LINEAR PROGRAMMING WITH NONLINEAR
PARAMETRIC OBJECTIVE FUNCTIONS

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by

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P R E F A C E

This thesis is an outgrowth of a term paper written in a course for Dr. R. E. Greenwood of the Mathematics Department of The University of Texas. The problem of linear programming is that of maximizing or minimizing a linear functional, $f(\lambda) = \sum_{j=1}^{n} c_j \lambda_j$, subject to a set of constraints,

$$\sum_{j=1}^{n} \lambda_j a_{ij} = b_i \qquad (i = 1,2,\ldots,m),$$

where $n > m$ and a_{ij}, b_i, and c_j are constants. The simplex method of Dantzig and the dual method of Lemke are methods of solving this problem. Saaty and Gass have solved the problem in which each c_j is a linear function of a parameter t. This paper formulates and solves the problem (1) in which each c_j is a parabolic function of a parameter t and (2) in which each c_j is a periodic function of a parameter t of the form $c_j = h_j \sin t + g_j \cos t$, where h_j and g_j are constants.

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M. M. H.

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CHAPTER I
THEORETICAL BACKGROUND
A. INTRODUCTION

The concept of linear programming is a relatively new one; and the development of the techniques used to attack it and related problems is occupying the time of a great many contemporary mathematicians, particularly those who are interested in economic and industrial applications of mathematics. Physically, the problem is concerned with "planning a complex of interdependent activities in the best possible (optimal) fashion" [1]. Scheduling trains, blending aviation gasolines, and allocating labor are only a few examples of the type of problems which have been formulated as linear programming problems. Mathematically, the problem is the maximization or minimization of a linear "functional" of a set of non-negative variables which are subject to a set of linear inequalities or linear equations, that is, finding $x = (x_1, x_2, \ldots, x_n)$, such that $f(x) = \sum_{j=1}^{n} c_j x_j$ is a maximum (or minimum), $x_j \geq 0$ $(j = 1, 2, \ldots, n)$, and $\sum_{j=1}^{n} a_{ij} x_j \leq b_i$ $(i = 1, 2, \ldots, m)$ or $\sum_{j=1}^{n} a_{ij} x_j = b_i$ $(i = 1, 2, \ldots, m)$.

In order to understand the development and application of the methods of handling such problems, one must first become familiar to some extent with the theory of convex sets in n-dimensional space and the theory of linear transformations. One purpose of this chapter is to present the theory from these two fields necessary to an understanding of the linear programming problem. Following this discussion and based on it will be a brief analysis of the simplex method of solution. A more detailed presentation of the simplex method and its procedure will be reserved until Chapter II. Finally, the last section of Chapter I will be a statement and proof of the "duality theorem," a basic theorem for linear programming.

B. DEFINITION OF TERMS

The first step in presenting any mathematical theory is to define the terms used. Points or vectors will mean points or vectors in n-dimensional space. Addition of points is defined only for sets of points, each member of which has the same number of components. The sum of two points $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ is then $x + y = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$. Multiplication of a point x by a real number a is defined by the following relation: $ax = (ax_1, ax_2, \ldots, ax_n)$.

The ray through the point x is defined as the set of all points ax such that $a \geq 0$. Charnes $\boxed{1}$ uses the symbol

$\left[x \mid P\right]$ to mean the set of all points x having the property P. This notation will be used henceforth in this paper. For example, the ray through the point x is $\left[ax \mid a \geq 0\right]$. If $x^{(1)}$ and $x^{(2)}$ are two distinct points, the segment joining $x^{(1)}$ and $x^{(2)} = \left[\surd x^{(1)} + (1 - \surd)x^{(2)} \mid 0 \leq \surd \leq 1\right]$.

A convex set is a "collection of points such that, if x and y are any two points in the collection, the segment joining them is also in the collection" $\boxed{1}$. An extreme point of a convex set is a "point in a convex set which does not lie on a segment joining some two other points of the set" $\boxed{1}$. If x is a point of the convex set K and $x^{(1)}, x^{(2)}, \ldots, x^{(m)}$ are the extreme points of the convex set K, then x may be expressed as a "convex" linear combination of the extreme points; that is,

$$x = \lambda_1 x^{(1)} + \lambda_2 x^{(2)} + \ldots + \lambda_m x^{(m)}$$

where $$0 \leq \lambda_i \leq 1 \quad \text{and} \quad \sum_{i=1}^{m} \lambda_i = 1.$$

A convex set may also be defined as the locus of all points which may be expressed as convex linear combinations of the extreme points. A convex polyhedron is a "convex set which may be generated from a finite number of points" $\boxed{1}$.

A cone is a collection of points such that if x is in the collection, so is the ray through x. A convex

<u>polyhedral cone</u> is a "cone generated by a convex polyhedron" [1]. A <u>convex polyhedral cone</u> is obviously the set of all linear combinations $\left[\lambda_1 x^{(1)} + \lambda_2 x^{(2)} + \ldots + \lambda_m x^{(m)} \middle| \lambda_i \geq 0\right]$ where $x^{(1)}, x^{(2)}, \ldots, x^{(m)}$ are the extreme points of the convex polyhedron. In the first place, every point in the convex polyhedron can be represented by an expression, $\mu_1 x^{(1)} + \mu_2 x^{(2)} + \ldots + \mu_m x^{(m)}$, where $0 \leq \mu_i \leq 1$ and $\sum_{i=1}^{m} \mu_i = 1$. Every point on a ray through any such point may be expressed as $a(\mu_1 x^{(1)} + \mu_2 x^{(2)} + \ldots + \mu_m x^{(m)})$ and hence is a linear combination of $x^{(1)}, x^{(2)}, \ldots, x^{(m)}$. Also any positive linear combination of these points may be expressed as a positive multiple of a point in the convex polyhedron and hence is a point of the convex polyhedral cone, as

$$x = \lambda_1 x^{(1)} + \lambda_2 x^{(2)} + \ldots + \lambda_m x^{(m)}$$

$$= \sum_{i=1}^{m} \lambda_i \left[\frac{\lambda_1}{\sum_{i=1}^{m} \lambda_i} x^{(1)} + \frac{\lambda_2}{\sum_{i=1}^{m} \lambda_i} x^{(2)} + \ldots + \frac{\lambda_m}{\sum_{i=1}^{m} \lambda_i} x^{(m)} \right]$$

where $\lambda_i \geq 0$.

"T is said to be a linear transformation if: $T(ax + bw) = aT(x) + bT(w)$ for all points x, w, and real numbers a, b" $\boxed{1}$. A linear transformation is completely determined by what it does to the unit vectors. The notation $e^{(i)}$ will be used to denote the unit vectors.

$e^{(1)} = (1,0,0,\ldots,0)$; $e^{(2)} = (0,1,0,\ldots,0)$; $e^{(n)} = (0,0,\ldots,1)$.

Now $x = (x_1, x_2, \ldots, x_n) = x_1 e^{(1)} + x_2 e^{(2)} + \ldots + x_n e^{(n)}$.

Therefore, $T(x) = x_1 T(e^{(1)}) + x_2 T(e^{(2)}) + \ldots + x_n T(e^{(n)})$.

Consequently, if $T(e^{(i)})$ $(i = 1,2, \ldots, n)$ are known, the transformation will be uniquely determined.

"$$T\left[e^{(1)}\right] = y^{(1)} = a_{11}e^{(1)} + a_{12}e^{(2)} + \ldots + a_{1n}e^{(n)}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$T\left[e^{(j)}\right] = y^{(j)} = a_{j1}e^{(1)} + a_{j2}e^{(2)} + \ldots + a_{jn}e^{(n)}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$T\left[e^{(n)}\right] = y^{(n)} = a_{n1}e^{(1)} + a_{n2}e^{(2)} + \ldots + a_{nn}e^{(n)}$$" $\boxed{1}$.

Every such linear transformation T is associated with a matrix A,

$$\text{where } A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix}.$$

It is also possible to have a linear transformation from n-dimensional space into m-dimensional space. For these transformations, the associated matrix has n rows and m columns. For example,

$$A = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

is the matrix (column vector in this case) associated with a linear transformation from n-dimensional space into one-dimensional space. (A convenient way of indicating column vector A is to write A', its "transpose," a row vector. The transpose A' of a matrix A is the matrix obtained by interchanging the rows and columns of A.) To every point x in n-dimensional space there corresponds a unique point w in m-dimensional space into which x is transformed by the linear transformation whose associated matrix has n rows and m columns.

The statements in the next three paragraphs are from matrix or vector theory and will be made and used in this paper without proof.

"A <u>linearly independent</u> set of <u>vectors</u> (or points) $P_1, \ldots, P_\ell$ is a set such that $c_1P_1 + \ldots + c_\ell P_\ell = \overrightarrow{0}$, the 'null vector' (0, ..., 0), for real numbers $c_1, \ldots, c_\ell$, <u>only</u> if $c_1 = c_2 = \ldots = c_\ell = 0$. Vectors not linearly independent are <u>linearly dependent</u>.

"A consequence of linear independence is that, if a point $P_0 = a_1P_1 + \ldots + a_\ell P_\ell$ (where $P_1, \ldots, P_\ell$ is a linearly independent set), then the $a_1, \ldots, a_\ell$ are the <u>only</u>

coefficients of the P_i's in terms of which one can express P_o as a sum of multiples of the P_i's, i.e., as a 'linear combination.'

"If the points are in an m-dimensional space, then there are at most m points in a linearly independent set. Such a set of m points (vectors) is called a basis of the space. Every point of the space may be written (uniquely) as a linear combination of the points of a basis" [1].

The following theorem is of importance in linear programming:

"A linear transformation L from an n-dimensional space U to an m-dimensional space W takes a convex polyhedron K into a convex polyhedron L(K), the image of K" [1]. The method of proof is (1) to show that the convexity of L(K) follows from the convexity of K and (2) to show that L(K) has at least one extreme point and no more extreme points than K has. For a detailed proof one may consult the Bibliography, Reference [1], or other references on linear transformations and convex sets.

A linear functional f(x) is a linear transformation which takes the points $x = (x_1, x_2, \ldots, x_n)$ of an n-dimensional space into one-dimensional space. For example, $f(x) = \sum_{i=1}^{n} c_i x_i$ is a linear functional. At the beginning of this paper the problem of linear programming was stated as the problem of finding $x = (x_1, x_2, \ldots, x_n)$ such that (1) $f(x) = \sum_{j=1}^{n} c_j x_j$ is a maximum (or minimum), (2) $\sum_{j=1}^{n} a_{ij} x_j \leq b_i$ $(i = 1,2, \ldots, m)$ or $\sum_{j=1}^{n} a_{ij} x_j = b_i (i = 1,2, \ldots, m)$, and (3) $x_j \geq 0$ $(j = 1,2, \ldots, n)$. If the restrictions $\sum_{j=1}^{n} a_{ij} x_j \leq b_i$ are in the form of inequalities instead of equations, they may easily be converted into equations by adding to each inequality another positive unknown whose coefficient is unity. Then $\sum_{j=1}^{n} a_{ij} x_j \leq b_i$ and $x_j \geq 0$ becomes $\sum_{j=1}^{n} a_{ij} x_j + x_{n+i} \equiv b_i$ and $x_j \geq 0$ and $x_{n+i} \geq 0$.

To every solution to the set of inequalities there corresponds a solution to the set of equations for which the value of the linear functional is the same. Hence one may work with the set of equations only. Henceforth this paper will use the notation of Charnes, and the unknowns will be denoted by $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ instead of x.

In order to clarify the discussion which follows the set of equations will be written out in greater detail here:

$$\begin{array}{l}
\lambda_1 a_{11} + \lambda_2 a_{12} + \ldots + \lambda_n a_{1n} = b_1 \\
\lambda_1 a_{21} + \lambda_2 a_{22} + \ldots + \lambda_n a_{2n} = b_2 \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\lambda_1 a_{m1} + \lambda_2 a_{m2} + \ldots + \lambda_n a_{mn} = b_m.
\end{array}$$

The column vectors $(b_1, b_2, \ldots, b_m)$, $(a_{11}, a_{21}, \ldots, a_{m1})$, ..., $(a_{1n}, a_{2n}, \ldots, a_{mn})$ will be regarded as vectors in m-dimensional space. They will be denoted by $P_o, P_1, \ldots, P_n$ respectively. The problem then is to find $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ such that $\lambda_1 P_1 + \lambda_2 P_2 + \ldots + \lambda_n P_n = P_o$ and for which $f(\lambda)$ is a maximum (or minimum). The set of equations,

$$\sum_{j=1}^{n} a_{ij} x_j = b_i \quad (i = 1, 2, \ldots, m),$$

may be considered a linear transformation from n-space into m-space if each b_i is replaced by w_i. Then the point $x = (x_1, x_2, \ldots, x_n)$ is transformed into $w = (w_1, w_2, \ldots, w_m)$. The matrix associated with this linear transformation is the transpose of the matrix of coefficients of the equations. This is true because the n unit vectors are transformed into the n column vectors $P_1, P_2, \ldots, P_n$. For example, $e^{(1)} = (1, 0, \ldots, 0)$ is transformed into P_1. To understand this, recall that if x is an n-dimensional point, it is transformed into w, where $w = x_1 P_1 + x_2 P_2 + \ldots + x_n P_n$. Therefore, $e^{(1)} = (1, 0, \ldots, 0)$ is transformed into the point, $(1)P_1 + (0)P_2 + \ldots + (0)P_n = P_1$. In general $e^{(i)}$ is transformed into P_i. Now the matrix associated with a linear transformation is the matrix $\left[a_{ij}\right]$ whose element a_{ij} is the jth component of the transform of the ith unit vector. In other words the ith row of the matrix is simply the components of the transform of $e^{(i)}$. Hence the matrix associated with the transform $x_1P_1 + x_2P_2 + \dots + x_nP_n = w$ is the n x m matrix whose <u>rows</u> are the P_i's. This is obviously the transpose of the matrix of coefficients of the set of equations

$$x_1P_1 + x_2P_2 + \dots + x_nP_n = P_o.$$

Henceforth, the following notation will be used: U will denote n-dimensional space; W will denote m-dimensional space; and L will denote the linear transformation whose associated matrix is the transpose of the matrix of coefficients.

"Now the set of all vectors in U having non-negative coordinates forms a convex polyhedral cone" [1]. The linear transformation L takes this convex polyhedral cone of U into a convex polyhedral cone of W. "A <u>representative convex set</u> associated with a convex polyhedral cone is a convex set generated by points, one on each edge of the cone" [1]. An edge of a convex polyhedral cone is the ray through one of the extreme points of the convex polyhedron generating the cone. Hence, any representative convex set may be thought of as having generated the cone.

In the linear programming problem there is usually more than one point λ such that $\sum_{i=1}^{n} \lambda_i P_i = P_o$ and $\lambda_i \geq 0$.

The set of all such points will be denoted henceforth by Λ. Such points are called "feasible" solutions. They do not necessarily maximize or minimize the linear functional. It will now be shown that Λ is a convex set. If $\lambda^{(1)}$ and $\lambda^{(2)}$ are any two distinct points in Λ, then $L(\lambda^{(1)}) = P_o$ and $L(\lambda^{(2)}) = P_o$. If $0 \leq \mu \leq 1$, then $L\left[\mu\lambda^{(1)} + (1 - \mu)\lambda^{(2)}\right] = \mu L(\lambda^{(1)}) + (1 - \mu)L(\lambda^{(2)}) = \mu P_o + (1 - \mu)P_o = P_o$. Hence Λ is a convex set.

C. STATEMENT AND PROOF OF A FUNDAMENTAL THEOREM

A theorem will now be proved which is basic to the solution of the problem at hand:

"Theorem. A linear functional $f(u)$ defined on a convex polyhedron K takes on its max. (or min.) at an extreme point of the convex set. If it takes on the max. (or min.) at more than one point, then it takes the same value over the whole convex set generated by those particular points.

"Proof. Suppose that x is a point of the convex set K for which $f(x)$ is a max. (or min.). If x is an extreme point, the first statement of the theorem is true. Suppose that x is not an extreme point and that $f(x)$ is a max. (or min.). Then we may express x as a 'convex' combination of extreme points of K, say $A_1, A_2, \ldots, A_r$. Thus: $x = \sum_{i=1}^{r} \checkmark_i A_i$ where $0 \leq \checkmark_i \leq 1$ $(i = 1,2,\ldots,r)$, and $\sum_{i=1}^{r} \checkmark_i = 1$.

Then, because f is a linear functional, we have:

$$f(x) = f\left[\sum_{i=1}^{r} \sqrt{}_i A_i\right] = \sum_{i=1}^{r} \sqrt{}_i f(A_i) = \text{max.}$$

Now, since the $\sqrt{}_i$ are all non-negative, we do not decrease the sum $\sum_{i=1}^{r} \sqrt{}_i f(A_i)$ if for each $f(A_i)$ we substitute the greatest of the values $f(A_i)$, say $f(A_k)$ for some fixed k. But then

$$f(x) \leq f(A_k) \sum_{i=1}^{r} \sqrt{}_i = f(A_k)$$

so that, if f(x) is a max., the equals sign must hold so that $f(A_k) = f(x)$ is a max. also.

"(If $f(x) = \sum_{i=1}^{r} \sqrt{}_i f(A_i) = \text{min.}$, then we do not increase the sum if we substitute the smallest of the values $f(A_i)$, say $f(A_m)$, for some fixed m. Hence,

$$f(x) \geq f(A_m) \sum_{i=1}^{r} \sqrt{}_i = f(A_m). \text{ Hence, } f(x) = f(A_m).)$$

"If f takes on a max. (or min.) for more than one extreme point, say $A_1, A_2, \ldots, A_k$, i.e., $f(A_1) = f(A_2) = \ldots = f(A_k) = M = \text{max. (or min.)}$, consider the convex set formed from these points. If x is any point in this set, then $x = \sum_{i=1}^{k} \sqrt{}_i A_i$ where $0 \leq \sqrt{}_i \leq 1$ and $\sum_{i=1}^{k} \sqrt{}_i = 1$ so that

$$f(x) = \sum_{i=1}^{k} \sqrt{}_i f(A_i) = \sum_{i=1}^{k} \sqrt{}_i M = M \sum_{i=1}^{k} \sqrt{}_i = M$$

This completes the proof of the theorem" $\boxed{1}$.

D. THE SOLUTIONS SPACE AND REQUIREMENTS SPACE

If K denotes the convex polyhedron in U generated by the unit vectors, then "L(K), the set in W of all transforms of points of K, is a convex polyhedron generated by the points P_i" $\boxed{1}$. The cone generated by the P_i's in W is the transform of the positive orthant of U. Charnes refers to U as the "solutions space" and W as the "requirements space."

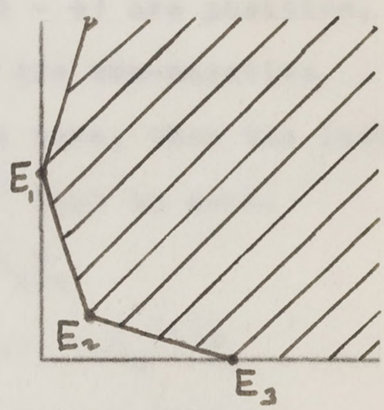
If λ is a point in Λ, then $\lambda_1 P_1 + \lambda_2 P_2 + \ldots + \lambda_n P_n = P_o$. Also

$$\frac{\lambda_1}{\sum_{i=1}^{n} \lambda_i} P_1 + \frac{\lambda_2}{\sum_{i=1}^{n} \lambda_i} P_2 + \ldots + \frac{\lambda_n}{\sum_{i=1}^{n} \lambda_i} P_n = \frac{1}{\sum_{i=1}^{n} \lambda_i} P_o.$$

Now if $\frac{1}{\sum_{i=1}^{n} \lambda_i}$ is denoted by α, there exists a point αP_o on the ray through P_o and which is in L(K), the convex set determined by the P_i's. Also if Λ is a bounded set, then $\alpha = \frac{1}{\lambda_1 + \lambda_2 + \ldots + \lambda_n}$ is bounded and hence has a maximum value α_M and a minimum value α_m. To each solution λ there corresponds a value α such that αP_o is in L(K) and such that $\alpha_m \leq \alpha \leq \alpha_M$.

E. ASSURING THAT THE SOLUTION SET IS BOUNDED

It has been shown that Λ is a convex set and that a linear functional defined on a convex polyhedron takes on its maximum or minimum at an extreme point of the convex polyhedron. The next step is to show that Λ is not only a convex set but that it also has a finite number of extreme points. Then if the solution set is known to be bounded, it is a convex polyhedron. If the solution set is not bounded, it may have a finite number of extreme points and still not be a convex polyhedron. For example, the shaded space in the drawing is a convex set having a finite number of extreme points but not "generated by" a finite number of points. Boundedness of the convex set is assured by appending the equation $\sum_{i=1}^{n+1} \lambda_i = B$ where λ_{n+1} is a new non-negative variable and B is an unspecified constant larger than the sum of the coordinates of any extreme point. B does not need to be specified beyond this. If the solution set is not bounded, the addition of this new equation will add new extreme points to the solution set, and these new extreme points will involve B.



F. THE EXTREME POINTS OF THE SOLUTION SET

Another theorem basic to an analysis of the linear programming problem is the following: "$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is an extreme point of Λ if, and only if, the non-zero λ_i are the coefficients of linearly independent vectors P_j" $\boxed{1}$. Its proof follows.

If $\lambda_1 P_1 + \lambda_2 P_2 + \ldots + \lambda_q P_q = P_o$ and the P_i's ($i = 1,2, \ldots, q$) are linearly independent, then $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_q, 0, \ldots, 0)$ is an extreme point of Λ. To prove this, one assumes that there exist two points $\lambda^{(1)}$ and $\lambda^{(2)}$ in Λ and some number c such that $0 < c < 1$ and $\lambda = c\lambda^{(1)} + (1 - c)\lambda^{(2)}$ (in other words that λ is not an extreme point of Λ). Now since c and $(1 - c)$ are positive, all the coordinates of $\lambda^{(1)}$, $\lambda^{(2)}$, and λ are non-negative, and the last $(n - q)$ coordinates of λ are zero, then the last $(n - q)$ coordinates of both $\lambda^{(1)}$ and $\lambda^{(2)}$ must be zero. Therefore,

$$\begin{aligned} P_o &= \lambda_1 P_1 + \lambda_2 P_2 + \ldots + \lambda_q P_q \\ &= \lambda_1^{(1)} P_1 + \lambda_2^{(1)} P_2 + \ldots + \lambda_q^{(1)} P_q \\ &= \lambda_1^{(2)} P_1 + \lambda_2^{(2)} P_2 + \ldots + \lambda_q^{(2)} P_q. \end{aligned}$$

Since the q P_i's are linearly independent, then the expression for P_o in terms of them is unique. Hence $\lambda_i = \lambda_i^{(1)} = \lambda_i^{(2)}$ ($i = 1,2, \ldots, q$), and λ is an extreme point of Λ. It should be noted that $q \leq m$, where m is the number of equations, that is, the dimensionality of the P_i vectors. This is true because more than m vectors in m-dimensional space are necessarily linearly dependent.

If λ is an extreme point of Δ, then in the expression $\sum_{i=1}^{n} \lambda_i P_i = P_o$ the non-zero coordinates of λ are coefficients of linearly independent vectors P_i. Assume that there are q non-zero coordinates. Then $\sum_{i=1}^{q} \lambda_i P_i = P_o$. Assume also that the q vectors P_i are linearly dependent. Then there exist numbers c_i such that $\sum_{i=1}^{q} c_i P_i = \overrightarrow{0}$ where not all of the c_i are zero. For any positive constant k,

$$P_o = \sum_{i=1}^{q} \lambda_i P_i \pm k \sum_{i=1}^{q} c_i P_i = \sum_{i=1}^{q} (\lambda_i \pm kc_i) P_i .$$

Since all $\lambda_i > 0$, there exists a positive number k such that $\lambda_i + kc_i$ and $\lambda_i - kc_i$ are positive (i = 1,2, ..., q).

Then $\lambda^{(1)} = (\lambda_1 + kc_1, \lambda_2 + kc_2, \ldots, \lambda_q + kc_q)$

and $\lambda^{(2)} = (\lambda_1 - kc_1, \lambda_2 - kc_2, \ldots, \lambda_q - kc_q)$

are points of Δ. Also $\lambda = \frac{1}{2}(\lambda^{(1)} + \lambda^{(2)})$. Therefore, the λ_i (i = 1,2, ..., q) must be coefficients of linearly independent vectors because this is a contradiction of the assumption that λ is an extreme point. It should be noted, as above, that $q \leq m$ because more than m vectors in m-dimensional space are necessarily linearly dependent. Now, since n is a finite number, the number of sets of linearly independent vectors and hence the number of extreme points is finite. Therefore, if the solution set is bounded, it is a convex polyhedron. The functional takes on its optimal value (maximum or minimum) at one of its extreme points.

G. FINDING A SECOND EXTREME POINT SOLUTION FROM A KNOWN EXTREME POINT SOLUTION

The foundation has now been constructed upon which the simplex procedure is based. The first step in this procedure is to assume that a solution λ, involving exactly m non-zero components which are coefficients of linearly independent vectors, is readily available. This is actually no restriction on the problem, as will be explained in Chapter II. The components of this solution will be denoted by $\lambda_1, \lambda_2, \ldots, \lambda_m$ and the corresponding column vectors by $P_1, P_2, \ldots, P_m$. Since these column vectors are linearly independent, they form a basis of W (m-dimensional space). All of the P_j's $(j = 1, 2, \ldots, n)$ may be expressed in terms of them. For example, $P_j = \sum_{i=1}^{m} x_{ij} P_i$. The values x_{ij} are unique. The value of the functional determined by this first "trial" solution $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ is $z_o = \sum_{i=1}^{m} c_i \lambda_i$.

This solution is an extreme point of Λ, and what is sought is another extreme point of Λ which, if possible, increases the value of the functional. Let P_k be some vector not in the basis, $P_1, P_2, \ldots, P_m$. Then

$$P_o = \sum_{i=1}^{m} \lambda_i P_i + \theta P_k - \theta P_k.$$

$$P_o = \sum_{i=1}^{m} \lambda_i P_i - \theta \sum_{i=1}^{m} x_{ik} P_i + \theta P_k.$$

(1-1) $$P_o = \sum_{i=1}^{m} (\lambda_i - \theta x_{ik}) P_i + \theta P_k.$$

If $\theta > 0$ and each $(\lambda_i - \theta x_{ik}) \geq 0$, then this will give another solution. If $\theta > 0$, all $(\lambda_i - \theta x_{ik}) \geq 0$, and at least one $(\lambda_i - \theta_{ik}) = 0$, then this is another extreme point of the solution set because it involves a different set of linearly independent vectors. To prove that these vectors are linearly independent, let $\lambda_r - \theta x_{rk} = 0$. Now assume the set of vectors P_i $(i = 1,2, \ldots, m;\ i \neq r;\ i = k)$ is linearly dependent. Then there exists a set of numbers d_i and a number d_k, not all zero, such that $\sum_{\substack{i=1 \\ i\neq r}}^{m} d_i P_i + d_k P_k = 0$. Now $d_k \neq 0$; for if $d_k = 0$, then $\sum_{\substack{i=1 \\ i\neq r}}^{m} d_i P_i = 0$ and at least one of the $d_i \neq 0$. This is not possible because the P_i's $(i = 1,2,\ldots,m;\ i \neq r)$

are linearly independent. Thus $d_k \neq 0$, and

$$d_k P_k = \sum_{\substack{i=1 \\ i \neq r}}^{m} (-d_i) P_i ;$$

$$P_k = \sum_{\substack{i=1 \\ i \neq r}}^{m} \left(\frac{-d_i}{d_k}\right) P_i .$$

Let $\frac{-d_i}{d_k} \equiv b_i$; $P_k = \sum_{\substack{i=1 \\ i \neq r}}^{m} b_i P_i$.

But $$P_k = \sum_{i=1}^{m} x_{ik} P_i .$$

Subtracting the expression involving b_i's from the expression involving x_{ik}'s gives:

$$0 = x_{rk} P_r + \sum_{\substack{i=1 \\ i \neq r}}^{m} (x_{ik} - b_i) P_i .$$

Now since the vectors $P_1, P_2, \ldots, P_m$ are linearly independent, all the coefficients must vanish. In particular $x_{rk} = 0$. However, it has been assumed that $\lambda_r > 0$, $\theta > 0$, and $\lambda_r - \theta x_{rk} = 0$. Therefore, $\frac{\lambda_r}{\theta} = x_{rk} > 0$. Hence the new set of vectors P_i ($i = 1,2, \ldots, m$; $i \neq r$; $i = k$) is a linearly independent set, and the new solution is an extreme point of Λ.

The problem now is to find a way of choosing θ so that $\theta > 0$, all $(\lambda_i - \theta x_{ik}) \geq 0$, and at least one value $\lambda_i - \theta x_{ik} = 0$.

The procedure follows:

1. The solution $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ such that all $\lambda_i > 0$, $\sum_{i=1}^{m} \lambda_i P_i = P_o$, and $P_1, P_2, \ldots, P_m$ are linearly independent is assumed known. These m P_i's form a basis of W.

2. All the P_j's $(j = 1,2, \ldots, n)$ are expressed in terms of the basis vectors. $(P_j = \sum_{i=1}^{m} x_{ij} P_i.)$

3. Assume that for some value of j, say j = k, some x_{ij}'s > 0; that is, $P_k = \sum_{i=1}^{m} x_{ik} P_k$ and some x_{ik}'s are greater than zero. For every $x_{ik} > 0$, λ_i / x_{ik} is calculated.

4. θ is assigned the smallest of these positive values. If λ_r / x_{rk} is the smallest of the set, λ_i / x_{ik} for $x_{ik} > 0$, then $\theta = \lambda_r / x_{rk}$. Then all values $(\lambda_i - \theta x_{ik})$ will be ≥ 0. Since $\lambda_i > 0$ and $\theta > 0$, then if $x_{ik} \leq 0$, then $(\lambda_i - \theta x_{ik}) > 0$. On the other hand, if $x_{ik} > 0$, then, because of the choice of θ as the smallest of those values λ_i / x_{ik} which are positive, every value of λ_i / x_{ik} (for $x_{ik} > 0$) is $\geq \theta$. Hence, if $x_{ik} > 0$,

$$\frac{\lambda_i}{x_{ik}} \geq \theta,$$

$$x_{ik}\frac{\lambda_i}{x_{ik}} \geq \theta x_{ik},$$

$$\lambda_i \geq \theta x_{ik},$$

$$(\lambda_i - \theta x_{ik}) \geq 0.$$

It will now be assumed that if P_k (the replacing vector) has been chosen such that some $x_{ik} > 0$, then $\theta = \lambda_i/x_{ik}$ for only one value of i; that is, that $(\lambda_i - \theta x_{ik}) = 0$ for only one value of i, namely, i = r. Then a new basis has been uniquely determined in which P_k replaces P_r in the old basis. The case where $\lambda_i - \theta x_{ik} = 0$ for more than one value of i is the case of "degeneracy" and will be discussed in the next chapter.

5. Then $\sum_{i=1}^{m} (\lambda_i - \theta x_{ik})P_i + \theta P_k$ provides another extreme point λ' of the solution set Λ.

$$\lambda_i' = \lambda_i - \theta x_{ik} \quad (i = 1, 2, \ldots, m).$$

$$\lambda_k' = \theta.$$

$$\lambda_i' = 0 \quad (i = m + 1, m + 2, \ldots, n;\ i \neq k).$$

Notice that $\lambda_r' = 0$.

Let z_o be the value of the functional determined by λ; and z_o^*, the value determined by λ'. Also define the symbol

z_j to mean $\sum_{i=1}^{m} x_{ij}c_i$ and notice that these values depend upon the solution being used or, more particularly, the set of m linearly independent vectors (the basis) corresponding to the solution.

$$z_o = \sum_{i=1}^{m} c_i\lambda_i; \quad z_o^* = \sum_{i=1}^{m} (\lambda_i - \theta x_{ik})c_i + \theta c_k$$

$$= \sum_{i=1}^{m} \lambda_i c_i + \theta(c_k - \sum_{i=1}^{m} x_{ik}c_i).$$

(1-2) $$z_o^* = z_o + \theta(c_k - z_k).$$

H. DETERMINING WHETHER A GIVEN EXTREME POINT SOLUTION IS OPTIMAL

The method outlined above allows one to find a second extreme point of Λ when one extreme point is given. The only restriction is that for some P_j, not a basis vector, one of the values of $x_{ij} > 0$. It will be shown shortly that if this is not the case, then either the given solution is optimal or there is no optimal solution (the value of the functional is unbounded). When a new extreme point has been found by the method outlined above, it is a "better" solution than the first if, and only if, $(c_k - z_k) > 0$ where k is the subscript of the new vector introduced into the basis.

It is now advantageous to backtrack to the point where it was assumed that a solution λ had been found involving exactly m non-zero components which were coefficients of m linearly independent vectors. These vectors form a basis of W, and hence all of the column vectors P_j may be expressed in terms of them. This is done, and all of the values x_{ij} ($i = 1,2, \ldots, m$; $j = 0,1,2, \ldots, n$) are tabulated. Each z_j ($\equiv \sum_{i=1}^{m} x_{ij}c_i$) is then calculated and recorded, and then $z_j - c_j$ is also calculated and recorded for each value of j. The table of values is examined, and there are three possibilities.

I. For <u>some</u> j, $c_j - z_j > 0$, and for this j and <u>every</u> i, $x_{ij} \leq 0$. It should be recalled here that

$$\sum_{i=1}^{m} (\lambda_i - \theta x_{ij})P_i + \theta P_j$$

is a solution for any θ such that all $(\lambda_i - \theta x_{ij}) \geq 0$ and $\theta \geq 0$ and that the value of the functional for this solution is $z_o + \theta(c_j - z_j)$. Hence since all $x_{ij} \leq 0$ and $c_j - z_j > 0$, any positive value of θ will give a "feasible" solution. The functional has no bound. (Appending the equation $\sum_{i=1}^{n+1} \lambda_i = B$, where B was discussed in Section E of this chapter, automatically eliminates the possibility of this case.)

II. For <u>all</u> $j = 1,2, \ldots, n$, $c_j - z_j \leq 0$. Then z_o is the maximum value. Let λ' be any another solution.

Then $P_o = \sum_{i=1}^{n} \lambda_i' P_i$. Let z_o^* be the value of the functional determined by λ'; that is, $z_o^* = \sum_{i=1}^{n} c_i \lambda_i'$. It will now be shown that $z_o \geq z_o^*$. By assumption,

$$z_j \geq c_j \text{ for all } j.$$

Also $\qquad \lambda_i' \geq 0.$

Hence $$z_o^* = \sum_{i=1}^{n} \lambda_i' c_i \leq \sum_{i=1}^{n} z_i \lambda_i' = \sum_{i=1}^{n} \lambda_i' \left(\sum_{j=1}^{m} x_{ji} c_j \right)$$

$$= \sum_{j=1}^{m} \left(\sum_{i=1}^{n} \lambda_i' x_{ji} \right) c_j .$$

Now $$P_o = \sum_{i=1}^{n} \lambda_i' P_i = \sum_{i=1}^{n} \lambda_i' \left(\sum_{j=1}^{m} x_{ji} P_j \right) = \sum_{j=1}^{m} \left(\sum_{i=1}^{n} \lambda_i' x_{ji} \right) P_j .$$

Also $P_o = \sum_{j=1}^{m} \lambda_j P_j$. Since $P_1, P_2, \ldots, P_m$ form a basis of W, the expression of P_o in terms of them is unique.

Therefore, $$\left(\sum_{i=1}^{n} \lambda_i' x_{ji} \right) = \lambda_j .$$

Substituting this in the last expression for the right-hand side of the inequality above yields $z_o^* \leq \sum_{j=1}^{m} \lambda_j c_j = z_o$.

Charnes (after proving the above) states the following optimality test. "Let $\sum_{i=1}^{p} \lambda_i P_i = P_o$ be a solution on p linearly independent vectors. Add m - p more P_i's to obtain a basis for W. Express the remaining P_j's in terms of this basis. Then $\lambda = (\lambda_1, \ldots, \lambda_p, 0, \ldots, 0)$ is optimal, i.e., yields the finite maximum of $f(\lambda)$ if all $c_j - z_j \leq 0$" $\boxed{1}$.

III. For <u>some</u> j, $c_j - z_j > 0$; for every such j and some i, $x_{ij} > 0$. This is the case which is dealt with by introducing a new vector P_k into the basis and removing one of the old basis vectors. (See preceding section.) If only one vector is replaced by P_k, that is, if there is only one vector, say P_r, such that $\lambda_r/x_{rk} = \theta$, then the new solution will be based on m linearly independent vectors, which also form a basis of W. In this case the value of the functional for the new solution is larger than the value of the functional for the first solution. Since the new solution involves vectors which form a basis of W, the whole process may be repeated. If, at each stage, a solution involving m linearly independent vectors is obtained, the process may be continued. When I or II occurs, the problem is solved. The third possibility cannot recur indefinitely because (1) there are only a finite number of sets of m linearly independent vectors in the set $P_1, P_2, \ldots, P_n$, and (2) the possibility of cycling or repeating any basis is eliminated because each solution is assured different from all preceding ones by the fact that each solution determines a value of the functional larger than those determined by all preceding solutions in the procedure. Hence I or II will eventually occur. A difficulty which arises in practice is that at some stage there will be fewer than m vectors involved in a solution, and hence the other vectors cannot be expressed in terms of them. This is called "degeneracy" and will be discussed fully in Chapter II.

I. STATEMENT OF THE DUALITY THEOREM OF LINEAR PROGRAMMING

One of the fundamental theorems of this relatively new branch of mathematics — linear programming — is known as the "duality theorem." Consider the following linear programming problem: minimize

$$f = \sum_{j=1}^{n} c_j x_j \text{ where } \sum_{j=1}^{n} a_{ij} x_j \geq b_i \quad (i = 1,2, \ldots, m)$$

and $x_j \geq 0$ $(j = 1,2, \ldots, n)$.

Its "dual" problem then is this: maximize

$$g = \sum_{i=1}^{m} b_i w_i \text{ where } \sum_{i=1}^{m} a_{ij} w_i \leq c_j \quad (j = 1,2, \ldots, n)$$

and $w_i \geq 0$ $(i = 1,2, \ldots, m)$.

The duality theorem states that if there is a finite solution to either problem, then there is a finite solution to the other and that the minimum of f is equal to the maximum of g.

The proof which will be presented here is essentially the proof given by Charnes in An Introduction to Linear Programming. It will be necessary before presenting the proof to clarify some of the matrix notation which will be used in it.

J. EXPLANATION OF MATRIX NOTATION

Small letters without subscripts will be used to designate column vectors. For example,

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

The same small letter with a prime will indicate the corresponding row vector; that is, $x' = (x_1, x_2, \ldots, x_n)$. More generally, a prime on any symbol for a matrix indicates the transpose of the matrix. Capital letters without subscripts will denote matrices with more than one column and more than one row. I will be used to mean the square matrix with entries of unity in the ith row and column and entries of zero everywhere else. $I = \begin{bmatrix} 100 \\ 010 \\ 001 \end{bmatrix}$ or a similar matrix of another order, so long as it is square. If A and B are two matrices with the same number of rows, $\begin{bmatrix} A, B \end{bmatrix}$ means the matrix obtained by writing A to the left of B; that is,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix},$$

$$\begin{bmatrix} A, B \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & b_{11} & b_{12} \\ a_{21} & a_{22} & a_{23} & a_{24} & b_{21} & b_{22} \\ a_{31} & a_{32} & a_{33} & a_{34} & b_{31} & b_{32} \end{bmatrix}.$$

Multiplication of matrices is not commutative, and the product AB is defined only when the number of columns in A is equal to the number of rows in B. When this is the case, the element in the ith row and jth column of the product matrix is the sum of the products of corresponding elements in the ith row of A and the jth column of B. If AB = C and $(a_{i1}, a_{i2}, \ldots, a_{in})$ is the ith row of A and $\begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix}$ is the jth column of B, then

$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$. Obviously the number of rows in A is the same as the number of rows in C, and the number of columns in B is the same as the number of columns in C. If X is any square matrix whose rows (or columns) are linearly independent, then X^{-1} will mean the square matrix of the same order such that $XX^{-1} = I = X^{-1}X$. In matrix theory it is proved that if the rows of a square matrix are linearly independent, the columns are linearly independent. Also it is proved that only matrices which are square and have linearly independent rows and columns have inverses. The inverse of a matrix is unique. In the proof which follows, P_i will mean the same thing which it has meant throughout the discussion. An m x n matrix means a matrix having m rows and n columns. To say $x \geq 0$ means that each component $x_i \geq 0$.

Now the linear programming problem may be stated as: minimize $f = c'x$, where x and c are n x 1 and x is subject to $Ax \geq b$ and $x \geq 0$. A is m x n and b is m x 1. Its dual may be stated as: maximize $g = w'b$, where w is m x 1 and w is subject to $w'A \leq c'$ and $w \geq 0$.

K. PROOF OF THE DUALITY THEOREM OF LINEAR PROGRAMMING

The first step in proving the theorem is to convert the set of inequalities $Ax \geq b$ to the equivalent system of equations: $Ax - I\alpha = b$. I is the m x m unit matrix; α is m x 1; $\alpha \geq 0$. Now if $\bar{x} = (x_1, x_2, \ldots, x_n, \alpha_1, \alpha_2, \ldots, \alpha_m)$, then $\begin{bmatrix} A, & -I \end{bmatrix}\bar{x} = b$ and $\begin{bmatrix} A, & -I \end{bmatrix}$ has $(n + m)$ columns. In the simplex procedure one finds $\lambda \geq 0$ such that $\sum_{j=1}^{q} P_j\lambda_j = P_o$, where $q = m + n$. The column vectors in $-I$ have been denoted by $P_{n+1}, P_{n+2}, \ldots, P_{n+m}$. A solution involving precisely m P_j's can be obtained by the simplex method. This solution will involve the m column vectors $P_{q_1}, P_{q_2}, \ldots, P_{q_m}$; that is, $P_o = \sum_{j=1}^{m} P_{q_j} \lambda_j$. It is assumed that these column vectors are linearly independent. Hence every column vector may be expressed in terms of them. The values x_{ij} in the following expressions are calculated:

$$P_j = \sum_{i=1}^{m} x_{ij} P_{q_i} \qquad (j = 1,2, \ldots, n + m).$$

Then the values $z_j = \sum_{i=1}^{m} c_{q_i} x_{ij}$ are then calculated.

$$\text{Now} \qquad \begin{bmatrix} A, & -I \end{bmatrix} = \begin{bmatrix} P_1, P_2, \ldots, P_n, P_{n+1}, \ldots, P_{n+m} \end{bmatrix}$$

$$= \begin{bmatrix} B \end{bmatrix} \begin{bmatrix} X, & Y \end{bmatrix},$$

where X is the $m \times n$ matrix whose (i,j) element (ith row and jth column) is x_{ij} $(i = 1,2, \ldots, m;\ j = 1,2, \ldots, n)$.

Y is the $m \times m$ matrix whose (i,j) element is $x_{i,j+n}$ $(i = 1,2, \ldots, m;\ j = 1,2, \ldots, m)$.

B is the matrix whose columns are $P_{q_1}, P_{q_2}, \ldots, P_{q_m}$. These are the basis vectors.

$$\text{Now} \qquad \begin{bmatrix} -I \end{bmatrix} = \begin{bmatrix} P_{n+1}, P_{n+2}, \ldots, P_{n+m} \end{bmatrix}$$

$$= \begin{bmatrix} P_{q_1}, P_{q_2}, \ldots, P_{q_m} \end{bmatrix} Y = BY.$$

Therefore, $Y = -B^{-1}$, and $\left[X, Y\right] = \left[X, -B^{-1}\right]$.

Also $$z_j = \sum_{i=1}^{m} c_{q_i} x_{ij} \qquad (j = 1,2, \ldots, n),$$

$$z_j - c_j = \sum_{i=1}^{m} c_{q_i} x_{ij} - c_j \qquad (j = 1,2, \ldots, n),$$

$$z_{j+n} = \sum_{i=1}^{m} c_{q_i} x_{i,n+j} \qquad (j = 1,2, \ldots, m),$$

and $$c_{n+j} = 0 \qquad (j = 1,2, \ldots, m).$$

Let $$c^{*\prime} \equiv (c_{q_1}, c_{q_2}, \ldots, c_{q_m}).$$

Let $$\surd' \equiv (z_1 - c_1, z_2 - c_2, \ldots, z_n - c_n) = c^{*\prime} X - c'$$

and $$-w' \equiv (z_{n+1}, z_{n+2}, \ldots, z_{n+m}) = -c^{*\prime} B^{-1}.$$

Let $$x^{*\prime} \equiv (\lambda_{q_1}, \lambda_{q_2}, \ldots, \lambda_{q_m}) \text{ and } P_o \equiv b.$$

Then $$b = \sum_{i=1}^{m} P_{q_i} \lambda_i = Bx^*.$$

For <u>any</u> feasible solution x of the first problem (not necessarily optimal) and <u>any</u> feasible solution w of its dual problem (not necessarily optimal) the following things are true.

Since $Ax \geq b$ and $w \geq 0$, then $w'Ax \geq w'b$. However, $w'A \leq c'$, so $c'x \geq w'Ax \geq w'b$ (since $x \geq 0$). This is true of <u>all</u> solutions x to the first problem and <u>all</u> solutions w to its dual. Therefore, min $c'x \geq$ max $w'b$. If, for some x and some w, $w'b = c'x$, then for this x and this w both $c'x$ and $w'b$ are optimal (minimal for $c'x$ and maximal for $w'b$). This is true because for these values min $c'x \leq c'x = w'b \leq$ max $w'b \leq$ min $c'x$, and the equality sign holds throughout. Hence min $c'x$ = max $w'b$.

If it can be shown that in an optimal tableau for the first problem, $w \equiv -(z_{n+1}, z_{n+2}, \ldots, z_{n+m})$ satisfies the conditions of the dual problem ($w \geq 0$ and $w'A \leq c'$) and that $w'b = c'x$, then it will have been proved that if a finite solution to the first problem exists, then a finite solution to the dual problem exists and min $c'x$ = max $w'b$. It will then follow that if a solution to the dual problem exists and is finite, then a solution to the original problem exists and is finite and also min $c'x$ = max $w'b$. In order to understand this, one needs only to rewrite the two problems as follows:

(1) In place of the first problem write the equivalent one, "maximize $(-c'x)$ where $-Ax \leq -b$ and $x \geq 0$."

(2) In place of the dual problem write "minimize $(-w'b)$ where $-w'A \geq -c'$ and $w' \geq 0$."

It should also be noted that min $(-w'b)$ = max $w'b$ and max $(-c'x)$ = min $c'x$.

Hence if the above conditions can be shown to hold, the theorem will be proved. In an optimal tableau for the first problem,

$$z_j - c_j \leq 0 \quad (j = 1, 2, \ldots, n + m).$$

(In the earlier text the optimal tableau was characterized by the fact that $z_j - c_j \geq 0$ for all values of j. The discussion there was for maximizing a functional. It is obvious that if the functional is to be minimized then $z_j - c_j \leq 0$ for all values of j.)

Therefore, $\surd \leq 0$ and $-w \leq 0$, hence $w \geq 0$.

Now
$$\begin{aligned} w'b &= (c^{*\prime}B^{-1})b = (c^{*\prime}B^{-1})(Bx^*) \\ &= c^{*\prime}x^* \\ &= \min c'x. \end{aligned}$$

Also $\surd' = c^{*\prime}X - c'$ and $-\surd \geq 0$ so that (since $A = BX$ and hence $B^{-1}A = B^{-1}BX = X$):

$$c' = c^{*\prime}X - \surd' = c^{*\prime}B^{-1}A - \surd' = w'A - \surd' \geq w'A.$$

This completes the proof.

To reduce a linear programming problem to the problem of finding <u>any</u> solution to a larger set of inequalities, write the problem in the form:

minimize $f(x) = c'x$ where $Ax \geq b$ and $x \geq 0$.

Then write down its dual and the added restriction that $w'b \geq c'x$. From this one may conclude that any method of solving inequalities may be applied to linear programming problems. However, such methods are usually more cumbersome than the methods developed specifically for the linear programming problem.

L. DUALITY IN OTHER FIELDS OF MATHEMATICS

The theorem just proved is called the duality theorem of linear programming because of its similarity to the duality principle in other fields of mathematics. One classic illustration is the duality principle of projective geometry. According to this principle, "all the theorems of projective geometry occur in pairs, each similar to the other, and, so to speak, identical in structure" [2]. A theorem may be constructed from its "dual theorem" by interchanging "dual elements." For example, if point and line are dual elements, then interchanging them in one theorem will give its dual theorem. Pascal's Theorem states that, "If the <u>vertices</u> of a hexagon <u>lie alternately on</u> two <u>straight lines</u>, the <u>points where opposite sides meet</u> are <u>collinear</u>." (See Reference [2], page 191.) Its dual, Brianchon's Theorem, states that, "If the <u>sides</u> of a hexagon <u>pass alternately through two points</u>, the <u>lines joining opposite vertices</u> are <u>concurrent</u>" [2].

The algebra of sets also has a duality principle. (See Reference [2], pages 108-112.) If, in any one of the laws of the algebra of sets, the expressions, "is a subset of," "the empty set," and "the union of," are interchanged respectively with the expressions, "contains as a subset," "the universal set," and "the intersection of," then another one of the laws results.

CHAPTER II
TWO METHODS OF SOLUTION
A. INTRODUCTION

In Chapter I a theoretical background was developed for attacking the linear programming problem. It will be the purpose of this chapter to go into two of the present methods of solution in some detail. The first method, which was introduced in Chapter I, is the simplex method. Its discussion will be followed by a discussion of the "dual" method of C. E. Lemke $\boxed{3}$.

B. FINDING AN INITIAL FEASIBLE EXTREME POINT SOLUTION

In Chapter I the theory behind the simplex method was presented without going into the details of computation. It was assumed that an initial feasible extreme point solution to the linear programming problem was readily available. Recall that such a solution $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ involves precisely m non-zero components, which are the coefficients of linearly independent vectors. (See Chapter I, Section G.) In many problems the set of constraints is stated in terms of inequalities of the form: $\sum_{j=1}^{n} a_{ij}\lambda_j \leq b_i$ (i = 1,2, ..., m) where all b_i's ≥ 0. This set of inequalities is converted to a set of equations by adding a new non-negative variable with coefficient unity to each inequality. The set then becomes $\sum_{j=1}^{n} a_{ij}\lambda_j + \lambda_{n+i} = b_i (i = 1,2, \ldots, m)$. Then in the matrix of coefficients the last m column vectors are the unit vectors of W (m-dimensional space). They are linearly independent. Also, since all b_i's are ≥ 0, then — denoting these last m columns by $P_{n+1}, P_{n+2}, \ldots, P_{n+m}$ — the following is true: $\sum_{i=1}^{m} b_i P_{n+i} = P_o$. P_o is the m-dimensional column vector whose ith component is b_i; that is, the corresponding row vector $P_o' = (b_1, b_2, \ldots, b_m)$. This is an initial solution, and all column vectors may easily be expressed in terms of the vectors $P_{n+1}, P_{n+2}, \ldots, P_{n+m}$. If the constraints are not so conveniently expressed, it is still possible to get an initial solution without having to determine a set of m linearly independent vectors from the set $P_1, P_2, \ldots, P_n$, where $P_j (j = 1,2, \ldots, n)$ is the m-dimensional column vector whose components are $(a_{1j}, a_{2j}, \ldots, a_{mj})$. Suppose that the set of constraints is expressed as equations; that is,

$$\sum_{j=1}^{n} a_{ij}\lambda_j = b_i (i = 1,2, \ldots, m).$$

A related problem is now considered. A new variable is attached to each equation. All the coefficients of the new variables are either 1 or -1. If $b_i \geq 0$, then $a_{i,n+i}$ (the coefficient of the variable attached to the ith equation) $= 1$. If $b_i < 0$, then $a_{i,n+i} = -1$. The set of equations may then be expressed $\sum_{j=1}^{n+m} a_{ij}\lambda_j = b_i$ where $P_{n+1}, P_{n+2}, \ldots, P_{n+m}$ are the unit vectors of W, some of them perhaps negative unit vectors. Obviously, $\sum_{i=1}^{m} |b_i| P_{n+i} = P_o$. Hence one is provided with a solution in terms of m linearly independent vectors (a "basic" solution) to this related problem. The question arises of the exact relationship between the two problems. Any solution to the original problem automatically provides a solution to the related problem (with $\lambda_{n+1} = \lambda_{n+2} = \ldots = \lambda_{n+m} = 0$). Assume that, by some manner, it is possible to construct a new functional, f', for the related problem, such that, for any solution involving only the variables of the original problem, f'= f (the functional of the original problem) and, for any solution involving at least one of the "attached" variables, f' is less than the minimum value of f (if f is to be maximized) or greater than the maximum value of f (if f is to be minimized). Maximum and minimum values of f exist since by appending $\sum_{j=1}^{n} \lambda_j \leq B$ to the original problem one can assure boundedness of Λ, the solution set. (See Chapter I, Section E.) Then, if any solution to the original problem exists, the optimal solution to the original problem will be the optimal solution to the related problem. If $f = \sum_{i=1}^{n} c_i\lambda_i$, then f' will satisfy the above conditions if $f' = f - M \sum_{i=n+1}^{n+m} \lambda_i$, where M is a value so large that its appearance in the functional makes it smaller than the minimum value of f. (Obviously, this is for maximizing f; if f is to be minimized, then $f' = \sum_{i=1}^{n} c_i\lambda_i + M \sum_{i=n+1}^{n+m} \lambda_i$.) There is no question of the existence of values of M large enough because (1) the solution set to the related problem can be made bounded (by the addition of $\sum_{i=1}^{n+m} \lambda_i \leq B'$) and hence will have a finite number of extreme points (some of them possibly involving B'), and (2) the functional (this is <u>not</u> the functional to be maximized or minimized but is the multiplier of M in f') $\sum_{i=n+1}^{n+m} \lambda_i$ has a finite minimum over all extreme points involving at least one positive $\lambda_i (i > n)$. M then has some value such that $M(\sum_{i=n+1}^{n+m} \lambda_i)_{min}$ is larger than

$$(\sum_{i=1}^{n} c_i\lambda_i)_{max} - (\sum_{i=1}^{n} c_i\lambda_i)_{min}.$$

There is always a solution to this related problem; and if a solution exists to the original problem, the optimal solution to the original problem will be the optimal solution to the related problem.

It should be noted that, by the methods discussed in the preceding paragraph, it is possible to find a basis of W and a solution in terms of it. This solution does not necessarily involve precisely m non-zero coordinates. When it does not, it is a "degenerate" solution; and this will be resolved now.

C. RESOLUTION OF DEGENERACY

The next difficulty to be resolved in the simplex method is that of "degeneracy." In using the simplex method one starts with a solution in terms of a set of m linearly independent vectors in W, that is, in terms of a basis of W. The process is carried out by replacing one vector P_r in the set by a vector P_k not in the set and obtaining a new solution in terms of the new set of vectors. This is done as follows: if $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is the original solution and $\lambda_i > 0$ $(i = 1, 2, \ldots, m)$ and $\lambda_i = 0$ $(i = m + 1, m + 2, \ldots, n)$, then $\sum_{i=1}^{m} \lambda_i P_i = P_o$ and $P_1, P_2, \ldots, P_m$ are linearly independent vectors forming a basis of W, that is, m-dimensional space. Hence any m-dimensional vector may be expressed in terms of them. In particular $P_k = \sum_{i=1}^{m} x_{ik} P_i$.

Therefore,

$$P_o = \sum_{i=1}^{m} \lambda_i P_i + \theta P_k - \theta P_k$$

$$= \sum_{i=1}^{m} \lambda_i P_i - \theta \sum_{i=1}^{m} x_{ik} P_i + \theta P_k$$

$$= \sum_{i=1}^{m} (\lambda_i - \theta x_{ik}) P_i + \theta P_k .$$

(See Chapter I, Section G, Equation 1-1.) If $\theta > 0$ and all $(\lambda_i - \theta x_{ik}) \geq 0$, then this provides a solution. If θ is taken to be the smallest positive value of λ_i/x_{ik} and there is only one value of λ_i/x_{ik} which is equal to θ, then the new solution will have m positive coordinates; they will be the coefficients of linearly independent P_i's. Then the process may be repeated. (Recall that P_k is chosen to increase the functional if possible. If z_o is the value of the functional determined by the first solution, then the value determined by the solution obtained when P_k is brought into the basis is $z_o^* = z_o + \theta(c_k - z_k)$, where $z_k = \sum_{i=1}^{m} x_{ik} c_k$. (See Chapter I, Section G, Equation 1-2.) Two assumptions have been made without being justified: (1) that the original solution has precisely m non-zero coordinates and (2) that at each stage there exists only one smallest value λ_i/x_{ik}.

Again a related problem is considered. The related problem will satisfy the above assumptions and will yield the solutions to the original problem. P_o in the original problem is replaced by $(P_o + \sum_{j=1}^{n} e^j P_j)$, where e is unspecified but positive. The rest of the problem remains unchanged. Now if $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m, 0, \ldots, 0)$ is a solution to the original problem in terms of m linearly independent vectors, $P_1, P_2, \ldots, P_m$, then

$$\sum_{i=1}^{m} (\lambda_i + \sum_{j=1}^{n} x_{ij} e^j) P_i = P_o + \sum_{j=1}^{n} e^j P_j .$$

This is true because

$$P_j = \sum_{i=1}^{m} x_{ij} P_i ,$$

$$e^j P_j = e^j \sum_{i=1}^{m} x_{ij} P_i ,$$

$$\sum_{j=1}^{n} e^j P_j = \sum_{j=1}^{n} e^j \sum_{i=1}^{m} x_{ij} P_i ,$$

$$\sum_{j=1}^{n} e^j P_j = \sum_{i=1}^{m} (\sum_{j=1}^{n} e^j x_{ij}) P_i .$$

Also $$P_o = \sum_{i=1}^{m} \lambda_i P_i .$$

Hence $\quad P_o + \sum_{j=1}^{n} e^j P_j = \sum_{i=1}^{m} (\lambda_i + \sum_{j=1}^{n} e^j x_{ij}) P_i.$

Now, if $f_e(\lambda)$ denotes the value of the functional for a solution to the related problem where λ is a solution to the original problem, then

$$f_e(\lambda) = \sum_{i=1}^{m} c_i\lambda_i + \sum_{i=1}^{m} c_i \left(\sum_{j=1}^{n} e^j x_{ij} \right)$$

$$= \sum_{i=1}^{m} c_i\lambda_i + \sum_{j=1}^{n} e^j \sum_{i=1}^{m} x_{ij}c_i$$

$$= \sum_{i=1}^{m} c_i\lambda_i + \sum_{j=1}^{n} e^j z_j$$

$$= z_o + \sum_{j=1}^{n} e^j z_j$$

where z_o is the value of the functional determined by λ.

Consider now the polynomial $\sum_{j=1}^{n} e^j x_{ij}$. Any polynomial in e is dominated by its lowest power if e is sufficiently small. Consider the two polynomials in e:

$$a(e) = a_1e + a_2e^2 + \dots + a_qe^q$$

$$b(e) = b_1e + b_2e^2 + \dots + b_qe^q$$

where q is some finite number. Then

$$a(e) - b(e) = (a_1 - b_1)e + (a_2 - b_2)e^2 + \ldots + (a_q - b_q)e^q.$$

Let $(a_s - b_s)$ be the first non-zero coefficient in this last line. Then

$$a(e) - b(e) = (a_s - b_s)(e^s)(1 + c_{s+1}e^1 + c_{s+2}e^2 + \ldots + c_q e^{q-s})$$

where

$$c_i \equiv \frac{a_i - b_i}{a_s - b_s}.$$

Now let

$$C(e) \equiv (1 + c_{s+1}e^1 + c_{s+2}e^2 + \ldots + c_q e^{q-s}).$$

Then

$$a(e) - b(e) = (a_s - b_s)(e^s)C(e).$$

Now the limit of $C(e)$ as e approaches zero from the positive side is unity. Hence there exists some positive number e_o such that if $0 \leq e \leq e_o$, then $C(e) > 0$. Therefore, for $0 < e \leq e_o$ the sign of $\left[a(e) - b(e)\right]$ is the same as the sign of $(a_s - b_s)$, and

$$a(e) > b(e) \quad \text{if and only if} \quad a_s > b_s.$$

This result will be used later.

Now the first assumption which was made and not justified was that the first solution had exactly m non-zero coordinates. It was shown in the preceding section of this chapter how it is possible to get a basis of W, but there was no guarantee that, based on these vectors, all $\lambda_i > 0$. In the corresponding e-problem, however, each component in the solution will be greater than zero. Consider that $P_1, P_2, \ldots, P_m$ are the original basis vectors and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ the original solution. Then a solution to the corresponding e-problem is

$$\mu = (\mu_1, \mu_2, \ldots, \mu_m, 0, \ldots, 0)$$

where

$$\mu_i = \lambda_i + \sum_{j=1}^{n} e^j x_{ij} \qquad (i = 1,2, \ldots, m).$$

Also

$$\lambda_i + \sum_{j=1}^{n} e^j x_{ij} = \lambda_i + e^i + \sum_{j=m+1}^{n} e^j x_{ij}.$$

This last statement is true because

$$P_i = (0)P_1 + (0)P_2 + \ldots + (1)P_i + \ldots + (0)P_m.$$

Hence $x_{ij} = 0 \quad (j = 1,2, \ldots, m;\ j \neq i)$ and $x_{ii} = 1$. Since in the expression, $e^i + \sum_{j=m+1}^{n} e^j x_{ij}$, e^i is the smallest power of e, then for e sufficiently small (since e is positive) $(e^i + \sum_{j=m+1}^{n} e^j x_{ij})$ is positive. Also all $\lambda_i \geq 0$; therefore,

all $\quad (\lambda_i + e^i + \sum_{j=m+1}^{n} e^j x_{ij}) = (\lambda_i + \sum_{j=1}^{n} e^j x_{ij}) > 0$

$(i = 1,2, \ldots, m)$.

Hence the first solution to the e-problem has precisely m non-zero components.

The next assumption was that at each stage there exists only one smallest value λ_i/x_{ik} where $x_{ik} > 0$. In the e-problem $(\lambda_i + \sum_{j=1}^{n} e^j x_{ij}) \div x_{ik}$ corresponds to λ_i/x_{ik} in the original problem. If λ_r/x_{rk} and λ_s/x_{sk} are two distinct values of λ_i/x_{ik} and $\lambda_r/x_{rk} > \lambda_s/x_{sk}$, then obviously there exist values of e small enough such that

$$(\lambda_r + \sum_{j=1}^{n} e^j x_{rj}) \div x_{rk} > (\lambda_s + \sum_{j=1}^{n} e^j x_{sj}) \div x_{sk}.$$

If $\lambda_r/x_{rk} = \lambda_s/x_{sk}$ then there exist only a finite number of values of e such that

$$(\lambda_r + \sum_{j=1}^{n} e^j x_{rj}) \div x_{rk} = (\lambda_s + \sum_{j=1}^{n} e^j x_{sj}) \div x_{sk}.$$

There exist only a finite number of solutions to the equation:

$$(\sum_{j=1}^{n} e^j x_{rj}) \div x_{rk} - (\sum_{j=1}^{n} e^j x_{sj}) \div x_{sk} = 0$$

unless all the coefficients are identically zero. This is not true in this case because the coefficients of e^r and e^s are $1/x_{rk}$ and $-1/x_{sk}$ respectively. Hence if there is a set of values λ_i/x_{ik} $(i = 1,2, \ldots, q)$ which are all equal, there exist values of e smaller than any preassigned value such that in the set $(\lambda_i + \sum_{j=1}^{n} e^j x_{ij}) \div x_{ik}$ there exists a unique minimum. Therefore, in the e-problem, the simplex process replaces only one vector of the old basis by the new vector being brought into the basis; at every stage the solution will involve <u>precisely</u> m non-zero components.

An optimal solution to the e-problem automatically provides an optimal solution to the original problem. Assume that this is not true; that is, that

$$f_e(\lambda) = z_o + \sum_{j=1}^{n} e^j z_j$$

is an optimal value of the functional for the e-problem, but z_o is not the finite maximum of $f(\lambda)$. Then there exists a solution λ' to the original problem such that $z_o' > z_o$ where z_o' is the value of the functional determined by λ'. If all z_j''s are based on the solution λ' (λ' is necessarily an extreme point and hence may be expressed in terms of m linearly independent vectors), then $f_e(\lambda') = z_o' + \sum_{j=1}^{n} e^j z_j'$. Since $f_e(\lambda) = z_o + \sum_{j=1}^{n} e^j z_j$ is the optimal value of f_e, then

$$z_o + \sum_{j=1}^{n} e^j z_j \geq z_o' + \sum_{j=1}^{n} e^j z_j'.$$

If e is sufficiently small, $z_o \geq z_o'$, but $z_o < z_o'$, which is a contradiction. Therefore, an optimal solution to the e-problem automatically provides an optimal solution to the original problem, and one may deal with this non-degenerate e-problem. Actually, as will be demonstrated, it is not ever necessary to specify the value of e.

D. DEVELOPMENT OF THE COMPUTATIONAL PROCEDURE

The first step in the actual simplex procedure is to obtain a basis of W and a solution in terms of it to the original problem. Then, in terms of this basis, x_{ij} for all values of i and j are calculated, as well as all values of z_j and $z_j - c_j$. All of these calculations are tabulated in the first "tableau" as follows:

		Column Vectors								
Unit Values	Basis Elements	P_o	P_1	...	P_j	...	P_k	...	P_n
c_1	P_1	λ_1	x_{11}	...	x_{1j}	...	x_{1k}	...	x_{1n}
...	...	...	...	...	...	...	...	...	...
c_i	P_i	λ_i	x_{i1}	...	x_{ij}	...	x_{ik}	...	x_{in}
...	...	...	...	...	...	...	...	...	...
c_r	P_r	λ_r	x_{r1}	...	x_{rj}	...	x_{rk}	...	x_{rn}
...	...	...	...	...	...	...	...	...	...
c_m	P_m	λ_m	x_{m1}	...	x_{mj}	...	x_{mk}	...	x_{mn}
		$f(\lambda)$	z_1	...	z_j	...	z_k	...	z_n
	Net Differences	$f(\lambda)$	z_1-c_1	...	z_j-c_j	...	z_k-c_k	...	z_n-c_n

In the section headed "Column Vectors" each column represents a vector, and its elements are its coefficients in terms of the basis vectors. The element in the ith row is the coefficient of the ith vector, as indicated to the left of each row in the column headed "Basis Elements."

The next step is to examine the $(z_j - c_j)$ row. If case I or case II is found to be true, the problem is solved. (See Chapter I, Section H.) If case III is true, a new "tableau" must be calculated. A new vector P_k must be chosen to enter the new basis and replace some vector P_r. Since $z'_o = z_o + \theta(c_k - z_k)$ and a maximum value of the functional is being sought, the logical choice for P_k is the vector whose net difference $(c_k - z_k)$ is largest, hence whose $(z_k - c_k)$ is most negative. The next step is to determine P_r. Each λ_i is divided by its corresponding x_{ik}. If this set of values λ_i/x_{ik} has a unique minimum among the ones such that $x_{ik} > 0$, then the vector corresponding to this value is P_r. Suppose, however, that there are several values (such that $x_{ik} > 0$) of λ_i/x_{ik} which are equal and smaller than all the others. As the reader has already seen, their corresponding values in the e-problem are not equal. If each element in the ith row is divided by x_{ik}, then these values will be the coefficients of the polynomial $(\lambda_i + \sum_{j=1}^{n} e^j x_{ij}) \div x_{ik}$ in ascending powers of e. It has been assumed that the λ_i/x_{ik} are equal. The second elements in each row, that is, the x_{i1}/x_{ik} are compared. If they have a unique smallest value, its vector is P_r. If not, the vectors corresponding to all except these smallest values are eliminated and then the third elements are compared, and the process is continued until a unique smallest value is obtained. Such a value must exist in the first m elements. When corresponding elements are compared, then vectors corresponding to any except the smallest values may be eliminated, but the matrix of the first m columns in the tableau is the m x m unit matrix. Hence if P_1 has one of the minimum λ_i/x_{ik}, it will be eliminated the first time because $x_{11}/x_{1k} = 1/x_{1k} > 0$ and $x_{i1}/x_{ik} = 0$ $(i = 2,3, \ldots, m)$. Similarly, if P_2 has a minimum λ_i/x_{ik}, it will be eliminated on the second comparison. In this first tableau the value λ_i/x_{ik} which is a minimum and has the largest value of i will determine P_r.

Now P_k and P_r have been determined, and a new tableau based on the new basis must be calculated. If x'_{ij} is the new value based on the new basis, then:

$$P_j = \sum_{\substack{i=1 \\ i \neq r}}^{m} x'_{ij} P_i + x'_{kj} P_k \quad ; \quad z'_j = \sum_{\substack{i=1 \\ i \neq r}}^{m} x'_{ij} c_i + x'_{kj} c_k$$

$(j = 1,2, \ldots, n)$.

$$P_k = x_{rk}P_r + \sum_{\substack{i=1 \\ i \neq r}}^{m} x_{ik}P_i.$$

Also $$P_j = \sum_{i=1}^{m} x_{ij}P_i \qquad (j = 1,2, \ldots, n).$$

(2-1) $$P_j = \sum_{\substack{i=1 \\ i \neq r}}^{m} x_{ij}P_i + x_{rj}P_r \qquad (j = 1,2, \ldots, n).$$

But, from above, $$x_{rk}P_r = P_k - \sum_{\substack{i=1 \\ i \neq r}}^{m} x_{ik}P_i;$$

(2-2) $$P_r = \frac{1}{x_{rk}} P_k - \sum_{\substack{i=1 \\ i \neq r}}^{m} \frac{x_{ik}}{x_{rk}} P_i.$$

Substituting this expression for P_r in the expression for P_j gives:

$$P_j = \sum_{\substack{i=1 \\ i \neq r}}^{m} x_{ij}P_i + \frac{x_{rj}}{x_{rk}} P_k - x_{rj} \sum_{\substack{i=1 \\ i \neq r}}^{m} \frac{x_{ik}}{x_{rk}} P_i$$

$(j = 1,2, \ldots, n)$;

(2-3) $$P_j = \sum_{\substack{i=1 \\ i \neq r}}^{m} (x_{ij} - x_{rj} \frac{x_{ik}}{x_{rk}})P_i + \frac{x_{rj}}{x_{rk}} P_k$$

$(j = 1,2, \ldots, n)$.

E. THE SIMPLEX ALGORITHM

Since the expression for P_j in terms of the new basis is unique, the following algorithm is provided for calculating the new tableau:

(2-4) $$x'_{ij} = x_{ij} - \frac{x_{rj}}{x_{rk}} x_{ik}$$

$(i = 1,2, \ldots, m;\ i \neq r;\ j = 1,2, \ldots, n)$;

(2-5) $$x'_{kj} = \frac{x_{rj}}{x_{rk}} \qquad (j = 1,2, \ldots, n).$$

Now $\sum_{i=1}^{m} (\lambda_i + \sum_{j=1}^{n} e^j x_{ij} - \theta x_{ik})P_i + \theta P_k = P_o + \sum_{j=1}^{n} e^j P_j$

and $$\theta = \frac{\lambda_r + \sum_{j=1}^{n} e^j x_{rj}}{x_{rk}};$$

$$\therefore \sum_{i=1}^{m} \left(\lambda_i + \sum_{j=1}^{n} e^j x_{ij} - \frac{\lambda_r + \sum_{j=1}^{n} e^j x_{rj}}{x_{rk}} x_{ik}\right)P_i + \frac{\lambda_r + \sum_{j=1}^{n} e^j x_{rj}}{x_{rk}} P_k$$

$$= \sum_{i=1}^{m} \left[\lambda_i - \lambda_r \frac{x_{ik}}{x_{rk}} + \sum_{j=1}^{n} e^j \left(x_{ij} - \frac{x_{rj}}{x_{rk}} x_{ik}\right)\right] P_i + \left(\frac{\lambda_r}{x_{rk}} + \sum_{j=1}^{n} \frac{x_{rj}}{x_{rk}} e^j\right)P_k$$

$$= \sum_{i=1}^{m} (\lambda'_i + \sum_{j=1}^{n} e^j x'_{ij})P_i + (\lambda'_k + \sum_{j=1}^{n} e^j x'_{kj})P_k,$$

and if λ_i' is denoted by x_{io}', then the same algorithm may be used for computing it as for computing the other x_{ij}'. Also, as may be easily proved, this same algorithm may be used to compute the $z_j' - c_j = (z_j - c_j) - \frac{x_{rj}}{x_{rk}}(z_k - c_k)$.

When the new tableau is computed, it is again examined to see whether case I, case II, or case III applies. If it is case III, the process is repeated. Since there are only a finite number of sets of m linearly independent vectors and since, in the e-problem, each solution provides a value of the functional larger than values of all preceding solutions, there is no possibility of returning to the same basis and "cycling"; hence case I or case II will eventually be reached. Case I will be ruled out by appending the equation $\sum_{i=1}^{n+1} \lambda_i = B$. If the problem has no maximum, then at some stage some λ_i other than λ_{n+1} will involve B, and the problem will be solved. (See Chapter I, Section E.)

F. DETERMINING ALL OPTIMAL SOLUTIONS

The problem now arises of determining <u>all</u> optimal solutions when one optimal solution has been found. If the solution set is bounded, the set of optimal solutions is the set of all extreme points which yield this optimal value of the functional and the convex set determined by them. Denote by $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m, 0, \ldots, 0)$ the optimal solution reached through the simplex method, so that

$$\lambda_1 P_1 + \lambda_2 P_2 + \ldots + \lambda_m P_m = P_o.$$

Let $$\theta_k P_k + \theta_q P_q + \ldots + \theta_p P_p = P_o$$

denote any other solution. Now

$$P_k = \sum_{i=1}^{m} x_{ik} P_i \text{ and } P_q = \sum_{i=1}^{m} x_{iq} P_i \text{ and } \ldots \text{ and } P_p = \sum_{i=1}^{m} x_{ip} P_i.$$

Therefore,

$$\theta_k \sum_{i=1}^{m} x_{ik} P_i + \theta_q \sum_{i=1}^{m} x_{iq} P_i + \ldots + \theta_p \sum_{i=1}^{m} x_{ip} P_i = P_o.$$

However, the expression for P_o in terms of $P_1, P_2, \ldots, P_m$ is unique. Hence,

$$\theta_k x_{1k} + \theta_q x_{1q} + \ldots + \theta_p x_{1p} = \lambda_1$$

$$\theta_k x_{2k} + \theta_q x_{2q} + \ldots + \theta_p x_{2p} = \lambda_2$$

. . . .

. . . .

. . . .

$$\theta_k x_{mk} + \theta_q x_{mq} + \ldots + \theta_p x_{mp} = \lambda_m.$$

Multiplying each equation for λ_i by c_i gives:

$$\theta_k c_1 x_{1k} + \theta_q c_1 x_{1q} + \ldots + \theta_p c_1 x_{1p} = c_1\lambda_1$$

$$\theta_k c_2 x_{2k} + \theta_q c_2 x_{2q} + \ldots + \theta_p c_2 x_{2p} = c_2\lambda_2$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\theta_k c_m x_{mk} + \theta_q c_m x_{mq} + \ldots + \theta_p c_m x_{mp} = c_m\lambda_m.$$

Adding up the equations on both sides gives

$$\theta_k z_k + \theta_q z_q + \ldots + \theta_p z_p = z_o,$$

where z_o is the value of the functional determined by λ and the z_j's are based on $P_1, P_2, \ldots, P_m$. If z_o' is the value of the functional determined by $(\theta_k, \theta_q, \ldots, \theta_p)$, then

$$\theta_k c_k + \theta_q c_q + \ldots + \theta_p c_p = z_o'.$$

Subtracting z_o from z_o' gives

$$z_o' - z_o = \theta_k(c_k - z_k) + \theta_q(c_q - z_q) + \ldots + \theta_p(c_p - z_p),$$

$$\text{(2-6)} \quad z_o' = z_o + \theta_k(c_k - z_k) + \theta_q(c_q - z_q) + \ldots + \theta_p(c_p - z_p).$$

Now because all $\theta_j > 0$ and all $(c_j - z_j) \leq 0$, the only vectors which may be entered into the basis with positive coefficients are vectors for which $(c_j - z_j) = 0$. Each of these vectors is entered into the basis, and after each such entry the new basis is "followed up"; that is, each other vector whose $(c_j - z_j) = 0$ is entered to form still another basis. When all possible combinations of m linearly independent vectors of the vectors, $P_1, P_2, \ldots, P_m$ and the vectors P_j for which $(c_j - z_j) = 0$, have been considered as bases, then the extreme points of the optimal set will have been found. The assumption has been made that the optimal set is bounded, that is, has a finite number of extreme points. This may be assured by appending the equation $\sum_{i=1}^{n+1} \lambda_i = B$. If the optimal solution set is not bounded, one of its extreme points will involve B.

G. AN EXAMPLE SOLVED BY THE SIMPLEX METHOD

A simple example and its solution will be presented.

Problem: find x and y such that:

$$
\begin{array}{ll}
(1) & -y - 3x \leq 6 \\
(2) & y + 3x \leq 15 \\
(3) & y - x \leq 2 \\
(4) & y + x \leq 7 \\
(5) & -18y - 2x \leq 27 \\
(6) & x \geq 0 \\
(7) & y \geq 0
\end{array}
$$

and such that $z = y + 2x$ is a maximum. The problem will first be solved by the simplex method, and then it will be solved geometrically.

The first step is to convert the inequalities to equations by adding a new non-negative variable to each inequality. These new variables will be denoted by λ_1, λ_2, λ_3, λ_4, λ_5, and y and x will henceforth be denoted by λ_6 and λ_7 respectively. The equations then are:

$$\begin{aligned}
\lambda_1 - \lambda_6 - 3\lambda_7 &= 6 \\
\lambda_2 + \lambda_6 + 3\lambda_7 &= 15 \\
\lambda_3 + \lambda_6 - \lambda_7 &= 2 \\
\lambda_4 + \lambda_6 + \lambda_7 &= 7 \\
\lambda_5 - 18\lambda_6 - 2\lambda_7 &= 27.
\end{aligned}$$

The matrix of coefficients then is:

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_0
1	0	0	0	0	-1	-3	6
0	1	0	0	0	1	3	15
0	0	1	0	0	1	-1	2
0	0	0	1	0	1	1	7
0	0	0	0	1	-18	-2	27.

If one uses $P_1, P_2, \ldots, P_5$ as the first basis, he may easily calculate the first tableau. Recall that $z_j = \sum_{i=1}^{m} x_{ij}c_i$.

On page 59 are the three tableaus in the problem. Notice that, in the first, P_7 has the most negative value of $z_j - c_j$ and hence becomes P_k. The positive values of x_{i7} are $x_{27} = 3$, $x_{47} = 1$, and then $\lambda_2/x_{27} = 15/3 = 5$ and $\lambda_4/x_{47} = 7$. Hence $\theta = 5$ and P_r is P_2. The second tableau may now be calculated. The elements x'_{ij} in it (where i is the subscript of the basis vector and j of the column vector) are calculated from the elements in the first tableau by the algorithm:

$$x'_{ij} = x_{ij} - \frac{x_{rj}}{x_{rk}} x_{ik}$$

and

$$x'_{kj} = \frac{x_{rj}}{x_{rk}} .$$

Since the x'_{kj} are used in computing the other elements, it is convenient to calculate them first. A check on the computation is to calculate $z'_j - c_j$ by the algorithm and then by $z'_j = \Sigma\, x'_{ij} c_i$.

In the second tableau the only negative $z_j - c_j$ is $z_6 - c_6 = -1/3$. Hence P_6 becomes the new P_k and P_4 the new P_r. When the third tableau is completed, all $z_j - c_j$ are non-negative; hence, the solution is optimal. Since $z_j - c_j = 0$ only for the basis vectors, there are no other optimal solutions.

Tableau I — Column Vectors

Unit Values	Basis Elements	P_0	P_1	P_2	P_3	P_4	P_5	P_6	P_7
0	P_1	6	1	0	0	0	0	-1	-3
0	P_2	15	0	1	0	0	0	1	3
0	P_3	2	0	0	1	0	0	1	-1
0	P_4	7	0	0	0	1	0	1	1
0	P_5	27	0	0	0	0	1	-18	-2
	z_j	0	0	0	0	0	0	0	0
Net Differences	$z_j - c_j$	0	0	0	0	0	0	-1	-2

Tableau II — Column Vectors

Unit Values	Basis Elements	P_0	P_1	P_2	P_3	P_4	P_5	P_6	P_7
0	P_1	21	1	1	0	0	0	0	0
2	P_7	5	0	1/3	0	0	0	1/3	1
0	P_3	7	0	1/3	1	0	0	4/3	0
0	P_4	2	0	-1/3	0	1	0	2/3	0
0	P_5	37	0	2/3	0	0	1	-52/3	0
	z_j	10	0	2/3	0	0	0	2/3	2
Net Differences	$z_j - c_j$	10	0	2/3	0	0	0	-1/3	0

Tableau III — Column Vectors

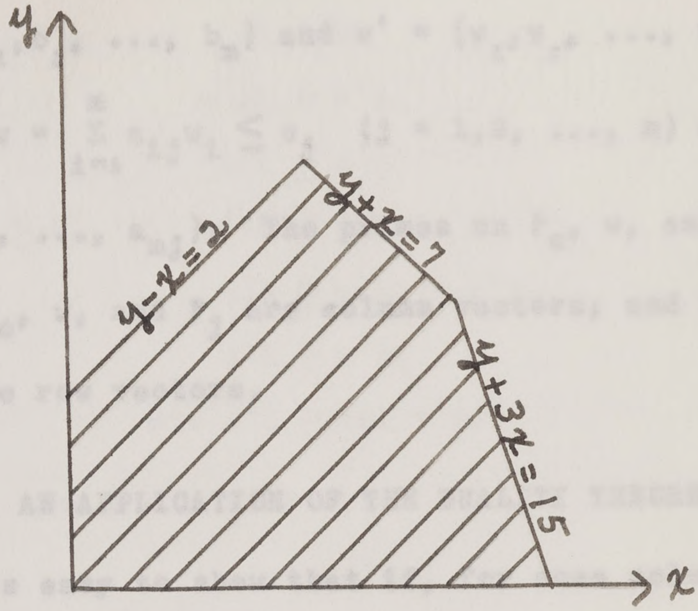
Unit Values	Basis Elements	P_0	P_1	P_2	P_3	P_4	P_5	P_6	P_7
0	P_1	21	1	1	0	0	0	0	0
2	P_7	4	0	1/2	0	-1/2	0	0	1
0	P_3	3	0	1	1	-2	0	0	0
1	P_6	3	0	-1/2	0	3/2	0	1	0
0	P_5	89	0	-8	0	26	1	0	0
	z_j	11	0	1/2	0	1/2	0	1	2
Net Differences	$z_j - c_j$	11	0	1/2	0	1/2	0	0	0

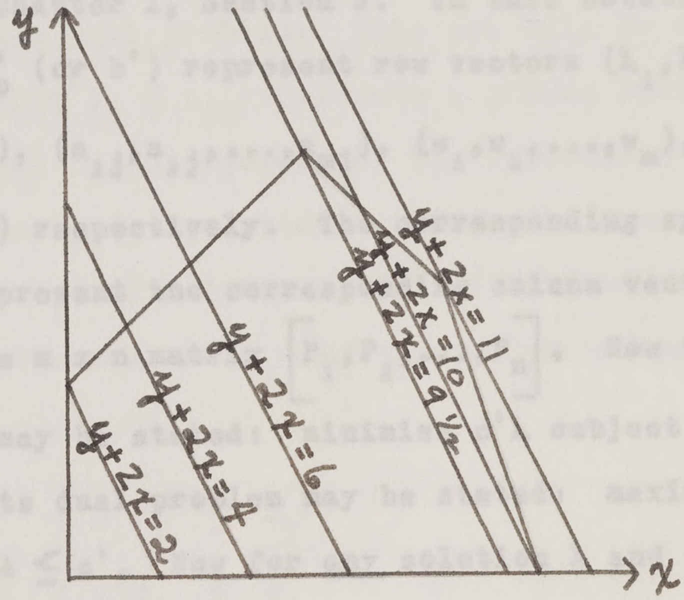
H. THE GEOMETRIC SOLUTION

To solve the problem geometrically, one starts by drawing the two-dimensional graph on page 61. The shaded area is the convex set determined by the inequalities. The first extreme point solution corresponds to the origin. The next extreme point solution, to (5,0); and the final one, to (4,3). The family of lines $y + 2x = z$ is the family of lines with slope -2. Maximizing the form $y + 2x$ over the convex set determined by the inequalities corresponds to finding that point (or points) of the convex set which lies on the line (of the family $y + 2x = z$) which has the largest possible y-intercept and still contains at least one point of the convex set.

I. STATEMENT OF THE DUAL PROBLEM FROM THE LINEAR PROGRAMMING PROBLEM

C. E. Lemke has devised a method of solving a linear programming problem based on the duality theorem, a proof of which was presented in Chapter I, Section K. Assume that the linear programming problem is to minimize $z = \sum_{j=1}^{n} \lambda_j c_j$ subject to $P_o = \sum_{j=1}^{n} \lambda_j P_j$ and $\lambda_j \geq 0$ $(j = 1,2,\ldots,n)$. Then its dual problem may be stated: to maximize $P_o' w = \sum_{i=1}^{m} b_i w_i$,


y
y−x=2
y+x=7
y+3x=15
x


y
y+2x=11
y+2x=10
y+2x=9½
y+2x=6
y+2x=4
y+2x=2
x

where $P_o' = (b_1, b_2, \ldots, b_m)$ and $w' = (w_1, w_2, \ldots, w_m)$, subject to $P_j'w = \sum_{i=1}^{m} a_{ij}w_i \leq c_j$ $(j = 1, 2, \ldots, n)$ and where $P_j' = (a_{1j}, a_{2j}, \ldots, a_{mj})$. The primes on P_o, w, and P_j indicate that P_o, w, and P_j are column vectors; and hence, P_o', w', and P_j' are row vectors.

J. AN APPLICATION OF THE DUALITY THEOREM

It is easy to show that if, for some solution λ to the programming problem and some solution w to its dual problem, $P_o'w = \sum_{j=1}^{n} \lambda_j c_j$, then w and λ are optimal solutions to the two problems. To do this, return to the vector notation explained in Chapter I, Section J. In this notation λ', c', P_j', w', and P_o' (or b') represent row vectors $(\lambda_1, \lambda_2, \ldots, \lambda_n)$, $(c_1, c_2, \ldots, c_n)$, $(a_{1j}, a_{2j}, \ldots, a_{mj})$, $(w_1, w_2, \ldots, w_m)$, and $(b_1, b_2, \ldots, b_m)$ respectively. The corresponding symbols without primes represent the corresponding column vectors. A represents the $m \times n$ matrix $\begin{bmatrix} P_1, P_2, \ldots, P_n \end{bmatrix}$. Now the programming problem may be stated: minimize $c'\lambda$ subject to $A\lambda = b$ and $\lambda \geq 0$. Its dual problem may be stated: maximize $w'b$ subject to $w'A \leq c'$. Now for <u>any</u> solution λ and <u>any</u> solution w, not necessarily optimal,

$$A\lambda = b.$$

Hence $$w'A\lambda = w'b.$$

Also $$w'A \leq c'.$$

Then, since $\lambda \geq 0$, $$w'A\lambda \leq c'\lambda.$$

Therefore
$$c'\lambda \geq w'A\lambda = w'b,$$
$$c'\lambda \geq w'b,$$
$$\min c'\lambda \geq \max w'b.$$

Hence, if for some λ and some w,

$$c'\lambda = w'b,$$

this λ and this w <u>are</u> optimal solutions to both problems.

K. THE SOLUTION SET OF THE DUAL PROBLEM

The next step is to examine the set of "feasible" solutions to the dual problem. This set will be called Ω. Each inequality $w'P_j \leq c_j$ represents a closed half space in W (m-dimensional space) bounded by the hyperplane $w'P_j = c_j$. Ω is then the convex polyhedron bounded by the n hyperplanes, $w'P_j = c_j$ $(j = 1,2,\ldots,n)$. It is assumed that $n > m$. Ω is convex because, if w_1 and w_2 are two points in Ω, then

$$\left[\begin{array}{l} w_1'P_j \leq c_j \\ w_2'P_j \leq c_j \end{array}\right. \qquad (j = 1,2,\ldots,n).$$

Hence if $0 \leq \theta \leq 1$,

$$\left[\theta w_1' + (1 - \theta)w_2'\right]P_j = \theta w_1'P_j + (1 - \theta)w_2'P_j$$
$$\leq \theta c_j + (1 - \theta)c_j = c_j$$

$(j = 1,2, \ldots, n)$.

Ω is a polyhedron because its extreme points are among those points in W which are the intersection of m linearly independent hyperplanes of the n hyperplanes. Obviously some of these points of intersection of m linearly independent hyperplanes may not lie in Ω. Those points which do are called "extreme point solutions" of the dual problem. Now it has been proved (Chapter I, Section C) that a linear functional defined over a convex polyhedron takes on its optimal value at an extreme point of the convex polyhedron. Hence, if, for some w and some λ, $w'b = c'\lambda$, then w is an extreme point solution of Ω.

L. THEORETICAL DEVELOPMENT OF THE DUAL METHOD

In presenting his dual method Lemke makes several assumptions. They are:

(1) that every extreme point solution of Ω represents the intersection of precisely m hyperplanes, that is, that if for some set, P_{r_1}, P_{r_2}, ..., P_{r_m}, of m linearly

independent vectors from the set, $P_1, P_2, \ldots, P_n$, the unique point satisfying $w'P_{r_i} = c_{r_i}$ $(i = 1,2, \ldots, m)$ is denoted by w_o, then, if w_o is in Ω, for any vector P_j of the set, $P_1, P_2, \ldots, P_n$, and not in the set, $P_{r_1}, P_{r_2}, \ldots, P_{r_m}$, of vectors, $w_o'P_j \neq c_j$ but $w_o'P_j < c_j$ (this restriction can be removed by a procedure similar to the resolution of degeneracy in the simplex method);

(2) that Ω has at least one extreme point, that is, that there exists at least one set of linearly independent vectors, $P_{r_1}, P_{r_2}, \ldots, P_{r_m}$, among the set, $P_1, P_2, \ldots, P_n$, such that $w_o'P_j \leq c_j$ $(j = 1,2, \ldots, n)$ where equality holds if (and, from assumption 1, only if) $j = r_1, r_2, \ldots, r_m$;

(3) that this set has been determined;

(4) that there exists a finite maximum value of $w'b$.

Proceeding from these assumptions, Lemke writes down the set of linearly independent vectors, $P_{r_1}, P_{r_2}, \ldots, P_{r_m}$, described in assumption 2. He relabels this set of vectors as $a_1, a_2, \ldots, a_m$, where $a_i = (a_{1r_i}, a_{2r_i}, \ldots, a_{mr_i})$. These vectors form a basis of W (m-dimensional space). Hence the vectors $P_o, P_1, \ldots, P_n$ may be expressed in terms of them.

If $\quad P_j = \mu_1 a_1 + \mu_2 a_2 + \dots + \mu_m a_m$

and $\quad \mu' = (\mu_1, \mu_2, \dots, \mu_m)$,

then $\quad P_j = \begin{bmatrix} a_1, a_2, \dots, a_m \end{bmatrix} \mu.$

Now if $\begin{bmatrix} a^1 \\ a^2 \\ \vdots \\ a^m \end{bmatrix}$ is the inverse of $\begin{bmatrix} a_1, a_2, \dots, a_m \end{bmatrix}$ where

$a^1, a^2, \dots, a^m$ are row vectors, then $a_i a^j = 0$ if $i \neq j$ and

$a_i a^j = 1$ if $i = j$.

Also $\begin{bmatrix} a^1 \\ a^2 \\ \vdots \\ a^m \end{bmatrix} \begin{bmatrix} a_1, a_2, \dots, a_m \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \dots \\ 0 & 1 & 0 & \dots \\ \vdots & & \ddots & \end{bmatrix}$, the $m \times m$

unit matrix, so that

$$\begin{bmatrix} a^1 \\ a^2 \\ \vdots \\ a^m \end{bmatrix} P_j = \begin{bmatrix} a^1 P_j \\ a^2 P_j \\ \vdots \\ a^m P_j \end{bmatrix} = \begin{bmatrix} a^1 \\ a^2 \\ \vdots \\ a^m \end{bmatrix} \begin{bmatrix} a_1, a_2, \dots, a_m \end{bmatrix} \mu$$

$$= \begin{bmatrix} I \end{bmatrix} \mu = \mu.$$

Therefore $\quad a^1 P_j = \mu_1$

$$a^2 P_j = \mu_2$$
$$\vdots$$
$$a^m P_j = \mu_m.$$

Finally $$P_j = (a^1P_j)a_1 + (a^2P_j)a_2 + \dots + (a^mP_j)a_m$$

and $$P_j = \sum_{i=1}^{m} (a^iP_j)a_i.$$

In particular $$P_o = \sum_{i=1}^{m} (a^iP_o)a_i.$$

Recall now that the a_i are m linearly independent vectors of the set $P_1, P_2, \dots, P_n$. Therefore, if

$$a^iP_o \geq 0 \qquad (i = 1,2, \dots, m),$$

then one is provided with a solution to the original programming problem. Its corresponding value of the functional is

$$z_o = \sum_{i=1}^{m} (a^iP_o)c_{r_i}.$$

But $$c_{r_i} = w_o'P_{r_i} = w_o'a_i.$$

Hence $$z_o = \sum_{i=1}^{m} (a^iP_o)(w_o'a_i).$$

Now $$P_o = \sum_{i=1}^{m} (a^iP_o)a_i = (a^1P_o)a_1 + (a^2P_o)a_2 + \dots + (a^mP_o)a_m.$$

Notice that each quantity in parentheses is simply a scalar.

$$\text{Hence } w_o'P_o = w_o'\left[(a^1P_o)a_1 + (a^2P_o)a_2 + \ldots + (a^mP_o)a_m\right]$$

$$= (a^1P_o)w_o'a_1 + (a^2P_o)w_o'a_2 + \ldots + (a^mP_o)w_o'a_m$$

$$= \sum_{i=1}^{m} (a^iP_o)(w_o'a_i) = z_o.$$

Therefore, a solution w_o to the dual problem and a solution λ (such that $\lambda_i = 0$ if i is not in the set, $r_1, r_2, \ldots, r_m$, and $\lambda_i = a^iP_o$ if i is in that set) have been found where $w_o'P_o = w_o'b = c'\lambda$. Hence, these are optimal solutions to both problems.

Now suppose that, in $P_o = \sum_{i=1}^{m} (a^iP_o)a_i$, $a^iP_o < 0$ for some i, say $i = s$; that is, $a^sP_o < 0$. Now let $\overline{w}' = w_o' - \theta a^s$. Since $a^ia_j = 0$ for $i \neq j$ and $a^ia_j = 1$ for $j = i$, then for any value of θ it is true that

$$\overline{w}'a_i = w_o'a_i - \theta a^sa_i = w_o'a_i = c_{r_i} \quad \text{if } i \neq s.$$

Also, for any positive value of θ, one may say that

$$\overline{w}'a_s = w_o'a_s - \theta a^sa_s = c_{r_s} - \theta < c_{r_s}.$$

By assumption, if P_j is not an a_i, then $w_o'P_j < c_j$.
If for some such P_j, $a^sP_j \geq 0$, then for this value of j and any positive value of θ

$$\overline{w}'P_j = w_o'P_j - \theta a^sP_j \leq w_o'P_j < c_j.$$

If $a^sP_j \geq 0$ for all P_j's, then $\overline{w}'$ would be a solution to

$$w'P_j \leq c_j \qquad (j = 1,2, \ldots, n)$$

for any positive θ; and $\overline{w}'P_o = w_o'P_o - \theta a^sP_o$ could be made arbitrarily large (since $a^sP_o < 0$), contrary to assumption. Hence there exists a value of j such that $a^sP_j < 0$. Then θ is chosen as the minimum of the values:

$$\frac{w_o'P_j - c_j}{a^sP_j} \qquad (a^sP_j < 0).$$

(Obviously, if $a^sP_j < 0$, then P_j is not an a_i because $a^sa_i = 0$ for $i \neq s$ and $a^sa_s = 1$.) Notice that all of the values described above are positive because if P_j is not an a_i, then $w_o'P_j - c_j < 0$ by assumption.

Suppose then that $\theta = \dfrac{w_o'P_k - c_k}{a^sP_k}$.

Then $$\overline{w}'P_k = w_o'P_k - \left(\frac{w_o'P_k - c_k}{a^sP_k}\right)a^sP_k = c_k.$$

For any other $a^sP_j < 0$

$$\theta = \frac{w_o'P_k - c_k}{a^sP_k} \leq \frac{w_o'P_j - c_j}{a^sP_j} .$$

Hence, multiplying through by the negative number a^sP_j,

$$\theta a^sP_j \geq w_o'P_j - c_j$$

and $$w_o'P_j - \theta a^sP_j = \overline{w}'P_j \leq c_j.$$

It has already been shown that if $a^sP_j \geq 0$, then $\overline{w}'P_j \leq c_j$. Hence $\overline{w}'$ is a solution to the dual problem. Also, $\overline{w}'a_i = c_{r_i}$ $(i = 1,2, \ldots, m;\ i \neq s)$ and $\overline{w}'a_s < c_{r_s}$ (from above).

Summarizing, one may say that:

$$\overline{w}'a_s < c_{r_s};$$

$$\overline{w}'a_i = c_{r_i} \qquad (i = 1,2, \ldots, m;\ i \neq s);$$

$$\overline{w}'P_k = c_k;$$

and for any j, not in the set, $r_1, r_2, \ldots, r_m$, and not k,

(1) if $a^sP_j \geq 0$, then $\overline{w}'P_j < c_j$,

(2) if $a^sP_j < 0$, then $\overline{w}'P_j \leq c_j$.

(The equality sign can hold if, and only if, for some $j \neq k$

$$\frac{w_o'P_j - c_j}{a^sP_j} = \theta.$$

However, this possibility has been ruled out by assumption 1 and the fact that $\overline{w}'a_i = c_{r_i}$ $(i = 1,2, \ldots, m;\ i \neq s)$ and $\overline{w}'P_k = c_k$, if the a_i's $(i = 1,2, \ldots, m;\ i \neq s)$ and P_k are linearly independent.) Also, if these m vectors are linearly independent, a new extreme point solution has been found to the dual problem, and the process may be repeated. It is a simple matter to show that P_k and the a_i's $(i = 1,2, \ldots, m;\ i \neq s)$ are linearly independent.

(1) $$P_k = \sum_{i=1}^{m} (a^i P_k) a_i.$$

(2) Assume that they are linearly dependent.

Then, for some numbers d_i not all zero,

$$d_k P_k + \sum_{\substack{i=1 \\ i \neq s}}^{m} d_i a_i = 0.$$

(Note that the d_i are scalars and the a_i vectors.) Now since the a_i's are linearly independent, if $d_k = 0$, all other $d_i = 0$. Hence $d_k \neq 0$. Therefore, rewriting:

$$P_k = -\sum_{\substack{i=1 \\ i \neq s}}^{m} \left(\frac{d_i}{d_k}\right) a_i.$$

Subtracting this expression for P_k from the one given by (1):

$$\sum_{i=1}^{m} (a^i P_k) a_i + \sum_{\substack{i=1 \\ i \neq s}}^{m} \left(\frac{d_i}{d_k}\right) a_i = 0;$$

$$(a^sP_k)a_s + \sum_{\substack{i=1 \\ i \neq s}}^{m} (a^iP_k + \frac{d_i}{d_k})a_i = 0.$$

Since the a_i's $(i = 1,2, \ldots, m)$ are linearly independent, all coefficients must vanish. In particular $a^sP_k = 0$, contradicting the choice of a^sP_k as negative. Hence, the vectors P_k and a_i $(i = 1,2, \ldots, m;\ i \neq s)$ are linearly independent, and a new basis of m-dimensional space has been found.

Now, with the new basis, the process may be repeated. Also $\overline{w}'P_o = w_o'P_o - \theta a^sP_o > w_o'P_o$ since $\theta > 0$ and $a^sP_o < 0$. Hence, at each stage the functional is increased. There are only a finite number of extreme point solutions, and, due to the increase of the functional at each stage, there can be no repetition or cycling of extreme point solutions. A stage will finally be reached where all $(a^iP_o) \geq 0$, and the problem will be solved.

M. RESOLUTION OF ASSUMPTIONS MADE IN SECTION L

It now remains only to clear up the assumptions of Section L. Assumptions 2 and 4 may be disposed of easily. By referring to the proof of the duality theorem in Chapter I, Section K, one may see that if the programming problem has an optimal solution, then its dual problem has an optimal solution. Each of these solutions will be extreme points of their respective convex sets. Hence, if there exists no extreme point solution of the dual problem, there exists no extreme point solution of the programming problem. Also if there exists no finite maximum to $w'P_o$, there exists no finite minimum to $c'\lambda$.

Assumption 3 was that a set of m linearly independent vectors, $P_{r_1}, P_{r_2}, \ldots, P_{r_m}$, had been determined from the set, $P_1, P_2, \ldots, P_n$, such that $w_o'P_j \leq c_j$ and equality holds at least for $j = r_1, r_2, \ldots, r_m$. This assumption is somewhat difficult to resolve. Lemke suggests a possible method. Starting with <u>any</u> set of m linearly independent vectors, $P_{r_1}, P_{r_2}, \ldots, P_{r_m}$, from the set, $P_1, P_2, \ldots, P_n$, he first solves the set of m equations for the m-dimensional point w_o:

$$w_o'P_{r_i} = c_{r_i} \qquad (i = 1, 2, \ldots, m).$$

Now, the convex set of solutions Ω to the dual problem is independent of the vector P_o in the functional $w'P_o$ to be maximized. Hence Lemke suggests that one minimize $\bar{z} = \sum_{j=1}^{n} \lambda_j c_j$, where $\lambda_j \geq 0$ and $\bar{P} = \sum_{j=1}^{n} \lambda_j P_j$, and $\bar{P}$ is any vector which may be expressed as a positive linear combination of the P_{r_i}. Then this provides a basic solution to the new simplex problem. Finding an optimal solution to the new simplex problem will provide an extreme point solution to Ω.

The final unresolved assumption (number 1) was that when $w_o'a_i = c_{r_i}$ $(i = 1,2, \ldots, m)$ and the m a_i's are linearly independent and w_o is a point of Ω, then $w_o'P_j < c_j$ if j is not in the set, $r_1, r_2, \ldots, r_m$. The difficulty which arises when this restriction is removed is that there is then a possibility of "cycling" in the process. Now suppose that for some value of j, say q, not of the set, $r_1, r_2, \ldots, r_m$,

$$w_o'P_q = c_q$$

and

$$a^sP_q < 0.$$

The increase in $w'P_o$ obtained by the dual method was $(-\theta a^sP_o)$

where $$\theta = \min \frac{(w_o'P_j - c_j)}{a^sP_j} \qquad (a^sP_j < 0).$$

Hence one would be forced to choose

$$\theta = (w_o'P_q - c_q) \div a^sP_q = 0.$$

Hence there is no increase in the functional and no assurance that "cycling" is ruled out. In order to resolve this situation Lemke poses an equivalent problem in which this situation cannot occur. The procedure is analogous to the procedure used to resolve degeneracy in the simplex method.

Ω is the set of all solutions to $w'P_j \leq c_j$ $(j = 1,2, \ldots, n)$. If $e > 0$, let $\Omega(e)$ be defined as the set of all solutions to $w'P_j \leq c_j + e^j$ $(j = 1,2, \ldots, n)$. It is possible to find $e_o > 0$ such that for $0 < e \leq e_o$ the extreme points of $\Omega(e)$ lie on precisely m of the hyperplanes $w'P_j = c_j + e^j$ $(j = 1,2, \ldots, n)$.

Let $a_1, a_2, \ldots, a_m$ be linearly independent column vectors forming a basis of W and such that each a_i is one of the P_j's $(j = 1,2, \ldots, n)$. Suppose also that $w_o'(e)$ is the solution to $w'a_i = c_{r_i} + e^{r_i}$ $(i = 1,2, \ldots, m)$.

If P_j is not an a_i, then $P_j = \sum_{i=1}^{m} (a^i P_j) a_i$.

Hence
$$w_o'(e)P_j = \sum_{i=1}^{m} (a^i P_j) \left[w_o'(e) a_i \right]$$
$$= \sum_{i=1}^{m} (a^i P_j) \left[c_{r_i} + e^{r_i} \right].$$

This last expression can be equal to $c_j + e^j$ for at most n values of e. If every P_j which is not an a_i is expressed in this form and this is done for all possible bases, then it can be seen that there exists a finite number of values of e for which an extreme point of $\Omega(e)$ lies on more than m of the hyperplanes $w'P_j = c_j + e^j$. Hence if e_o is some positive number smaller than the smallest such value of e, then for $0 < e \leq e_o$ maximizing $w'P_o$ subject to

$$w'P_j \leq c_j + e^j \qquad (j = 1,2, \ldots, n)$$

is a "non-degenerate" problem. Also if $w_o(e)$ is a maximum extreme point solution to the non-degenerate dual problem, $w_o(0) = w_o$ is a maximum extreme point solution to the original dual problem.

In the actual computation it is not necessary to specify e. When an extreme point solution w_o to the dual problem has been found, associated with the basis, $a_1, a_2, \ldots, a_m$, then the coefficients of P_o and the P_j's $(j = 1,2,\ldots,n)$ in terms of this basis are recorded, along with the values $w_o'P_j - c_j$, in a tableau analogous to the simplex tableau. This tableau is shown on page 77. The problem then becomes degenerate if the value,

$$\theta = \min \frac{w_o'P_j - c_j}{a^sP_j} \qquad (a^sP_j < 0),$$

is taken on for more than one value of j such that $a^sP_j < 0$.

It should be noticed that this tableau is almost the same as the corresponding simplex tableau in Section D of this chapter; the entry a^iP_j is the same as x_{ij}. Hence, the simplex algorithm <u>could</u> be used for calculating these entries in the next tableau. However, Lemke suggests that using the inverse of each new basis and the original data has some computational advantages in that errors are not cumulative.

$c_j \longrightarrow$			c_1	$\cdots$	c_k	$\cdots\cdots$	c_n
$\downarrow$	a_i	P_o	P_1	$\cdots$	P_k	$\cdots\cdots$	P_n
c_{r_1}	a_1	a^1P_o	a^1P_1	$\cdots$	a^1P_k	$\cdots\cdots$	a^1P_n
c_{r_2}	a_2	a^2P_o	a^2P_1	$\cdots$	a^2P_k	$\cdots\cdots$	a^2P_n
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\cdots\cdots$	$\vdots$
c_{r_s}	a_s	a^sP_o	a^sP_1	$\cdots$	a^sP_k	$\cdots\cdots$	a^sP_n
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\cdots\cdots$	$\vdots$
c_{r_m}	a_m	a^mP_o	a^mP_1	$\cdots$	a^mP_k	$\cdots\cdots$	a^mP_n
$w_o'P_j - c_j$		$w_o'P_o$	$w_o'P_1 - c_1$	....	$w_o'P_k - c_k$	....	$w_o'P_n - c_n$

Now a scheme will be demonstrated whereby the replacing vector P_k will be uniquely determined, and no cycling will occur. First, since the a_i's are linearly independent vectors, then the a^i's are also linearly independent. (The a_i's, remember, are m-dimensional column vectors, and the a^i's are the row vectors of the inverse of the matrix

$$\left[a_1, a_2, \ldots, a_m\right].$$

The independence of the a^i's may be proved by matrix theory.) Therefore, any m-dimensional row vector x' may be expressed in terms of them as: $x' = \sum_{i=1}^{m} (x'a_i)a^i$. In particular:

$$w_o' = \sum_{i=1}^{m} (w_o'a_i)a^i = \sum_{i=1}^{m} c_{r_i} a^i.$$

Also

$$w_o'(e) = \sum_{i=1}^{m} (w_o'(e)a_i)a^i = \sum_{i=1}^{m} (c_{r_i} + e^{r_i})a^i$$

$$= \sum_{i=1}^{m} c_{r_i} a^i + \sum_{i=1}^{m} e^{r_i} a^i;$$

$$w_o'(e) = w_o' + \sum_{i=1}^{m} e^{r_i} a^i.$$

For $0 < e \leq e_o$ degeneracy cannot occur. If $w_o(e)$ is not an optimal solution, then for some s,

$$a^s P_o < 0.$$

Also there exists another extreme point solution given by

$$\bar{w}'(e) = w_o'(e) - \theta_k(e)a^s$$

where

$$\theta_k(e) = \left[w_o'(e)P_k - c_k - e^k \right] \div a^s P_k$$

for some unique value of k such that $a^s P_k < 0$.

Using the value of $w_o'(e)$ in terms of w_o',

$$\theta_k(e) = \frac{\left[w_o'P_k - c_k\right]}{a^sP_k} + \frac{\left[\sum_{i=1}^{m} (a^iP_k)e^{r_i} - e^k\right]}{a^sP_k} .$$

Now degeneracy occurs in the original problem if for some value, say q:

$$\theta = \frac{w_o'P_q - c_q}{a^sP_q} = \frac{w_o'P_k - c_k}{a^sP_k} .$$

However, in the e-procedure, the corresponding value for q, say $\theta_q(e)$, becomes:

$$\theta_q(e) = \frac{w_o'P_q - c_q}{a^sP_q} + \frac{\left[\sum_{i=1}^{m} (a^iP_q)e^{r_i} - e^q\right]}{a^sP_q}.$$

Again, as in the e-procedure in the simplex method, the two polynomials, $\theta_q(e)$ and $\theta_k(e)$, can be compared term by term. The lowest power of e for which the coefficients in the two polynomials differ determines whether P_k or P_q is to be the replacing vector. No matter how many values of j exist for which

$$\frac{w_o'P_j - c_j}{a^sP_j}$$

has the same value, there will always be a unique vector determined by this method. The reasons are entirely analogous to the reasons why there will be a unique vector replaced in the simplex method when the e-procedure is employed, as in Section C of this chapter.

N. AN EXAMPLE SOLVED BY THE DUAL METHOD

The problem worked in a preceding section by the simplex method will now be worked by the dual method. Starting from the statement of the problem in terms of equations, one may rephrase it as follows: minimize $-\lambda_6 - 2\lambda_7$ subject to

$$\begin{aligned} \lambda_1 - \lambda_6 - 3\lambda_7 &= 6 \\ \lambda_2 + \lambda_6 + 3\lambda_7 &= 15 \\ \lambda_3 + \lambda_6 - \lambda_7 &= 2 \\ \lambda_4 + \lambda_6 + \lambda_7 &= 7 \\ \lambda_5 - 18\lambda_6 - 2\lambda_7 &= 27. \end{aligned}$$

The coefficient matrix is again written out and the dual problem may be stated from it.

Coefficient Matrix

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_0
1	0	0	0	0	-1	-3	6
0	1	0	0	0	1	3	15
0	0	1	0	0	1	-1	2
0	0	0	1	0	1	1	7
0	0	0	0	1	-18	-2	27

The dual problem is: maximize $6w_1 + 15w_2 + 2w_3 + 7w_4 + 27w_5$ subject to:

$$\begin{array}{lrcl}
(1) & w_1 & \leq & 0 \\
(2) & w_2 & \leq & 0 \\
(3) & w_3 & \leq & 0 \\
(4) & w_4 & \leq & 0 \\
(5) & w_5 & \leq & 0 \\
(6) & -w_1 + w_2 + w_3 + w_4 - 18w_5 & \leq & -1 \\
(7) & -3w_1 + 3w_2 - w_3 + w_4 - 2w_5 & \leq & -2.
\end{array}$$

By trial and error one may find an extreme point solution to this problem. The first obvious possibility would be the origin, that is, the solution to the first five constraints stated as equations. It may be easily verified that this point does not satisfy the inequalities (6) and (7). The next trial could be the solution to any four of the first five constraints and one of the last two — expressed as equations. Again, if one takes the first four constraints and the sixth one and expresses them as equations, he can easily verify that their solution does not satisfy the remaining two constraints. Continuing in this manner, one can see that the solution $w_o' = (0, -1, 0, 0, 0)$ to

$$
\begin{aligned}
&(1) \quad w_1 &&= 0\\
&(3) \quad w_3 &&= 0\\
&(4) \quad w_4 &&= 0\\
&(5) \quad w_5 &&= 0\\
&(6) \quad -w_1 + w_2 + w_3 + w_4 - 18w_5 &&= -1
\end{aligned}
$$

does satisfy (2) $w_2 < 0$

and (7) $-3w_1 + 3w_2 - w_3 + w_4 - 2w_5 < -2.$

Hence the original basis consists of P_1, P_3, P_4, P_5, and P_6; that is, in the first stage, $a_1 = P_1$, $a_2 = P_3$, $a_3 = P_4$, $a_4 = P_5$, and $a_5 = P_6$. The next step is to calculate the inverse of $\left[a_1, a_2, a_3, a_4, a_5\right]$. Calculation of inverses will be omitted here. Any method may be used.

STAGE I

$$
\begin{array}{ccccc}
a_1 & a_2 & a_3 & a_4 & a_5
\end{array}
\qquad\qquad \text{Inverse}
$$

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & -1\\
0 & 0 & 0 & 0 & 1\\
0 & 1 & 0 & 0 & 1\\
0 & 0 & 1 & 0 & 1\\
0 & 0 & 0 & 1 & -18
\end{bmatrix}
\qquad
\begin{array}{c}
a^1 =\\ a^2 =\\ a^3 =\\ a^4 =\\ a^5 =
\end{array}
\begin{bmatrix}
1 & 1 & 0 & 0 & 0\\
0 & -1 & 1 & 0 & 0\\
0 & -1 & 0 & 1 & 0\\
0 & 18 & 0 & 0 & 1\\
0 & 1 & 0 & 0 & 0
\end{bmatrix}
$$

The values a^iP_o are now calculated. Since a^2P_o and a^3P_o are negative, the problem is not finished. A new basis must be decided upon and the process repeated. Either a_2 or a_3 may be the "replaced vector." Since the functional of the dual problem is to be maximized and its increase in the next stage is $(-\theta a^sP_o)$, the logical choice for a_s is that vector for which a^sP_o is most negative.

$a^1P_o = 21$
$a^2P_o = -13$
$a^3P_o = -8$
$a^4P_o = 297$
$a^5P_o = 15$

Hence $a_s = a_2 = P_3$. Deciding the "replacing vector" entails calculating the quantities,

$$\frac{w_o'P_j - c_j}{a^sP_j} \qquad (a^sP_j < 0).$$

Only those values of j for which P_j is not in the present basis need to be considered. (If P_j is in the basis, $a^sP_j = 0$ or 1.)

j	a^sP_j	$w_o'P_j - c_j$	$\frac{w_o'P_j - c_j}{a^sP_j}$
2	-1	-1	1
7	-4	-1	$\frac{1}{4}$

Now θ is the minimum of the quantities in the last column, and P_k — the replacing vector — is the vector which corresponds to this value. Hence θ = 1/4 and $P_k = P_7$. A new basis is now determined in which P_7 replaces P_3 in the old basis. The process may now be repeated.

STAGE II

$a_1 = P_1 \quad a_2 = P_7 \quad a_3 = P_4 \quad a_4 = P_5 \quad a_5 = P_6$

The new value of $w_o' = (0, -3/4, -1/4, 0, 0)$, as may be easily verified.

$$\begin{array}{c}
\begin{array}{ccccc} a_1 & a_2 & a_3 & a_4 & a_5 \end{array} \\
\begin{bmatrix} 1 & -3 & 0 & 0 & -1 \\ 0 & 3 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & -2 & 0 & 1 & -18 \end{bmatrix}
\end{array}
\qquad
\begin{array}{c}
\text{Inverse} \\
\begin{array}{c} a^1 = \\ a^2 = \\ a^3 = \\ a^4 = \\ a^5 = \end{array}
\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & -\frac{1}{4} & 0 & 0 \\ 0 & -\frac{1}{2} & -\frac{1}{2} & 1 & 0 \\ 0 & 5 & 13 & 0 & 1 \\ 0 & \frac{1}{4} & \frac{3}{4} & 0 & 0 \end{bmatrix}
\end{array}$$

$a^1P_o = 21$

$a^2P_o = 3\frac{1}{4}$

$a^3P_o = -1\frac{1}{2}$

$a^4P_o = 128$

$a^5P_o = 5\frac{1}{4}$

$\therefore\ a^s = a^3$

and $a_s = a_3 = P_4$.

j	a^sP_j	$w_o'P_j - c_j$	$\dfrac{w_o'P_j - c_j}{a^sP_j}$
2	$-\frac{1}{2}$	$-\frac{3}{4}$	$\frac{3}{2}$
3	$-\frac{1}{2}$	$-\frac{1}{4}$	$\frac{1}{2}$

Hence $\theta = \frac{1}{2}$ and $P_k = P_3$.

$a_1 = P_1 \quad a_2 = P_7 \quad a_3 = P_3 \quad a_4 = P_5 \quad a_5 = P_6$

The new value of $w_o' = (0, -1/2, 0, -1/2, 0)$.

$$\begin{array}{ccccc} a_1 & a_2 & a_3 & a_4 & a_5 \end{array} \qquad\qquad \text{Inverse}$$

$$\begin{bmatrix} 1 & -3 & 0 & 0 & -1 \\ 0 & 3 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & -2 & 0 & 1 & -18 \end{bmatrix} \qquad \begin{array}{c} a^1 = \\ a^2 = \\ a^3 = \\ a^4 = \\ a^5 = \end{array} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & -8 & 0 & 26 & 1 \\ 0 & -\frac{1}{2} & 0 & \frac{3}{2} & 0 \end{bmatrix}$$

$a^1P_o = 21$ Hence an optimal solution to the original problem is:

$a^2P_o = 4$

$a^3P_o = 3$

$a^4P_o = 89$

$a^5P_o = 3$

$\lambda_1 = 21$

$\lambda_2 = 0$

$\lambda_3 = 3$

$\lambda_4 = 0$

$\lambda_5 = 89$

$y = \lambda_6 = 3$

$x = \lambda_7 = 4.$

The ease with which this problem was solved is deceiving. In most problems the calculation of the inverses will be very time-consuming unless done on electronic computers. However, there are straightforward methods of calculating inverses which may be adapted to electronic computers. This method, as noted before, has some computational advantages over the simplex method.

CHAPTER III

THE PROBLEM IN WHICH THE COST COEFFICIENTS ARE LINEAR FUNCTIONS OF A PARAMETER

A. INTRODUCTION

The problem so far considered has been that of minimizing (or maximizing) a linear functional subject to a set of constraints. It has been shown that these constraints may always be expressed as a set of m linear equations in n unknowns, where $n > m$. The linear functional is expressed in the following form: $f(\lambda) = \sum_{j=1}^{n} c_j\lambda_j$, where the c_j's are constants and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is subject to the constraint $\sum_{j=1}^{n} \lambda_j P_j = P_o$. $P_o, P_1, P_2, \ldots, P_n$ are m-dimensional vectors. Saaty and Gass have studied the problem $\boxed{4}$ in which the cost coefficients, that is, the c_j's, are linear functions of one or more parameters $(t_1, t_2, \ldots, t_q)$. The one-parameter case has been completely resolved. When parameters are introduced into the cost coefficients, the matrix of coefficients $\left[P_1, P_2, \ldots, P_n\right]$ of the constraint equations is unaffected. Hence there are still a finite number of feasible extreme point solutions; therefore, there are a finite number of possible optimal solutions. If the solution set is bounded, then for <u>every</u> value of the parameter t or <u>every</u> set of values of the parameters $(t_1, t_2, \ldots, t_q)$ the optimal solution or solutions will be a subset of the set of feasible extreme point solutions, which has a finite number of members.

B. THE ONE-PARAMETER CASE SOLVED BY THE SIMPLEX METHOD

The one-parameter case will be considered first. With every value of the parameter there will be associated a set of one or more feasible extreme point solutions which optimize the functional if the functional has an optimum value. In the discussion non-degeneracy will be assumed. The degenerate problem may be handled by introducing the corresponding "e-problem," as in the simplex method. Now the problem may be stated as follows: for every value of the parameter t find the solution to

$$\sum_{j=1}^{n} \lambda_j P_j = P_o$$

which maximizes $\quad f(\lambda) = \sum_{j=1}^{n} c_j \lambda_j$

where $\qquad c_j = g_j + h_j t \qquad (j = 1, 2, \ldots, n)$

and all values of g_j and h_j are constants.

Now assume that, for some fixed value t_o of the single parameter t, a maximal solution has been found by the simplex method. Also assume that the basis vectors of this maximal solution are $P_1, P_2, \ldots, P_m$. Then if P_j is any vector in the set $P_1, P_2, \ldots, P_n$, then

$$P_j = \sum_{i=1}^{m} x_{ij} P_i$$

and $z_j = \sum_{i=1}^{m} x_{ij} c_i$. Now if the solution involving the vectors $P_1, P_2, \ldots, P_m$ is maximal, then

$$z_j - c_j \geq 0 \qquad (j = 1,2, \ldots, n),$$

$$z_j = \sum_{i=1}^{m} x_{ij} c_i \qquad (j = 1,2, \ldots, n),$$

$$z_j - c_j = \sum_{i=1}^{m} x_{ij} c_i - c_j \qquad (j = 1,2, \ldots, n),$$

$$z_j - c_j = \sum_{i=1}^{m} x_{ij}(g_i + h_i t) - (g_j + h_j t) \qquad (j = 1,2, \ldots, n),$$

$$z_j - c_j = \left(\sum_{i=1}^{m} x_{ij} g_i - g_j\right) + \left(\sum_{i=1}^{m} x_{ij} h_i - h_j\right)t \quad (j = 1,2, \ldots, n).$$

Now let

$$(3\text{-}1) \qquad \alpha_j \equiv \sum_{i=1}^{m} x_{ij} g_i - g_j \qquad (j = 1,2, \ldots, n),$$

$$(3\text{-}2) \qquad \beta_j \equiv \sum_{i=1}^{m} x_{ij} h_i - h_j \qquad (j = 1,2, \ldots, n).$$

Then $\quad z_j - c_j = \alpha_j + \beta_j t \qquad (j = 1,2, \ldots, n),$

and

(3-3) $$\alpha_j + \beta_j t \geq 0 \qquad (j = 1,2, \ldots, n)$$

if the solution based on $P_1, P_2, \ldots, P_m$ is maximal.

The set of inequalities (3-3) defines a convex set of values of t. Let t_1 and t_2 be two points in one-dimensional space which satisfy the inequalities (3-3).

Then $$\alpha_j + \beta_j t_1 \geq 0 \qquad (j = 1,2, \ldots, n)$$

and $$\alpha_j + \beta_j t_2 \geq 0 \qquad (j = 1,2, \ldots, n).$$

If $0 < \theta < 1$, then

$$\theta\alpha_j + \beta_j(\theta t_1) \geq 0 \qquad (j = 1,2, \ldots, n),$$

$$(1 - \theta)\alpha_j + \beta_j(1 - \theta)t_2 \geq 0 \qquad (j = 1,2, \ldots, n),$$

$$\theta\alpha_j + (1-\theta)\alpha_j + \beta_j(\theta t_1) + \beta_j(1-\theta)t_2 \geq 0 \qquad (j = 1,2, \ldots, n),$$

$$\alpha_j + \beta_j\left[\theta t_1 + (1 - \theta)t_2\right] \geq 0, \qquad (j = 1,2, \ldots, n).$$

Hence the set of points in one-dimensional space defined by (3-3) is convex. It may take one of three forms if it contains at least one point, and it has been assumed that it contains one point.

1. It may be a single point.

2. It may be a closed interval; that is, it may consist of all the points t such that $\underline{t} \leq t \leq \bar{t}$, where $\underline{t}$ and $\bar{t}$ are two fixed points and $\bar{t} > \underline{t}$.

3. It may be a "ray" or "half line"; that is, it may consist of all points t such that, for some point $\bar{t}$, $t \leq \bar{t}$ or of all points t such that, for some point $\underline{t}$, $t \geq \underline{t}$. Notice that the meaning of "ray" employed here is different from the definition of a ray in Chapter I, Section B. Hence the term "half line" will be used.

Now each of the inequalities in (3-3) imposes either an upper or a lower bound on t if $\beta_j \neq 0$. If $\beta_j = 0$, then $\alpha_j \geq 0$ because it has been assumed that the solution is maximal for some value of t. Obviously, if P_j is a basis vector, $\alpha_j = \beta_j = 0$.

$$\text{If } \beta_j > 0, \text{ then } t \geq -\frac{\alpha_j}{\beta_j} .$$

$$\text{If } \beta_j < 0, \text{ then } t \leq -\frac{\alpha_j}{\beta_j} .$$

Now the upper limit of t imposed by the inequalities (3-3) is the minimum of $(-\alpha_j/\beta_j)$ over the values of j for which $\beta_j < 0$. The lower limit of t is the maximum of $(-\alpha_j/\beta_j)$ over the values of j for which $\beta_j > 0$.

Let $\overline{t} \equiv$ minimum of $(-\alpha_j/\beta_j)$ for $\beta_j < 0$

and

$\underline{t} \equiv$ maximum of $(-\alpha_j/\beta_j)$ for $\beta_j > 0$.

Then if $\overline{t} = \underline{t}$, the set of points satisfying (3-3) is this single point. If $\overline{t} > \underline{t}$, then all points t such that $\underline{t} \leq t \leq \overline{t}$ satisfy (3-3). If $\beta_j \leq 0$ $(j = 1,2, \ldots, n)$, then all points t such that $t \leq \overline{t}$ satisfy (3-3); and if $\beta_j \geq 0$ $(j = 1,2, \ldots, n)$, then all points t such that $t \geq \underline{t}$ satisfy (3-3). Since it has been assumed that for some value of t the solution based on the vectors $P_1, P_2, \ldots, P_m$ is maximal, $\overline{t}$ must be greater than or equal to $\underline{t}$. If $\overline{t}$ were less than $\underline{t}$, there would be no value of t which satisfied (3-3) and hence no value of t for which the solution is maximal.

If $\alpha_j > 0$ for all values of j such that P_j is <u>not</u> in the basis and $\beta_j = 0$, then for $\underline{t} < t < \overline{t}$ the solution is a <u>unique</u> maximal solution. This may be clarified by recalling that introducing any vector P_q into the solution with positive coefficient θ increases the functional by $\theta(c_q - z_q)$, that is, decreases it by $\theta(z_q - c_q)$. If $\underline{t} < t < \overline{t}$ and $\alpha_j > 0$ for all values of j such that P_j is not in the basis and $\beta_j = 0$, then $\theta(z_j - c_j) > 0$ for all values of j such that P_j is not in the basis and all positive values of θ. If $\alpha_j = 0$ for some value of j such that $\beta_j = 0$ and P_j is not a basis vector, then introducing this vector into the basis to form a new basis (by the simplex method) will yield another optimal solution for the same interval of values of t.

The next step is to find the optimal solutions for values of t greater than $\overline{t}$ and less than $\underline{t}$. Let k be that value of j such that

$$\min_{\beta_j < 0} \left(-\frac{\alpha_j}{\beta_j}\right) = -\frac{\alpha_k}{\beta_k} = \overline{t}.$$

Unless the set of values of t for which the known solution is maximal has no upper bound, there will always be at least one such value. If there are several such values of j, then k may be any one of them. The following argument is for finding the maximal solutions for values of t greater than $\overline{t}$. An entirely analogous argument applies to finding the maximal solutions for values of t less than $\underline{t}$.

Now $$\overline{t} = -\frac{\alpha_k}{\beta_k}$$

and $\beta_k < 0$.

Hence $$\alpha_k + \beta_k \overline{t} = 0.$$

There are two possibilities. Either P_k cannot be introduced into the basis by the simplex method because $x_{ik} \leq 0$ $(i = 1, 2, \ldots, m)$, or P_k can be introduced into the basis by the simplex method to form a new basis.

If P_k cannot be introduced into the basis because $x_{ik} \leq 0$ $(i = 1,2, \ldots, m)$, then for $t > \bar{t}$ there is no maximal solution; that is, the value of the functional is unbounded. To understand this, recall that this is case I of the simplex method. (See Chapter I, Section H.) The solution which maximizes the functional for $\underline{t} \leq t \leq \bar{t}$ is a feasible solution (though not optimal) for $t > \bar{t}$. The x_{ik} are unaffected by the value of t. Hence $x_{ik} \leq 0$ $(i = 1,2, \ldots, m)$. Also if $t > \bar{t}$,

then $\qquad \beta_k t < \beta_k \bar{t}$ (since $\beta_k < 0$)

and $\qquad \alpha_k + \beta_k t < \alpha_k + \beta_k \bar{t} = 0.$

However, $\qquad z_k - c_k = \alpha_k + \beta_k t < 0,$

$$z_k - c_k < 0,$$

$$c_k - z_k > 0.$$

Hence if $t > \bar{t}$ and $x_{ik} \leq 0$ $(i = 1,2, \ldots, m)$, then the conditions for case I of the simplex method are fulfilled; and the functional has no maximal solution.

If P_k can be introduced by the simplex method into the basis to form a new solution, then for $t = \bar{t}$ this new solution involving P_k is an alternate maximal solution. This is true since $\alpha_k + \beta_k \bar{t} = 0$. Now if $t < \bar{t}$, then $\beta_k t > \beta_k \bar{t}$ (since $\beta_k < 0$),

$$\alpha_k + \beta_k t > \alpha_k + \beta_k \bar{t} = 0,$$

$$z_k - c_k = \alpha_k + \beta_k t > 0,$$

$$z_k - c_k > 0,$$

$$c_k - z_k < 0.$$

Hence the feasible solution obtained by introducing P_k into the basis is not optimal for $t < \bar{t}$. It has been shown that the range of t for which any solution is optimal is either a point, a closed interval of the real line, or a half line. Therefore, $\bar{t}$ is the lower limit of the range of t for which the new extreme point solution (obtained by introducing P_k into the basis) is maximal. If $\bar{t}$ is also the upper limit of this new range of values, then this new solution is maximal only for the point $t = \bar{t}$. If there is another upper limit, say $\bar{t}'$, then the new solution is optimal for $\bar{t} \leq t \leq \bar{t}'$. If there is no upper limit, then the new solution is optimal for $t \geq \bar{t}$. The new range of values is obtained by constructing the new simplex tableau and calculating the new values of $z_j - c_j$ $(j = 1,2, \ldots, n)$.

If the set of values of t for which the new basis is maximal is the one point $t = \bar{t}$, then in the new set of inequalities $\alpha_j + \beta_j t \geq 0$:

$$\max_{\beta_j > 0} \left(-\frac{\alpha_j}{\beta_j}\right) = \min_{\beta_j < 0} \left(-\frac{\alpha_j}{\beta_j}\right) = \bar{t}.$$

If the set of values of t for which the new basis is maximal is the closed interval $\bar{t} \leq t \leq \bar{t}'$, then in the new set of inequalities $\alpha_j + \beta_j t \geq 0$:

$$\bar{t} = \max_{\beta_j > 0} \left(-\frac{\alpha_j}{\beta_j}\right) = \text{new lower limit}$$

and $$\bar{t}' = \min_{\beta_j < 0} \left(-\frac{\alpha_j}{\beta_j}\right) = \text{new upper limit.}$$

In either case the <u>new</u> vector P_k is again that vector such that

$$-\frac{\alpha_k}{\beta_k} = \min_{\beta_j < 0} \left(-\frac{\alpha_j}{\beta_j}\right).$$

This vector is introduced to form another basis and to give another feasible solution, and the simplex tableau is again calculated. The process may be repeated. Each repetition gives a new basis. There will be no cycling of bases because the interval of values of t for which a particular basis is maximal is convex, that is, has no discontinuities or is continuous. There are a finite number of possible bases, and for every value of t either there is a maximal solution or the functional is unbounded. Hence finally one of two eventualities will occur. Either for some half line $t \geq t_1$ some extreme point solution will be maximal, or for some such half line there will be no maximal solution; that is, the functional will be unbounded.

In order to complete the problem one finds, by an entirely similar procedure, the maximal solutions for values of $t < \underline{t}$.

C. EQUIVALENCE OF THE DUAL AND SIMPLEX METHODS IN THE ONE-PARAMETER CASE

It is not difficult to show that solving this parameter problem by the dual method reduces to solving the same set of inequalities to which the solution of the problem by the simplex method reduces. Now if the linear programming problem is to maximize

$$f(\lambda) = \sum_{j=1}^{n} c_j\lambda_j$$

subject to $\sum_{j=1}^{n} P_j\lambda_j = P_o$, where $P_o, P_1, \ldots, P_n$ are m-dimensional column vectors, then its dual problem is to minimize $w'P_o$ subject to $w'P_j \geq c_j \quad (j = 1,2, \ldots, n)$, where w' is an m-dimensional row vector. (See Chapter II, Sections I and J.)

In the dual method it is assumed that an initial feasible extreme point solution has been found to the dual problem. (See Chapter II, Sections K and L.) In other words it is assumed that a set of m linearly independent vectors $P_1, P_2, \ldots, P_m$ has been found such that $w_o' = (w_1, w_2, \ldots, w_m)$, the unique solution to the set of m linearly independent equations $w'P_i = c_i \quad (i = 1,2, \ldots, m)$, also satisfies the constraints $w'P_j \geq c_j \quad (j = 1,2, \ldots, n)$ to the dual problem. Now recall that in the dual method the column vectors $P_1, P_2, \ldots, P_m$ are denoted by $a_1, a_2, \ldots, a_m$, respectively, and that the inverse of the matrix $\left[a_1, a_2, \ldots, a_m\right]$ is calculated. (See Chapter II, Section L.) The rows of this inverse are denoted by $a^1, a^2, \ldots, a^m$. It was also shown in Chapter II, Section L, that

$$P_o = \sum_{i=1}^{m} (a^i P_o) a_i$$

and that if $a^i P_o \geq 0 \quad (i = 1,2,\ldots,m)$, then an optimal solution to the original programming problem is $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ where

$$\lambda_i = a^i P_o \qquad (i = 1,2, \ldots, m)$$

and $$\lambda_i = 0 \qquad (i = m + 1, m + 2, \ldots, n).$$

Now in solving the parameter problem by the dual method one would assume that a set of m linearly independent vectors $P_1, P_2, \ldots, P_m$ had been found (1) such that $a^i P_o \geq 0$, where the a^i's have been defined above, and (2) such that — for some value of t, say t_o — the solution w_o' to the m equations

$$w'P_i = c_i = g_i + h_i t \qquad (i = 1,2, \ldots, m) \tag{3-4}$$

also satisfies $w'P_j \geq c_j = g_j + h_j t \qquad (j = 1,2, \ldots, n)$.

Notice that this is an extreme point solution to the original programming problem and that it is optimal only for those values of t for which the second condition is true. Now w_o' is a function of t, and equation (3-4) will be solved explicitly for t. The vector and matrix notation of Chapter I, Section J, will be employed. Then

$$w'P_i = c_i = g_i + h_i t \qquad (i = 1,2, \ldots, m)$$

becomes

$$w'\begin{bmatrix} P_1, P_2, \ldots, P_m \end{bmatrix} = (c_1, c_2, \ldots, c_m).$$

So $$w'\begin{bmatrix} P_1, P_2, \ldots, P_m \end{bmatrix} \begin{bmatrix} a^1 \\ a^2 \\ \vdots \\ a^m \end{bmatrix} = (c_1, c_2, \ldots, c_m) \begin{bmatrix} a^1 \\ a^2 \\ \vdots \\ a^m \end{bmatrix};$$

$$w' = w'\begin{bmatrix} I \end{bmatrix} = (c_1, c_2, \ldots, c_m) \begin{bmatrix} a^1 \\ a^2 \\ \vdots \\ a^m \end{bmatrix}.$$

Now the constraints $w'P_j \geq c_j \quad (j = 1,2, \ldots, n)$ may be written:

$$(c_1, c_2, \ldots, c_m) \begin{bmatrix} a^1 \\ a^2 \\ \vdots \\ a^m \end{bmatrix} P_j \geq c_j \qquad (j = 1,2, \ldots, n),$$

$$(3\text{-}5) \qquad (c_1, c_2, \ldots, c_m) \begin{bmatrix} a^1 P_j \\ a^2 P_j \\ \vdots \\ a^m P_j \end{bmatrix} \geq c_j \qquad (j = 1,2, \ldots, n).$$

Recall (Chapter II, Section L) that

$$P_j = \sum_{i=1}^{m} (a^i P_j) a_i = \sum_{i=1}^{m} (a^i P_j) P_i$$

and $$P_j = \sum_{i=1}^{m} x_{ij} P_i.$$

Since $P_1, P_2, \ldots, P_m$ are linearly independent, the expression for any m-dimensional vector (in particular P_j) in terms of them is unique.

Therefore, $$a^i P_j = x_{ij}$$

and the inequalities (3-5) may be written:

$$(c_1, c_2, \ldots, c_m)\begin{bmatrix} x_{1j} \\ x_{2j} \\ \vdots \\ x_{mj} \end{bmatrix} \geq c_j \qquad (j = 1, 2, \ldots, n),$$

$$c_1 x_{1j} + c_2 x_{2j} + \ldots + c_m x_{mj} \geq c_j \qquad (j = 1, 2, \ldots, n).$$

Hence $\quad z_j \geq c_j \qquad (j = 1, 2, \ldots, n)$

and $\quad z_j - c_j \geq 0 \qquad (j = 1, 2, \ldots, n).$

This is precisely the same set of inequalities which must be satisfied in solving the problem by the simplex method. Hence the two methods are equivalent.

D. THE TWO-PARAMETER AND MULTI-PARAMETER PROBLEM

In Reference [5] Saaty and Gass discuss the solution to the multi-parameter problem, with particular emphasis on the two-parameter case. The one-parameter problem was to find the optimal solution or solutions for every point t in one-dimensional space. The two-parameter problem is to find the optimal solution or solutions for every point, (t_1, t_2), in two-dimensional space. The n-parameter problem is to find the optimal solution or solutions for every point in n-dimensional space.

By the same method as that used in the one-parameter case, it may be shown that the region in n-dimensional space for which a particular extreme point solution is optimal is a convex region. In two-dimensional space these regions are convex polygons defined by a set of inequalities. If the basis of the extreme point solution being considered is $P_1, P_2, \ldots, P_m$, then the inequalities which define the convex region for which this solution is maximal are:

$$z_j - c_j = \alpha_j + \beta_j t_1 + \gamma_j t_2 \geq 0 \qquad (j = m + 1,\ m + 2,\ \ldots,\ n).$$

The boundaries of this polygon are:

$$\alpha_j + \beta_j t_1 + \gamma_j t_2 = 0 \qquad (j = m + 1,\ m + 2,\ \ldots,\ n).$$

Along the particular boundary $\alpha_k + \beta_k t_1 + \gamma_k t_2 = 0$ the solution obtained by introducing P_k into the basis by the simplex method is an alternate optimal solution. By introducing P_k into the basis and calculating the new simplex tableau, one will arrive at a new set of defining inequalities and hence determine a new convex region. However, even if this is done for all boundaries and then for all the boundaries of the new convex polygons at each stage, it is possible to "reach an impasse without having considered all possible bases and their associated regions" $\boxed{5}$. The only completely general method available now for solving the n-parameter case is that of considering each possible basis separately and the convex regions in n-dimensional space determined by the inequalities

$$z_j - c_j \geq 0 \qquad (j = m + 1, m + 2, \ldots, n)$$

for each basis. Obviously the length of time required in most such problems would be prohibitive.

E. SOLUTION OF A ONE-PARAMETER PROBLEM

A one-parameter problem will now be solved. The problem is to maximize $(2 + t)y + (3 - 2t)x$ for all values of t and subject to:

$$
\begin{array}{ll}
(1) & y - x \leq 2 \\
(2) & y + x \leq 6 \\
(3) & y + 4x \leq 16 \\
(4) & -33y + 9x \leq -11.
\end{array}
$$

The inequalities are first converted into equations by adding a new non-negative variable to each inequality. The matrix of the resulting set of equations is written below. Each column is labeled as a vector. The column of coefficients of y is P_1, and the column of coefficients of x is P_2. Henceforth, y and x will be denoted by λ_1 and λ_2 respectively; the other variables will be λ_3, λ_4, λ_5, and λ_6.

P_1	P_2	P_3	P_4	P_5	P_6	P_0
1	-1	1	0	0	0	2
1	1	0	1	0	0	6
1	4	0	0	1	0	16
-33	9	0	0	0	1	-11

The next step is to assign t some particular value and to find the maximal extreme point solution for this value of t. For example let t = 1. Then by the simplex method one may show that the following solution is maximal:

$$\lambda = (4, 2, 0, 0, 4, 103).$$

Tableau I is the tableau based on P_1, P_2, P_5, and P_6 and corresponding to this solution. From this tableau one may also determine the range of values of t for which this solution is maximal. This range is determined by the inequalities:

$$z_3 - c_3 \geq 0,$$

$$z_4 - c_4 \geq 0.$$

Now
$$z_3 - c_3 = -\tfrac{1}{2} + \tfrac{3}{2}\, t,$$

$$z_4 - c_4 = \tfrac{5}{2} - \tfrac{1}{2}\, t$$

from Tableau I.

Hence $\quad \alpha_3 = -\frac{1}{2}, \qquad \beta_3 = \frac{3}{2}, \qquad -\frac{\alpha_3}{\beta_3} = \frac{1}{3};$

and $\quad \alpha_4 = \frac{5}{2}, \qquad \beta_4 = -\frac{1}{2}, \qquad -\frac{\alpha_4}{\beta_4} = 5.$

Therefore, if $1/3 \leq t \leq 5$, the solution

$$\lambda = (4, 2, 0, 0, 4, 103) \text{ is optimal.}$$

Introducing P_4 into the basis gives a solution which is an alternate optimal solution for $t = 5$. Calculating the new tableau, Tableau II, allows one to determine the range of values of t for which this solution

$$\lambda = (2, 0, 0, 4, 14, 55) \text{ is optimal.}$$

It may be easily verified from the inequalities

$$z_2 - c_2 \geq 0,$$
$$z_3 - c_3 \geq 0$$

that this solution is optimal if $t \geq 5$.

Optimal solutions have now been found for all values of t greater than or equal to 1/3. One may find optimal solutions for values of t less than 1/3 by returning to Tableau I and introducing the vector P_3 into the basis to form a new solution by the simplex method. Upon this new basis Tableau III is calculated, and from the inequalities

$$z_4 - c_4 \geq 0,$$
$$z_5 - c_5 \geq 0$$

one may see that this solution, that is,

$$\lambda = (\tfrac{8}{3}, \tfrac{10}{3}, \tfrac{8}{3}, 0, 0, 47), \text{ is optimal if } -\tfrac{5}{6} \leq t \leq \tfrac{1}{3}.$$

Introducing P_4 into this last basis gives still another extreme point solution

$$\lambda = (\tfrac{4}{3}, \tfrac{11}{3}, \tfrac{13}{3}, 1, 0, 0).$$

It may be easily shown by calculating Tableau IV and the inequalities

$$z_5 - c_5 \geq 0,$$
$$z_6 - c_6 \geq 0$$

that this solution is optimal if $t \leq -\frac{5}{6}$. Hence the problem is solved. Below is a table of extreme point solutions with corresponding values of t for which each solution is optimal.

Tableau Number	Solution	From t =	To t =
I	$\lambda = (4, 2, 0, 0, 4, 103)$	$\frac{1}{3}$	5
II	$\lambda = (2, 0, 0, 4, 14, 55)$	5	$+\infty$
III	$\lambda = (\frac{8}{3}, \frac{10}{3}, \frac{8}{3}, 0, 0, 47)$	$-\frac{5}{6}$	$\frac{1}{3}$
IV	$\lambda = (\frac{4}{3}, \frac{11}{3}, \frac{13}{3}, 1, 0, 0)$	$-\infty$	$-\frac{5}{6}$

$c_1 = 2 + t$ $\qquad c_2 = 3 - 2t$ $\qquad c_3 = 0$

$c_4 = 0$ $\qquad c_5 = 0$ $\qquad c_6 = 0$

Tableau I — Column Vectors

Unit Values	Basis Elements	P_0	P_1	P_2	P_3	P_4	P_5	P_6
2 + t	P_1	4	1	0	$\frac{1}{2}$	$\frac{1}{2}$	0	0
3 - 2t	P_2	2	0	1	$-\frac{1}{2}$	$\frac{1}{2}$	0	0
0	P_5	4	0	0	$\frac{3}{2}$	$-\frac{5}{2}$	1	0
0	P_6	103	0	0	21	12	0	1
Net Differences	$z_j - c_j$ —	14	0	0	$-\frac{1}{2}$ $+\frac{3}{2}t$	$\frac{5}{2}$ $-\frac{1}{2}t$	0	0

Tableau II — Column Vectors

Unit Values	Basis Elements	P_0	P_1	P_2	P_3	P_4	P_5	P_6
2 + t	P_1	2	1	-1	1	0	0	0
0	P_4	4	0	2	-1	1	0	0
0	P_5	14	0	5	-1	0	1	0
0	P_6	55	0	-24	33	0	0	1
Net Differences	$z_j - c_j$ —	4 +2t	0	-5 +t	2 +t	0	0	0

Tableau III — Column Vectors

Unit Values	Basis Elements	P_0	P_1	P_2	P_3	P_4	P_5	P_6
$2 + t$	P_1	$\frac{8}{3}$	1	0	0	$\frac{4}{3}$	$-\frac{1}{3}$	0
$3 - 2t$	P_2	$\frac{10}{3}$	0	1	0	$-\frac{1}{3}$	$\frac{1}{3}$	0
0	P_3	$\frac{8}{3}$	0	0	1	$-\frac{5}{3}$	$\frac{2}{3}$	0
0	P_6	47	0	0	0	47	-14	1
Net Differ-ences	$z_j - c_j$ —	$\frac{46}{3} - 4t$	0	0	0	$\frac{5}{3} + 2t$	$\frac{1}{3} - t$	0

Tableau IV — Column Vectors

Unit Values	Basis Elements	P_0	P_1	P_2	P_3	P_4	P_5	P_6
$2 + t$	P_1	$\frac{4}{3}$	1	0	0	0	$\frac{3}{47}$	$-\frac{4}{141}$
$3 - 2t$	P_2	$\frac{11}{3}$	0	1	0	0	$\frac{11}{47}$	$\frac{1}{141}$
0	P_3	$\frac{13}{3}$	0	0	1	0	$\frac{8}{47}$	$\frac{5}{141}$
0	P_4	1	0	0	0	1	$-\frac{14}{47}$	$\frac{1}{47}$
Net Differ-ences	$z_j - c_j$ —	$\frac{41}{3} - 6t$	0	0	0	0	$\frac{39}{47} - \frac{19}{47}t$	$-\frac{5}{141} - \frac{6t}{141}$

CHAPTER IV

THE PROBLEMS IN WHICH THE COST COEFFICIENTS ARE NONLINEAR FUNCTIONS OF A PARAMETER

A. INTRODUCTION

In Chapter III there was a discussion of the linear programming problem in which the cost coefficients, that is, the c_j's $(j = 1,2,\ldots,n)$, are linear functions of a single parameter t. In other words an optimal solution is sought for every value of t, and $c_j = g_j + h_j t$ $(j = 1,2,\ldots,n)$ where g_j and h_j are constants. In this chapter a similar problem will be discussed; however, the cost coefficients will not be linear functions of t. In Section B the problem in which the cost coefficients are parabolic functions of a single parameter will be discussed, while Section C will contain a discussion of the problem in which the cost coefficients are periodic functions of a single parameter.

B. THE PARABOLIC CASE

In many ways the parabolic case is similar to the linear case. The first step is to assign a particular value to t and for this value to find the maximal solution by the simplex method. The next step is to determine for what set of values of t the known extreme point solution is optimal. This latter step is accomplished by solving the set of inequalities,

$$z_j - c_j \geq 0 \qquad (j = 1,2, \ldots, n),$$

corresponding to the extreme point solution being considered. Since the z_j's $(j = 1,2, \ldots, n)$ are linear functions of the c_j's and the c_j's are parabolic functions of t, then $z_j - c_j$ may be written in the form,

$$(4\text{-}1) \qquad z_j - c_j = \alpha_j + \beta_j t + \gamma_j t^2 \qquad (j = 1,2, \ldots, n).$$

Now if the problem is to find a maximal solution for every value of t and if a simplex tableau has been calculated which gives a maximal solution for some particular value of t, then the set of values of t for which this solution is maximal is the set of values of t satisfying the inequalities,

$$(4\text{-}2) \qquad z_j - c_j = \alpha_j + \beta_j t + \gamma_j t^2 \geq 0 \qquad (j = 1,2, \ldots, n).$$

The set of values of t satisfying any <u>one</u> of the inequalities (4-2) takes one of four possible forms if it contains at least one value. The particular form depends upon the constants α_j, β_j, and γ_j. The possibilities may be divided into two groups, depending on whether γ_j is positive or negative. Let $y_j(t) \equiv \alpha_j + \beta_j t + \gamma_j t^2$. Then the graph of $y_j(t)$ plotted against t will be concave upward if γ_j is positive and concave downward if γ_j is negative. If the graph of $y_j(t)$

is concave upward, it either intersects the t-axis in two places, is tangent to the t-axis, or lies entirely above the t-axis. If the graph of $y_j(t)$ is concave downward, it either intersects the t-axis in two places or is tangent to the t-axis. The possibility that the graph of $y_j(t)$ is concave downward and lies entirely below the t-axis is ruled out by the assumption that, for <u>some</u> value of t and every value of j, $y_j(t) \geq 0$. If $y_j(t)$ is concave upward and intersects the t-axis at two places, say at $t = t_1$ and $t = t_2$, where $t_1 < t_2$, then $y_j(t) \geq 0$ for every value of t except those values between t_1 and t_2. If $y_j(t)$ is concave upward and is either tangent to the t-axis or entirely above it, then $y_j(t) \geq 0$ for every value of t. If the graph of $y_j(t)$ is concave downward and intersects the t-axis at two places, say at $t = t_1$ and $t = t_2$, where $t_1 < t_2$, then $y_j(t) \geq 0$ for every value of t such that $t_1 \leq t \leq t_2$. If the graph of $y_j(t)$ is concave downward and is tangent to the t-axis, say at $t = \bar{t}$, then $y_j(t) = 0$ at $t = \bar{t}$ and $y_j(t) < 0$ for every other value of t. Obviously if P_j is in the basis upon which the simplex tableau is based, then $\alpha_j = \beta_j = \gamma_j = 0$. (See Chapter III, Section B.) To every value of j there corresponds a set of values of t such that $y_j(t) \geq 0$. Let T_j denote the set of values of t such that $y_j(t) \geq 0$. The table below is a list of all possible forms of T_j and the direction of concavity and number of points of intersection with the t-axis of each corresponding $y_j(t)$. The last column in the table is a list of the "complements" of the T_j's, that is, of the portions of the t-axis not contained in T_j.

Direction of Concavity of $y_j(t)$	Number of Intersection Points of $y_j(t)$ With t-axis	Description of T_j	Description of Complement of T_j
Upward	One or None	The entire t-axis	The empty set
Upward	Two	The entire t-axis except a finite open interval	A finite open interval
Downward	One	One point	The entire t-axis except one point
Downward	Two	A finite closed interval	The entire t-axis except a finite closed interval

For those values of t which are common to all of the T_j's ($j = 1,2, \ldots, n$) the extreme point solution under consideration is optimal. For those values of j such that P_j is a basis vector, $y_j(t) = 0$ for all values of t.

Hence if P_j is a basis vector, T_j is the entire t-axis. If $\sqrt{}_j = 0$ for some value of j such that P_j is not a basis vector, then T_j is the half line defined by $\alpha_j + \beta_j t \geq 0$. Those points t (points in one-dimensional space) which are the "intersection" of the T_j's (j = 1,2, ..., n) are the points for which the given solution is optimal. Let T be defined as this intersection of the T_j's (j = 1,2, ..., n). Now T may take one of several forms. It may be only a single point. Obviously it cannot be larger than the smallest of the sets T_j. T is assumed to contain at least one point. If, for some value of j, T_j is a single point, then T is also this point. T may contain a single closed interval or several closed intervals. It may contain all of the t-axis or all of the t-axis except one or more finite open intervals; this is obviously a case in which $\sqrt{}_j \geq 0$ (j = 1,2, ..., n). If $\sqrt{}_j < 0$ for any value of j, then for this value of j, $y_j(t)$ is concave downward and T_j contains at most a finite closed interval. Hence T contains no points outside this interval. The reader may determine all the possibilities of "intersections" of the T_j by examining the table of possible forms of T_j.

Upon determining T for a given extreme point solution, one may proceed to determining the set of values of t for which another extreme point solution is optimal. The set of values for which the first solution considered is optimal is either the entire t-axis (in which case the problem is solved), or it contains at least one end point. At this end point $y_j(t) = 0$ for some value of j, say $j = k$. If P_k may be introduced into the basis by the simplex method, then introducing P_k into the basis to give a new extreme point solution gives an alternate optimal solution at this end point. (See Chapter III, Section B.) Then by constructing the new simplex tableau one may determine the set of values for which the new solution is optimal. The process may be repeated until for each value of t either an optimal solution has been determined or the functional is known to have no optimum value. If P_k cannot be introduced into the basis by the simplex method because for every value of i, $x_{ik} \leq 0$, then for those values of t such that $z_k - c_k = \alpha_k + \beta_k t + \gamma_k t^2 < 0$ the problem has no optimal solution. By referring to the column labeled "Description of Complement of T_j" in the table on page 111, one may see for what set of values the problem has no maximal solution. In other words T_k is the set of values of t for which $z_k - c_k = \alpha_k + \beta_k t + \gamma_k t^2 \geq 0$, and the complement of this set of values is the set of values for which $z_k - c_k = \alpha_k + \beta_k t + \gamma_k t^2 < 0$ and hence for which there is no maximal solution. Recall that this is case I of the simplex method. (See Chapter I, Section H, and Chapter III, Section B.) An example of this type of problem will be worked in Section D of this chapter.

C. THE PERIODIC CASE

The next case to be considered is the case in which the cost coefficients — the c_j's — are periodic functions of a single parameter. In other words, one such case could be

$$c_j = g_j \sin t + h_j \cos t \qquad (j = 1,2, \ldots, n).$$

The problem is to determine an optimal solution to a linear programming problem for every value of t. Again one starts by assigning a value to t and determining an optimal solution for this particular value of t by the simplex method. Then, as before, one must determine for what set of values of t this solution is optimal by solving the inequalities

$$z_j - c_j \geq 0 \qquad (j = 1,2, \ldots, n).$$

This set of inequalities may be rewritten:

$$(4\text{-}3) \quad z_j - c_j = \alpha_j \sin t + \beta_j \cos t \geq 0 \quad (j = 1,2, \ldots, n).$$

The inequalities (4-3) correspond to the inequalities (4-2) of the parabolic case. (See Section B of this chapter.)

Solving the inequalities (4-3) entails first solving the equations

(4-4) $\qquad \alpha_j \sin t + \beta_j \cos t = 0 \qquad (j = 1,2, \ldots, n).$

It is easily verified that the solution to these equations is $t = \arctan(-\beta_j/\alpha_j)$. The solution is multivalued, and the values are spaced at intervals of π along the t-axis. Let t_1 be defined as the solution to (4-4) such that $0 \leq t_1 < \pi$, and let t_2 be defined as the solution to (4-4) such that $\pi \leq t_2 < 2\pi$. In a manner similar to the parabolic case let $y_j(t) \equiv \alpha_j \sin t + \beta_j \cos t \quad (j = 1,2, \ldots, n)$. Now $y_j(t)$ is a continuous function of t; therefore, between t_1 and t_2 it is either everywhere positive or everywhere negative. Except for the special case in which $t_1 = 0$ and hence $t_2 = \pi$, the value $t = \pi$ is between t_1 and t_2. Therefore, the sign of $y_j(\pi)$ determines the sign of $y_j(t)$ throughout the interval $t_1 < t < t_2$. Now

$$y_j(\pi) = \alpha_j \sin \pi + \beta_j \cos \pi$$
$$= \alpha_j(0) + \beta_j(-1)$$
$$= -\beta_j.$$

So, except for the special case mentioned above, the sign of $y_j(t)$ between t_1 and t_2 is the sign opposite to the sign of β_j. The sign of $y_j(t)$ between 0 and t_1 and between t_2 and

2π is opposite to the sign of $y_j(t)$ between t_1 and t_2. The special case of $t_1 = 0$ and $t_2 = \pi$ implies that $\beta_j = 0$. In this case the sign of $y_j(t)$ between t_1 and t_2 is the same as the sign of α_j, and the sign of $y_j(t)$ between t_2 and 2π is the opposite to the sign of α_j. In other words the solution between 0 and 2π to any one of the inequalities (4-3) is either an interval of the form $t_1 \leq t \leq t_2$ or two intervals of the forms $0 \leq t \leq t_1$ and $t_2 \leq t < 2\pi$. Over the entire axis there are touching intervals of length π. For example, the interval, $t_1 \leq t \leq t_2$, is followed by the interval, $t_2 \leq t \leq t_2 + \pi$; and this interval is followed by the interval, $t_2 + \pi \leq t \leq t_2 + 2\pi$. If one starts with the interval, $t_1 \leq t \leq t_2$, and calls it interval number one and then proceeds to the right, numbering the succeeding intervals in numerical order, then the solution to the inequality is either the set of values of t in the even-numbered intervals or the set of values of t in the odd-numbered intervals.

It is only necessary to determine the solution to (4-3) in the interval $0 \leq t < 2\pi$ because $y_j(t)$ is a periodic function of t with a period of 2π for all values of j. After the solution to each individual inequality of the set (4-3) has been found, then the solution must be found to the entire set. Let T_j, as in the parabolic case, be defined as the set of values of t such that $0 \leq t < 2\pi$ and such that $y_j(t) \geq 0$, in other words, the solution to one of the inequalities (4-3). Then again T will denote the set of values of t common to all of the T_j's or the "intersection" of the T_j's. If P_j is a basis vector for the extreme point solution under consideration, then $\alpha_j = \beta_j = 0$. (See Chapter III, Section B.) T may be only one point; it may be one closed interval. In the parabolic case it is possible for T to be any number of closed intervals. However, in the periodic case it cannot be more than two intervals. This statement will now be proved. The sets T_j are of two possible forms. Either T_j is a closed interval of length π, whose left end point is greater than or equal to zero and whose right end point is less than 2π; or T_j is a pair of intervals, the sum of whose lengths is π and such that (1) the first interval has zero for a left end point and is closed on both ends and (2) the second interval is closed on the left end and is open on the right end and approaches 2π as a limit. In other words, let $t_{1j} = \arctan(-\beta_j/\alpha_j)$ and such that

$$0 \leq t_{1j} < \pi$$

and let $t_{2j} = \arctan(-\beta_j/\alpha_j)$ and such that

$$\pi \leq t_{2j} < 2\pi.$$

Then $t_{2j} = t_{1j} + \pi$; and T_j is either the set of points t such that $t_{1j} \leq t \leq t_{2j}$, or T_j is the set of points t such that <u>either</u> $0 \leq t \leq t_{1j}$ or $t_{2j} \leq t < 2\pi$. Assume without loss of generality that, for $j = m + 1, m + 2, \ldots, q$, T_j is one closed interval and that, for $j = q + 1, q + 2, \ldots, n$, T_j is the pair of intervals as described above. (Recall that for those values of j ($j = 1,2, \ldots, m$) such that P_j is a basis vector $y_j(t)$ is identically zero and T_j is the entire interval, $0 \leq t < 2\pi$.) Now T is that set of values of t which satisfy the following:

$$(4\text{-}5)\quad \left[\begin{array}{l} t_{1,m+1} \leq t \leq t_{2,m+1} \\ t_{1,m+2} \leq t \leq t_{2,m+2} \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ t_{1q} \leq t \leq t_{2q}, \end{array}\right.$$

$$(4\text{-}6)\quad \left[\begin{array}{l} 0 \leq t \leq t_{1,q+1} \\ 0 \leq t \leq t_{1,q+2} \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ 0 \leq t \leq t_{1n}, \end{array}\right.$$

$$(4\text{-}7)\quad \left[\begin{array}{l} t_{2,q+1} \leq t < 2\pi \\ t_{2,q+2} \leq t < 2\pi \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ t_{2n} \leq t < 2\pi. \end{array}\right.$$

Let t_{1r} be the maximum of the values

$$t_{1,m+1},\ t_{1,m+2},\ \ldots,\ t_{1q}$$

and let t_{1s} be the minimum of these values. Then, in order to satisfy the inequalities (4-5), a point t must satisfy the condition: $t_{1r} \leq t \leq t_{2s}$.

Let t_{1v} be the minimum of the values

$$t_{1,q+1},\ t_{1,q+2},\ \ldots,\ t_{1n}$$

and let t_{1u} be the maximum of these values. Then, in order to satisfy the inequalities (4-6), a point t must satisfy the condition: $0 \leq t \leq t_{1v}$.

In order to satisfy the inequalities (4-7), a point t must satisfy the condition: $t_{2u} \leq t < 2\pi$.

If all of the T_j's are one-interval sets, that is, may be defined by inequalities like the inequalities (4-5), then T is simply the closed interval, $t_{1r} \leq t \leq t_{2s}$.

If all of the T_j's are two-interval sets, that is, may be defined by two sets of inequalities like (4-6) and (4-7), then T is simply the two intervals, $0 \leq t \leq t_{1v}$ and $t_{2u} \leq t < 2\pi$.

Now assume that some of the T_j's are one-interval sets and some of the T_j's are two-interval sets. Then either $t_{1r} \leq t_{1v}$ or $t_{2u} \leq t_{2s}$ if T contains at least one point.

Both of these inequalities cannot be true except in the special case in which $t_{1r} = t_{1s} = t_{1v} = t_{1u}$. This is true because

$$t_{1r} \leq t_{1v}$$

implies that $t_{2r} \leq t_{2v}$,

but $t_{2s} \leq t_{2r}$

and $t_{2v} \leq t_{2u}$.

Hence $t_{2s} \leq t_{2r} \leq t_{2v} \leq t_{2u}$

and the inequality $t_{2u} \leq t_{2s}$ is satisfied only if the equality sign holds throughout. Similarly

$$t_{2u} \leq t_{2s}$$

implies that $t_{1u} \leq t_{1s}$,

but $t_{1s} \leq t_{1r}$

and $t_{1v} \leq t_{1u}$.

Hence $t_{1v} \leq t_{1u} \leq t_{1s} \leq t_{1r}$

and the inequality $t_{1r} \leq t_{1v}$ is satisfied only if the equality sign holds throughout. In this special case T is the two points t_{1r} and t_{2r}. In all other cases if $t_{1r} \leq t_{1v}$, then $t_{2u} > t_{2s}$ and T is the set of points t such that $t_{1r} \leq t \leq t_{1v}$; and if $t_{2u} \leq t_{2s}$, then $t_{1r} > t_{1v}$ and T is the set of points t such that $t_{2u} \leq t \leq t_{2s}$.

When one has determined T, the set of points for which a particular extreme point solution is optimal, he is ready to determine the set of values of t for which another extreme point solution is optimal. The set of values of t for which the first solution considered is optimal has at least one end point unless it is the entire interval $0 \leq t < 2\pi$, in which case the problem is solved. At this end point $y_j(t) = 0$ for some value of j, say $j = k$. If P_k may be introduced into the basis by the simplex method, then introducing P_k into the basis to give a new extreme point solution gives an alternate optimal solution at the end point. Then by constructing the new simplex tableau one may determine the set of values for which the new solution is optimal. If P_k cannot be introduced into the basis by the simplex method because for every value of i, $x_{ik} \leq 0$, then for those values of t such that

$$z_k - c_k = \alpha_k \sin t + \beta_k \cos t < 0$$

the problem has no optimal solution. The set of these values such that $0 \leq t < 2\pi$ will be either an open interval of length π or two intervals, the sum of whose lengths is π. In other words it will be all those values such that $0 \leq t < 2\pi$ and which are not contained in T_k. By repeating the processes outlined above, one will eventually determine an optimal solution for every value of t or determine for what values of t there is no optimal solution. In Section E of this chapter a problem of the type outlined in this section will be worked.

D. SOLUTION OF A PARABOLIC PROBLEM

The best way to understand the problems discussed in Sections B and C of this chapter is to see the solution of numerical problems. For example, consider the problem:

$$\text{maximize } f(\lambda) = \sum_{j=1}^{6} c_j \lambda_j$$

subject to

$$\begin{aligned} \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 \qquad\qquad &= 10 \\ -2\lambda_1 - 6\lambda_2 - 3\lambda_3 \qquad + \lambda_5 \qquad &= -6 \\ -15\lambda_1 + 20\lambda_2 - 12\lambda_3 \qquad\qquad + \lambda_6 &= 120. \end{aligned}$$

The coefficient matrix is

P_1	P_2	P_3	P_4	P_5	P_6	P_0
1	1	1	1	0	0	10
-2	-6	-3	0	1	0	-6
-15	20	-12	0	0	1	120.

As an example of the parabolic problem one may define the cost coefficients as follows:

$$c_1 = t^2 + 2t + 1$$
$$c_2 = -t^2 - 3t - 2$$
$$c_3 = 2t^2 + 4t - 1$$
$$c_4 = 3t^2 - t + 2$$
$$c_5 = -4t^2 + 2t - 3$$
$$c_6 = -2t^2 + t + 3.$$

The first step is to assign a value to t and to solve the problem for this particular value. Let $t = 1$. Then: $c_1 = 4$; $c_2 = -6$; $c_3 = 5$; $c_4 = 4$; $c_5 = -5$; $c_6 = 2$. In order to start solving for an optimal solution for this particular value of t it is necessary first to find any extreme point solution. The first paragraph of Section B, Chapter II, provides the method for obtaining the solution. A non-negative solution cannot be obtained in terms of the unit vectors P_4, P_5, and P_6. If, however, P_5 is replaced by $-P_5$, then a non-negative solution is available. Let $-P_5$ be denoted by P_7 and added to the coefficient matrix. Then let c_7 be $-M$, a number discussed in Section B, Chapter II. Then, an entirely equivalent problem is the following:

$$\text{maximize } f(\lambda) = \sum_{j=1}^{6} c_j\lambda_j - M\lambda_7$$

subject to

$$\begin{array}{ccccccccc} & P_1 & & P_2 & & P_3 & & P_4 & & P_5 \\ \lambda_1 & \begin{bmatrix} 1 \\ -2 \\ -15 \end{bmatrix} & + \lambda_2 & \begin{bmatrix} 1 \\ -6 \\ 20 \end{bmatrix} & + \lambda_3 & \begin{bmatrix} 1 \\ -3 \\ -12 \end{bmatrix} & + \lambda_4 & \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} & + \lambda_5 & \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \end{array}$$

$$\begin{array}{cccccc} & P_6 & & P_7 & & P_0 \\ + \lambda_6 & \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} & + \lambda_7 & \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} & = & \begin{bmatrix} 10 \\ -6 \\ 120 \end{bmatrix}. \end{array}$$

Then the first extreme point solution is the solution in terms of P_4, P_6, and P_7. Tableau I, on page 131, is the tableau corresponding to this solution. Proceeding by the simplex method, one may calculate Tableaux II, III, IV, and V. Since, in Tableau V, $z_j - c_j \geq 0$ $(j = 1,2,\ldots,7)$, Tableau V provides the optimal solution for t = 1. In order to find the entire set of values for which this solution is optimal it is necessary to solve the inequalities:

$$z_2 - c_2 \geq 0$$

$$z_3 - c_3 \geq 0$$

$$z_4 - c_4 \geq 0.$$

Notice that $z_7 - c_7 \geq 0$ is not included. This is true because P_7 is not a vector in the original problem but was only introduced to provide the initial extreme point solution. Solving these inequalities entails first solving the corresponding equations:

$$z_2 - c_2 = x_{52}c_5 + x_{62}c_6 + x_{12}c_1 - c_2$$

$$= -4c_5 + 35c_6 + c_1 - c_2$$

$$= -52t^2 + 32t + 120 = 0;$$

$$z_3 - c_3 = x_{53}c_5 + x_{63}c_6 + x_{13}c_1 - c_3$$

$$= -c_5 + 3c_6 + c_1 - c_3$$

$$= -3t^2 - t + 14 = 0;$$

$$z_4 - c_4 = x_{54}c_5 + x_{64}c_6 + x_{14}c_1 - c_4$$

$$= 2c_5 + 15c_6 + c_1 - c_4$$

$$= -40t^2 + 22t + 38 = 0.$$

Denoting $z_j - c_j$ by y_j, one may make the following statements.

The solution to $y_2 = 0$ is $t = \dfrac{4 \pm \sqrt{406}}{13} \doteq 1.86, -1.22.$

The solution to $y_4 = 0$ is $t = \dfrac{11 \pm \sqrt{1641}}{40} \doteq 1.29, -0.74.$

The solution to $y_3 = 0$ is $t = 2, -2\frac{1}{3} \doteq 2.00, -2.33$.

Also y_2, y_3, and y_4 are all concave downward.

Therefore, T_2 is the set of points t such that

$$-1.22 \leq t \leq 1.86.$$

T_3 is the set of points t such that

$$-2.33 \leq t \leq 2.00.$$

T_4 is the set of points t such that

$$-0.74 \leq t \leq 1.29.$$

Notice that approximate values are used. Finally T for Tableau V is the set of values t such that

$$-0.74 \leq t \leq 1.29.$$

Since, at both end points of T for Tableau V, $y_4 = 0$, the new tableau is constructed by introducing P_4 into the basis by the simplex method. This gives a basis involving P_4, P_6, and P_1. This is the basis of Tableau IV. Hence no new tableau needs to be constructed.

From Tableau IV the following may be calculated:

$$z_2 - c_2 = -132t^2 + 76t + 196$$

$$z_3 - c_3 = -23t^2 + 10t + 33$$

$$z_5 - c_5 = 20t^2 - 11t - 19.$$

Also $y_2 = 0$ at $t = \dfrac{19 \pm \sqrt{6829}}{66} \doteq 1.54, -0.96$.

And $y_3 = 0$ at $t = -1, \frac{33}{23} \doteq -1.00, 1.43.$

And $y_5 = 0$ at $t = \frac{11 \pm \sqrt{1641}}{40} \doteq 1.29, -0.74.$

Now y_2 and y_3 are concave downward, and y_5 is concave upward.

Hence T_2 is the set of points t such that

$$-0.96 \leq t \leq 1.54.$$

T_3 is the set of points t such that

$$-1.00 \leq t \leq 1.43.$$

T_5 is the set of points t such that

$$\text{either} \quad t \geq 1.29 \quad \text{or} \quad t \leq -0.74.$$

For Tableau IV, T is the set of points common to T_2, T_3, and T_5. In other words it is those points t such that

$$\text{either} \quad 1.29 \leq t \leq 1.43 \quad \text{or} \quad -0.96 \leq t \leq -0.74.$$

At $t = -0.96$, $y_2 = 0$, so at this value of t introducing P_2 into the basis of Tableau IV will give a basis for an alternate optimal solution. This basis turns out to be P_4, P_6, and P_2, the basis of Tableau II. Then from Tableau II one may determine the set of values of t for which its extreme point solution is optimal.

$$z_1 - c_1 = 44t^2 - \tfrac{76}{3}t - \tfrac{196}{3}.$$

$$z_3 - c_3 = 43t^2 - 28t - 65.$$

$$z_5 - c_5 = -2t^2 + \tfrac{2}{3}t + \tfrac{41}{3}.$$

Also $y_1 = 0$ at $t = \frac{19 \pm \sqrt{6829}}{66} \doteq 1.54, -0.96$;

and $y_3 = 0$ at $t = \frac{14 \pm \sqrt{2991}}{43} \doteq 1.60, -0.95$;

and $y_5 = 0$ at $t = \frac{5 \pm \sqrt{1009}}{12} \doteq -2.23, 3.06$.

Since y_1 and y_3 are concave upward and y_5 is concave downward, then T_1 is the set of points t such that

$$\text{either } t \geq 1.54 \text{ or } t \leq -0.96.$$

T_3 is the set of points t such that

$$\text{either } t \geq 1.60 \text{ or } t \leq -0.95.$$

T_5 is the set of points t such that

$$-2.23 \leq t \leq 3.06.$$

Finally, for Tableau II, T is the set of points t such that

$$\text{either } -2.23 \leq t \leq -0.96 \text{ or } 1.60 \leq t \leq 3.06.$$

At $t = 1.60$, $y_3 = 0$. Therefore, introducing P_3 into the basis gives an alternate optimal solution at this point. Introducing P_3 into Tableau II gives the basis P_4, P_6, and P_3, which is the basis of Tableau III. The following calculations are from Tableau III.

$$z_1 - c_1 = \tfrac{46}{3}t^2 - \tfrac{20}{3}t - 22.$$

$$z_2 - c_2 = -86t^2 + 56t + 130.$$

$$z_5 - c_5 = \tfrac{37}{3}t^2 - \tfrac{23}{3}t - 8.$$

Then $y_1 = 0$ at $t = \frac{33}{23}, -1.00 \doteq 1.43, -1.00;$

and $y_2 = 0$ at $t = \dfrac{14 \pm \sqrt{2991}}{43} \doteq 1.60, -0.95;$

and $y_5 = 0$ at $t = \dfrac{23 \pm \sqrt{4081}}{74} \doteq -0.55, 1.17.$

Also y_1 and y_5 are concave upward, and y_2 is concave downward. Hence T_1 is the set of points t such that

$$\text{either} \quad t \geq 1.43 \quad \text{or} \quad t \leq -1.00.$$

T_2 is the set of points t such that

$$-0.95 \leq t \leq 1.60.$$

T_5 is the set of points t such that

$$\text{either} \quad t \geq 1.17 \quad \text{or} \quad t \leq -0.55.$$

Then T for Tableau III is the set of points t such that

$$1.43 \leq t \leq 1.60.$$

Reviewing the various sets of values for which each extreme point solution is optimal, one may see that for every value of t except those values less than -2.23 and those values greater than 3.06 an optimal solution has already been obtained. At t = -2.23, the optimal solution is the one based on Tableau II, and at t = 3.06, the optimal solution is the same. Also, based on this tableau, $y_5 = 0$ at $t = -2.23, 3.06$. Hence introducing P_5 into Tableau II gives a new basis, which also determines an optimal solution for these two points. The new basis is P_4, P_5, and P_2, and the new tableau is Tableau VI (see page 133). The following calculations are based on this tableau.

$$z_1 - c_1 = 31t^2 - \frac{29}{2}t + \frac{47}{2}.$$

$$z_3 - c_3 = \frac{149}{5}t^2 - 17t + \frac{126}{5}.$$

$$z_6 - c_6 = \frac{3}{5}t^2 - \frac{1}{2}t - \frac{41}{10}.$$

Also $y_6 = 0$ at $t = \dfrac{5 \pm \sqrt{1009}}{12} \doteq -2.23,\ 3.06.$

It is not difficult to verify the fact that there is no real solution to $y_1 = 0$ or to $y_3 = 0$. Since y_1, y_3, and y_6 are all concave upward, T_6 and T are the same and are that set of points t such that

$$\text{either} \quad t \geq 3.06 \quad \text{or} \quad t \leq -2.23.$$

To summarize the results, the table on page 131 has been constructed. Approximate values have been used. The exact values may be found in the text in fractional or radical form.

Left End Point of the Interval	Right End Point of the Interval	Tableau Which Provides the Optimal Solution
$t = -\infty$	$t = -2.23$	Tableau VI
$t = -2.23$	$t = -0.96$	Tableau II
$t = -0.96$	$t = -0.74$	Tableau IV
$t = -0.74$	$t = 1.29$	Tableau V
$t = 1.29$	$t = 1.43$	Tableau IV
$t = 1.43$	$t = 1.60$	Tableau III
$t = 1.60$	$t = 3.06$	Tableau II
$t = 3.06$	$t = +\infty$	Tableau VI

Tableau I — Column Vectors

Unit Values	Basis Elements	P_0	P_1	P_2	P_3	P_4	P_5	P_6	P_7
4	P_4	10	1	1	1	1	0	0	0
2	P_6	120	-15	20	-12	0	0	1	0
-M	P_7	6	2	6	3	0	-1	0	1
Net Differences	$z_j - c_j$	280 -6M	-30 -2M	50 -6M	-25 -3M	0	5 +M	0	0

Tableau II — Column Vectors

Unit Values	Basis Elements	P_0	P_1	P_2	P_3	P_4	P_5	P_6	P_7
4	P_4	9	$\frac{2}{3}$	0	$\frac{1}{2}$	1	$\frac{1}{6}$	0	$-\frac{1}{6}$
2	P_6	100	$-\frac{65}{3}$	0	-22	0	$\frac{10}{3}$	1	$-\frac{10}{3}$
-6	P_2	1	$\frac{1}{3}$	1	$\frac{1}{2}$	0	$-\frac{1}{6}$	0	$\frac{1}{6}$
Net Differences	$z_j - c_j$	230	$-\frac{140}{3}$	0	-50	0	$\frac{40}{3}$	0	$-\frac{25}{3}$ +M

Tableau III Column Vectors

Unit Values	Basis Elements	P_0	P_1	P_2	P_3	P_4	P_5	P_6	P_7
4	P_4	8	$\frac{1}{3}$	-1	0	1	$\frac{1}{3}$	0	$-\frac{1}{3}$
2	P_6	144	-7	44	0	0	-4	1	4
5	P_3	2	$\frac{2}{3}$	2	1	0	$-\frac{1}{3}$	0	$\frac{1}{3}$
Net Differences	$z_j - c_j$	330	$-\frac{40}{3}$	100	0	0	$-\frac{10}{3}$	0	$\frac{25}{3}$ +M

Tableau IV Column Vectors

Unit Values	Basis Elements	P_0	P_1	P_2	P_3	P_4	P_5	P_6	P_7
4	P_4	7	0	-2	$-\frac{1}{2}$	1	$\frac{1}{2}$	0	$-\frac{1}{2}$
2	P_6	165	0	65	$\frac{21}{2}$	0	$-\frac{15}{2}$	1	$\frac{15}{2}$
4	P_1	3	1	3	$\frac{3}{2}$	0	$-\frac{1}{2}$	0	$\frac{1}{2}$
Net Differences	$z_j - c_j$	370	0	140	20	0	-10	0	15 +M

Tableau V Column Vectors

Unit Values	Basis Elements	P_0	P_1	P_2	P_3	P_4	P_5	P_6	P_7
-5	P_5	14	0	-4	-1	2	1	0	-1
2	P_6	270	0	35	3	15	0	1	0
4	P_1	10	1	1	1	1	0	0	0
Net Differences	$z_j - c_j$	510	0	100	10	20	0	0	5 +M

Tableau VI Column Vectors

Basis Elements	P_0	P_1	P_2	P_3	P_4	P_5	P_6	P_7
P_4	4	$\frac{7}{4}$	0	$\frac{8}{5}$	1	0	$-\frac{1}{20}$	0
P_5	30	$-\frac{13}{2}$	0	$-\frac{33}{5}$	0	1	$\frac{3}{10}$	-1
P_2	6	$-\frac{3}{4}$	1	$-\frac{3}{5}$	0	0	$\frac{1}{20}$	0

E. SOLUTION OF A PERIODIC PROBLEM

The same problem will be used to illustrate the case in which the cost coefficients are periodic functions of t except that of course the cost coefficients will be defined differently. For example, one may define them as follows:

$$c_1 = \sin t + \cos t$$
$$c_2 = 3 \sin t - 2 \cos t$$
$$c_3 = -\sin t + 5 \cos t$$
$$c_4 = -3 \sin t + 4 \cos t$$
$$c_5 = -\sin t - 3 \cos t$$
$$c_6 = 7 \sin t + 3 \cos t.$$

Again the first step is to assign a value to t and to determine an optimal solution for this particular value of t. Let $t = 0$. Then: $c_1 = 1$; $c_2 = -2$; $c_3 = 5$; $c_4 = 4$; $c_5 = -3$; $c_6 = 3$. Using these values of $c_1, c_2, \ldots, c_6$, one may easily verify that Tableau V provides an optimal solution for $t = 0$

by calculating the values, $z_2 - c_2$, $z_3 - c_3$, and $z_4 - c_4$, which are all greater than zero. To find the entire set of values of t for which Tableau V provides the optimal solution, one again solves the inequalities:

$$z_2 - c_2 \geq 0$$
$$z_3 - c_3 \geq 0$$
$$z_4 - c_4 \geq 0.$$

As before, the corresponding equations must be solved first. Then T_2, T_3, and T_4, the solutions to the individual inequalities, will be determined, and from them T, the solution to the entire set of inequalities, can be determined.

$$z_2 - c_2 = x_{52}c_5 + x_{62}c_6 + x_{12}c_1 - c_2$$
$$= 247 \sin t + 120 \cos t = 0.$$
$$z_3 - c_3 = x_{53}c_5 + x_{63}c_6 + x_{13}c_1 - c_3$$
$$= 24 \sin t + 8 \cos t = 0.$$
$$z_4 - c_4 = x_{54}c_5 + x_{64}c_6 + x_{14}c_1 - c_4$$
$$= 107 \sin t + 36 \cos t = 0.$$

Again denoting $z_j - c_j$ by y_j, one may summarize the results as follows:

the solution to $y_2 = 0$ is $t = \arctan\left(-\frac{120}{247}\right)$

$$= \arctan(-0.4858)$$
$$= 2.689, 5.831;$$

the solution to $y_3 = 0$ is $t = \arctan(-\frac{1}{3})$

$= \arctan(-0.3333)$

$= 2.820, 5.962;$

the solution to $y_4 = 0$ is $t = \arctan(-\frac{36}{107})$

$= \arctan(-0.3364)$

$= 2.817, 5.959.$

At $t = \pi$, $y_2 < 0$, $y_3 < 0$, and $y_4 < 0$. Therefore, T_2 is the set of points t such that

either $0 \leq t \leq 2.689$ or $5.831 \leq t < 2\pi$.

T_3 is the set of points t such that

either $0 \leq t \leq 2.820$ or $5.962 \leq t < 2\pi$.

T_4 is the set of points t such that

either $0 \leq t \leq 2.817$ or $5.959 \leq t < 2\pi$.

Hence, for Tableau V, T is the set of points t such that

either $0 \leq t \leq 2.689$ or $5.962 \leq t < 2\pi$.

At $t = 2.689$, $y_2 = 0$; and at $t = 5.962$, $y_3 = 0$. Introducing P_2 into Tableau V gives Tableau VII, and introducing P_3 into Tableau V gives Tableau VIII. The calculation of T for each of these and for succeeding tableaux will be given in tabular form on pages 136, 137, and 138.

Results from Tableau VII

j	$y_j = z_j - c_j$	Solutions to $y_j(t) = 0$	Conditions on Points t Belonging to T_j
3	$\frac{99}{35} \sin t - \frac{80}{35} \cos t$	$t = 0.680$, 3.821	$0.680 \leq t \leq 3.821$
4	$\frac{8}{7} \sin t - \frac{108}{7} \cos t$	$t = 1.497$, 4.638	$1.497 \leq t \leq 4.638$
6	$-\frac{247}{35} \sin t - \frac{120}{35} \cos t$	$t = 2.689$, 5.831	$2.689 \leq t \leq 5.831$

For Tableau VII, T is the set of points t such that

$$2.689 \leq t \leq 3.821.$$

Results from Tableau VIII

j	$y_j = z_j - c_j$	Solutions to $y_j(t) = 0$	Conditions on Points t Belonging to T_j
1	$-24 \sin t - 8 \cos t$	$t = 2.820$, 5.962	$2.820 \leq t \leq 5.962$
2	$223 \sin t + 112 \cos t$	$t = 2.676$, 5.818	$0 \leq t \leq 2.676$ $5.818 \leq t < 2\pi$
4	$83 \sin t + 28 \cos t$	$t = 2.816$, 5.958	$0 \leq t \leq 2.816$ $5.958 \leq t < 2\pi$

For Tableau VIII, T is the set of points t such that

$$5.958 \leq t \leq 5.962.$$

In the results for Tableau VII, $y_3 = 0$ at $t = 3.821$; and introducing P_3 into Tableau VII gives Tableau IX. In the results for Tableau VIII, $y_4 = 0$ at $t = 5.958$; and introducing P_4 into Tableau VIII gives Tableau III.

Results from Tableau IX

j	$y_j = z_j - c_j$	Solutions to $y_j(t) = 0$	Conditions on Points t Belonging to T_j
1	$-\frac{99}{32} \sin t + \frac{80}{32} \cos t$	$t = 0.680$, 3.821	$0 \leq t \leq 0.680$ $3.821 \leq t < 2\pi$
4	$-\frac{5}{8} \sin t - \frac{112}{8} \cos t$	$t = 1.615$, 4.757	$1.615 \leq t \leq 4.757$
6	$-\frac{223}{32} \sin t - \frac{112}{32} \cos t$	$t = 2.676$, 5.818	$2.676 \leq t \leq 5.818$

Results from Tableau III

j	$y_j = z_j - c_j$	Solutions to $y_j(t) = 0$	Conditions on Points t Belonging to T_j
1	$-\frac{155}{3} \sin t - \frac{52}{3} \cos t$	$t = 2.818$, 5.960	$2.818 \leq t \leq 5.960$
2	$306 \sin t + 140 \cos t$	$t = 2.713$, 5.854	$0 \leq t \leq 2.713$ $5.854 \leq t < 2\pi$
5	$-\frac{83}{3} \sin t - \frac{28}{3} \cos t$	$t = 2.816$, 5.958	$2.816 \leq t \leq 5.958$

For Tableau IX, T is the set of points t such that

$$3.821 \leq t \leq 4.757.$$

At $t = 4.757$, $y_4 = 0$; and introducing P_4 into Tableau IX gives Tableau VI. For Tableau III, T is the set of points t such that

$$5.854 \leq t \leq 5.958.$$

At $t = 5.854$, $y_2 = 0$; and introducing P_2 into Tableau III gives Tableau II.

Results from Tableau VI

j	$y_j = z_j - c_j$	Solutions to $y_j(t) = 0$	Conditions on Points t Belonging to T_j
1	$-2 \sin t + 27 \cos t$	$t = 1.497$, 4.638	$0 \leq t \leq 1.497$ $4.638 \leq t < 2\pi$
3	$\sin t + \frac{112}{5} \cos t$	$t = 1.615$, 4.757	$0 \leq t \leq 1.615$ $4.757 \leq t < 2\pi$
6	$-7 \sin t - \frac{21}{5} \cos t$	$t = 2.601$, 5.743	$2.601 \leq t \leq 5.743$

Results from Tableau II

j	$y_j = z_j - c_j$	Solutions to $y_j(t) = 0$	Conditions on Points t Belonging to T_j
1	$-\frac{461}{3} \sin t - \frac{192}{3} \cos t$	$t = 2.747$, 5.888	$2.747 \leq t \leq 5.888$
3	$-153 \sin t - 70 \cos t$	$t = 2.713$, 5.854	$2.713 \leq t \leq 5.854$
5	$\frac{70}{3} \sin t + \frac{42}{3} \cos t$	$t = 2.601$, 5.743	$0 \leq t \leq 2.601$ $5.743 \leq t < 2\pi$

For Tableau VI, T is the set of points t such that

$$4.757 \leq t \leq 5.743.$$

For Tableau II, T is the set of points t such that

$$5.743 \leq t \leq 5.854.$$

Summarizing results, as in the parabolic case, one may see that for every value of t an optimal solution has been determined. The results are shown in the table on page 139.

Left End Point of the Interval	Right End Point of the Interval	Tableau Which Provides the Optimal Solution
t = 0	t = 2.689	Tableau V
t = 2.689	t = 3.821	Tableau VII
t = 3.821	t = 4.757	Tableau IX
t = 4.757	t = 5.743	Tableau VI
t = 5.743	t = 5.854	Tableau II
t = 5.854	t = 5.958	Tableau III
t = 5.958	t = 5.962	Tableau VIII
t = 5.962	t = 2π	Tableau V

These results are, of course, approximate values.

Tableau VII — Column Vectors

Basis Elements	P_0	P_1	P_2	P_3	P_4	P_5	P_6	P_7
P_5	$\frac{314}{7}$	0	0	$-\frac{23}{35}$	$\frac{26}{7}$	1	$\frac{4}{35}$	-1
P_2	$\frac{54}{7}$	0	1	$\frac{3}{35}$	$\frac{3}{7}$	0	$\frac{1}{35}$	0
P_1	$\frac{16}{7}$	1	0	$\frac{32}{35}$	$\frac{4}{7}$	0	$-\frac{1}{35}$	0

Tableau VIII Column Vectors

Basis Elements	P_0	P_1	P_2	P_3	P_4	P_5	P_6	P_7
P_5	24	1	-3	0	3	1	0	-1
P_6	240	-3	32	0	12	0	1	0
P_3	10	1	1	1	1	0	0	0

Tableau IX Column Vectors

Basis Elements	P_0	P_1	P_2	P_3	P_4	P_5	P_6	P_7
P_3	$\frac{5}{2}$	$\frac{35}{32}$	0	1	$\frac{5}{8}$	0	$-\frac{1}{32}$	0
P_5	$\frac{93}{2}$	$\frac{23}{32}$	0	0	$\frac{33}{8}$	1	$\frac{3}{32}$	-1
P_2	$\frac{15}{2}$	$-\frac{3}{32}$	1	0	$\frac{3}{8}$	0	$\frac{1}{32}$	0

B I B L I O G R A P H Y

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