve misconceptions" since they appeared to at least partially be artifacts of students' previous learning. Use of language appeared as an especially significant influence with words like "limit" suggesting an actual bound, as in "speed limit" (one student even drew a picture of a fence around a group of sheep). In addition, they identified the predominant use of monotonic sequences and sequences represented with specific formulas throughout the course as potentially contributing to these misconceptions.

As mentioned earlier, conflicting aspects of a concept image may be evoked by even the surface features of mathematically similar questions, leading to varying results. Tall (1977, 1990) reported that 14 out of 36 students entering university given a survey for their study responded that $\lim_{n\to\infty}(1+\frac{9}{10}+\frac{9}{100}+\cdots+\frac{9}{10^n})=2$, while elsewhere on the same survey stating that $0.\overline{9}<1$. One week later, the students were asked to write each of the decimals .25, .05, .3, $.\overline{3}$=.333…, and $.\overline{9}$=.999… as fractions. This time, 13 of the 14 students who had previously indicated that $0.\overline{9}<1$ now confirmed that they were equal. In addition, Tall and Vinner (1981) report that in response to the last two decimals in this list, "several [of the remaining 22 students] now experienced *actual* cognitive conflict." It is interesting to note that students were content to apply contradictory aspects of their concept images when the representations differed (a limit and a repeating decimal) but experienced conflict when the representations were the same (both repeating decimals). There are different reasons suggested for this focusing on the notation for repeated decimals. Monaghan (1986), for example, provided evidence that students view repeating decimals as a dynamic process that never ends and not as "*proper* numbers." Others researchers (Tall & Schwarzenberger, 1978; Tall & Vinner, 1981) have found students giving static interpretations and couching their explanations in infinitesimal terms.

Difficulties Focusing on Relevant Features in Complex Situations

Many situations involving limits, such as differentiation and integration, add several layers of complexity to a situation that can cause students to become overloaded and lose track of relevant information (Orton, 1983; Tall, 1992; Thompson, 1994b). In a study on students' understanding of differentiation, Orton (1983) found that 43 out of 110 students were unable to interpret a drawing of a sequence of secant lines as approaching a tangent (see Figure 1a), focusing instead on a vanishing chord. Similarly when asked to interpret the expression $\frac{dy}{dx} = \lim_{h \to 0} \frac{k}{h}$ for the diagram in Figure 1b, a common error was to state that it gave the rate of change over the entire interval. They were equally unable to interpret the meaning of related symbols, e.g., 47 of the 110 students could not explain the meaning of dy/dx with 17 answering in terms of a ratio, and 78 of the 110 students were unable to explain the relationship between $\delta y/\delta x$ and dy/dx.

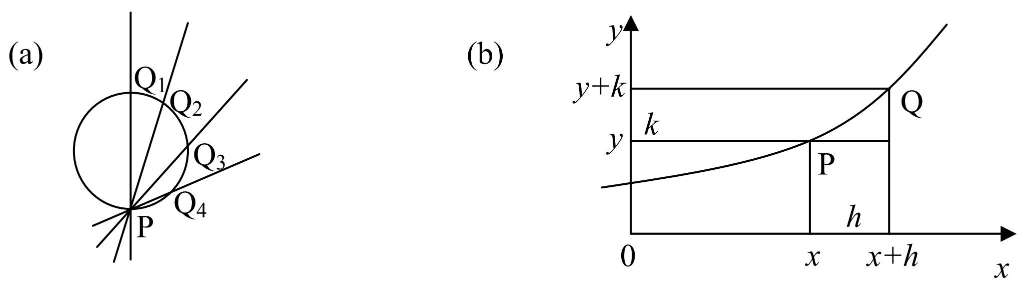


Figure 1. Diagrams from Orton's study of students' understandings of differentiation.

Steve Monk (1987) found that similar misinterpretations while reading graphs were related to whether the students were responding to questions about a function acting at a single point or "across-time," i.e., for a dynamically changing location. The students had little difficulty with pointwise tasks such as finding the slope of the secant and value of v in Figure 2a when M and N have coordinates (1,6) and (4,12) or determining the areas $A(1)$ and $A(3)$ in Figure 2b. When asked

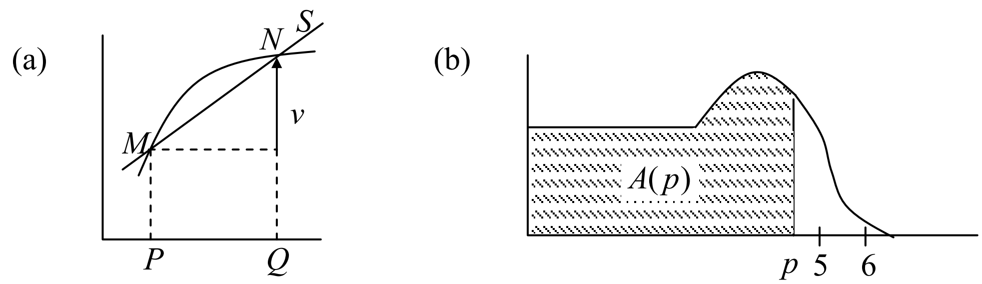


Figure 2. Diagrams from Monk's study of calculus students' understanding of functions.

across-time questions, however, the students were unable to keep all of the information straight. When asked what happens as the point Q moves toward P in Figure 2a, students regarded the secant line as moving with the slope increasing but the vertical distance v remaining fixed. Similarly, when asked about the behavior of $A(p)$ as p moves, they responded as if $A(p)$ was to be found by simply looking at the height of the graph. Monk suggests that for students to be successful interpreting standard diagrams such as these, they must be able to evoke an image of the diagram that can be made to move and be able to draw conclusions from mental experiments performed on them.

Dynamic Images of Limits

Tall and Vinner (1981) gave a questionnaire to 70 first-year students in which they were asked to provide a definition of $\lim_{x\to a} f(x) = c$ if they knew one. Most of the students who attempted to give a formal definition were incorrect, while those who responded with a dynamic definition were usually correct (see Table 1).

Table 1. Students' Definitions of $\lim_{x \to a} f(x) = c$.

	Correct Definition	Incorrect Definition
Dynamic	27	4
Formal	4	14

Further, the same survey asked students to explain what is meant by $\lim_{x \to 1} \frac{x^3 - 1}{x - 1} = 3$. Most of the students who could not give a definition of a limit were still able to make an attempt at explaining the meaning of this particular limit statement (see Table 2).

Table 2. Explanations of the Meaning of $\lim_{x \to 1} \frac{x^3 - 1}{x - 1} = 3$ by Students not Giving a Definition of a Limit.

	Correct Explanation	Incorrect Explanation
Dynamic	11	1
Formal	6	3

Students' informal dynamic language about limits can also lead to erroneous reasoning. Twenty-two of the students in Tall and Vinner's (1981) study were asked the following question two years later when they were in their final year of study and had dealt with the formal epsilon-delta definition of limits for two years:

True or false: Suppose as $x \to a$ then $f(x) \to b$
and as $y \to b$ then $g(y) \to c$
then it follows that
as $x \to a$ then $g(f(x)) \to c$.

All but 1 of these 22 students responded "true" and refused to change their answer, even when pressed, relying on the false syllogism set up by the dynamic language: "If x approaches a then $f(x)$ approaches b. If y approaches b then $g(y)$ approaches c." In other words, if the first premise holds, then $f(x)$ satisfies the hypothesis of the second premise, i.e., it qualifies as a y that approaches b. Thus, the dynamic language leads to the false conclusion that $g(f(x))$ approaches c.

Cognitive Obstacles

Another line of research has sought to identify the *cognitive obstacles* to the development of limit concepts. One approach is to identify *epistemological obstacles*, or difficulties inherent in the mathematical concept of limit, typically based on the historical development of the concept. A second type of cognitive obstacle is *genetic* and occurs in the personal psychological development of the student. Finally, *didactical obstacles* occur as a result of specific instructional methods.

Epistemological Obstacles

Bernard Cornu (1991) approached this question from the historical perspective and identified the 4 following epistemological obstacles:

1) *The failure to link geometry with numbers*. This obstacle is essentially what prevented the ancient Greeks from translating their "method of exhaustion" to a modern notion of limit. Euclid connected the idea that given two unequal lengths, one may be reduced by half a finite number of times so that it is

smaller than the other to the idea that the ratio of the areas of polygons inscribed in a circle is equal to the ratio of the squares of the diameters. This idea essentially mirrors the logic of an epsilon-*N* proof that the area of a circle is proportional to its diameter. The gap, however, lies in the fact that Greeks' conceptualizations were grounded in geometrical magnitudes rather than numbers.

2) *The notion of the infinitely large and infinitely small*. Newton, Leibniz, Euler, and Cauchy all used notions of the infinitely small or infinitesimal to great effect, but also with unease. Newton, for example, spoke of the "soul of departed quantities," and sought to overcome the need for infinitesimals by developing his theory on the intuitive basis of motion (fluents and fluxions). Before Cauchy established the modern formulation of limit definitions, he defined a function as continuous when "within given limits if between these limits an infinitely small increment i in the variable x produces always an infinitely small increment $f(x+i)-f(x)$, in the function itself" (quoted in Cornu, 1991). Cornu notes that this "idea of an 'intermediate state' between that which is nothing and that which is not is frequently found in modern students."

3) *The metaphysical aspect of the notion of limit*. Associated with the difficulty of rigorously conceptualizing limits is a corresponding metaphysical dialogue, which has been despised by historical figures such as D'Alembert and Lagrange. Ruminations about the existential meaning of the infinite, infinitesimal, or limits were seen as antithetical to the scientific mathematical

discourse. Students may likewise see this aspect as beyond computation and "not really mathematics."

4) *The question of whether or not the limit is attained.* Cornu quotes two opposing historical views of this question. On the one hand Robins (1697-1751) asserted, "We give the name ultimate magnitude to the limit which a variable quantity can approach as near as we would like, but to which it cannot be absolutely equal (as quoted in Cornu, 1991)." Providing the opposing view, Jurin (1685-1750) said that the "ultimate ratio between two quantities is the ratio reached at the instant when the quantities cancel out (as quoted in Cornu, 1991)." Cornu notes that statements such as "When *n* tends to zero, isn't *n* equal to zero?" are common among students today.

Genetic Decomposition

A significant portion of the research literature on students' understanding of function uses a theoretical perspective based on Piaget's theory of genetic epistemology outlined in Dubinsky, 1992. This perspective is generally called APOS theory for its categorization of mental constructions into actions, processes, objects, and schemas. An *action* is a physical or mental transformation on objects that is externally motivated and requires explicit instructions to be carried out. Such a conception cannot be conceptualized without the actual performance of the action and thus is not generalizable. When an individual reflects on an action he or she may establish control over it. This interiorization results in a *process* which the individual is able to reflect upon or describe without actually performing the individual steps. A process can be transformed in various ways, such as by reversing it or combining it with other processes. As an individual reflects on such transformations, the process may be seen as a coherent whole, resulting in an encapsulation of the process into an *object*. An object may be the recipient or product of actions and processes, leading to the development of higher-level mental constructions. Anna Sfard (1992) describes encapsulation as occurring in two stages. First, condensation occurs when a process is viewed as a self-contained whole, a perspective that is gradually developed over time with cognitive assistance. Reification on the other hand reflects a sudden ontological shift to view the underlying process as an object in its own right. Sfard notes that this final encapsulation is often the most difficult aspect of encapsulation. Once they are constructed, actions, processes, and objects can be interconnected in various ways in a *schema*, and brought to bear on a problem situation in a coherent way.

A group of researchers has recently begun to investigate the mental constructions necessary to understand the limit of a function using the APOS theory (Cottrill, et. al., 1996). Based on preliminary research, they have suggested that the standard dynamic conceptualization of limits is more complicated than generally characterized in the literature because it is a coordinated pair of processes, which is a schema. Specifically, for $\lim_{x \to a} f(x) = L$, there is a domain process ($x \to a$) and a range process ($f(x) \to L$) coordinated by the action of *f*. Not only is this difficult, but to understand the formal definition, Cottrill et. al. suggest that students need to reconstruct this schema into a new process sending $0<|x-a|<\delta$ to $|f(x)-L|<\varepsilon$, encapsulate this into an object, then apply a 2-level quantification schema. Their detailed genetic decomposition of the limit is as follows:

1. The action of evaluating *f* at a single point *x* that is considered to be close to, or even equal to, *a*.
2. The action of evaluating the function *f* at a few points, each successive point closer to *a* than was the previous point.
3. Construction of a coordinated schema as follows:
   (a) Interiorization of the action of Step 2 to construct a domain process in which *x* approaches *a*.
   (b) Construction of a range process in which *y* approaches *L*.
   (c) Coordination of (a) and (b) via *f*. That is, the function *f* is applied to the process of *x* approaching *a* to obtain the process of $f(x)$ approaching *L*.
4. Perform actions on the limit concept by talking about, for example, limits of combinations of functions. In this way, the schema of Step 3 is encapsulated to become an object.
5. Reconstruct the processes of 3(c) in terms of intervals and inequalities. This is done by introducing numerical estimates of the closeness of approach, in symbols $0<|x-a|<\delta$ and $|f(x)-L|<\varepsilon$.
6. Apply a quantification schema to connect the reconstructed process of the previous step to obtain the formal definition of a limit.
7. A completed *ε*-*δ* conception applied to specific situations.

(Cottrill, et. al., 1996)

This genetic decomposition has not been tested for a match to the actual processes that students undergo in developing an understanding of the limit of a function. Cottrill et. al., however, have developed computer based activities to guide each of these constructions for use in further research to refine the decomposition and as a beginning to the design of instruction based on their work.

Students' Spontaneous Models

Much of the "New Math" reform during the 1960s was based on a focus on rigor so that students were introduced to concepts through formal definitions and theorems. While this approach was very successful in preparing the most talented students for further studies in advanced mathematics, it left the vast majority of students with little more than a procedural understanding and an impression of mathematics as personally incomprehensible (Tall, 1992; Davis, 1986; Tucker & Leitzel, 1995). What this approach ignored is the mediation of formal concepts by the learners' spontaneous concepts. Several research studies have attempted to determine what spontaneous models students use and to develop instruction around these ideas.

Local Linearity and Qualitative Calculus

Students' difficulties with limits often result in misconceptions of later calculus topics. For example, Vinner's (1982) study on concept images of the tangent showed that first-year calculus students avoided drawing tangents which cross the graph, avoided drawing vertical tangents, sometimes drew multiple tangents, indicated the existence of infinitely many tangents at a cusp or at an abrupt (but still smooth) curve, and found horizontal tangents confusing. Even though the definition in the course was in terms of limits, 35% of the students still responded to a request for a definition (incorrectly) based on their global intuitions. Thompson (1994b) cites the response of a class of advanced mathematics students to a transcript from an interview with a 7th grader. In the interview, the young student spontaneously constructed the rudiments of a definite integral in response to a question about the distance traveled while smoothly accelerating from 50 mph to 60 mph over one hour. When the advanced class was shown this solution, they found it clumsy and wanted to have the student "discover" that she could just multiply mean speed by time. These students missed the connection between average speed and instantaneous speed that allows the definite integral to generalize.

David Tall (1992) proposes the use of *cognitive roots* to guide calculus instruction. These are conceptualizations that play the dual role of providing familiarity on one hand and a basis for later mathematical development on the other. Thus, they are good starting points for curriculum and may differ from mathematical foundations, which are intended to be starting points for the logical development of the subject. Rather than expecting student to make sudden, cognitively unaided conceptual shifts to understand the structure of the mathematics, Tall suggests we must help students slowly build these understandings from a base in their everyday knowledge.

Tall (1986, 1990, 1992) outlines historical and cultural conceptualizations and student acquisition of function, limit, infinity, and proof concepts. In his discussion of student difficulties and misconceptions related to limits, he suggests that a better cognitive root for calculus might be "local straightness." Students are introduced to tangents via magnification of a function's graph at a point. This approach, he suggests, allows for the investigation of a rich source of concepts: different left and right gradients, functions that are locally straight nowhere, etc.

Students taught with this approach were much better at recognizing, drawing, and reasoning about graphical information for derivatives than students in a control group. On the other hand, they tended to describe a tangent as passing through two or more very close points on the graph.

At least part of these students' difficulties seems to be conflation of the tangent line and the actual graph caused by the appearance of the graph as a straight line after sufficient magnification. An overexposure to linear relationships results in a view of proportionality as a privileged kind of relation and a serious epistemological obstacle for students' general understandings of functions (Sierpinska, 1992). An interesting alternative intuitive approach to teaching calculus that avoids this difficulty has been proposed by James Kaput (1994) and Walter Stroup (1996). Both researchers have proposed viewing the mathematics of change as a content strand to be addressed throughout the middle school and high school curriculum rather than as just the content of a capstone calculus *course*.

Kaput (1994) designed an interactive computer environment called MathCars that graphically simulates the motion in a vehicle with the user controlling an accelerator. The visual display presents a rich array of coordinated information such as, audio and visual feedback for passage of time and distance, and numerical and graphical representations of time, distance traveled, and velocity, to name only a portion of the options. The user would typically only have a few representations active at a time, but Kaput describes their presentation as "carefully coordinated in order to link the phenomenology of the experience of motion with its formal representations. More globally, the entire environment is intended to make a connection to the intuitive notions of motion (fluxions of fluents) used by Newton in his development of the calculus of change. Once these conceptualizations have been built with strong connections among the various domains and a deep understanding of the structure of graphical representations, Kaput suggests encouraging the translation to action representations.

Kaput describes an *action system* as a notation system that provides "systematic means for the user to act on it physically." An example would be Leibniz's differential notation that syntactically guides the relevant conceptual operations. This differs from a *display system* that primarily serves "either to display information for the user to read or respond to." The graphical representation of position or velocity with respect to time provides primarily information in this sense. Kaput suggests teaching students methods of analyzing the graphs they generate "in ways that reflect the insights of the masters." A graph as in Figure 3, for example, addresses the interpretation of first differences over uniform intervals, their relationship to the average velocities, the constriction of an approximation to the velocity graph (by translating each vertical line down to the time axis), increasing the accuracy of the approximations by having the computer generate more dots, etc. The process can also be used to create a change in velocity graph (i.e., acceleration) and can be reversed using an area interpretation to yield antiderivatives.

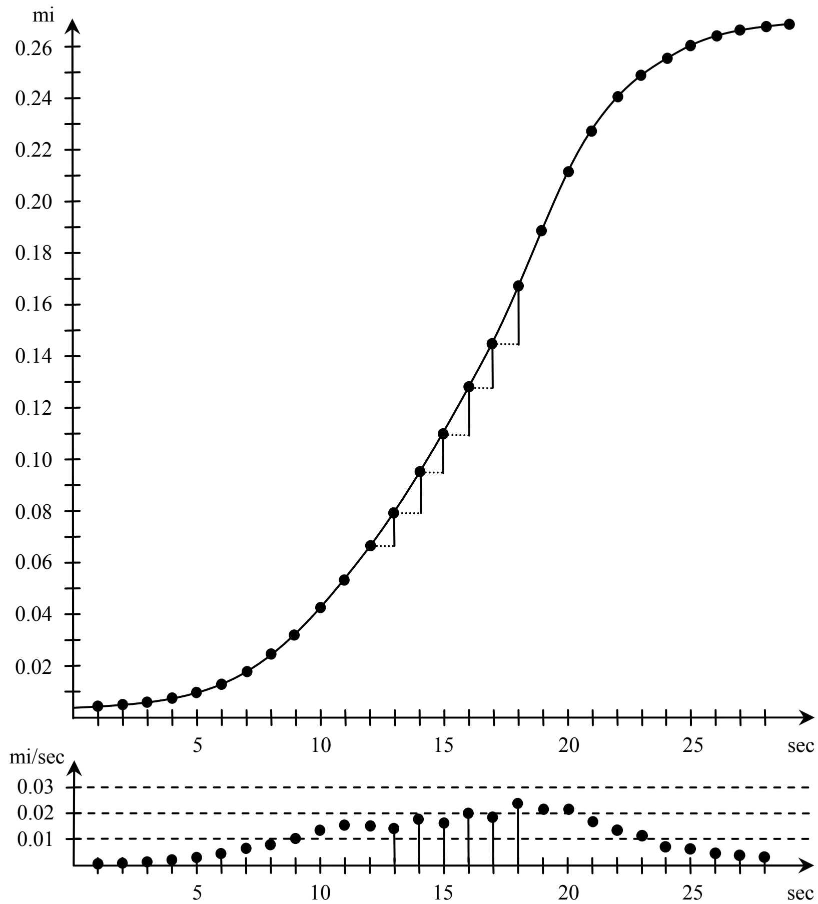


Figure 3. Kaput's example of an action representation for position and velocity.

Stroup (1996) notes that the structure involved in the activity of translating among the types of graphical representations generated in Kaput's MathCars simulation reflect's the Piaget's notion of universal structure. Specifically, the group codifies an integral relation between objects and processes in the requirements of an identity, inverses (or an inverse operation), and associativity.

Stroup describes the foundation of Piaget's constructivism as based on engaging conceptual systems that involve

> 1. the condition that a "return to he starting point" always be possible (via the "inverse operation");
> 2. the condition that the same "goal" or "terminus" be attainable by alternative routes and without the itinerary affecting the point of arrival ("associativity"). (Stroup, 1996)

Stroup describes how reversibility is "embodied" in the types of activities involved in translating between position ("how much" in Stroup's study) and velocity ("how fast") graphs:

> Given a graph of how much, can the learner sketch a graph of how fast? And given a graph of how fast, can the learner sketch a graph of how much? The idea of inverse is 'embodied' in the integral of a derivative resulting in something like the original graph ('off', of course, by the constant of integration) and vice versa. One 'gets back' to something *like* what one began with. Steepness becomes a central concept in this approach. The second major way of understanding the relation between integral and derivative centers on the area under the graph of the derivative (rate graph) representing the change in the how much quantity. The area represents an accumulation of the amount.

Later, he describes path independence in terms of the acquisition of constructs of how much and how fast:

> If how much and how fast ideas *are both differentiated and integrated in such a way as to form structure* (in the sense in which Piaget meant), then understanding the mathematics of change depends on the *interaction of these constructs.* Additionally, the path one travels (including where it begins) in arriving at this integration of how much and how fast ideas loses its significance. For individual learners the paths may vary, but the nature of the understanding arrived at – under the Piagetian analysis – is not seen to be impacted by the particularities of the path traveled. How much and how fast constructs interact to create powerful cognitive structure people can (and do) use in making sense of their lived experiences.

Stroup worked with groups of eighth- and ninth-grade students developing the interpretation of "how much" and "how fast" information from graphical environments similar to Kaput's MathCars, treating rate in terms attending to global information about "steepness" related to ideas of speed. Stroup found that linear examples were cognitively degenerate for the students in that they did not provide sufficient structure (in the Piagetian sense) on which to base reasoning about rate of change. In contrast, activities involving more complex, nonlinear graphs allowed students to operate on notions of rate and changing rate and to perform well at moving back and forth between notions of "how much" and "how fast."

Infinitesimals

Though a continuum composed of infinitesimals was the image held by both Newton and Leibniz, most modern calculus instruction is based on real analysis, which is often viewed as the only rigorous option. Nonstandard analysis, however, was developed by Abraham Robinson (1966) to give a logical foundation for infinitesimal concepts. His results allow one to embed the real numbers into a bigger ordered field which has an element, N, larger than any real number. Thus, in the order relation, $1/N$ is smaller than any positive real number and in this sense is "infinitesimal." A line segment can then be thought of as composed of infinitely many line segments of nonzero, infinitesimal size.

Sandra Frid (1994) notes that Robinson's work can be translated into a form suitable for an introductory calculus course. Her study investigated students' learning in calculus courses taught by three different methods: a traditional approach, a reformed approach, and an infinitesimal approach. In the infinitesimal approach, computations are performed using an infinitesimal element, ε, and standard algebra extended to the infinitesimals. This process is followed by rounding off infinitesimal terms, so that an expression like $2x + \varepsilon$ is replaced by $2x$. Tangents are then treated by magnification as described above with the addition that the graph is magnified to an infinitesimal scale. Frid describes this approach as differing from the traditional approach in a number of conceptually important ways: 1) it is a dynamic rather than static method for interpretation of graphs, 2) magnification makes the limit concept of "close to" accessible, 3) students have a well-developed intuition about rounding that can be exploited, and 4) students tend to believe in the existence of infinitesimal elements regardless of the approach used. She found that although students that were given instruction with infinitesimals did not perform significantly better on standard computations, they did use the language and notation of rounding as an integral part of their explanations. Students in the traditional and reformed classes did not use limit notation in their responses, indicating a possible lack of integrating limit concepts with the rest of their calculus knowledge.

Michèle Artigue (1991) conducted a study with 85 third-year university students (enrolled in multivariable calculus and physics courses) to investigate their understanding and use of differential elements. In their course, students were provided with a tangent linear approximation definition, which dominated their declarative descriptions of differentials. At the procedural level, however, they reverted to treating differentials algebraically in algorithms involving partial derivatives and Jacobian matrices. Students were also not able to identify necessitating conditions in specific contexts for the use of differentials and gave justifications about convergence of approximations based on convergence of the geometric "slices."

Infinity

In a study closely analyzing the structure of the arguments students provided about infinite sequences, Anna Sierpinska (1987) identified several distinct attitudes about infinity that were related to various attitudes toward the limit of a sequence. Her typology first made the distinction between a conception of the infinite as an actual state (labeled *infinitist*) and a conception of the infinite as something that is never completed (labeled *finitist*). The finitist perspective asserts that infinity does not exist or, at best, is a mathematical abstraction with no real meaning. The limit of an infinite sequence is seen to be "the last term." This attitude can be coupled with either a *definitist* or *indefinitist* view. The definitist view describes everything as not only finite but also determinate. Thus, the number of terms in a sequence can be established and the value of the last term, and hence the limit, found. The indefinitist view asserts that while everything is finite, the exact number of elements cannot always be determined. If such is the case for a sequence, then one agrees on a stopping point for approximation. In the case that an abstract mathematical unboundedness is acknowledged, it is seen as unrealistic, and only bounded sequences are considered to have limits.

Sierpinska outlines three different infinitist attitudes, with the first two placing a heavy emphasis on the role of temporal construction. First, a *potentialist* view suggests that infinity is never actually achieved, but that a process or temporal construction may have no end. A sequence infinitely approaches its limit without ever reaching it, and the impossibility of actually reaching the limit is implied by the impossibility of running through infinity in a finite time. A *potential actualist* view suggests that an infinite process can be actualized if an infinite amount of time is allowed. The limit of a sequence is viewed as the "ultimate term" or the "next term" after the sequence is finished. Finally an *actual actualist* view ignores any temporal aspect of construction and considers only the result of a mathematical construction; infinity is reached. There are two possible models for limit in this view. First, a *boundist* model focuses on the supremum and infimum of a sequence as fixed. Second, an *infinitesimalist model* views the sequence as an infinitely small distance away from the limit.

No student was classified with a single label; rather various portions of their arguments were classified, with students shifting from one perspective to another. In a separate study (Fischbein et. al., 1979) with 470 fifth- through ninth-grade students, younger students were seen to provide highly inconsistent arguments in response to different problem contexts. Interestingly, none of the categories emerging from Sierpinska's study showed students thinking about either a concept of infinity or the limit of a sequence completely consistent with their mathematical definitions. Nevertheless, the students in the study were able to use these perspectives to make and evaluate claims about sequences and to engage in debate with one another, refining their ideas.

George Lakoff and Rafael Núñez's recent book, *Where Mathematics Comes From: How the Embodied Mind Brings Mathematics into Being* (Lakoff & Núñez, 2000), has received broad attention among mathematicians and mathematics educators along with both positive and highly critical reviews (Goldin, 2001; Madden, 2001). In their book, Lakoff and Núñez have proposed that all concepts relating to infinity, including limits, are understood via a "Basic Metaphor of Infinity." Briefly, they describe a metaphor as consisting of a well-understood source domain, an abstract target domain, and a mapping that carries objects, structures, and implications from the source to the target. The source for the Basic Metaphor of Infinity is the domain of completed iterative processes, and the target is the domain of iterative processes that never end. The final resultant state of the completed process is mapped onto a final "infinite" state, which is unique and follows every non-final state. Lakoff and Núñez suggest that monotonic sequences $\{x_n\}$ are understood as

1) an initial state $S_1 = \{x_1\}$ and $R_n = (x_1, L)$ for $\{x_n\}$ increasing (or $R_n = (L, x_n)$ for $\{x_n\}$ decreasing)
2) the iterative process $S_{n-1} \rightarrow S_n = S_{n-1} \cup \{x_n\}$
3) intermediate resultant states S_n and $R_n = (x_n, L)$ for $\{x_n\}$ increasing (or $R_n = (L, x_n)$ for $\{x_n\}$ decreasing)
4) and a final, metaphorical state $S_\infty = \cup S_n$, $R_n = \varnothing$, and the limit L.

They describe the understanding of arbitrary sequences in a similar way except that the stages of the iterative process correspond to terms in a subsequence of "critical elements," terms for which all remaining terms are closer to the limit. Finally the limit of a function at a point $\lim_{x \to a} f(x) = L$ is described as understood through the sequential definition: for every sequence $r_n \to a$, the resulting sequence $f(r_n) \to L$.

Lakoff and Núñez suggest that metaphors often combine in "conceptual blends." For example, a common metaphor in mathematics is the treatment of numbers as if they were points on a line. Another linguistic and psychological effect, referred to as Talmy's *fictive motion,* describes a static path in terms of either the motion that it affords (e.g., a road or path "running" through the woods) or the perception of motion created by sequentially highlighting adjacent points (e.g., a point "moves" along a curve). Combining the Basic Metaphor of Infinity with Numbers as Points on a Line and Talmy's fictive motion yields the familiar motion-based metaphor of "approaching a limit."

Whether or not this type of purely linguistic analysis (it is not based on data collected from subjects in a study) accurately describes the embodied structure for the precise understanding of experts has not been examined. More important to education research, this analysis has not been shown to relate to the emerging understanding of someone engaged in learning limit concepts, and no attempt is made by Lakoff and Núñez to address the potential role of metaphor in concept formation. Their program is not about mathematics education, however. Instead, their goal is to construct a plausibility argument for preconceptual structures containing sufficient richness to serve as a metaphorical foundation for abstract mathematical concepts. Their discussion of limits, as a typical example, is an attempt to characterize a sophisticated understanding of the concept by finding similar structures in everyday experience and more basic mathematics then construct explicit maps showing the relationships between these structures and the abstract ideas. (See the following section of this chapter for a discussion of the details of Lakoff and Núñez's perspective on metaphor in context with other perspectives.)

Limits

In a review of the literature on students' understanding of limits, Bernard Cornu (1991) notes that students' spontaneous conceptions of limits likely have much to do with their non-mathematical images of words. As noted earlier, "limit" is likely to connote some type of boundary or constraint. The phrase "tends to" has also been shown to have varying meanings for students, including 1) to approach without reaching, 2) to approach just reaching, and 3) to resemble without any variation (as in "this blue tends towards violet"). Cornu reports a classification of students models for limits observed by Roberts (1982a, b), consisting of the following views: *stationary* (final term language), *barrier* (values cannot pass the limit), *monotonic* (bounded above), *dynamic-monotonic* (increasing toward), *dynamic* (approaches), *static* (sequence terms grouped around the limit), and *mixed*.

Williams (1989, 1991) conducted a study to explore, at great depth, students' spontaneous models of limits and possible means by which these models can be altered and made more rigorous. He first presented 341 second-semester calculus students with the questionnaire given in Figure 4, designed to roughly categorize their views of limits.

A. Please mark the following six statements about limits as being true or false:
   1. T F A limit describes how a function moves as x moves toward a certain point.
   2. T F A limit is a number or point past which a function cannot go.
   3. T F A limit is a number that the y-values of a function can be made arbitrarily close to by restricting x-values.
   4. T F A limit is a number or point the function gets close to but never reaches.
   5. T F A limit is an approximation that can be made as accurate as you wish.
   6. T F A limit is determined by plugging in numbers closer and closer to a given number until the limit is reached.

B. Which of the above statements best describes a limit as you understand it? (Circle one)

   1 2 3 4 5 6 None

C. Please describe in a few sentences what you understand a limit to be. That is, describe what it means to say that the limit of a function f as $x \to c$ is some number L.

Figure 4. Williams' initial questionnaire.

The responses to the first two questions in this survey are shown in Table 3. Although the numbers indicate that these students possessed complex combinations of the six informal models, Williams selected 10 students most clearly representing various perspectives for an in-depth analysis of the students models: 4 were judged to view a limit as dynamic, 4 as unreachable, 1 as a bound, and 1 as an approximation.

Table 3. Percentage of Subjects Indicating Each Statement as True, False, or Best on Williams' Initial Questionnaire.

Question Number	Statement Type	Percentage Response		
		True	False	Best
1	Dynamic-Theoretical	80	19	30
2	Boundary	33	67	3
3	Formal	66	31	19
4	Unreachable	70	30	36
5	Approximation	49	50	4
6	Dynamic-Practical	43	57	5

Each of the 10 selected students participated in a sequence of five interviews, beginning with a session to establish their working definition and operational model of limits. The second through the fourth interviews presented students with opposing viewpoints on the issues of a limit being reachable, limits involving motion, and a limit as a bound. The students were asked to explain each viewpoint and whether they agreed with one or the other. They were then given a series of problems and asked to discuss their work from each viewpoint. In the fifth interview, students were asked to respond to each of the three viewpoints and asked whether or not their views had changed. At the end of each session, the students were given the opportunity to change their definition of a limit.

Williams found students' viewpoints to be extremely resistant to change, even in response to extremely explicit discussions about contradictory examples. Basically no change, for example, was observed in students' dynamic view of limits. To varying degrees, most came to agree limits are reachable, although with notable exceptions, such as the response to a question about the limit of the position of a train coming to a stop in which one student was willing to even question her own physical experience saying, “Do trains really stop?”

Williams also identified several metaphorical aspects of the students’ reasoning. He termed the students’ faith in graphing and formulae their “generic metaphor.” Specifically, while the students often believed that they were incapable of determining what a function was doing near a point, they placed great faith in their ability to produce a graph that somehow magically accounted for “the problem of continuity, topological properties of the real line, and a myriad of other difficulties which they realize might arise in taking limits.” In addition, Williams suggests that several students exhibited what he calls a “base metaphor,” essentially a strongly held set of beliefs typically surrounding the contexts in which they were first exposed to limits. These included

1) an image of Zeno’s paradox in which one takes steps successively decreasing the distance to a point but never surpassing it,
2) the division of an interval “into infinite subintervals” to obtain the limit of a function,
3) the function values squeezing in on each side of the limits (with an implicit assumption of monotonicity),
4) the limit as an asyomptote where motion along the graph is determined by the *x*-values and *y*-values moving at different rates.

In addition, the students viewed counterexamples as minor exceptions rather than reasons to abandon an incomplete concept. They evaluated the appropriateness of any particular conceptualization based on its usefulness in a given setting rather than on its rigor, consistency, or correctness.

In a subsequent analysis of this data, Williams (2001) investigated the specific relationships of these models to one another in each student's conceptual structure. Williams used a method of establishing the predicational structure among students' various models, that is, the implications among the various constructs in the students overall concept image. An example of one student's developing predicational structure is shown in Figure 5. Arrows indicate the direction of implication between concepts and boxes cluster together concepts that

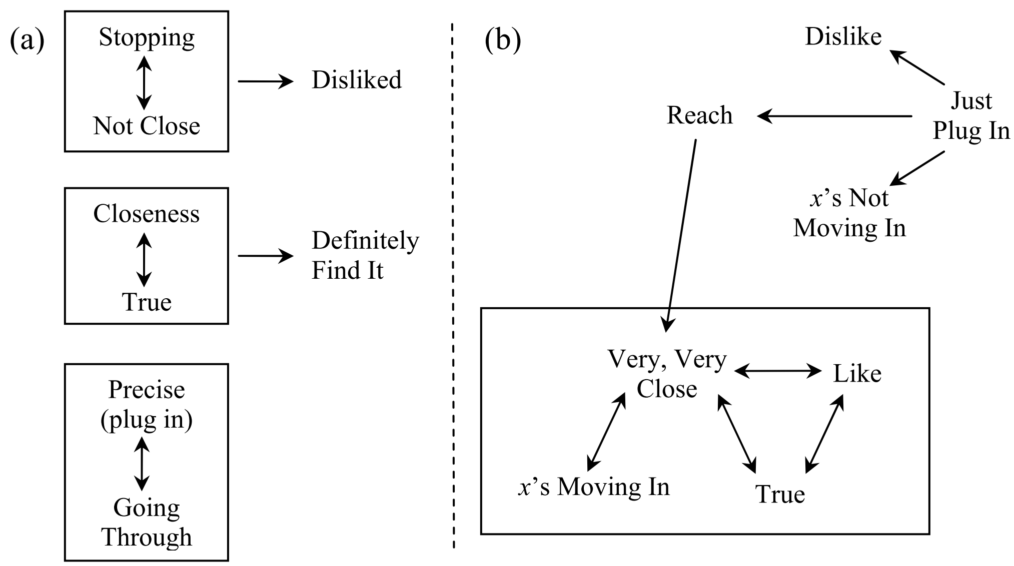


Figure 5. A students' predicational structure for limit concepts (a) at the beginning of the 7-week study and (b) at the end of the study.

are (directly or indirectly) mutually implicative. In the initial interview (Figure 5a), this student identified truth statements with ideas about "closeness," which he perceived as leading to the ability to "definitely find [the limit]." He equated the opposite of closeness with "stopping at the limit," an idea that he disliked. Finally, he equated finding the value of a limit by "plugging in the value" for a continuous function with the idea of the graph "going through" the point instead of just getting near.

In the final interview, the centrality of this student's image of closeness was clarified. He viewed closeness as being implicitly related to the requirement that a function be monotonic so that the values of the function are seen as "sandwiching in" on the limit when the "*x*'s are moving in" on the point. Williams suggests that his foundational concepts appear in this interview to have "pulled together into a tighter structure." In addition, a second structure has emerged related to the students' ambivalence about whether or not a limit can be reached. Here, Williams suggests that "'just plugging in' numbers is… disliked and is seen as antithetical to the *x'*s moving in construct" so that it "is not really a part of [the students'] core concept of what a limit is – it doesn't involve 'sandwiching.'"

In a similar way, most of the students in Williams' study retained or strengthened core beliefs about limits. Most of their conceptual structure was built around a dynamic view; apart from the conceptualization of evaluating a function at sequentially selected points, they did not have an alternative framework to develop stronger limit concepts.

Williams' studies represent the most in-depth research, to date, on students' working conceptualizations of limits. They have provided an across-time view of the definitions given by students, the nature of convictions and misconceptions, factors influencing the origins of their conceptions, and the resilience of students' personal metaphors. The main drawback to these studies, however, is that they are somewhat restrictive in the responses allowed by the students. By asking students to respond to specific statements, Williams is able to generate the clear categories of responses needed for his modes of analysis. The cost is that they do not capture the students' completely spontaneous thoughts in the process of inquiry. Through my study, I hope to build on the work of students' emerging understandings of limits presented in this section. In particular, using Williams' findings as a base, I intend to engage students in open-ended problem solving activities involving limit concepts to elicit the metaphors that they use in the process. I hope to characterize, in detail, both the conceptual structure and the functional aspects of their application.

Linguistic Considerations in Learning Mathematics

Many studies have documented the difficulties students encounter in understanding the meaning of language and symbols in mathematics. Specifically related to calculus, students have trouble interpreting even the most basic mathematical symbols and often fall back on nonsensical language when mathematically describing situations (Orton, 1983; White & Mitchelmore, 1996). They have difficulty reading and understanding basic mathematical text such as definitions, theorems, and problem statements, and these difficulties can be extremely resistant to direct instruction (Ferguson, 1980). Aside from being difficult for students to understand, mathematical language can also be the source of misconceptions because most of this language (e.g. “limit” or “tangent” in calculus) has prior, nonmathematical meaning for students (Davis & Vinner, 1986; Frid, 1994; Rubin & Nemirovsky, 1991; Tall, 1992; Tall & Vinner, 1981).

Though language is often a source of difficulty for students, it is also a powerful tool with which students at all levels organize and access various modes of mathematical analysis. Hinsley, Hayes, and Simon (1976) demonstrated that students’ understandings of algebra word problems are highly schematized, that these schemas are triggered early in the problem solving process (often after reading only a phrase or two of the problem), and that students use these schemas to structure their analyses in solving the problems. In a study of calculus learning, Rubin and Nemirovsky (1991) interviewed their subjects working with toy cars, air pumps, and a computer spreadsheet to model concepts of rate of change and accumulation. They showed that for mathematically identical features, students’ uses of language would vary greatly among the different environments. Each environment allowed students to see different features of the structure but also generated their own unique misconceptions. Finally, Steven Williams (1991) showed that the presence of analogies and metaphors to aid understanding is an important factor in determining the ability of students to make conceptual shifts in learning calculus concepts such as limit.

Prototypes are referent systems that have been shown to be important for students' uses of mathematical ideas. Schwarz & Hershkowitz (1999) showed that there are certain example functions that are central to the learning of functional concepts. They demonstrated that prototypical functions are those examples which most reflect the "redundancy structure of the category." Students judge examples by their distance from the prototype and extensively use these prototypes as analytical tools in describing functional situations rather than relying on abstract definitions.

A number of researchers have used linguistic tools to interpret students' work, revealing deep effects of language on students' thought processes. Silver and Cai (1996) were able to evaluate the complexity of the products of students' mathematical problem posing using a semantic analysis developed by Sandra Marshall (1995). This analysis groups the steps of a solution to a problem based on the nature of the mental operations involved and is dependent on their linguistic as well as mathematical structure. In another study, Patrick Thompson (1994) carefully analyzed students' learning in a computer environment designed to teach the concept of rate. As a result, he found a distinction between students use of the words "ratio" and "rate" in terms of the different mental operations necessary to conceptualize the two, even though they are mathematically equivalent. Carpenter, Hiebert, & Moser (1981) used semantic categories to analyze the mathematical structure of addition and subtraction problems. They then showed that the wording of these problems affected the method of analysis used by young children.

As we have seen, the ability for abstract reasoning within a domain is the key for students to recognize inconsistencies and make correct evaluations about the tools they are selecting. White & Mitchelmore (1996) investigated the role of abstraction in learning calculus. They suggest that solving typical calculus problems involves translating from the context to the abstract level of calculus symbolization, solving the abstract problem, and finally translating the solution back to the context. Although ignoring the possibility of the various conceptualizations of the problem interacting with one another, their results are still quite interesting. In solving problems requiring various levels of modeling, students had the most difficulty with aspects involving translation between the concrete and abstract. As a means of coping, students often fell back on a "manipulation focus" such as failure to distinguish a general relationship from a specific value, searching for symbols to which to apply known procedures regardless of their meaning, or remembering procedures solely in terms of symbols used when they were first learned.

This type of surface understanding does not reflect the true structure of the mathematics we are trying to teach to our students, yet a grasp of the actual mathematical structures is crucial if systems of concepts are to emerge. In explaining how these understandings develop, James Hiebert and Thomas Carpenter (1992) begin with two theoretical assumptions: 1) there is a relationship between internal and external representations and 2) internal representations can be related or connected to one another in useful ways. They define understanding in terms of the number and strength of appropriate connections between internal representations. Understanding abstract entities such as mathematical symbols requires them to be represented internally as mathematical objects rather than just marks that stand for other things. Meanings are then developed by making connections with other forms of representation (where the source of meaning is derived from preexisting internal networks) or within the representation (which occurs by recognizing patterns within the system).

Hiebert and Carpenter noted that students are often asked to memorize seemingly isolated bits of information. As a result, they then come to believe that mathematics is mainly a matter of following rules, that it consists mostly of symbols on paper, and that these symbols and rules are disconnected from other things they know about mathematics. It is then from this perspective that they approach other mathematical learning. On the other hand, if students are asked to construct connections between pieces of information, they will begin to make crucial connections within the structure of mathematics.

Sandra Frid's (1994) study on the impacts of different approaches to teaching calculus corroborates these assertions. Frid notes that though students often do not realize it, their mathematical knowledge is grounded in 1) linguistic knowledge, conventions, and rules, 2) social process by which individual, subjective knowledge becomes shared, external, objective knowledge, and 3) objectivity viewed as public, social acceptance rather than an inherent property of the content of knowledge. In her study, Frid categorized students according to their epistemological beliefs about mathematics. "Collectors" and "technicians" did not see the subject as something they could personally understand, while the rare “connectors” in the study attempted to draw together various information and make sense of the material as a cohesive whole. Both of these types of beliefs can be self-propagating.

Frid found that whether the students’ use of everyday language was of help or a hindrance depended on the extent to which they integrated that informal language with technical language or symbols in ways congruent with the corresponding concepts. For example, the lower performing students in her study were not able to work abstractly with calculus concepts. They did not see symbols as personally meaningful but rather as objects to be manipulated according to memorized rules. Even when asked to do so, they were unable to use symbols in their explanations. They used mathematical language solely in ways consistent with their prior knowledge rather than with the structure of the mathematics, and the meaningful explanations that they were able to give relied on visually oriented language. Such observations are all strong indications, according to a Vygotskian perspective, that these students’ thinking is not scientific. On the other hand, students in Frid’s study who did make connections between various concepts were able to use correct mathematical symbols as an integral part of their conceptual explanations. Developing this ability is crucial to learning mathematics, as symbols provide more than just a way of talking about ideas. They alleviate working memory and processing load by reifying complex ideas into conceptual entities and syntactically guiding important mental operations on those entities (Harel & Kaput, 1991).

PHILOSOPHICAL PERSPECTIVES ON METAPHORS

The study of metaphor has received considerable attention since Aristotle first laid out his theory over two thousand years ago. Although many students of language since that time have been intrigued by the mystery of the form, others in philosophy and the sciences have disdained and warned against its imprecise character. Consequently, very little rigorous treatment has been given to metaphor from the Greek period until the past century. Renewed attention in the philosophy of language began in earnest in the mid twentieth century, but there is still little consensus except on the admission that we have only the beginning of a theory for metaphor. In this section, although far from providing a comprehensive review of either historical or modern thought on metaphor, we will examine several views and shifts in thought that are particularly relevant to the research in this study.

Metaphor and Categorization

We begin with Aristotle's definition of metaphor and its direct connection to his theory of categories. This link is important for the purpose of this study because we are concerned with students' uses of metaphor to understand mathematics, a field in which the structure and logic has largely been built on Aristotelian categorization. Before considering modern perspectives on metaphor, we examine a radical shift in our understanding of how humans naturally categorize the world and their experiences in it that was precipitated by groundbreaking research by Eleanor Rosch in the 1960s.

Two Millennia of Aristotelian Thought

Perhaps the first theory of metaphor was offered by Aristotle in *The Poetics*, which included a concise definition that continues to influence modern thought:

> Metaphor is the application of an alien name by transference either from genus to species, or from species to genus, or from species to species, or by analogy, that is, proportion. Thus from genus to species, as: "There lies my ship"; for lying at anchor is a species of lying. From species to genus, as: "Verily ten thousand noble deeds hath Odysseus wrought;" for ten thousand is a species of large number, and is here used for a large number generally. From species to species, as: "With blade of bronze drew away the life," and "Cleft the water with the vessel of unyielding bronze." Here arusai, "to draw away" is used for tamein, "to cleave," and tamein, again for arusai- each being a species of taking away. Analogy or proportion is when the second term is to the first as the fourth to the third. We may then use the fourth for the second, or the second for the fourth. Sometimes too we qualify the metaphor by adding the term to which the proper word is relative. Thus the cup is to Dionysus as the shield to Ares. The cup may, therefore, be called "the shield of Dionysus," and the shield "the cup of Ares." (Aristotle, Trans. Butcher, 1929)

Here Aristotle outlines four types of metaphor, three based on the transference among and between genera and species, and one based on proportion. This definition relies heavily on a hierarchy of types in which individual things ("primary realities"), such as Socrates or a specific robin are members of "secondary realities," genera and species. Socrates is a human, and the robin is a bird (species). Both are animals (genus). Species such as human and bird are differentiated from other species of animals through defining characteristics such as "two-footed" and between each other by further categorical differentiae such as "feathered." Still further distinctions lead to subspecies such as robin.

Aristotle's theory of categorization is a crucial foundation for the modern sciences in their concern for systematic and hierarchical classification of types (e.g., types of knowledge, abstract structures, biological specimen, etc.). In the classical theory of categorization, necessary and sufficient criteria determine category membership, the basis of definition. Distinctions between categories are sharp, as the Aristotelian principle of contradiction asserts that the same attribute cannot simultaneously belong to, and be absent from, the same subject. In propositional calculus, it is impossible for both *p* and not *p* to be true, and in set theory, the law of the excluded middle requires an element either to be in a set or not in that set. Classical categorization is the foundation for the paradigm of an objective scientific and mathematical language, independent of individual interpretation.

If the Aristotelian theory of categorization has thrived in science, mathematics, and philosophy for two millennia, then the use of metaphor has been largely avoided in these fields. In Aristotle's definition, metaphor is an explicit crossing of categories; acceptable perhaps as an art form, but a grave error in sciences that require a rigorous discourse. In fact, it appears that in Aristotle's view, the purpose of metaphor was largely decorative. In *Rhetoric*, he wrote

> Metaphor is of great value both in poetry and in prose. Prose-writers must, however, pay specially careful attention to metaphor, because their other resources are scantier than those of poets. Metaphor, moreover, gives style clearness, charm, and distinction as nothing else can: and it is not a thing whose use can be taught by one man to another. Metaphors, like epithets, must be fitting, which means that they must fairly correspond to the thing signified: failing this, their inappropriateness will be conspicuous: the want of harmony between two things is emphasized by their being placed

> side by side. It is like having to ask ourselves what dress will suit an old man; certainly not the crimson cloak that suits a young man. And if you wish to pay a compliment, you must take your metaphor from something better in the same line; if to disparage, from something worse. (Aristotle, Trans. Roberts, 1954)

Prior to the twentieth century, the disconnect between conceptions of rigorous science and artistic forms of natural language relegated even the study of metaphor to the realm of literary criticism. The classical view was such an integral part of the scientific paradigm that it was even taken to be the basis of rational thought. Describing the extent to which this perspective held sway, Eleanor Rosch wrote

> The processor was assumed to be rational, and attention was directed to the logical nature of problem-solving strategies. The "mature western mind" was presumed to be one that, in abstracting knowledge from the idiosyncrasies of particular everyday experience, employed Arsitologian laws of logic. When applied to categories, this meant that to know a category was to have an abstracted clear-cut, necessary, and sufficient criteria for category membership. If other thought processes, such as imagery, ostensive definition, reasoning by analogy to particular instances, or the use of metaphors were considered at all, they were usually relegated to lesser beings such as women, children, primitive people, or even to nonhumans. (Rosch, 1978)

Eleanor Rosch's Studies of Human Categorization

Beginning in the late 1960's, Eleanor Rosch, with various collaborators, conducted a series of studies that unexpectedly showed humans actually build conceptual categories differently than the classical theory suggests (see, for example, Rosch & Mervis, 1975; Rosch, 1976; and Rosch, et. al., 1978). Her initial groundbreaking research was conducted with a Stone Age tribe in New Guinea, the Dani, who had only two words for color, essentially "light" and "dark." Despite this linguistic feature, the Dani in the study recognized colors in the same ways that Americans did in similar studies, pointing to universal aspects of human perception and interpretation. Over the following 20 years, Rosch and other researchers extended these findings to many other categories, developing a systematic treatment of human conceptual categorization in the process.

These findings revealed several main departures from the classical theory of categorization. First, conceptual categories are built around prototypical members rather than from necessary and sufficient criteria for category membership which allow no room for "best examples." Prototypes are central elements that exemplify the category, typically maximizing the properties shared with other category members while minimizing characteristics shared with objects external to the category. For example, a robin is a more prototypical example of a bird than a penguin, and is consequently more likely to be used in reasoning about birds than either penguins or a categorical definition such as "feathered biped." Prototype effects thus account for a wide variety of cognitive activity including judgments based on concrete carriers rather than a complete and coherent set of defining criteria, development and application of mental images, and analysis according to central features and an accumulation of past experience.

Second, categories contain internal structure with certain "basic levels" at which objects are most readily identified. The work of Roger Brown, Brent Berlin, and Eleanor Rosch (Gardner, 1985; Rosch, et. al., 1976) established that humans most easily classify objects at a middle level of generalization rather than building perceptions up from atomic images. We do this because the most conceptually simple categories are not necessarily the most structurally simple. This basic level is determined by the convergence of gestalt perception, bodily movement, and an ability to form mental images. Specifically, in terms of perception, it is the level of categorization at which members have similarly perceived overall shapes and are easily perceived as a single unit. The use of similar motor actions for interacting with members is another physical stimulus establishing a natural conceptual category. Finally, when a single mental image can reflect the entire category, the act of conceptualizing that category becomes more cognitively simple.

As a result of their conceptual simplicity, basic level categories are the easiest to identify and are the level at which children first understand and name concepts. They are often marked linguistically with the shortest primary lexemes and by their use in neutral contexts. Prior to Rosch's work, fundamental objects of perception were taken to be atomic, but a key feature of basic level categories is that they are, in fact, highly structured. Categories such as "dog" or "chair" are composed of subcategories, individual members have parts and configurations, and these categories are interrelated through superordinate categories such as "animal" and "furniture." Consequently, even after complex hierarchical concepts develop, basic level categories provide structure for the organization of most knowledge.

Finally, boundaries between categories are fuzzy rather than fixed. Members may exist peripherally to a category or even on the boundary of otherwise conflicting categories. In the classical theory this prospect would render meaning impossible, however, choices about categories are not arbitrary but based on our experiences and interactions with our environment. Without being reductionistic, Rosch recognized that the cognitive mechanisms behind these choices flow naturally from physiological and sociological principles, thus giving human experiential meaning to these natural categories.

Do Metaphors Carry Cognitive Content?

The rejection of a classical theory for human categorization opens the possibility to explain some aspects of cognition through non-propositional forms such as metaphor. In so doing, it is first necessary to establish the ability of such forms to, in fact, carry cognitive content.

Davidson's Critique

Donald Davidson (1978) suggests that metaphors do not carry any cognitive content apart from the literal meanings of the constituent words and that any additional role is simply to "nudge," "intimate," or "provoke" attention toward some similarity. Max Black (1979) challenges this perspective by noting that the literal meaning in a metaphor is almost always patently false. Thus, the claim that nothing more is meant by a metaphor amounts to claiming it says nothing at all. If the focus is placed on the role of directing attention, Black argues that a soliloquizing thinker using metaphor cannot convincingly be described as trying to nudge or provoke himself to notice a likeness when he clearly must have already made such an observation. Instead, Black asserts, although much of what is being said with a metaphor may not be propositional in nature, it may certainly provide a "vision" or "view" which "is compatible with its also *saying* things that are correct or incorrect, illuminating or misleading, and so on (Black, 1979 original emphasis)."

Lakoff's Embodiment and Metaphorical Projection

From George Lakoff's perspective, abstract concepts get their meaning from structure projected metaphorically from the domain of embodied experience (Lakoff, 1987; Lakoff & Johnson, 1980). Lakoff's theory of meaning is grounded in a philosophical perspective he calls "experiential realism." He offers this perspective as an alternative to objectivism while casting both as special cases of basic realism, by which he means that both are committed to the existence of a real world, external to human beings and including the reality of human experience. Both perspectives address matters of cognition by presuming a link between human conceptual systems and other aspects of reality. For objectivism, this link is a direct representation of the world in classical terms. Lakoff rejects the possibility of direct representation, favoring "embodied experience," which is not purely internal but is constrained at every instant by the need to interact with the world in a manner to successfully function within it both physically and socially. Truth, according to Lakoff, is based on coherence with constant, real experience as opposed to an objectivist perspective where internal human factors must be excluded from any theory of meaning. Even though these experiences do not have a unique internal interpretation and cannot completely determine conceptual structure, he also rejects total relativism in maintaining a commitment to the existence of a real world external to human beings and its central role in conceptual formation.

Direct experience leads to a direct understanding of many concepts. Aspects of a situation are directly experienced if they play a causal role in the experience, and these aspects are directly understood if they are preconceptually structured. Although not all understanding is direct, Lakoff claims that embodied concepts still lie at the heart of all meaning. To support this claim, he must show that preconceptual, bodily experience is sufficiently structured to give rise to all conceptual structure. Lakoff proposes two sources for this structure: basic level categories (based on Rosch work) and kinesthetic image-schemas. The latter are deeply held (almost physically perceived) psychological images reflecting structures that recur in everyday life. They, in turn, structure perception and bodily movement. For example, our many repeated interactions with containers of various sorts lead us to perceive various aspects of many situations and concepts in terms containers (e.g., our bodies, homes, social groups, mental states, and mathematical sets). Kinesthetic image-schemas consist of key structural elements and are ordered by a basic logic, which flows from their configuration. For the container schema, the structural elements are an interior and an exterior separated by a boundary, and according to this configuration, the logic of the schema is based on the observation that everything is either inside or outside of a container.

Once Lakoff has established the existence of rich structure in embodied human experience, he then argues that all human understanding is based on the transfer of this structure to abstract concepts. Primarily, this argument is accomplished through metaphorical projection. A structured, experiential domain serves as a source to understand a more abstract target through mapping elements and relationships. There is an experiential basis for the metaphors themselves as well as for their source domain. For a particular metaphor to be natural and motivated by the structure of our experience, the source must be understood independent of the metaphor. Structural correlations in our daily experience must also motivate the details of the metaphorical mapping as well as the selection of that domain over other possibilities. In explicit uses of metaphor, analysis may occur within the context of the source and implications transferred to the target domain. In more unconscious use of metaphor, the effects are produced through subtle linguistic connections, with mental imagery, or by other indirect means.

It is important to recognize that this body of work was not intended to comprise a theory of learning. Lakoff, Johnson, and Núñez do not explicitly attempt to account for the specific cognitive mechanisms involved either in the interpretation and use of conceptual metaphors or in the development of complex understandings based on these metaphors. This work is based purely on linguistic analyses in which the evidence is drawn from standard imagery and word usage. It is not based on studies involving actual individuals either developing or using metaphors for any specific purpose. From this methodology, they are not able (and, for the most part, do not attempt) to address idiosyncratic versions of new learners' metaphors and their replacement with the standard ones. Instead, they focus on explicating the structures of fully developed mathematical concepts in terms of mappings to other domains. Although Lakoff and Johnson (1980) suggest that "metaphors are capable of giving us a new understanding of our experiences" and provide several examples of what they call "imaginative and creative" metaphors, they offer no suggestions about what cognitive processes might be involved. Lakoff and Núñez (Lakoff, 1987; Lakoff & Núñez, 2000) addresses the matter by alluding to a process of mixing antecedently existing schemas in a "conceptual blend" to form something new, but the details are similarly vague.

How Metaphors Create and Convey Meaning

In this section, we discuss various perspectives about the types of content that may be conveyed metaphorically, the mechanisms by which it is accomplished, and whether metaphor can create new perspectives rather than simply reporting antecedently existing ones. First, we address three particularly problematic aspects of the pervasive wisdom about metaphor, which characterize it as carrying only emotive content, as primarily based on intuition, and as being an oblique form of comparison. We then turn to three attempts to provide a coherent treatment of the mechanisms involved in metaphor. First, we consider an approach offered by Monroe Beardsley's (1958, 1978) attempt to explain how metaphors manage to communicate which associations are to be transferred from one domain to another through a treatment of language as possessing layered meanings. Next, we present Israel Scheffler's (1979, 1986) view that is based on the role of contextual cues. Finally, we give a more extended treatment to the theory of Max Black (1962a, 1977, 1979), who seeks to understand metaphor as a dynamic "interaction" between conceptual domains in which new perspectives are literally created.

Intuition Does Not Account for Metaphorical Meaning

As argued above, the meaning of a metaphor cannot grow out of the literal meanings of its components. Quite to the contrary, metaphorical meaning is capable of outstripping what is available within the standard lexicon. Faced with these strong claims, it is tempting to conclude that because no literal analysis is possible to explicate a metaphor, the gap between the new meaning and past literal applications must then be bridged by an act of intuition. Beardsley labels this perspective "supervenience" and Scheffler calls it "intuitionistic," both referring to the mystical nature of the implied process by which one divines metaphorical meaning. The two main theses supported by this view are that metaphors cannot be replaced by literal equivalents and that there cannot be a formula to guide their interpretation.

Scheffler (1979) points out that the theses of anti-replaceability and anti-formula are independent of one another. Even if there is no systematic method for deriving the metaphorical meaning of an expression, that meaning may still be shared by a literal equivalent. More consequentially, anti-replaceability does not imply the anti-formula thesis as suggested in the logic that leads to the intuitionistic conclusion. A formula may indeed exist to provide meaning in an extended version of the language in question, as is done in the coining of new literal terms. The greatest difficulty, however, with intuitionism as a basis for a theory of metaphor is that it merely offers a name to the mystery of how metaphors carry meaning and provides no explanation of the mechanisms involved.

Metaphors Convey More Than Emotion

One commonly espoused view of metaphors is that they do not convey literal information but are powerful only to the extent that they evoke emotion. Versions of this "emotive" perspective either claim that metaphors carry no cognitive (non-emotional) content or allow for the metaphor to be replaced by some literal equivalent. Either way, the force of meaning is seen as contained in the arousal of feelings. Some versions concede that through their emotive power, metaphors may add to the cognitive power of a language. Such concessions, however, render any claim that the primary function of metaphor is emotive difficult to defend. As Scheffler (1979) notes, "Once distinctive cognitive effect is conceded to metaphor, how can it be said that emotive function is in every case dominant in metaphorical expression?"

Furthermore, Scheffler (1979, 1986) points out that there are no clear lines between emotive and non-emotive cognitive content for any type of statement. Although metaphorical expressions may arouse emotions, they are not necessarily unique in this regard. Literal terms such as "neutron bomb" or "leukemia" may be more emotionally powerful than certain metaphorical expressions such as "a sparkling intelligence." Finally, Scheffler adds that even though metaphorical expressions may convey emotion, they can also carry substantive cognitive content. As a simple example, he suggests that while the phrase "sharp wind" might arouse negative feelings associated with sharp objects, it is important to notice that this reference is extensionally divergent from "wind."

Metaphor is not Elliptical Simile

In opposition to the intuitive and emotive approaches, several theorists have asserted that metaphors do, in fact, carry cognitive content and that their interpretation is guided by a set, although perhaps very complex, cognitive process. These claims probably originate in Aristotle's definition, which essentially lays out four different ways that one set of ideas may replace another. The various labels given to this perspective emphasize several distinct aspects of the claims in this category. Specifically, there is a deterministic mechanism (Scheffler's term is "formulaic") that replaces the metaphorical statement (Black calls this "substitution") with some type of literal equivalent (Beardsley refers to a "literalist" perspective).

The standard model for determining the literal substitute is through comparison. Aristotle suggests metaphor is an expression of likeness in his description of simile as differing "only in the way it is put; and just because it is longer it is less attractive. Besides, it does not say outright that 'this' *is* 'that,' and therefore the hearer is less interested in the idea" (Aristotle, 1929). Likewise, Davidson, while denying any meaning beyond the literal sense of the words, suggests that “a metaphor producer is drawing attention to a resemblance between two or more things” (Davidson, 1978).

Black (1962a) places this type of comparison as part of a more general view of “figurative” language in which the author provides a transform of the intended meaning; the listener’s task is to apply the inverse of that transform to obtain the original meaning. In irony, the transform yields a phrase with opposite meaning; in hyperbole, it is exaggeration; in metaphor, the basis for substitution is similarity or analogy. He offers a familiar illustrative example of interpreting metaphor as comparison,

> When Schopenhauer called a geometrical proof a mousetrap, he was, according to such a view, saying (though not explicitly): “A geometrical proof is like a mousetrap, since both offer a delusive reward, entice their victims by degrees, lead to disagreeable surprise, etc.” This is a view of metaphor as a condensed or elliptical simile. (Black, 1962a)

The goal in any “formulaic” approach is to offer an analytic method of determining a literal equivalent for a metaphor, eliminating any need (or possibility) for ingenuity in interpretation, that is, to place the workings of metaphor on sound scientific ground. Many current philosophers argue, however, that instead of being problematic, creativity is essential to metaphor. Black, for example, claims that “We need metaphors in precisely the cases where there cannot be a scientific answer to an objective question of similarity” (Black, 1962). He elaborates on this point:

> Suppose we try to state the cognitive content of an interaction-metaphor in ‘plain language.’ Up to a point, we may succeed in stating a number of the relevant relations between the two subjects (though in view of the extension of meaning accompanying the shift in the subsidiary subjects

> implication system, too much must not be expected of the literal paraphrase). But the set of literal statements so obtained will not have the same power to inform and enlighten as the original. For one thing, the implications, previously left for a suitable reader to educe for himself, with a nice feeling for their relative priorities and degrees of importance, are now presented explicitly as though having equal weight. The literal paraphrase inevitably says too much - and with the wrong emphasis. One of the points I most wish to stress is that the loss in such a case is a loss in cognitive content; the relevant weakness of the literal paraphrase is not that it may be tiresomely prolix or boringly explicit (or deficient in qualities of style); it fails to be a translation because it fails to give the insight that the metaphor did. (Black, 1962)

Ironically, the failure of similarity to provide a scientific basis of determination is probably the greatest weakness of the substitution view. Specifically, similarity yields a trivial relation, as any two things are similar in some way. Thus, this approach offers no explanation of how specific similarities in objects, attributes, and implications are selected in a metaphor for transfer. A fix requiring attention to intent, salience, or importance is contradictory to a formulaic perspective because it necessarily requires an appeal to context (although, as we shall see later, an appeal to context is precisely the approach taken by Scheffler after rejecting the possibility and need for a formula).

Beardsley's Controversion Theory

Monroe Beardsley (1958, 1978) attempts to solve the problem of how attribution works in metaphor in an approach he calls "controversion." This approach is based on a view of language as having layered interpretations. When primary readings are blocked by obvious self-contradiction or falsehood (as when claimed that man is a wolf), metaphorical readings based on peripheral properties may emerge. Beardsley proposes two principles that guide the emergence of these properties. First, the Principle of Congruence asserts that only properties fitting to the primary subject are applied. This principle would exclude, for example, anything attributable to wolves that cannot be applied to man, such as having four legs. Second, the Principle of Plentitude associates with the primary subject all possible fitting connotations. Thus, all attributes that can be applied together form the meaning of a metaphor. This point of view is particularly powerful in the case of poetic metaphors, in which readers may observe levels of meaning not explicitly considered by the author. In addition, it is very much in line with suggestions that a poem "means" everything that it can mean.

The potential range of interpretation for either a conceptual or literary metaphor may well encompass much of the vast ground between Beardsley's two principles. Scheffler (1979), however, argues that this approach, which he calls "intensionalism," is still too broad. A wolf is neither a tree nor identical with Aristotle, and insisting on including such meanings as part of the metaphor "man is a wolf" borders on the vacuity of the comparison view. One amendment to the theory of intensionalism might rule out attributions which are either tautological or absurd. In response to this possibility, Scheffler asks, once the author's mind is introduced to make such decisions, how one can restrict it to the mere exclusion of the non-sensible. Further difficulties for this view are that a metaphorical reading may be prompted by some condition other than self-contradiction or falsehood and that literal meaning is not always superseded and may be retained, as in the case of multiple meanings.

Scheffler's Contextualism

Israel Scheffler (1979, 1986), drawing on the ideas of Nelson Goodman (1968), outlined an approach to the question of how attribution is accomplished in metaphors. Given his objections to the previously described views, his perspective stresses the role of ingenuity through the use of old language to solve a new problem and to break new ground. Arguing that an appeal to context is not a defect in a theory that rejects the idea of a formula, Scheffler places primary emphasis on characteristics that are salient or important in the context in question. He notes that,

> even constancy of literal application is usually relative to a set of labels: what counts as red, for example will vary somewhat depending upon whether objects are being classified as red or non-red, or as red or orange or yellow or green or blue or violet. What the admitted alternatives are is of course less often determined by declaration than by custom and context.

Guidance in the process of attending to such contextual cues derives from salience and past applications and exemplifications.

This approach, as the others described above, attempts to explain how metaphors work largely in terms of the mechanisms for determining what attributes of the metaphorical subject are to be applied to the literal subject. This approach still does not sufficiently account for the internal dynamics involved when using metaphor to explore new ideas. Scheffler (1979) raises this issue only briefly by identifying its surface features. In these cases, he says, the utterance serves as an invitation to author and listener to explore the context for significant shared characteristics. A metaphor may lead one to rethink old material in light of new categorization or to consider newly discovered phenomena in available terms. In addition, a metaphor "does not simply report isomorphisms but calls them forth afresh to direct, and be tried by, further investigations" (Scheffler, 1979).

Other than this brief allusion to the role of metaphor in exploration, Scheffler does not develop the mechanisms for how metaphors generate new perspectives. In contrast, Max Black focuses his "interaction" theory on precisely the question of how metaphors manage to generate new insights. Though much of Black's work significantly precedes Scheffler's, it provides a far richer functional approach to metaphor.

Black's Interactionism as a Generative Theory of Metaphor

Black's central claim, based on his extension of Ivor Richards' (1936) earlier work, is that one must regard the literal and metaphorical subjects together, as an interacting system (Black, 1962a, 1977, 1979). Specifically, attribution in a metaphor is achieved through what Black calls the system of associated commonplaces, characteristics of the metaphorical subject that are readily and freely evoked and applied in a dynamic process of mutual influence with the literal subject. This process requires two levels in which thoughts of different things must be held active together. First, the meaning of the metaphorical subject is distinct with and without the context of the metaphor. Second, this system imposes an extension of meaning on the literal subject, and one must simultaneously attend to both the new and old meanings. Black stresses that the implications arise from the interaction itself and that different applications produce different interplays with the metaphorical subject (e.g., plowing one's memory vs. plowing through a meeting).

Strong metaphors, such as those that would be necessary for supporting creative thinking, according to Black, require conceptual innovation and are ontologically creative. In other words, he means that the relevant concepts involved are required to change in response to one another through the application of the metaphor. The resulting perspective that is created is one that would not otherwise have existed. This claim can be illustrated with a distinction that Black (1977) draws between "seeing one thing as another thing" and "metaphorical thinking." An example of the former is imagining the Star of David, as in Figure 1, "seen as" being composed of (i) two equilateral triangles, (ii) a regular hexagon with an equilateral triangle attached to each of the six sides, or (iii) three congruent parallelograms. Seeing the Star of David in these different ways may

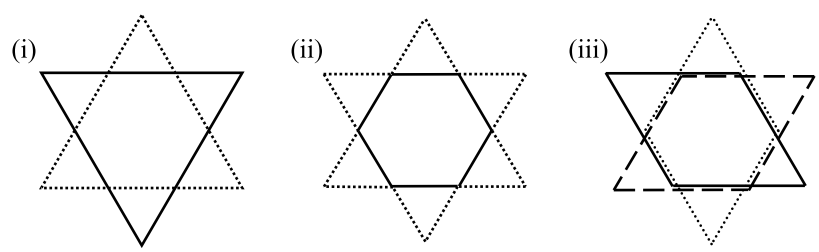


Figure 6. The Star of David "seen as" composed of different geometric figures.

yield discovery, but lacks conceptual innovation. It may support the development of implications among the concepts involved, but not in ways that change one's conceptualizations. The concepts of triangle, hexagon, parallelogram, and congruence, for example, are all applied as they previously existed with nothing new or creative demanded of their independent conceptual status.

This example differs from what Black refers to as metaphorical thinking. He suggests considering what is involved in thinking of the following diagrams (illustrated in Figure 7) as triangles: (i) three curved segments, (ii) a single line segment, and (iii) a base segment connecting the origins of two parallel rays. In

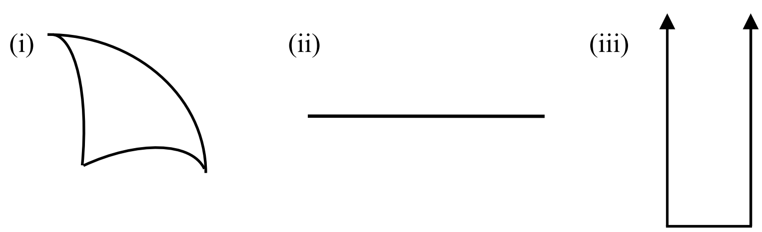


Figure 7. Metaphorical triangles.

so doing, one cannot simply apply an antecedently formed concept of triangle as-is; something new and actively responsive to the situation is required of all concepts involved. Certain aspects of the concept of triangle are highlighted while others are suppressed in the process of applying the metaphor, and the ways in which they are relevant are highly context-dependent. If pursued, the implications can support a degree of discovery that leads far beyond one's original thoughts, in this case perhaps, leading even someone familiar only with Euclidean geometry to ideas reminiscent of spherical or projective geometries.

At the same time, the metaphor creates a perspective of the three images that would not have otherwise existed, that is, it is ontologically creative. A metaphor used as a cognitive tool in this fashion can generate a new way of thinking. Black (1977) notes that the producer of a metaphor

> is employing conventional means to produce a nonstandard effect, while using only the standard syntactic and semantic resources of his speech community. Yet the meaning of an interesting metaphor is typically new or 'creative,' not inferable from the standard lexicon.

To emphasize this generative aspect of metaphor, he draws the analogy to the role of slow motion photography in the creation of the slow motion appearance of a galloping horse. Did such a perspective exist, he asks, prior to the invention and application of these human tools?

Black (1962a, 1977) distinguishes two key characteristics of *strong metaphors* that provide the type of cognitive power described above: emphasis and resonance. *Emphasis* refers to the commitment of the author to the metaphorical domain, reflecting whether only that subject may serve the purpose of the metaphor or other alternatives may serve as well. Black suggests, "Emphatic metaphors are intended to be dwelt upon for the sake of their unstated implications: their producers need the receiver's cooperation in perceiving what lies behind the words used." Aristotle's description of metaphor as lending "style, clearness, charm, and distinction" to an otherwise drab statement would qualify as the opposite of emphatic since these are decorative and expendable roles. The second key characteristic of a strong metaphor, *resonance,* is the degree to which the metaphor supports implicative elaboration. Resonant metaphors provide the complexity and richness of background implications necessary for generating new ways of perceiving the world.

Scheffler criticizes interactionism for allowing too broad an interpretation for the metaphorical implications to be transferred. The requirement that characteristics be obvious relative to the metaphorical subject but non-obvious with respect to the principal subject allows for too many obvious characteristics. For example, a wolf has four legs, but this is not likely part of the meaning of the metaphor "man is a wolf." Strengthening the requirement to exclude any characteristics not commonly denied to the literal subject is also insufficient: simply note that a wolf is not able to whistle a tune or speak French. Further, requiring spontaneity of thought for the characteristics still falls short, as a wolf has sharp teeth and can run fast, yet these would not normally be considered part of the meaning of the metaphor "man is a wolf."

Black's interactionism probably allows for more of an influence from contextual cues than Scheffler recognizes. The dialectic between literal and metaphorical domains, for example, occurs in a context from which the new features directly emerge. Black (1962a) notes that "The new context [of a metaphorical term] imposes extension of meaning upon the focal term." Any act of inquiry involving a metaphor is situated in some context which must be reflected in any such resulting broadened understandings. Black provides an example,

> When Churchill, in an infamous phrase, called Moussolini "that *utensil,*" the tone of voice, the verbal setting, the historical background, helped to make clear *what* metaphor was being used… This is an example, though still a simple one, of how recognition and interpretation of a metaphor may require attention to the *particular circumstances* of its utterance. (Black, 1962a original emphasis)

Black also explicitly notes that deviant implications for the literal uses of a metaphorical reference may be established ad-hoc prior to using it metaphorically, a matter of establishing context.

The Relationship between Metaphors and Models

Various researchers in cognition have posited descriptions of "models" or "mental models" to account for human reasoning in ways similar to the descriptions of metaphor given above. James Greeno (1991), for example, describes knowledge, in contrast to an information-processing perspective, as an ability to find and use resources within the landscape of a conceptual environment. An important type of non-propositional reasoning from Greeno's perspective is the use of a mental model with properties and behavior similar to an object or situation being represented. Once the model is constructed, it provides affordances to reasoning through implicit constraints and symbolized features so that inferences may be based on simulations or enactments. Rather than consisting of a network of schemata, concepts are built up from implications of the spatial properties of the conceptual environment discovered through interaction.

Greeno's description of constructing and mentally manipulating imagery is similar to that of Johnson-Laird, which is based on his groundbreaking research on an array of cognitive activities ranging from reasoning about syllogisms, to inference, word meaning, grammar, and comprehension of discourse (Johnson-Laird, 1983). By constructing specific "mental models" for each of these topics, he was able to show that reasoning could proceed along lines that did not use classical logic. Based on the type and complexity of the model required to solve specific problems, Johnson-Laird was able to predict which ones would pose the greatest difficulty to subjects, to differentiate among least skilled subjects according to performance of specific manipulations of the models, and to identify factors that would help subjects improve. For example, in the case of syllogisms, the process involved three steps: 1) constructing a representation of the first premise, ideally with as few entities as possible, 2) add the information of the second premise to the representation accounting for all possibilities in separate models, and 3) searching each model to check for inconsistencies with the premises.

This type of representation is different from what Black described as something that provides a flash of insight. These mental models are iconic in nature in that they denote by virtue of their own characteristics (e.g., an image of an entity satisfying both premises of a syllogism). They do not support the exploration of implications beyond what was intentionally built into their structure. Thus, while providing for efficient processing of structures and affording the creation of new knowledge, they do not generate a fundamentally new way of seeing a subject.

A type of model that is more closely related to metaphor is what Black (1962b) called a "theoretical model." This is the type of model used by Clerk Maxwell in imagining an electrical field as an "imaginary" incompressible fluid. If treated as a "heuristic fiction" in which ontological disbelief is merely suspended, the model provides a well-understood domain that is imagined to be isomorphic with respect to certain structures and properties so that inferences may be transferred. This description of a model is similar to Lakoff's description of metaphor in which structures and relations are mapped from one domain to another.

Such a perspective, however, offers no explanation of the original phenomenon. Thus it becomes compelling to risk making existential claims (and "the dangers of self-deception by myths") to attain that explanatory power. Black traces, for example, Maxwell's own transition from treating his fluid model for electrical fields as a heuristic model to treating it as reality, quoting him as saying that Faraday's lines of force "must not be regarded as mere mathematical abstractions. They are directions in which the medium is exerting a tension like that of a rope, or rather, like that of our own muscles." In terms of emphasis and resonance, Maxwell's model, at this point, has become very similar to a "strong metaphor." Black notes the types of admonitions against such figurative thinking in scientific endeavors,

> the crucial question about the autonomy of the method of models is paralleled by an ancient dispute about the translatability of metaphors. Those who see a model as a mere crutch are like those who consider metaphor a mere decoration or ornament. (Black, 1962b)

There is, however, a key distinction between Black's descriptions of metaphorical reasoning and the existential use of theoretical models. First, while the theoretical model may eventually be interpreted *as* reality, it begins intentionally as a model. The analogue is not necessarily true of metaphor, as a producer may unconsciously apply a domain metaphorically, even in cases of strong metaphor. Second, Black himself points out that,

> Metaphor operates largely with *commonplace* implications. You need only proverbial knowledge, as it were, to have your metaphor understood; but the maker of a scientific model must have prior control of a well-knit scientific theory if he is to do more than hang an attractive picture on an algebraic formula. Systematic complexity of the source of the model and capacity for analogical development are of the essence. (Black, 1962b)

This description of scientific systematicity is far stronger than what most calculus students are likely to display. Consequently, this aspect of Black's perspective is not necessary to a theoretical perspective on the development students' understanding of limit concepts.

No Complete Theory of Metaphor

It is important to point out that the philosophers cited above generally warn that we do not possess an adequate general theory of metaphor. Scheffler (1979) acknowledges the inherent vagueness in an appeal to context saying, "We may be able to piece together some plausible histories of metaphors, but no general theory of guidance can be offered." Black (1977) refers to his own work as only preliminary suggestions toward a theory, and is later self-critical on this point:

> In my opinion, the chief weakness of the 'interaction' theory, which I still regard as better than its alternatives, is lack of clarification of what it means to say that in a metaphor one thing is thought of (or viewed) *as* another thing. Here, if I am not mistaken, is to be found a prime reason why unregenerate users of appropriate metaphors may properly reject any view that seeks to reduce metaphors to literal statements of the comparisons with the structural analogies which *ground* the metaphorical in sight. To think of God *as* love and to take the further step of identifying

> the two is emphatically to do something more than to *compare* them as nearly being alike in certain respects. But what that 'something more' is remains tantalizingly elusive: we lack an adequate account of metaphorical thought. (Black, 1979)

Although we do not possess a complete characterization of what happens cognitively as one thinks with a metaphor, this section has outlined a number of features of metaphorical thought relevant to the development of new ideas. The following chapter draws heavily from the interactionist perspective to develop a framework for analyzing student language and thinking. We expect to see features of the students' thought processes that would otherwise be hidden by organizing data according to contexts employed by various students, creative statements made within those contexts, interaction between intuitive and abstract domains, etc. If students' conceptualizations of important mathematical concepts are highly influenced by thinking within various nonmathematical contexts (appropriate or not), an understanding of metaphorical reasoning is likely to help uncover these features.

Chapter 3: Theoretical Perspective

What is metaphorical reasoning, and how is it related to students mathematical reasoning? This chapter first describes an exploratory study with second-semester calculus students investigating their refinement of knowledge about content learned in the previous semester. Two themes emerged from this study that led to the framing of the current research in terms of metaphors: 1) students were observed to take cues from linguistic structures in the problems and their own memory and 2) students built non-mathematical schemas to use in reasoning about the mathematics. After the discussion of the exploratory study, we outline the details of theoretical perspective used to guide the development of the study and the subsequent data analysis.

THE EXPLORATORY STUDY

During the spring semester of 2000, we conducted task-based interviews with 15 students enrolled in a second-semester calculus course at the University of Texas. All of the students were also members of intensive workshops that met 6 hours per week to work on challenging calculus problems under the supervision of an experienced graduate student teaching assistant. Eight interviews, lasting 2-2.5 hours each, were conducted with pairs of students. (One student participated twice.) Audio-tapes of the interviews were transcribed for analysis of emergent themes.

Purpose

The exploratory study was used to identify themes for further research. Our focus was to watch students reflecting on their own work to give an account of small-order building and restructuring of knowledge. Specifically, we were interested in the processes by which students prepare for larger conceptual shifts in their understanding and through which they reorganize their knowledge after these shifts have occurred.

Interviews

The interviews began with a brief introduction of our study and a loosely structured opening interview on the students' mathematical backgrounds. We then separated the students for approximately one hour to work on the following problem taken directly from the 1997 Advanced Placement® Calculus AB exam:

> A particle moves along the y-axis with velocity given by $v(t) = t\sin(t^2)$ for $t \geq 0$.
>
> (a) In which direction (up or down) is the particle moving at time $t = 1.5$? Why?
> (b) Find the acceleration of the particle at time $t = 1.5$. Is the velocity of the particle increasing at $t = 1.5$? Why or why not?
> (c) Given that $y(t)$ is the position of the particle at time t and that $y(0) = 3$, find $y(2)$.
> (d) Find the total distance traveled by the particle from $t = 0$ to $t = 2$.

During this portion of the interview, we encouraged students to verbalize all of their thoughts as they worked, and we frequently asked them to clarify the meaning of various statements. On some occasions, we interrupted students' work to directly address a misconception if a student became unable to make any progress, correct or otherwise. After the students were finished working on the problems, we brought them together to collaborate on writing a solution incorporating the best aspects of each of their individual work. Finally, we conducted a brief exit interview to obtain the students' thoughts about the activity.

Exploratory Study Results

The two main themes to emerge from the data were that 1) students used several linguistic cues to access their memory when trying to solve the problem or to explain the work they did and 2) students built or accessed conceptual schemas to organize their thinking about the problem which resulted in highly context-dependent thinking.

Linguistic Cues

One of the most striking examples of a linguistic trigger was the students' uses of the word "direction" in Part (a) of the problem to cue the (incorrect) decision to try to find position. An appropriate response would be to evaluate $v(1.5)$ and conclude that since the velocity is positive, the particle is moving up. Even though this question is routine and the solution does not actually involve the use of any calculus, almost all students had extreme difficulty with it. Several students began by taking the antiderivative of $v(t)$. Some first took the derivative, reconsidered their options, and then proceeded to take the antiderivative. This behavior on a simple problem asking about the relationship between position and velocity was baffling, especially since Part (b) of the problem asks the analogous question about the relationship between velocity and acceleration, and none of the students had difficulty answering that.

Several possible explanations could be offered for this. First, the students expected a “calculus problem” to necessarily begin with the computation of a derivative or antiderivative. A similar analysis could be offered based on students’ anticipations developed from cues while reading the problem. Specifically, students may have been operating with prematurely chosen schemas as were the subjects in the study of Hinsley, Hays, and Simon (1976). A third explanation, offered in terms of interpretations in the literature on calculus learning (e.g. Monk, 1987; Monk, 1992), suggests that under an overburdening cognitive load, many students confuse a function with its derivative.

Since students consistently did not make this error on Part b, and as the question is straightforward, the latter suggestion seems unlikely. That students were operating from certain anticipations about what is typically done in solving a calculus problem might explain why some students started off taking a derivative or antiderivative. It does not, however, offer any explanation as to why most students eventually decided that the antiderivative was most appropriate.

The most common paraphrasing of the word “direction” from the problem statement was in terms of “where the particle is going.” Focusing on the adverb “where,” a word about position or location, may have miscued students to compute an antiderivative of the velocity function. Consider the following excerpt:

Tara: OK. I know we need to take the derivative of a function. The first derivative is velocity. The second derivative is acceleration. So I know that if they give you velocity here, that if I take the derivative once I would get acceleration. If I integrate it possibly - yeah if I integrate it I should get direction I think - yeah. - yeah. - yeah, OK - direction – [*computes* $d(t) = \frac{1}{2}\cos(t^2)$ *and* $d(1.5) = 0.314$]

I: So, what does - what does $d(t)$ tell you?

Tara: The direction. It's - I don't - yeah. It should be direction.

I: Can you say what you mean by that? So, what is the direction - so what does the 0.314 represent?

Tara: [*pause*] that OK - at time $t = 1.5$, it's moved maybe this far along the *y* axis - 0.314 units. It's moved up, because it's positive. I think that's what it means. I think I'm wrong though.

I: So, do you remember - do you remember anything about the position function?

Tara: No.

I: Does that sound familiar?

Tara: It sounds familiar, but I don't remember anything about it.

I: OK. So, when you integrate velocity, what you end up with is a position function.

Tara: Right. So this tells me the position, like where it is.

I: So, how does that differ from the direction?

Tara: Direction tells you where it's going.

I: OK. Can you tell where it's going to from the position function?

Tara: [*pause*] No. You can't.

I: Why not?

Tara: Because it's just where it is. It's stationary. I have to plug this into the velocity function. I think so. Because the velocity is direction and speed. Right? Right?

Here, Tara begins by looking for the derivative of *v* then decides instead to integrate. It is at this point that she starts talking about direction. Her statement interpreting direction, "at time $t = 1.5$, it's moved maybe this far along the *y*-axis - 0.314 units," indicates she is analyzing "direction" as a position.

It is interesting to note that Tara does posses all of the correct information she needs to tackle this problem. As soon as the word position is associated to the computation she has just performed, an entirely different schema seems to be triggered for her. She correctly states that position is “where it is,” that “direction tells you where it's going,” that you can’t infer direction from position, that she needs “to plug this [1.5] into the velocity function,” and that velocity is composed of both “direction and speed.” After this point in the interview she did not display any confusion about position and direction.

Another student, Maria, was not able to clear up this confusion so easily. She alternated through the entire interview between confusing position and direction and seemingly understanding the difference. At the end of the interview I asked her about her confusion, and she self-reported identifying position and direction and that the word direction that triggered her thought processes:

> I: By the way, I just want to ask you to think back as you were solving the problem. Was there a point where you were kind of going down the wrong path?
>
> Maria: Part a. I was going down the wrong path. And I went down the wrong path anyways. I didn't make as stupid a mistake I guess because obviously - what was I doing? I was taking the derivative of velocity to get position, then I figured out to take the integral. Like just after a while - I don't know how. I just - it just connected to me a second ago that we just did that, and it's the position. The derivative - the velocity - position is velocity - there you go. It just kind of clicked in my mind that you take the integral.
>
> I: And what was it that made you change your thoughts?
>
> Maria: I honestly don't - what made me change my thoughts? Or what made me do the whole problem wrong in - [*stops speaking*]
>
> I: What made you start going down - like trying to find - see you were working on an antiderivative here to find position instead of putting in for velocity. What do you think it was that made you go down that road?
>
> Maria: To go - because of the word direction - I just thought direction is position. Right or left. That's what made me think of position. Like I thought of position like right as I read the problem.

Notice at the beginning of this excerpt that Maria reports her error on Part (a) as "taking the derivative of velocity to get position, then I figured out to take the integral." Thus she is still confused, thinking that taking the antiderivative was necessary, despite having cleared this up for herself a number of times throughout the interview. In contrast to Tara's swift and complete recovery from the error, Maria's confusion between position and direction was much more persistent.

In one of Tara's moments of confusion, she had just noticed from looking at a graph of the position function that the particle turns around at time 1.7. She then made the claim "So x is 0 where x is 1.7." It is possible that she is able to make such an obviously contradictory statement because, in her confusion between position and direction, the phrase "x is 0" refers to direction. Specifically, she seems to be pattern matching "$x = 0$" with "derivative equals zero" indicating a possible turning point. The subsequent phrase "x is 1.7" then refers to a time (or possibly position) where this turning occurs. Thus her statement "x is 0 where x is 1.7" is really an encoded version of "at $x = 1.7$, there is a critical point at which the particle is turning around" and contains no contradiction for her.

Interestingly, students often did not remember the things that triggered certain thoughts for them, even if the thought itself was important for their understanding. For example, while working on Part d, Maria first noticed that the particle changes direction when looking at a graph of the position function. She then immediately drew the graph on her paper as part of her explanation. Later, she did not credit the graph as being useful:

I: Alright. Maria you mentioned the graph you drew. This graph. When you were grading yourself you said you decided that it should not count as a helpful diagram.
Maria: Yeah, I don't think it was helpful at all.
I: Eduardo, you seem to indicate that you thought it was [helpful] when you were going through it - so how is this helpful?
Eduardo: It's just showing that the graph - starts at 3, goes up and then comes back down. It goes down. It's going up, then it goes down before 2. So, that graph was helpful because she found *x* was - you know that there was a max point, like a maximum. Where *x* like actually turned around you know. Because the particle basically turned around and went back so that was at like *x* 1.7, she found it was here.
I: Is that different than how you actually used it?
Maria: No, that's - well I just threw in a graph because I had to. But, my thing was I think it's the same exact thing as my number line. I just drew it.

Even after Eduardo explained how he thought the graph was useful, exactly in the same way it had triggered Maria's correct thinking about the turning point, she still denied that it was a useful diagram for her.

Some pattern matching attempts were made obvious to us by their failure. For example, several students tried to recall some version of the statement "acceleration is the derivative of velocity which is the derivative of position." Typically a statement of this form was recited as if from memory, but with the words position, velocity, and acceleration seemingly randomly interchanged. The students apparently knew the basic template but did not understand the relationships. Another pattern that students often incorrectly tried to apply was some version of "positive direction is up, negative direction is down; positive derivative indicates increasing, and negative derivative indicates decreasing." In both cases, they fumbled around with the words as if just getting them in the right place would help them understand. One student, Sandra, was so focused on remembering the relationship between position and velocity that when she finally read it in the book, she tried to change Part a of the problem so that she could use her new fact: "that means that, OK, position function… If $t\sin(t^2)$ were to be $x(t)$ instead of $v(t)$ then… this would be a position function, and its derivative would help you find velocity." She did not realize that the velocity was already given in the problem so there was no need to compute it.

Other examples of triggers were based on extra-linguistic signs such as graphs or visual hierarchies. Mental images of portions of graphs (such as those showing a maximum) seemed to encapsulate sets of ideas that allow new possibilities of understanding to quickly emerge (such as the relationship of positive, zero, and negative velocity to a turning point of the particle). These cues behave slightly differently than the description of algebra students' schematization given by Hinsley, Hayes, & Simon (1976). In their study, students appeared to use contextual cues to access entire structures of knowledge. The students in this study have not yet encapsulated the mathematical structures surrounding the concept of rate of change. Consequently, they retrieved fragmented bits of information from memory which they then had to organize before applying them in productive inquiry.

Building Schemas

The six most identifiable schemas used by students in the exploratory study were labeled as slope, rate of change, automobile, motion detector, motion on the graph, and vertical motion. Three factors characterized the use of these schemas. First, most students alternately employed two or three of these schemas but rarely attempted to do so at the same time. Relatively clear boundaries in their thoughts seemed to exist separating their analysis according to each one. Second, each schema came with its own set of language, signs, and images. When using an automobile schema, for example, students would discuss the speedometer, the accelerator, brakes, reverse gear, and even the transmission in trying to explain their ideas about calculus and the motion of the particle in the problem. The third and most interesting aspect of students' schema use was that their mode of analysis changed consistently with the schema they employed.

The following excerpts of a student using three different schemas (vertical motion, automobile, and motion on the graph) were chosen to illustrate the different modes of analysis used within each schema. In the first excerpt, she uses a vertical motion schema:

Julie: I don't know if this is right, but all I did was I figured since it says for *t* is greater than or equal to zero, I said, I might as well start at zero. And then they're asking for *t* is equal to 1.5 which is greater than or equal to - so that satisfies the equation, or the inequality. Then I just plugged in zero for *t,* and I got zero, and then I plugged in 1.5 for *t* and I got 1.2. So that's why I concluded that the particle is moving up, because at *t* is greater than or equal to zero - or I made it equal to zero - at *t* is equal to zero, the answer - the output was zero, and then at $t = 1.5$ the output was

1.2, so I figured it was just going up. It goes from zero to 1.2. I don't know if that's right, though.

I: So, can you explain to me then why at 1.5 - what's happening to the particle at that point?

Julie: At 1.5?

I: I mean at time 1.5.

Julie: OK. It's higher than at time zero.

I: So, are you saying - so are you saying - so where - Do you know where the particle is at time 1.5?

Julie: At 1.2 - or approximately.

Later in the interview she is analyzing the same velocity data in terms of an automobile schema before abruptly switching to a motion on the graph schema:

Julie: Yeah. Well that helped me picture it better. The thing with a car. So on your y axis, if your velocity is increasing then you're just - you keep moving forward. And if your velocity is decreasing, you're either stopping or you're moving backward. If you have a negative velocity then you're - like (inaudible) velocity obviously you're reversing. So that would be velocity moving down.

I: OK. So - can you say anything about the position without doing other work?

Julie: Well that's I think that's - well if your velocity decreases, it could be either that you're in reverse or that you're just gonna stop. Or that you're just slowing down. So, I can't really tell then. If you're gonna go back. Because you don't - you're not necessarily going back. You can just stop but your velocity equals - it gets smaller and smaller and smaller.

I: OK. So maybe let's go back to this problem. So you got a particle moving with that velocity - and so when you plugged in - into $v(t)$, can you explain your answer in terms of what we were talking about?

Julie: OK. I plugged zero in to $v(t)$ and I got zero out of it, so it's not - at time equals zero they're not moving. Which is - makes sense. So - and then I plugged in 1.5 to the equation, and the output was 1.2. So 1.5 seconds later it was speeding up by 1.2 miles or kilometers or whatever per hour.

I: Can you say what it - so what it's going at 1.2 -

Julie: Yeah. It's going… but then again if you - I'm just picturing it like several you know - if it curves with a different maximums and minimums. Then again - it could be at the very very top of the curve, and then it could just you know let go and because of gravity, it speeds up. But it's moving down. Is that what -

I: Well, I didn't quite get it, but maybe you could say it again.

Julie: OK. All right. I'm just picturing - you know let's pretend like the sine curve, and it's at the very very top at - at one. And then the velocity would increase if it's not you know - the velocity isn't given at a certain rate then it's going down. It might just go down faster since it's going downhill instead of going uphill.

Notice Julie's analysis of velocity under the influence of each schema. While talking about vertical motion, she is confusing the velocity for position. Her assertion is that since $v(1.5) = 1.2 > 0 = v(0)$, the particle is higher at time 1.5 than it is at time zero. These values of v represent height for her. When using the automobile metaphor, her analysis of this same data is quite different. She now thinks of $v(0) = 0$ indicating that "they're not moving" and $v(1.5) = 1.2$ as representing a speed of "1.2 miles or kilometers or whatever per hour." She describes negative velocity as reversing and decreasing velocity as reversing or slowing down. Finally her sudden heartbreaking switch to thinking about motion on the graph is almost visible in the words "…but then again if you…." Here she drew a sine curve and asserted that the velocity increases because gravity pulls on it as it goes downhill. Julie does not realize that these three analyses all contradict one another since they represent different worlds to her. They are not all subordinate to a single structure that would allow her to make revealing comparisons. Throughout the interview she alternated between these three schemas, producing consistent analyses within each one but never drawing them together.

Students used the automobile schema most spontaneously. They often resorted to this set of language when working on Part b and talking about acceleration. The mathematical structure of a second derivative and its relationship to the derivative and to the original function involves at least one step of greater abstraction. Considering that many of our students were having difficulties even thinking about changes in distance or time, it seems unlikely that they were prepared to coordinate these images, grasp the structure involved in moving to a limit, and apply the result to understand acceleration mathematically. See Carlson, Jacobs, Coe, & Hsu (under review) for an analysis of this type of "covariational" reasoning. On the other hand, students have had a lifetime of experience riding in automobiles and talking about motion using the very words given in the problem. They were able to spontaneously apply that experience to talk powerfully about the problem. They were not, however, able to talk about acceleration abstractly, suggesting that the scientific concept of acceleration had not yet emerged.

Impact of the Exploratory Study on the Theoretical Perspective

Initial attempts to understand how student's were organizing and using their knowledge about limit concepts in terms of schemas did not provide any analytic power other than categorizing the contexts students discussed. It was clear, however, that the students were using these extra-mathematical schemas to try to understand the mathematical problems posed in the study. In addition, the analysis of how students were using specific language stood apart from the characterization of the students' schemas. Combining the two themes of linguistic cues and conceptual schemas led to the investigation and development of a theoretical perspective of metaphors for the design and analysis of data of the main study described in this dissertation.

THEORETICAL PERSPECTIVE

Not only do we have an incomplete understanding of how metaphor works, but even characterizing what metaphor is has proven difficult. One only need note the few definitions of metaphor in the related literature, which are offered are often couched in metaphorical terms themselves. Nonetheless, in order to develop a theoretical perspective that provides a framework capable of guiding research and interpreting results, we must begin with, at least, a characterization of the phenomenon that we wish to study.

The central aspect of the perspective used in the main study is a version of Max Black's interaction theory of metaphors augmented with both a perspective on schemas from cognitive psychology and John Dewey's description of inquiry as the application of mental tools against a problematic situation. In this section, we outline the key aspects of each of these perspectives then proceed with a discussion of their combination for the theoretical perspective of this study and their subsequent impact on methodology.

Characterization of the Phenomenon under Study

For the purposes of this study we characterize metaphorical reasoning as *the active application of a schema of spontaneous concepts as a mental tool against a problematic situation and the reciprocal evaluation of that tool against the problem.* Within the theoretical framework outlined below, the potential for generativity with respect to one's perspective of both conceptual domains lies precisely in this dialectic interaction. The remainder of the chapter is devoted to exploring the meaning of this statement and its translation into an operable theoretical perspective to guide the research in this study.

We may delineate between two types of descriptions of knowledge. On one hand, a *structural description of knowledge* includes a characterization of the objects of knowledge, their relationships to one another, the ways in which they are stored in and recalled from memory, knowledge about strategies and applications of the knowledge, etc. In this study, we apply a perspective on schemas from cognitive psychology to elaborate the structures of the metaphorical domains being constructed for particular situations. In addition to the organization of knowledge, one might also account for its use. Thus, a complementary *functional description of knowledge* includes a characterization of how ideas are applied to address specific problems, the activities involved in understanding something new, the details of each step in a process of inquiry, etc. John Dewey's description of inquiry as the application of mental tools against a problematic situation provides the framework for the actual processes of exploring mathematics with metaphors used in this study.

Schemas

In an attempt to capture the current usage of various terms in the literature on the psychology of learning, Alexander, Schallert, and Hare (1991) noted that the term schema did not have a consistent meaning among researchers. These interpretations vary widely from graph theory metaphors of connected nodes of information to computer science metaphors of artificial intelligence and fuzzy logic to notions of explicit maps and procedures for solving a problem. Alexander, Schallert, & Hare proposed their own interpretation of this term that fits into the broader framework resulting from their analysis of the literature (a small portion of the framework is depicted in Figure 8). Specifically, *explicit knowledge* refers to any knowledge, conceptual or metacognitive, currently residing in a person's working memory; it is conscious knowledge (represented by the two intersecting planes in the figure). *Tacit knowledge* on the other hand denotes all knowledge that resides in a person's memory that is not currently the object of thought (represented by the space between the planes). Tacit knowledge is not organized in a rigid, well-connected set of "schemas" to be recalled as whole chunks. It is instead an "unrealized and unanalyzed" collection of ideas, senses, connections, etc. from which

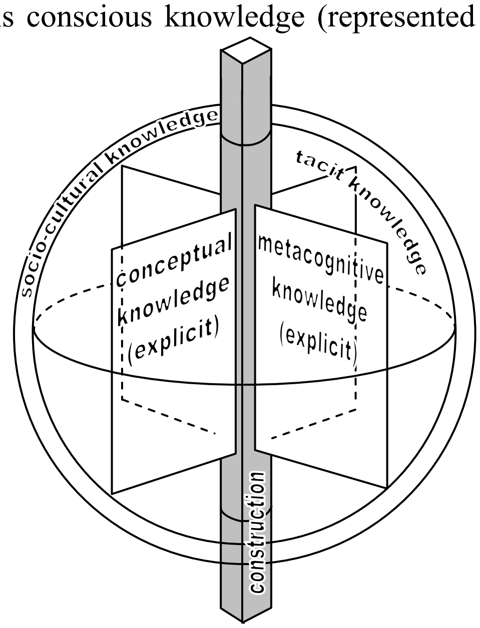


Figure 8. Alexander, Schallert, & Hare's knowledge structures.

one may retrieve bits of knowledge in different forms each time it is accessed.

It is at this stage, as explicit knowledge, that a schema is *constructed* to fit the needs of the individual situation. In this sense, it is an "instantiation," or "particularized model of the physical, social, or mental world that is constructed at the interface of prior knowledge and ongoing processing demands." The construction of such schemas occurs at "the point of contact between the learner's prior knowledge and other human processes," and is thus shown "as extending beyond the confines of the prior knowledge sphere." That is, new knowledge can be created in this dynamic.

In this study, we investigate students' applications of precisely this type of schema. In the exploratory study, prior knowledge about cars, motion detectors, and other non-mathematical domains were cobbled together in ways that allowed students to respond to a particular mathematical problem.

Instrumentalism

While it is important to account for the ways in which concepts are interrelated, human knowledge cannot be characterized by its structure alone. Equally important are the *functional* ways in which those knowledge structures are applied against specific problems. In order to address aspect of knowledge in the theoretical perspective of this study, we turn to John Dewey's "instrumentalism." Larry Hickman (1990) describes Dewey's commitment to functionalism by giving a physical example:

> Eye, arm, and hand may be treated structurally, as objects: in use, however, they function as tools for grasping and handling. But grasping is

> an activity that when actively engaged resists attempts to separate that which grasps from that which is grasped: "…Whenever they are in action they are cooperating with external materials and energies. Without support from beyond themselves the eye stares blankly and the hand moves fumblingly. They are means only when they enter into organization with things which independently accomplish definite results." What is grasped and what grasps may be analyzed *after* grasping has been attempted or accomplished, and *on the basis of* that functional activity. But Dewey argues that to say (as is common in philosophical treatments of technology) that there exists *before* that activity takes place something essentially grasping and something essentially grasped, is to commit what he terms "the philosophic fallacy": the taking the results of inquiry as prior to it.

Thus, to understand a human activity, it is crucial to examine the function of applying the relevant tools against problematic aspects of the situation. Dewey describes such an active role of tools in the process of inquiry as *technological*. In its modern use, the word technology typically refers to physical inventions rather than cognitive tools used in mental activity. Dewey argued, however, that such Cartesian lines between environment and organism and between mind and body are not so definite. The same principles that apply to human physical tool use also apply to productive mental activity. Hickman traces Dewey's interpretation of technology to the Greek *techne* referring to the knowledge of an artisan or craftsman and deeply rooted in human activity with a productive purpose: "*Techne* was for the Greeks a pro-duction, a leading toward, and a con-struction, a drawing together, of various parts and pieces in order to make something novel" (Hickman, 1990).

For Dewey, describing tool use as technological meant that it is active, testable, and productive. A cognitive tool is selected and applied in a dynamic process which actively engages the attention of the individual. It is used to perform tests upon the problem that gave rise to its selection, and reciprocally, the tool is itself tested against the problem and evaluated for appropriateness (see Figure 9). Thus, a dialectic interaction between the tool and problem is

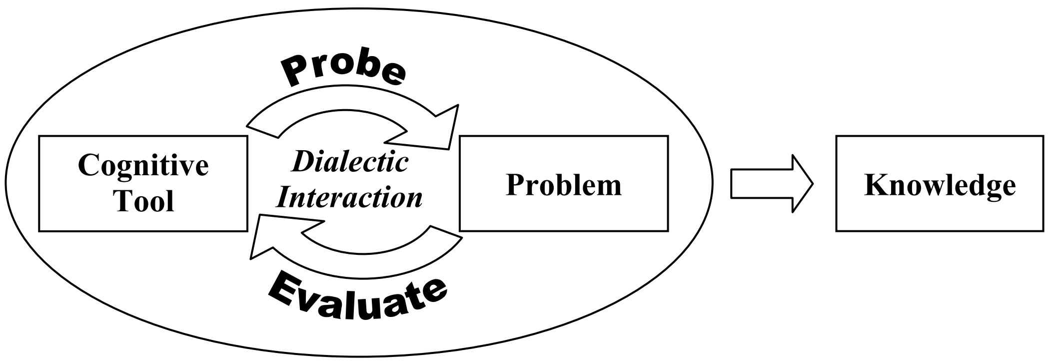


Figure 9. Dewey's instrumentalism: production of knowledge in active, testable tool use.

established, effecting change in both by bringing together a variety of their aspects (both human and environmental). The artifacts of this dialectic are knowledge. As new meanings arise, they present new situations, which may themselves become the object of further inquiry. According to Dewey, meanings

> …copulate and breed new meanings. There is nothing surprising in the fact that dialectic… generates new objects; that in Kantian language, it is "synthetic," instead of merely explicating what is already had. All discourse, oral or written, which is more than a routine unrolling of vocal habits, says things that surprise the one that says them, often indeed more than they surprise any one else. (Hickman, 1990)

Prawat and Floden (1994) present structural coupling in evolutionary theory as an analogy for this view of human understanding. Structural coupling represents a fortuitous coming together of structures emergent in the organism and environment which interact in a complex, reciprocal, and implicative relationship. The truth of a hypothesis, for example, is determined as the object of perception talks back, forcing the individual to rethink their initial expectations. In this process, the original idea becomes more "coherent" and "densely textured." Since such inquiry is situated and ongoing, one cannot separate knowledge from the context of its origins. It is bound to the unique circumstances and processes through which it was created; truth is emergent, not located externally. Consequently, Dewey's focus is on the process of inquiry rather than knowledge itself. Meaning for a proposition, symbol, or metaphor is defined in terms of the object's function in productive activity, just as it is for a physical tool such as a computer or hoe.

At this point, one distinction from, and one similarity to, the literature discussed in the previous chapter should be noted. First, the functional role of a cognitive tool should not be confused with perspectives that include "operational knowledge" (or "process view" in APOS terminology). An example of the latter perspective applied to mathematical metaphors is Anna Sfard's (1997) distinction between operational metaphors (e.g., thinking of a rational number, p/q, as dividing an object into q equal parts and taking p of them) and structural metaphors (e.g., p/q equated with a certain concrete piece of the object). To understand a concept in operational terms is to explicitly imagine it as a process rather than treating it as a reified entity. The conceptual construct of a process, however, is a part of the *structure* of one's knowledge. It does not imply the application of that process in inquiry against some problematic situation.

The similarity that should be noted is with Max Black's interactionism theory of metaphor (Black, 1962a, 1977). The dialectic interaction he describes between domains in a metaphor that creates new meaning is precisely the same as Dewey's characterization of the tool and problem being tested against one another to produce new knowledge. Instrumentalism adds to the perspective of metaphor in this study the necessity of a problematic situation to engage active inquiry. Both perspectives are also compatible with Alexander, Schallert, & Hare's description of construction at the interface of prior knowledge and the external demands of a particular situation.

The inclusion of reciprocal influences is a crucial point of convergence, as it is what affords the creation of new ideas according to each of these theories. An account of metaphor merely in terms of inference preserving mappings between domains cannot account for such creativity because only pre-existing structures may be mapped. For example, Lakoff (1987) and Lakoff & Núñez (2000) suggest that sets are understood metaphorically as containers, and they provide a detailed correspondence between the objects and logic of containers and the objects and logic of sets. Unfortunately, this cannot explain why a set that contains another set as an element does not necessarily contain all of the elements of that element-set. From an instrumentalist perspective, an idea that is new (e.g., a set as an object that contains elements and that may be contained as a single element-set rather than as multiple elements in another set) instead arises in response to particular demands of inquirential problem solving activity. Blends of multiple mappings also fail to produce anything new except, possibly, for the simultaneous evocation of multiple pre-existing ideas. For example, as mere mappings, Lakoff & Núñez's (2000) Basic Metaphor of Infinity coupled with the container metaphor for sets cannot account for an understanding of the inequality of $P\left(\bigcup_{n\in\mathbb{N}}\{1,2,\ldots,n\}\right)$ and $\bigcup_{n\in\mathbb{N}} P\left(\{1,2,\ldots,n\}\right)$, where $P(A)$ denotes power set of A. In contrast, instrumentalism views these metaphors as conceptual tools which evolve as they are used (e.g., containers and their contents are reified into a single whole in response to a need to perform actions on sets) and which produce effects on the problems to which they are applied (e.g., providing multiple conceptual options for a metaphorical final state which can each be tested against conditions of the problem for the selection of an appropriate result). In vetting such possibilities, new ways of thinking may emerge.

Interactionism

Though Dewey's instrumentalism parallels many of the aspects of Black's interaction theory of metaphor, there are several aspects of interactionism that we will emphasize. We mention them only briefly here and refer to Chapter 2 for a more in-depth discussion. Specifically, we are interested in reasoning that impacts the development of students' concepts, and Black's work is concerned with characterizing precisely the types of metaphors that are "ontologically creative." Thus the characteristics of a strong metaphor, being resonant and emphatic, should be included in the perspective. Recall that a metaphor is resonant if it supports a high degree of implicative elaboration and is emphatic if the author is decidedly committed to the particular metaphorical context being invoked.

The Metaphor of Metaphor

As illustrated by the similarities between Black's theory of interactionism and Dewey's theory of instrumentalism, the domains of generative metaphor and human knowledge share much in common. Certainly, one mutual trait is the difficulty in rigorously defining or explaining either one. Both are, consequently, often described in metaphorical terms themselves. In fact, the perspectives we have outlined ("abstract thought as metaphor," "schematic structures of explicit knowledge formed from relatively unstructured tacit knowledge," and "inquiry as tool use") are themselves metaphors for understanding, knowledge, and the learning process. We now briefly consider the implications of choosing these particular metaphors over others.

First, we acknowledge some of the common alternative metaphors for learning. Anna Sfard (1998) describes two: the *acquisition metaphor* and the *participation metaphor.* The acquisition metaphor views the human mind as a container to be filled with information. Once knowledge is acquired by a learner, either by transmission from another source or by construction, it is an owned commodity and is inherently private. In the participation metaphor, the learner is seen to engage in communal activities and discourse. Knowledge is contained not in the mind, but in social norms and linguistic practices and is therefore a public construct. Sfard argues that neither metaphor represents "the correct view" and that both are needed to enrich our understanding of learning. The acquisition metaphor, for example, cannot explain Plato's "learning paradox," essentially, the question of how new knowledge can ever be generated from a state of ignorance of that knowledge. (Plato concluded that both knowledge and virture were God-given gifts, and thus inherent rather than formed by humans.) The participation metaphor avoids this difficulty by refusing to objectify knowledge, but encounters other problems such as an inability to account for the transfer of knowledge from one context to another. Consequently, it is in a creative tension between the two metaphors, Sfard suggests, that we are best able to guide research and account for various learning phenomena.

The metaphors we have chosen for the theoretical perspective of this study (schema, metaphor, and tool use) carry their own challenges, but as Sfard suggests, each may help reconcile central difficulties of the other. Although instrumentalism uses essentially a personalized participation metaphor, one problematic entailment is its reification of thought into extant "instruments" or "tools" that can be selected for use in various applications. Similarly, theories of metaphor suggest an interaction between pre-existing conceptual domains. Both perspectives also posit an influence of the particular setting on the nature of the subsequent tool or metaphorical domain. The resulting difficulty is that the applications of "a metaphor" in different settings would have to be treated separately, because they would necessarily involve different tools. We would, however, like to be able to account for commonalities among various metaphors used by students in different problem contexts. The theory of schemas presented above explicitly denies the prior existence of such concrete objects of thought. Instead, it suggests that an instantiation is constructed from elements within tacit knowledge in response to the ongoing demands of the situation. Similarities between schemas applied by a student in different contexts may then be seen to draw upon similar collections of ideas, only constructing them in slightly different ways to create particular metaphors.

On the flip side, while the theory of schemas suggests that new knowledge is created as a result of prior knowledge being reconstructed in a manner responsive to external demands, it does not suggest a mechanism for *how* this might occur. The treatment of knowledge as a schema is essentially an acquisition metaphor, and this difficulty is a variation of the learning paradox. Here instrumentalism is more explicit, mirroring the approach taken by the participation metaphor. By treating meaning in a pragmatic sense, a tool is endowed with new meaning when it is applied in a new setting. Its effects are different. Similarly, the problem responds in different ways to being probed by various tools and thus requires equally differing reactions from the individual.

Throughout the following discussions of this study, we will characterize various aspects of students' reasoning in terms corresponding to each of the three metaphors presented here. The particular language will depend on which perspective that best represents the relevant aspects of students' reasoning we wish to discuss. They are all interwoven, however, in the underlying treatment of students metaphorical reasoning as "the active application of a schema of spontaneous concepts as a mental tool against a problematic situation."

Implications for Data Collection and Analysis

The above perspectives are used in this research to guide both the design of the study and the analysis of the data. Interviews, questionnaires, and writing assignments are developed to provide various types of problematic settings to elicit the students' metaphorical reasoning. Data is analyzed to identify clusters of similar metaphor usage including the underlying schemas. These metaphor clusters are intended to capture global patterns in students' responses, and consequently, they present an amalgam of different students' actual metaphors. In order to remain faithful to the individual character of students' thoughts, however, we rely heavily on individual data as well. In this section, we discuss the nature of and motivation behind the methods of data collection and analysis for the study.

Presentation of Problematic Situations

Questions such as "What is a limit?" and "Is a limit an approximation?" will elicit only structural responses, students descriptions of what they happen to associate the concept with at the time. These responses may be very different from the nature of students' actual use of the relevant concepts. To observe functional aspects of students' thought and how their structural knowledge develops in the process of its application, data collection instruments and settings must be designed to allow for the technological application of the students' metaphors against actual problems. Questions that are perceived by the students to be routine will elicit a non-inquirential response. That is, the students will invoke an established procedure or description. If, on the other hand, they perceive a genuine puzzle that they are willing to engage in a process of inquiry, then their metaphors may emerge as technological tools in Dewey's sense (i.e., their use is active, testable, and productive). In this research study, we present students with a range of problems. Some begin as routine computations with problematic aspects eventually called out by the interviewer for resolution by the students. Other problems ask students to discuss simply stated "paradoxes" whose resolutions are likely beyond the students' current mathematical capabilities. These difficult problems are not intended to test their understandings of the subtleties of the mathematics, but rather to establish active problem solving situations in which the students are likely to apply the cognitive tools under study.

A key feature of both interactionism and instrumentalism is that reciprocal influences change the metaphorical domain (or tool) in the process of inquiry. Thus, afterwards, some information about the nature of a student's emerging understanding may be reflected in their subsequent structural characterization of that domain. Some of the questions we will pose to the students directly address these adaptations. It is difficult, however, to access the nature of a student's conceptualization of a metaphorical domain before it is applied (if such a thing can even be said to exist). On the one hand, the instantiation of a specific domain or tool is only cobbled together by the student in response to a specific situation. Thus, without presenting that situation, the particular concept is not at hand. Then again, as soon as the situation is engaged as a problem, the tool undergoes change. Some early statements made during this process might reveal aspects of the initial conceptual structure, but the present research interest lies in their application and the subsequent impact on mathematical understanding. Thus for these portions of the data, following Dewey, we will focus on processes of inquiry and their resulting products rather than descriptions of the initial tools.

Finally, since any given schema will emerge from the specifics of the problem for which it was constructed, we cannot set up a priori structures such as "the Basic Metaphor of Infinity," a model of "limit as a boundary," or "a coordination of processes view." This restriction has implications for both the selection of prompts for data collection and the analysis of data. First, most prompts should be open ended, allowing the students to reveal the particularized metaphors they construct and use. Multiple choice or other types of questions that allow only certain kinds of responses will not establish the conditions in which these instantiations can be observed. Second, the analysis must begin with an "open-coding" that takes its' initial models from the details revealed in the data rather than a coding based on a predetermined template.

Metaphor Clusters

As mentioned above, we will a look for generalizable characterizations of the students' metaphorical reasoning by generating *metaphor clusters* from the data. Each metaphor cluster will be based on a generalized set of related constructs used by a large number of students in various problem settings, which we will call the *underlying schema* for the cluster. These schemas differ from the description provided earlier in that they reflect a broad rather than particularized use. A metaphor cluster will also include a characterization of the way it is applied by students in various problem contexts and the resulting conclusions drawn about limit concepts. The details of each of these aspects of metaphor clusters are discussed below.

In attempting to find and characterize general patterns in the data, the cost is an inability to completely fit the characterization to the reasoning of any individual student. The major gain, however, is that we are able to more readily identify what Black referred to as strong metaphors, those that are likely to be influencing students' conceptual development. To capture resonance, we will require that a potential metaphor satisfy two criteria related to its use to support extended reasoning by the students. First, some minimum number of students should be observed using a coherent set of ideas from a generalized schema (i.e., generalized from multiple students but conceptually compatible and relating to a common context) while reasoning about a given problem. Satisfying such a criteria will not necessarily indicate that a metaphor is widespread in a given context (although it may be), but we are most interested in the indication that some *reason* exists for the students committing to that particular metaphor. Second, we will require that a schema be used by students in a variety of contexts. This is an attempt to ensure that 1) the language or imagery evoked by a particular question does not lead students to respond in a certain way and 2) students' willingness to respond in a certain way in an isolated context is not overgeneralized. Though a minimum cut-off will be required, the choice of a particular threshold is necessarily an arbitrary decision. Thus, some care is required to guard against overlooking important aspects of students metaphorical reasoning and, conversely, against giving undue weight to extraneous or weak metaphorical language. Finally, multiple types of problem settings will be used to allow students to respond with varying levels of commitment to the particular descriptions they choose. More spontaneous responses are expected from initial questions on interviews and short quizzes, while writing assignments and follow-up discussions during interviews will allow for more reflection.

Resonance is a slightly more difficult phenomenon to observe. How can one tell if a student (or group of students collectively) is using a schema in ways that supports their implicative investigation into a specific problem? Students may use signs peripheral to the ideas they are exploring, such as phrases picked up from their professors or imagery used to express certain ideas that they have already developed. Although, the details of language use are a standard source of evidence for purely linguistic analyses, these signs are not necessarily a part of the students' active inquiry into a problem. At the risk of excluding some cases of students' metaphorical reasoning, we will not count language, imagery, or any other signs as an instance of a particular metaphor if it is mentioned only in passing by a student. We will require the relevant ideas to be discussed with some link to the student's reasoning about the limit concepts involved in the particular problem context being discussed.

Schemas

Most prior research has focused on students' structural descriptions of limit concepts. Although this study is not intended to duplicate those efforts, it is important to investigate and report the structural details of the tools being used. Thus some questions posed to the students will attempt to reveal how they have organized specific concepts, and the subsequent analysis will look for underlying relationships. For the most part, however, the structure of student's concepts will be encoded by the underlying schemas of the metaphor clusters used by the students. In this study, we will not attempt to develop a complete characterization of any one student's underlying schemas for their limit metaphors.

This information will be taken mostly from students' descriptions of the metaphorical domain during or after its application in solving a problem. In interviews and email correspondence, students' responses to follow-up questions asking for clarification of potential aspects of a schema may also be included. A particular student may not construct all aspects of an underlying schema that are available in their tacit knowledge, and if they do, it may not be revealed in the data collected. Thus, we will look for commonalities and logical coherence across different problem contexts from individual students' instantiations of the ideas within a generalized schema assuming that students who reveal only part of such a schema are also likely to have used, or at least had at their disposal, some of the other aspects. On the flip side, we will also allow for differences between individual students' instantiations in the recognition that students' thoughts are idiosyncratic and built up from differing experiences.

A description of the underlying schema for a metaphor cluster will thus include the objects that students describe for a given metaphor cluster, the language of those descriptions, the structural and logical relationships mentioned or implied between the objects, and any differing versions of these used by some students.

Characterizations of Metaphors Applied in Specific Problem Contexts

Although metaphor clusters are built up from a confluence of students' similar responses, we will also require that they faithfully portray the reported characteristics of individual students' thought processes. In order to capture some of the individual character of metaphor use, we will heavily augment a description of the generalities of a metaphor cluster with the ways in which students used the underlying schema in different problem contexts. For each context in which students apply an underlying schema, we will develop a detailed account of how the specific ideas were applied with multiple accounts from students. We will consider these descriptions as an integral aspect of the metaphor cluster rather than simply illustrations of their generalized characteristics.

Chapter 4: Methodology

What are the base metaphors students use as they are learning limit concepts? How do they apply those metaphors to specific problems involving various limit concepts and when trying to understand subsequent material? How does students' spontaneous reasoning affect their interpretation of content presented in the classroom or by the textbook? We seek descriptive answers to these questions in terms of the perspective of metaphorical reasoning outlined in the previous chapters.

This study was conducted in three major phases, where the ongoing analysis of data from each phase was used to inform the construction and implementation of research instruments in the following phases. See Figure 10 for a schematic diagram of the goals and products of each phase of the study. The first two phases of the study were intended to generate a coherent set of metaphors used widely by the students to understand and work with limit concepts. Phase I involved a series of open-ended interviews and writing assignments to have first-semester calculus students describe how they thought about specific limit concepts and problems. The initial themes emerging from the interpretations and metaphors given by these students informed the construction of a series of non-standard writing assignments for Phase II of the study in a second-semester calculus course. The data from both of these phases were used to construct a set of metaphor clusters and their underlying schemas as used by the students. For Phase III, two of these schemas were modified by the researcher, removing extraneous and potentially misleading aspects and developing others to obtain schemas that more closely resembled epsilon-delta and epsilon-*N* definitions of limits. Students in the third phase of the study were given activities which prescribed the use of these revised metaphors. Data was collected through interviews to see how the students interpreted various aspects of these explicitly presented metaphors.

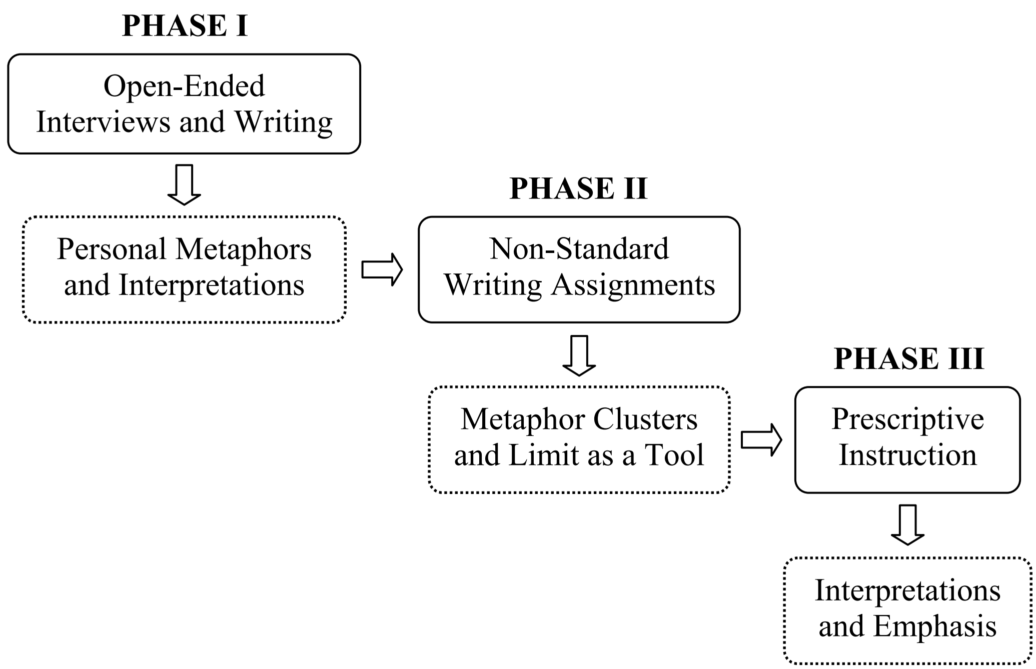


Figure 10. Global design of the study. Data from each phase informed the design of the next phase.

This research employs some techniques from Strauss & Corbin's (1990) description of grounded theory in which the design of the research instruments and the models used to understand the data are allowed to emerge through iterative processes of analysis and refinement. In this study, all data was initially coded as liberally as possible for various features (contexts and imagery described, strategies employed, specific groups of words used, etc.). Then tentative models for the students' usage of metaphors were built based on the convergence of specific features and relationships among them. These models were tested and by recoding the original data and evaluating whether the description of each code was adequate to determine application to specific passages, only compatible codes were assigned to similar passages, all passages with a given code shared significant common features and were distinct from other passages in a significant trait. The models were then refined by looking for further relationships among and within the codes, building a new model and repeating the process until the coding system organized the data into clusters of similar schemas and metaphors.

In addition, throughout the data analysis, the researcher used reflexive techniques as described in Foley (1998, 2000). First, the lens of metaphor was applied to the data interpretation, recognizing that a description of students' thinking as metaphorical is itself a metaphor. Consequently, it is necessary to consider the implications of this choice as only one of many possible metaphors for the types of thinking displayed by the students. (Other popular metaphors for thought include, for example, the mind as a computer in various information processing models, or thought as a biological species developing through a complex evolutionary process in an organismic view). Second, the impact of both the specific study design and the role of the researcher on the nature of the data collected is acknowledged (e.g., a student's need to appear knowledgeable in front of the researcher, the unnatural setting of an interview to observe a student's process of inquiry, etc.). Finally, in order to present a sense of the complexities of students' developing thinking rather than an abstracted, coherent whole, an attempt was made to acknowledge and account for idiosyncratic and conflicting modes of thinking among the students in the study and even within each student's own thoughts.

DESCRIPTION OF THE SETTING

The research for this study was conducted in a series of first- and second-semester calculus classes at a large southwestern public university. Calculus at this university is taught in a two-semester sequence covering the content listed in the Mathematics Department syllabi (Appendix A). These classes meet in large lectures (with approximately 120 students) with a professor for 3 hours per week and in small discussion sections (with approximately 40 students each) with a graduate student teaching assistant for an additional 2 hours per week. The researcher conducted interviews and collected written work from students in 2 first-semester courses (Fall 2000 and Fall 2001) and in 1 second-semester course (Spring 2001). These three classes were all taught by the same professor, referred to throughout this dissertation simply as "the professor." Each semester, a group of approximately 20 students in his class also enrolled in an honors-style freshman calculus program instead of attending the regular class discussion sections. In the workshop for this program, these students worked in small groups for 6 hours per week on challenging problems related to the course.

The Professor

The professor for the course described various limit concepts in similar ways throughout the three semesters of this study. In the first-semester course, he presented the epsilon-delta definition for the limit of a function at a point and expected students to be able to do simple proofs involving linear and quadratic functions. Throughout all three courses, he continually referred to epsilon-delta or epsilon-*N* ideas to describe other concepts such as the definite integral or Taylor series. In doing so, he often paired these rigorous descriptions with informal ones saying things such as "you can make the difference [between a series and its limit] arbitrarily small by adding enough terms." He also regularly invited students to compare relative sizes of various terms, as when investigating $x^2 = 1 + 2(x-1) + (x-1)^2$ when x is near 1 (i.e., when $x-1$ is small). For example, if $x = 1.01$, then $(x-1)^2$ is 100 times smaller than $x-1$, thus the graph of $y = x^2$, the professor would conclude, is going to be virtually indistinguishable from the line $y = 1 + 2(x-1)$. He often punctuated these comparisons with an extreme value, describing, for example, 10^{-20} as "smaller than the width of a single electron" or 10^{100} as "more than the number of molecules in the entire universe."

The professor referred to an expansion of $x^2 = a^2 + 2a(x-a) + (x-a)^2$ as the "a-centric" view of x^2 to discuss tangent lines as approximations to functions in small intervals around *a*. He used the same type of analysis of relative scales as discussed above to describe the size of second and third order corrections. In the generalization to Taylor series, he referred to the role of the limit in the derivatives with comments such as "by knowing a lot about the function on any small interval around *a*, you can tell exactly what the function will do everywhere else, even at points far away."

The professor emphasized that a limit of a function depends on the values of the function on entire intervals (or neighborhoods), rather than at individual points. Specifically, the limit of a function at a point, *c*, is a number, *L*, if any neighborhood of *L* contains the image of some deleted neighborhood of *c*. To symbolize this type of verbal definition, the professor regularly used notation such as $D_{c,r} = (c-r, c+r)$ and $\hat{D}_{c,r} = D_{c,r} - \{c\}$, verbally describing a "disk," "interval," or "neighborhood with center *c* (deleted in the second case) and radius *r*." He expected students to interpret $f(D_{c,r})$ and $f(\hat{D}_{c,r})$ as sets in the range of *f*. Thus when he presented the standard epsilon-delta definition of the limit of a function, he was also able to write and explain the parallel statement using set notation (e.g., replacing "if $0 < |x-a| < \delta$, then $|f(x)-L| < \varepsilon$" with "$f(\hat{D}_{a,\delta}) \subset D_{L,\varepsilon}$"). More informally, the professor might describe the function *f* as sending all of the points near *c* to points near *L*, or as "preserving closeness."

The concepts of continuity and the derivative were also treated in terms of functions acting on intervals, in parallel with their standard definitions. The professor described the continuity condition as the same as the definition of the limit with the exception that the function acts on the entire interval rather than on a deleted one. Specifically, a function is continuous at *x* if any neighborhood of $f(x)$ contains the image of some neighborhood of *x* (i.e., $f(D_{x,\delta}) \subset D_{f(x),\varepsilon}$). The derivative was described as a measure of local interval magnification by a function in order to give concepts such as the chain rule a relatively straightforward intuitive interpretation (multiplication of magnifying factors). Specifically, a differentiable function magnifies (or shrinks) very small intervals around a point by roughly the same scale factor (i.e., $\text{diameter}(f(D_{x,c})) \approx f'(x)\bullet\text{diameter}(D_{x,c})$). Notice that the ratio of the diameters of these intervals is quite different from arbitrarily small intervals of values around $f'(x)$. Thus, unlike their role in the definitions of continuity and the limit of a function, the role of the intervals here is not to make the limit in the definition of the derivative more precise. Instead they are emphasized to show the structure of slope in a way that can easily be followed through operations such as composition of functions.

The Textbook

The textbook used for all calculus courses at the university in this study (Salas, Hille, and Etgen, 1999) treated limits slightly differently than the professor. It introduced "the idea of limit" intuitively in a chapter with several examples of functions and graphs and using language such as "the function values $f(x)$ approach *L* as *x* approaches *c*" and "$f(x)$ is close to *L* whenever *x* is close to *c* but different from *c*." The definition of the limit of a function is presented by the textbook with an informal statement, "$|f(x)-L|$ can be made arbitrarily small simply by requiring that $|x-c|$ be sufficiently small but different from zero." Epsilon-delta proofs were presented as a two-step process of "finding a delta" and "showing that the delta 'works.'" Similar, but much shorter definitions were provided for infinite limits, limits at infinity, and limits of sequences. Other than these definitions, however, the only places where epsilon-delta or epsilon-*N* occurred in the text were in theorem proofs, without any accompanying conceptual discussion. (For a survey of concepts discussed in the text using limit concepts, see Table # in Chapter 2: Review of the Relevant Literature.)

The Researcher

The researcher was introduced to the three classes in this study at the beginning of each semester in order to briefly describe the research project and obtain permission from the students to use their work for the study. He attended over 95% of the lectures for the classes during the first two phases of the study to observe the presentation of limit concepts by the professor. Since many of the writing assignments were turned in directly to the researcher and involved his feedback, most students recognized him as another TA for the class and felt free to ask him questions about the material being covered, assignments, and grades. He assisted the professor and TAs for the class in preparing and grading some of the regular class assignments and exams, especially when the content was related to limit concepts. Throughout the study, the researcher conducted all of the interviews except for the series of two interviews with one of the students in Phase I of the study. For simplification, the person conducting the interviews will always be referred to as "the interviewer."

ELICITING STUDENTS' METAPHORS

The first two phases of the study were designed for three purposes: 1) to educe the metaphors about limits used by students in a first-year calculus sequence and 2) to provide data on the ways in which those metaphors were used in solving problems, understanding new material, and 3) to investigate students' interpretations of language about limits used in lecture and by the textbook. These two phases coincided with the two-semester calculus sequence taught by the professor described above.

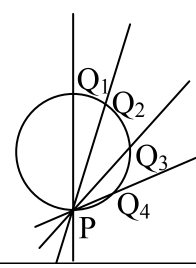
Phase I Data Collection

The first phase of the study was conducted in the Fall of 2000 in a first semester calculus course. Most of the students had just arrived at the university as freshmen, and slightly over half (64 out of 120) had taken a calculus course in high school. The data from this phase of the study consists of students' responses to a pre-course survey covering basic pre-calculus and calculus concepts, interview transcripts from two different clinical interviews that were conducted individually with 9 students during the semester, and students' written work from weekly writing assignments given to the entire class.

Pre-course Survey

Three of the seven questions on the pre-course survey addressed limit concepts (see Table 4). The surveys were completed by 116 students, but the number of responses on each item varied. As with many of the other short-response items used in this study, two of these items were taken directly from previous research on students' understandings of limit concepts. Other instruments that borrow items from previous research are the short writing assignments in this phase of the study and the pre- and post-course surveys for Phase II, both discussed below. These items were intended to allow comparisons between, first, the students in this study and students in other studies and, second, between students' responses on these items and their responses on more in-depth instruments such as interviews and writing assignments.

Table 4. Limit Problems on the Phase I Pre-Course Survey.

1. What is between 0.999… (The nines repeat.) and 1? (a) Nothing because 0.999…=1 (b) An infinitely small distance because 0.999…<1 (c) You can't really answer because 0.999… keeps going forever and never finishes. (d) If you don't agree with any of the above, circle (d) and give your own answer. (Szydlik, 2000)
2. The diagram shows a circle and a fixed point P on the circle. Lines PQ are drawn from P to points Q on the circle and are extended in both directions. Such lines across a circle are called *secants*, and some examples are shown in the diagram. As Q gets closer and closer to P what happens to the secant? (Orton, 1983) 
3. Have you previously been introduced to the concept of a limit? ___Yes ___No If yes, please describe the meaning of a limit. Include intuitive ideas as well as a definition if you know one.

Clinical Interviews

During Phase I, a group of 9 first-semester students participated in clinical interviews at two points during the semester. The first interview was conducted shortly after students were introduced to the limit of a function at a point and probed their understanding of the meanings of informal descriptions of limits, the epsilon-delta definition and related proofs, and the computational methods typically covered in a first calculus course. The second interview was conducted shortly after the definition of the derivative was introduced in lecture and focused on the students' interpretations of the limit in this definition and the meaning of an instantaneous rate. The root question for each of these interviews was taken from routine problems used in previous research and questions were added to probe students' reasoning about their work. See Table 5 for the interview protocols for these interviews.

Table 5. Phase I Interview Protocols.

Interview A – The limit of a function at a point

1. What is a limit?
2. Explain what is meant by $\lim_{x \to 1} \frac{x^3 - 1}{x - 1} = 3$. (Tall & Vinner, 1981)
3. Why does the method you used to compute this limit work?
4. What is the meaning of saying this limit is 3?
5. How do you think of the symbols "lim" and "$x \to 1$?"
6. The only place where $\frac{x^3 - 1}{x - 1}$ and $x^2 + x + 1$ differ is at $x = 1$. Why is it acceptable to interchange these two functions even though we are trying to find the limit at $x = 1$?
7. When you use words like "approaching" and "tends to," what do you mean? They seem to imply motion. Do you think of something moving?

Interview B – The definition of the derivative

1. Let $f(x) = x^2 + 1$. What interpretation do you have for the expression $\lim_{h \to 0} \frac{f(3+h) - f(3)}{h}$? (Frid, 1994)
2. What is the meaning of the computation? Why does it work? What does it mean for h to cancel out?
3. What is the role of the limit in this expression?
4. How are the graph, the slope, the secant, and the tangent related to one another?
5. How does this expression give the slope of a tangent line?
6. Suppose $f(x)$ were replaced by $p(t)$, the position of a moving object as a function of time. Now explain the meaning of this limit and its various parts.
7. When you use words like "approaching" and "tends to," what do you mean? They seem to imply motion. Do you think of something moving?

Clinical interviews were designed to obtain fine-grained data on a small number of participants' understandings of a particular concept. Each subject was interviewed individually and asked to solve a specific problem, explaining their thoughts as they worked. For the interviews conducted during this portion of the study, the problem focus was two-fold. Initially the students were asked to work the limit problems given in the protocols and to explain their methods. Gradually, they were asked to explore and attempt to understand various interpretations of these limits. Most of the interview time and questions were devoted to the latter focus, first, since this task was the most difficult for the students and, second, because the main interest in this study is students' interpretations and uses of limit concepts.

Throughout an interview, the interviewer attempted to follow up on each of the student's main ideas by asking them for more in-depth explanations.

Incorrect lines of reasoning were followed as vigorously as correct ones, and the interviewer did not offer correct explanations of the concepts nor provide feedback about the accuracy of the student's explanations. The interviewer occasionally shifted the direction of the discussion when a student became stuck and seemed unlikely to make further progress or when a line of thinking became too divergent from the students' thinking about limit concepts. In the rare cases when a student seemed unable to develop any line of reasoning on their own, the interviewer provided minimal suggestions to get a discussion started (e.g., asking them to identify aspects of a graph that relate to the various expressions in the difference quotient).

The students who participated in these interviews were also questioned about other key concepts (covariation, rate of change, and accumulation of rate) from the first semester calculus course. Small portions of the data from these other interviews containing references to limits were also used for this study.

Writing Assignments and Quizzes

At various points throughout the fall semester, students were given short (roughly 1 page) writing assignments and in-class quizzes, each with 80 to 110 responses. Problems for these instruments were either adapted from the existing research literature or developed to address students' interpretations of a specific concept presented during the lectures. See Table 6 for the writing assignment problem statements.

Table 6. Phase I Writing Assignments.

1. Find The limit $\lim_{n\to\infty}(1+\frac{9}{10}+\frac{9}{100}+\cdots+\frac{9}{10^n})$. (Tall & Vinner, 1981)
2. Decide on intuitive grounds whether the limit $\lim_{x\to 1}\begin{cases} 2x, & x \text{ rational} \\ 2, & x \text{ irrational}\end{cases}$ exists, and evaluate the limit if it does exist.
3. One of the ideas we have discussed is that “zooming in” on a graph of a nice function at a point results in what appears to be a straight line. A. How would you explain such “zooming in” to someone who has never seen this before? In your explanation use your favorite images and be sure to explain how they are the same and how they are different from what is actually meant by "zooming in" in calculus. B. Compare and contrast what you can see if you “zoom in” on a graph by a) Using a graphing calculator b) Putting the graph under a microscope, c) Plugging in ever smaller numbers, and d) Shrinking yourself to a very small sizes and walking around the graph.
4. Suppose that *f* and *g* are functions (not necessarily continuous), *a*, *b* and *c* are real numbers and that $\lim_{x\to a} f(x) = b$ and $\lim_{y\to b} g(y) = c$. Does it follow that $\lim_{x\to a} g(f(x)) = c$? (Tall & Vinner, 1981)
5. What does dx in $\int_a^b f(x)dx$ mean?

6. Please mark the following six statements about limits as being true or false:
   1. T F A limit describes how a function moves as x moves toward a certain point.
   2. T F A limit is a number or point past which a function cannot go.
   3. T F A limit is a number that the y-values of a function can be made arbitrarily close to by restricting x-values.
   4. T F A limit is a number or point the function gets close to but never reaches.
   5. T F A limit is an approximation that can be made as accurate as you wish.
   6. T F A limit is determined by plugging in numbers closer and closer to a given number until the limit is reached.

   Which of the above statements best describes a limit as you understand it? (Circle one)

   1 2 3 4 5 6 None

   Please describe in a few sentences what you understand a limit to be. That is, describe what it means to say that the limit of a function *f* as $x \to c$ is some number *L*.

(Williams, 1991)

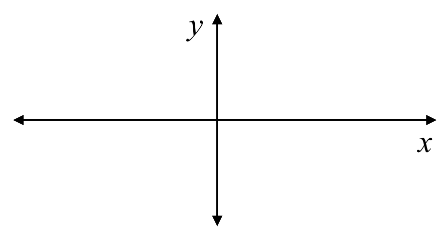
Phase II Data Collection

Students were presented with mostly routine problems and explicit limit statements in the first phase of this study. The writing assignments and interview questions were straightforward inquiries about how the students conceptualized the meanings of those problems and statements. The goal in the second phase of the study was to observe how students applied their understanding of limits from the first-semester course toward making sense out of challenging concepts developed during the second semester course. Written data were collected from students in the second-semester calculus course through identical pre- and post-course surveys and through a series of six writing assignments.

Pre- and Post-Course Surveys

Students in the second-semester course were given a survey of 12 conceptual questions at both the beginning and end of the course, with 4 of these questions specifically addressing limit concepts (see Table 7). Each question stated a fact covered in the first semester course and asked for a couple paragraphs of explanation. A total of 104 students responded to either one or both of these surveys (exact numbers varied for each of the 12 problems).

Table 7. Limit Questions on the Phase II Pre- and Post-Course Surveys.

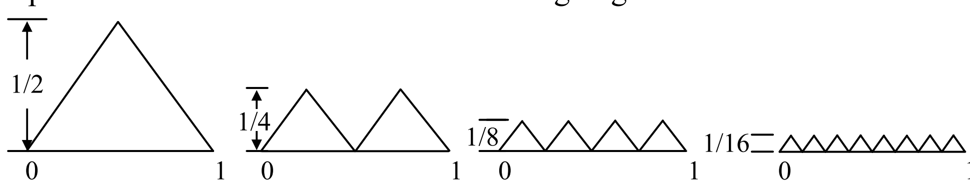
2. Explain why the repeating decimal $0.\overline{9}$ is equal to one.	
8. Explain why the limit $\lim_{x\to\infty} \sin x$ does not exist	
10. On the axes to the right, graph the function $f(x) = x\sin\frac{1}{x}$ paying careful attention to the behavior near $x = 0$. Explain why the limit of *f* as *x* approaches zero is zero.	
11. Explain why the derivative $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ gives the instantaneous rate of change of *f* at *x*.	

Writing Assignments

During Phase II, weekly writing assignments were posted to the class website for the students to earn extra-credit points toward their grades in the class. The questions in these assignments ostensibly covered current class content, however the research focus was to determine the involvement of the application or generalization of limit concepts covered in the first-semester course. Of these weekly assignments, 6 dealt with the limit concepts involved in the following contexts: L'Hospital's rule, an unbounded volume of revolution, the limit comparison test, the Taylor series of $\sin x$, the topological limit of an infinite sequence of sets, and multivariable continuity (see Table 8 for the problem statements). Prompts for all of these questions stated a non-routine fact and asked the students to justify the statement and discuss their understanding of the relevant concepts. The data for these problems consist of initial and follow-up email responses from between 20 to 35 students.

Table 8. Phase II Web Problem Writing Assignments.

Web Problem 1 In certain cases, L'Hospital's Rule connects the limit of a quotient (say f/g) to the limit of the quotient of the derivatives (f'/g'), but you may have noticed that our textbook does not describe how to understand why this works. Pick either the case where *f* and *g* both go to zero or the case where *f* and *g* both go to infinity and explain why L'Hospital's Rule works for that case. Make sure to describe explicitly how you think of the roles of the limit, the derivative, and the quotient.
Web Problem 2 In lecture last week, we saw that the area between the *x*-axis and the graph of $y = 1/x$ beyond $x = 1$ is infinite. We also saw that if this infinite area is revolved around the *x*-axis, we get a finite volume. Furthermore, this finite volume is bounded by an infinite surface area! Discuss how such a phenomenon can occur. Here are some things you may choose to focus on: the exact meaning of these statements, resolving their seemingly contradictory nature, intuitive explanations for why they are true, the role of limits in these ideas, the nature of area and volume, etc.

Web Problem 3 Theorem 11.2.6 in the text is the limit comparison test. It states that for two series $\sum a_k$ and $\sum b_k$ with positive terms, if the limit of a_k / b_k exists and is not zero, then either both of the series converge or they both diverge. Explain why this works. In addition, what can you say if the limit of a_k / b_k is zero? What can you say if the limit is infinity? Explain your reasoning.
Web Problem 4 In lecture last week, we saw that sin(x) can be represented by a power series $$x - \tfrac{1}{3!}x^3 + \tfrac{1}{5!}x^5 - \tfrac{1}{7!}x^7 + \cdots$$ Suppose one of your classmates made the statement that this means sin(x) is a polynomial. While this is not technically correct, there is a good reason for your classmate to think of sin(x) in this way. Write an explanation to this classmate about i) why this way of thinking can be useful and ii) how it can be made more accurate.
Web Problem 5 In the diagram below, the line segment [0,1] is approximated by successively smaller jagged lines. After this process repeats about 20 times, the distance from the horizontal segment [0,1] to the jagged line would be less than the width of a single electron, so they would be virtually indistinguishable! The length of the jagged line, however, is always $\sqrt{2}$ (check it out yourself). A classmate says that something must be wrong since the limit of the jagged lengths would have to be 1, because that is the length of [0,1]. Write an explanation to this classmate about what is going on here. 
Web Problem 6 Describe in as many ways as you can (geometrically, algebraically, in terms of some physical situation, etc.) what it means for a function $f(x, y)$ of two variables to be continuous at a point, say at the origin (0,0). How does this compare to your understanding of continuity for a function $g(x)$ of one variable?

Recall that the purpose of these assignments was to explore the use of concepts learned during the first semester of calculus in the process of understanding other, challenging concepts (rather than to evaluate the students' understanding of the concepts). Consequently, all problem statements were intentionally chosen so that fully correct explanations would be slightly beyond the ability of most students in a first-year calculus course. In addition, for each assignment, the students were given the following instructions for writing a response:

> You may appeal to intuitive concepts and real-world experiences as well as formal mathematics. Also, feel free to be philosophical, especially when it comes to contemplating your own understanding. The purpose of these exercises is for YOU to explore ways to conceptualize the ideas in this course, not for us to assess whether you can correctly answer the questions. Consequently, even if you are unsure about the material, you can still score extra-credit points for deep reflection about your understanding. Although there is no length requirement, it will probably take around 400-500 words to reasonably explore these ideas.

Phase I and II Data Analysis

The full text of all interview transcripts and the written responses to the web problems were coded and analyzed as described below using NUD*IST (a software package for qualitative data analysis, Qualitative Solutions & Research Pty Ltd, Markham, ON, Canada). Brief descriptions of students' responses to the pre-course survey and writing assignments from Phase I and the pre- and post-course surveys from Phase II were also entered into NUD*IST, with indices for reference to the actual documents throughout the coding process.

Initial Coding

The initial open coding consisted of several direct readings of each piece of data immediately after it was collected. During these readings, any noticeable contexts, images, strategies, and language usage were marked, regardless of whether or not they seemed relevant to the student's limit concepts. Though very liberal criteria were used for coding statements and exchanges at this point in the analysis, ancillary uses of phrases (such as "approach" in "the square of a variable will approach a finite number") were ignored. Only uses of words and phrases within some context providing a possible interpretation were coded.

Loosely defined response categories (such as "appeal to definition," "image of a molecule getting stuck," "focus on computation," and "approximation language") were established by reviewing all of the coding for patterns, and all text was recoded according to these categories. Passages where coding decisions were ambiguous were marked and reviewed later to see if an adjustment to the category descriptions could resolve the ambiguity. As additional data was collected, an initial open coding was conducted as described previously, followed by coding for the emerging categories.

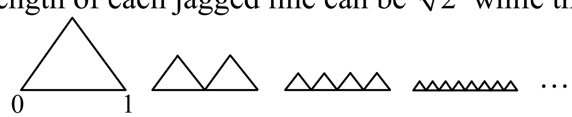
Metaphor Clusters

After the end of the second semester (Phase II), all of the text coded in each of the preliminary categories was examined, in detail, for common language, logic, images, and applications to specific limit concepts. Several of these categories (e.g., collapse, approximation, practical limit, closeness, physical limitation, infinity as a number, motion, and zooming) contained sufficient commonalities to describe an underlying schema consisting of a non-mathematical context, objects and structures in the context, and relationships and a logic among the objects and structures. All of these schemas were applied metaphorically in students' explanations about limits.

The theoretical perspective in this study characterizes metaphorical reasoning as the *application* of a schema against a conceptually problematic situation such that one's perspectives on both the problematic situation and the metaphorical schema are altered. To capture this dynamic, metaphor clusters were developed from the preliminary categories. A metaphor cluster is a characterization of the application of a particular schema (such as approximation) in various problem contexts. A cluster consists of a common schema, the various contexts in which it was applied, the details of each application, and the conclusions drawn by the students about both the schema and the problem context.

Development and characterization of the metaphor clusters required 1) several students responding to any given problem context, 2) in sufficient depth to reveal aspects of the structures and usage of their metaphors, 3) in ways that were also observed in other problem contexts. Items in the study were not all intended to elicit responses for this level of analysis and were not all used in the development of the metaphor clusters (see discussion below). In addition, none of the items were pilot-tested, so some provoked responses from students that could not be used for this purpose. For example, questions that sounded simple to the students received uniform responses which tended to be too brief to reveal any depth in the students' thinking. Questions that were too difficult or worded in ways that caused confusion tended to provoke unintelligible responses for which decisions could not be made about coding. Responses from 10 of the problem contexts (2 interview settings and 8 writing assignments) from Phases I and II of this study contained sufficiently rich data to be used in this process (See Table 9).

Table 9. Problem Contexts Used for Development and Characterization of Metaphor Clusters.

Context	**Paraphrased Problem Statement***
Interview A	Explain the meaning of $\lim_{x\to 1} \frac{x^3-1}{x-1}$.
Interview B	Let $f(x)=x^2+1$. Explain the meaning of $\lim_{h\to 0} \frac{f(3+h)-f(3)}{h}$.
Pre/Post #2	Explain why $0.\overline{9}=1$.
Pre/Post #11	Explain why the derivative $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ gives the instantaneous rate of change of *f* at *x*.
Web #1	Explain why L'Hospital's Rule works.
Web #2	Explain how the solid obtained by revolving the graph of $y=1/x$ around the *x*-axis can have finite volume but infinite surface area.
Web #3	Explain why the limit comparison test works.
Web #4	Explain in what sense $\sin x = 1-\frac{1}{3!}x^3+\frac{1}{5!}x^5-\frac{1}{7!}x^7+\cdots$.
Web #5	Explain how the length of each jagged line can be $\sqrt{2}$ while the limit has length 1. 
Web #6	Explain what it means for a function of two-variables to be continuous.

*See Table 5 for Interview A and B protocols, Table 7 for exact statements of the pre- and post-course survey questions, and Table 8 for the full web problem statements.

In students' written work, several students did provide an in-depth discussion of their thinking about these questions. In some cases, however, shorter descriptions of a student's ideas could be counted as an application of a specific schema and used in its characterization if that description was both explicit and central to their justification or illustration of their reasoning. A slightly stronger requirement was used for the interview data since there is a greater possibility that the interviewer's questions could have led a student to respond with a certain metaphor rather than students generating metaphors on their own. To be included in the characterization of a schema, a response required an extended discussion involving a particular application initiated by the student. In all interviews, the interviewer attempted to only ask questions that either probed for more details about statements the student had already made or to refocus their attention on aspects of ideas already discussed. The interviewers' effect, however, can never be completely eliminated. Thus, even if it was mildly provoked by the interviewer, a student's adoption and strong use of a metaphor was noted. Thus, no attempt was made to exclude cases where the student took the initiative to continue a significant discussion of a metaphor inadvertently introduced by the interviewer.

Furthermore, in the characterization of the underlying schemas and their metaphorical applications, only problem contexts in which a minimum of 10% of the students responded with that schema were used. To establish this cut-off, the researcher initially picked a frequency that clearly included all of the examples that were helpful in the earlier rounds of coding. He then went back and reexamined applications in contexts that were close to 10% on either side to determine if there might be a reason to question inclusion or exclusion in any case. For example, one might argue for a possible exception for including an infrequent application in the characterization of a cluster based on an exceptionally strong usage in a small number of instances. On the other hand, it might be desirable to exclude a common application if a majority of the codes are potentially attributable to a convergence of dead metaphors (i.e., metaphors which have become convention and do not generate a new perspective). In this examination, however, no compelling cases for exceptions were found. Some minor cases were, however, used to further illustrate some of the features in the presentation of data in Chapter 5.

Of the original categories, several (e.g., motion and zooming) failed to meet any of the three criteria of being applied in consequential ways in multiple problem contexts by more than 10% of the students. Others (e.g., approximation and practical limit) could not be distinguished from one another by their schema structures or their applications and were combined into one cluster. The five resulting metaphor clusters were collapse, approximation, closeness, physical limitation, and infinity as a number. (These clusters are described in detail in Chapter 5). Once these 5 metaphor clusters were fully developed and characterized, the data was recoded for the tabulation of frequencies in various problem contexts and for the selection of archetypical examples. (A large portion of Chapter 5 is comprised of discussions of excerpts from these selected examples).

Interpretations of Special Words and Images

Although the data in this study could not support the development of students' uses of ideas that fell into the categories of motion and zooming, the interviewer did explicitly ask students how they interpreted the corresponding language and imagery. In all interviews during Phase I and Phase III (the third phase is discussed below), students were asked how they interpreted words such as "approaching" and whether they thought of motion of any kind (see Table 5 and Table 14 for the interview protocols). During Phase I, the professor heavily used imagery about zooming in on the graph of a function to give an intuitive description of a tangent line, and the researcher asked students to discuss their interpretations of this imagery in one of the writing assignments (see Table 6 for the problem statement). In addition, during the interviews in Phase III of the study, students were asked how they interpreted the words "arbitrarily" and "sufficiently" (see Table 14 for the interview protocols).

The data from each of these questions were used to develop categories of responses similar to the preliminary categories to the metaphor clusters. The difference between these categories and a metaphor cluster is the development of an underlying schema and the characterization of its use in various problem contexts. Nonetheless, these categories reveal aspects of how the students interpreted these key ideas relevant to limit concepts and are described in a short section in Chapter 5.

EXPLORING STUDENTS' INTERPRETATIONS OF EXPLICITLY PRESENTED METAPHORS

Although portions of the data collected in the first two phases of the study represent some very strong ways in which students understood limit concepts, certain aspects revealed serious deficiencies in these students' conceptual metaphors. The most obvious (though probably not the most effective) method to influence students' use of metaphors toward the development of more standard interpretations is to explicitly teach the desired versions. Phase III of the study was designed to investigate students' interpretations of explicitly presented versions of the approximation and closeness metaphor clusters. These clusters were modified to more closely resemble the formal limit definitions and presented to the students in a very prescriptive set of exercises and writing assignments.

The formal limit definitions are notoriously difficult for students (Cornu, 1991; Sierpinska, 1987; Orton, 1983; Tall, 1992; Davis & Vinner, 1986) and may often be omitted from standard first-year calculus courses. Consequently, there are probably several aspects of students' uses of limit metaphors that might be more important to address than making them more closely resemble the formal definitions. There were two reasons behind this choice, however. First, as discussed earlier, the class in which this study was conducted focused heavily on both formal and informal epsilon-delta and epsilon-*N* ideas. Second, the changes to the structures and logic of the observed approximation and closeness metaphors, required to make them more closely resemble the formal definitions, were relatively unambiguous.

As a point of emphasis, the intention of this phase of the study was to explore how students responded to external attempts at influencing their use of certain metaphors. This objective contrasts with the intention of teaching students these metaphors, for which the many decisions about the design of the activities and the data collection would differ significantly. Consequently, it is important to focus on the students' interpretations of the concepts presented to them rather than on the evaluation of pedagogical effectiveness.

Phase III Data Collection

While all of the metaphor clusters observed in this study contained some aspects divergent from formal limit concepts, the schemas for the approximation and closeness clusters most directly resembled the structures in limit definitions. By suppressing the misleading aspects of the observed metaphor clusters and adding other features to their schemas, it is possible to create metaphors that are nearly isomorphic to various limit definitions. The schema modifications for the approximation and closeness clusters are described below, followed by specific metaphors based on these schemas for various limit concepts.

The Modified Approximation Schema

The approximation schema as observed in Phase I and Phase II of this study did not contain parallels for the complete logic of epsilon-*N* and epsilon-delta definitions. Specifically, the number of students who actually discussed bounding the error (i.e., an equivalent to "…then $\left|f(x)-L\right|<\varepsilon$" in the case of the limit of a function) during the first two phases of the study was quite small, so this aspect of the original schema needs to be emphasized to the students. Although some students did discuss bounding errors, there was no mention of a need to obtain *any* predetermined degree of accuracy (i.e., something corresponding to "For any $\varepsilon>0$ …"). Furthermore, there was little discussion of the way in which acceptable approximations could be generated (i.e., something corresponding to

"…there exists a δ such that whenever $0<|x-a|<\delta$…"). Linking these structures together gives the practical statement of being able to find a suitable approximation for any degree of accuracy on the one hand and the epsilon-delta definition on the other. See Table 10 for the resulting revised modified approximation schema.

Table 10. The Modified Approximation Schema.

Structural Elements:

- An actual value to be approximated
- Approximations
- Errors
- Bounds on errors

Logic:

- Each approximation is associated with an error:
  error=|value-approx|.
- The actual value is often unknown (or unknowable).
- A bound on the error allows you to use an approximation to restrict the range of possibilities for the actual value:
  approx-bound < value < approx+bound.
- Accuracy: One can always make the error as small as desired.
- Importance of a bound: An approximation is useless without knowledge of a bound on the error.
- Practicality: For any bound, there is a method to find an approximation with an error smaller than that bound.

Note that precision can be mapped to Cauchy convergence (e.g., for sequences, informal statements made by students such as "There will not be a significant difference among the approximations after a certain point," may be mapped to the condition "If $n>N$, then $|a_m-a_n|<\varepsilon$"). This topic, however, is not covered in the calculus course in which this study was conducted, thus precision was removed from the schema to minimize possible confusion. As described above, the similarities of the revised schema to the epsilon-delta definition are fairly straightforward, as are the corresponding connections to the limit of a sequence and the definition of the derivative.

The Modified Closeness Schema

For the closeness metaphor cluster, the strategy was to tie ideas about regions in space to the language used by the professor about functions acting on intervals. The basic schema involves point-locations in a metric space, downplaying aspects suggesting a hyperreal or other nonstandard metric, while building on the intuitive experience with continuous properties of space in which nearby points have similar properties. Glossing over the $0 < |x-a|$ (i.e., deleted neighborhood) case, the intuition behind the continuity condition can then be used as a definition of the limit of a function. For cases such as the definition of the derivative or limits at points not in the domain of the function, the difference is a matter of ignoring the fact that a function may not be defined on the entire region. The modified closeness schema is given in Table 11.

Table 11. The Modified Closeness Schema and Metaphors.

Structural Elements:
• One-, two-, or three-dimensional space composed of point-locations
• Distance between points and sizes of regions of space
• Continuous properties of space
• Successively selected points (or sets of points) in space
Logic:
• Two points in space are "close" if the distance between them is small.
• A region in space is small if the distance between any two points in that region is small.
• A point (center, c) and a distance (radius, r) define a region in space (interval, disc, ball, or neighborhood, $D_{c,r}$). Thus if r is small, then $D_{c,r}$ is small.
• "Local" features are those that are present on every "small region."
• For continuous properties of space:
  o Preservation of Closeness: Small changes in initial physical locations result in small changes in properties of those locations.
  o Continuity Condition: Differences in properties can be made as small as you want by selecting locations sufficiently close together.
• For successively selected points in space:
  o Cluster point: Points may cluster around some special fixed point in space.
  o Cluster point condition: The region in space around a cluster point may be made as small as desired and will still contain all points beyond some selection.

The Activities Developing the Modified Schemas

Recall that in the perspective of this study, a metaphor is much more than a map from one domain to another. In fact, although the construction of isomorphisms from these schemas to the formal limit definitions is quite straightforward, such mappings were not presented explicitly or implicitly to the students. Instead, due to the instrumentalist framework of this study, it is important for these ideas to emerge through their active application in actual problem situations. At the same time, fostering the use of the modified schemas required some type of prescriptive activities. To attempt to balance these two needs, the modified schemas (their language, structures, and logic) were first developed through a set of fairly straightforward problems. Then students were given several problems, implicitly involving limit concepts that required the use these schemas. See Appendix B for the exact assignments given.

Clinical Interviews

For data collection, students were given a final writing assignment containing three problems which also later served as the initial prompts for the clinical interviews. The three problem contexts in these assignments were to explain the equality $0.\overline{9}=1$, the meaning of $\lim_{x\to 4} 2x+1=9$, and the reason the slope of $y=x^2$ at $x=1$ is 2. Each student was given these problems, matched in some permutation with the modified approximation schema, the modified closeness schema, and ideas about epsilon-delta or epsilon-*N* definitions as presented in the text. The exact problem statements are given in Table 12, where each blank was filled in with one of the schemas: "approximation," "arbitrarily small intervals" (for closeness), or "ideas about limits as presented in Chapter 2.2 in the textbook" (for epsilon-delta or epsilon-*N*).

Table 12. Phase III Final Writing Assignment and Initial Interview Prompts.

1. Refer to your lecture notes from last week when we discussed why $0.\overline{9}=1$ using the Archimedean principle. Write a one-page explanation to a classmate of how you understand this fact using ideas about ___________.
2. Last week in lecture, we discussed ε-δ proofs for limits of linear functions. Write one page to a classmate explaining how you understand the proof that $\lim_{x\to 4} 2x+1=9$ using ideas about ___________.
3. In the writing assignment and lecture last week, we discussed slopes of tangent lines. Write one page to a classmate explaining how you understand why the slope of the tangent line to the graph of $y=x^2$ at $x=1$ is 2 in terms of the ideas about ___________.

Only 16 students submitted responses to this writing assignment and 11 agreed to participate in a 90-minute interview over the assignment. See Table 13 for the specific versions of the writing assignments given to each of these students (i.e., which schemas were matched with each problem context).

Table 13. Students Responding to each of the Phase III Final Writing Assignment Configurations.

Problem Context			**Students**
$\mathbf{0.\overline{9}=1}$	**Tangent of $y=x^2$**	$\mathbf{\lim_{x\to 4} 2x+1=9}$	
Approximation	Closeness	Epsilon-delta	Bob, Enrique, Karen, Sandra, Nina, Steve (+ 3 others)
Closeness	Epsilon-delta	Approximation	Jacob, Marty, Cindy (+ 2 others)
Epsilon-delta	Approximation	Closeness	Janice
Closeness	Approximation	Epsilon-delta	Cheryl

The interviews for this phase of the study were slightly more directed than the clinical interviews conducted during Phase I. While students were initially asked to approach the problems in the manner of their choosing, they were also asked to give explanations in terms of each of the three schemas and to directly discuss their interpretations for the various phrases and images associated with each schema. The extended interview protocol in Table 14 reflects this additional direction. After the second of the 11 students was interviewed, it became clear that the students were using the words "arbitrarily" and "sufficiently" in ambiguous ways, so questions about interpretations of these words were added to the protocol.

Table 14. Phase III Interview Protocol.

General Questions (revisit at the end of the interview) • Describe how you think about the meaning of limits. • Is this similar to your previous understanding of limits (say from high school) or has your understanding changed? • What do you see as the purpose of all of the hard work we've done to explore the limit concept rigorously? • Can you think of examples where your previous understanding would not work? If so, how would you explain those examples using your current ideas?
$0.\overline{9} = 1$ • Disregarding what we learned in class for a moment, do you really believe that $0.\overline{9} = 1$ or do you think that they are actually different numbers in some sense? Explain. • What does it mean to say they are equal? • Describe how you think of the real numbers. What are they? What images come to mind? What does it mean to say there are no gaps or holes? How does that compare with saying that the "points" don't actually touch? • How do you think of the Archimedean Principle?

Slope of a Tangent
• What is a tangent line?
• How do you get its slope?
• How do you think of
    o Approximation of slopes within some degree of accuracy?
    o Arbitrarily small sets of possible slopes?
    o A slope m so that $\left\lvert \frac{f(x+h)-f(x)}{h} - m \right\rvert < \varepsilon$?

$\lim_{x \to 4} 2x+1=9$ • How do you think about limits like this? • Do phrases like "x approaches 4" or "2x+1 approaches 9" indicate motion to you? • Is there a difference between your intuitive understanding of this and what you would consider a formal proof?
Approximation • Tell me what you consider to be important aspects of approximation. Would your answer have been different before this class? • What mental images do you have associated with these ideas about approximation? • What are similarities and differences between approximations and limits? • Explain your understanding on bounding errors in approximation. • How do you think about accuracy of approximations? What about arbitrary degrees of accuracy? • Explain as carefully as you can the logic involved in the statement "We can make the error less than any predetermined bound, no matter how small."
Closeness/Intervals • How would you describe the important ideas about small intervals that we've discussed in this class? • What mental images do you have associated with these ideas about small intervals? • How are these small intervals related to limits? • What does "arbitrarily close" mean? • What does "sufficiently close" mean? • How are these two related? (Give example if necessary. • Explain as carefully as you can the logic involved in the statement "For an arbitrarily small interval (call it *I*) centered at *L* in the range, there is an interval in the domain centered at *c* that is mapped entirely into *I*."

ε-δ from the Book
• How would you describe the important ideas about ε's and δ's that we've learned about in this class? • What mental images do you have associated with these ideas? • How are ε's and δ's related to limits? • What does the book mean by saying that a particular value of δ "works"? • What is the relationship between ε's and δ's? • Explain as carefully as you can the logic involved in the statement "for every $\varepsilon > 0$ there is a $\delta > 0$ such that whenever $0 < \lvert x - c \rvert < \delta$, then $\lvert f(x) - L \rvert < \varepsilon$."
All Three Contexts
• We've discussed approximation, small intervals, and ε's and δ's. What are the similarities across these three things? Why did we discuss all three of these in connection with limits?

Phase III Data Analysis

The initial analysis of the interview transcripts from Phase III of the study was an open coding for contexts, images, strategies, and language usage (as in the analysis for Phases I and II). After this open coding, the transcripts were also coded for the 5 metaphor clusters developed from the first phases of the study as well as for applications of the revised closeness and approximation schemas.

Since an attempt was made in this portion of the study to prescribe the ways in which certain language was used, these words served as markers for potentially interesting exchanges in the transcripts. After the initial coding was completed, the text was inspected with a simple program written by the researcher that simultaneously displayed a moving excerpt of the transcript of fixed length and a visual representation of the contexts being discussed in the excerpt. Specifically, the program computed a moving average of word separation for key phrases in each of the approximation, closeness, and epsilon-delta contexts.

These averages were presented visually in comparison to the average separation between each of the three contexts. This comparison was used to locate portions of a transcript 1) where a student was discussing a single context (indicated by a small average separation within one context), 2) where they were discussing multiple contexts in different ways (small average separation for multiple contexts with larger separation between the contexts), and 3) where they were either comparing or interchanging words from multiple contexts (small average separation for multiple contexts with small separation between the contexts).

This quick first pass at the data made it possible to examine large portions of all of the interviews to locate interesting passages for closer scrutiny. Once these passages were identified, the usage of each key phrase in the revised approximation schema (approximation, error, bound on error, and accuracy) and in the revised closeness schema (point-locations, distance, region, clustering, and preservation of closeness) were examined. Portions of the interview where this language was used were then coded as 1) matching the revised schemas, in which case the specific aspects used were noted, 2) divergent from the revised schemas, in which case the alternate structures were noted, or 3) indeterminate. The last section in Chapter 5 presents the data from this portion of the study describing how students interpreted the language from the revised schemas and how various understandings of these concepts coexisted within each student.

Chapter 5: Summary of the Data

Phase I and Phase II of this study were designed to educe the metaphors used by students in the first and second semesters, respectively, of a year-long freshman calculus sequence. The first section of this chapter describes the five metaphor clusters observed in this portion of the study (Collapse, Approximation, Closeness, Infinity as a Number, and Physical Limitation). This is followed by two shorter sections based on the data from phases I and II. The first reports on students' interpretations of three common sets of language surrounding the teaching of limit concepts (zooming, motion words like "approaching," and the words "arbitrarily" and "sufficiently"). Following this is a case study of a single interview to provide a sense of the ways in which her limit concepts emerged and were applied in an act of inquiry. Phase III of this study was conducted the following Fall to explore first-semester students' responses to explicit attempts to influence their use of metaphors. The final section of this chapter reports on students' interpretations of the revised schemas for the approximation and closeness metaphor clusters presented to them during this phase of the study.

Since the raw data for this study is textual and qualitative, any presentation of that data necessarily involves some level of analysis. (In both qualitative and quantitative studies, no report of data or act of data collection is free of interpretation. At some point decisions are made, consciously and unconsciously, about countless factors such as the questions asked, categories of responses that are allowed, the medium for presentation of results, etc.) In this chapter, there are two levels of analysis which must be acknowledged. The first level involved the development and characterization of the metaphor clusters. All data was initially processed with an open coding scheme to identify potential metaphorical contexts and word usage, and was then recoded through iterative cycles using the emerging structures as the basis for coding in each pass. When new features ceased to emerge for any context, the result was a metaphor cluster consisting of a source schema with a structure and logic and specific metaphors resulting from the application of the schema to specific problem contexts. See Chapter 4 for a detailed explanation of data analysis.

The second level of analysis involved in the presentation of data in this chapter is the manner of presentation, itself, specifically the choices of examples to illustrate each metaphor and the corresponding commentary. Although this study was not designed to collect data for quantitative analysis, numbers are often provided to give a general sense of the frequency of occurrence of various types of reasoning. Throughout this chapter, an effort is made to allow the students' voices to be heard through the data, by providing numerous examples and using the students' own language whenever possible in descriptions of the data. The analysis accompanying various excerpts in this chapter is kept at the level of calling attention to specific features of the students' responses, with all further analysis left for Chapter 6.

An important observation at this point is that much of what the students say, as reported in this chapter, is mathematically incorrect! We will not, however, treat their statements as mere misconceptions. Instead we are looking for the roots of growth toward a future, deeper understanding of the corresponding concepts. As will be argued later, the nonstandard interpretations presented by these students are, at least, fertile sites for positive discussions. Recognizing that potential requires an effort to see past the errors.

STUDENTS' METAPHORS FOR LIMITS

For each of the five metaphor clusters observed in this study, this section presents a general overview of the cluster, a description of the underlying schema (the structure and logic of the source domain), and archetypical examples of specific metaphors occurring in various problem contexts. The five clusters observed are labeled "collapse," "approximation," "closeness," "infinity as a number," "and physical limitation." The collapse metaphor cluster involves imagining a limit as the collapse of one or more dimensions of a physical or spatial referent. In the approximation metaphor cluster, limits are viewed as a process of estimating some quantity with various degrees of accuracy. The closeness metaphor cluster is based on the metaphor of numbers as points on a line and on spatial proximity measured in space. The infinity as a number metaphors treat infinite quantities as numbers, extending algebraic and functional properties of the real numbers. The last cluster, physical limitation, assumes that there is some smallest physical scale beyond which nothing exists providing a cut-off point for conceptualizing limits. After the five metaphor clusters are discussed, we present an example of 'mixed metaphors" to illustrate the blurry lines between these categories.

Data from the 2 interviews in Phase I of the study, 2 questions from the pre- and post-course surveys of Phase II, and 6 web problems also from Phase II were used to develop and characterize the metaphor clusters. See Table 8 in Chapter 4 for statements of these problem contexts with an accompanying discussion of how the metaphor clusters were developed.

In this chapter, a frequency table for each metaphor cluster provides the number of students who were coded as responding to these 10 problem contexts with the schema for that metaphor cluster. In the data tables, the problem contexts are referenced by two-part labels briefly indicating the setting and content of the problem (see Table 15). For example, "Web #3: Limit Comparison Test" refers to the writing assignment posted on the class website asking for an explanation of the limit comparison test. Note that in the frequency tables, a student is counted as responding with a specific schema to the pre- and post course survey questions once whether they provided that response on either of the surveys or on both. Occasionally, students' responses from other problem contexts are used for the purpose of illustrating various metaphors, but are not included in any of the frequency tables.

Table 15. Brief Labels for Problems Used to Develop the Metaphor Clusters.

Data Collection Context	**Problem Content**
Interview A	Limit of a Function
Interview B	Derivative Definition
Pre/Post #2	$0.\overline{9} = 1$
Pre/Post #11	Derivative Definition
Web #1	L'Hospital's Rule
Web #2	Volume of Revolution
Web #3	Limit Comparison Test
Web #4	Taylor Series of $\sin x$
Web #5	Sequence of Sets
Web #6	Multivariable Continuity

Collapse Metaphors

The first metaphor cluster we will discuss is one in which the concept is mathematically incorrect, yet which, as I shall argue later, did appear to afford the students some ability to reason quite powerfully. In the Collapse Metaphor Cluster, students characterized a limiting situation by imagining a physical referent for the changing quantity collapsing along one of its dimensions, yielding an object that was one dimension smaller. An example of this type of reasoning was observed by Thompson (1994b) in his study of advanced students' conceptualizations of the fundamental theorem of calculus. When explaining why the rate of change of the volume of water in a container as a function of height is equal to the surface area of the water, one student erroneously explained that you could consider a thin slice of water at the top. As you let that slice get thinner, this student claimed, it's height would eventually become zero, leaving only a surface area (see Figure 11). Thompson described this students' error as

> …thinking about an increment in volume unrelated to any increment in height. Moreover, he began to think of a limiting process whereby, figurally, when you diminish the accrual's incremental thickness, you *get* an area. [He] seemed to be thinking of making the cylinder shorter and shorter, until top meets bottom. His image could be described formally as $\lim_{\Delta h \to 0} V(h+\Delta h) - V(h) = A(h)$.

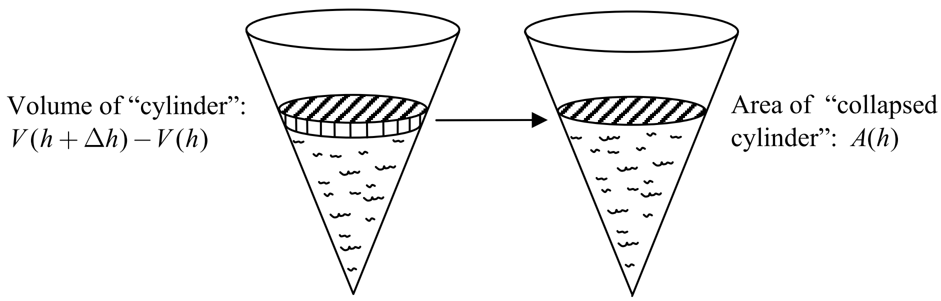


Figure 11. An incorrect image of the fundamental theorem of calculus.

Another version of this metaphor, anecdotally familiar to most calculus teachers, is the student's fallacious justification of the fundamental theorem imagining the incremental change given by "the area function." The basis of this argument is the claim that the difference quotient gives a final thin rectangle of area underneath the curve. The limit as the width "becomes" zero is then imagined to cause that slice to become the one-dimensional height of the graph.

Structural Elements and Logic of the Collapse Schema

Students' collapse schemas that were used to understand limits all involved the common structure of a one-, two-, or three-dimensional object that varies in size along some measurement of the object (such as a width or radius).

The variable is determined by the indexing variable in the limit. The variable measurement is decreased in value so that the dimensions associated with the vanishing measurement "collapse out," which results in a "collapsed" object of smaller dimension. Some property of the collapsed object is measured by the function or sequence value in the limit. Upon collapse the properties of the object associated with the collapsed dimensions may either persist or cease to exist.

Observed Collapse Metaphors

The collapse metaphor was observed in two main versions involving descriptions of the definition of the derivative and volumes of unbounded solids of revolution. In both the interviews and written assignments about the definition of the derivative, approximately one third of the responses involved significant use of a collapse metaphor. While describing the volume of a solid of revolution, nearly one sixth of the students used a collapse metaphor. (See Table 16 for exact numbers.)

Table 16. Frequency of Collapse Metaphors in Various Problem Contexts.

Question	**Brief Description**	**Total Responses**	**Collapse Responses**	**Percent**
Interview B	Derivative Definition	9	3	33.3%
Pre/Post #11	Derivative Definition	98	36	36.7%
Web #2	Volume of Revolution	31	5	16.1%

A Collapse Metaphor for the Definition of the Derivative

When considering the limit of the difference quotient from the definition of the derivative, students would describe a dynamic secant line through two points with the base and height of a right triangle as in a standard slope illustration. The variable dimension is the base of this triangle measured by the length h or $x - x_0$ in the difference quotient. Moving these points closer together yields secant lines closer to the tangent, and the collapsed object is achieved when the two points are moved to the same location. The result is the tangent line at that point (see Figure 12).

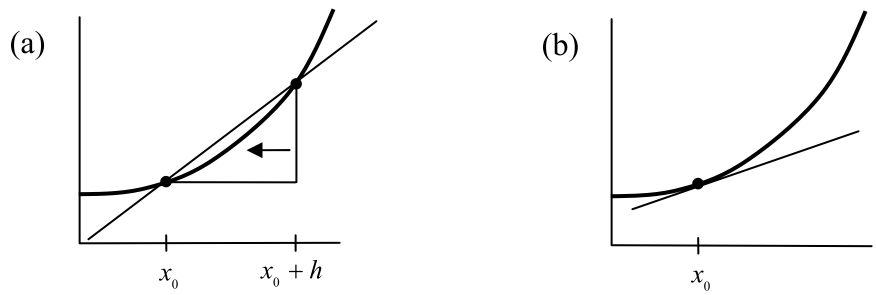


Figure 12. A secant line collapsing to a tangent. (a) Before collapse: a secant line between two points. (b) After collapse: a tangent line through "two points" at a single location.

Students were split on their assertions about whether or not the existence of two points persisted. Some described taking a slope at a single point while others reported thinking of the slope between two points at the same location.

Consider the following interview excerpt in which Amy wrestles with the role of $h \to 0$ in the definition of the derivative, and comes to the conclusion the two points become one. Amy is noticeably unsure about her claims here but writes off not being able to make sense of her own statements simply because "it works."

Amy: As you take the limit, the value h is going to be getting continually smaller until it reaches zero at which point you'll be finding - the slope - of the line between [3,10] and [3,10]. It doesn't make se-, oh it doesn't make sense but it works. I'm not, yeah, I know. But - [*pause*]

I: Well do you think that's kind of weird? Or does it make sense to you?

Amy: Well it makes sense but when you say it, it doesn't. Because you're dealing with the rate of change of the line in general. And so it's a limit process - And so it just sounds funny. [*laughs*] …What you're doing is taking the limit of the slope - of what is - actually it's the slo-, it's not the slope of the tangent line, it's just what it ends up being, but you're taking the limit, you're taking the slope of two points. It only - and the limit is involved to allow you to eventually phase out the other point - and it just becomes to be, it would be just become the slope of the original po-, of the line at the original point - Does that make sense? …Okay - so basically it would be the change in the *y* direction – between - you know - it's basically y_1 [laughs] and y_2 minus y_1 over x_2 minus x_1, is what it basically comes down to. But I - it involves - taking x_2 and, and making it gradually closer to x_1 - until x_2 is equal to x_1. Which - um - which you would also you know y_2 would be equal to y_1. And so - basically what you're doing is you're taking the slope of two points that are infinitely close together - so that they become the same point.

Two aspects of Amy's response were typical of other students who used this metaphor during the interviews. First, students often (but not always) sounded very unsure of themselves when saying something like, "You'll be finding the slope between (3,10) and (3,10)." Amy's discomfort with such a nonsensical idea is visible in her questioning and laughter at the idea. The second aspect shared by other students is the attempt to explain it away by some mystical means. Here Amy claims that "It doesn't make sense but it works." Other students described this phenomenon as "magical."

In written responses, which are not as rich as the interview data, the students lack of confidence and mystical attribution were not observable. Rather, students provided very matter-of-fact claims such as

> First given two points on a curve, its slope (or also known as the rate of change between the two points) can be found. Now if the distance

> between those two points were to begin to decrease (or as $h \to 0$, where h is Δx) they would eventually be one in the same point. At this point, the slope depends only on one point rather than two and gives an instantaneous rate of change rather than an average.

In these written explanations, 11 students responded with the collapse metaphor on both the pre- and post-course surveys. In these cases, responses tended to be textually very similar. For example on her pre-course survey, one student wrote "As the distance gets to zero, the rate of change becomes instantaneous. The tangent line shows where *h* becomes 0 and the distance between *x* and *h* is unmeasurable." In the post-course survey, the same student wrote, "h is the distance from a point. As this distance gets smaller and smaller the amount of time that the rate is taken over gets smaller until it is 0 and the rate is instantaneous." This student focuses on the same ideas in both explanations: 1) distance, *h*, between two points, 2) *h* becoming zero, and 3) the resulting instantaneous rate.

During the interviews, I also asked students to give an interpretation of the definition of the derivative for the position of a car as a function of time. The students all struggled with this new context and did not ostensibly refer to their previous work, but several gave mathematically similar accounts. (Such a similarity will be explored further as part of the case study provided at the end of this section.) In the following excerpt, Eloise explains instantaneous velocity as a collapsed average velocity, and like Amy, becomes unsure of herself in the process.

Eloise: Because we're wanting the instantaneous velocity, not - like at that one point. Because it's always increasing. The velocity is always increasing, so in order to find it - how fast the car's going, you have to just isolate it and look as - as the difference - the functional values - as the difference gets closer and closer to just that one point.

I: Is there a way to describe that it in the language of the car?

Eloise: [*laughs*] In the language of the car? OK.

I: So you were talking about function values, [not a car].

Eloise: OK. Yes. So the car's driving along this parabola, and we're saying the distance that the car has traveled between $x = 3$ and $x =$, or time 3 and time $3 + t$. So as the interval between the two times, 3 and $3 + t$, as t is getting closer and closer to zero, you want to know how fast the car is going at point t, so if you take - evaluate it as it gets closer to zero. If you only evaluate it at the one point, it would give you the velocity.

Eloise began this final statement sounding very confident, but toward the end she slowed down as she spoke and a puzzled expression came over her face. The multiple referents for t (i.e., its alternate use as elapsed time and the time-point of interest) occurs before this, so it appears that her confusion is about the validity of evaluating the velocity "at the one point."

A Collapse Metaphor for the Volume of Solids of Revolution

The second version of this metaphor appeared with students attempting to explain how the volume of a solid of revolution could be finite (Web Problem #2). Here the dynamic object is a cross-sectional disk produced from revolving a point on the curve and varying in the dimension of its radius (see Figure 13). The radius is imagined to decrease to zero at some definite point (possibly but not necessarily infinity) so that the two

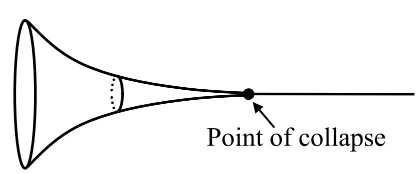


Figure 13. A collapsing solid of revolution.

dimensions of the disk collapse to a point. This collapse is imagined to occur somewhere before "the end" of the object so that what exists at any location beyond that point is also a disk collapsed to a point. Simultaneously imagining all of the collapsed "disks" beyond this point, one imagines the three dimensional solid pinching off to a one-dimensional line with no volume. The following is one student's explanation for how the volume of a solid of revolution collapses in this manner:

> At every x value, there is a disc that is rotated around the x-axis. As x goes to infinity, the area of that disc gets closer and closer to zero which I assumed implied that the volume just eventually stops. Eventually, the surface area gets so small that eventually it seems as though the "funnel" turns into just a one-dimensional line, which in a way that is what it is really turning into. It almost seems as though the volume would just have to stop whenever it turned into this line.

Some students talked about properties of the disks persisting beyond the point of collapse. For them this persistence explained, for example, how a solid of revolution could have infinite surface area but finite volume; while the property of the disks contributing volume collapsed out, some other aspect of the disks still exists so as to continue contributing surface area. In the following excerpt, it is this contradiction that alerts the student, Karrie, that something might be wrong:

> The finite volume is not really finite in the same way that familiar containers such as bowls and ice cream cones are finite. The volume is the result of a line which stretches off into infinity into the x direction. Thus, we cannot actually imagine it pinching off and ending like an ice cream cone does. Rather, the radius of the disks in the volume gets so small as the x values get extremely large that at infinity the radius becomes zero in the same way that .9999→ is actually exactly the same as 1. This progressively smaller disks actually add up to a finite amount. I imagine this "pinching off" as the two-dimensional volume (looking only

> at the disks, and taking two dimensions at a time) wrapping more and more closely around the one-dimensional line that is the *x*-axis, and then, at infinity, losing that radius entirely to zero and becoming one-dimensional, like the line. This is where volume ends, but surface area continues to exist in that single dimension. This is the only illustration I could think of, as infinite surface area and finite volume seem to imply surface area somewhere that is not associated with volume. This makes me uncomfortable, however, since the previous discussion about infinite area and finite volume doesn't seem to lend itself to the same construct. Area under the curve that is not associated with any volume does not make any sense.

For Karrie, the collapse occurs "at infinity" but the object continues to exist beyond this where "volume ends, but surface area continues to exist in that single dimension." Again, this causes some concern for Karrie, and her explanation is full of hedges to soften her commitment to a complete idea collapse. Karrie's reference to the similarity of this phenomenon with the equality of $0.\overline{9}$ and 1, can be explored through her response to the pre-course survey:

> If you keep on adding .9+.09+.009, etc. on to infinity, you will each time be adding a smaller fraction of the distance between the previous number and one. At infinity you will eventually fill that space *with infinitely small parts* (original emphasis). The series $9/10^n$ converges to zero, because the denominator increases very quickly, causing the fraction to increase to an infinitely small number. If you add together all the elements in the series, you start at .9 and keep adding smaller and smaller pieces. This does not add up to an infinitely large number, because the numbers you are adding eventually become zero at infinity.

Here Karrie expresses the belief that the size of the terms, and thus the length that they fill on the number line, "eventually become zero at infinity." She does not explicitly mention whether she imagines null terms continuing in the sum or whether that represents an end for her. Either way, however, her reason for concluding that the series is convergent is that the terms collapse to zero.

Approximation Metaphors

The most common metaphor cluster for limits that emerged from the data involved ideas about approximation. The strength and frequency of approximation ideas among calculus students is not surprising since much of the subject is historically motivated by needs for numerical estimation techniques, which still influence our classroom and textbook presentation and language today. For example, the students in this study used a textbook that described infinitely repeating decimals primarily in terms of approximation:

> If we stop the decimal expansion of a given number at a certain decimal place, then the result is a rational number which approximates the given number. For instance, 1.414 = 1414/1000 and 3.14 = 314/100 are common rational number approximations for $\sqrt{2}$ and π, respectively. More accurate approximations can be obtained by taking more decimal-places in the expansions. (Salas, Hille, & Etgen, 1999).

In its definition for the derivative, the text declares that the difference quotient gives the slopes of "approximating secant lines" and that the resulting tangent "is the line that best approximates the graph of *f* near the point $(x, f(x))$." Approximation is presented as a major application for the concept of continuity and for improper integrals, and after carefully reading the five sections in the text devoted to Taylor series and power series, it would be difficult for a student to conclude that these topics were about anything other than approximation.

Initial codes for the approximation cluster were based on simple occurrences of words traditionally associated with approximation such as “estimate,” “error,” “accuracy,” etc. Examination of the usage of the related passages of text then revealed that the students were using these words with the standard meanings; so for example, one would want to decrease the error and improve the accuracy. Consequently, the schema structure given below will appear very straightforward. When responses were recoded for occurrences of the approximation schema, several passages originally coded as approximation were then excluded because language was present without evidence of the schema structure and logic.

Initially, responses using phrases such as “virtually the same,” “negligible difference,” and “considered equal for all practical purposes” were coded as a separate metaphor cluster dealing with practical versus theoretical or immeasurable differences. The subsequent schema, however, had a structure and logic similar to the approximation schema. As we will discuss later, an examination of word usage also showed that these phrases were being used in similar ways as phrases like “approximately equal.” Furthermore, words from each of the clusters were often used in conjunction with words from the other. Since distinctions in the structures of a schema, the logic of their relationships and usage of words describing that structure and logic are the main features for making coding decisions, the “Approximation” and “Practical Limit” clusters were subsequently merged and labeled “Approximation.”

Structural Elements and Logic of the Approximation Schema

The main components of the approximation schema are an unknown actual quantity and known approximations that are close in value to the actual quantity. For each approximation, there is an associated error,

error = | actual value – approximation |.

Consequently, a bound on the error allows you to use an approximation to restrict the range of possibilities for the actual value as in the inequality

approximation – bound < actual value < approximation + bound.

An approximation is contextually judged to be accurate if the error is small, and a good approximation method allows one to improve the accuracy of the approximation so that the error is as small as desired. An approximation method is precise if there is not a significant difference among the approximations after a certain point of improving accuracy. (Note: the students did not differentiate their use of the words "precise" and "accurate," using both to refer to small error. Several students did, however, discuss the concept of precision, so for clarity, we will use the words according to their standard meanings.)

While there are several aspects of this approximation schema that are structurally similar to epsilon-delta or epsilon-*N* definitions of limits, several of the students' logical entailments are unfortunately divergent. Specifically, typical claims about approximations made by the students in this study are: 1) in practice, the actual value may be replaced with the approximation; 2) the error becomes negligible and may be ignored if it is small in comparison to the actual value; 3) there cannot be a final answer because you can always find something more accurate; 4) infinitely many refinements to an approximation gives the exact value (or the approximations can be made so close that it equals the actual value); and 5) accuracy improves with each successive step/term.

Observed Approximation Metaphors

As indicated in Table 17, students' discussions about the equality $0.\overline{9} = 1$ and the Taylor series of sin(x) emerged as the two contexts in which they most heavily used the approximation schema. These were the only two instances where over half of the students responding to a particular question used a given metaphor cluster. Over a third of the students discussed the definition of the derivative in terms of approximation in the pre- and post-course surveys as did one student in her interview on the derivative (see Shawna's interview, the case study at the end of this section). Around one fourth of the students used approximation to talk about the volume of a solid of revolution, and one of the nine interviewees described the limit of a function as an approximation.

Table 17. Frequency of Approximation Metaphors in Various Problem Contexts.

Question	**Description**	**Total Responses**	**Approximation Responses**	**Percent**
Interview A	Limit of a Function	9	1	11.1%
Interview B	Derivative Definition	9	1	11.1%
Pre/Post #2	$0.\overline{9} = 1$	103	72	69.9%
Pre/Post #11	Derivative Definition	98	34	34.7%
Web #2	Volume of Revolution	31	8	25.8%
Web #4	Taylor Series of sin(x)	35	26	74.3%

An Approximation Metaphor for Infinite Series (Infinite Decimals and Taylor Series)

The two problem contexts in which approximation metaphors were most predominant both involved infinite series. For the pre- and post- course survey questions about the equality $0.\overline{9} = 1$, students were more likely to use the approximation schema (72 out of 103) than they were to even mention limits or infinite series (59 out of 103, with only 17 doing so correctly). When discussing the Taylor series of sin(x), a larger percentage of students used an approximation metaphor than any other metaphor or problem context in this study. In addition, the students used the approximation metaphor prolifically. For example, these students dedicated an average of 15 out of 31 lines of text specifically describing approximation ideas, more than any other metaphor or problem context. (On average, each instance of a metaphor was given 10 lines out of a 36 line response.) These students dedicated an average of 15 out of 31 lines of text specifically describing approximation ideas, also more than any other metaphor and problem context. (On average, each instance of a metaphor was given 10 lines out of a 36 line response.)

The application of the approximation schema in the contexts of infinite decimals and Taylor series was similar in many ways. Students described partial sums as approximations, the limit as the value being approximated, and the difference between the two as the error. Discussions of accuracy were abundant in both cases, but students only described trying to bound the error for the Taylor series (using the LaGrange remainder).

Another difference between the two cases occurred when students considered whether the infinite series as a whole would equal its limit. Even though the surveys asked the students to explain why $0.\overline{9}=1$, almost every student described the infinite decimal $0.\overline{9}$ with words indicating that, however close $0.\overline{9}$ might be to one, it is not exactly one. For example, Roland claimed "The repeating decimal $0.\overline{9}$ expands to infinity the number 9. For all practical purposes this number is equal to one because the difference between $0.\overline{9}$ and one is a marginal infinite difference." Thus the difference is marginal, but the implication is that there is still a difference. Other students referred to "irrelevant" and "negligible differences," often mixed with language more directly associated with approximations such as "infinitely small errors" that "don't matter." Janet provides a similar response as Roland's using slightly different language, "The difference between $0.\overline{9}$ and 1 is so negligible that it is accurate enough to call it equal to 1…. $0.\overline{9}$ is so very close to 1 that it is practically equal to 1."

Notice that Roland and Janet both treat the infinitely repeating decimal as an object, specifically as a number that can be subtracted from one, as a point that is near one, or as some blend of the two. Other students described $0.\overline{9}$ as an infinite process of approximating the value one, varying in terms of whether they indicated an ability to control the process at hand. In the following excerpt, Nadia explicitly indicates agency by suggesting that one's action of "adding nine" drives the process:

> If you keep adding 9, the infinitely closer you get to 1, therefore $0.\overline{9}$ is pretty much equivalent to 1. The repeating decimal $0.\overline{9}$ keeps getting infinitely closer to 1 for the repeated 9 makes the decimal infinitely closer to 1. Therefore, this makes $0.\overline{9}$ approximately equal to one.

The process described by another student, Valerie, is similar except that the agent of control appears external to her, "[the partial sum] a_n becomes so incredibly close to 1 that it is more accurate and convenient to write $a_n = 1$ as opposed to the repeating decimal $0.\overline{9}$."

Again, notice that implicit in all of the students' explanations of why $0.\overline{9} = 1$ is the assertion that they are not actually equal. This claim was nearly universal, and some students even stated this as an explicit objection to the question and did not write anything else. For the Taylor series, the equality of the infinite sum and sin(x) was not typically mentioned, and the students who did discuss the issue were not in general agreement as they were in the infinite decimal context. Explanations that the infinite series is equal to sin(x) typically focused on the error becoming zero (essentially as a collapse metaphor), as in "If n goes to infinity then the polynomial becomes the exact value of the function because there would be no remainder. As n approaches infinity there is less error, therefore the remainder will be equal to zero." Other students, in contrast, claimed that "You can approximate the sine curve using the Taylor polynomial approximation. If you add up an infinite number of the terms you get from Taylor's theorem, you will have a very close approximation of the sine curve." Such an approximation "can become infinitely close to being equal to the function that it is trying to mimic if taken out to an infinite number or degrees, but it will never actually be exactly equal."

Students also used more of the aspects of the approximation schema reflecting the structure of the epsilon-N definition when talking about the Taylor series compared to the infinite decimal. In the following excerpt, a student describes the length of the approximating polynomial as "depending on how close you want your value to come to the value of sin x."

> To think of sin x as a polynomial would be incorrect, because although an approximation of its value can be determined by a polynomial, the sin x itself is a function who will technically never equal the polynomial exactly. It can however be useful to think of sin x as equal to this value though, because although the power series for sin x and sin x are 2 different functions, their values are very close to one and other. So for every day use of values for sin x, their values will be close enough to think of as equal. In fact the power series for sin x will approximate a value infinitely close to the value of sin x and even a remainder can be calculated…. The power series of sin x continues forever depending on how close you want your value to come to the value of sin x, and since it would be impossible to have infinite time to calculate a value, the values for sin x and its power series could never be technically equal, or could they? This is where I get a bit lost. Dealing with the concept of infinity and the definition of equal seems very abstract to me. The remainder is designed to show how much a power series deviates from the value of a function at a particular point, so I guess they will never be equal, but since their values can come infinitely close to each other, its easy to think of a function like sin x and its power series which is a polynomial as the same thing. I guess a more accurate statement would be to say that the power series or polynomial for sin x is an approximation of its value that can be as close of value as you want it to be.

Some students described specific methods for being able to bound the error by a certain amount using the fact for alternating series that "the maximum error given a polynomial is the is the next term" or by using the Lagrange formula.

An Approximation Metaphor for the Definition of the Derivative

Aspects of the approximation schema reflecting the structure of the epsilon-delta definition of a limit did not surface when students applied the approximation schema to the definition of the derivative. The slope of the tangent line was the unknown quantity to be approximated and the approximations were values of the difference quotient $\frac{f(x+h)-f(x)}{h}$ for different values of *h*. There was little mention of error or the difference between the slope of the tangent line and a secant line. If students tried to deal with the limit directly, they tended to use a collapse metaphor, as described above.

There was very little variation in the nature of the approximation metaphor for the definition of the derivative. The following excerpt from a response to the post-course survey question is typical.

> If you want to find the slope of the tangent line at the point *x* of the graph $f(x)$, a good approximation would be the line between the two points $(x, f(x))$ and $(x+h, f(x+h))$, with *h* being a small number. The slope of that line would be $\frac{f(x+h)-f(x)}{(x+h)-x}$ or $\frac{f(x+h)-f(x)}{h}$. The smaller you make your *h*, the better an approximation you would have, since the two points would be get closer and closer. So if you just did $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ you would have the slope of the tangent line at $(x+h, f(x+h))$, or the instantaneous rate of change of *f* at *x*.

Notice there is no indication of exactly what is meant by "the better an approximation," or how the limit turns "a good approximation" into "the instantaneous rate of change." The relationship between the approximation and collapse metaphors for the definition of the derivative are discussed further in the case study at the end of this section.

Two students tried to use the tangent line as an approximation to a function in order to explain L'Hospital's rule in response to Web Problem 1. These attempts appeared to mimic a proof found, perhaps, in an alternate textbook. (The text for the class did not prove L'Hospital's rule nor treat differentiation or tangent lines in the ways described by most of these students.) As illustrated in the excerpt that follows, the responses of these two students were confused and incorrect in several of their details. It is interesting, however, to see how the students interpreted and re-presented the ideas.

> A function is said to be differentiable at x if and only if $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ exists. In other words, a function is differentiable at $x=a$ if $f(x)$ is very close to its tangent line $y=f'(a)*(x-a)+E_1(x)$. $E_1(x)$ is considered an error term which goes to zero as x goes to a. $E_1(x)$ must approach zero so fast that $\lim_{x\to a}\frac{E_1(x)}{x-a}=0$ because $\frac{E_1(x)}{x-a}=\frac{f(x)-f(a)}{x-a}-f'(a)$. From the definition of derivative, we know that this quantity has the limit zero. Similarly, if g is differentiable at $x=a$, $g(x)=g(a)+g'(a)*(x-a)+E_2(x)$. When you are computing $\lim_{x\to a}\frac{f(x)}{g(x)}$, the numerator becomes indistinguishable from $f(a)$ and the denominator from $g(a)$, so the limit is $\frac{f(x)}{g(x)}=\frac{f(a)}{g(a)}$. If both $f(a)$ and $g(a)$ are zero, then you can use the tangent approximations to say that:
>
> $$\begin{aligned}\frac{f(x)}{g(x)} &= \frac{f(a)+f'(a)*(x-a)+E_1(x)}{g(a)+g'(a)*(x-a)+E_2(x)}\\ &= \frac{f'(a)+E_1(x)/(x-a)}{g'(a)+E_2(x)/(x-a)}.\end{aligned}$$

> In other words, when both function values approach zero as x approaches a, the ratio of the function values just reduces to the ratio of the slopes of the tangents.

Here, taking a limit at a point makes a function "indistinguishable from" it's value at that point. A function is considered differentiable at a point if it "is very close to its tangent line" and the difference is an "error" that goes to zero "fast" enough, i.e., $E_1(x)/(x-a)$ also goes to zero. Thus, a tangent line is an approximation to the function that can be used in computations, possibly taking appropriate care with the error (the other student simply replaced the functions with the equations for their tangent lines).

An Approximation Metaphor for the Volume of Unbounded Solids of Revolution

Often, the practicality of approximation was mentioned by students. When the error is "insignificant," it is no longer worth keeping track of the distinction between an approximation and the actual value. Eight of the 32 students responding to the web question about the finite volume of a solid of revolution described approximating volume of the entire unbounded solid by using a very long, but bounded solid. These students then claimed either that the remaining volume was so small that it is "practically negligible" or at least "that it can be ignored compared to the large portion of volume near to $x=1$." Notice how one student combines the use of several senses of the word "practical."

> Even though there is a tiny hole in the end of it, at the extreme values of x for the graph of $y=1/x$, the hole becomes practically negligible. Since there will not be a significant increase in volume after a certain point down the x-axis though values continue to increase infinitely, a volume

> can be estimated. This approximation is accurate enough for most practical purposes, and can be mathematically derived.

Initially, "practically negligible" indicates that the "tiny hole" *nearly* or *almost* vanishes. Later, "practical purposes" means that there is some *realistic use* intended for the volume. In addition, there is an implicit use of "practicality" in the final suggestion indicating that, *in practice,* only finite volumes can be calculated.

An Approximation Metaphor for the Limit of a Function

Several students justified using approximations for the practical reason that "they are much easier to deal with" than the actual value. Students in this study described using polynomials instead of sin(x), writing 3.14 instead of π, or rounding to a given decimal place as approximations that made calculations simpler. In the case of rounding, the roles of approximation and actual value were conflated for the students. For example a long decimal number might be seen as approximating some value, but rounding to a decimal place is justified by treating the new number as an acceptable approximation.

When asked about the meaning of a computation she performed during the interview on limits, one student, Jessica, said that the limit was not reached and gave an explanation using the approximation schema based on rounding. She considers a number very close to the limit and claims that the difference is small and that it is acceptable to round for practical purposes:

> I think of it as if something is approaching a certain number, like as x approaches one, at that point I guess the y value approaches three, and the limit is three but I don't think it actually gets to three but it's so small like I

> was saying and that problem like the difference - it's like 2.9999. It gets so close to three that you can just kind of round it up to three. And it can't go past that point because I don't know why. It just can't…. Yeah. Like it can't - it can't - yeah. Like it can't be three or something, but it's so close to three that you just kind of like round it because like it's kind of like probably like a millionth or a thousandth close, but it's not going to be that big of a difference for - for what ever you're using calculus for.

Later, Jessica used the approximation schema to describe the purpose for making the difference as small as possible: "If you have some like physics problem or some engineering problem, you need to find out like how close - if you plug in your input, like what your output is gonna be, like within really close."

Closeness Metaphors

Based on the metaphor of points on a line for the real numbers, spatial representations of concepts in calculus are abundant. These images support a cluster of metaphors for limits bases on spatial proximity or "closeness" and "clustering." This approach is roughly metric-topological, in the sense that one imagines sets of points, with some sets being closer to the limit than others. If conceptualized as a coordination of becoming closer in two separate spaces, this schema can resemble the structures of epsilon-delta or epsilon-*N* definitions. The schema can also appear in a much more intuitive form with very physical language about one object getting closer to another. One student, for example, stated that for large values of *n*, the graph of the Taylor series "wraps very tightly around the graph of $\sin(x)$."

Small changes in initial states of physical experiences often result in small changes in properties of those states. That is, most physical situations that can be modeled with functions are continuous. This experience often leads students make claims such as if two points x and y are close together, then the resulting function values $f(x)$ and $f(y)$ will also be close. These function values can be interpreted as locations themselves or as properties of the space containing x and y. For any point in the domain and a point nearby, there is always a point even closer. For this closer point, the function will be even closer (higher, etc.) than the other point.

Other common claims using this schema are that a function is differentiable if the graph of f is "close to the tangent line;" the limit of a sequence is the "closest value" or the value "infinitely close" to the terms; and the purpose of limits is to look at the value of a function near a certain point.

Structural Elements and Logic of the Closeness Schema

The observed closeness schema consists of a one-, two-, or three-dimensional space composed of point-locations, measurement of distances between points and of sizes of regions in space, continuous properties of space, and successively selected points (or sets of points) in space. Two points in space are "close" if the distance between them is small, although for any two points, there are infinitely many that are points closer. In most cases the topology of the space resembled hyperreal (rather than real) lines, planes, and spaces because "infinitesimal distances" could be measured. Points in space may have numerically measured properties; if so, small changes in initial physical locations result in small changes in the properties of those locations. Finally, successively selected points (or sets of points) may cluster around some special fixed point (or set of points) in space.

Observed Closeness Metaphors

The closeness schema was observed in six of the problem contexts in this study. Numerical tallies for the frequencies in each context are provided in Table 18. Even in cases where the structure and logic of this schema are not involved, much of the language used in a calculus course is language about closeness. Thus, while almost every student used this language in nearly every response, it was typically not accompanied by an explicit discussion involving the structure and logic of the closeness schema. In some cases, students were probably using the schema, but did not provide direct evidence to code their response as such. For example, several of the 103 responses to the pre- and post-course survey question about the equality $0.\overline{9} = 1$ were ambiguous in this respect, and only 11 could be coded as a closeness metaphor. Responses in this context were similar to those for the web problem about the Taylor series of sin(x), and these will be discussed together below. Likewise, the usage of the closeness schema in the contexts of continuity and the limit of a function were also similar and will be discussed as a unit. A small number of students used a closeness metaphor for the definition of the derivative, so this is discussed only briefly. Finally, there was no separate closeness metaphor for L'Hospital's rule, only applications of the metaphors to the limit of a function and the definition of a derivative in the context of using L'Hospital's rule, so these responses will not be discussed separately.

Table 18. Frequency of the Closeness Schema in Various Problem Contexts.

Question	Brief Description	Total Responses	Closeness Responses	Percent
Interview A	Limit of a Function	9	4	44.4%
Interview B	Derivative Definition	9	2	22.2%
Pre/Post #2	$0.\overline{9} = 1$	103	11	10.7%
Web #1	L'Hospital's Rule	28	4	14.3%
Web #4	Taylor Series of sin(x)	35	6	17.1%
Web #6	Multivariable Continuity	25	4	16.0%

A Closeness Metaphor for Infinite Series.

Students discussed both the infinite decimal $0.\overline{9}$ and the Taylor series for sin(x) using a closeness metaphor involving the locations of objects to represent values of the partial sums and the limits. In the infinite decimal context, the objects were points on the number line, whereas in the Taylor series context, they were the graphs of the sin(x) and the Taylor polynomials in the coordinate plane. Distance was measured in a way that resembled the standard metric on either the real or hyperreal numbers.

In the following excerpt, a student lays out the elements of the closeness schema then uses a statement about small distances to explain why $0.\overline{9} = 1$:

> If we thought about this on a number line, we would know that there are infinitely many numbers between integers, and even if we drug out $0.\overline{9}$ to the 10 millionth place, we can still add one more 9 between the number and one. Pretty soon, we'll have added so many nines that the distance between our number and the number one will [be] so arbitrarily small that we assume our number just equals one.

It is difficult to know exactly what this student meant by “so arbitrarily small” (see the discussion in the section on students interpretations of key phrases about limits), but there is a hint of approximation in this since “we assume our number just equals one.” Several students described an “infinitely small” or “infinitesimal” distance, while others explained that $0.\overline{9}$ would be “the next number” or that it “would touch one.” The more physical versions of this language is illustrated by a student who describes the graphs of the Taylor polynomials for sin(x) in this way: “the more polynomials we use to approximate the original function, the closer the polynomials will wrap themselves around the original function.”

Some students explicitly discuss closeness as in this student’s description of how the idea connects her ideas of getting closer to the point, one, with her algebraic strategy of converting $0.\overline{9}$ to fraction of integers:

> The repeating decimal of $0.\overline{9}$ is equal to one b/c as you add more nines, the number infinitely gets closer to 1. If you go far enough, the decimal will eventually be so close to one that it will equal one. If you use the method of putting repeating decimals into fractions, the repeating $0.\overline{9}$ will equal one. Also, like I said before if you get closer to one it will eventually become so close that it can equal one. The connection is closeness. Just like on a graph if you have a function and a tangent line, you can [look] infinitely close to the graph and the tangent line will have the same graph as the function.

A Closeness Metaphor for the Limit of a Function and Continuity

For continuity and the limit of a function, the closeness metaphor typically involves two separate spaces, one for the domain and one for the range. The range space may be understood either spatially or as a set of possible properties for the locations in the domain space.

Andrea begins her discussion about the continuity of a function of two variables by first describing the one-variable case:

> For the function of one variable $g(x)$, the domain of the function is contained within the *x*-axis (a line). Any value of *x* input into the function will give a value representing the height above the graph (i.e., the *y* value). If we are looking at the graph according to the axis, the coordinates to describe the position of these points will have 2 variables (*x* and *y*). This graph is therefore, on a plane. To see if the graph is continuous at a point (let's say the origin), the *y* points corresponding to an *x* value that are sufficiently close to zero must also be sufficiently close to each other. Another way of stating this is saying that the right hand limit as *x* approaches zero must equal the left hand limit as *x* approaches zero. There are only two directions to look at (right and left) because geometrically, $g(x)$ is the picture of some 2 dimensional curve.

Here, Andrea describes the *x*-axis "a line," the value of $g(x)$ as a "height," and making decisions about continuity based on looking in "two directions" to determine whether certain values are "sufficiently close to each other." Specifically, points that are close to each other must be mapped to points that are also close. As Andrea begins to discuss the case of a function of two variables, she extends this language:

> For the function of two variables $f(x,y)$, the domain of this function is the *x,y*-plane. Therefore, any point on this plane (ie *x* and *y* input in) will give a single output. This output represents the height (*z*) above the plane at that particular point. Because, the domain is a plane and each point in that plane has a corresponding height, the graph of the $f(x,y)$ function will be that of a curved plane over the *x,y*-plane. As a result to determine if the function is continuous at a particular point (let's say the origin), the *z* values corresponding to the *x,y*-values within a sufficiently small radius around the origin, must also be sufficiently close to each other. Another

way of saying this is that the limits as *x,y*-values within this radius approach (0,0) must be equal to each other. Since the graph of $f(x,y)$ is a 3 dimensional curve, rather than having only a right an left hand limit, there are an infinite number of limits (i.e. a circle around the point in question) such that the limits must be equal to one another.

The "infinite number of limits" here is not described as an infinite number of 1-dimensional paths moving toward the origin. Rather, Andrea implies that there is a "circle around the point in question" and indicates that all of the values in an entire region of space "values within a sufficiently small radius" must be considered. The function values for this region "must also be sufficiently close."

Over the course of multiple lectures, the professor presented the concept of functions acting on entire neighborhoods or deleted neighborhoods of a point. This appeared to influence several students' closeness metaphors, but the important points were clouded by other concerns. For example, after graphing $f(x) = x^2 + x + 1$ and finding the image of the interval $(0,2)$ (see Figure 14), Darlene worried about why the image wasn't symmetric about $3 = f(1)$:

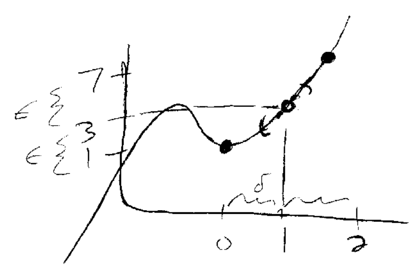


Figure 14. Darlene's graph of a function acting on an interval.

> The neighborhood should be - they should be the same distance here and here [*points at image on y-axis*]. Because it is the same distance here and here [*points at interval on x-axis*]. I don't know. Are my numbers wrong? [*pause*] it should be the same distance.

Instead of separate physical spaces for the domain and range, some students imagined a single product space containing the graph of the function. In this case, closeness was described in terms of a region around a point on the graph. This was typical of students who described continuity as being able to "trace the graph with your pencil" and who referred to the limit $L = \lim_{x \to c} f(x)$ as the point (c, L) on the graph. A less dynamic version was Lindsay's image of a graph as train tracks:

> I look at where the track ends…[to see if] they're at the same area…. if like there's like a hole on the graph, then right outside the hole - if they are - well I mean, you know, on my graph, like they are just so close to it. You know? If there was a point there, would the two tracks meet up? And that's what, you know, I'm thinking of that area right there…. But if it didn't meet up, then there wouldn't be a limit for this.

A Closeness Metaphor for the Definition of the Derivative

Two students used aspects of the closeness schema in their interview about the definition of the derivative. Similar versions were mentioned by students responding to other questions, e.g., when explaining why L'Hospital's Rule works. In this metaphor, students focus on secant lines moving to a limiting position of the tangent line. Thus, the range space is the coordinate plane where the "distance" of a secant line to the tangent can be measured. This conceptualization consisted of either visually comparing the slopes or comparing the separation between the lines within some region containing the point of tangency (such as the region visible in a graph drawn on paper). The domain space could either be the number line or the coordinate plane, depending on whether or not the student considered the step of obtaining two points on the graph of a function. The full process, for example, could be described as follows: For $i = 1, 2$ use two nearby points, x_i, to find two nearby points on the graph $x_i \to (x_i, f(x_i))$, then the secant line between these points will be "close" to the tangent line at any selected point in the region of $(x_i, f(x_i))$. Some students even defined a function to be differentiable if its graph was “indistinguishable from” or “close to its tangent line.”

As mentioned earlier, often students use a version of the closeness schema that omits the domain process. In the context of the definition of the derivative, this omission means ignoring the selection of nearby points for the secant line. The focus is then solely on the range process, in this case, the tendency of a set of lines toward a “tangent.” This is precisely what Shawna does when she draws Figure 15 in which none of her lines are shown to pass through more than one point on the graph.

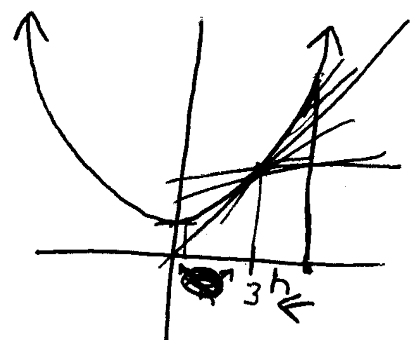


Figure 15. Shawna’s graph of a tangent as a limiting position.

Shawna: I'm trying to think of what happens as this approaches zero. We just like keep tracing this graph. Oh, I guess it. As this approaches zero, my tangent keeps getting like - keeps getting smaller until it gets to zero. [*draws lines through* $(3, f(3))$ *successively flatter*]

I: The tangent gets - flattens out?

Shawna: Yeah. *m* equals zero. That's what happens as it approaches.

I: Can you tell me what you’re thinking about there?

Shawna: Because I'm just assuming - like I know this is like I guess like a derivative, because that's how it was introduced to me, and so as *h* approaches zero, the derivative keeps getting I guess smaller, or the slope keeps getting smaller and smaller until it gets zero. As this get smaller [*points at h*], this line [*points at lines*] gets traced in more, and so it like comes closer and closer until it's here [*points at flattest line*].

Later Shawna describes that she was attending to the steepness of the lines to determine closeness.

Shawna: I was just looking at, you know, well, this one goes up [*points at steep line*] and this one goes to the side [*points at horizontal line*] more. This one's more like undefined [*points at steep line*] and this one's more like zero [*points at horizontal line*].
I: Kind of steepness?
Shawna: Yeah. I was just looking at that.

Infinity as a Number Metaphors

In the extended number system $\mathbb{R} \cup \{-\infty, \infty\}$, certain arithmetic operations can be generalized to make sense for infinite quantities in ways that reflect corresponding arithmetic properties of limits involving infinity. However, operations lacking a corresponding limit property are nonsensical. The students in this study often treated infinity as a number that could be used in calculations. Many of these students displayed a basic understanding of the subtleties involved by being able to unpack the relevant limit concepts when necessary. Although most students were at least weary of the indeterminate cases for the arithmetic, some were led to serious misinterpretations such as claiming that infinity could occur in various places on the number line.

In addition to performing algebraic operations with infinite quantities, students also plugged infinity into functions (e.g., $\ln(\infty) = \infty$), typically doing so in a way consistent with the limit of the function. Students also treated infinity as just a really big number, for example comparing the size of real numbers of infinity or describing very large numbers as approximations to infinity. Dividing by infinity, one is led to consider infinitesimal quantities, which are often described ambiguously by students as being nonexistent in size, yet not zero. Extending the metaphor of numbers as points on a line, students also represented infinity as a point, which led to a compactification of sorts in cases where this point was considered an endpoint.

Structural Elements and Logic of the Infinity as a Number Schema

The schema for these metaphors involves the real numbers, their arithmetic operations, and functions of real numbers. Representations of numbers, arithmetic operations, and functions (notably the number line and graphs of functions) may also serve as structural elements for this schema. The logic is based on the standard properties of numbers, arithmetic operations, and functions. Intuitions associated with large numbers (such as when drawing a graph or number line, the largest numbers cannot be seen because they are beyond the scope of the page) are especially important for this schema. Strong visual images are often dealt with intuitively in pre-calculus courses, introducing concepts such as asymptotes and "end behavior" without using limits.

Observed Infinity as a Number Metaphors

Not surprisingly, students used this schema in contexts involving infinite limits or limits at infinity. Between a fourth and a third of the students responded to each of the four web problems about these limits with an infinity as a number metaphor. Exact numbers are provided in Table 19. The interview contexts did not elicit observable usage of this schema. Consequently, the data does not contain probes of the subtleties and variations of students' thinking with this schema, so the four contexts are discussed together.

Table 19. Frequency of Infinity as a Number Metaphors in Various Problem Contexts.

Question	**Description**	**Total Responses**	**∞=Number Responses**	**Percent**
Web #1	L'Hospital's Rule	28	8	28.6%
Web #2	Volume of Revolution	31	8	25.8%
Web #3	Limit Comparison Test	34	11	32.4%
Web #4	Taylor Series of sin(x)	35	10	28.6%

An Infinity as a Number Metaphor for Limits Involving Infinity

One way of treating infinity as a number is to simply replace it with something very large. Ignoring the circular logic in the following excerpt, we see infinity described in this way ("infinitely large… and eventually out of bounds") before the student applies arithmetic.

> If $\lim a_k / b_k = 0$, we can infer that a_k is convergent at the same time as b_k is divergent. We can surmise this with the understanding that if a_k converges, then it has a finite value which it will eventually approach where as b_k will get infinitely large since it's divergent and eventually out of bounds. When this happens, what we have is a finite number over infinity which will be some number extremely close to zero, and we can therefore state $\lim a_k / b_k = 0$.

Dividing a finite number by infinity does not yield zero, according to this student, but "some number extremely close to zero," precisely what happens when dividing a number by something that is extremely large in comparison. In fact, using infinity as an ultimate point of comparison is described in the following excerpt, expressed in the claim that "20 times is just about nothing when compared to infinity."

> For [the limit of] a_k / b_k to exist, both a_k and b_k must be changing at a relatively the same rate. (When I use the word relative, I mean relative to things like infinity.) For example if the limit of a_k / b_k is equal to 20, the top function must be changing twenty times as faster than the bottom function. 20 times is just about nothing when compared to infinity. And so is about every other conceivable positive number that can be expressed.

When explaining how L'Hospital's Rule works to resolve the indeterminate form ∞/∞, students typically treated infinity as a number by imagining that functions grow at different rates yielding different sizes of infinity. At this point, they could think of dividing the actual limits as numbers. The following is a typical use of infinity in these responses.

> The bottom is becoming so huge so quickly compared to the top, that it is effectively dividing a small number by a huge number which is zero. If the top goes to infinity much more quickly then the bottom does, the bottom is effectively a constant as an unimaginably large number is divided by a small number, which due to the size of the top, has no appreciable effect. In this case the whole thing goes to infinity and is divergent.

Other students were amazingly creative in their application of the infinity as a number schema to L'Hospital's Rule. In the following excerpt, Fred attempts a proof by extending the Mean Value Theorem to cover rate of change over an interval $[0, x]$ on which the function is unbounded, beginning with the acknowledgment that this is a metaphorical argument.

> Although infinity is not a set number, but more of an idea, let's temporarily imagine that it is. What I mean is that at let's imagine that at some value x that is plugged in to a diverging equation, we get the value of infinity…. Let's also say the graph starts at the origin and goes to this

> "point" (x,∞)…. The derivative of a function, in a sense, shows how the curve is changing. It also shows us the slope of the curve. By extending the Mean Value Theorem to the theoretical (x,∞) point, we can assume that there is point on the domain where the slope of the line at that point is equal to the slope of the initial point and the (x,∞) point. Because the (x,∞) point is moving to infinity (y value that can't be reached), no matter how steep a slope on the curve is, if it is extended, it will be approached by the (x,∞) point. Therefore, when dealing with infinity, the derivative of a function is not just a good approximation of what the limit approaches but actually is what the infinity approaches. Therefore, when we have a case where there is infinity over infinity, we can take the derivatives of both the top and the bottom to get the limit of the function.

While completely wrong as stated, the idea behind Fred's proof can be adapted to give a correct proof. It is possible that he found such a proof in a book (although L'Hospital's Rule was not discussed in this way in the class text) and re-interpreted the application of the Mean Value Theorem to the entire interval $[0,x]$ rather than nested intervals $[0,x_n]$ with $x_n \to \infty$.

The eight students using an infinity as a number metaphor for Web Problem 2 about the volume of revolution either divided one by infinity or plugged infinity into a function (the function was the natural log in all but one case). Students using the schema to discuss Taylor series used cardinal infinity to describe the series as an actual infinite sum rather than a limit: "If we were to use an infinitely large polynomial we could write the function as a polynomial literally, but… a function with an infinite number of terms is useless for calculating values practically…"

Physical Limitation Metaphors

A set of ideas about very small-scale physical objects phenomena such as fundamental particles and quantum mechanics were described by students as relevant to limit concepts. Typically, this schema involved stating that we cannot observe or measure quantities beyond a certain scale, and even in some cases that nothing could exist beyond that scale. This line of reasoning led to two different types of claims. Some students suggested that small differences do not exist so that the two quantities, points, or graphs were actually the same if their "difference" was smaller than, say, an atom. Several other students argued the opposite claiming that real differences may exist, such as the difference between $0.\overline{9}$ and 1, that are beyond the power of mathematics to measure.

Structural Elements and Logic of the Physical Limitation Schema

The Physical Limitation Schema consists of a limiting object representing the smallest physical size possible (e.g., a molecule, electron, or quark) and other objects composed of, interacting with, or measured against the limiting object. Logically objects must be composed of something in order to exist, thus nothing smaller than the limiting object exists.

Observed Physical Limitation Metaphors

A physical imitation metaphor was observed strongly (i.e., as the main idea in several students' arguments) in students' discussions about the volume of an unbounded solid of revolution (see Table 20). This prevalence was probably partially a result of remarks made by the professor during lecture as discussed below. The schema was also used when students discussed how a sequence of sets (jagged lines) of length $\sqrt{2}$ could have a limit of length 1. Other contexts elicited low levels of responses with the physical limitation schema (small numbers of students), some of which are used below to illustrate students' thinking.

Table 20. Frequency of Physical Limitation Metaphors in Various Problem Contexts.

Question	**Brief Description**	**Total Responses**	**Collapse Responses**	**Percent**
Web #2	Volume of Revolution	31	13	41.9%
Web #5	Sequence of Sets	22	8	36.4%

The most striking use of a physical limitation metaphor occurred in students' responses to Web Problem 2, where they were asked to explain how the solid obtained by revolving the graph of $y = 1/x$ around the x-axis could have infinite surface area but finite volume. Prior to this assignment, the professor had presented this example in lecture as an interesting paradox for the students to consider. He illustrated the paradox by describing the surface of revolution as a can which one could fill with paint (because it has finite volume) but could not paint the surface of the can (because it has infinite surface area). Making what he probably intended as a throw-away comment, the professor then added that "Of course, you could never actually fill the can with paint, because at some point, the diameter of the can gets smaller the diameter of a single molecule of paint."

Thirteen of the 31 students responding to this question repeated this compelling imagery, but with a different interpretation than intended by the professor. In the students' versions, the substance varied from paint to water to ice cream, but they all claimed that the volume is finite because a single molecule would *plug up* the end of the container, allowing the rest to fill (see Figure 16). The following excerpt is typical of these responses.

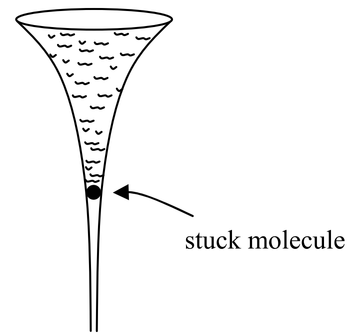


Figure 16. An image of physical limitation in a solid of revolution.

> The volume can be proved as finite by looking at a water molecule. Take a conical cup, drinking end up, and pour a single water molecule in. It slides down the side until eventually the sides get so close together that the molecule gets stuck there. Pour some more in, and it starts to fill up. Eventually, you fill the max number of water molecules and you get the volume.

One student developed a slightly different version of this argument focusing on his experience with the rate at which water flows through a funnel depending on the size of opening.

> In real life when you use a funnel, the water flows out of the funnel at the bottom. If the funnel was short and had a large radius at the bottom the water would flow out quickly. If the funnel was longer and therefore had a smaller hole, the water would flow out slower and slower the longer the funnel got. The function $1/x$ is in the shape of a funnel when it is rotated about the x-axis. The farther it goes down the x-axis, the longer the funnel gets, and the bottom radius gets smaller and smaller…. When we try to run water through the funnel, it flows out slower almost to the point of no flow at all. If the water does not flow out [of] the, funnel the volume is finite.

In Web Problem 5, students were asked to explain how a sequence of sets (jagged lines) of length $\sqrt{2}$ could have a limit of length 1. In their responses to this question, the physical limitation schema was not as essential to their main arguments as it was in the case of the volume of revolution. The students made two different types of arguments in this context, both resting on the idea that a mathematical limit somehow transcends a physical limit. One of these types of arguments was that

> It's irrelevant if the jaggeds are less than the width of a single electron. In math, you can't always visualize what you're working with (especially when dealing with the concept of infinity), but as long as you can prove it mathematically, you can establish that it's true.

Thus, students seemed to be questioning the validity of an argument based on a physical limitation schema. Another student gave more detail on why this was the case for the sequence of jagged lines.

> Please recall we are not dealing with a triangle made of wood, or plastic, or anything that contains electrons, for that matter. We are dealing with a perfect triangle made of imaginary material. The widths of the lines do not only virtually, but absolutely measure zero. They are so zero that they do not recognize a second dimension, only length.

The other physical limitation argument made by students in this problem context involved arguing that a mathematical limit transcends any physical limitation. In such cases, they also viewed the limit as retaining all of the properties of the terms of the sequence. For example, since all of the sets in the sequence in Web Problem 5 were jagged, they believed it must be the case that the limit is also jagged.

> The height of a single electron going to be invisible, so someone might say the jagged line is parallel to the base line. But is that true? If [you] look in to the microscope it would be still jagged! And obviously the total length

> of the sides [is] always $\sqrt{2}$, that means the length of jagged line is not 1, by guess, but it is $\sqrt{2}$.

This student argues that while one might "guess" that the limit is a straight line, this is only an illusion. Another version of this type of argument appeared when one student claimed that the reason sin(x) has infinitely many zeros is because he expected an n^{th} degree Taylor polynomial to have n zeroes so that the limit would have infinitely many.

Mixed Metaphors

The metaphor clusters described in this chapter do not precisely represent the thinking of any individual student in the study. Rather, they are intended to emphasize common themes and magnify structural elements and the logic involved in the students' reasoning. Most students only expressed a portion of the ideas within any metaphor cluster, and several mixed aspects of two or more. The following excerpt is an example of a student, Karrie, drawing on collapse, approximation, and closeness metaphors.

> When calculating a Taylor polynomial, the accuracy of the approximation becomes greater with each successive term. This can be illustrated by graphing a function such as sin(x) and its various polynomial approximations. If one such polynomial with a finite number of terms is centered around some origin, the difference in y-values between the points along the polynomial and the points along the original curve (sin x) will be greater the further the x-values are from the origin. If more terms are added to the polynomial, it will hug the curves of the sin function more closely, and this error will decrease. As one continues to add more and more terms, the polynomial becomes a very good approximation of the curve. Locally, at the origin, it will be very difficult to tell the difference between sin(x) and its polynomial approximation. If you were travel out away from the origin, however, you would find that the polynomial

> becomes more and more loosely fitted around the curve, until at some point it goes off it it's own direction and you would have to deal once again with a substantial error the further you went in that direction. Adding more terms to the polynomial in this case increases the distance that you have to travel before it veers away from the approximated function, and decreases the error at any one *x*-value. Eventually, if an infinite number of terms could be calculated, the error would decrease to zero, the distance you would have to travel to see the polynomial veer away would become infinite, and the two functions would become equal. This is a very important and useful characteristic, as it allows for the approximation of complicated functions. By using polynomials with an appropriate number of terms, one can find approximations with reasonable accuracy.

Karrie's language from these three metaphors is inextricably interwoven in this excerpt. The very physical language of the closeness metaphor, for example, is used to justify ideas about approximation. When the Taylor polynomials "hug the curves of the sin function" then the "error will decrease," but when they are "more loosely fitted around the curve," there will be "substantial error." Then Karrie's description shows an understanding of the difference between pointwise and uniform convergence indicating that every polynomial will eventually "veer away" from sin(*x*) but that "adding more terms… decreases the error at any one *x*-value." Finally, when there are an "infinite number of terms," the collapse occurs with multiple ramifications: "the error would decrease to zero, the distance you would have to travel to see the polynomial veer away would become infinite, and the two functions would become equal."

STUDENTS' INTERPRETATIONS OF OTHER KEY PHRASES ABOUT LIMITS

The language and imagery used to express the concepts in calculus contains several words and phrases that have divergent mathematical and everyday meanings. These differences have been shown to cause particular conceptual difficulties for students (Frid, 1994; Orton, 1983; Tall, 1990; Tall & Vinner, 1981). In this section, we investigate the ways in which the students in this study interpreted three commonly used terms and images especially relevant to limit concepts: motion language such as "approaching," "zooming in" on the graph of a function, and the terms "arbitrarily" and "sufficiently."

Motion Imagery and Interpretations of "Approaching"

Several researchers have found that a dynamic conceptualization of functions and variables in crucial to students' understanding of key concepts in calculus such as limits (Monk, 1987, 1992; Tall, 1992; Thompson, 1994b). Unexpectedly, motion metaphors were quite rare among the students in this study. While they heavily used words such as "approaching" or "tends to," they were not often accompanied by any description of something actually moving. When asked specifically about their use of the word "approaches," students almost always denied thinking of motion and gave an alternate explanation. Motion for these students was something more "literal" as suggested here by Karen:

> I guess with motion I think of - with motion I'm thinking force and work. I'm thinking of actual, like, locomotion. I don't necessarily think that that's what's happening when you're talking about a limit or talking about a number. I don't know that that's - I guess for me motion is a more literal

> term, like cars moving along the ground or I'm walking. That's more what I'm thinking than on the number line.

Only for web problem 6 about the continuity of functions of two variables were at least 10% of the students observed to discuss motion. In response to this question, 6 out of 25 students explicitly described something moving. Another 11 of the 25 respondents used motion language, but without applying it to an actual object. These students said things like

> There are [an] infinite number of ways to approach to the origin. Well now, how should we prove the continuity of this function? Well, just showing a few paths that x and y tend along are not enough. We have to generalize the cases by checking all the paths!

Here "ways to approach the origin" and "paths" seem synonymous, and while the student might be thinking of motion, they might also be thinking of static paths or of the strategy of parametrizing curves to create limits of single-variable functions for easier computation.

In the cases that something was imagined to be moving, that motion tended to be simply superimposed on another conceptual image that actually carried the structure and logic of their thinking. For example, all 6 of the motion references in responses to Web Problem 6 were to an object (an ant, a mouse, a moving truck, a baseball, the tip of a pencil, and a generic "you") moving along the graph of the function. For both the single- and two-variable cases, these students described the function as continuous if the object could move freely along the graph without having to traverse a jump or hole. In the following excerpt, a student describes continuity in terms of moving on the graph of a function of two variables.

> A good example is the surface of a big wooden board. What does it mean for this to be continuous? Imagine a tiny mouse is on the board. If the board was continuous, the cute little mouse could venture all over the board without falling to its death. If the board wasn't continuous, maybe [it] contains a hole in the center.

Thus, the concepts about discontinuity for these students were presented as topological features of the surface (holes, cliffs, breaks, etc.). The addition of motion may add visual effect or drama, but not conceptual structure or functionality.

During the first Phase I interview and the Phase III interview, students were asked how they interpreted language they used for limits that implied motion. In every case, the word "approaches" was the main point of discussion. At several times during these interviews when students used such words, the interviewer would ask them to discuss whether they thought about something moving. Of the 20 students interviewed, only eight ever agreed that they thought of motion when using a variation of the word "approaches" (see Table 21 for the tallies of all responses.) Five of these students described the motion occurring on the graph of the function, one described motion along the x-axis, and two gave explanations in which it was impossible to tell what object was imagined to move. None of these students mentioned explicit motion other than during these exchanges initiated by the interviewer.

Of the 12 students who denied imagining any type of motion, six explained that they thought of “approaches” as indicating closeness, five described picking points sequentially, and one student thought it meant that changing the value of input caused the output to change. Below are brief descriptions and examples of the responses from each of these categories.

Table 21. Students’ Interpretations of “Approaches.”

Interpretation	**Phase I Responses**	**Phase III Responses**	**Total Responses**
Motion on the graph	3	2	5
Motion on the x-axis	1	0	1
Motion: vague object	0	2	2
Static Closeness	4	2	6
Sequential	1	4	5
Input Affects Output	0	1	1

Motion on the Graph

Like those responding to Web Problem 6 about continuity, most of the interviewees who associated motion with the word “approaches” described some object traversing the graph of a function. In the following excerpt, Jennifer describes thinking about the visual effect of her graphing calculator tracing out the graph of a function.

I: When you say x is going to some number, are you thinking of something moving? Or how are you thinking about that?

Jennifer: Oh OK. Well, I would think about like ... yeah. I guess you could say moving. It's approaching. They like to use this word approaching a lot so that's always making me think of moving towards a spot at ... at the certain number.

I: You said that makes you think of moving. What exactly is moving in your mental image?

Jennifer: The only thing I can relate to ... the closest would be like the graphing calculator when it moves. I mean when it's graphing.

I: When it's graphing?

Jennifer: Yeah. When it's moving. That's the only ... that's the closest thing I can picture of moving.

Shortly after this exchange, Jennifer used the phrase "x approaching to one" when reading the symbols $\lim_{x \to 1} \frac{x^3-1}{x-1}$. In the next excerpt, the interviewer comments that this phrase implies that x is moving and asks her how that implication matches her previous description of motion along the graph.

I: When you say that, you say x approaches one. So if you take that apart grammatically, when you look at the subject for the word approaches, it's x that's approaching something. So, how would you think of that in terms of x being the thing that's actually moving?

Jennifer: I never thought of that before.

I: I'm just curious because earlier you were talking about the graph moving, like how it moves on the calculator, but the way you say it is like x is approaching something.

Jennifer: Well it's because x is, you know, - everybody says approaching or going to.

I: So, whenever you see the word - that arrow, you just replace it with the words "approaching" or "going to?"

Jennifer: Yeah. I've never thought of x actually as an object going to the place. I always thought of the graph. It's the graph moving. I mean not the graph, but the line of the function, you know - well, not the function but the graph.

Motion on the x-Axis

Only one student, Jessica, reported thinking of any motion occurring along the x-axis, but this was not accompanied by corresponding motion on the y-axis. Instead she imagined moving to the point in question then "looking up" at the function value (or "the hole" where the function should be.)

I: You said traveling. Do you actually think about something moving?

Jessica: Yeah. Probably. I don't think I would say traveling on the actual graph because I wouldn't feel like I was traveling to one. You would be traveling this distance rather than going to one just straight on the x axis.

I: You think of this on the x axis as going to one. So do you also think about something moving on the y axis going to three?

Jessica: Well it's just kind of like you go to one and you look up and where are you? OK. You're at three.

Later in the interview, Jessica reports that it is the *horizontal* arrow in the limit symbol that makes her think of motion along the x-axis.

Jessica: Well it's like taking a limit of this function. And I think a lot of the time, like I said, I travel along the x-axis, and I think it's also - I think – I look at x and it's traveling to 1, and so that's horizontal as well [*points at* $x \to 1$] so like the limit on your x-axis as you're traveling to one. [*pause*] I don't know.

I: When you're saying it's horizontal, you're talking about this part of the expression here [*points at* $x \to 1$]?

Jessica: Yeah. I think that's part of what makes me think of like it's traveling to one, because it's like - I don't know, x is going to one so it's going over here [*points at* $x = 1$ *on the graph*]. So I think of like walking along the x-axis.

I: Like that horizontal arrow there?

Jessica: Yeah.

Static Closeness

Half of the students (6 out of 12) who claimed to not think about motion when using words like "approaches" described something similar to a static closeness schema. Lindsay's metaphor of two train tracks meeting in the same place was already discussed in the section on closeness metaphors. She described meeting up in terms of being located in the same region in space. Karen, quoted earlier in this section drawing a distinction between "literal motion" and "approaching," described the latter as meaning "close" in a very static sense.

Karen: I don't think that I necessarily picture motion, but picture that idea that you may have a value that your points are really close to that - so close that they - like in the first problem that they're almost that point but they're not quite that point, so I guess the way I think of approaches is that it's not necessarily moving from 3 to 2½ to 2. You know, it's not moving, but it's the idea behind that it may not be - it may not be 2, but it's really close to 2…. I can't explain it any other way than to say in a similar way to the first problem, as point nine nine nine nine nine nine and on and on and on and on, but as the nines are added and they get - you put more and more nines exist after the decimal point, it's gonna - that value is closer to one. And I don't think that this is something that is happening as we're working on the problem, but all the nines are like - you know, somewhere out there and they're getting added on and on. [*laughs*] But just the way that you have to kind of think about it is that the [*value*] that you're looking at is really really close to what ever number that is, but it's not quite that number and the only way that you can maybe articulate that in a quick way when you talking about it and trying to write about it is to say that it approaches to but it's not like it's - you know, someone out there is like, you know, in some number factory adding on twos at the end of the decimal factory [*laughs*].

I: So, it's not happening in time somehow?

Karen: No. I don't think so. I think that it just - I just think that's how you say - that's how you articulate that particular value.

Karen struggles with the disconnect between her static image and the dynamic language using awkward phrases like, "you put more and more nines exist after the decimal point." In the end, she confirms the static nature of her imagery by emphasizing that the nines are not actually being added as if through a process in "some number factory." Instead, they already exist "somewhere out there."

Sequential

In contrast, five students specifically described a process of sequentially selecting points closer to the point at which the limit was being evaluated. Here, Darlene describes this as picking numbers.

> I: OK. The word "approaches" has a lot of - it sounds kind of like something is moving. Do you think of motion at all?
> Darlene: No.
> I: No?
> Darlene: That's just the way it's always been explained to me.
> I: OK. So, people have used that word before?
> Darlene: Yeah. The book uses that word, too. [laughs] … I don't really think about it that way. I just, you know, pick numbers. [*points at several distinct points on the x-axis successively closer to 1*]

Another student gave a similar explanation, making the distinction between "moving motion" and his approach of "I take a point, then I take this point, then I take this point."

> I'm not saying like a car approaches point *a*. I don't think of it as like that. I think of it as like, OK, I'm gonna take this value [*points at the x-axis near a*]. The next time I'm going to take this value [*points at a spot closer to a*], so it's approaching - approaching in intervals basically. I don't - yeah. I'm not thinking - that's what I'm thinking of. I'm not thinking of it like moving motion, like that. Like I take this interval - like I take a point, then I take this point, then I take this point, then it's approaching - yeah.

Input Affects Output

Only one student, Nina, explained that "approaching" meant that changing the input of a function caused a change in the output. In discussing the definition of the derivative, she described two points that "both approach the same limiting position" but denied that these points actually moved. Instead, she said, "We produce that motion by changing the input. Like for example, like making x smaller or bigger and depending on the change in the input." According to Nina, points on the graph "move if we change the input. Like if we make a closer to x…. If we change the input, they're gonna approach - like they're gonna be even closer." Thus, motion in this description is more like Talmy's fictive motion, an appearance of motion created by focusing one's attention on different points.

Zooming Imagery and Interpretations of Local Linearity

Some researchers have suggested that an intuitive description of "local straightness" should be used to introduce calculus concepts, thus avoiding the difficulties of limits (Artigue, 1991; Tall, 1986, 1990, 1992). In this approach students are introduced to "practical tangents" through "zooming in" at a point on the graph of a function with a computer or calculator graphing application. If the function is differentiable, at some scale, the graph will appear to be a straight line.

The professor for the course in which this study was conducted regularly discussed zooming imagery to supplement a standard presentation of differentiation. In different lectures, the professor described various tools with which one could imagine magnifying the graph of a function, such as a graphing calculator or a microscope, and also likened the idea to shrinking oneself to a small size and walking on the graph. After several lectures in which these types of illustrations were predominant, students were asked in one of their regular writing assignments to explain what you see when zooming in on a graph using the various methods mentioned in class.

Beyond repeating that zooming "results in what appears to be a straight line" from the problem statement, there was great variation in the 77 responses received. First, students gave four different types of reasons for why you would see a straight line 1) after zooming, only part of the graph is visible, 2) zooming is similar to moving in close from far away in a landscape where we know from experience that curves seem to straighten (see Figure 17(a) for a student's illustration of viewing the earth from these two perspectives), 3) curves do not

Figure 17. Tanya's illustrations of zooming in. (a) Two views of the Earth. (b) Two views of the graph of a function.

occur at a small scale, and 4) over a small portion of the domain, there can only be a small vertical change. (See Table 22 for frequencies and brief examples.) The students were asked to describe this phenomenon for "a nice function." If one interprets "nice" to mean "differentiable," all but the last of these justifications can be considered at least reasonable. In addition to the misinterpretation in the fourth justification, however, students gave several descriptions that focused on extraneous aspects of the metaphorical zooming context rather than the intended mathematics. The two unintended interpretations held by several students were that by zooming in you would see 1) a very thick or blurry line (see Figure 17(b) for a student's illustration) and 2) the fundamental elements composing the graph (e.g., pixels on a calculator or computer screen, atoms on a piece of paper, or individual points on a theoretical graph).

Table 22. Frequencies and Examples of Statements about Zooming In on the Graph of a Function.

Statement Category/ ***Example***	**Responses (out of 77)**
Only see part of the graph *When you're at that specific point, you are not able to extend your vision beyond a few spaces in any direction. Hence, you are not able to see the rest of the graph.*	42
Extrapolate from example *We know the earth is round, so why does the horizon appear flat? You can say it is because we have "zoomed in" on the curve of the earth.*	22
Small scale *If you were to shrink yourself and walk around on the graph, you would be walking in a straight line, because you are too small to tell the curves of the graph.*	14
Small vertical change *When you view a very small portion of the function… you're making the x values so small that the y values are given little chance to change.*	10
Thicker/blurrier *The line would be magnified therefore appearing thicker. Putting the graph under a microscope would magnify the line and focus on a segment or maybe a particular point. Eventually, it would get too close and become blurry.*	16
Pixels/Points/Atoms *Ultimately, the graph will be so pixelated that it will be of no use.*	10

Other Misinterpretations *Say that you only read one page of an entire book and use this one-page to gather your opinion on the whole book…There is no possible way for you to assume the plot of the entire story. In order to understand the entire book, you must "zoom out" and look at the entire picture.*	18

Interpretations of "Arbitrarily" and "Sufficiently"

The words "arbitrarily" and "sufficiently" appear regularly throughout the data for this study in students' phrases such as "arbitrarily precise" and "sufficiently close." Students' interpretations of these two words have not been discussed in previous research literature on limit concepts. The words "arbitrarily" and "sufficiently" have specific meanings in mathematical contexts, as reflected in the frequent use by both the textbook and the professor when providing intuitive versions of definitions of limits. For example, the definition of the limit of a function at a point might be expressed as "The limit $\lim_{x \to a} f(x) = L$ means that you can make $f(x)$ arbitrarily close to L by choosing x sufficiently close to a." As in other similar informal paraphrases of the epsilon-delta definition, here "arbitrarily" captures the meaning of the quantifier "for every" from definition "for every epsilon greater than zero…" indicating *all possible degrees of* closeness between $f(x)$ and L. "Sufficiently" (together with the phrase "by making") captures the condition on delta for any specific (rather than arbitrary) value of epsilon. That is, a degree of closeness is sufficiently close if *by making x* that close to *a*, the specific closeness in the range is achieved. In most cases, students' usage of "arbitrarily" and "sufficiently" in this study did not reflect these standard mathematical meanings. They were, however, consistent with students' usage of more ambiguous phrases such as "infinitely close" or "very precise." With the exception of the data discussed in this section, none of the students' responses contained an explicit description of their intended meanings for either of the words "arbitrarily" or "sufficiently."

After the second interview during Phase III of this study, questions about students' interpretations of these words were added to the interview protocol for the 9 remaining students. Only 1 of these students gave an explanation compatible with the standard mathematical meaning, while 7 described both in terms of extremity as in "Arbitrarily small means not much of a distance. It's something that's hard to observe…. It gets so small that it's immeasurable." The remaining student provided a nonstandard interpretation based on her understanding of bounding errors, which will be discussed in the following section on influencing students' metaphors.

Steve was the only student who provided descriptions for their interpretations of "arbitrarily" and "sufficiently" that were compatible with their mathematical usage. In the first excerpt, he describes his interpretation of "arbitrarily."

> An arbitrary degree of accuracy - [*pause*] - so any amount of accuracy that you want, you can get - that would mean to me that - that you could - well, basically the range could be anything. That your range could be as close - like you're talking about an arbitrary degree of accuracy to a certain value, right? OK, that means your range - and that means to me that your range of the value can be as close to - as close to that value as you want it, so your degree of accuracy could be very small or it could be as large as you want. Because "arbitrary," you're saying it could be any form of accuracy. It can be very, very accurate, it means - like take an interval very close. It

means it could be not accurate at all. So you could take a very large interval.

In this passage, Steve's usage of "arbitrarily" matches it's mathematical sense fairly well. As shown in the next excerpt, his understanding of "sufficiently" does capture a connection between closeness in the domain and range, but only in the incorrect sense that they should be "just as close." His understanding does not include the nature of the condition on closeness in the domain.

I: And what is sufficiently small then?

Steve: OK. What I mean there is that it - if you take - like if your $f(x)$ is going to be - is gonna be as small as you want it to, then your *x* has to be like - *x* has to be - if your *y* - if this is gonna be like sufficiently close to this, then your *x* is also gonna be like - in comparison to that, it's gonna be just as close. Do you see what I mean?

I: No.

Steve: So like when this is - when this is really far away, your *x* is gonna be really far away. And when this is really close to this, your *x* is gonna be really close to that. I mean that's the point of what I'm saying, like if this - if your *y* value is really close to this, your *x* can't be out here. It has to be sufficient to it. It has to be really close to it as well.

The 7 students who described "arbitrarily" and "sufficiently" without any of their logical relationships typically expressed their meaning simply as modifiers of degree. Thus "arbitrarily small" could mean an extreme version of "very small" as expressed by Nina in the following excerpt.

Nina: Arbitrarily accurate - arbitrarily accurate for me is a very very accurate, but not exactly accurate.

I: Do you think of a similar thing for arbitrarily small or arbitrarily close?

Nina: Arbitrarily small?

I: What would you say - how would you describe to somebody what that meant? Arbitrarily small.

Nina: Very, very, very, very small.

These students seemed to treat “sufficiently” in the same way, attributing to the word the simplest possible meaning as a modifier. At the same time they recognized that the term “sufficiently” carries an implication of moderation, as something which is sufficient must only meet but not necessarily exceed a certain threshold. This implication apparently caused some students to create a hierarchy of sorts, applied to size, for example, going from “small” to “sufficiently small” to “arbitrarily small.” Jacob describes the difference between the latter two in this excerpt.

Jacob: I guess that's how I used it in my example. It would be so small that for practical purposes it doesn't really matter, you know?

I: Another phrase that [the professor] has used in class is the phrase “sufficiently small.” How would you interpret that phrase?

Jacob: I guess larger than arbitrarily - the way I think of arbitrarily small. So it doesn't have to be so microscopically small, it would just be small like a decimal point number, you know? Which is sufficient.

METAPHORS AS THEY UNFOLD: THE CASE OF SHAWNA

The data presented to this point has been clustered according to types of metaphors and only small portions of students’ responses have been shown. In order to explore the nature of a single student’s unfolding thoughts as they wrestle to understand a concept, a brief case study will now be discussed. The following excerpts are taken from the transcript of an interview covering the definition of the derivative. The student, Shawna, was a freshman mathematics major planning on becoming a math teacher and had taken a calculus course in high school. During the beginning portions of the interview, Shawna’s descriptions did not explicitly rely on metaphors, but we include a discussion of the entire transcript in order to place her later use of metaphors in the larger context of an entire process of inquiry.

The interview began with the problem "Let $f(x) = x^2 + 1$. Explain the meaning of $\lim_{h \to 0} \frac{f(3+h) - f(3)}{h}$." Shawna immediately drew the graph of f (as in Figure 18 but, at this point, consisting of only the graph and axes). She recognized $\lim_{h \to 0} \frac{f(3+h) - f(3)}{h}$ as "the formula that we had to memorize" for derivatives, but couldn't "remember how we used it." When asked how she thought of derivatives, Shawna referred to the computational process of finding the derivative of a monomial, "I think of pulling down the exponent. I never understood what a derivative is." The interviewer then asked whether she associated derivatives with slope, to which she replied:

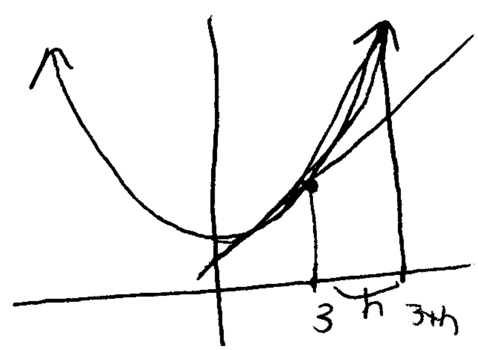


Figure 18. Shawna's graph identifying parts of the definition of the derivative.

> Well, yeah. Because like the slope of the tangent is, you know, the derivative - like the - yeah the slope. The *m* is the derivative of the function, but - OK. Maybe that'll - maybe that's coming back to me. I knew that like from high school, but I never really paid attention to it. [*pause*] OK. Umm. I know that when you do the limit, you're like moving the slope until it gets undefined or zero or something. That kinda - that ties into it somehow. [*laughs*] umm. I don't know. Oh man. This is bad. I know the derivative is the slope of the tangent at a certain point, and I guess *x* equals three would be the point, right? Because [*pause*] $f(3+h)$ - what

would *h* be? *h* would have to be over here [*points to the left of* 3] then, huh? Well, *h* could be over here [*points to the right of* 3]. I don't know.

Thus, Shawna correctly remembered that the derivative is the slope of the tangent line, which appears to have triggered a memory from her high school calculus class of what was likely a standard description of secant lines approaching a limiting position at the tangent.

After this exchange, Shawna correctly identified the locations of $3+h$, $f(3)$, and $f(3+h)$ as well as the standard referents for *h* and $f(3+h)-f(3)$ on her graph (Figure 18). She then tries to make sense of dividing the length $f(3+h)-f(3)$ by *h*.

> Yeah. See, $3+h$ would be right here. That's why I kinda drew that up there. I don't know why you take… We're taking $f(3+h)$ which is f of here, this point [*points at* $f(3+h)$] minus this point [*points at height* $f(3)$ *above the point at* $3+h$] divided by the span between them [*points at h*]. Why would we do that? I don't know. OK. Let me think. I don't know why we do that.

Shawna's thinking during this period might be schematically represented as shown in Figure 19. This diagram indicates the influence of specific thoughts on others with directional arrows. In this case, Shawna took a cue from the expressions in the difference quotient to identify corresponding locations and

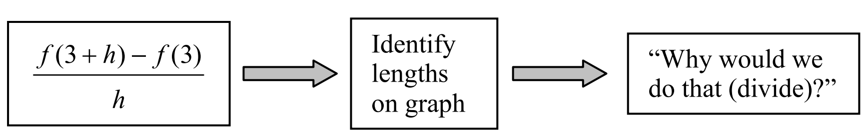


Figure 19. Schematic of Shawna's initial discussion about the relationship between the difference quotient and the graph.

lengths on the graph. With those referents in mind, she then asked why we would divide $f(3+h)-f(3)$ by *h*. Becoming stumped by her own question, Shawna temporarily suspended her inquiry into the difference quotient. Thus, the sequence of three boxes in this diagram represent a relatively complete portion of Shawna's thought processes during this interview.

At this point, Shawna shifted her focus to the limit in the definition of the derivative, saying "As *h* approaches zero, so *h* is getting smaller. This [*points at vertical line for* $f(3)$] is moving towards three then, right? Yeah." The interviewer was still interested in how she was thinking about the slope and asks her for an explanation. In the following excerpt, Shawna responds to this question, pauses, then returns to thinking about the limit. The result is her drawing of a sequence of lines through the point $(3, f(3))$ with successively smaller slope (see Figure 20).

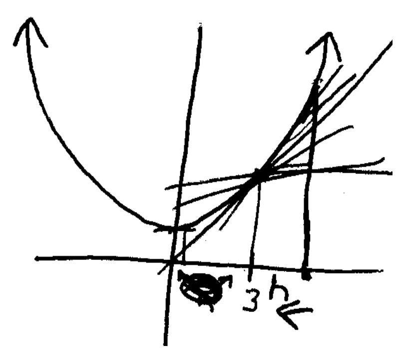


Figure 20. Shawna's picture of lines rotating to a limiting position.

Shawna: Umm. The slope is whatever the change in *y* over the change in *x* after every unit. So like even if the *y* goes down one, you know, that's like a negative change. So it's the change in *y* over the change in *x* over every unit interval. That's what I think slope is. [*pause*] I'm trying to think of what happens as this approaches 0. We just like keep tracing this graph. Oh, I get it. As this approaches 0, my tangent keeps getting like - keeps getting smaller until it gets to 0. [*draws several lines*]

I: The tangent gets - flattens out?

Shawna: Yeah, $m = 0$. That's what happens as it approaches.

I: Can you tell me what you're thinking about there?

Shawna: Because I'm just assuming - like I know this is like I guess like a derivative, because that's how it was introduced to me, and so as *h* approaches 0, the derivative keeps getting, I guess smaller, or the slope keeps getting smaller and smaller until it gets 0. As this gets smaller, this line [*points at would-be secant lines*] gets traced in more, and so it likes come closer and closer until it's here. And so would it be 0, though? No. What would it be? Yeah. I guess so. Or, no, it wouldn't be 0. Dork! That would be like here [*points at vertex*]. I'm thinking about the bottom. It would - it would be the derivative at 3. Or derivative at 3 again. What am I doing?

Recall that a portion of this description was discussed in the section on closeness metaphors. Shawna treats the coordinate plane as a physical space in which the closeness of points and lines can be measured, and follows the logic that a small change in location (change in *h*) produces a small change in features associated with those points (positions of the lines). The result is a conflation of the two which leads her to claim that the slope (rather than *h*) is zero, depicted schematically in Figure 21 with arrows pointing from both "rotating tangent" and "$h \to 0$" to the statement "$m = 0$, so it flattens." Shawna realizes that this argument is not correct when she compares the statement to the graph and

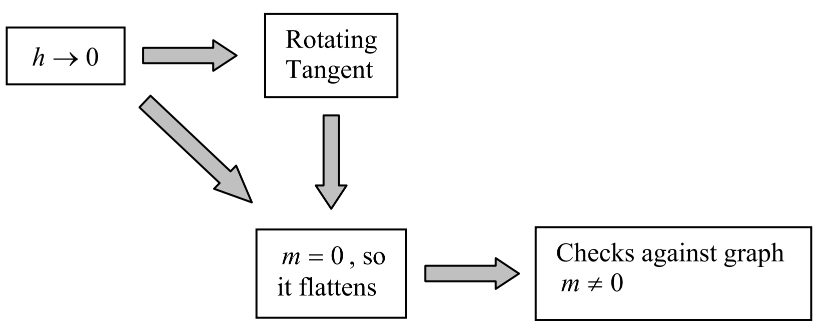


Figure 21. Schematic of Shawna's discussion about lines rotating to a limiting position.

visually seeing that the slope at is zero at the vertex of the parabola rather than at $(3, f(3))$. The recognition of this contradiction brings an end to this line of reasoning.

Immediately following this exchange, Shawna started over trying to understand how the limit was involved in the expression $\lim_{h\to 0}\frac{f(3+h)-f(3)}{h}$. This time, she brought in her understanding of the derivative as involving the tangent line at a point and drew the graph in Figure 22.

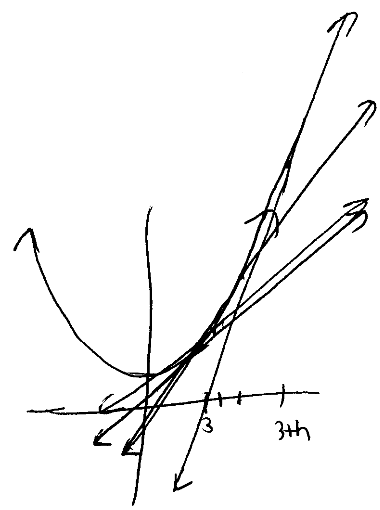


Figure 22. Shawna's picture of multiple tangents.

Shawna: Ok. And this is my *h*, this little space in here [*points between* 3 *and* $3+h$]. And this is my $3+h$. As this get smaller [*points between* 3 *and* $3+h$], the derivative - let's just put the derivative on here for a second, like a tangent line. It would be like right about here, huh? [*draws tangent line at* $(3, f(3))$] As this gets closer and closer - as *h* approaches zero, this line would just keep coming in more and more [*mock sketches several short tangent lines between* $3+h$ *and* 3] until it's like the derivative - or the, excuse me, tangent at $x=3$.

I: So, you're drawing lots of little tangent lines?

Shawna: Yeah. On each - on $3+h$ [*draws large tangent line at* $(3+h, f(3+h))$]. And between 3 and $3+h$ [*draws two more tangent lines*], so - yeah. So I guess that's why - I don't know.

I: So what do you think all of this has to do with that limit?

Shawna: That's what I'm trying to figure out. [*pause*]

In this excerpt, the symbols $h\to 0$ draw Shawna's attention to moving toward 3 along the graph. Having drawn a tangent line at $(3, f(3))$, she then thinks of several tangent lines on the graph. The influence of these two signs is depicted schematically in Figure 23 with arrows pointing from each to her graph drawn with multiple tangents. Symbolically, Shawna's explanation might be represented as $\lim_{h \to 0} f'(3+h) = f'(3)$, with the exception that she is really describing tangent lines rather than their slopes. When the interviewer questions her about the exact role of the limit in this explanation, Shawna is unsure, and once again, a line of inquiry ends.

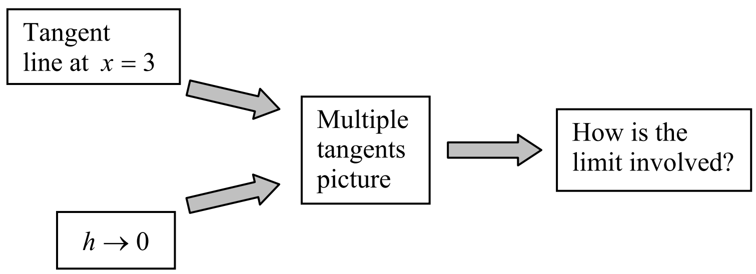


Figure 23. Schematic of Shawna's discussion of multiple tangents to the graph.

Even though Shawna had talked about slope as "the change in y over the change in x" and had identified $f(3+h) - f(3)$ and h as such changes, she had still not recognized that the difference quotient $\frac{f(3+h)-f(3)}{h}$ expressed a slope. At this point, the interviewer redrew Shawna's picture as in

Figure 24 adding the two darkened line segments (the two short vertical lines were added later by Shawna). The following excerpt begins with Shawna's immediate, albeit tentative, recognition of the difference quotient as a slope, which leads to a burst of discovery.

Shawna: So, that's kind of like slope? I don't know. Yeah. Because that's kind of like the *x* value, and that's the difference, which is the *y* value.

I: Can you say more about what you mean by that's the slope?

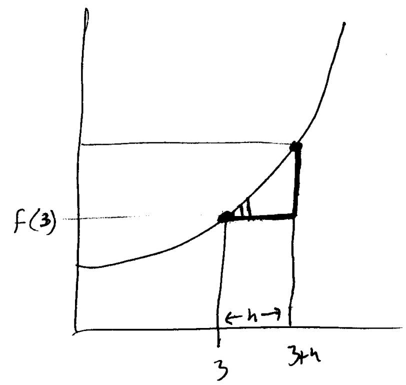


Figure 24. Shawna's image of collapse for the derivative.

Shawna: Hmm. let me think. [*pause*] Yeah. That makes sense. Because I mean if you didn't know how to differentiate, you could do this [*points at* $\frac{f(3+h)-f(3)}{h}$], and if you take $f(3+h)$ and subtract $f(3)$ - like *h* could be any number like, I don't know, just a number - and if you subtract them, you get your *y* - your change in *y*. And then when you divide by this [*points at h on the graph*], you - what do we do? Oh, I had it there. [*pause*] Ok. This kind of makes sense. OK. As this approaches zero [*makes motion from right vertical line to left vertical line*], you divide - I still have to remember that it's going to zero. I can't **JUST** use this part [*points at* $\frac{f(3+h)-f(3)}{h}$], because I mean that wouldn't be the derivative or anything. That would just be a number. As this gets smaller [*points at h on the graph*], this comes down [*points at* $f(3+h)$]. OK. [*pause*] that kind of makes sense. Because it's a limit and it can only go so far until it reaches the point. As this comes smaller, that's your *y* value divided by your *x* value which is a slope. And so - OK. That makes sense. As you bring *h* towards three, your *y* - your $f(3+h)-f(3)$ gets smaller, because you're tracing down the graph. Well, that is if - if of course if the graph looks like this, but it does, so I'm going to say that [*laughs*]. The *y* value gets smaller, and this value gets smaller [*points at h on the graph*]. It gets smaller. So you're dividing *y* over *x* which is actually - that's the slope. And so you get so small until you can go no more and that gives you the slope at three. Magically. I don't know. [*laughs*] That make sense though, because I mean, I really don't know how to explain limits like as a professor or anything or a really intelligent person because I just - that's how I understand limits to be. You know? You take something and - and I don't mean to go on that tangent. [*pun intended?*] You take your values and you squish them really small until you can get - until you can go no more, and magically that's the limit. I don't know why it gives you that, though. I mean I kind of do, but I don't know how you get a number out

> of that. You take - I couldn't explain it to too many people. As this gets smaller and this get smaller [*points at the darkened vertical and horizontal segments*], your - the difference between these two gets closer and closer. Say you get like here and here, and here and here [*draws the two short vertical lines*], and so you're getting really really close to the rise over run of this. And when you reach your limit, that's what the rise over run of this is [*points at* $(3, f(3))$] so I guess that's the tangent which is the derivative. Yeah. That does make sense. Because that's what happens on a limit. Like when you - on a graph, you get smaller and smaller until you get to the point that you want, and that's what your value is. And so I guess this would be - if you could see these two little lines down here, your tangent - or your slope - or yeah your tangent would be smaller and smaller until you finally hit this point at three which gives you like **THE** tangent. So if you have like a really small *h* like a 0.001 and you did this, and you just found the rise over run - or if you just take that divided by that - hold on. If you take – yeah, if you just take $f(x)$ and divide by change in $f(x)$ - like the change in *y* and you divide by the *h*, that would be like really close to the tangent, and so the smaller you go, the closer and closer to the tangent you get, and that's why you **GO TO** zero, because you can't divide by zero, but that's why it's the tangent. [*Shawna's emphasis throughout*]

Shawna uses a collapse metaphor in this excerpt to deal with the jump from considering secant lines between two separate points and a tangent line passing through a single point: "And so you get so small until you can go no more and that gives you the slope at three. Magically." Although her very expressions of this metaphor convey her uncertainty (she uses "magically" twice and also says explicitly "I don't know why it gives you that"), she continually returns to the idea and explores its implications. The centrality of this metaphor to her reasoning is represented in the schematic diagram in Figure 25. Notice that the collapse metaphor emerges from her converging considerations of the slope of a secant line and the motion from $(3+h, f(3+h))$ to $(3, f(3))$. After this, Shawna begins

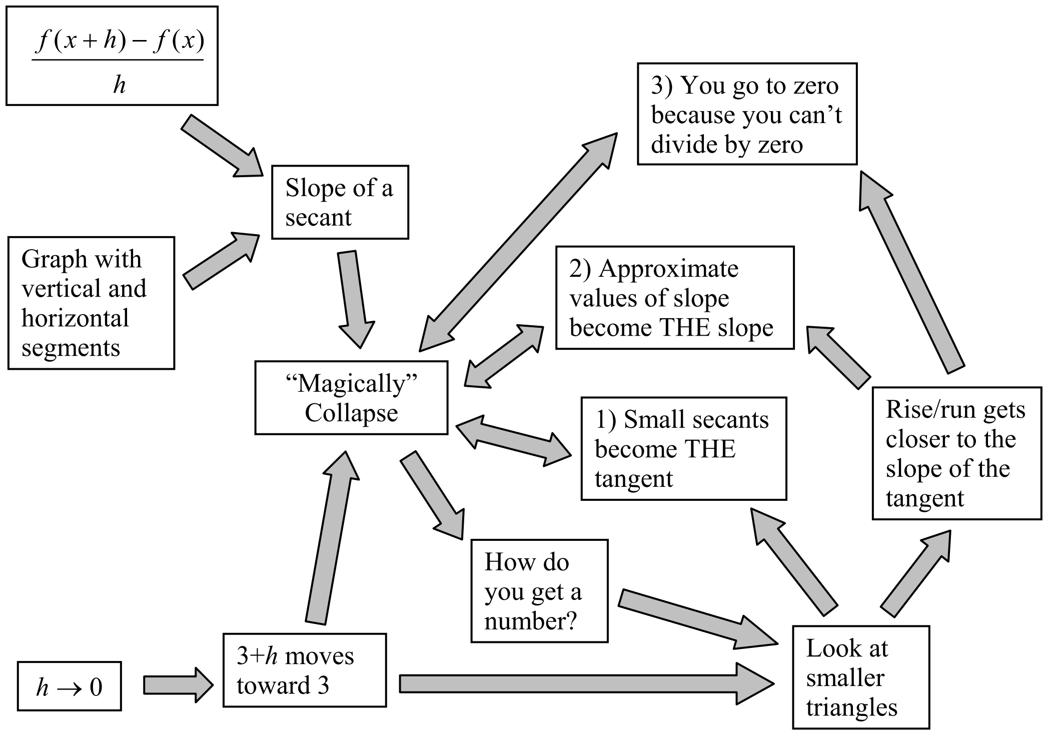


Figure 25. Schematic of Shawna's thinking related to her collapse schema.

to ask questions of her new idea, most notably when she wonders "how you get a number out of that." This question leads her to three separate conclusions all feeding back into her collapse metaphor: 1) small secants become "**THE** tangent," 2) approximate values of slope become the slope, and 3) "you **GO TO** zero, because you can't divide by zero."

This last conclusion could indicate some sense of the subtleties involved in limits, but regardless, Shawna continued to use collapse metaphors throughout the remainder of the interview. When she was reflecting back on her burst of ideas from the previous excerpt, she also reported thinking of approximation to translate her geometrical understanding into a numerical understanding. In this excerpt Shawna's final comments ("the limit takes you as small as possible until you reach that point") indicate that her approximation metaphor contains ideas about collapse in a manner consistent with her treatment of numerical values in the previous excerpt.

> I was looking at more like this gets smaller and smaller and so like when you draw a line here [*draws a secant line*], it's gonna slice right through your x^2 graph. But as you get smaller and smaller, I was thinking no, it's gonna come further and further to the edge. And then when you get to the perfect point, you know it's going to be on the edge [*points at tangent line*] and I was like yeah, that's what it is. Because I was looking at that [*points at* $\frac{f(x+h)-f(x)}{h}$] thinking like that's a slope, but that's like really off. Like I was saying, if you just - if you didn't take the limit - and if you just did that, the smaller your *h* is, the more accurate the limit would be. You can like estimate if you just plug in like a number for *h*, but the closer it is to the number you're looking for, the better it would be. So like if I drew from here to here [*points at smallest triangle*], that would be a much closer limit than the one from here to here [*points at largest triangle*]… so I was just thinking the smaller *h* you got, the more accurate the limit would be, and then the more accurate the slope would be… And so then that brought me to the conclusion, I was like yeah, that's why you take the limit. Because the limit takes you as small as possible until you reach that point, so that makes sense. I never really thought about it like that before, but now I see it and I won't forget it.

In the second portion of this interview, students were asked to give an interpretation of the same limit in the context that the function represented position as a function of time, $\lim_{h\to 0}\frac{p(3+h)-p(3)}{h}$. Shawna went through the same process as she did during the first part of the interview of identifying referents for the various expressions in this limit. With a small amount of help from the interviewer, she eventually located each of the times ($t=3$, $t=3+h$, and $\Delta t=h$)

and distances ($p(3)$, $p(3+h)$, and $p(3+h)-p(3)$) on a straight line. representing a road along which a car was traveling (see Figure 26). Shawna's initial inability to figure out what the quotient, $\frac{p(3+h)-p(3)}{h}$, represented was similar to her thought process while interpreting this limit in the context of the graph of a function. In the new car context, she made comments like "So that would be like the ratio of how they convert together - how one affects the other? Maybe?" Finally, once again, it was remembering that the expression $\lim_{h\to 0}\frac{p(3+h)-p(3)}{h}$ is related to derivatives and that derivatives give velocity, she is able to recognize the quotient as "a good estimate of" velocity.

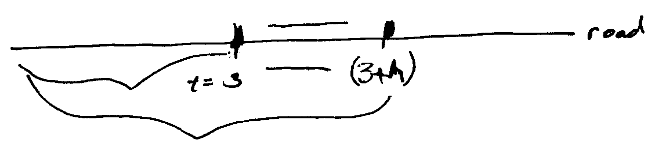


Figure 26. Shawna's picture of a road showing distances and times from the difference quotient, $\frac{p(3+h)-p(3)}{h}$.

Shawna: The distance traveled in h time. I think. Yeah. It would be. It would be the distance traveled in - but that would be the same thing as here divided by that [*points at the region between the marks on the line*]. So I don't know. But you have to think about as *h* goes to zero, so as *h* gets smaller, we travel less. Hmm. This is hard. Not thinking about a pretty little tangent line and stuff. [*laughs*] OK. OK. [*pause*] This would be a good estimate of - I don't want to say - what is this in terms of? Like the derivative? What would that be called? The – like $p'(t)$? That wouldn't be like the velocity of it?

I: So you're looking at - trying to interpret what the derivative of *p* would be?

Shawna: Yeah. That would be velocity I think. Yeah, because second derivative is acceleration. So like this - if you just divided these two, that would be a good measure - I mean like not good. It depends on how close your *h* is. But it would be like an estimate of what the velocity is. But the lower

> your *h* - the smaller your *h* gets, the closer you get to a real point with a real velocity, so you would - you have smaller and smaller numbers to divide until you got to - until you made *h* zero and you got your velocity at *t* equals three. So velocity is [*pause*] - what is velocity? Velocity is speed? Right? So - yeah. That would be your speed and, because you traveled so much distance in some amount of time, but - I mean, that's like an estimate, because it's not gonna be exact, but the closer and closer you get to a real point, that's gonna be your speed. OK. That kinda makes sense. So this would determine how fast you were going at a certain time *t*. Yeah. Because the closer you get - the lower your numbers get, the more accurate your rise over run division would be to what it really is at I guess $t = 3$. So yeah, that makes sense. [*pause*] What else can I add to that?

This explanation is given primarily in terms of approximation, using phrases such as "it would be like an estimate of what the velocity is," "that's like an estimate, because it's not gonna be exact," and "the lower your numbers get, the more accurate your rise over run division would be to what it really is." Shawna also treats the limit for this approximation with a collapse metaphor as she did in the graphical context. Here she describes the collapse as "the smaller your *h* gets, the closer you get to a real point with a real velocity, so you would - you have smaller and smaller numbers to divide until you got to - until you made *h* zero and you got your velocity at *t* equals three."

STUDENTS' INTERPRETATIONS OF EXPLICITLY PRESENTED METAPHORS

In Phase III of the study, students were exposed to a series of problem sets and writing assignments that developed versions of the approximation and closeness metaphor clusters more closely resembling formal aspects of limit definitions. In these assignments, the students were asked to respond using the language and logic of the revised metaphors, and 11 students were interviewed about their subsequent interpretations of their work and the ideas of approximation, closeness, and limit definitions. See Chapter 4 for a detailed discussion of the presentation of these assignments and the protocols or the interviews.

Even after the series of very prescriptive activities and while being continually prompted to talk about specific aspects of the approximation and closeness schemas in the interviews, the students often responded with their own personal versions of metaphors based on these contexts. Table 23 gives a list of the types of personal metaphors based on approximation and closeness used by the students with an illustrative quote from the interviews. Some of these metaphors involve the application of only a portion of the modified schemas, for example, thinking of an error or distance as something that is "so small it doesn't matter." Others, however, are based on entirely different domains, such as color wheels, or on the schemas for other metaphor clusters, such as physical limitation.

Table 23. Contexts Influencing Interpretations of the Modified Metaphors.

Ruler Measurements (Sandra, Karen) *I can think about like if you're trying to measure it with a ruler. If I try to put it in visual terms, like each - like an inch, there's always like a smaller unit that isn't being measured and so that limit can go on and on and on. Like how close you can actually measure it.*
Function Input Determines Output (Nina) *I think that no matter how small delta is, you can find an epsilon and for which all points can fit into the band… because the value that changes is the input and then the function depends on the input.*

So Close/Small It Doesn’t Matter (Cheryl, Bob, Cindy, Marty) [$0.\overline{9}$ *is*] *so close that it might as well be one. It's just so minute that we just think of it as 1, but I don't know how that would work in something like aerospace engineering or something when the decimals would definitely matter. Would they say point nine repeating is equal to one or would they say, no we can’t do that, its point nine repeating.*
The Closest Value (Janice, Marty) *I don't think I wouldn’t take a limit, but I guess they're involved. If these two points get as close together as possible, then you want to take the slope between those two points I guess, but I don't think of it as a limit.*
Collapse (Karen, Janice) *I think of it as a tangent line because it's basically that point, close enough to where it's basically the two points make up one. Actually here I think of them as one point… when you get two points that are so close to the point that you’re looking at, they’re essentially the same point and you get your slope.*
There's always infinitely many numbers between any two numbers (Enrique) $0.\overline{9}$ *never equals one because there’s still always going to be a number in between. It's either one or it's not one. Because there's going to be a slight difference that is going to go on forever… Even when you get the smallest number you can think of, in between there's these really small numbers. And you have more numbers inside of that and you have more numbers inside of that. So you can always make it smaller.*
Zeno’s Paradox: Theoretically vs. In Actuality (Jacob) *I just went back to something my eighth grade algebra teacher showed us during - before class when we were just goofing around or something. And he said - said something like if you're standing at one end of the room and you start walking towards the door, and each time you just take half the distance, then theoretically you would never actually get to the door because no matter how close you are, there's still some microscopic, you know, space between you and the door, so you would never get there, but in actuality you're there, you know? You can actually go past the door. So I just thought about that the same way, like no matter how - if you take, you know numbers that are microscopic - arbitrarily close to one, you know? Then you're not actually on one, but you're so close in reality, it's actually equal to one. That's the way I think of it.*

To illustrate in greater detail the nature of some of these personal versions of the approximation and closeness metaphors, we now present data from three separate students. Bob began responding to questions about his interpretations of phrases such as “degree of accuracy” and “bounds on error” in terms of being able to ignore errors, then suddenly shifted to applying the modified approximation schema precisely as presented in the problem sets and writing assignments. Sandra applied a version of the closeness metaphor based on her image of the number line as a ruler and making measurements in order to think about the problems and even to spontaneously describe fairly sophisticated concepts such as “connected” and “continuum.”

Bob’s Interpretation of Error: “So Small it Doesn’t Matter”

Bob had not previously taken calculus and did not learn about limits in his high school pre-calculus class. During the first half of the interview, Bob uses a blend of the physical limitation and the approximation schemas in discussing the problems about the equality $0.\overline{9} = 1$ and derivative computations. He begins with the following explanation of the equality:

> Like 0.9999 is equal to 1 because there’s such little difference that in the approximation it really doesn’t matter. If your error is wrong then you can always go out another 9 or 1000 more 9’s, and you can get so small that there’s practically no difference between it and 1.

This explanation includes a small portion of the modified approximation schema. Specifically, it includes the terms of the sequence as approximations to one with the difference as the error. In his phrase “if the error is wrong,” Bob could be describing an unsuccessful attempt at bounding the error, but it is unclear whether this interpretation is what he intends. The phrase "because there's such little difference that in the approximation it doesn't matter" might simply reflect thinking about a small error, but the following excerpts suggest otherwise.

When asked about the meaning of his phrase "you can get so small that there's practically no difference," Bob responds,

> You know, whenever you get that small - maybe geneticists will get that small with DNA strands. No one else like a banker or something with money - a country doing its budget - no one is actually going to know the difference between those two numbers. They're just like the same.

Here Bob has clearly departed from the modified approximation schema, and is instead describing something so small that it will go unnoticed. The interviewer then asks where he would plot $0.\overline{9}$ on the number line, and Bob responds with, "If I was just drawing it, I would probably just make the line really big and I would probably just like put that practically on it - so close that you can't tell the difference no matter how close you look at it." Again, Bob is describing such a small difference, this time a spatial one, that can be ignored for practical purposes.

Bob began to mention derivatives in one of his explanations about "margin of error, but pauses, then moves on to another thought. The interviewer later asked what he was about to say.

> I: When you were talking about a margin of error with the derivatives could you say more about that?
>
> Bob: Usually there will be like h^2 is the difference and as h approaches zero, it's so small that it doesn't matter, so all our derivatives have that margin of error. It's usually so small, that you get very small numbers. I'm in computer science. Like whenever you're using huge numbers that you're trying to sort, so, you know, like say you're sorting all of the letters in the dictionary then looping through that - if you loop through it

2n times, 2 really doesn't matter. It's the n that matters. That's the dominant thing. The 2 just - it would be your margin of error.

Here Bob describes what amounts to round-off error, an amount that is negligible in comparison to some other quantity of reference. Initially, he seems to be referring to computations involving the definition of the derivative for polynomials in which an h^2 term in the numerator of the difference quotient is "so small that it doesn't matter." Bob's reference to polynomial time algorithms is similar, in this case with the (would-be) higher order terms dominating.

Shortly after Bob responds to this question about "margin of error," he referred to a television show that he watched on The Learning Channel (TLC) about quantum mechanics, which developed into the application a physical limitation schema to the infinite decimal, $0.\overline{9}$. The implication drawn from this metaphor is the same as Bob's previous reasoning, that differences on this scale do not matter and can be ignored. Immediately after this the interviewer's question about approximation and error, although similar to several of the previous questions, elicits a very different use of this language.

Bob: I used to watch TLC all the time [*laughs*]. It's just something I like, and I remember them talking about - like the smaller and smaller on the scope of things you get, like eventually - it will get down to, you know, like atoms and then electrons and get smaller and smaller until you get to like this [*draws a grid*] - it's been so long. I think I was in like six or seventh grade when I was watching, but like this is somewhere - I guess maybe where electrons are. This is somewhere where you can't really be. You have to go from here to here [*points at adjacent intersection points*]. You can be either here or here but you can't really be here in this, you know, this area right here [*points at region in between*]. You can't - it's so small that - I don't know what level that would be on the but it's just an extremely small physical level.

I: And you see that as being connected to limits?

Bob: Yeah. In a way. Yeah. I mean, well I was using that - the only person who might think that there's a difference between point 999 and one is a physicist who maybe has to deal at this level, right? That kind of smallness.

I: In your response, close to the end, you were talking about bounding and approximation inside an interval that yields an acceptable error. Could you explain that a little bit more?

Bob: Well, I was using the point nine as an example. If say your margin of error is - you know, you have point 99 then your margin of error is point 001.… If your error cannot be bigger than that, you just throw on some more nines and you can get smaller than 0.0001 when you subtract it from one. That was the example that I was using. You can always find something smaller than what they give you. Like this is pretty much our epsilon delta proofs. You can always find a spot closer than where you need to be. Your margin of error is here [*holds up hands facing each other to indicate a distance*] and here's your limit [*waves one hand*] and you have to be at least in so far closer to it [*waves other hand across the space in between*]. You can always get closer to it, you know? That's the way I was looking at bounding. You can always get closer to it.

In this passage Bob uses the phrase "margin of error" to alternately mean error and a bound for the error. Nevertheless, he does put together nearly all of the components of the modified approximation schema, referring to the difference between $0.\overline{9}$ and 1 as error, describing the need to achieve a pre-specified bound in a number of ways (e.g., "Your margin of error is here and here's your limit and you have to be at least in so far closer to it"), explaining how this can be achieved ("If your error cannot be bigger than that, you just throw on some more nines and you can get smaller"), and finally, noting this can be done for any bound ("You can always find something smaller than what they give you."

After this point in the interview, Bob did not return to describing error as round off error or as being something that is "so small that it doesn't matter." As hinted at in the previous excerpt, he noticed the similarity between this description and the epsilon-delta definition which he later accurately explained on a graph, but when he tried to explain the meaning of the definition of the derivative in terms of epsilon-delta language, he became extremely confused as shown in the first part of the following excerpt. Interestingly, he was able to give the corresponding approximation schema explanation when asked to do so by the interviewer.

Bob: Your epsilon - this - the slope of this tangent line. You want to pick a set of x's, and that's here. This x, it's barely changing such that it's equal to or less than this tangent line. That would be your delta. The slope - oh, OK. The slope of this tangent line - that's epsilon. The slope of this line that you're making is your delta at 2. Take a delta - a slope of this line less - such that it is less than the slope of this tangent line.

I: OK. What if you were talking about it in terms of approximations?

Bob: Approximations? OK. [*pause*]

I: If that doesn't make sense that's fine.

Bob: It kind of makes sense. I'm just not sure how to do it. That's the thing.

I: What are you thinking about?

Bob: OK well the way I'm - you're saying how can we make this [*points at secant line*] approximately equal to this [*points at tangent line*], is that correct?

I: Yeah, describe what you were talking about with the tangent lines in terms of approximations and making errors small. Use that kind of language.

Bob: [*pause*] There will be - there could be a difference in the slopes of these lines. You could say that the slope of this line is approximately equal to this with a margin of error of such and such, and that margin of error can be less than that. You can choose a slope that's less than the margin of error - less than what ever you need it to be.

Sandra's Interpretation of Closeness: "Ruler Measurements"

Sandra began the interview responding to a question about her general impression about limits by saying she imagined using a ruler to measure distances on the number line.

> I just - I think of how - I guess - sometimes I can actually picture it as like, I don't know, trying to approach an actual object and I can think about, like if you're trying to measure it with a ruler - if I try to put it in visual terms, like each - like an inch, there's always like a smaller unit that isn't being measured and so that limit can go on and on and on. Like how close you can actually measure it.

Sandra represents her limit concept here by describing always having "a smaller unit" with which to measure. She hints at a conflict in this schema by saying that this "limit can go on and on and on," but also suggesting in the phrase "how close you can actually measure it" that there is an end to this process at the point where anything smaller cannot *actually* be measured.

In the following excerpts, Sandra continually returned to similar ideas. Shortly after this, for example, she said the difference between $0.\overline{9}$ and 1 would be infinitely small, and when the interviewer asked her about this, she responded,

> You're trying to like measure something, like you could never actually like - anything that you try to measure would be bigger than - I always visualize like someone just actually like measuring it, and anything that you measure with will end up being bigger than the actual difference, because that just goes out to zero.

So now it is clearer that Sandra is imagining the infinite decimal $0.\overline{9}$ as being so close to 1 that no instrument is sufficiently sensitive to measure a difference at that small of a scale. When asked where she would plot $0.\overline{9}$ on the number line, she gave a similar example, this time explicitly treating the number line itself as a ruler:

> I guess my number line, when I visualize it in my head it looks like - it looks like a ruler in a way, and - so my first inclination would be to make - like I want to make point nine repeating fall ahead of one, but then again you can't actually put a distance between the two because - between the two numbers because like any difference that I would put - whether I mark $0.\overline{9}$ to be right here and one to be here, the distance would have to be smaller than that, so it would end up - the limit of the distance would end up making them fall in the same place.

Thus, any markings that can be made on this number line/ruler are larger than the distance between $0.\overline{9}$ and 1. It is possible that she is thinking of these ideas in a way very similar to the definition of the limit of a sequence, even referring slightly to this possibility by saying "the limit of the distance would end up making them fall in the same place." The details for how she might intend that statement are unclear, however.

Sandra did not use approximation language as naturally as she used this very physical, spatially oriented language. In fact, spatial representations of concepts about approximations surfaced in ways that appeared to cause serious difficulties for her. For example, she described an approximation as "something that is essentially as close to being equal as you could calculate or measure." She also viewed a bound on the error as a barrier to achieving greater accuracy, the opposite of the intended meaning in the revised approximation schema. Sandra introduced this interpretation with the physical imagery of a "boundary" or "fence."

> If you put a limit as to - well, I guess I shouldn't use limit - a bound to how precise an approximation is and [the approximation] isn't carried out all the way, then there will be an error… A bound would be a fence or something that stops the limit from being - like from going out on further. Like a boundary is what stops - like inhibits you from like getting an approximation that's closer.

Consistent with this interpretation, Sandra's written responses described a limit as existing if "the difference does not have a limit as to how small it actually is and is, therefore, essentially equal to 0."

In the portion of the interview on the definition of the derivative, Sandra had difficulties even identifying what the approximations would be. In the following excerpt, she suggests approximating the slope of a linear function and doing so by looking at the value of the function rather than the slope of a secant line.

> Well, you're trying to approximate that rate on that function. That slope. Like as y is changing and x is changing, and so - like that - this. You can continually get a better approximation - a closer approximation for - when 4 is applied to this function [*points at graph of* $y = 2x + 1$], it's going to also yield a better approximation, 9. Then in terms of the slope will also be better approximated.

Later Sandra conflated use of the tangent line as an approximation to the function and secant lines as approximations to the tangent. Of course a derivative can be defined in either of these terms, but it appears that Sandra is more focused on where tangent and secant lines "touch the function" and secants approaching a limiting position of a tangent than the subtlety of the connection between these two definitions.

> The tangent line of a point is like a line that - well in the book it says that it best approximates of the function at the point. It has the same slope at that point and will fit the function at that point, and then at least in this interval, like local points, it won't touch that function again, and so when you're comparing two points that are away from this point to, you'll just get secants, but as you get closer and closer, like you restrict the distance away from this point, these will get closer to that tangent line.

Sandra combined her spatial language (using words like "space," "distance," and "gap") with images of a color wheel from her chemistry class in an interesting way to describe her understanding of the structure of the real numbers. In this description, she used words like "continuum," "connected," and "disconnected" in ways that are similar to their informal mathematical meanings and even potentially consistent with their mathematical definitions.

Sandra: Like - I guess it would have to be some kind of *continuum* where they aren't *disconnected*. Because if you have 0.9998, there will - there's some *space*, like *distance* between those. But there are other real numbers that fall in those - in that *distance* - in that *gap*, and so I guess all real numbers are essentially *connected* to each other.

I: When you think of the word continuum, what do you imagine? What do you picture?

Sandra: Well actually, the first thing that I picture is actually like the color wheel and how it just - I guess because I like chemistry. That is the first thing that comes to my mind, but - and so it always just like - each color leads into everything else.

I: Right. So when you're thinking about the color wheel, that's like perfectly blended, there's not like a sudden change from one color to another? Is that what you're picturing?

Sandra: Right. And it doesn't just have - like how it doesn't have a sudden change, it just all fades into each other.

I: Alright then when you talk about connected or the opposite of that, disconnected would you think of?

Sandra: Well if I use the same analogy, like numbers can't - on the number line, real numbers can't be disconnected as colors can't be disconnected because there's always a color that is a shade closer to that next color on the color wheel, just as there is in numbers that falls in between the interval between two points on the number line.

Jacob's Interchanging of Language and Ideas: "I Think of Them as Sort of the Same Thing"

The students who were able to correctly interpret the modified schemas in this phase of the study, were often unable, however, to distinguish between the schemas. Jacob began his interview by describing most of his ideas in terms of Zeno's Paradox and his distinction between "theoretically" and "in actuality." Later, when he was asked to explain the definition of the derivative in terms of approximation, he easily gave an appropriate interpretation of the full modified schema. The following excerpt occurred several minutes into this discussion, where Jacob began to interchange language about approximation and epsilon-delta ideas.

Jacob: No matter how small you made that approximate - bound of error - the amount that we're allotting ourselves - if you had a good method of approximating, then you could stay within that bound no matter how small that bound is.

I: And what would you have to do to get it within that bound?

Jacob: Choose deltas arbitrarily close to c.... So you know you could pick delta here, but that wouldn't - maybe that wouldn't get you a within your range that you want - your bound of error, so then you would get closer. You just keep getting closer. That lets you be within that bound of error

When asked to describe the relationship between approximation and epsilons and deltas, Jacob was able to describe both appropriately and to identify the corresponding structures. Later the interviewer asked how he would describe the differences, and he gave the following response,

> Ok. Well, the second one [epsilon-delta] is saying that we can pick any *x* within this range - that'll give us a value of - a value in between this range [*points at epsilon- and delta-bands in a standard diagram*]. And the first one [approximation] is saying that no matter how small we make this range, we'll get - we can find a limit - or we can be a certain distance from the actual value of the limit that's within a bound we allow ourselves - so that we pick. So say we pick, you know, our allowed error to be this much, then we can find a delta that would allow us to be within there.

In this response, Jacob uses similar words and phrases to describe both the approximation and epsilon-delta ideas, eventually beginning to interchange words. He continues to do this throughout the interview as in the following portion of his description of the definition of the derivative:

> This would be like your - the value you got. That would be - alright, that's your error, right? That would be like what we were talking about being an error. The secant line minus this actual slope. [*pause*] I guess - if you were given a - if you were given epsilon - sort of like I was saying, you're given a bound of error to be within, and you would get this value to be within that.

Although the language about closeness and functions acting on intervals seemed somewhat awkward for him, Jacob was also able to accurately interpret their meaning and tended to translate the ideas into epsilon-delta language. At the conclusions of the interview, the interviewer again asked Jacob how he thought of the relationships between each of these schemas, to which he replied, "I guess - now I think of them as sort of the same thing."

Chapter 6: Discussion and Conclusions

This study used a "micro-ethnographic" approach by closely analyzing the detailed data from spoken and written communications from a relatively small number of students. The aim of the study was to provide a descriptive account of these students' spontaneous reasoning and its relationship to their developing scientific (in Vygotsky's sense) understanding of limit concepts. Thus, the results presented in this chapter are intended to faithfully represent the nature of the reasoning displayed by these students, given the theoretical perspective outlined in Chapter 3. Although the results do not automatically generalize to calculus students reasoning and learning more broadly, by all measures observed by the researcher, the participants in this study were fairly typical of calculus students at Research I institutions where most similar research has been conducted. Most notably, their responses to items from the research literature were comparable to those of the students in the original studies and in their exposure to calculus content and grades compared to other students at the university where the study was conducted. In addition, the types of responses that are represented in the metaphor clusters were used by a large number of students in a variety of problem contexts. Consequently, in many cases, the following results also represent a plausible characterization of the thinking of similar students under similar circumstances.

This chapter presents a discussion of the features of the metaphor clusters that students used to reason about limit concepts in this study and of the general characteristics of students' metaphorical reasoning with a focus on aspects that have not been observed in previous research. We then consider the implications for both teaching and research that may be drawn from these results. Finally we provide suggestions for further research to investigate methods for helping students better develop metaphorical reasoning that is supportive of learning the relevant mathematical concepts.

THE NATURE OF STUDENTS' METAPHORICAL REASONING

The methodology employed in this study, built from theoretical perspectives of Black's interaction theory of metaphor, Alexander, Schallert, & Hare's theory of knowledge organization and conceptual schemas, and Dewey's instrumental view of inquiry, has allowed for several observations to be made about the nature of students' metaphorical reasoning not previously documented in the research literature. Five metaphor clusters were identified as central to students reasoning about limits, and their structures and uses in various problem contexts were characterized. The same methods were used to investigate students' interpretations of two constructs that are typically considered as active metaphors for students thinking about limits, motion and zooming, but neither appeared to be used by students in the ways generally assumed. Likewise, students' interpretations of the words "arbitrarily" and "sufficiently" did not match the standard mathematical interpretations of the words. After discussing these in turn, we will focus on the characteristics of students' metaphorical reasoning more generally then examine what allowed some students to use metaphors in ways that helped them better understand the mathematics.

The Metaphor Clusters

One of the main results of this study is the detailed characterization of five metaphor clusters used by students to understand limit concepts and of the interpretations that students construct for commonly used language and imagery associated with limits. The students in this study often reasoned metaphorically about new concepts involving limits and interpreted statements about limits based on more intuitively understood ideas, images, and experiences. Common examples of their metaphorical reasoning included ideas about approximation, closeness, infinity as a number, collapse in dimension, and physical limitation.

The relationships between limit concepts and some of these sets of ideas (approximation, closeness, and infinity as a number) have been noted in previous research (e.g., Cottril et. al., 1996; Williams, 1991, 2001; Sierpinska, 1987; Tall, 1992) but the metaphorical nature of students' thought involving these ideas has not been explored (with the exception of Williams' observation of approximation as a metaphor for limits). Of the others, only collapse has been mentioned in the research literature (Thompson, 1994b), but only in passing. One important set of findings from this research was the nature of students' uses of various metaphors in different problem contexts. Previously, students' metaphors have been treated as singular constructs. This research, however, provides details about the different, context-dependent manifestations of these ideas as mental tools. Thus, much of the characterizations of these clusters presented in Chapter 5 represents new information on the spontaneous concepts that students use as their limit conceptualizations are developing. While the rich descriptions provided in the previous chapter are essential to capture the nature of these metaphors, their newly discovered aspects are summarized below.

Collapse Metaphors

As mentioned above, the only mention of a collapse metaphor for limits is in Thompson's (1994b) article on the fundamental theorem of calculus (although he characterizes the students' thinking as a misconception rather than a metaphor). The current study, however, indicates that it may be a fairly common way that students reason about certain limit concepts. A collapse metaphor involves imagining one or more dimensions of a geometric object decreasing to zero so that it "collapses out" leaving a lower dimensional object which is perceived as the limit.

Students used collapse metaphors to

1) interpret the passing from secant lines to a tangent line in the definition of the derivative as involving the two points on the graph defining the secant becoming a single point and
2) conceptualize an unbounded solid of revolution with finite volume by imagining the radius to collapse to zero at some point leaving a one-dimensional line beyond that point.

In addition, based on Thompson's (1994b) observation of a collapse metaphor and similar informal observations made during this research, it is plausible that students often use a collapse metaphor to understand the fundamental theorem of calculus $\frac{d}{dx}\int_a^t f(t)dt = f(x)$ as involving a thin slice of area, corresponding to $\int_x^{x+h} f(t)dt$, eventually becoming a line of height $f(x)$ as $h \to 0$. To fully make this claim, however, transcripts of students explaining their understanding of specific problems involving the use the fundamental theorem would need to be collected and analyzed. The collapse metaphors were not widely observed for students talking about the limit of a function at a point, $\lim_{x\to c} f(x)$. In this context, a collapse might involve the line segments on the graph from the point $(c,0)$ horizontally to $(x,0)$ and from $\left(c, f(c)\right)$ vertically to $\left(c, f(x)\right)$ diminishing to single points. Although collapse metaphors were used widely by the students in this study for more complex contexts involving limits (e.g., the definition of the derivative), few explicitly described such thinking related directly to a simple limit of a sequence or function. Recall as one example, however, that Karrie repeatedly used collapse metaphors in her responses throughout the second-semester course, including those about the infinite decimal $0.\overline{9}$ and the Taylor series of $\sin(x)$. She discussed the repeating decimal as additional terms collapsing to zero so that their further addition would not push the limit beyond a finite amount. Likewise, her discussion about the Taylor series (used to illustrate students' "mixed metaphors") involved a collapse metaphor to bring a finality to the indefinite nature of her approximation and closeness metaphors for the graphs representing the sequence of partial sums. Unlike Karrie, other students may not have explicitly described a collapse metaphor that they were using to understand a function or sequence eventually reaching or becoming its limit. Again, further research would be necessary to determine whether there is a more widespread use of collapse metaphors in these simpler cases involving only the limit of a function at a point or the limit of a sequence.

Approximation Metaphors

Students' spontaneous imagery about and metaphorical use of concepts involving approximation resulted in one of the two metaphor clusters that most closely resembled actual mathematical definitions of limit concepts. The epsilon-delta and epsilon-*N* definitions are mimicked structurally by concepts such as error, bounding the allowable error, arbitrary levels of accuracy, and the requirements for obtaining such accuracy. The approximation metaphor cluster also involved a significant number of ideas about the practicality of dealing with very small quantities. Specifically, errors or differences were considered to be "negligible" or "insignificant" if they were either extremely small by some standard perceived by the student or small in comparison to some other stated quantity or scale. In most cases, these ideas were used interchangeably with ideas about being able to make an error as small as one wants. In some cases, however, it was sufficiently developed in a different sense to suggest an actual cut-off level, beyond which differences may be ignored. In this sense, the practical limit precludes any investigation of error and conflicts with important aspects of mathematical limits.

Students used approximation metaphors to

1) consider the addition of each term of an infinite series as refining the accuracy of the approximation,
2) select successive pairs of points defining secant lines that provide better approximations to a tangent line in the definition of the derivative,
3) argue that the volume of a solid of revolution is finite if you are able to disregard a “negligible” portion of the volume beyond a certain point, and
4) estimate the limit of a function by evaluating it at successively chosen points to provide more accurate approximations to the “true” function value.

Closeness Metaphors

In addition to approximation, the other metaphor cluster that most resembled formal aspects of limit definitions was the closeness cluster. This metaphor treated numbers as points on a line (or as coordinates for points in space) and attended to the behavior of a function or sequence with respect to a metric. A judgment about what constitutes a small region on the line (in space) was then made by the student (either explicitly or implicitly) to use as a measure of convergence.

Students used closeness metaphors to

1) locate a limit on a number line as the point around which partial sums are clustering,

2) apply continuity criteria of a function in descriptions of limits as involving either preservation of closeness (nearby points are sent to nearby points) or small changes in properties of a space (e.g., temperature) over small regions of space, and
3) interpret the definition of the derivative as implying that secant lines move to the limiting position of a tangent line, decreasing "the space between them."

Although some of these uses of closeness metaphors did resemble aspects of the structure and logic of epsilon-delta or epsilon-*N* definitions through ideas like making something "as close as you want," most were of a more informal form. The excerpts highlighted in Chapter 5 reveal students' use of a very physical sense of closeness. Their language of sequence points "clustering" around the limit or of a partial sum for a Taylor series that "wraps very tightly around the graph" are suggestive of actual objects coming into near contact. Thus, these objects and their visual configurations seem to be the most salient aspects of students thinking. Actual distances in the space and sizes of regions are left in the background, and closeness is intuited more than analyzed. For example, when students suggested that a secant line was "close" to a tangent line then confronted by the interviewer with the fact that far enough from their intersection point the two lines are actually quite far apart, the students were baffled and unable to even explain what they meant by "close." In addition, as described in Chapter 5, many students used closeness ideas with reference to only the range process of the limit. They might consider the proximity of sequence points to its limit or of a secant line to the tangent, but not mention or coordinate the corresponding domain process.

The students who did make some connection between their closeness metaphors and more formal aspects of limit definitions were mostly those who attached closeness language to the professor's descriptions of functions acting on intervals and epsilon-delta proofs (or perhaps more likely, they picked up the professor's language). Typically this resulted in statements about nearby input resulting in nearby output. This directionality of the function process caused difficulties for some students understanding the formal definition and the dependence of delta on epsilon.

Infinity as a Number Metaphors

The limited extension of algebraic operations on the real numbers to include $\pm\infty$ is not mentioned in the textbook used by the students in this study, nor was it discussed by the professor. Abuses of notation such as writing $\int_a^\infty f(x)dx = F(x)\Big|_a^\infty$ instead of $\int_a^\infty f(x)dx = \lim_{K\to\infty} F(x)\Big|_a^K$ or, even worse, writing something like $F(\infty) - F(a)$ are generally considered in bad form. Nevertheless, some students adopt these and other conventions. One might argue that the students are just using a shorthand notation to avoid writing out the corresponding limits. This study suggests, however, that students' treatment of infinity as a number or as a point on the number line is often more than notational convention. Comparing the number 20 to the size of infinity or trying to extend the mean value theorem to argue that the slope of the function with a vertical asymptote at

$x=c$ is somewhere equal to the slope of the line between $\left(0, f(0)\right)$ and (x,∞) involves more than applying a shorthand notation for limits.

Students used infinity as a number metaphors to

1) compute by extending the algebra of the real numbers and the domain and range of functions to include infinity,
2) make intuitive size comparisons between different "products" of various rates of growth or decay, and
3) extend ideas about finite intervals to the entire set of real numbers by geometrically imagining a compactification of the number line yielding $[-\infty,\infty]$.

Physical Limitation Metaphors

Students' physical limitation metaphors were derived from imagining a smallest physical size beyond which nothing exists. This metaphor cluster was observed in students responses to only two of the problem contexts presented to the students. It was, however, used by a large number of students (41%) in response to the web question about a solid of revolution, and these students based their entire argument on the metaphor.

Students used physical limitation metaphors to

1) reason about the finite volume of a solid of revolution in terms of the radius becoming so small as to no longer allow the smallest existing particle with the property of volume to fit beyond that point and

2) ascribe to a limit properties of the sequence elements or function values by imagining the limit as achieved once it transcends the smallest attainable sizes possible in the physical world.

Students' Interpretations

Students' spontaneous reasoning in interpreting ideas presented in class was also investigated with the techniques used to elicit their metaphors about limits. The interpretations revealed through this aspect of the research revealed that students assumed that "arbitrarily small" and "sufficiently small" refer to a hierarchy of smallness, that motion language does not necessarily imply motion for the students, and that they understood zooming in on a graph in ways irrelevant to the mathematics of local linearity. Although the details are provided in Chapter 5, in this section we discuss how these characterizations provide new insights or invite reconsideration of previous findings on our understanding of the nature of students' learning of limit concepts.

Dynamic Imagery

The use of metaphorical reasoning involving motion was not observed in this study. Certainly the language used for limit concepts often involves metaphorical applications of words about motion. When students do think about actual motion, they typically imagine some object traveling along the graph of a function and arriving at a limit point. This type of description, however, was not actually used by students to reason about limit concepts. Instead, students typically described sequentially selecting points. It may be argued that this description represents a form of motion, especially if one incorporates Talmy's fictive motion. The students themselves suggested otherwise, however, drawing a clear distinction between selecting points and actual motion. The specific procedure of selecting these points was crucial to their thought process, involving intentions about making the selections for certain purposes, and was seen as an inherently discrete process.

When researchers argue that a dynamic view of functions is necessary to understand limits, my research suggests that "dynamic" should be interpreted as involving change rather than literal motion. Certainly, students did find value in changing choices of input points or numbers and in rearranging geometric configurations to check the affect of these changes on output, but such manipulations differ from actual motion. When they did describe motion, it was not used in their reasoning in an integral way. For example, they often described moving along the graph of a function to illustrate continuity, but the critical feature was whether or not one encountered some kind of gap, hole, or break in the process. Thus, the image of a gap, hole, or break is the operative concept and the motion toward that feature was ancillary.

This issue is slightly more subtle in problem contexts that actually involve motion. A distinction must be made between the motion of the context (e.g., a car traveling along a straight road) and the motion that would correspond to changes in relevant quantities in a limit (e.g., the motion corresponding to $x \to c$ and $f(x) \to L$ in the limit $\lim_{x \to c} f(x) = L$). The only case in which they are the same is when one considers the limit of the position of a moving object at some time, but this case was not treated as problematic by the students in this study. On the other hand, a limit such as the one arising in the definition of the derivative $\lim_{t \to t_0} \frac{p(t) - p(t_0)}{t - t_0}$, where p is position and t is time, did cause students to struggle. In this case, however, the motion of the car does not correspond to either of the two changing values in the limit, average velocity $\frac{p(t) - p(t_0)}{t - t_0}$ and time $t \to t_0$. The distinction for the latter may seem like a mere technicality of distinguishing between changing time on one hand and motion or changing values of position on the other, but it is actually more significant. If one imagines time flowing continuously from t to t_0, the motion of the object cannot simply be followed along during that time to understand the limit. If one is to understand the role of the limit, the motion during the interval $\left[t, t_0\right]$ must be considered for every t in order to arrive at changing values of average velocity. That such is the case can be seen in the interviews with the students on this concept. Instead of trying to imagine the limit implying motion of the object, they considered changing (successively smaller) time intervals over which to compute average velocity. The motion of the object is then important in the concept of average velocity and in providing an intuitive, physical referent for the resulting instantaneous velocity. It is not critical in the students understanding of the limit. Having an image which can be made to change while considering various cases is important in such an analysis.

The Terms "Arbitrarily" and "Sufficiently"

Only nine students were asked about their interpretation of the words "arbitrarily" and "sufficiently," so generalizations are difficult to make. Striking, however, was that only one student described an interpretation of "arbitrarily" consistent with its mathematical meaning when used in a phrase such as "$f(x)$ can be made arbitrarily close to L by choosing x sufficiently close to a." While his interpretation of "sufficiently close" in such a phrase indicated a connection to the size of any particular choice for "arbitrarily close," the connection was that the two sizes should be the same. Also striking was that seven of the other students interpreted these phrases as giving a hierarchy of smallness with "sufficiently small" being smaller than plain "small" and "arbitrarily small" being even smaller.

Several students confessed that they had not really thought about the meanings of these words, so that when the professor used them, they just allowed whatever image first came to mind determine the meaning. When modifying words like small and close, it is easy to see then how they could be interpreted as indicators of degree. When there were no circumstances that forced them to reconsider their interpretations, these first impressions were allowed to stand.

Zooming Imagery

Zooming imagery was not used to solve any problem perceived by the students. They were readily able to intuitively recognize a tangent line or the steepness of a graph without having to see it "straighten out" under successive magnification. This may be likened to Black's characterization of "seeing as" rather than metaphorical reasoning. Specifically, students' were already comfortable which the intuitive concept of a tangent line when it was presented in a new way. They then *saw* this previously understood concept *as* a small segment of the actual graph through zooming. Beyond this, however, the concepts of zooming and local straightness were not used to investigate problematic questions about functions and graphs, to explore the relationship between rate and accumulation of rate, or even to connect to the standard limit definition of the derivative in a way that added new insights. Note that the research instrument did not ask students to use zooming instrumentally either, it just asked them to explain what one would see. When the students engaged this task many of them wrote multiple pages explaining their ideas, including pictures and anecdotes. The compelling imagery of zooming had clearly remained with them from class and their own experimentation, but the details were interpreted in ways that did not help them properly address the relevant mathematical concepts. Instead, through their creativity, many imagined seeing a line composed of large, blocky pixels, a line becoming thick and blurry, or reading a single page of a book being analogous.

Some students did consider more mathematical aspects, but did not seem to know how to explore those ideas. Students who described seeing a straight line because only a small part of the graph or only small-scale features would be visible were at least focusing on the graph and its local behavior. They did not recognize that this is not an explanation, however. For example, the same would be true of a nondifferentiable, continuous curve, which would not appear as a straight line under sufficient zooming. In fact, the appearance of indefinitely small-scale features is precisely what would prevent it from being straight, but the students were focusing on curviness as an inherently large-scale feature. Other students argued that over a small portion of the domain, there can only be a small vertical change, and most concluded that the line would appear horizontal. These students were actually considering amounts of change, but not in proportional terms.

These students' ideas about both the nature of small and large scale features and small amounts of vertical change could be used as excellent conversation or debate topics for the class to consider. If used to explore situations for which these ideas actually help solve a real problem, through proper guidance, a zooming metaphor could become very powerful for the students, both as an intuitive and a mathematical tool. Using the image of the round earth appearing flat from close range, many of the students even spontaneously generalized zooming in on a graph to the two-dimensional case to imagine a "practical tangent plane" (extending Tall's language from "practical tangent"). If the students' zooming imagery related to graphs of single-variable functions had been well connected to an understanding of properties of linearity and proportional amounts of change, such a generalization may have provided a valuable basis for an understanding of tangent planes. Otherwise, their understanding is not likely to extend beyond the simple image of a plane "just touching" the graph or the definition provided by their textbook as, "the plane through x_0 that best approximates the surface in a neighborhood of x_0," with no further explanation of what this means.

Characteristics of Students' Metaphors

In addition to the specific metaphors that students use to reason about limits and the interpretations they bring to specific language and imagery, several conclusions may also be drawn about the general nature of these students' metaphorical reasoning. Their metaphors were idiosyncratic in that they were either nonstandard versions of common metaphors or involved domains not typically formally or informally associated with limit concepts. The more standard aspects of students' metaphors appeared to have been picked up from or at least influenced by the professor and developed through imitation. Finally, students' strong (resonant and emphatic) metaphors provided both powerful ways for them to reason as well as misleading aspects.

Idiosyncrasy

Although there were sufficient commonalities to groups students' main metaphors into clusters, individual uses of these metaphors tended to be somewhat personalized. Within a cluster, each student revealed only portions of the underlying generalized schema, different students often applied various aspects of the schemas in opposite ways, and different contexts elicited metaphors from different clusters from each student. Thus students interpreted and used certain metaphorical ideas in different ways. They also developed their own highly personalized metaphors involving round-off error, reading a single page from a book, color wheels, and concentrations of molecules in solution.

Most of the metaphors used by the students were based on similar metaphors widely accepted in mathematics, but with a nonstandard interpretation. For example, describing numbers as points on a line is such a ubiquitous metaphor in mathematics that it is considered literal for most purposes. The students in this study, however, were prone to interpret the number line as containing infinitesimal elements, points that are "next to" each other or that "overlap." When these ideas become the basis of a closeness metaphor for limits, there is no surprise that the result conflicts with the standard mathematical meanings for limits. Similar nonstandard interpretations of common metaphors were generated by the students in the approximation and infinity as a number metaphor clusters, and an extreme divergence from a standard metaphor was revealed in students' interpretation of zooming imagery.

Students also created their own metaphors for use in various situations. These metaphors were often based on concepts they had learned in non-mathematics classes. From their chemistry classes, students used their image of the smooth blending in the color wheel to think about the continuum and used their understanding of procedures to track significant figures to think of limits as rounding off. They thought of comparing different polynomial time processes, learned in computer science, to think of errors in approximations as being small relative to the estimation being made. From physics, they conjured images of smallest possible particles such as electrons or quarks to use as a physical limitation on the differences one could measure. These ideas were all used in the students' exploration of various limit concepts, and in turn, impacted their understanding of the mathematics.

Often, various versions of a student's different metaphors conflicted with one another. For example, several students reasoned that the volume of a solid of revolution could be finite if no particle would fit into the portions with a very small radius. Many of the same students also argued that the surface area could be infinite because the thin sheet of infinitely many particles is revolved around the axis. Some students recognized such contradictions, but many did not. Previous research has shown that students can learn formal concepts without impacting their existing informal knowledge and their ways of using it (e.g., many of the studies using concept image and concept definition, such as Vinner, 1982; Davis & Vinner, 1986; and Tall & Vinner, 1981).

Imitation

In both the cases of the approximation and closeness clusters, it is likely that the students were imitating similar usage observed in their professor, teaching assistant, or textbook, each a plentiful source of such language. While imitation may be seen as indicative of a feeble understanding, Vygotsky gives it a much higher intellectual status, arguing that it is only possible when a concept is within a learner's zone of proximal development and they have grasped the fundamental structure of the task. This differs markedly from training in which a person (or even animal) learns a behavior through repeated action and a conceptual understanding is not required. According to Vygotsky, imitation is the action which builds conscious awareness and volitional use, hallmarks of a scientific concept. Although the questions did not prompt for the use of formal definitions, many students applied their approximation and closeness language in ways equivalent to the logic and structure of epsilon-delta and epsilon-*N* definitions of limits. Thus, even though these definitions are often considered to be conceptually too difficult for first-year calculus students, several chose to actually apply an informal equivalent to problems they encountered. Some calculus learning environments, however, lack a critical element of Vygotsky's protocol for learning: the subjection of the student's imitative efforts to supportive supervision. Students' development of the standard versions of metaphors is unlikely to occur without guided critical evaluation and revision.

Resonance and Emphasis

Requirements of resonance and emphasis were built in to the data collection and analysis, so the five resulting metaphor clusters necessarily posses these properties to some degree. It is instructive, however, to examine the various ways in which both resonance and emphasis are actually manifested in the data.

Resonance is the degree to which a metaphor supports elaborative implication. It may be seen in the connections that students develop between the metaphorical and mathematical domains, their ability to resolve perceived problems with specific metaphors, and the conclusions that they draw about both the metaphors and the mathematics. The students in this study used metaphors in ways that were highly implicative for their understanding. For example, the use of a physical limitation metaphor led many students to develop an understanding of volume as inherently tied to an amount of an actual substance. In this case, although contradictory to the mathematical definition, the metaphor actually provided more fundamental conceptual support than the textbook, where the meaning of volume is simply treated as existing a priori to be explored (not defined) with the tools of calculus. Other examples of students' metaphors strongly influencing understanding are collapse ideas providing students a way to conceive of the actual passing to a limit and treating infinity as a number in order to conceptualize the slope of a line through a point (x,∞) and apply the mean value theorem.

One argument might be that students aren't using metaphors to think, and that instead, they only use such informal language because they do not posses the proper language to express their ideas correctly. This research suggests the opposite is most likely the case. In the examples cited above, the actual images described play a central role in the arguments being made. They are not accidental artifacts of words chosen by students who are trying to describe a different idea. Furthermore, the students do seem to pick up the technical language, but often have little idea what it means. Consider, for example, the terms "arbitrarily" and "sufficiently." Students readily used these words in phrases throughout their descriptions of limit concepts, but to describe their own spontaneous interpretation rather than what was intended by the textbook or professor.

Emphasis, the degree to which the user of a metaphor is committed to the specifics of that metaphor, was observed in two main ways in this study. First, students chose metaphors that were compelling to them in particular contexts or that were personally meaningful, and second, students' personal versions of metaphors persisted even after explicit instruction on the use of standard mathematical versions.

By design, the five metaphor clusters were developed to represent students' reasoning that appeared in multiple problem contexts in which several students provided descriptions in terms of these metaphors. These metaphors were introduced by the students into problem contexts which were not phrased in such terms. They were both readily available for use by the students (who invoked them spontaneously in interviews) and still considered relevant after reflection (students also used these metaphors after time for consideration in the writing assignments). Thus, some aspects of the problem contexts, of students' understanding of limit concepts, or of students' past experiences made the application of these specific metaphors particularly compelling.

Students' personal metaphors also appeared highly emphatic in the third phase of the study in the sense that the students still presented nonstandard interpretations of metaphors that were developed through a series of very prescriptive writing assignments. Although most of the students eventually revealed that they did accommodate many aspects of these explicitly presented metaphors, their personal versions were typically dominant in the early parts of the interviews, if not throughout, even when they were directly asked to explain a concept in terms of the ideas presented in the writing assignments. For example, even though "error" was consistently presented to the students as the difference between an estimate and the value being approximated, a few still interpreted it as "round-off error." Even more significant was their difficulty interpreting a "bound on the error," which was repetitively characterized in the writing assignments as a largest acceptable value for the error. Regardless, some students confused it with "error," others suggested that it would *prevent* an error from being reduced beyond a certain degree, while even others described thinking in terms of precision rather than accuracy.

Thus, introducing new metaphors for a context does not necessarily alter students' existing idiosyncratic metaphors. In Vygotsky's terms, their thinking is extrasystemic. Ideas that are seen as relating to different phenomena and are not part of a coherent system tend not to interact with one another. For a student to reconcile differences, or even recognize a contradiction, they must see the relevant concepts as pertaining to the same thing. Students who did so made quick progress in understanding both the metaphors and the mathematics, often coming to identify the two. Recall that as Jacob began to understand the logic of the limit definition in terms of approximation, closeness, and epsilons and deltas, he began to interchange language from the three domains. In the end, he explicitly identified them saying "I guess - now I think of them as sort of the same thing." Like the "connectors" in Frid's (1994) study, Jacob saw mathematics as a set of interrelated ideas that had could have personal meaning. The process of seeking that meaning, was as much a part of his mathematical understanding as the resulting understandings.

Deficiencies

As discussed in chapter 5 and throughout this chapter, all of the metaphor clusters included several entailments that were misleading for the students. Several of these deficiencies have already been described in other sections: the mathematical inaccuracy of the entire premise of the collapse metaphor, the loss of ability to develop actual limit concepts through the treatment of infinity as a fixed number or point, various idiosyncratic versions of standard metaphors that lose the original meaning and pick up new, incorrect meanings, etc.

Students' metaphors also negatively affected students' epistemological beliefs. For example, in both the physical limitation and approximation metaphor clusters, limits were sometimes seen as allowing a computation to be made down to either some smallest known size or cut-off level of accuracy. In the first case, any other differences that a mathematician might try to describe would be irrelevant to the real world. The second view, often conversely led students to claim that mathematics pertains only to the practical endeavor of describing large scale phenomena and that it is up to particle physics or microbiology to provide understanding for what mathematics could not probe.

What Makes Students' Metaphors Helpful

Finding fault with a metaphor is not difficult; after all, they are almost always false in a literal sense. Consequently, their use by students and teachers is easily criticized for a lack of precision. A metaphor may also be viewed as an evasion of rigor or as “a mere crutch” in coping with it, hindering true understanding. At best in such a view, it may be seen as only helping someone think about a concept they already understand. The results of this study suggest that even metaphors that portray mathematical concepts incorrectly may, in some cases, be beneficial to students’ understanding and reasoning. Before making this argument, it will be helpful to discuss other corroborating research. After doing so, we will turn to an exploration of the aspects of metaphor use that enables students’ to develop stronger mathematical understandings.

Even Mathematically Incorrect Metaphors May Be Useful

Recent research on experts’ reasoning indicates that treating nonstandard interpretations simply as misconceptions may miss many important aspects of the learning process. Roth & Bowen (2001) presented research scientists with graphs that were unfamiliar but intended for undergraduate instruction in their own field. While interpreting the information contained in the graphs, these scientists made errors strikingly similar to those of students such as conflating interpretations of slope and height, ignoring changes in one quantity as a result of change in another, and improperly attributing physical properties of a graph with related features of the represented phenomenon (iconic translation). In similar studies, experienced professors have been shown to extend analogy beyond the applicable structure of a situation (Roth & Bowen, 1999; Roth, Tobin, & Shaw, 1997) and struggle with concepts and experience frustration and anxiety during problem solving (Carlson & Bloom, under review).

When observed in students, such errors and behaviors are typically treated with deficit arguments, but these studies suggest that powerful reasoning is still possible. Carlson & Bloom's study of research mathematicians' problem solving strategies suggests that struggle, frustration, and anxiety are an integral part of larger metacognitive and affective cycles, leading to greater control and motivation. Scientists in the studies of Roth et al. were able to use even nonstandard interpretations to produce and test hypotheses that moved their thinking forward in positive ways.

Such active ways of engaging with ideas is similar to the students' productive uses of metaphors observed in this study. Shawna, in her interview about the definition of the derivative, went through at least three cycles of developing an idea, applying it to the situation, and eventually rejecting or revising it through testing for its entailments. These ideas included trying to make sense of various algebraic expressions in terms of lengths on the graph, imagining lines through a single point on the graph rotating to a limiting position, and lines tangent to the graph at various points sliding along the curve to the point of interest. In these cycles she begins to experiment with putting various lengths in relationship to one another, makes points move and line segments change length, and compares the slopes at different parts of the graph to her intuition about the slope at the point in which she is interested.

This process is crucial for Shawna's developing thinking, and arguably, without the availability of these "misconceptions," she may have never arrived at a point to be able to make the progress she did. Her fourth iteration of the cycle of developing and applying an idea involved the collapse metaphor with an unspoken (at the time) influence of approximation when translating to a numerical domain. While the idea of secant lines collapsing to a tangent is mathematically incorrect, it is very productive for Shawna, helping her understand some powerful mathematical ideas about the definition of the derivative. Minimally, it provides her with a way to connect the secant lines with the tangent through attention to the appropriate limiting process. More than this, however, it becomes a tool with which she is able to ask some interesting questions and develop good connections. As shown in the schematic of this portion of her reasoning (Figure 15 in Chapter 5), she uses the collapse metaphor as a central point around which to organize her reasoning. Through this process, she is able to see that the slopes of the secants become closer to the slope of the tangent, and she wonders about and finds the connection between graphical, algebraic, and numerical representations of the definition. Later, she is even able to connect these ideas to her interpretation of the derivative in the physical context of the changing position of a car through her collapse metaphor.

In the complex process of conceptual development, accuracy of ideas is not always the only concern. With respect to mathematical notation, Harel and Kaput (1991) argue that the strength of a symbol system often relies on its capacity to syntactically guide important mental operations. Thus, students may often prefer an elaborated notation (reflecting the structure of a conceptual entity and mental actions that may be performed on it) over a more tacit notation even if it is not considered appropriate. For example, the use of the symbol $f(x)$ rather than *f* to refer to a function emphasizes a salient aspect (a "function machine" accepting input and producing output) of the function that may help provide meaning for a student. This can be especially appealing when any complexity is added as in considering composition, where students often choose the notation $f(g(x))$ over $f \circ g$. The former relies on the familiar conceptualization of *f* as a function machine with $g(x)$ serving as the input, while the latter emphasizes the perhaps less familiar aspect of the functions *f* and *g* as elements of a set with the binary operation of composition producing a new function-element.

Even mathematicians will often favor an "abuse of notation" if it is conceptually advantageous, and the resulting interpretations can become part of the concept. Leibniz's notation for the calculus gained wide preference over Newton's, precisely because of (not in spite of) the fact that it evoked conceptual ideas that were mathematically flawed – ideas that were nevertheless central to the mathematics. Leibniz spoke to the power of such symbolization by saying, "In signs one observes an advantage in discovery which is greatest when they express the exact nature of a thing briefly, and, as it were, picture it; then indeed the labor of thought is wonderfully diminished" (quoted in Harel & Kaput (1991). Even after Weierstrass's epsilon-delta formulation of Cauchy's definition formally made infinitesimal interpretations unnecessary, they (along with

Leibniz's notation) remained a central part of the informal concept images of the corresponding concepts.

Roy Pea (1985) argues that the essential characteristic that makes such notation conceptually powerful is that it is technological, and thus may serve to qualitatively reorganize thought. Vygotsky observed that "the psychological tool alters the entire flow and structure of mental functions." Thus, what one *does* when using such a tool is fundamentally different, not simply more powerful, and one's relationship to the concepts and the process of inquiry are changed as well. Generalizing Harel & Kaput's arguments from notation to other signs, mental models, or metaphors, these cognitive tools are powerful not simply to the degree that they accurately reflect the mathematical detail, but rather to the extent that they afford an engaged process of mathematical inquiry.

Consequently, in analyzing students' responses to research instruments, we should not rush to label all nonstandard interpretations as misconceptions. We must consider how these ideas might affect and be part of a larger learning process. For example, it may be perfectly reasonable for a student to claim that $0.\overline{9}$ is a number smaller than one given their understanding of what numbers are (which may or may not be the same as their understanding of what real numbers are). Certainly in this study, students were willing to suggest that, at some level, numbers could be "next to" each other or that the small differences "become irrelevant." Several of these students displayed useful ways of reasoning about numbers, limits, or other mathematical processes and objects that are tied to such claims. While they may not have understood the subtleties of the properties of the real number system, these students were able to use these conceptualizations to make sense of certain limit statements. In some systems (e.g., nonstandard analysis), such reasoning is even mathematically correct and students' ideas are likely to reflect some of the structure of these systems (Frid, 1994; Tall, 1981, 1992).

In general, students' nonstandard interpretations are likely to be sites of rich discussion and for important conceptual development. Even when the standard interpretations are not fully developed in this process, we must better understand what types of thinking and learning are possible. We are not arguing that misconceptions do not exist, but rather that treating them simply as misconceptions that must be eliminated in students' thinking overlooks a great deal of positive potential. Powerful mathematical reasoning is possible for our students even while working with nonstandard interpretations of concepts. The question then becomes how can we recognize and encourage aspects that are productive.

Ownership

In this study, we have seen several instances where the professor's metaphors were misinterpreted by students who did not themselves go through the thought process of creating the metaphor to understand or explain a specific concept. Thus the features that were important to the professor were not necessarily important, perhaps not even salient, to the students. This makes it easy for them to focus on the wrong things or to interpret them in ways not informed by the process by which the metaphor was created. When students who noticed the symmetry of an interval $I = (c - r, c + r)$ around its center, *c*, decided that this was the most important feature of intervals, they became confused when the image $f(I)$ was not symmetric around $f(c)$. Similarly when they focused on the compelling imagery of the width of a container becoming smaller than a single molecule, this became a part of their operative model for volume. When these students did not participate in the creation of the metaphors they adopted, they had no way of knowing what was relevant and what was extraneous.

Mikhail Bakhtin characterized the basic unit of speech as the *utterance* that lies on the boundary between the constancy and systematicity of language on the one hand and its situated nature and dependence on social context on the other (Wertsch, 1991). The important characteristic of utterances is that they are dialogic. They mutually reflect one another and cannot stand alone; their meaning is tied to their interdependence. According to Bakhtin, "When the listener perceives and understands the meaning… of speech he simultaneously takes an active, responsive attitude toward it. He either agrees or disagrees with it… augments it, applies it, prepares for its execution, and so on… Any understanding of live speech, a live utterance, is inherently responsive, although the degree of this activity varies extremely. Any understanding is imbued with response and necessarily elicits it in one form or another; the listener becomes the speaker."

In unquestioningly adopting metaphors unmotivated by real problems, the students in this study treated mathematics as a monologic language. Mathematics is often seen by students as composed of definites that are divorced from concrete life context. The role of authority for determining mathematical truth and even for determining what constitutes mathematical activity is seen to lie within the formal structure of the subject. At best, it is determined by others without any room for the students' personal interaction or dialogue. There are, however, many social, intuitive, and context-dependent aspects to mathematics. Axioms, definitions, and accepted rules of logic are determined by a community. The acceptance of a proof and the determination of what constitutes valid mathematics are socially situated and may shift with time or social context.

As described earlier in this chapter, metaphors that are not seen to solve a perceived problem for the students will not be interpreted in personally or mathematically meaningful ways. Students must develop a sense of ownership over the tools they will use. This occurs through solving problems with them to gain new insights and participating in their construction. This is dialogical activity: saying something with a metaphor while simultaneously listening to what it has to say about the mathematics.

Connections to the Mathematics

Reinterpreting Ricoeur's hermeneutic circle to mathematical meaning-making, Tony Brown (1996) suggests that this activity requires the ability to alternate between understanding and appreciating the inherited tools and formal structure of mathematics on the one hand and being able to operate on this structure to create new meaning and interact socially with its tools on the other.

In the previous section we discussed students' need to develop ownership over the metaphors they use. The flip side to the hermeneutic circle, however, suggests that personal, idiosyncratic metaphors that are never exposed to the rigors of the corresponding mathematics will not provide meaningful conceptual insights.

Students who had a strong connection between their intuitive ideas and the mathematics were able to use those ideas to help them understand the concepts. For example, students' thinking about approximation or physical proximity in ways similar to epsilon-delta ideas were able to use those concepts effectively in understanding traditionally difficult ideas. Often, for these students, the intuitive and abstract ideas became identified with each other with language being freely interchanged between them. As a negative example, even if students had used zooming metaphors more frequently, they would not likely have been very useful as they were not connected in any way to the structures, logic, and applications of specific limit concepts.

Critical Evaluation and Modification

An important aspect of instrumentalism is the testability of a tool used in inquiry. Whether or not students actively engaged in critically evaluating the metaphors they were using was important for their ability to recognize problems and make helpful modifications. The case study of Shawna's work on the definition of the derivative illustrates this claim. She continually monitored the statements she was making for consistency between one another and with her intuitions. This allowed her to abort lines of reasoning that were becoming problematic and look for other alternatives. When she started to make progress, she kept checking herself (repeatedly saying things like "Yeah. That does make sense.") and asking questions *of* the metaphors she was using (such as "I don't know how you get a number out of that" and going on to investigate the implications numerically.)

The formative nature of this type of self-assessment is important. Carlson & Bloom (under review) observed that self evaluation is a critical part of the problem solving behaviors of research mathematicians on at least two levels, both of which helped move their thinking forward. Their activities could be organized into a global cycle of *planning-executing-checking* which is repeated as various strategies are tried. Within each planning stage they also carried out an internal sub-cycle of *conjecture-imagine-evaluate* to determine potentially productive approaches. Both of these cycles rely on the assessment of ideas and progress against goals, not simply as a final act but as one that drives the process through successive cycles.

Perhaps the first step in self-evaluation that leads to the development of new knowledge is conscious reflection on a variety of possible meanings as well as the initially presumed meaning. Several students in the study confessed that they had not actually thought about the meaning of the words "arbitrarily" and "sufficiently," which never allowed them the opportunity to consider whether that meaning or another might be most appropriate. Consequently these words remained largely useless for them.

IMPLICATIONS

Below, we discuss the implications of these results for both research and teaching. In terms of research, we argue that functional aspects of students' reasoning have largely been ignored in previous research in undergraduate mathematics education and discuss features of this study that may be adapted to other research. Further research is required to allow these results to significantly inform teaching practices, but we consider some preliminary implications involving an awareness of some important possible metaphors students may use for reasoning about limits.

Structure and Function in Research on Learning and Cognition

One of the major emphases of this research was to account for the students' functional application of the ideas being investigated as well as their structure. This aspect of the study represents a nascent methodology for this purpose, comprised simply of placing students in a variety of situations in which the concepts being studied are likely to be used as tools to solve a problem, then characterizing the use of the related metaphors in each context.

A Focus on Conceptual Structure in the Research Literature

Such a functional approach is rare in research on undergraduate mathematics education. Below, we detail the structural focus in some of the main perspectives discussed in this dissertation. APOS theory attempts to capture Piagetian structure through classifying conceptualizations. Lakoff's theory of metaphor is focused on demonstrating that preconceptual concepts embodied in basic level categories and kinesthetic image schemas contain sufficient structure to serve as a metaphorical basis for abstract concepts. Finally various constructs of concept image and definition and of conceptual schemas are models for the organization of knowledge in the mind. None of these approaches has as a main feature of the theory a consideration of how knowledge is actually used in specific instances.

APOS theory focuses on the type of conceptualization a student possesses. At most, it takes into consideration different types of thought being used in different circumstances. Student thought is categorized based on the criteria of their ability to conceive of a particular concept as an action, process, object, or schema. Some of these criteria are based on what a student can do (e.g., imagine operations being performed on functions to give new functions), but how such conceptualizations are actually used by the individual is rarely considered in this framework. The standard reference for outlining the essential ideas of this theory is Dubinsky, 1991, in which APOS is essentially derived from Piaget's theory of reflective abstraction. (At this time, actions were not treated fully as a separate type of concept, and schemas were interpreted in a slightly different way than described in Chapter 2. Despite these changes, the theory is largely the same.) Dubinsky quote's Piaget on the nature of building a complex mathematical concept successively as action, process, object, and schema:

> The whole of mathematics may therefore be thought of in terms of the construction of structures,… mathematical entities move from one level to another; an operation on such 'entities' becomes in its turn an object of the theory, and this process is repeated until we reach structures that are

> alternately structuring of being structured by 'stronger' structures. (Piaget, quoted in Dubinsky, 1991)

Dubinsky later acknowledges that

> It is not possible to observe directly any of a subject's schemas or their objects and processes. We can only infer them from our observations of individuals who may or may not bring them to bear on problems – situations in which the subject is seeking a solution or trying to understand a phenomenon. But these very acts of recognizing and solving problems, of asking new questions and creating new problems are the means (in our opinion, essentially the *only* means) by which a subject constructs new mathematical knowledge. (Dubinsky, 1991, original emphasis)

Thus, a functional role for a concept is seen as crucial, but it is relegated to only the creation of the concept and not extended to the nature of the actual concept itself. Dubinsky suggests that the need to consider such mental actions performed in relation to a concept is "where reflective abstraction comes in." Furthermore, he provides a list of five relevant actions that may be performed: interiorization, coordination, encapsulation, generalization, and reversal (see Chapter 2 for a brief description of these types of reflective abstraction in APOS theory). Note that these actions are all things done *to* a concept (action, process, object, or schema) aiding in the movement from one level to another and development of more sophisticated conceptual structure. None refer to what one might do *with* a concept in solving a problem.

In the sense thus illustrated, proponents of APOS theory claim that it is based on Piaget's constructivism which is itself rooted in his notions of structure. Stroup (1996), however, notes that such actions, processes, objects, and schemas do not account for the full nature of structure as intended by Piaget. Specifically, a full account of reversibility and path independence as modeled by the prototypical structures of inverse and associativity in the algebraic group are lacking. (See the discussion in Chapter 2 on Stroup's use of Piaget's notion of structure to develop early-age curriculum on calculus concepts.) Stroup suggests that the standard APOS approach has thus "collapsed Piaget's dynamic notion of structure – as a coherent coordination of element *and* operation having important constructivist properties – to 'an object… a static structure, existing somewhere in space and time.'" Even if APOS theory were elaborated to account for a coherence between its operational and elemental aspects, it would still only refer to conceptual structure (although structure as conceived by Piaget).

George Lakoff's characterization of conceptual metaphors is subject to similar criticism. His emphasis is on characterizing mappings between and within different contexts when a person reasons indirectly. Special significance is given to embodied concepts, which are those that bear a direct relationship to an object or experience. In the big picture, they may play a pedagogical role of establishing a foundation from which other, more abstract notions may emerge, however, the role they play for the student is typically not considered. The influence of metaphors, metonymies, and prototypes in reasoning is considered from a perspective in which relationships between objects, experiences, and thoughts are seen as determining the perception of concepts. As noted by Walter Stroup (personal communication, 2001), the danger in such an approach is that the heavy emphasis on *mappings* from preconceptual to abstract structures reduces all metaphors to literal statements. The nature of metaphor in such a theory is reduced to a theory of explicit maps.

According to Lakoff, preconceptual structures are founded in basic level categories and kinesthetic image schemas. Rosch's work on prototypes and human conceptual categorization has not been widely applied in research on mathematics education (see Schwarz & Hershkowitz, 1999, for a notable exception). Some researchers have applied the relevant aspects of imagery and kinesthetic image schemas, however. Dörfler (1991), for example, addresses the cognitive functions of a problem solver by positing the *protocol of an action,* a "cognitive process which produces as observable output a concrete carrier for the intended image schema." While Dörfler observes that the resulting "icons," "symbols," and "linguistic signs" have no meaning apart from the pertinent actions, he stops short of giving either the protocol or the actions any significance beyond that of creating a knowledge structure. For Dörfler, the purpose of a protocol is to produce "a structural system of perceivable and manipulatable objects," a representational system of the cognitive activity and its products.

The widely applied constructs of concept image and concept definition makes its distinction between two types of conceptual structure. Both are characterized as types of knowledge that the individual possesses and are defined in terms of the content of a student's knowledge. Specifically, a students' concept image is the collection of all ideas they posses related to the concept, and a concept definition is the verbal definition a student will give. Tall and Vinner (1981) describe the combination of the two as separate cells within a larger "cognitive structure." No mention is made of the students' application of a concept image or concept definition with respect to being an essential part of either.

Finally the various schema models all essentially provide different metaphors for the nature of knowledge structures. As described in Chapter 4, the word schema is not well defined in the psychological literature, and a variety of models show up in the mathematics education literature. These models are mostly based on computer science metaphors such as nodes and connections between them (Hiebert & Carpenter, 1992), explicit maps and procedures for solving problems (Marshall, 1995; Hinsley, Hayes, & Simon, 1976), and even fuzzy logic and neural networks (Zadeh, 1965). Work in this area is concerned almost exclusively with building a model for the organization of human thought and the cognitive implications for such a model; different ways in which these structures might function in thought are typically not considered.

Functional Aspects of the Current Study

As found in this study, students' reported structural organization of limit concepts often did not account for their actually use of those ideas. When they were asked to explain what limits are, they typically described a graphical setting accompanied by language about motion. Determining what concepts about limits a student possesses and how they are connected is a very structurally oriented approach to their understanding. In this study, when they were asked to use limit concepts to think about something new or approach a difficult problem, motion language tended to remain at the level of language and did not enter their descriptions referring to their thinking about anything actually moving. Instead, other metaphors surfaced.

Such results suggest that research cannot fully uncover the nature of students' metaphors by examining only their surface language and responses to direct questions about their conceptualizations of the topic. Not only does this methodology miss the different structures that might appear in such problem solving contexts, but it also lacks the important characterizations of how those metaphors are actually applied, of the questions the metaphors are used to ask and the resulting answers, and of the changes the conceptual tools undergo in the process. One must look at richer data on these functional aspects in addition to their structure and logic.

This research found students using specific metaphors as organizers of ideas and touchstones for reasoning. For example, the case study on Shawna's understanding of the definition of the derivative showed how she repeatedly returned to the idea of a collapse from two points determining a secant line to a tangent line through a single point in graphical, algebraic, numerical, and physical contexts. She used this metaphor in asking questions such as wondering "how you get a number out of that," referring to the graphical version of the collapse metaphor. She answered this question using a numerical version, and continually checked her ongoing work against her ideas about various versions. These ideas became central to her developing understanding of the definition of the derivative and eventually even limits, holding together the thought of a collapse happening “magically” with the reasoning “that's why you go to zero, because you can't divide by zero.” Similarly, students who appealed to a physical limitation to argue that the volume of a solid of revolution is finite if the radius goes to zero were mathematically incorrect. This compelling imagery, however, appeared to become a major aspect of their understanding of solids of revolution, the meaning of volume, and limits.

The Value of a Micro-Ethnographic Approach in Accessing Functional Aspects of Students’ Understanding

The methodology in this study could be termed micro-ethnographic in the sense that it focused heavily on the collection of large amounts of written and verbal responses to open-ended questioning. While such qualitative data is neither suitable for standard statistical analyses nor supportive of broad generalization, its main advantage is that it provides a detailed look at students reasoning. The collection and analysis of data in this study was intended to develop metaphor clusters with such detail in the characterization of their structural and functional aspects.

In previous research, students’ understandings of limit concepts have been organized along dichotomies such as “static vs. dynamic,” “formal vs. informal,” “reachable vs. unreachable,” “infinitist vs. definitist,” and “correct vs. incorrect.” Such a categorization of responses is typically referred to as a description of the students’ “models for limits” or their “beliefs about limits.” While it is important to understand such models and beliefs, these categories represent more the product of students’ thinking about limits rather than the character of the thinking itself. For example, simply because two students view a limit as unreachable or a boundary does not mean that they have similar foundational understandings and experiences that led them to this conclusion. Their conceptualizations and reasoning may be very different depending on whether it is a result of a physical limitation metaphor, visualizing motion toward a point on a graph, a version of one of Zeno's paradoxes, a static image of a graph with a topological feature such as a jump, or some other metaphor or image. Even further differences may arise as these constructs probe and respond to the varying problem contexts in which they are applied, especially for strong metaphors (in Black's sense) as their support for elaborative implication allows great opportunity for a variety of interaction between problem and metaphor.

Williams' research was focused on investigating the predicational relationships between different types of statements students make about limits (Williams, 1989, 1991) and on the changes over a semester in those statements (Williams, 1989, 2001). Consequently, it was necessary to restrict the students' responses to reasoning about those particular statements. The open ended nature of the instruments used in this study did not structure students' responses as heavily, but in turn, provided a richer, more naturalistic characterization of the resulting metaphor clusters and of the ways that students used them. As a result, we are able to say far more than a student used an approximation metaphor or that they described a limit dynamically. We are able to provide a rich description of what this means: the ways in which approximation or motion ideas are related to the relevant limit concept, how the related details (such as error and accuracy for approximation or the relationship of sequentially chosen points to actual motion) are involved, and their different instantiations in a variety of problem contexts. Other metaphors that have not been previously investigated, like physical limitation and collapse, are also allowed to emerge through this open-ended approach. This study thus complements existing, more structured research by providing essential details of students spontaneous reasoning to the more general classifications previously established.

Some Preliminary Implications for Teaching

Although the primary purpose of this research was not to develop or evaluate specific teaching methods or curricula, the details about students' spontaneous reasoning that were revealed may inform teaching practices to a small degree.

The most obvious application of this research to teaching is to provide an awareness of some of the metaphors students use to reason about limit concepts. A teacher with this information will not be able to categorically determine which metaphors their students are using, but they will know what some of the important possibilities are. This chapter has argued that these metaphors can be very powerful tools in the development of students' mathematical understanding. Being able to recognize and take advantage of productive aspects of students reasoning can thus help a teacher support that growth. In addition, we have seen that many aspects of students' metaphors can be misleading. An awareness of the possibilities can help a teacher to avoid, or at least to be prepared for, problems that might arise.

In addition to an awareness of these metaphors, teaching may be informed by some of the more general results about students metaphorical reasoning derived from this study. For example, great care is needed when intentionally introducing a metaphor to a class. First, the metaphor should solve some problem perceived by the students, who should be allowed to develop, under guidance, as many of the details as possible. A teacher should also be aware of possible interpretations the students might construct and provide opportunities for the students to intentionally reflect on their own interpretation, consider other students interpretations, and evaluate them against various mathematical problems and requirements.

For most students, the learning of calculus concepts occurs over long periods of time, up to years (Carlson, 1998; Monk & Nemerovsky, 1994; Simonsen, 1995). The second-semester students in this study were still developing their understanding of even basic limit concepts as various topics such as improper integrals, infinite sequences and series, and continuity and differentiability of multi-variable functions arose. We should pay more attention to our students' process of constant refinement of knowledge rather than focusing solely on the large conceptual leaps. Much calculus instruction, especially as typically embodied in large lecture classes, attempts to build one concept upon another. If learning is truly a matter of continually evaluating and refining ideas, then a different approach must be taken. Students should be allowed to revisit ideas often, encouraged to investigate how common themes run through the material they are learning, and given the support necessary to compare their own modes of analyses under different schemas.

Finally, conceptual assessment requires more than asking direct questions or having students solve problems, even if they are conceptually based questions and problems. First this will evoke only a small portion of the students' understanding. Second, students' incorrect responses can only be interpreted as misconceptions and correct responses can only be interpreted as a complete understanding; there is no ability to determine what the students' can actually do with their ideas. These types of questions and problems are important, but a more complete assessment must also include opportunities for students to explore new situations with the concepts being assessed. Methods to accomplish this might include brief task-based interviews or extended writing assignments.

FUTURE RESEARCH

There are several lines along which future research may supplement and build on the findings in this study. First, students' metaphors for other key concepts in calculus may be investigated. Below, we outline some initial prospects for concepts related to the derivative and the definite integral. Further, research should be conducted that will apply these results to more directly impact teaching practices. We discuss several types of activities that may be developed through a research cycle to help students evaluate and refine their spontaneous concepts and metaphorical reasoning.

Rate of Change and Covariation

Students' concepts involving rate of change are very complex and is influenced by personal experiences beginning at a young age. Thus spontaneous concepts and metaphorical reasoning are likely to play a large part in students' understanding of rate. The exploratory study discussed in Chapter 3 revealed a number of potential metaphors related to the analysis of rate of change including steepness, automobile, motion detector, and motion on the graph. More detailed data on students' metaphorical understanding of rate of change was also collected concurrently with the main study on students' understanding of limits. With a small amount additional data collection and analysis as outlined in Chapter 4, metaphor clusters related to rate of change could be developed.

There has also been a greater amount of research conducted on students' understanding of rate of change. Significantly, this has resulted in a number of interesting decompositions of an understanding of rate as encapsulated by the derivative, and a connection between these frameworks and students' spontaneous reasoning would be interesting. For example, Carlson, Jacobs, Coe, & Hsu (under review) and Carlson & Jacobs (2000) found six levels of mental actions involved in conceptualizing covariation between two variables: operating with 1) an image of two variables changing, 2) a loosely coordinated image of direction, 3) an image of an amount of change for contiguous intervals, 4) an image of rate/slope for contiguous intervals, 5) an image of continuously changing rate, and 6) an image of increasing and decreasing rate. In a similar study, Michelle Zandieh (1997) decomposed students' understanding of the derivative along three independent dimensions: 1) into three layers corresponding to the ratio of the difference quotient, the limit at a point, and the derivative as a function, 2) distinguishing each layer between process, object, or pseudo-object conceptualization, and 3) in various contexts of graphical, verbal description, kinematic, and symbolic.

Accumulation of Rate and the Fundamental Theorem of Calculus

There has been very little research conducted on students' understanding of definite integrals and the fundamental theorem of calculus. Thompson's (1994b) study of advanced students' understanding is a notable exception, and hypothesizes about conceptual origins based on his more extensive work on young children's understanding of rate. In an interesting bridge between the two, he describes a 7th grade students' spontaneous construction of the ideas behind a definite integral. Similarly other work with children suggests that these understandings may be developed at a young age (Kaput, 1994; Stroup, 1996; Roschelle, Kaput, & Stroup, 2000). For analyzing calculus students' understanding, one might begin by observing that Zandieh's decomposition of the derivative concept has a dual decomposition in terms of accumulation in the definite integral. Understanding the metaphors that students develop to understand accumulation of rate and the fundamental theorem can inform both efforts with college students and students at an early age to foster appropriate conceptual tools early.

Activities for Students

A different line of research may also be pursued to synthesize the findings of this and other research on students' understanding of limits to develop and evaluate new research-based curricular activities and tools for teachers. Such products could be developed through a series of refinements in small-scale teaching experiments, then implemented and evaluated in a large-scale project. Based on the research in this study, initial attempts could occur in stages, starting with building conscious awareness and volitional use of students' metaphors, so that it becomes possible for the students to evaluate the metaphorical concepts as a part of their process of inquiry. Recognition of areas of good and poor fit and the replacement or buttressing of inaccurate parts and refinement of accurate parts of the metaphor could then follow.

First, activities that help students identify the metaphors they use should be developed. This could be accomplished through a series of writing assignments and directed discussions with other students. These activities should include challenging problems related to the concept for which metaphors are to be elicited followed by explicit discussions of the informal concepts used and their relationship to the mathematics. By discussing the results of their individual efforts with other students, they will be encouraged to reflect on and compare alternate conceptualizations.

Once students become consciously aware of their and other students' metaphors, they may begin to evaluate and refine their use of these metaphors. Toward this purpose, activities could be designed to have students identify aspects that match or don't match the mathematical concepts and evaluate successful and unsuccessful applications. These activities would involve scientific debate among the students to allow them to participate in the process of building their metaphors mediated through the goals and standards of the classroom community.

According to Dewey, it is through social interaction that a discourse community learns to "carve out" the world in similar ways and begins to develop anticipations about external reality (Prawat & Floden, 1994). Through productive inquiry, we test our anticipations and communication against one another. Thus, discourse is also technological in the sense that it is actively productive, and testable as a cognitive tool. We should give students the opportunity to use mathematical language in ways that make it clear that they are participating in mathematical meaning-making while working within the rich inherited formal structure of mathematics. Carl Bereiter's (1994) notion of science as progressive discourse and the implications for instruction may be very useful here. He suggests that we should bring students into a scientific discourse with commitments to work toward common understanding satisfactory to all, to frame questions and propositions in ways that allow evidence to be brought to bear on them, to expand the body of collectively valid propositions, and to allow any belief to be subjected to criticism if it will advance the discourse.

Appendix A: Calculus Syllabus

FIRST SEMESTER CALCULUS

Chapter 1 - Introduction (1.3 days)

1.1 What is Calculus? (assign to read)
1.2 Notations and Formulas from Elementary Mathematics (assign to read)
1.3 Inequalities (assign to read)
1.4 Coordinate Plane; Analytic Geometry (assign to read)
1.5 Functions (important)
1.6 The Elementary Functions (important, especially trig!!)
1.7 Combinations of Functions (cover composition in class; assign the remainder to be read)

Chapter 2 - Limits and Continuity (2.7 days)

2.1 The Idea of Limit
2.2 Definition of Limit (optional)
2.3 Some Limit Theorems
2.4 Continuity
2.5 The Pinching Theorem; Trigonometric Limits (cover lightly)
2.6 Two Basic Theorems (cover lightly)

Chapter 3 - Differentiation (4 days)

3.1 The Derivative
3.2 Some Differentiation Formulas
3.3 The d/dx Notation; Derivatives of Higher Order
3.4 The Derivative as a Rate of Change
3.5 The Chain Rule
3.7 Implicit Differentiation; Rational Powers
3.8 Rates of Change per Unit Time (optional)

Chapter 4 - The Mean Value Theorem (4 days)

4.1 The Mean Value Theorem (cover lightly)
4.2 Increasing and Decreasing Functions
4.3 Local Extreme Values
4.4 Endpoint and Absolute Extreme Values
4.5 Some Max-Min Problems (optional)
4.6 Concavity and Points of Inflection

4.7 Vertical and Horizontal Asymptotes; Vertical Tangents and Cusps
4.8 Some Curve Sketching

Chapter 5 - Integration (4.7 days)
5.1 An area Problem; A Speed-Distance Problem (cover lightly)
5.2 The Definite Integral of a Continuous Function
5.3 The Function $F(x) = \int_a^x f(t)\,dt$
5.4 The Fundamental Theorem of Integral Calculus
5.5 Some Area Problems
5.6 Indefinite Integrals
5.7 The u-Substitution; Change of Variables
5.8 Some Further Properties of the Definite Integral
5.10 The Integral as the Limit of Riemann Sums (cover lightly)

Chapter 6 - Some Applications of the Integral (1.3 days)
6.1 More on Area
6.2 Volume by Parallel Cross Sections (optional)
6.3 Volume by the Shell Method (optional)

Chapter 7 - The Transcendental Functions (4 days)
7.1 One-to-One Functions; Inverses (cover lightly)
7.2 The Logarithm Function, Part I
7.3 The Logarithm Function, Part II
7.4 The Exponential Function
7.5 Arbitrary Powers; Other bases; Estimating e
7.6 Exponential Growth and Decay (optional)
7.7 More on the Integration of the Trigonometric Functions
7.8 The Inverse Trigonometric Functions
7.9 The Hyperbolic Sine and Cosine (optional)

Chapter 8 - Techniques of Integration (4 days)
8.1 Review
8.2 Integration by Parts
8.3 Powers and Products of Sine and Cosine
8.4 Other Trigonometric Powers
8.5 Trigonometric Substitutions
8.6 Partial Fractions (repeated quadratic functions may be omitted)

Second Semester Calculus

Chapter 10 - Sequences; Indeterminate Forms (3.3 days)

10.2 Sequences of Real Numbers
10.3 Limit of a Sequence
10.4 Some Important Limits
10.5 The Indeterminate Form (0/0)
10.6 Indeterminate Forms
10.7 Improper Integrals

Chapter 11 - Infinite Series (6.7 days)

11.1 Sigma Notation
11.2 Infinite Series
11.3 The Integral Test; Comparison Theorems (go light on the comparison theorems)
11.4 The Root Test; The Ratio Test
11.5 Absolute and Conditional Convergence; Alternating Series
11.6 Taylor Polynomials in x; Taylor Series in x
11.7 Taylor Polynomials and Taylor Series in $x-a$ (emphasize numerical applications)
11.8 Power Series
11.9 Differentiation and Integration of Power Series

Chapter 12 - Vectors (3.3 days)

12.1 Cartesian Space Coordinates
12.2 Displacements; Forces and Velocities; Vectors
12.3 The Dot Product
12.4 The Cross Product
12.5 Lines
12.6 Planes

Chapter 9 - Polar Coordinates (2 days)

9.2 Polar Coordinates
9.3 Graphing in Polar Coordinates
9.5 The Intersection of Polar Curves
9.6 Area in Polar Coordinates
9.7 Curves Given Parametrically
9.8 Tangents to Curves Given Parametrically

Chapter 13 - Vector Calculus (1.3 days)

13.1 Vector Functions
13.2 Differentiation Formulas
13.3 Curves
13.4 Arc Length

Chapter 14 - Functions of Several Variables (2 days)

14.1 Elementary Examples
14.2 A Brief Catalog of the Quadratic Surfaces; Projections (assign to read or cover quickly)
14.3 Graphs; Level Curves and Level Surfaces
14.4 Partial Derivatives
14.6 Limits and Continuity; Equality of Mixed Partials (emphasis on mixed partials)

Chapter 15 - Gradients; Extreme Values; Differentials (4.7 days)

15.1 Differentiability and Gradient
15.2 Gradients and Directional Derivatives
15.4 Chain Rules
15.5 The Gradient as a Normal; Tangent Lines and Tangent Planes
15.6 Maximum and Minimum Values
15.7 Second Partials Test
15.8 Maxima and Minima with Side Conditions

Chapter 16 - Double and Triple Integrals (2.7 days)

16.1 Multiple Sigma Notation (assign to read)
16.2 The Double Integral Over a Rectangle
16.3 The Double Integral Over a Region
16.4 The Evaluation of Double Integrals by Repeated Integrals

Appendix B: Phase III Writing Assignments

APPROXIMATION PROBLEMS

In the following three problems, you will approximate the slope of the tangent line to a curve at various locations. There are several important ideas about approximation that are embedded in these exercises that have a close relationship to the limit concept. Here we use ε to denote a bound on the size of errors that is acceptable for some given purpose, and δ to denote the size of the set input values that result in an approximation with that degree of accuracy. You will need a graphing calculator or a graphing program on a computer.

1. Graph $y = 2^x$ on a calculator or computer over the interval $[-3, 3]$ and take careful note of the general shape of the curve. Now zoom in on the graph at $x = -1$, that is, change the window to show the graph over a smaller interval around $x = -1$, like $[-2, 0]$. Notice that the graph appears less curved and more like a straight line. If you keep zooming in around $x = -1$, the graph will appear more and more like a straight line. This is called the tangent line to the graph of $y = 2^x$ at $x = -1$. The details of tangent lines will be developed more fully later in this course. For now, you will approximate the slope of the tangent line.
   a. Look at a region of the curve where it appears fairly straight but still has a slight, noticeable curvature, e.g., on $[-2, 0]$. Take a point on the curve to the *right* of the point at $x = -1$, and find the slope between these two points. (This is best done by using something equivalent to a "trace" function where you slide a point along the graph and the calculator/computer displays the *x*- and *y*-values of the point. Also make sure to keep as many decimal places in your calculation as possible since this exercise will require precision.)
   b. Take a point on the curve to the *left* of the point at $x = -1$, and find the slope between these two points.
   c. Are the two slopes from parts a and b both underestimates, both overestimates, or one of each? Explain how you know. (Hint: Use the fact that the graph of $y = 2^x$ is concave up, i.e., it curves upwards.)

d. Using your work from above, give a range of possible values for the slope of the tangent line. Using the center of this range as an approximation, what is a bound on the size of the error?
e. Explain why your error bound is just an *upper bound* for the error and not *exactly* the error.
f. Zoom in and use points to the left and right of $x=-1$ to find an approximation of the slope of the tangent line with error less than $\varepsilon = 0.0001$. Record your work for each computation you do.
g. Explain why any points between $x=-1$ and the points you used in Part f would result in an approximation with error less than $\varepsilon = 0.0001$.
h. Find a value, δ, such that if the approximation is computed using x-values no more than δ units from $x=-1$, then the error will be less than $\varepsilon = 0.0001$.

2. Now zoom in on the graph of $y=2^x$ at $x=0$. Repeat the process from problem 1 to find an approximation for the slope of the tangent line at $x=0$ with error less than $\varepsilon = 0.0001$. For each computation you do, record the x-values chosen, the approximation resulting from taking a slope between the underestimate and overestimate, and the bound on the error for that approximation. Once your bound is less than $\varepsilon = 0.0001$, find the value, δ, such that if the approximation is computed using x-values no more than δ units from x=0, then the error will be less than $\varepsilon = 0.0001$.

3. Finally, zoom in on the graph of $y=2^x$ at $x=1$. Repeat the process from problem 1 to find an approximation for the slope of the tangent line at $x=1$ with error less than $\varepsilon = 0.0001$. For each computation you do, record the x-values chosen, the approximation resulting from taking a slope between the underestimate and overestimate, and the bound on the error for that approximation. Once your bound is less than $\varepsilon = 0.0001$, find the value, δ, such that if the approximation is computed using x-values no more than δ units from $x=1$, then the error will be less than $\varepsilon = 0.0001$.

Note: You may have noticed that the approximations you obtained for these slopes are related to each other by factors of two. In fact, the slopes of the tangent lines at $x=-1$, $x=0$, and $x=1$ are $\frac{1}{2}\ln 2$, $\ln 2$, and $2\ln 2$, respectively. Later in this course, you will be able to prove this. You will also see why the slope doubles for each unit to the right that we move (a pattern that continues in both directions along the x-axis).

Now you will approximate the distance traveled by an object moving at a varying velocity. Recall that for constant velocity, v, the distance traveled is $d = vt$, where t is time. There are several important ideas about approximation that are embedded in this exercise that have a close relationship to the limit concept. Here we use ε to denote a bound on the size of errors that is acceptable for some given purpose, and N to denote minimum number of steps required to achieve that accuracy.

4. Suppose a toy car travels with velocity given by $v = \sin\sqrt{\frac{\pi t}{15}}$ meters per second for ten seconds after it is released at time $t = 0$ seconds.
   a. Divide the ten seconds into one-second intervals. Find an overestimate for the distance traveled by the toy car by approximating the distance traveled during each second using the maximum velocity for that second. (Hint: the car is always speeding up during the ten seconds.)
   b. Find an underestimate for the distance traveled by the toy car by approximating the distance traveled during each second using the minimum velocity for that second.
   c. Use the distances found in Parts a and b to come up with an approximation for the total distance traveled and a bound on the error.
   d. Explain why dividing the time into 20 half-second intervals would result in a better approximation.

In the following exercise, you will approximate the value of π to varying degrees of accuracy. There are several important ideas about approximation that are embedded in this exercise that have a close relationship to the limit concept. Here we use ε to denote a bound on the size of errors that is acceptable for some given purpose, and N to denote minimum number of steps required to achieve that accuracy.

5. Recall from the first lecture that
$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots$$
   By using $\pi \cong 4(1) = 4$ as a first approximation, $\pi \cong 4(1 - \frac{1}{3}) = \frac{8}{3}$ as a second approximation, and so on, we can obtain better and better estimates for π. For this exercise, we will be concerned with the details of approximation, so refrain from using your knowledge about the value of π or any value for π given by your calculator. Use only the fact that $\pi/4$ is equal to the alternating sum given above.

a. Plot the first five approximations for π on a number line.

b. Explain why, for any approximation, the size of the error will be smaller than the next term to be added or subtracted.

c. Find the value of the sixth approximation. What is a bound on the error? Use this bound to determine the range of possible values for π. (Reminder: Use only the alternating sum to determine the bound on the error.) Given this approximation and error bound, what is the range of possible values for π?

d. Find an approximation that is accurate to within $\varepsilon = 0.1$ and indicate how many terms were added to obtain this approximation. Explain why any approximation obtained by adding/subtracting more terms would also be accurate to within

e. How many terms would be needed to make the approximation accurate to within $\varepsilon = 0.005$ (i.e., two decimal places)? Explain why any approximation obtained by adding/subtracting more terms would also be accurate to within $\varepsilon = 0.005$.

f. For an arbitrary value of ε, determine a number N such that if at least N terms are added, the approximation will be accurate to within ε.

INTERVAL PROBLEMS

For the following six problems, the notation $N_{c,r}$ refers to the neighborhood with center *c* and radius *r*. In interval notation this is the same thing as $(c-r,c+r)$. In these exercises, you will examine what happens to entire intervals under linear transformations. In the process, you will see why we use the neighborhood notation $N_{c,r}$ since it reveals behavior about the linear function that is more difficult to see using interval notation.

1. Let $f(x)=x+4$.
   a. Find $f\left(N_{1,2}\right)$. Write the answer in neighborhood notation **AND** interval notation.
   b. Find $f\left(N_{1,1/2}\right)$. Write the answer in neighborhood notation **AND** interval notation.
   c. For any radius *r,* express $f\left(N_{1,r}\right)$ in neighborhood notation.
   d. Find a neighborhood $N_{1,r}$ such that $f\left(N_{1,r}\right)$ fits inside of $N_{5,0.002}$.
   e. Rewrite *f* in the form $f(x)=f(1)+m(x-1)$. (Hint: you just need to replace $f(1)$ and *m*.)

2. Let $f(x)=3x-5$.
   a. Find $f\left(N_{2,1}\right)$. Write the answer in neighborhood notation **AND** interval notation.
   b. Find $f\left(N_{2,1/9}\right)$. Write the answer in neighborhood notation **AND** interval notation.
   c. For any radius *r,* express $f\left(N_{2,r}\right)$ in neighborhood notation.
   d. Find a neighborhood $N_{2,r}$ such that $f\left(N_{2,r}\right)$ fits inside of $N_{1,0.0001}$.
   e. Rewrite *f* in the form $f(x)=f(2)+m(x-2)$.

3. Let $f(x) = -7x + 4$.
   a. Find $f\left(N_{0,3}\right)$. Write the answer in neighborhood notation **AND** interval notation.
   b. Find $f\left(N_{0,1}\right)$. Write the answer in neighborhood notation **AND** interval notation.
   c. Find $f\left(N_{0,1/2}\right)$. Write the answer in neighborhood notation **AND** interval notation.
   d. For any radius *r,* express $f\left(N_{0,r}\right)$ in neighborhood notation.
   e. Find a neighborhood $N_{0,r}$ such that $f\left(N_{0,r}\right)$ fits inside of $N_{4,\varepsilon}$ where ε is a variable representing a small positive radius.
   f. Rewrite *f* in the form $f(x) = f(0) + m(x-0)$.

4. Let $f(x) = 4 + 1/7\,(x-3)$.
   a. Find $f\left(N_{3,1}\right)$. Write the answer in neighborhood notation **AND** interval notation.
   b. Find $f\left(N_{3,1/10}\right)$. Write the answer in neighborhood notation **AND** interval notation.
   c. For any radius *r,* express $f\left(N_{3,r}\right)$ in neighborhood notation.
   d. Find a neighborhood $N_{3,r}$ such that $f\left(N_{3,r}\right)$ fits inside of $N_{4,\varepsilon}$ where ε is a variable representing a small positive radius.

5. Let $f(x) = \begin{cases} 2x+1, & x \le 1; \\ 4x-1, & x > 1. \end{cases}$
   1. Find $f\left(N_{1,1}\right)$. Write the answer in neighborhood notation **AND** interval notation.
   2. Find $f\left(N_{1,1/600}\right)$. Write the answer in neighborhood notation **AND** interval notation.
   3. Find a neighborhood $N_{1,r}$ such that $f\left(N_{1,r}\right)$ fits inside of $N_{3,0.001}$.
   4. Is *f* a linear function? Explain.

6. Let $f(x)$ be a linear function, that is $f(x) = f(a) + m(x - a)$ for some constant *m* which we call the slope.
    a. Explain in a few sentences what happens to the **CENTER** of a neighborhood $N_{a,r}$ when it is transformed by *f*. Make sure you say what the center of the neighborhood $f\left(N_{a,r}\right)$ is.
    b. Explain in a few sentences what happens to the **RADIUS** of a neighborhood $N_{a,r}$ when it is transformed by *f*. Make sure you say what the radius of the neighborhood $f\left(N_{a,r}\right)$ is.

For the following four problems, the notation $N_{c,r}$ refers to the neighborhood with center c and radius r. In interval notation this is the same thing as $(c-r, c+r)$. In these exercises, you will examine how the actions of a function on small intervals are related to limits. The connection is the preservation of closeness, a condition we can place on a function that makes rigorous the idea that nearby points are mapped to nearby points. The crucial issue will be to look at how close together all of the points in $f\left(N_{c,r}\right)$ are.

1. Consider the function $f(x)=\dfrac{\sin x}{x}$ which is undefined at $x=0$. Suppose we are interested in defining $f(0)$ so that the function "preserves closeness" in the sense discussed in class.
    a. Find $f\left(N_{0,r}\right)$ where r is a small radius.
    b. Suppose we define $f(0)=1/2$. How small can you make the diameter of $f\left(N_{0,r}\right)$? Explain why this function does not preserve closeness.
    c. Find a different value to assign $f(0)$ such that the points in $f\left(N_{0,r}\right)$ can be made arbitrarily close together (i.e., the diameter of $f\left(N_{0,r}\right)$ can be made arbitrarily small). This value is called the limit of $f(x)$ at 0.
    d. Given that $f(0)$ equals the value you found in Part c, pick a very small value for ε then find a radius r such that the diameter of $f\left(N_{0,r}\right)$ is smaller than ε. Explain why this function does preserve closeness.

2. Let $g(x)=\dfrac{(x+3)^2-9}{x}$.

    a. Find a value to assign $g(0)$ so that g preserves closeness (i.e., $g\left(N_{0,r}\right)$ can be made arbitrarily small).

    b. Prove that $g\left(N_{0,r}\right)$ can be made arbitrarily small. That is, for an arbitrarily small diameter ε, find a radius r so that the diameter of $g\left(N_{0,r}\right)$ is less than ε.

3. For each of the following limits $\lim_{x \to c} f(x)$, describe the effect of *f* on small intervals around *c*. That is, describe what $f\left(N_{c,r}\right)$ looks like for small radii, *r*. Explain how this relates to the value of the limit.

a) $\lim_{x \to 2} 3x + 7$ c) $\lim_{x \to -1} \dfrac{1}{x+1}$

b) $\lim_{x \to 0} x^2$ d) $\lim_{x \to 1} f(x), \quad f(x) = \begin{cases} x, & x < 1; \\ x+2, & x \geq 1. \end{cases}$

4. For the functions in 3c and 3d, what is the smallest possible diameter for $f\left(N_{c,r}\right)$? Explain why no value of $f(c)$ could make *f* preserve closeness.

References

Alexander, P., Schallert, D., & Hare, V. (1991). Coming to Terms: How Researchers in Learning and Literacy Talk About Knowledge. *Review of Educational Research, 61,* 315-343.

Aristotle. (1929). *Poetics*. Trans. S. H. Butcher. London: Macmillan

Aristotle (1954). *Rhetoric.* Trans. W. Rhys Roberts. New York: Modern Library.

Artigue, M. (1991). Analysis. In D. Tall (Ed.) *Advanced mathematical Thinking*. Boston: Kluwer, 167-198.

Beardsley, M. (1958). *Aesthetics.* New York: Harcourt Brace & World, Inc.

Beardsley, M. (1978). Metaphorical Senses. *Noûs*, *12*, 3-16.

Black, M. (1962a). Metaphor. in M. Black. *Models and Metaphors*, Ithaca, NY: Cornell University Press, Chapter 3, 25-47.

Black, M. (1962b). Models and Archetypes. in M. Black. *Models and Metaphors*, Ithaca, NY: Cornell University Press, Chapter Chapter 13, 219-243.

Black, M. (1977). More about Metaphor. *Dialectica*, *31*, 433-57.

Black, M. (1979). How Metaphors Work: A Reply to Donald Davidson. *Critical Inquiry*, *6*, 131-143.

Breidenbach, D., Dubinsky, E., Hawks, J. and Nichols, D. (1992). Development of the Process Conception of Function, *Educational Studies in Mathematics*, 247-285.

Brown, T. (1996). Intention and Significance in the Teaching and Learning of Mathematics. *Journal for Research in Mathematics Education. 27*, 52-66.

Carlson, M. (1998). A Cross-Sectional Investigation of the Development of the Function Concept. *Research in Collegiate Mathematics Education III*, Conference Board of the Mathematical Sciences, Issues in Mathematics Education, Volume 7; American Mathematical Society, 114-163.

Carlson, M. & Bloom, I. (under review). A Multi-Dimensional Framework for Describing and Analyzing Problem Solving Behavior.

Carlson, M., Jacobs, S., Coe, T., & Hsu, E. (under review). Applying Covariational Reasoning While Modeling Events: A Framework and a Study.

Carlson, M. & Jacobs, S. (2000). *A General Framework for Describing Covariational Reasoning: Drawing from Piaget's Work*. Presentation at the Fifth Annual Conference of the MAA Special Interest Group on Research in Undergraduate Mathematics Education.

Carpenter, T., Hiebert, J., & Moser, J. (1981). Problem Structure and First Grade Children's initial Solution Processes for Simple Addition and Subtraction Problems. *Journal for Research in Mathematics Education, 12*, 27-39.

Cole, M. & Scribner, S. (1974). *Culture and Thought: A Psychological Introduction.* New York: John Wiley & Sons, Inc.

Cornu, B. (1991). Limits. In D. Tall (Ed.), *Advanced mathematical Thinking*. (pp. 153-166). Boston: Kluwer.

Cottrill, J., Dubinsky, E., Nichols, D., Schwingendorf, K., Thomas, K. & Vidakovic, D. (1996). Understanding the Limit Concept: Beginning with a Coordinated Process Schema, *Journal of Mathematical Behavior, 15*, 167-192.

Davis, R. (1986). Calculus at University High School. in Douglas, R.G. (Ed.). *Toward a Lean and Lively Calculus*. Washington, DC: Mathematical Association of America.

Davis, R. & Vinner, S. (1986). The Notion of Limit: Some Seemingly Unavoidable Misconception Stages. *Journal of Mathematical Behavior, 5*, 281-303.

Dörfler, W. (1991). Meaning: Image Schemas and Protocols. In F. Furinghetti (Ed.) *Proceedings of the Fifteenth International Conference for the Psychology of Mathematics Education, Vol. 1,* 17-32.

Dubinsky, E. (1991). Reflective Abstraction in Advanced Mathematical Thinking. In D. Tall (Ed.), *Advanced mathematical Thinking*. (pp. 95-126). Boston: Kluwer.

Ferguson, D. L. (1980). *The Language Of Mathematics: How Calculus Students Cope With It*. Doctoral Dissertation, the University of California at Berkeley, Berkeley, CA, Dissertation Abstracts International, 42-01, 121.

Foley, D. (1998). On Writing Reflexive Realist Narratives. In G. Shacklock and J. Smyth (Eds.), *Being reflexive in critical educational and social research*, London: Falmer Press. 110-129.

Foley, D. (2000). Making Critical Ethnography More Reflexive in the Postcritical Moment. in G. Noblit (Ed.), *The PostCritical Moment in Critical Ethnography.*

Frid, S. (1994). Three Approaches to Undergraduate Calculus Instruction: Their Nature and Potential Impact on Students' Language Use and Sources of Conviction. *Research in College Mathematics Education I,* Conference Board of the Mathematical Society, Issues in Mathematics Education, Volume 4, American Mathematical Society, 69-100.

Gardner, H. (1985). *The Mind's New Science*. New York: Basic Books.

Goldin, G. (2001). Counting on the Metaphorical. *Nature, 413*, 18-19.

Goodman, N. (1968). *Languages of Art*. Indianapolis: Hackett Publishing Co., Inc.

Greeno, J. (1991). Number Sense as Situated Knowing in a Conceptual Domain. *Journal for Research in Mathematics Education, 22*, 170-218.

Harel, G. & Kaput, J. (1991). Conceptual Entities and their Symbols in Building Advanced mathematical Concepts. In D. Tall (Ed.) *Advanced mathematical Thinking*. Boston: Kluwer, 82-94.

Hiebert, J. & Carpenter, T. (1992). Learning and Teaching with Understanding. In Grouws, D. (Ed.), *Handbook of Research on Mathematics Teaching and Learning* (pp. 65-97). New York: Macmillan.

Hickman, L.A. (1990). *John Dewey's Pragmatic Technology.* Bloomington: Indiana University Press.

Hinsley, D., Hayes, J., & Simon, H. (1976). From Words to Equations: Meaning and Representation in Algebra Words Problems. (CIP Working Paper No. 331).

Johnson-Laird, P. (1983). *Mental Models: Towards a Cognitive Science of Language, Inference, and Consciousness*. Cambridge: Harvard University Press.

Kaput, J. (1994). Democratizing Access to Calculus: New Routes to Old Roots. In A. H. Schoenfeld (Ed.) *Mathematics and Cognitive Science*, Washington, D. C.: Mathematical Association of America, 77-156.

Lakoff, G. (1987). *Women, Fire, and Dangerous Things: What Categories Reveal about the Mind*. Chicago: University of Chicago Press.

Lakoff , G. & Johnson, M. (1980). *Metaphors We Live By.* Chicago: The University of Chicago Press.

Lakoff, G. & Núñez, R. (2000). *Where Mathematics Comes From: How the Embodied Mind Brings Mathematics into Being*. New York: Basic Books.

Madden, J. (2001). Where Mathematics Comes From: How the Embodied Mind Brings Mathematics into Being (Book Review), *Notices of the American Mathematical Society, 48,* 1182-1188.

Marshall, S. (1995). *Schemas in Problem Solving*. New York: Cambridge University Press.

Monk, G. (1987). *Students' Understanding of Functions In Calculus Courses*. Unpublished Manuscript, University of Washington, Seattle.

Monk, G. (1992). Students' Understanding of a Function Given by a Physical Model. In G. Harel & E. Dubinsky (Eds.), *The Concept of Function: Aspects of Epistemology and Pedagogy*. MAA Notes, Volume 25. Washington, DC: Mathematical Association of America.

Monk, G. & Nemirovsky, R. (1994). The Case of Dan: Construction of a Functional Situation Through Visual Attributes. *Research in Collegiate Mathematics Education. I. Issues in Mathematics Education*, *4*, 139-168.

Orton, A. (1983). Students' Understanding of Differentiation. *Educational Studies in Mathematics, 14*, 235-250.

Pea, R. (1985). Beyond Amplification: Using the Computer to Reorganize Mental Functioning. *Educational Psychology, 20*, 167-182.

Prawat, R., & Floden, R. (1994). Philosophical Perspectives on Constructivist Views of Learning. *Educational Psychology*, *29*, 37-48.

Richards, I. (1936). *The Philosophy of Rhetoric*, New York: Oxford University Press.

Rosch, E. (1978). Principles of Categorization. In E. Rosch and B. B. Lloyd (eds.), *Cognition and Categorization.* Hillsdale, N.J.: Erlbaum.

Rosch, E. & Mervis, C. (1975). Family Resemblances: Studies in the Internal Structure of Categories, *Cognitive Psychology, 7*, 573-605.

Rosch, E., Mervis, C., Gray, W., Johnson, D., & Boynes-Brian, P. (1976). Basic Objects in Natural Categories. *Cognitive Psychology. 8*, 382-439.

Roschelle, J., Kaput, J. J., & Stroup, W. (2000). SimCalc: Accelerating Students' Engagement with the Mathematics of Change. In M. Jacobson & R. Kozma (Eds.), *Innovations in Science and Mathematics Education*. Mahwah, NJ: Lawrence Erlbaum Associates.

Roth, W. –M., & Bowen, G. (2001). Professionals Read Graphs: A Semiotic Analysis. *Journal for Research in Mathematics Education, 32*, 159-194.

Roth, W. –M., & Bowen, G. (1999). Complexities of Graphical Representations During Ecology Lectures: An Analysis Rooted in Semiotics and Hermeneutic Phenomenology. *Learning and Instruction, 9*, 235-255.

Roth, W. –M., Tobin, K., & Shaw, K. (1997). Cascades of Inscriptions and the Re-presentation of nature: How Numbers, Tables, Graphs, and Money Come to Re-present a Rolling Ball. *International Journal of Science Education, 19*, 1075-1091.

Rubin, A. & Nemirovsky, R. (1991). Cars, Computers, and Air Pumps: Thoughts on the Roles of Physical and Computer Models in Learning the Central Concepts of Calculus. In R.G. Underhill (Ed.), *Proceedings of the Thirteenth International Conference for the Psychology of Mathematics Education -- North American Chapter Conference* (pp. 168-174), Virginia.

Salamon, G. & Perkins, D. (1997). Individual and Social Aspects of Learning. in *Review of Research in Education*, *23*, 1-24.

Scheffler, I. (1979). *Beyond the Letter: A Philosophical Inquiry into Ambiguity, Vagueness and Metaphor in Language*. London: Routledge & Kegan Paul.

Scheffler, I. (1986). Ten Myths of Metaphor. *Communication & Cognition*, *19*, 389-394.

Schwarz, B. & Hershkowitz, R. (1999). Prototypes: Brakes or Levers in Learning the Function Concept? The Role of Computer Tools. *Journal for Research in Mathematics Education, 30*, 362-389.

Sfard, A. (1992). Operational Origins of Mathematical Objects and the Quandary of Reification – The Case of Function. In G. Harel & E. Dubinsky (Eds.), *The Concept of Function: Aspects of Epistemology and Pedagogy*. MAA Notes, Volume 25. Washington, DC: Mathematical Association of America.

Sfard, A. (1994). Reification as the Birth of Metaphor. *For the Learning of Mathematics*, 14, 44-55.

Sfard, A. (1997). Commentary: On Metaphorical Roots of Conceptual Growth. In L. English (Ed.) *Mathematical Reasoning : Analogies, Metaphors, and Images*. Mahwah, NJ: Lawrence Erlbaum Associates.

Sfard, A. (1998). On Two Metaphors for Learning and the Dangers of Choosing Just One. *Educational Researcher, 27,* 4-13.

Sierpinska, A. (1987). Humanities Students and Epistemological Obstacles Related to Limits. *Educational Studies in Mathematics, 18,* 371-397.

Sierpinska, A. (1992). On Understanding the Notion of Function, In G. Harel & E. Dubinsky (Eds.), *The Concept of Function: Aspects of Epistemology and Pedagogy*. MAA Notes, Volume 25. Washington, DC: Mathematical Association of America.

Simonsen, L. (1995 Dissertation). *Teachers' Perceptions of the Concept of Limit, the Role of Limits, and the Teaching of Limits in Advanced Placement Calculus*. Oregon State University, Corvallis, Oregon, Dissertation Abstracts International, 56-06, 2158.

Steinbring, H. (1989). Routine and Meaning in the Mathematics Classroom. *For the Learning of Mathematics, 9*, 24-33.

Strauss, A. & Corbin, J. (1990). *Basics of Qualitative Research: Techniques and Procedures for Developing Grounded Theory*. Sage Publications.

Stroup, W. (1996). *Embodying a Nominalist Constructivism: Making Graphical Sense of Learning the Calculus of How Much and How Fast*. Doctoral Dissertation, Harvard University, Cambridge, MA, Dissertation Abstracts International, 57-07, 2928.

Szydlick, J. (2000). Mathematical Beliefs and Conceptual Understanding of the Limit of a Function. *Journal for Research in Mathematics Education, 31,* 258-276.

Tall, D. (1986). *Graphic Calculus.* London: Glentop Press.

Tall, D. (1990). Inconsistencies in the Learning of Calculus and Analysis. *Focus on Learning Problems in mathematics, 12,* 49-62.

Tall, D. (1992). The Transition to Advanced Mathematical Thinking: Functions, Limits, Infinity, and Proof. In Grouws, D. (Ed.), *Handbook of Research on Mathematics Teaching and Learning* (pp. 495-511). New York: Macmillan.

Tall, D. & Schwarzenberger, R. (1978). Conflicts in the learning of real numbers and limits. *Mathematics Teaching*, *82*, 44-49.

Tall, D. & Vinner, S. (1981). Concept Image and Concept Definition in Mathematics with Particular Reference to Limits and Continuity. *Educational Studies in Mathematics*, *12*, 151-169.

Thompson, P. (1994a). The Development of the Concept of Speed and its Relationship to Concepts of Rate. In G. Harel & J. Confrey (Eds.), *The Development of Multiplicative Reasoning in the Learning of Mathematics*. Albany, NY: SUNY Press, 179-234.

Thompson, P. (1994b). Images of Rate and Operational Understanding of the Fundamental Theorem of Calculus. *Educational Studies in Mathematics*, *26*, 229-274.

Thompson, P. (1994c). Students, Functions, and the Undergraduate Curriculum. *Research in Collegiate Mathematics Education I, Issues in Mathematics Education*, *4*, 21-44.

Tirosh, D. (1991). The Role of Students' Intuitions of Infinity in Teaching the Cantorian Theory. In D. Tall (Ed.) *Advanced Mathematical Thinking*. Boston: Kluwer, 167

Tucker, T. (Chair), (1986). Calculus syllabi: Report of the content workshop. In R. G. Douglas (Ed.), *Toward a Lean and Lively Calculus* (pp. vii-xiv). Washington, DC: The Mathematical Association of America.

Tucker, A.C. & Leitzel, J.R. (Eds.). (1995). *Assessing Calculus Reform Efforts: A Report to the Community*. Washington, DC: Mathematical Association of America.

Vinner, S. (1982). Conflicts Between Definitions and Intuition - The Case of Tangent. In A. Vermandel (Ed.), *Proceedings of the Sixth International Conference for the Psychology of Mathematics Education* (pp. 24-28). Universitaire Instelling, Antwerpen.

Vygotsky, L. (1978). *Mind in Society: The Development of Higher Psychological Processes.* Cambridge, Massachussetts: Harvard University Press.

Vygotsky, L. (1987). The Development of Scientific Concepts in Childhood. In R. W. Rieber and A. S. Carton (Eds.) *The Collected Works of L. S. Vygotsky*. Vol. 1, 167-241.

Wertsch, J. (1991). A Sociocultural Approach to Socially Shared Cognition. In L. Resnick, J. Levine, & S. Teasley (Eds.), *Perspectives on Socially Shared Cognition,* Washington D.C: American Psychological Association, 85-100.

White, P. & Mitchelmore, M. (1996). Conceptual Knowledge in Introductory Calculus. *Educational Studies in Mathematics*., *26*, 229-274.

Williams, S. (1989). *Understanding of the Limit Concept in College Calculus Students.* Doctoral Dissertation, The University of Wisconsin, Madison, WI, Dissertation Abstracts International, 50-10, 3174.

Williams, S. (1991). Models of Limit Held by College Calculus Students. *Journal for Research in Mathematics Education*, *22*, 219-236.

Williams, S. (2001). Predications of the Limit Concept: An Application of Repertory Grids. *Journal for Research in Mathematics Education, 32,* 341-367.

Zadeh, L. (1965). Fuzzy Sets. *Information and Control, 8,* 338-353.

Zandieh, M. (1997). *The Evolution of Student Understanding of the Concept of Derivative*. Doctoral Dissertation, Oregon State University, Corvallis, Oregon, Dissertation Abstracts International, 58-8, 3056.

Vita

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