LIMIT CYCLES IN LINEAR SYSTEMS
WITH A RELAY IN THE
FEEDBACK LOOP

APPROVED:

Supervisor: ____________________

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LIMIT CYCLES IN LINEAR SYSTEMS WITH A RELAY IN THE FEEDBACK LOOP

by

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REPORT

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ABSTRACT

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A survey of different methods to investigate the existence of limit cycles in linear systems with nonlinear gain elements in the feedback loop is presented. The complete analytical solution of the problem in the case of asymptotically stable second order linear time invariant systems with an ideal relay in the feed back loop is also derived.

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Chapter 1

Classical Methods

1.1 Describing Function Method

Some of the methods used in dealing with nonlinear systems are based on the construction of a linear "approximate" model of the original system. The basic assumption required for the describing function method is that the closed loop system acts like a low-pass or band-pass filter. Consider the system shown in Figure 1.1, where L is a stable linear time invariant causal system and N is a nonlinear gain element whose output q(t), in terms of its input p(t) is given as

$$q(t) = N(p(t), \dot{p}(t))$$

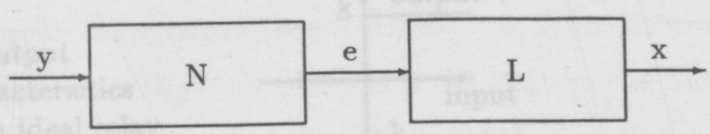


Figure 1.1: Block diagram of the system

Now if a sinusoidal input is applied at the input, $y(t) = A \sin u$, where $u = \omega t$, then $e(t)$ is also periodic and can be expanded in Fourier series as

$$e(t) = a_0 + \sum_{n=1}^{\infty} (a_n \sin nu + b_n \cos nu) \; ,$$

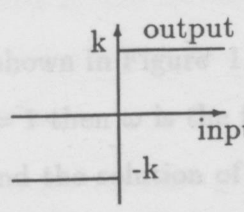
$$
\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_0^{2\pi} N(A\sin u, A\omega\cos u)du \ , \\
a_n &= \frac{1}{\pi}\int_0^{2\pi} N(A\sin u, A\omega\cos u)\sin nu \, du \ , \\
b_n &= \frac{1}{\pi}\int_0^{2\pi} N(A\sin u, A\omega\cos u)\cos nu \, du \ .
\end{aligned}
\tag{1.1}
$$

Now if the linear system is such that it passes the fundamental frequency and also blocks all of the harmonics, then the output x is a sinusoid of the same frequency as the input. In this case, if the loop is closed, the conditions for sustained oscillation are essentially the same as those for the system when N is replaced with a gain element $\frac{a_1+jb_1}{A}$ as given by (1.1).

In general the describing function for a nonlinearity is defined as the ratio of the complex amplitude (amplitude and phase) of the fundamental frequency in the output of the nonlinear element, when subjected to a sinusoidal input, to the amplitude of input.

Example: If N is an ideal relay then

Input-Output
characteristics
of an ideal relay



$$
a_0 = \frac{1}{2\pi}\int_0^{2\pi} N du = \frac{k}{2\pi}\left(\int_0^{\pi} du - \int_{\pi}^{2\pi} du\right) = 0 \ ,
$$

$$
\begin{aligned}
a_1 = \frac{k}{\pi}\int_0^{2\pi} N\sin u \, du &= \frac{k}{\pi}\left(\int_0^{\pi}\sin u \, du - \int_{\pi}^{2\pi}\sin u \, du\right) = \\
&= \frac{k}{\pi}\left(-\cos u|_0^{\pi} + \cos u|_{\pi}^{2\pi}\right) = \frac{4k}{\pi} \ ,
\end{aligned}
$$

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} N \cos u \, du = \frac{k}{\pi}(\int_0^{\pi} \cos u \, du - \int_{\pi}^{2\pi} \cos u \, du) = 0 \, ,$$

$$\Longrightarrow D(A) = \frac{a_1 + jb_1}{A} = \frac{4k}{\pi A} \, .$$

1.1.1 Finding Limit Cycles Using DF

As already mentioned, in DF method it is assumed that the output of the linear system in self-sustained oscillation is sinusoidal and based on this assumption the nonlinearity is replaced with a linear gain element in the feedback loop.

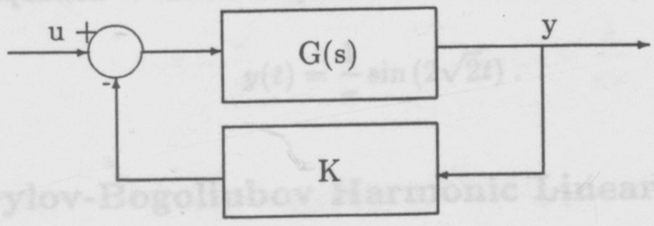


Figure 1.2: Block diagram of the equivalent linearized system.

Consider the system shown in Figure 1.2. From linear system theory it is known that if $-KG(j\omega) = 1$ then ω is the frequency of oscillation of the system. So it is sufficient to find the solution of the equation $-D(A) = \frac{1}{G(j\omega)}$ to find the amplitude and the frequency of the limit cycle. One way is to plot the locus of $\frac{1}{G(j\omega)}$ and $-D(A)$ on the same polar plot and read A and ω directly from the intersection of the two loci.

Example: Let the nonlinearity be an ideal relay and let

$$G(s) = \frac{-s+1}{s^2+4s+4} \, .$$

We have

$$for\ k = 1\ ,\ D(A) = \frac{4}{\pi A}\ ,\ and \quad G(j\omega) = \frac{-j\omega + 1}{(j\omega + 2)^2}\ ,$$

$$\frac{1}{G(j\omega)} = \frac{(-5\omega^2 + 4) + j\omega(8 - \omega^2)}{\omega^2 + 1}\ ,$$

$$\omega_1^2 - 8 = 0 \Longrightarrow \omega_1 = 2\sqrt{2}\ ,$$

$$\frac{1}{G(j2\sqrt{2})} = -4\ ,$$

$$D(A_1) = \frac{4}{\pi A_1}\ , \qquad A_1 = \frac{1}{\pi}\ .$$

Thus, the equation of the limit cycle is

$$y(t) = \frac{1}{\pi}\sin(2\sqrt{2}t)\ .$$

1.2 Krylov-Bogoliubov Harmonic Linearization

This method, devised by Krylov and Bogoliubov is also called the method of harmonic linearization or the method of harmonic balance. It is necessary to bear in mind that this method, like the DF method, gives approximate solutions to nonlinear systems.

Let the nonlinearity be described by $y = N(x, \dot{x})$, with y the output, x the input and $\dot{x}$ the time derivative of the input. Also let

$$x = a\sin\omega t\ , \qquad u = \omega t\ .$$

Then

$$y = N(a\sin u, a\omega\cos u)\ .$$

Expanding the right-hand side in Fourier series we get

$$
\begin{aligned}
y(t) = {} & \frac{1}{2\pi}\int_0^{2\pi} N(a\sin u, a\omega\cos u)du + \\
& + \left[\frac{1}{\pi}\int_0^{2\pi} N(a\sin u, a\omega\cos u)\sin u\, du\right]\sin\omega t + \\
& + \left[\frac{1}{\pi}\int_0^{2\pi} N(a\sin u, a\omega\cos u)\cos u\, du\right]\cos\omega t + \\
& + \text{Higher Harmonics}\,. \qquad (1.2)
\end{aligned}
$$

The first integral is the DC level in the output of the nonlinear element and for "symmetric" nonlinearities it is zero. Here it is generally assumed to be zero. If higher harmonics are neglected and we note that

$$
\sin\omega t = \frac{x}{a}\,, \qquad \cos\omega t = \frac{\dot{x}}{a\omega}\,,
$$

we get

$$
y(t) \approx q(a,\omega)x + \frac{q'(a,\omega)}{\omega}\dot{x}\,,
$$

where

$$
\begin{aligned}
q(a,\omega) &= \frac{1}{\pi}\int_0^{2\pi} N(a\sin u, a\omega\cos u)\sin u\, du\,, \\
q'(a,\omega) &= \frac{1}{\pi}\int_0^{2\pi} N(a\sin u, a\omega\cos u)\cos u\, du\,,
\end{aligned}
$$

Thus the nonlinearity, for sinusoidal input, is replaced with an amplitude and frequency dependent linear element. This method can be extended to find linear models which take second, third and any finite number of harmonics into account, but there is no guarantee that these complicated models would yield a better approximation.

Now consider the system in Figure 1.3. For $u = 0$ we have

$$
Le = x\,, \qquad e = -y\,, \qquad y = qx + \frac{q'}{\omega}\dot{x}\,,
$$

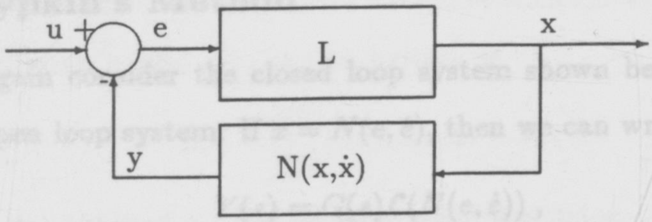


Figure 1.3: Block diagram of the closed loop system

$$-Ly = x\,, \qquad \Longrightarrow \qquad -L(qx + \frac{q'}{\omega}\dot{x}) = x\,.$$

If G(s) is the transfer function form of the linear operator L, we get

$$-qG(s)X - q'G(s)\mathcal{L}(\frac{\dot{x}}{\omega}) = X\,,$$

where $\mathcal{L}$ denotes the Laplace transform;

$$\Longrightarrow \qquad -qG(s)X - \frac{q'}{\omega}G(s)sX = X\,,$$

$$[(q + \frac{q'}{\omega}s)G(s) + 1]X = 0\,.$$

As usual the condition for having oscillation is that

$$(q + \frac{q'}{\omega}s)G(s) + 1 = 0$$

must have pure imaginary roots at $s = j\omega$. The solution can directly be read from the intersection of the polar plots of $\frac{1}{G(j\omega)}$ and $-(q + jq')$.

1.3 Tsypkin's Method

Again consider the closed loop system shown below and the corresponding open loop system. If $x = N(e, \dot{e})$, then we can write

$$Y(s) = G(s)\mathcal{L}(N(e, \dot{e})) ,$$

where G(s) is the transfer function of the linear system and $\mathcal{L}(N(e, \dot{e}))$ is the Laplace transform of the output of the relay when subjected to a periodic input. It should be noted that for a general e(t), X(s) is not necessarily defined, however if e(t) is periodic, the output of the relay is also periodic and its Laplace transform is well-defined.

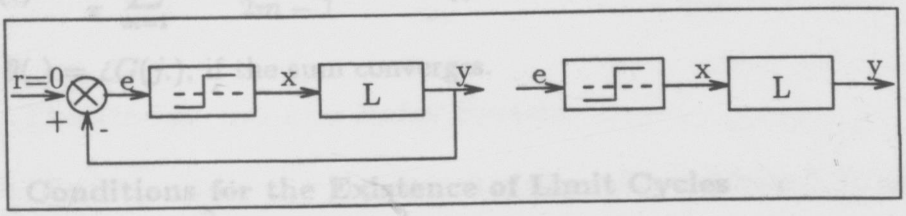


Figure 1.4: Block diagram of the system

In this case, the output of the relay is a sequence of rectangular pulses and can be expanded in Fourier series as

$$x(t) = \frac{1}{2} \sum_{n=-\infty}^{\infty} c_n e^{jn\omega t} ,$$

where c_n for piecewise constant periodic functions are defined as

$$c_n = \frac{1}{j\pi n} \sum_{i=1}^{m+1} \Delta x(t_i) e^{-jn\omega t_i} .$$

Here, $\Delta x(t_i)$ denotes the jumps of $x(t)$ at its points of discontinuity t_i for $i = 1, 2, \ldots, m+1$, i.e.

$$\lim_{\epsilon \to 0} [\Delta x(t_i) = x(t_i + \epsilon) - x(t_i - \epsilon)] .$$

If x(t) is symmetric; which is the case for relays with sinusoidal input, we have $c_n = 0$ for all even n, and x(t) can be written as

$$x(t) = \sum_{n=1}^{\infty} |c_n| \cos(n\omega t - \theta_n) ,$$

where θ_n is the phase of c_n. For an ideal relay this reduces to

$$x(t) = \frac{4k}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin[(2n-1)\omega t] .$$

To find the output of the linear system $y(t)$, due to input $x(t)$, we use the superposition principle and the fact that the input is a sum of sinusoids. Thus,

$$y(t) = \frac{4k}{\pi} \sum_{m=1}^{\infty} \frac{|G[j(2m-1)\omega]|}{2m-1} \sin\left[(2m-1)\omega t + \theta((2m-1)\omega)\right] , \tag{1.3}$$

where $\theta(.) = \angle G(j.)$, if the sum converges.

1.3.1 Conditions for the Existence of Limit Cycles

For the closed loop system we have

$$e(t) = -y(t) .$$

Clearly for the system to have oscillations of half-period $\frac{\pi}{\omega_0}$; refer to Figure 1.5; we need the following conditions to hold:

- *Condition of proper switching times*

$$y(t) = 0 \ , \quad for \ t = \frac{n\pi}{\omega_0} \ , \forall \, n \in \{0, 1, 2, \dots\} , \tag{C1}$$

- *Condition of proper switching direction*

$$\dot{y}(\frac{n\pi}{\omega_0}) \, (-1)^n < 0 \ , \ \forall \ n \in \{0, 1, 2, \dots\} \ , \tag{C2}$$

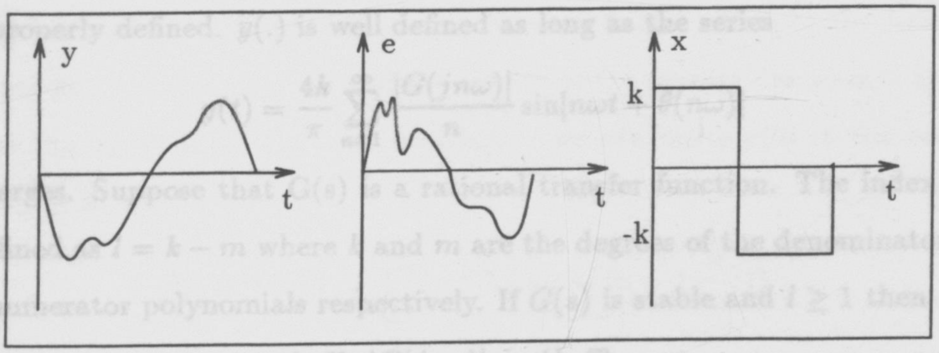


Figure 1.5: Typical signals at nodes

- *Condition of no additional switching*

$$y(t) < 0 \, , \forall \; 0 < t < \frac{\pi}{\omega_0} . \qquad (C3)$$

1.3.2 The Hodograph of a Relay System

If we can find an $\omega > 0$ such that y(t), as obtained from (1.3), satisfies conditions C1, C2 and C3, then the closed loop system will exhibit a limit cycle. Clearly this is not trivial, since $y(t)$ is given as an infinite series.

In this respect an important role is played by the concept of *hodograph*, which gives a graphical interpretation to the conditions of the proper switching time and direction. Tsypkin defines the hodograph for an ideal relay system as

$$J(\omega) = -\frac{1}{\omega}\dot{y}(\frac{\pi}{\omega}) - jy(\frac{\pi}{\omega}) \, , \qquad (1.4)$$

where $y(.)$ is the output of the linear part of the system due to a sinusoidal input of frequency ω at the relay input; refer to the open loop system in Figure 1.1; and $\dot{y}(.)$ is the derivative from left of this output.

It should be noted, that $J(\omega)$ is properly defined if both $y(.)$ and $\dot{y}(.)$

are properly defined. $y(.)$ is well defined as long as the series

$$y(t) = \frac{4k}{\pi} \sum_{n=1}^{\infty} \frac{|G(jn\omega)|}{n} \sin[n\omega t + \theta(n\omega)]$$

converges. Suppose that G(s) is a rational transfer function. The index of G is defined as $l = k - m$ where k and m are the degrees of the denominator and the numerator polynomials respectively. If $G(s)$ is stable and $l \geq 1$ then

$$y(t) \leq \frac{4k}{\pi} \sum_{n=1}^{\infty} \frac{|G(jn\omega)|}{n} \leq \frac{4k}{\pi} \sum_{n=1}^{\infty} |G(jn\omega)| \, .$$

Regarding $\frac{1}{\omega}\dot{y}(t)$ we have

$$\frac{1}{\omega}\dot{y}(t) = \frac{4k}{\pi} \sum_{n=1}^{\infty} |G(jn\omega)| \cos[n\omega t + \theta(n\omega)] \leq \frac{4k}{\pi} \sum_{n=1}^{\infty} |G(jn\omega)| \, .$$

We have $\sum_{n=1}^{\infty} |G(jn\omega)|$ converges if $l \geq 1$ and $G(s)$ is stable. Note that the convergece is preserved if $G(s)$ has a delay term in it. So the hodograph is a well defined function of ω as long as the linear part of the system is stable and has positive index.

For an ideal relay the first two conditions for self oscillations are satisfied at the intersection(s) of the plot of $J(\omega)$ with the negative real axis and the value of ω_0 obtained can be used to check for the third condition.

1.4 Determination of $J(\omega)$

The output of the linear system, $y(t)$, can be found in terms of $h(t)$, the step response of the linear system. Considering that $y(t)$ is symmetric and periodic, it suffices to find it in the interval $\left[0, \frac{\pi}{\omega}\right]$, where ω is the fundamental frequency of the input. Suppose the input is as shown in Figure 1.5. Then the output $y(t)$, in the interval $\left[0, \frac{\pi}{\omega}\right]$ is the sum of the outputs due to all pulses in the interval $(-\infty, 0]$ plus the input between 0 to t. Denote the unit step starting at $t = t_0$ by $u(t - t_0)$ and the output of the system due to $u(t)$ applied at the input, by $h(t)$. By time invariance, we can write $y(t)$ in the interval $0 \leq t \leq \frac{\pi}{\omega}$ as

$$y(t) = k\{h(t) + \sum_{n=1}^{\infty}(-1)^n \Delta h(t + (n-1)\frac{\pi}{\omega}))\} ,$$

where $\Delta h(t - t_0) = h(t) - h(t - t_0)$. Recalling that for systems with positive indexed rational transfer functions $h(0) = 0$, and that $y(\frac{\pi}{\omega}) = -y(0)$ we get

$$y(\frac{\pi}{\omega}) = -k\sum_{n=1}^{\infty}(-1)^n \Delta h((n-1)\frac{\pi}{\omega}) = k\sum_{n=0}^{\infty}(-1)^n \Delta h(n\frac{\pi}{\omega}) .$$

Hence,

$$Im\ J(\omega) = k\sum_{n=0}^{\infty}(-1)^n \Delta h(n\frac{\pi}{\omega}) .$$

To find $Re\ J(\omega)$ we have to find $\dot{y}(t)$ in the same interval. This simply implies using the impulse response in place of the step response to get the desired result.

$$\dot{y}(t) = k\{g(t) + \sum_{n=1}^{\infty}(-1)^n \Delta g(t + (n-1)\frac{\pi}{\omega}))\} ; \quad 0 \leq t \leq \frac{\pi}{\omega} .$$

We should notice that if we evaluate $\dot{y}(\frac{\pi}{\omega})$ this will yield the derivative from left of $y(t)$ at $t = \frac{\pi}{\omega}$ which is the desired value in calculating $J(\omega)$.

$$\dot{y}(\frac{\pi}{\omega}t) = k\{g(\frac{\pi}{\omega}) + \sum_{n=1}^{\infty}(-1)^n \Delta g(n\frac{\pi}{\omega})\} .$$

Now noting that $g(\frac{\pi}{\omega}) = \Delta g(0) + g(0)$ we obtain

$$ReJ(\omega) = -\frac{k}{\omega}\left[g(0) + \sum_{n=1}^{\infty}(-1)^n \Delta g(n\frac{\pi}{\omega})\right] .$$

Thus,

$$J(\omega) = -k[\frac{g(0)}{\omega} + \frac{1}{\omega}\sum_{n=1}^{\infty}(-1)^n \Delta g(n\frac{\pi}{\omega}) + j\sum_{n=1}^{\infty}(-1)^n \Delta h(n\frac{\pi}{\omega})] .$$

To obtain $J(\omega)$ in closed form it is easiest to assume that $G(s) = \frac{P(s)}{Q(s)}$ is with positive index and has only simple nonzero poles. In this case $G(s) = \sum_{i=1}^{N} \frac{r_i}{s-p_i}$, where N is the degree of P(s), p_i are the poles of G(s) and r_i are the residues at corresponding poles. Here we have

$$g(t) = \sum_{i=1}^{N} r_i e^{p_i t} ,$$

$$h(t) = \int_0^t g(\tau)d\tau = \int_0^t \sum_{i=1}^{N} r_i e^{p_i \tau} d\tau = \sum_{i=1}^{N} \frac{r_i}{p_i}(e^{p_i t} - 1) .$$

Then

$$\Delta h(n\frac{\pi}{\omega}) = h((n+1)\frac{\pi}{\omega}) - h(n\frac{\pi}{\omega}) = \sum_{i=1}^{N} \frac{r_i}{p_i}(e^{p_i \frac{\pi}{\omega}} - 1) ,$$

and

$$\Delta g(n\frac{\pi}{\omega}) = \sum_{i=1}^{N} r_i e^{p_i n \frac{\pi}{\omega}}(e^{p_i \frac{\pi}{\omega}} - 1) .$$

Using these results in the formula for $J(\omega)$ we obtain

$$Im\ J(\omega) = -k\sum_{i=1}^{N} \frac{r_i}{p_i}(e^{p_i \frac{\pi}{\omega}} - 1)\sum_{n=0}^{\infty}(-1)^n e^{p_i n \frac{\pi}{\omega}} =$$

$$= -k\sum_{i=1}^{N} \frac{r_i}{p_i} \frac{1 - e^{p_i n \frac{\pi}{\omega}}}{1 + e^{p_i n \frac{\pi}{\omega}}} = -k\sum_{i=1}^{N} \frac{r_i}{p_i} tanh(\frac{\pi p_i}{2\omega}) .$$

And

$$Re\ J(\omega) = -\frac{k}{\omega}[g(0) + \sum_{i=1}^{N} r_i\ tanh(\frac{\pi p_i}{2\omega})] .$$

Finally noting that $g(0) = \sum_{i=1}^{N} r_i$ we get

$$J(\omega) = -k\sum_{i=1}^{N} r_i\{\frac{1}{\omega}[tanh(\frac{\pi p_i}{2\omega}) + 1] + j\frac{1}{p_i} tanh(\frac{\pi p_i}{2\omega})\} .$$

Chapter 2

Limit Cycles in Second Order Systems With a Relay in The Feedback Loop

2.1 Introduction

In this chapter we consider a system described by an asymptotically stable, strictly proper, second order transfer function with an ideal relay in the feedback loop. We break the problem into three distinct cases according to the nature of the poles of the transfer function.

2.2 Some General Properties

In this section we establish some facts about linear systems with a relay in the feedback loop. These facts will later be used in investigating the existence of limit cycles in specific systems. Consider a linear single-input, single-output system:

$$\dot{x} = Ax + bu ,$$

$$y = cx ,$$

with a relay in the feedback loop, i.e.,

$$u = -sgn(y) .$$

The block diagram of the system is shown in Figure 2.1. It should be understood that the linear system is assumed to be minimal.

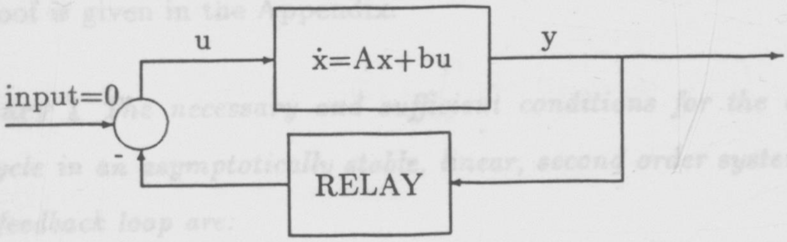


Figure 2.1: Block diagram of the system

Proposition 1 *If $bu = f(x)$ satisfying*

$$f(-x) = -f(x)\ ,$$

and $x(t)$ is a trajectory of the system, then $-x(t)$ is also a trajectory.

Proof: Introduce a set of new states $r = -x$. Then the state equation in terms of the new states becomes

$$-\dot{r} = -Ar + f(x) = -Ar - f(-x) = -Ar - f(x)\ .$$

Hence,

$$\dot{r} = Ar + f(r)\ .$$

Proposition 2 *Consider a second order linear system, described by an asymptotically stable, strictly proper, rational transfer function, with an ideal relay in the feedback loop. If the system has a limit cycle, then:*

i) The limit cycle is symmetric with respect to the origin of the phase plane.

ii) The limit cycle has exactly two switchings of the relay per period.

The proof is given in the Appendix.

Corollary 1 *The necessary and sufficient conditions for the existence of a limit cycle in an asymptotically stable, linear, second order system with a relay in the feedback loop are:*

(C1) Existence of a point $(0,\ z) \in \Re^2$ *and a* $T > 0$*, such that the trajectory* ϕ_t *satisfies* $\phi_T(0,\ z) = (0, -z)$*, and*

(C2) $y(\phi_t(0,\ z)) < 0,\ \forall\ t \in (0,\ T)$*. Where* $y(.)$ *is the output of the linear system.*

The period of the limit cycle is $2T$*.*

2.3 Second Order Systems

The analysis is based on Corollary 1. We distinguish the following cases:

1) $G(s) = \frac{b_1 s + b_0}{(s-p)^2}$

2) $G(s) = \frac{b_1 s + b_0}{(s-p_1)(s-p_2)}$

3) $G(s) = \frac{b_1 s + b_0}{[s-(\sigma+j\omega)][s-(\sigma-j\omega)]}$

Case 1 $G(s) = \frac{b_1 s + b_0}{(s-p)^2}$

The appropriate state space realization of the transfer function is

$$A = \begin{bmatrix} 2p & 1 \\ -p^2 & 0 \end{bmatrix},\quad b = \begin{bmatrix} b_1 \\ b_0 \end{bmatrix},\quad c = \begin{bmatrix} 1 & 0 \end{bmatrix},\quad p < 0.$$

Consider the trajectories

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}bu(\tau)d\tau \ , \tag{2.1}$$

where $x(0) = \begin{bmatrix} 0^- \\ x_2 \end{bmatrix}$. Here, we have

$$e^{At} = \begin{bmatrix} (1+pt)e^{pt} & te^{pt} \\ -p^2te^{pt} & (1-pt)e^{pt} \end{bmatrix} . \tag{2.2}$$

First we analyze condition (C1). Setting $x(T) = -x(0)$ in (2.1), we obtain

$$(e^{AT} + I)x(0) - A^{-1}(I - e^{AT})b = 0 \ . \tag{2.3}$$

where I is the identity matrix with appropriate dimension. Replacing the value of e^{AT} from (2.2) in (2.3) we get

$$\begin{bmatrix} x_2Te^{pT} \\ x_2 + (1-pT)x_2e^{pT} \end{bmatrix} = \begin{bmatrix} -\frac{b_0}{p^2}(1-e^{pT}) - \frac{b_1p+b_0}{p}Te^{pT} \\ (b_1 + \frac{2b_0}{p})(1-e^{pT}) + (b_1p+b_0)Te^{pT} \end{bmatrix} . \tag{2.4}$$

Eliminating x_2 in (2.4) yields:

$$\frac{T}{e^{-pT} + 1 - pT} = \frac{-\frac{b_0}{p^2}(1-e^{pT}) - \frac{b_1p+b_0}{p}Te^{pT}}{(b_1 + \frac{2b_0}{p})(1-e^{pT}) + (b_1p+b_0)Te^{pT}} ,$$

or

$$\frac{\sinh(pT)}{pT} = \frac{b_1}{b_0}p + 1 \ . \tag{2.5}$$

Noting that $\frac{\sinh v}{v} > 1$ and monotonically increasing for $v > 0$, we come to the conclusion that for the existence of a (unique) limit cycle it is necessary that $\frac{b_1}{b_0}p > 0$. Since p is assumed to be negative, we must have $\frac{b_1}{b_0} < 0$. Also solving for x_2 in (2.4), we get

$$Tx_2 = -\frac{b_0}{p^2}(e^{-pT} - 1) - (b_1 + \frac{b_0}{p})T = -\frac{b_0}{p^2}(e^{-pT} - 1) - \frac{b_0}{p}(\frac{b_1}{b_0}p + 1)T =$$

$$= -\frac{b_0}{p^2}(e^{-pT} - 1) - \frac{b_0}{p}\frac{\sinh pT}{p} = -\frac{b_0}{p^2}(\cosh pT - 1) \, .$$

Hence,

$$x_2 = -\frac{1}{T}\,\frac{b_0}{p^2}(\cosh pT - 1) \, . \tag{2.6}$$

The output of the system $y(t)$, subject to the initial condition $x(0) = \begin{bmatrix} 0^- \\ x_2 \end{bmatrix}$, can be found from (2.1) as

$$y(t) = x_1(t) = x_2 t e^{pt} + \frac{b_0}{p^2}(1 - e^{pt}) + \frac{b_1 p + b_0}{p} t e^{pt} \, , \tag{2.7}$$

provided that $y(t) < 0$, $\forall t \in (0,\ T)$. Utilizing (2.5) and (2.6), we can write the output as

$$y(t) = -\frac{b_0}{p} t e^{pt} \left[\frac{e^{-pt} - 1}{-pt} - \frac{e^{-pT} - 1}{-pT} \right] . \tag{2.8}$$

Let $f(v) = \frac{e^v - 1}{v}$, $v > 0$. Then $f'(v) = \frac{ve^v - e^v + 1}{v^2}$. Let $g(v) = ve^v - e^v + 1$. Since $g(0) = 0$ and $g'(v) = ve^v$, we deduce that $g(v) > 0$, $\forall v > 0$. Thus, $f'(v) > 0$, $\forall v > 0$ and $f(v)$ is monotonically increasing. Therefore $f(-pT) > f(-pt)$, for $0 < t < T$. Now from this observation and (2.8), we conclude that $b_0 > 0$ is both necessary and sufficient for condition (C2) of Corollary 1 to hold. Thus, the necessary and sufficient condition for the existence of a (unique) limit cycle in this case is: $b_1 < 0$ and $b_0 > 0$.

Case 2 $G(s) = \frac{b_1 s + b_0}{(s - p_1)(s - p_2)}$

The diagonal state space realization of the transfer function in this case greatly simplifies the analysis. Thus:

$$A = \begin{bmatrix} p_1 & 0 \\ 0 & p_2 \end{bmatrix}, \quad b = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}, \quad c = \begin{bmatrix} 1 & 1 \end{bmatrix},$$

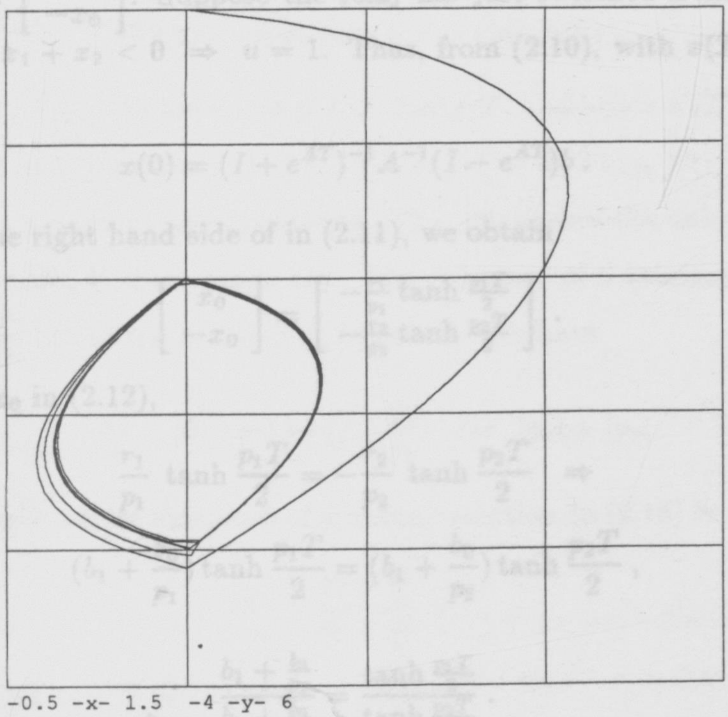


Figure 2.2: A trajectory for $G(s) = \frac{-s+4}{s^2+4s+4}$ with a relay in the feedback loop

where r_1 and r_2 are the residues of the transfer function at the corresponding poles, given by

$$r_1 = \frac{b_1 p_1 + b_0}{p_1 - p_2}\,, \qquad r_2 = \frac{b_1 p_2 + b_0}{p_2 - p_1}\,, \tag{2.9}$$

with both p_1 and p_2 negative. Note that in this particular realization of the system the output is the sum of the two states, $y = x_1 + x_2$, and, thus, the switching surface for the relay is $x_1 + x_2 = 0$. Consider

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)} b u(\tau) d\tau\,, \tag{2.10}$$

where $x(0) = \begin{bmatrix} x_0 \\ -x_0 \end{bmatrix}$. Suppose the relay has just switched from positive to negative. So $x_1 + x_2 < 0 \Rightarrow u = 1$. Thus, from (2.10), with $x(T) = -x(0)$, we get

$$x(0) = (I + e^{AT})^{-1} A^{-1} (I - e^{AT}) b \,. \tag{2.11}$$

Solving for the right hand side of in (2.11), we obtain

$$\begin{bmatrix} x_0 \\ -x_0 \end{bmatrix} = \begin{bmatrix} -\frac{r_1}{p_1} \tanh \frac{p_1 T}{2} \\ -\frac{r_2}{p_2} \tanh \frac{p_2 T}{2} \end{bmatrix}. \tag{2.12}$$

Eliminating x_0 in (2.12),

$$\frac{r_1}{p_1} \tanh \frac{p_1 T}{2} = -\frac{r_2}{p_2} \tanh \frac{p_2 T}{2} \quad \Rightarrow$$

$$(b_1 + \frac{b_0}{p_1}) \tanh \frac{p_1 T}{2} = (b_1 + \frac{b_0}{p_2}) \tanh \frac{p_2 T}{2} \,,$$

or

$$\frac{b_1 + \frac{b_0}{p_2}}{b_1 + \frac{b_0}{p_1}} = \frac{\tanh \frac{p_1 T}{2}}{\tanh \frac{p_2 T}{2}} \,. \tag{2.13}$$

Let

$$f(T) = \frac{\tanh \frac{p_1 T}{2}}{\tanh \frac{p_2 T}{2}} = \frac{\sinh \alpha T + \sinh \beta T}{\sinh \alpha T - \sinh \beta T} \,,$$

where $\alpha = \frac{p_1 + p_2}{2}$ and $\beta = \frac{p_1 - p_2}{2}$. Differentiating,

$$f'(t) = \frac{2(\beta \sinh \alpha T \cosh \beta T - \alpha \cosh \alpha T \sinh \beta T)}{(\sinh \alpha T - \sinh \beta T)^2} \,,$$

which can be simplified to

$$f'(T) = \frac{-p_2 \sinh p_1 T + p_1 \sinh p_2 T}{(\sinh \alpha T - \sinh \beta T)^2} \,.$$

Note that the denominator of $f'(T)$ is positive; therefore its sign depends on the sign of the numerator. Let

$$h(T) = -p_2 \sinh p_1 T + p_1 \sinh p_2 T \,.$$

Then

$$h'(T) = -p_1 p_2(\cosh p_1 T - \cosh p_2 T) \, .$$

Suppose $|p_1| > |p_2|$. We have $\cosh p_1 T > \cosh p_2 T$, and hence $h'(T) < 0$, $\forall\, T > 0$. Since $h(0) = 0$, we deduce that $h(T) < 0$, $\forall\, T > 0$. Thus, $f'(T) < 0$, $\forall\, T > 0$ and $f(T)$ is monotonically decreasing. This establishes the uniqueness of the solution of (2.13), if there exists any. The existence of a solution depends on whether $\frac{b_1 + \frac{b_0}{p_2}}{b_1 + \frac{b_0}{p_1}}$ lies in the range of $f(T)$ or not. We have

$$f(T) \; : \; (0 \; , \; \infty) \mapsto (1 \; , \; \frac{p_1}{p_2}) \;\; for \;\; |p_1| > |p_2| \, .$$

Then the condition for existence of a unique solution to (2.13) is

$$\frac{p_1}{p_2} > \frac{b_1 + \frac{b_0}{p_2}}{b_1 + \frac{b_0}{p_1}} > 1 \, ,$$

which simplifies to

$$\frac{b_1}{b_0} < 0 \, . \tag{2.14}$$

We now check for condition (C2) of Corollary 1. From the state transition equation, utilizing (2.13), the output of the system starting from $x(0)$ is

$$y(t) = \frac{2r_1}{p_1} \, \frac{e^{p_1 t} - e^{p_1 T}}{1 + e^{p_1 T}} + \frac{2r_2}{p_2} \, \frac{e^{p_2 t} - e^{p_2 T}}{1 + e^{p_2 T}} \, . \tag{2.15}$$

Hence,

$$\dot{y}(t) = \frac{2r_1 e^{p_1 t}}{1 + e^{p_1 T}} + \frac{2r_2 e^{p_2 t}}{1 + e^{p_2 T}} \, . \tag{2.16}$$

Setting $\dot{y}(t) = 0$, we find that the output has a unique extremum at

$$t' = \frac{1}{p_1 - p_2} \ln \left[-\frac{r_2(1 + e^{p_1 T})}{r_1(1 + e^{p_2 T})} \right] \, . \tag{2.17}$$

Note that for $b_1 b_0 < 0$, the term inside the brackets in (2.17) is positive. From (2.15), utilizing (2.13) we obtain $y(0) = y(T) = 0$. Observe that $t' = 0$ only if one of the following holds $p_1 = p_2$, $T = 0$ or $T = \infty$. Thus, $y(t) < 0$, $\forall\, t \in (0,\, T)$ if $\dot{y}(0) < 0$. Consider

$$\dot{y}(0) = \frac{2r_1}{1 + e^{p_1 T}} + \frac{2r_2}{1 + e^{p_2 T}} \,.$$

We obtain

$$sgn(\dot{y}(0)) = -sgn\left[\left(\frac{p_1 - \alpha p_2}{1 - \alpha}\right) b_1 + b_0\right] , \tag{2.18}$$

where, under the assumption $|p_1| > |p_2|$

$$1 > \alpha = \frac{1 + e^{p_1 T}}{1 + e^{p_2 T}} > 0 \,.$$

From (2.14) and (2.18) we conclude that the necessary and sufficient condition for the existence of a (unique) limit cycle in this case is: $b_1 < 0$ and $b_0 > 0$.

Case 3 $G(s) = \frac{b_1 s + b_0}{[s - (\sigma + j\omega)][s - (\sigma - j\omega)]}$

The state space realization of the transfer function is

$$A = \begin{bmatrix} 2\sigma & 1 \\ -(\sigma^2 + \omega^2) & 0 \end{bmatrix},\quad b = \begin{bmatrix} b_1 \\ b_0 \end{bmatrix},\quad c = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

with $\sigma < 0$. Following the same procedure as in cases 1 and 2 consider

$$x(0) = (I + e^{AT})^{-1} A^{-1} (I - e^{AT}) b \,, x(0) = \begin{bmatrix} 0^- \\ x_2 \end{bmatrix} . \tag{2.19}$$

Here

$$e^{AT} = \begin{bmatrix} \frac{\sigma}{\omega} e^{\sigma T} \sin \omega T + e^{\sigma T} \cos \omega T & \frac{1}{\omega} e^{\sigma T} \sin \omega T \\ \frac{\sigma^2 + \omega^2}{\omega} e^{\sigma T} \sin \omega T & -\frac{\sigma}{\omega} e^{\sigma T} \sin \omega T + e^{\sigma T} \cos \omega T \end{bmatrix} . \tag{2.20}$$

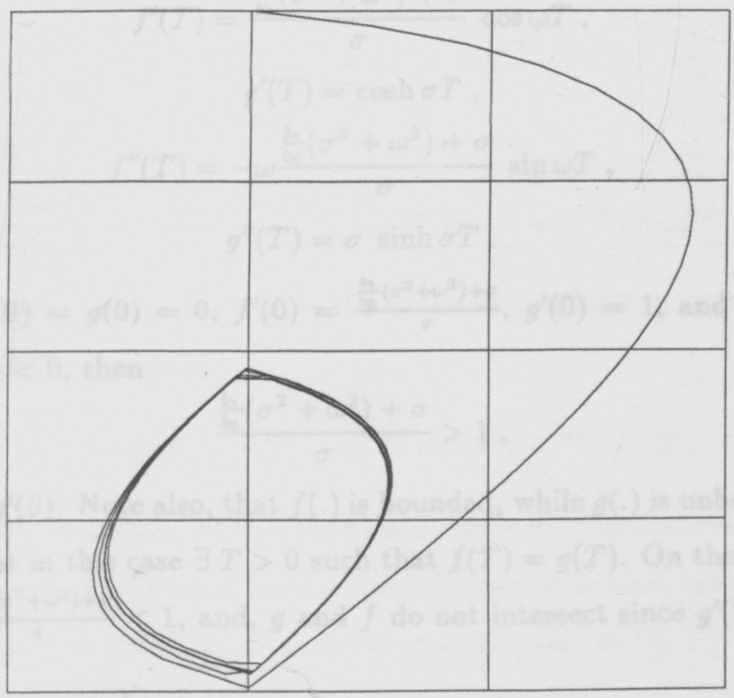


Figure 2.3: A trajectory for $G(s) = \frac{-s+4}{s^2+5s+4}$ with a relay in the feedback loop

From (2.20) and (2.19), we obtain

$$\left[\frac{\frac{b_1}{b_0}(\sigma^2+\omega^2)+\sigma}{\omega}\right] \frac{\tan\frac{\omega T}{2}}{1+\tan^2\frac{\omega T}{2}} = \frac{\tanh\frac{\sigma T}{2}}{1-\tanh^2\frac{\sigma T}{2}} ,$$

which simplifies to

$$\frac{\frac{b_1}{b_0}(\sigma^2+\omega^2)+\sigma}{\sigma} \, \frac{\sin\omega T}{\omega} = \frac{\sinh\sigma T}{\sigma} . \qquad (2.21)$$

Let

$$f(T) = \frac{\frac{b_1}{b_0}(\sigma^2+\omega^2)+\sigma}{\sigma} \, \frac{\sin\omega T}{\omega} .$$

and

$$g(T) = \frac{\sinh\sigma T}{\sigma} , \quad T \in [0\,,\,\infty) .$$

Then,

$$f'(T) = \frac{\frac{b_1}{b_0}(\sigma^2+\omega^2)+\sigma}{\sigma}\cos\omega T\ ,$$

$$g'(T) = \cosh\sigma T\ ,$$

$$f''(T) = -\omega\frac{\frac{b_1}{b_0}(\sigma^2+\omega^2)+\sigma}{\sigma}\sin\omega T\ ,$$

$$g''(T) = \sigma\ \sinh\sigma T\ .$$

Note that $f(0) = g(0) = 0$, $f'(0) = \frac{\frac{b_1}{b_0}(\sigma^2+\omega^2)+\sigma}{\sigma}$, $g'(0) = 1$, and $g''(T) > 0$, $\forall T > 0$. If $\frac{b_1}{b_0} < 0$, then

$$\frac{\frac{b_1}{b_0}(\sigma^2+\omega^2)+\sigma}{\sigma} > 1\ ,$$

and $f'(0) > g'(0)$. Note also, that $f(.)$ is bounded, while $g(.)$ is unbounded. We conclude, that in this case $\exists\, T > 0$ such that $f(T) = g(T)$. On the other hand if $\frac{b_1}{b_0} > 0$, $\frac{\frac{b_1}{b_0}(\sigma^2+\omega^2)+\sigma}{\sigma} < 1$, and, g and f do not intersect since $g'(T) > f'(T)$, $\forall\, T > 0$.

So condition (C1) of Corollary 1 is satisfied if and only if $b_1 b_0 < 0$. Note that in this case, due to the form of the two functions, more than one solution $T > 0$ can exist, satisfying (C1). We now show that the solution satisfying $0 < \omega T < \pi$ is the only one which satisfies (C2) of the Corollary.

Consider

$$x(t) = e^{At}x(0) - A^{-1}(I - e^{At})b\ . \tag{2.22}$$

Solving (2.22), yields

$$y(t) = x_1(t) = \frac{x_2}{\omega}e^{\sigma t}\sin\omega t + \alpha e^{\sigma t}\sin\omega t - \beta(e^{\sigma t}\cos\omega t - 1)\ , \tag{2.23}$$

where

$$\alpha = \frac{b_1(\sigma^2+\omega^2)+\sigma b_0}{\omega(\sigma^2+\omega^2)} \qquad \beta = \frac{b_0}{\sigma^2+\omega^2}\ . \tag{2.24}$$

From (2.19) and (2.21), utilizing (2.24), we obtain

$$x_2 = \frac{\omega}{e^{\sigma T}\sin\omega T}\left[-\alpha e^{\sigma T}\sin\omega T + \beta(e^{\sigma T}\cos\omega T - 1)\right]. \tag{2.25}$$

From (2.23) and (2.25),

$$y(t) = \frac{\beta e^{\sigma(T+t)}}{e^{\sigma T}\sin\omega T}\left[\sin(\omega(t-T)) - e^{-\sigma T}\sin\omega t + e^{-\sigma t}\sin\omega T\right]. \tag{2.26}$$

We show that the expression inside the brackets in (2.26) is negative for all $t \in (0,\ T)$, $\omega T \in (0,\ \pi)$. Let

$$m(t) = \sin(\omega(t-T)) - e^{-\sigma T}\sin\omega t + e^{-\sigma t}\sin\omega T.$$

Utilizing (2.21) we obtain $m(0) = m(T) = 0$. So if $m(.)$ is positive somewhere in the interval $(0,\ T)$, it must have a local maximum in that interval. Hence its second derivative must be nonpositive at some point of $(0,\ T)$. We have

$$m'(t) = \omega\cos(\omega t - \omega T) - \omega e^{-\sigma T}\cos\omega t - \sigma e^{-\sigma t}\sin\omega T,$$

$$m''(t) = -\omega^2\sin(\omega t - \omega T) + \omega^2 e^{-\sigma T}\sin\omega t + \sigma^2 e^{-\sigma t}\sin\omega T.$$

Clearly $m''(t) > 0$, and hence, $m(t) < 0$, $\forall t \in (0,\ T)$, $\omega T \in (0,\ \pi)$. So $y(t)$ *is, and remains negative for all* $0 < \omega t < \omega T < \pi$ if and only if

$$\beta > 0, \quad \textit{or equivalently} \quad b_0 > 0. \tag{2.27}$$

On the other hand if (2.21) has more than one solution, then this solution satisfies $\omega T > 2\pi$. Evaluating (2.26) at $\omega t = \pi$, yields

$$y\left(\frac{\pi}{\omega}\right) = \frac{\beta e^{\sigma(T+\frac{\pi}{\omega})}}{e^{\sigma T}\sin\omega T}\left[\sin(\pi - \omega T) + e^{-\sigma\frac{\pi}{\omega}}\sin\omega T\right]$$

$$= \frac{\beta e^{\sigma(T+t)}}{e^{\sigma T}\sin\omega T}\left[\sin\omega T + e^{-\sigma\frac{\pi}{\omega}}\sin\omega T\right] > 0.$$

Thus, any solution of (2.21) satisfying $\omega T > \pi$, violates condition (C2) of Corollary 1.

We conclude, that the necessary and sufficient condition for the existence of a (unique) limit cycle in this case is: $b_1 < 0$ and $b_0 > 0$.

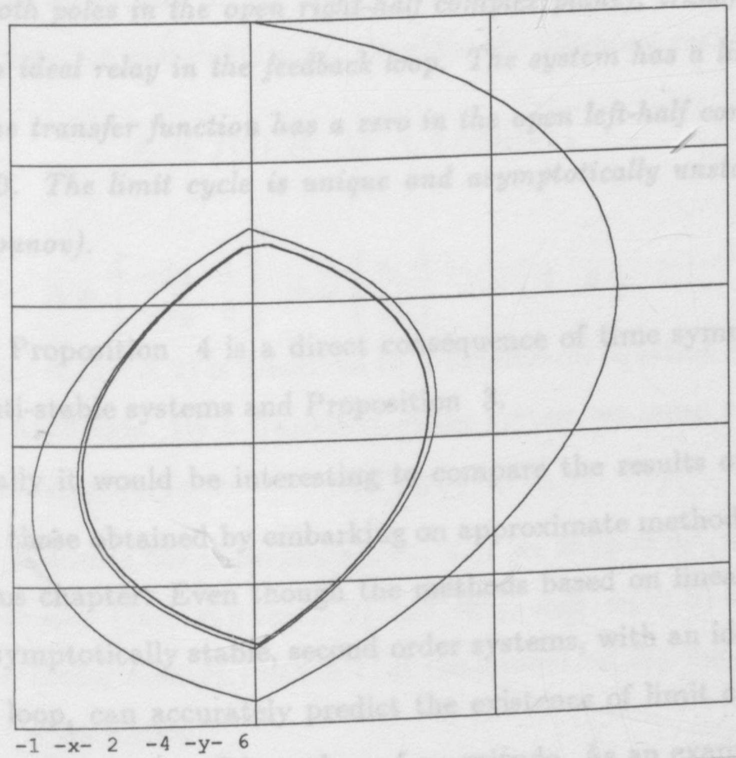


Figure 2.4: A trajectory for $G(s) = \frac{-s+4}{s^2+2s+4}$ with a relay in the feedback loop

We summarize our analysis in the two following Propositions:

Proposition 3 *Consider a second order system described by a strictly proper, asymptotically stable, transfer function $G(s)$, with an ideal relay in the feedback* *loop. The system has a limit cycle if and only if the transfer function has a zero in the open right-half complex plane and $G(0) > 0$. The limit cycle is unique and asymptotically stable (in the sense of Lyapunov).*

Proposition 4 *Consider a second order system described by a strictly proper, anti-stable (both poles in the open right-half complex plane), transfer function $G(s)$, with an ideal relay in the feedback loop. The system has a limit cycle if and only if the transfer function has a zero in the open left-half complex plane and $G(0) > 0$. The limit cycle is unique and asymptotically unstable (in the sense of Lyapunov).*

The proof of Proposition 4 is a direct consequence of time symmetricity of stable and anti-stable systems and Proposition 3.

Finally it would be interesting to compare the results of our exact analysis with those obtained by embarking on approximate methods discussed in the previous chapter. Even though the methods based on linearization, in the case of asymptotically stable, second order systems, with an ideal relay in the feedback loop, can accurately predict the existence of limit cycles, their quantitative results can be off by orders of magnitude. As an example, let

$$G(s) = \frac{-10000s + 1}{s^2 + 200s + 10001}$$

The Describing Function method predicts a limit cycle with a period of $T \approx \frac{2\pi}{100} \approx 63$ milliseconds, while the actual value of the period, as obtained from our analysis is approximately 347 milliseconds.

Appendix A

Before we embark on the proof of Proposition 2, we need to introduce some helpful notation and definitions. Consider the following realization.

$$\dot{x} = f(x) \equiv Ax - b\, sgn(cx)\,, \tag{A.1}$$

where:

$$A = \begin{bmatrix} -a_1 & 1 \\ -a_0 & 0 \end{bmatrix} \quad b = \begin{bmatrix} b_1 \\ b_0 \end{bmatrix} \quad c = \begin{bmatrix} 1 & 0 \end{bmatrix} \;. \tag{A.2}$$

The system in (A.1), (A.2) is described by a differential equation with discontinuous right hand side. By a trajectory of the system we mean a solution of the differential equation in the sense of Filippov [F1].

Let S denote the hyperplane $\{(x_1,\ x_2)|\ x_1 = 0\}$ in $\Re^2$, and H_+, H_- the open halfspaces $\{(x_1,\ x_2)|\ x_1 > 0\}$ and $\{(x_1,\ x_2)|\ x_1 < 0\}$ respectively. Note that the vector field $f(.)$ of (A.1) is continuous on H_+ and H_-. For a point $(0,\ z) \in S$ we define:

$$f_+(0,\ z) = \lim_{\substack{x \to (0,\ z) \\ x \in H_+}} f(x)\ , \tag{A.3}$$

and analogously for $f_-(0,\ z)$. We also let

$$F(0,\ z) = conv\{f_+(0,\ z),\ f_-(0,\ z)\}\ , \tag{A.4}$$

where " *conv* " denotes the convex hull.

We say that a connected set $U \subset S$ is a *local section in* S if for every $x \in U$ the trajectory $\phi_t(x)$ of (A.1) is uniquely defined for $t \in (-\epsilon, \epsilon)$, ($\epsilon > 0$ depending on x), and $\{\phi_t(x)| \ \ 0 < t < \epsilon\} \in H_+$, $\{\phi_t(x)| \ \ -\epsilon < t < 0\} \in H_-$, $\forall x \in U$, or vice-versa.

Note that U is a local section in S if $F(0, \ z) \bigcap S = \{\emptyset\}$, for all $(0, \ z) \in U$ (here S is identified with its tangent space).

The following Theorem can be established using the same argument as in ([H1], pp. 244-247).

Theorem: 1 *Let γ be a limit cycle in (A.1). Suppose that γ is a simple, closed curve in $\Re^2$. Then γ intersects any local section in S at not more than one point.*

The hypothesis that γ is simple, i.e., the homeomorphic image of a circle, is necessary here, since the solutions of (A.1) are not necessarily unique. Note though, that if the points of intersection of γ with S lie in local sections in S, then the hypothesis is clearly satisfied. We now proceed to prove Proposition 2.

Proof of Proposition 2

Let γ be a limit cycle of (A.1). Observe that $sgn(cx)$ can not be constant over γ since, otherwise, γ is a trajectory of an asymptotically stable, linear system driven by a step input, and hence can not be a limit cycle. Let $x_0 \in \gamma \bigcap H_-$ be an arbitrary point and let $(0, \ z_0)$ be the first crossing of S by the trajectory $\phi_t(x_0)$ of (A.1).

Since $f_-\ (0,\ z_0)\ =\ \begin{bmatrix} z_0 + b_1 \\ b_0 \end{bmatrix}$, it must be the case that $z_0 \geq -b_1$. We distinguish the following cases:

a) Suppose $b_1 < 0$. Observe that in this case, the sets $S' = \{(0,\ z)|\ \ z \geq -b_1\}$ and $S'' = \{(0,\ z)|\ \ z \leq -b_1\}$ are local sections in S. Clearly then $z_0 \in S'$ and using the symmetry argument of Proposition 1 we deduce that $\gamma \bigcap S \subset S' \bigcup S''$. In view of Theorem A1, assertion *ii)* of the proposition follows.

b) Now let $b_1 \geq 0$. In this case the sets $S' = \{(0,\ z)|\ z > b_1\}$ and $S'' = \{(0,\ z)|\ z < -b_1\}$ are local sections in S. Consider $W = \{(0,\ z)|\ |z| \leq b_1\}$. Following Filippov ([F1], p 206) we deduce that if $(0,\ z) \in W$ then $\phi_t(0,\ z) = (0,\ z(t))$, where $z(t)$ satisfies $\frac{d}{dt}z(t) = -\frac{b_0}{b_1}z(t)$ (as long as $(0,\ z(t)) \in W$). If $b_0 \geq 0$, this implies that W is positively invariant and, hence, $\gamma \bigcap W = \{\emptyset\}$. Therefore, $z_0 \notin W$, which implies that $z_0 \in S'$. The conclusion follows as in case a) .

On the other hand if $b_0 < 0$, then $z(t') = b_1$ (or $-b_1$) at some finite t'. Observe that the system has two equilibrium points: $x' = (\frac{b_0}{a_0},\ \frac{a_1}{a_0}b_0 - b_1) \in H_-$ and $x'' = (-\frac{b_0}{a_0},\ -\frac{a_1}{a_0}b_0 + b_1) \in H_+$. These equilibrium points are locally asymptotically stable. We claim that $(0,\ -b_1) \in S$ and $(0,\ b_1) \in S$ lie in the domain of attraction of x' and x'' respectively. To establish this fact, let $v(x) = x^T V x$ be a positive definite quadratic form such that $A^T V + VA$ is negative definite. Let $c_1 > 0$ be the supremum of the set of constant $c > 0$ such that

$$(x - x')^T V (x - x') < c \quad \Rightarrow \quad x \in H_- .$$

Clearly the level set $\{x|\ \ (x - x')^T V (x - x') = c_1\}$ is tangent to S at exactly one point, $(0,\ z')$. The direction of the vector field f_- implies that $z' < -b_1$. Therefore there exists a constant $c_2 > c_1$ such that the boundary of the set $E = \{x|\ (x-x')^T V(x-x') \leq c_2\}$ contains the point $(0,\ -b_1)$. Clearly $E \cap \overline{H}_-$ is a positively invariant set which lies in the region of attraction of x'. The claim follows.

Thus, $(0,\ -b_1) \notin \gamma$ which contradicts the hypothesis that $z_0 \in W$. Therefore $z_0 \in S'$, and the conclusion follows as in case **a)** .

Thus far, we established *ii)* of Proposition 2. Let $(0,\ z') \in S'$ and $(0,\ z'') \in S''$ be the unique points of intersection of γ and the local sections. If $z' = -z''$ assertion *i)* follows by symmetry. We argue by contradiction. Suppose that, without loss of generality, $z' > -z''$. Consider the trajectory $\phi_t(0,\ z')$. Evidently for some $t' > 0$ $\phi_{t'}(0,\ z') = (0,\ z'')$ and $\phi_t(0,\ z') \in H_+$, for all $0 < t < t'$. By Proposition 1, there is a $t'' > 0$ such that $\phi_{t''}(0,\ -z'') = (0,\ -z')$ and $\phi_t(0,\ -z'') \in H_+$, for all $0 < t < t''$. Since $z' > -z'' > z'' > -z'$, the trajectories $\phi_t(0,\ z')$ and $\phi_t(0,\ -z'')$ should intersect at some point in H_+, contradicting the uniqueness of solution of (A.1) in H_+. Q.E.D.

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