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Nonarchimedean Factorization Theorems Via Factorization Algebras

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Nonarchimedean Factorization Theorems Via Factorization Algebras

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We formulate an analogue of factorization algebras theory over a nonarchimedean field K, building on work of Costello and Gwilliam in the complex analytic case. Several constructions involved in factorization algebras theory, leading to a wealth of standard examples, are developed in the nonarchimedean setting. En route, we build aspects of Jacob Lurie's Verdier duality theory in the rigid analytic setting. Last, an analogue of the factorization theorems traditionally studied in rational conformal field theory, as in Faltings' work on the Verlinde Formula, is developed in the nonarchimedean setting by interpreting nodal degenerations of smooth algebraic curves in terms of nonarchimedean gluing.

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Chapter 1

Introduction

In this paper, we introduce a theory of factorization algebras in the setting of rigid analytic geometry, a version of complex analytic geometry for nonarchimedean fields. Our factorization algebras are defined in the style of Costello and Gwilliam, as in their volumes *Factorization Algebras in Quantum Field Theory.* A motivation for the paper was the appearance of nonarchimedean methods in Faltings' work on the factorization rules in classical rational conformal field theory in his *A Proof of the Verlinde Formula,* which correspond to Segal's theory of modular functors in his original work defining conformal field theories. In this spirit, our main results develop nonarchimedean versions of the examples considered by Costello and Gwilliam of factorization algebras associated to Lie algebras, and show that, specializing to rigid analytic curves, these satisfy analogues of the aforementioned factorization rules attached to semistable models of the rigid curves. An integral part of this contribution is defining nonarchimedean factorization algebras appropriately.

1.1 Factorization Algebras Basics

There are a few formalisms of factorization algebras, but they share the general theme of being related to (co)sheaves on spaces with points corresponding to finite collections of points of a given topological space or geometric object. Two major such formalisms are those of Beilinson-Drinfeld and Costello-Gwilliam. The first is defined in the setting of algebraic geometry, and is based on the geometry of diagonals (encoding the collision of points). Costello-Gwilliam style factorization algebras, on the other hand, are closer to algebras over operads of disks considered in algebraic topology, but can (particularly via the work of Dwyer-Stolz-Teichner) be considered in various geometric settings (for instance, the holomorphic one). They are essentially multiplicative versions of precosheaves on the given geometric object, satisfying a more restricted codescent condition than cosheaves.

Let us briefly review the ideas of Costello and Gwilliam most salient to our work. Here is a sketch of a definition of their notion of factorization algebra:

Definition 1.1.1. Let X be a topological space. A *factorization algebra* $\mathcal{F}$ is an assignment to opens U of X objects $\mathcal{F}(U)$ of a symmetric monoidal category $\mathcal{C}$ sending disjoint unions to the corresponding monoidal products, together with structure maps $\mathcal{F}(U) \otimes \mathcal{F}(V) \to \mathcal{F}(W)$ for U, V disjoint opens in W, so that the assignment satisfies codescent for *Weiss covers.*

Weiss covers are covers of the topological space X, usually a manifold for Costello and Gwilliam's purposes, so that every finite subset $S \subset X$ is contained in some open of the cover. This is a kind of proxy for referring to the Ran space of X, which is a topological space with underlying set the set of finite subsets of X. To generalize the Ran space to algebraic geometry, as is imaginable, we must replace finite collections of points in this set-theoretic sense with finite collections of points in a functor-of-points sense.

Costello and Gwilliam develop important examples of factorization algebras by showing how to attach a factorization algebra to a cosheaf (many times, notions like cosheaf can refer to a homotopical version). Namely, for a cosheaf $\mathcal{F}$, the assignment $U \mapsto Sym\mathcal{F}(U)$ defines a prefactorization algebra in a natural way, and at least if the underlying topological space X is a reasonable space like a manifold, and $\mathcal{F}$ arises by passing to compact supports from a reasonable sheaf (such as one underlying a vector bundle), Costello-Gwilliam demonstrate Weiss codescent is also satisfied. The case of the symmetric factorization algebra is used to also build an enveloping factorization algebra, whose Weiss-codescent can be reduced to that of the symmetric case.

There is also a rich literature on factorization algebras as certain cosheaves on Ran spaces, as alluded to above, and there is a particularly rich literature on factorization algebras in the context of topological field theory, where we are concerned most with the special case of disjoint disks inside a bigger disk (this leads to the theory of E_n-algebras). There are beautiful stories to tell on equivalences of categories between categories of E_n-algebras and those of locally constant factorization algebras. There is also a celebrated result satisfied by locally constant factorization algebras called *excision,* which is not unlike the factorization formulas the present work was motivated by. Roughly, this says the following: suppose $\mathcal{A}$ is a locally constant factorization algebra (where we can identify $\mathcal{A}(D_r) \cong \mathcal{A}(D_s)$ for $D_r \subset D_s$ disks of possibly different sizes) on a manifold M, and let N be a codimension 1 submanifold with a trivialization of its neighborhood $N \times D^1$ (D^1 denotes a 1-dimensional disk) so that M can be glued as $X \cup_{N\times D^1} Y$, where X, Y are submanifolds of M. Then, $\mathcal{A}(N \times D_1)$ has an E_1-algebra structure for which $\mathcal{A}(X)$ is a right module and $\mathcal{A}(Y)$ is a right module, and there is a natural equivalence

$$\mathcal{A}(M) \cong \mathcal{A}(X) \otimes^{\mathbb{L}}_{\mathcal{A}(N\times D^1)} \mathcal{A}(Y)$$

Results of this sort are close to gluing laws in topological field theory, and in functorial field theory in general; the version that inspired our work appears in Segal's notion of a modular functor.

Let us also briefly discuss the more Beilinson-Drinfeld flavor of factorization algebras. Commonly, this involves considering an algebraic scheme X and the fiber product spaces X^n for all positive integers n, since these help to classify points of X in the appropriate functor-of-points sense. A Beilinson-Drinfeld factorization algebra involves sheaves (in fact, $\mathcal{D}$-modules) $\mathcal{F}_n$ on X^n together with compatibilities like isomorphisms $i^!\mathcal{F}_n \cong \mathcal{F}_k$ for closed immersions $i : X^k \hookrightarrow X^n$, as well as compatibilities involving the off-diagonals. The idea is that the way these sheaves fit together along the diagonals is encoded using the famous operator product expansion in conformal field theory.

There are also analytic versions of this flavor of definition discussed by Frederic Paugam, involving, to the author's knowledge, intuitions about analyticity results for operator product expansions. To the author's knowledge, these flavors of factorization algebras involving sheaves (or cosheaves) on powers of a space with compatibilities tend to define cosheaves on a Ran space that should be regarded as endowed with a colimit topology, not quite (but still related to) the topologies on Ran spaces closest to what Costello and Gwilliam discuss.

Beilinson-Drinfeld factorization algebras are one of many equivalent formulations of the same idea: chiral algebras are another, and vertex algebras are yet another, if we restrict to affine spaces (also, a vertex algebra can, by a local-to-global procedure, define a Beilinson-Drinfeld factorization algebra; for details, see the book of Edward Frenkel and David Ben-Zvi called *Vertex Algebras and Algebraic Curves.*)

The reader should note that vertex algebras are equipped with maps $Y(-,z) : V \otimes V \to V((z))$, which should be thought of as analogous to maps $\mathcal{F}(D_1) \otimes \mathcal{F}(D_2) \to \mathcal{F}(D_3)$ (the D_i are complex analytic disks) arising from an operad of holomorphic disks. Costello and Gwilliam develop this point of view and show how to attach aa vertex algebra to one of their holomorphic factorization algebras.

1.2 Rigid Analytic Geometry

Rigid analytic geometry is one among a few different formulations of an analogue of complex analytic geometry for nonarchimedean spaces. Other major formulations include Raynaud's theory of formal models, Huber's adic spaces, and Berkovich geometry. The basic idea behind rigid analytic geometry is that nonarchimedean spaces, when defined in roughly the same way as ordinary real or complex analytic spaces, are totally disconnected, so Tate proposed the solution of treating the nonarchimedean space using a restricted, more abstract topology called a G-topology. The idea of the G-topology, whose building blocks are called affinoids, analogues of compact spaces like closed balls, is to ensure that the types of coverings considered are restricted so that affinoids behave like compact spaces despite the disconnectedness of the naive nonarchimedean topology.

The theory of Raynaud formal models of rigid analytic spaces describes the category of suitable rigid analytic spaces in terms of the birational geometry of formal schemes. We utilize Raynaud's theory eventually in the specific context of semistable formal models of rigid analytic curves, particularly when discussing factorization rules. Briefly, Huber's adic spaces are genuine topological spaces encoding roughly the same information as the G-topologies on rigid spaces. We can think of these adic spaces as arising from considering all possible admissible formal blowups of a formal model of a rigid space. We will not consider adic spaces in this work, but possibly in future work.

1.3 Summary of Paper

Here, we summarize the paper in some more detail. Chapter 2's major purpose is to define the appropriate notion of an admissible covering with respect to which our nonarchimedean geometric factorization algebras satisfy (a homotopical version of) codescent. The key difficulty is that we require an appropriate combination of the notion of admissibility and that of Weiss cover.

We then move in the third chapter to showing how to produce examples of homotopical cosheaves from homotopy sheaves. Here, we utilize the language of ∞-categories, and our treatment is quite different from that of Costello-Gwilliam. We opt to use Lurie's formalism of the Verdier duality functor (developing an appropriate analogue for our setting), as the more explicit arguments of Costello-Gwilliam are unavailable to us in the more formal setting of rigid spaces. One main theorem of the section is the following theorem:

Theorem 1.3.1. *Let $\mathcal{C}$ be a stable ∞-category admitting all small limits and colimits. Let X be a separated rigid analytic space. Then, the assignment given by sending $\mathcal{F} \mapsto (U \mapsto \Gamma_c(\mathcal{F}, U))$ defines a functor of ∞-categories $(-)_c : Shv(X, \mathcal{C}) \to CShv(X, \mathcal{C})$ from $\mathcal{C}$-valued sheaves to cosheaves.*

This defines a functor from sheaves to cosheaves, but, due to the fact that rigid sheaves are not necessarily determined by their behavior on wide opens, only affinoids, there are further restrictions that are natural to impose to attempt to get a genuine equivalence. We suspect the equivalence should hold either for all or a wide class of separated rigid analytic spaces, but we check the details of an equivalence for ambient affinoid spaces, after restricting from all rigid analytic sheaves to overconvergent ones (and appropriate analogues on the cosheaf side). Here is the main result:

Theorem 1.3.2. *Let $\mathcal{C}$ be as above, and suppose X is affinoid. Then, the above functor defines an equivalence of ∞-categories $OverShv(X, \mathcal{C}) \cong OverCShv(X, \mathcal{C})$ between overconvergent sheaves and co-overconvergent cosheaves.*

Chapter 4 covers examples of factorization algebras arising from homotopical cosheaves, and is motivated by the fact that the previous section provides a rich source of such cosheaves. These are the symmetric and enveloping factorization algebras. There are two main results here. The first tells us that taking the symmetric powers of a homotopy cosheaf is a rigid factorization algebra, and the second uses the first to say the following:

Theorem 1.3.3. *Let $\mathcal{L}$ be a Lie-structured cosheaf on a rigid space X (a precosheaf of dg Lie algebras that is a homotopy cosheaf of dg vector spaces, all over a nonarchimedean field K). Then, the prefactorization algebra $\mathcal{U}(\mathcal{L})$ given by the assignment $U \mapsto C_*(\mathcal{L}(U))$ of Chevalley-Eilenberg chains on $\mathcal{L}(U)$ to an admissible open $U \subset X$ is a rigid factorization algebra.*

Chapter 5 specializes to the case of rigid analytic curves, and is the construction of factorization rules for rigid factorization algebras. We use the notion of basic wide open pairs in rigid geometry as a version of rigid analytic curves with boundary annuli, akin to manifolds with boundary. To such a pair Σ with two boundary wide open annuli (open annuli in rigid geometry), and some rigid factorization algebra on Σ, we attach a bimodule over certain dg categories we attach to the annuli. The main result of the section states that, given two such basic wide open pairs Σ_1, Σ_2 (each with, for simplicity, at most two boundary annuli) appearing in a semistable covering with some extra marked structure (with two elements overlapping in a single wide open annulus *Ann*) attached to the underlying wide open (the underlying curve without specifying boundary annuli) of some basic wide open pair Σ, there is a certain dg-categorical factorization formula. Let the bimodule attached to Σ be given by $\mathcal{M}(\Sigma)$, and similarly for other basic wide open pairs. Let the dg category attached to a wide open annulus A be given by $\mathcal{A}(A)$. Then, we have the following:

Theorem 1.3.4. *The natural map* $\mathcal{M}(\Sigma_1) \otimes^{\mathbb{L}}_{\mathcal{A}(Ann)} \mathcal{M}(\Sigma_2) \to \mathcal{M}(\Sigma)$ *is a weak equivalence.*

Section 6 sketches how to build a locally constant prefactorization algebra on $\mathbf{R}_{\geq 0}$ (in particular, not a rigid factorization algebra but a standard one as in Costello-Gwilliam) attached to the vector space of a Kac-Moody vertex algebra, and show this prefactorization algebra densely approximates the cohomology of our Kac-Moody factorization algebras on wide open disks and annuli (defined in the appendices).

Last, we provide appendices on rigid geometry and on homotopical notions like the relation between homotopy (co)limits and ∞-categorical (co)limits, since we freely utilize the dictionary between model and ∞-categorical notions in this paper.

1.4 Motivation and Outlook

A motivation for our work arose from noting two rather different pictures in algebro-geometric conformal field theory. One is the Beilinson-Drinfeld theory of factorization algebras defined in terms of colliding points on algebraic curves and the geometry of diagonals. The other is also closely related to the geometry of colliding points on curves: the story of Deligne-Mumford compactifications of moduli of curves, where colliding points are received by the sprouting of Riemann spheres attached to the original curve at nodal singularities. This latter story plays a role in some algebro-geometric formulations of the factorization rules satisfied by spaces of conformal blocks of rational conformal field theories. In Faltings' *A Proof of the Verlinde Formula,* the factorization rules are formulated as a relation between spaces of vacua associated to generic and special fibers of a semistable model of a nonarchimedean curve, which expresses the degeneration of a smooth, projective curve to a curve with nodal singularities.

Our work may be seen as an attempt to cast this picturing involving semistable models in terms of the relation between rigid analytic geometry and the Raynaud theory of formal models. A potential outlook for the present work would be a proof using factorization algebras methods of the factorization rules considered by Faltings. It seems that rigid geometry is quite suited for the situation, as both spaces of vacua considered, strictly speaking, are associated to smooth curves. That is, the vacua attached to the special fibers above ultimately are attached to the normalizations of these singular nodal curves. The slogan that rigid spaces are generic fibers which remember a good amount about the special fibers thus seems particularly relevant. It is also worth noting that the rational conformal field theory situation considered by Faltings has also been considered in a more complex analytic context, involving degenerating families parametrized by holomorphic disks, so our considerations should relate to more standard conformal field theory when considering the base field to be (in general, an appropriate extension of) $K = \mathbf{C}((t))$, although an advantage of our more algebro-geometric/rigid analytic approach is also that we can replace $\mathbf{C}$ by more general fields.

We should also make some remarks about the place of our theory within theories of factorization algebras on nonarchimedean analytic spaces. We would expect one very natural definition in a style not so reminiscient of Costello and Gwilliam would involve simply considering families of cosheaves $\mathcal{F}_n$ on powers (fiber products) X^n of the rigid analytic space X in question, together with appropriate compatibilities (probably via a !-pullback, where the precise variety of ! may depend on whether we are working in the world of (co)sheaves of dg vector spaces, or if there is some $\mathcal{O}$-module or $\mathcal{D}$-module structure to consider) with respect to closed immersions of form $X^k \hookrightarrow X^n$, where $k < n$ (and also compatibilities with regard to the off-diagonals), and so on. It is to be noted that, as with scheme theory, there is a big gap between a naive product topology flavored definition and the actual Grothendieck topology considered on X^n. Further, even at the level of underlying sets, these X^n do not correspond to the n-fold products of the underlying set of X, unless the base field is algebraically closed. Last, but not least, definitions of the mentioned flavor tend in the topological setting to correspond to cosheaves on a colimit-topologized Ran space. All these potential subtle differences from our flavor of factorization algebras theory warrant some discussion of our work's place within the jungle of possible approaches.

It seems that the major advantage a Costello-Gwilliam style offers us is its reliance on the notion of Weiss covers to capture the topology of Ran spaces, because this enables us to relate decompositions of Ran spaces to those of X, the latter of which is precisely the sort of thing we may expect to encounter in a factorization theorem. Hence, it would be our hope that a theory of the above families-of-$\mathcal{F}_n$ flavor should still produce objects whose factorization homology on various admissible opens (global sections across the full Ran space, built from the global sections of each $\mathcal{F}_n$ on X^n) roughly defines a factorization algebra in our sense (that is, satisfies what we call admissible Weiss codescent) in the same sort of way the factorizable cosheaves considered by, for instance, Lurie in *Higher Algebra* can be used to define factorization algebras on manifolds. It seems to us that for there to be a hope of this in general, the base field may have to be algebraically closed, as then at least, direct products of rigid spaces set-theoretically agree with the products of the factors' underlying sets. In particular, should the families-of-$\mathcal{F}_n$ flavored factorization algebras satisfy a kind of admissible Weiss codescent, it seems it should arise from the codescent condition satisfied by $\mathcal{F}_n$ for admissible covers of X^n (here, unlike in the body of the paper, powers denote fiber products) of form $\{U_i^n\}_{i\in I}$, where the U_i admissibly cover X. That is, the U_i should have the property involving the U_i^n for any positive integer n. Roughly, our theory seems to correspond to considering cosheaves on a sort of Ran space $Ran(X)$ of finite collections of points of a rigid space X with coverings of $Ran(U)$ of form $\{Ran(U_i)_{i\in I}\}$ satisfying an admissibility condition involving pullbacks to subsets of form $Ran^{\leq n}K$, where $K \subset X$ is compact.

We end by remarking that our discussion suggests there may be three flavors of factorization algebras theory over a nonarchimedean field: one involving our present work/cosheaves on something roughly like a Grothendieck-topologized Ran space, the families-of-$\mathcal{F}_n$ living on fiber products of a rigid space X^n flavor, and last, the Beilinson-Drinfeld algebro-geometric theory for schemes X/K. Future work may endeavor to compare/contrast any of these.

Probably one of the main applications of this kind of work should, paralleling the reduction of Verlinde formula computations for higher genus curves to lower genus curves, involve the ability to reduce questions about factorization homology for complicated (probably algebro-geometric) objects to simpler ones. With progress in relating the Beilinson-Drinfeld and Costello-Gwilliam flavors, this suggests hope of some applications to settings where the former figures in prominent ways, such as the geometric Langlands program.

Remark 1.4.1. Throughout, we will refer to the work of Costello and Gwilliam, and we mention here (so we need not in the future) that we specifically are referring to their volumes *Factorization Algebras in Quantum Field Theory,* though also at times to Gwilliam's closely related thesis.

Remark 1.4.2. Throughout, K will denote a complete, discretely valued field of nontrivial valuation. Let R_K denote its ring of integers, and denote by m_K the maximal ideal of R_K. For some purposes, the reader might want to imagine K is algebraically closed (please see the discussion above on the place of our theory among possible nonarchimedean factorization algebras theories). Last, denote by $\mathbb{F}_K$ the residue field, which for us is of characteristic 0, since we are mainly concerned with the case of K being the field of Laurent series (or appropriate extensions) over the complex numbers, where curves over K can be thought of as families of curves over the complex numbers, and semistable models thought of as families of smooth curves degenerating to nodal curves. Also, we will assume our rigid spaces are separated to ensure finite intersections of affinoids remain affinoid. Whenever not specified, our (pre)factorization algebras and (pre)/(co)sheaves take values in dg vector spaces over the base field K, though the reader can freely imagine substituting cochain complexes in some appropriate quasi-abelian category or more general sort of category. When we refer to wide opens of curves in Chapter 5, these correspond to those defined by (for instance) Robert Coleman (see bibliography), but there is also a notion of a wide open used in Schneider's work *Points on Rigid Analytic Varieties* which we use in our discussion of Verdier duality, which the reader should beware not to confuse, though they are related.

Remark 1.4.3. When working ∞-categorically, the only concrete model we appeal to of ∞-categories is the theory of quasicategories in the style of Jacob Lurie's work.

Chapter 2

Nonarchimedean Factorization Algebras: The Basic Notions

Throughout this section, we will let $\mathcal{C} = dgVect_K$ denote the symmetric monoidal model category of dg vector spaces over K, endowed with the projective model structure and the usual tensor product (this is for simplicity of presentation, as many other stable monoidal model categories should work well for the constructions presented in this thesis with a little modification of the details, not least model categories of cochain complexes in quasi-abelian categories of functional-analytic interest, like bornological vector spaces). We sometimes refer to $\mathcal{C}$ simply as the model category of dg vector spaces. A compact subspace of a rigid space is defined to be an admissible open which admits a finite admissible covering by affinoids.

We will introduce some terminology to facilitate defining rigid analytic prefactorization algebras.

Definition 2.0.1. Let X be a rigid analytic space. We will term $V_1, ..., V_k$ an *admissible sequence* of admissible opens of X if the V_i are pairwise disjoint, their union is admissible open in X, and they constitute an admissible covering of their union.

Remark 2.0.1. In multicategories, it is important to be able to concatenate sequences of objects. As we will be dealing with structures very like functors out of multicategories whose objects are admissible opens as above, we make the following remark regarding concatenating admissible sequences. Given a finite set $I = 1, ..., n$ and finite sets J_i corresponding to each $i \in I$, and given an admissible sequence consisting of V_j indexed by $j \in J_i$ for each J_i, we can concatenate these to get a new sequence in the obvious way. It turns out that this concatenated sequence is an admissible sequence if and only if the sequence indexed by I given by $\cup_{j \in J_i} V_j$ for every i, is itself an admissible sequence. This is simply a consequence of basic properties of admissible coverings: if the covering $\{\cup_{j \in J_i} V_j\}_{i \in I}$ is admissible, then the concatenated cover given by all V_j ranging over all j in any J_i must be admissible by an axiom of G-topologies. In the other direction, if a concatenated cover is admissible, the covering $\{\cup_{j \in J_i} V_j\}_{i \in I}$ is as well, since it has an admissible refinement.

Definition 2.0.2. A *rigid prefactorization algebra* (sometimes referred to as nonarchimedean prefactorization algebra, nonarchimedean geometric prefactorization algebra, or prefactorization algebra when the nonarchimedean context is understood) on a rigid space X is

(1) an assignment $U \mapsto \mathcal{F}(U)$ to admissible opens U of X

(2) structure maps $m_{V,U} : \mathcal{F}(V) \to \mathcal{F}(U)$ for $V \subset U$ admissible opens ($m_{U,U} = id_{\mathcal{F}(\mathcal{U})}$), and $m_{U_1,...,U_n;U} : \mathcal{F}(U_1) \otimes \cdots \otimes \mathcal{F}(U_n) \to \mathcal{F}(U)$, also denoted $m_{I,U}$ if $I = 1, ..., n$ indexes the U_i, for any admissible sequence $U_1, ..., U_n$,

so that these structure maps satisfy the usual symmetry condition, and so that the following associativity condition is satisfied: for any U_i as above, and for J_i a finite set of indices corresponding to each i, given an admissible sequence of V_j with $j \in J_i$ (with union sitting in U_i) for each J_i, if the admissible sequence given by the concatenation of the V_j as described in the above remark is an admissible sequence indexed by $s \in S$, then the map $m_{S,U}$: $\otimes_i \otimes_{s \in J_i} \mathcal{F}(V_s) \to \mathcal{F}(U)$ coincides with the composition $\otimes_i m_{J_i,U_i} \circ m_{U_1,\ldots,U_n,U}$

Definition 2.0.3. A prefactorization algebra will be called *unital* if there is a unit map $1 \to \mathcal{F}(\emptyset)$ satisfying the following: for any admissible open U, the left unitor $1 \otimes \mathcal{F}(U) \to \mathcal{F}(U)$ equals $1 \otimes \mathcal{F}(U) \to \mathcal{F}(\emptyset) \otimes \mathcal{F}(U) \to \mathcal{F}(U)$, where this latter composition is given first by tensoring the unit map with the identity on $\mathcal{F}(U)$, and secondly by the structure map $m_{\emptyset,U;U}$ (and similarly for the right unitor and the structure map $m_{U,\emptyset,U}$). We will also assume that the empty admissible open is sent to the tensor unit.

We can also view such objects as functors out of multicategories of admissible opens.

Remark 2.0.2. We remark that the concatenation of the V_j above always produces an admissible sequence for the following reason: it suffices to show that the collection $\{\cup_{j \in J_i} V_j\}_{i \in I}$ admissibly covers its union, where the union is admissible open. This follows because the U_i constitute an admissible sequence, since this means $U_1 \coprod \cdots \coprod U_n$ has the rigid structure coming from gluing disjoint pieces.

Our multicategory of choice is the following:

Definition 2.0.4. Let X be a rigid analytic space. Define a multicategory $Disj_X$ as follows. The objects are connected admissible opens $U \subset X$. The maps are given as follows: $Maps(\{U_1, ..., U_n\}, U)$ is empty, unless the U_i constitute an admissible sequence with union sitting in U, in which case it is a singleton. Composition is defined in the obvious way.

Proposition 2.0.1. *Any nonarchimedean prefactorization algebra $\mathcal{F}$ defines a functor of multicategories $Disj_X \to dgVect_K$, where an admissible open U is once again sent to $\mathcal{F}(U)$.*

PROOF: If $\{U_1, ..., U_n\}$ is an admissible sequence whose union sits in U, the structure map $m_{U_1,...,U_n;U}$ provides us with a way of assigning to the singleton $Maps_{Disj_X}(U_1, ..., U_n; U)$ a map $\mathcal{F}(U_1) \otimes \cdots \otimes \mathcal{F}(U_n) \to \mathcal{F}(U)$, and the identity map $U \to U$ is sent to the identity in $dgVect_K$ (this tells us units are respected). That composition is respected follows from the associativity condition for nonarchimedean prefactorization algebras.

There is a further notion which we will require before moving to define factorization algebras (akin to defining cosheaves after defining precosheaves).

Definition 2.0.5. Let $\mathcal{F}$ be a prefactorization algebra on X. It is said to be *multiplicative* if it satisfies the following *factorization axiom:* for any distinguished sequence $\{U_1, ..., U_n\}$, the structure map $\mathcal{F}(U_1) \otimes \cdots \otimes \mathcal{F}(U_n) \to \mathcal{F}(U_1 \coprod \cdots \coprod U_n)$ is a weak equivalence.

We can now proceed to the final ideas needed to define factorization algebras, namely the ones having to do with the locality/gluing condition, analogous to a cosheaf gluing condition. *We make the note that, when we refer to products of rigid spaces, unlike in the introduction, we will simply mean, for this work, the set-theoretic product, not fiber product (unless otherwise specified) where the underlying factors' rigid analytic structure is still referenced in the normal way.* The reader who wishes a closer correspondence between these products and genuine fiber-products of rigid spaces is invited to consider the case where the base field is algebraically closed.

Remark 2.0.3. When we refer, without further clarification, to products of admissible opens in a product of rigid spaces $X_1 \times \cdots \times X_n$, we simply mean a product of n different admissible opens U_i of X_i of form $U_1 \times \cdots \times U_n$. The analogous remark applies for products of affinoids. Further, we refer to products of admissible coverings in a product of k rigid spaces when we mean a covering gotten from taking all combinations of products of elements from admissible coverings of the individual factors. More precisely, for a product of rigid spaces $X \times Y$, a product admissible cover is one of form $\{U_i \times V_j\}_{i \in I, j \in J}$ where $\{U_i\}_{i \in I}$ and $\{V_j\}_{j \in J}$ are admissible coverings of X and Y respectively.

Definition 2.0.6. We define a p-admissible covering (for any given positive integer $k \geq 2$) of a product of rigid analytic spaces $X_1 \times \cdots X_k$ to be a covering by products of admissible opens so that, for any product of k affinoids, the pullback of the covering to this product of affinoids admits a finite refinement consisting of products of affinoids.

It turns out that we can define a Grothendieck pretopology on the set-theoretic finite product of rigid spaces. For ease of notation, we just pursue the case of a product of two spaces X and Y. Define a poset category of subsets of $X \times Y$ of form $U \times V$, where $U \subset X, V \subset Y$ are admissible opens of X and Y respectively. The coverings of $U \times V$ will be the p-admissible coverings. That this defines a Grothendieck pretopology is quite straightforward. The identity $U \times V \subset U \times V$ is p-admissible, since for any product of affinoids (we are being loose here about distinguishing the underlying G-topologized space from the affinoid with a structure sheaf) $M(A) \times M(B) \subset U \times V$, the pullback has refinement $M(A) \times M(B)$ itself. Any two $U \times V$ and $U' \times V'$ intersect to yield a product $(U \cap U') \times (V \cap V')$. Further, given an inclusion $V_1 \times V_2 \subset U_1 \times U_2$, the pullback of a p-admissible cover of the latter to the former yields a p-admissible cover of the former, since any product of affinoids $M(A) \times M(B) \subset V_1 \times V_2$ is contained in $U_1 \times U_2$, and the p-admissibility condition for $U_1 \times U_2$ easily is seen to yield that for $V_1 \times V_2$. Last, given a p-admissible covering $\{U_i \times V_i\}_{i \in I}$ of $U \times V$, if we have p-admissible covers $\{U_j \times V_j\}_{j \in J_i}$ of $U_i \times V_i$ for every i, concatenating them yields a p-admissible covering of $U \times V$ as follows. Given a product $M(A) \times M(B) \subset U \times V$, the pullback of $\{U_i \times V_i\}_{i \in I}$ to this has a finite refinement consisting of products of affinoids. Let $M(A_k) \times M(B_k) \subset U_i \times V_i$ be one such product. We can then find a refinement of $\{U_i \times V_j\}_{j \in J_i}$'s pullback to $M(A_k) \times M(B_k)$ consisting of products of affinoids (finitely many of them). Putting together these refinements for each k, we obtain the desired one for $M(A) \times M(B)$.

The following fact will be very useful for us in the future.

Proposition 2.0.2. *Let $M(A_1) \times \cdots \times M(A_n) = X_1 \times \cdots \times X_k$ be a product of (admissible open subsets underlying) affinoids. Any p-admissible covering admits a refinement of a special form, where the refinement consists of all products of elements from individual admissible (finite) coverings by affinoids of each factor. Further, this refinement can be chosen so that each individual admissible covering of a factor is closed under finite intersections (and thus, so is the product cover).*

PROOF: Given a p-admissible covering of a product of affinoids (we mean the product of the underlying G-topologized spaces, but are abusing terminology for convenience) $X_1 \times \cdots \times X_n$, we can find a finite refinement consisting of products of affinoids: call this set $\{M(A_{i,1}) \times \cdots \times MA_{i,n})\}_{i \in I}$. Let $x \in M(A_k)$ for some k. We can consider the intersection of all the $M(A_{i,k})$ containing x, and call this $M(A_x)$. Note that this is a finite intersection, and is thus an affinoid admissible open. The upshot is that, for any point $(x_1, ...x_n) \in X_1 \times \cdots \times X_n$, we know it is contained in $M(A_{x_1}) \times \cdots M(A_{x_n})$, but is also contained in some $M(A_{i,1}) \times \cdots \times M(A_{i,n})$. By construction, we must have that $M(A_{x_k}) \subset M(A_{i,k})$ for each k. Thus, we have constructed a refinement (consisting of all products of form $M(A_{x_1}) \times \cdots M(A_{x_n})$)) of the original p-admissible cover (by constructing a refinement of its refinement) that is finite (since it is built out of products of finite intersections among finitely many possible open affinoids). Further, by construction, this is a product cover (that is, constructed as a product of covers of factors), so we are done.

Definition 2.0.7. Let X be a rigid analytic space. An *admissible Weiss cover* of X is a collection $\{U_i\}_{i\in I}$ of admissible opens satisfying the following: first, it is an admissible covering of X; second, for any integer $n \geq 2$, $\{U_i^n\}_{i\in I}$ constitutes a p-admissible covering of X^n by p-admissible opens.

Remark 2.0.4. The point of the notion of p-admissible covering is that we do not need to consider the more sophisticated rigid analytic topology on fiber products, and need something closer to the product topology for our purposes. This corresponds roughly to the fact that we are defining a Costello-Gwilliam flavored theory.

Remark 2.0.5. Given an admissible Weiss cover, we can form an associated one closed under finite intersections by throwing all finite intersections in. This will be such that the associated p-admissible covers of X^k for each k will be closed under finite intersection. Also, note that any refinements associated to p-admissible coverings consisting of products of affinoids can always be assumed to be closed under finite intersection. The same thing holds for admissible coverings and associated refinements consisting of affinoids.

Proposition 2.0.3. *A covering of X^n by products of admissible opens is p-admissible if and only if for any compact $K \subset X$, the pullback to K^n admits a finite refinement consisting of products of affinoids.*

PROOF: Suppose we have a p-admissible covering. Let K be a finite union of affinoids $M(A_1), ..., M(A_k)$ of X. Then, K^n is a finite union of the mutual n-fold products of these affinoids. The pullback of our p-admissible covering to each of these products of affinoids admits the desired type of finite refinement. Putting all these refinements together yields the first direction.

For the second, note that to show our covering is p-admissible, we consider some product of n affinoids $M(A_1) \times \cdots \times M(A_n)$. Consider $K = M(A_1) \cup \cdots \cup M(A_n)$. The pullback of the covering to K^n now has a refinement consisting of finitely many products of affinoids. Now, noting that $M(A_1) \times \cdots \times M(A_n) \subset K^n$, we can simply intersect the constructed refinement with $M(A_1) \times \cdots M(A_n)$ to produce the desired refinement. This completes the proof.

Remark 2.0.6. Given an admissible covering of a rigid space X, can we produce an admissible Weiss cover from it? One option would be to take all finite unions of elements of our covering. If we assume the elements of the admissible covering are compact, the resulting covering is in fact admissible Weiss. This is firstly because the separatedness hypothesis ensures these finite unions are admissible open. Secondly, we can use the criterion established later. Let us call our admissible covering $\{X_i\}_{i \in I} = \mathcal{U}$. Consider $K \subset X$ compact, and let $n \geq 2$. Denote by $\mathcal{W}$ the covering of X consisting of all finite unions of elements of $\{X_i\}_{i \in I}$. We note there is a finite subcover of $\mathcal{U}$ containing K. Further, taking n-fold unions of the elements of this finite subcover and calling the resulting collection $\{F_j\}_{j \in J}$, it is now clear that $\left\{F_j^n\right\}_{j \in J}$ contains K^n.

Here is perhaps the simplest, most lucid characterization of an admissible Weiss cover:

Proposition 2.0.4. *Let $\{U_i\}_{i\in I} = \mathfrak{U}$ be an admissible cover of U. It is admissible Weiss if and only if it is a Weiss cover in the ordinary sense and, for any $n \geq 2$, and any compact $K \subset U$, there is a finite refinement of $\mathfrak{U}$, called $\{K_j\}_{j\in J}$, consisting of K_j all compact, such that $\left\{K_j^n\right\}_{j\in J}$ has union containing K^n.*

PROOF: For one direction, assume that our cover satisfies the condition involving K_js. We must show that for every $n \geq 2$, the cover $\{U_i^n\}_{i\in I}$ of U^n is p-admissible. To this end, let us consider some $K^n \subset U^n$. The pullback of $\{U_i^n\}_{i\in I}$ to K^n admits a refinement consisting of elements of form K_j^n. Note now that every such K_j^n is a finite union of products of affinoids, so putting together all these products (given there are only finitely many K_j), we are done with one direction.

For the other direction: suppose given an admissible Weiss cover as in the proposition, and let $K \subset X$ be compact. Consider some power K^n. The pullback of $\{U_i^n\}_{i\in I}$ to this power has a finite refinement consisting of products of affinoids. For any given product of affinoids, say $M(A_1) \times \cdots \times M(A_n) \subset U_i^n \cap K^n$, note that we can consider the union $C = M(A_1) \cup \cdots \cup M(A_n)$. Now, $C^n \subset (U_i \cap K)^n$. Doing this for every product of affinoids occurring in aforementioned finite refinement produces the desired collection of compacts K_j.

Corollary 2.0.5. *An admissible covering $\{U_i\}_{i\in I}$ of $U \subset X$ is admissible Weiss if and only if, for any compact $K \subset X$, there is a refinement consisting* *of compacts of the pullback cover to K that is n-Weiss, in the sense that for any n points in K, we can find a refinement element containing all the n points.*

The corollary now makes it easy to check that admissible Weiss covers let us define a natural Grothendieck pretopology associated to a rigid space X. Namely, we can consider for each admissible open $U \subset X$ the coverings given by admissible Weiss coverings of U. It is clear the single $\{U\}$ is admissible Weiss, and of course, admissible opens are stable under finite intersection. That admissible Weiss covers are stable under pullbacks along inclusions of admissible opens is easy to see. Note also that, given an admissible Weiss cover $\{U_i\}_{i\in I}$ of U and ones $\{V_j\}_{j\in J_i}$ of U_i for each $i \in I$, we can see the concatenated cover of U is admissible Weiss as follows. Consider a positive integer n. Let $K \subset U$ be compact. The pullback $\{U_i \cap K\}_{i\in I}$ has a finite refinement of compacts that is a n-Weiss cover of K. Note that, for any given $K_v \subset U_i$, we have that the pullback of $\{V_j\}_{j\in J_i}$ to K_v has a finite refinement of compacts that is n-Weiss for K_v. Now, for any n points of U, it is clear they must be in some K_v, so in some element of the refinement of $\{V_j \cap K_v\}_{j\in J_i}$. Concatenating the refinements associated to each v produces the refinement needed to verify the admissible Weiss condition for K.

Definition 2.0.8. A multiplicative, unital rigid prefactorization algebra $\mathcal{F}$ on X is a *rigid factorization algebra* (also referred to as a nonarchimedean factorization algebra, nonarchimedean geometric factorization algebra, or just factorization algebra) if it satisfies locality with respect to admissible Weiss covers. That is, for $\{U_i\}_{i\in I} = \mathcal{U}$, where $\mathcal{U}$ is an admissible Weiss cover of U, an admissible open in X, the natural map from the associated Čech complex of $\mathcal{F}$ viewed as a precosheaf to $\mathcal{F}(U)$ is a weak equivalence. Equivalently, consider the simplicial object given in degree k by $\oplus_{j_1,\dots,j_k}\mathcal{F}(U_{j_1}\cap\cdots\cap U_{j_k})$ with the simplicial structure induced by the precosheaf extension maps. The geometric realization is the Čech complex, so locality is equivalently defined by saying, for $C(\mathcal{U},\mathcal{F})_*$ the aforementioned Čech simplicial diagram, the natural map $|C(\mathcal{U},\mathcal{F})_*| \to \mathcal{F}(U)$ is a weak equivalence.

Remark 2.0.7. Equivalently, we can define a rigid factorization algebra's locality condition as requiring that, for any admissible Weiss cover $\{U_i\}_{i\in I}$, the natural map $hocolim_S(\mathcal{F}(U_S)) \to \mathcal{F}(U)$ is a weak equivalence, where $S \in J$ range over the finite subsets of I, and U_S denotes the intersection of all U_i for $i \in S$. This follows, for instance, as in Costello and Gwilliam, Appendix 5, Definition 5.4.4 and Lemma 5.4.5, where the functor considered is the diagram in $dgVect_K$ indexed by J viewed as a poset category. Basically, the simplicial bar construction in this case is precisely the Čech simplicial diagram above whose geometric realization is the Čech complex.

Remark 2.0.8. Note that any prefactorization algebra has an underlying precosheaf, given by only considering the structure maps of form $m_{V;U}$, where $V \subset U$ are admissible opens in the ambient X.

Remark 2.0.9. We refer to a precosheaf which satisfies the analogue of the gluing/locality condition for factorization algebras for all admissible covers (not just admissible Weiss ones) as a homotopy cosheaf, and analogously for homotopy sheaves. The interested reader can see a discussion of these objects in Costello-Gwilliam.

Chapter 3

From Sheaves to Cosheaves: A Nonarchimedean Verdier Duality Functor

In order to have a good source of examples of factorization algebras of the sorts we are interested in, it will be important to know how to associate cosheaves to sheaves, since there is a natural way to then associate factorization algebras to these cosheaves. This procedure will supply a version of factorization algebras associated to Lie algebras in our nonarchimedean setting. Another application of knowing how to pass between sheaves and cosheaves is to future work on factorization algebras given by families of (co)sheaves on fiber products X^n of a rigid space: compatible families of sheaves can yield compatible families of cosheaves by applying our Verdier duality functor (this is especially salient in cases where there is a genuine duality/equivalence of sheaf/cosheaf categories).

The basic means of associating a cosheaf to a sheaf in the ordinary topological setting (in particular, in context of a locally compact space) is considering the functor of compactly supported sections associated to a sheaf, with Lurie's Verdier duality yielding in this way an equivalence of ∞-categories between $Shv(X, \mathcal{C})$ and $CShv(X, \mathcal{C})$ for X a locally compact, Hausdorff topological space and $\mathcal{C}$ a stable ∞-category with all small limits and colimits where our (co)sheaves take values, and $Shv, CShv$ denoting the categories of sheaves and cosheaves. Our aim will be to adapt this story to the rigid setting. We define a version of Lurie's Verdier duality functor for sheaves on separated rigid analytic spaces, but we only demonstrate it is an equivalence in the case of affinoid spaces, and between overconvergent versions of the sheaf and cosheaf categories. It seems such overconvergence may even be necessary to get a Verdier duality equivalence.

3.0.1 Defining a Verdier Duality Functor

The goal of this section, as in the title, is to define a functor from sheaves to cosheaves. We will begin by selecting appropriate analogues of Lurie's Verdier duality ingredients for our setting. The analogue of a compact subset of a locally compact, Hausdorff topological space will be an admissible open subset of a separated rigid analytic space that is a finite union of affinoids. This is sensible, because the affinoids are, by definition of the G-topology, compact in an appropriate sense. Another nice property of our setting is that the theory of what Lurie calls $\mathcal{K}$-sheaves (that is, a version of sheaves, and indeed, crucial to Lurie's proof, an equivalent model for ordinary sheaves, evaluated on compact subsets of the spaces he considers) is very clearly related to ordinary rigid analytic sheaves, given that rigid analytic spaces are built up from affinoids.

We recall the notion of $\mathcal{K}$-sheaf, a sheaf on compact subsets, in the rigid analytic setting. First, we recall the correct notion of compact subspace for our setting.

Definition 3.0.1. A compact subspace $K \subset X$ is an admissible open that is a finite union of affinoids. Denote the category given by the poset of compact subspaces by $\mathcal{K}(X)$.

Remark 3.0.1. We reiterate that such finite unions are automatically admissible coverings, due to the separatedness we are always assuming.

Definition 3.0.2. A $\mathcal{K}$-*sheaf* on a rigid analytic space X valued in a stable ∞-category $\mathcal{C}$ with small limits and colimits is a functor $\mathcal{F} : N(\mathcal{K}(X))^{op} \to \mathcal{C}$ satisfying the following:

(i) $\mathcal{F}(\emptyset)$ is final.

(ii) For every pair of compact subspaces K, K' of X, the diagram

$$\begin{tikzcd} \mathcal{F}(K \cup K') \arrow[r] \arrow[d] & \mathcal{F}(K) \arrow[d] \\ \mathcal{F}(K') \arrow[r] & \mathcal{F}(K \cap K') \end{tikzcd}$$

is a pullback square.

(Note that we do not need an analogue of Lurie's condition (iii) defining $\mathcal{K}$-sheaves; in fact, this can be seen to be because our situation is special, where the G-topology is generated by affinoids, and thus compact subspaces.)

Denote the $\mathcal{K}$-sheaves by $Shv_{\mathcal{K}}(X)$. We can define $\mathcal{K}$-cosheaves similarly, and denote them by $CShv_{\mathcal{K}}(X)$.

Remark 3.0.2. We will at times refer to $\mathcal{K}$-(co)sheaves also in the topological space setting, for instance when considering the underlying topological space of the Berkovich space of an affinoid rigid analytic space.

We state the main theorem of this section, whose proof is deferred until a little later. It is a partial analogue of Lurie's Proposition 5.5.5.10.

Theorem 3.0.1. *Let $\mathcal{C}$ be a stable ∞-category with all small limits and colimits. Let X be a separated rigid analytic space. Then, the assignment given by sending $\mathcal{F} \mapsto (\mathcal{F}_c : U \mapsto \Gamma_c(U, \mathcal{F}))$ defines a functor of ∞-categories $(-)_c : Shv(X, \mathcal{C}) \to CShv(X, \mathcal{C})$. Here, the functor sends $\mathcal{F}$ to the assignment $U \mapsto colim_{K \subset U} \Gamma_K(X, \mathcal{F})$, where K is compact in U.*

Remark 3.0.3. Note the above does *not* read $U \mapsto colim_{K \subset U} \Gamma_K(U, \mathcal{F})$, and this point is discussed further later.

Remark 3.0.4. The separatedness hypothesis will (among the other useful roles it is playing) ensure that, when we take complements of compact K in X, we get an admissible open set.

We will have a couple of lemmas first that are analogues of results in *Higher Topos Theory* , namely 7.3.4.8-9. The first shows that $\mathcal{K}$-(co)sheaves are determined locally. The second shows that, by right Kan extension, we get a sheaf from a $\mathcal{K}$-sheaf (that is, the extension to all admissible opens from merely compact ones by right Kan extension is a sheaf). This second lemma's analogue for cosheaves and left Kan extensions holds by the same arguments.

Lemma 3.0.2. *Let X be a rigid analytic space, $\mathcal{C}$ a stable ∞-category with small limits and colimits. Consider an admissible covering by admissible opens $\mathcal{U}$ of X. Denote by $\mathcal{K}_{\mathcal{U}}(X)$ the collection of compact admissible opens of X contained in a given element of $\mathcal{U}$. Consider $\mathcal{F}$ a $\mathcal{K}$-sheaf. Then this is a right Kan extension of its restriction to $N(\mathcal{K}_{\mathcal{U}}(X))^{op}$.*

PROOF: It is a fact that a covering $\mathcal{U}$ satisfies the conclusion of the lemma (henceforth, such a cover is called a *good cover*) if and only if for any compact $K \subset X$, the set of $\{K \cap U\}_{K \in \mathcal{K}(X)}$ for $U \in \mathcal{U}$ satisfies the analogous conclusion replacing X with K. Therefore, we may assume X is compact. This implies that $\mathcal{U}$ has a finite (admissible) refinement consisting of compact subsets. We will induct on the size of this refinement to show that $\mathcal{U}$ is a good cover if it has such a finite refinement.

Call the refinement $\{K_1, ..., K_n\}$. The case of $n = 0$ just involves showing $\mathcal{F}(\emptyset)$ is final (since in this case, X is empty), which follows from the definition of $\mathcal{K}$-sheaf.

To carry out the inductive step, we notice the following fact. Suppose that we have two admissible coverings with one, $\mathcal{U}$ a refinement of the other, $\mathcal{U}'$; additionally, suppose that for every $U' \in \mathcal{U}'$, $\{U' \cap U\}_{U \in \mathcal{U}}$ is a good covering of U'. Then, we note that we have $\mathcal{U}'$ is good if and only if $\mathcal{U}$ is. Here, we are appealing to *Higher Topos Theory,* 4.3.2.8 for the chain $\mathcal{K}_{\mathcal{U}}(X) \subset \mathcal{K}_{\mathcal{U}'}(X) \subset \mathcal{K}(X)$.

Now, suppose for the inductive step that we've shown the conclusion for $n-1$, and consider the admissible open $V = K_2 \cup \cdots \cup K_n$. We put $\mathcal{U}' := \mathcal{U} \cup \{K_1\} \cup \{V\}$. This is an admissible covering with admissible refinement $\mathcal{U}$. We wish to apply the above paragraph's fact of notice to each of the elements. This is easy for those of $\mathcal{U}$ and K_1. For V, notice by the inductive hypothesis that, since $\{W \cap V\}_{W \in \mathcal{U}}$ has an admissible refinement consisting of $n-1$ compact subsets, that this is a good cover of V. (To elaborate slightly, the refinement is produced by noting $\mathcal{U}$ has the refinement consisting of all the K_i, and we can intersect each of these with V to get a refinement of $\{W \cap V\}_{W \in \mathcal{U}}$, and of course, this has refinement simply consisting of all the K_i for $i \neq 1$.) Therefore, we have reduced to demonstrating the lemma for $\mathcal{U}'$ instead of for $\mathcal{U}$. Further, we can reduce to showing the lemma for the covering consisting of K_1 and V. This reduction also follows from the fact of notice from above: indeed, all we must know is that $K_1 \cap W, V \cap W$ for W admissible open in $\mathcal{U}'$ is a good cover of W. Notice that this can be demonstrated simply by demonstrating the analogous fact for $K \subset W$ compact. In this case, appealing to separatedness of X, each of $K_1 \cap K, V \cap K$ is compact, and we can set $\mathcal{R}$ equal to the admissible cover of K consisting of $K_1 \cap K, V \cap K$. This is certainly good, as our $\mathcal{F}$ evaluated on any compact K' inside K is realized as the appropriate right Kan extension by considering $\mathcal{R}'$ to be the admissible covering given by $K_1 \cap K', V \cap K'$ and noting the cofinality of $N(K_1 \cap K', V \cap K', K_1 \cap K' \cap V) \subset N(\mathcal{K}_{\mathcal{R}'}(K'))$, plus the fact that $\mathcal{F}$ is a $\mathcal{K}$-sheaf.

To complete the proof, we want to demonstrate that $\mathcal{F}(K)$, for K a compact subset of X, is the limit of $\mathcal{F}|(N(\mathcal{K}_{\mathcal{R}}(K))^{op})$. This once again follows by the aforementioned cofinality argument, exactly as above.

Remark 3.0.5. We note that the main significance of good coverings being *admissible* above is seen in the fact that we could reduce to the case of $\mathcal{U}$ being finite (of size 2, in particular). Indeed, it is unsurprising that a rigid sheaf would be determined by its behavior on affinoids. However, since $\mathcal{K}$-sheaves only exhibit a limited gluing property with respect to covers with two compact subspaces, some further argument (such as the above lemma) is crucial. A concise summary of the above lemma is that any *admissible* covering is good.

Let us now continue on to the second required lemma.

Lemma 3.0.3. *Let X be a rigid analytic space, and $\mathcal{C}$ be a stable ∞-category with all small limits and colimits. Let $\mathcal{G}$ be a $\mathcal{K}$-sheaf on X, and let $\mathcal{F} : N(\mathcal{U}(X))^{op} \to \mathcal{C}$ be a functor derived from $\mathcal{G}$ by right Kan extension. Then, $\mathcal{F}$ is a sheaf on the rigid space X.*

PROOF: Let $\mathcal{W}$ be a covering sieve. We must show that $N(\mathcal{W})^{\triangleleft} \to \mathcal{C}$ is a colimit diagram. Let $\mathcal{K}_{\mathcal{W}}(X)$ denote those members of $\mathcal{K}(X)$ so that each is contained in some element of $\mathcal{W}$. Noting that $N(\mathcal{W}) \subset N(\mathcal{W} \cup \mathcal{K}_{\mathcal{W}}(X))$ is cofinal, it suffices to demonstrate that, for any admissible U, the restriction of $\mathcal{F}$ to $N(\mathcal{W} \cup \mathcal{K}_{\mathcal{W}}(X))^{op}$ has right Kan extension the restriction to $N(\mathcal{W} \cup \mathcal{K}_{\mathcal{W}}(X) \cup \{U\})^{op}$. To show this, by appealing to *Higher Topos Theory,* 4.3.2.8 for the tower $\mathcal{K}_{\mathcal{W}}(X) \subset \mathcal{K}_{\mathcal{W}}(X) \cup \mathcal{W} \subset \mathcal{K}_{\mathcal{W}}(X) \cup \mathcal{W} \cup \{U\}$, it suffices to show that $\mathcal{F}|(\mathcal{K}_{\mathcal{W}}(X) \cup \mathcal{W} \cup \{U\})^{op}$ is a right Kan extension of $\mathcal{F}|(\mathcal{K}_{\mathcal{W}}(X))^{op}$. However, since we already know this is true at every element except U, it will suffice to show $\mathcal{F}|(\mathcal{K}_{\mathcal{W}}(X) \cup \{U\})^{op}$ is a right Kan extension of the corresponding restriction to $\mathcal{K}_{\mathcal{W}}(X)^{op}$.

This is straightforward, because of the previous lemma and another appeal to *Higher Topos Theory,* 4.3.2.8 for the tower $\mathcal{K}_{\mathcal{W}}(X) \subset \mathcal{K}(X) \subset \mathcal{K}(X) \cup \{U\}$. The Kan extension condition at U for our $\mathcal{K}_{\mathcal{W}}(X)$ can be shown as follows: as long as we can demonstrate the analogous one from left to center and center to right, we will be done. The left to center case follows from the previous lemma. The center to right case is clear by construction.

Remark 3.0.6. The analogue of the above with cosheaves instead of sheaves and left Kan extension instead of right holds by exactly analogous arguments.

We are now almost ready for the proof of Theorem 3.1. There is just one lemma to verify first:

Lemma 3.0.4. *Suppose that X is a separated rigid analytic space. Then, for any compact subspaces $K, K' \subset X$, $\{X \setminus K, X \setminus K'\}$ constitutes an admissible covering of its union $X \setminus (K \cap K')$.*

PROOF: Suppose first that X is affinoid. Now we can appeal to the results of Schneider's *Points on Rigid Analytic Varieties* in connection with the assignment $M(-)$, his version of the topological space underlying the Berkovich space of an affinoid. The statement of the lemma holds for X

precisely if $\{M(X) \setminus M(K), M(X) \setminus M(K')\}$ is a cover in the ordinary topological sense of $M(X \setminus K \cap K')$, since the complement of a compact subspace is a wide open in the sense of Schneider's article (for the rest of this argument, when we use the term wide open, we refer to Schneider's article, not to be confused with the use later based on Coleman's work on rigid analytic curves). However, due to the compatibilities of the assignment $M(-)$, we have that $M(X \setminus (K \cap K')) = M(X) \setminus M(K \cap K') = M(X) \setminus (M(K) \cap M(K'))$. This is in turn $(M(X) \setminus M(K)) \cup (M(X) \setminus M(K')) = M(X \setminus K) \cup M(X \setminus K')$, as desired.

We now suppose X is a general separated space and show we can reduce to the affinoid case. Let $\{X_i\}_{i \in I}$ be an admissible covering of X by open affinoids. We intersect it with $X \setminus (K \cap K')$ to obtain an admissible covering $\{X_i \setminus ((X_i \cap K) \cap (X_i \cap K'))\}_{i \in I}$ of this admissible open. Thus, to show that $\{X \setminus K, X \setminus K'\}$ constitutes an admissible covering of $X \setminus (K \cap K')$, it is sufficient to show intersecting the former with each $X_i \setminus ((X_i \cap K) \cap (X_i \cap K'))$ yields an admissible covering of it. So it is enough to show $\{X_i \setminus (X_i \cap K), X_i \setminus (X_i \cap K')\}$ constitutes an admissible covering of $X_i \setminus ((X_i \cap K) \cap (X_i \cap K'))$. However, this follows from the work on the affinoid case, applied to the compact subspaces $X_i \cap K, X_i \cap K'$ of X_i.

PROOF OF THEOREM 3.0.1: We produce our functor $Shv(X, \mathcal{C}) \to CShv(X, \mathcal{C})$ by way of a composition of two functors $Shv(X, \mathcal{C}) \to CShv_{\mathcal{K}}(X, \mathcal{C}) \to CShv(X, \mathcal{C})$. The first functor is given by sending $\mathcal{F} \mapsto (K \mapsto \Gamma_K(X, \mathcal{F}))$. The second is given by sending $\mathcal{G} \mapsto (U \mapsto colim_{K \subset U} \mathcal{G}(K))$. The composition, then, is given by taking sections over X with support in some compact of the given admissible open. Given the second lemma we proved applies equally well to left Kan extensions and $\mathcal{K}$-cosheaves with exactly the same sort of argument, there is nothing to do except to show that the first assignment produces not just a functor on $N(\mathcal{K}(X))^{op}$, but an actual K-cosheaf. We check each of the conditions. Condition (i) follows easily from noting that $\Gamma_{\emptyset}(X, \mathcal{F})$ is zero, given it is gotten from the kernel of an equivalence. Condition (ii) follows if we note that, if we denote the precosheaf we want to show to be a cosheaf by $\mathcal{G}$, the square

$$\begin{array}{ccc} \mathcal{G}(K \cup K') & \longrightarrow & \mathcal{G}(K) \\ \downarrow & & \downarrow \\ \mathcal{G}(K') & \longrightarrow & \mathcal{G}(K \cap K') \end{array}$$

is a pullback if and only if it is a pushout (the latter of which we want to be the case). The former is the case, because it arises as the fiber of a map of pullback squares

$$\begin{array}{ccc} \mathcal{F}(X) & \longrightarrow & \mathcal{F}(X) \\ \downarrow & & \downarrow \\ \mathcal{F}(X) & \longrightarrow & \mathcal{F}(X) \end{array}$$

and

$$\begin{array}{ccc} \mathcal{F}(X \setminus (K \cup K')) & \longrightarrow & \mathcal{F}(X \setminus K) \\ \downarrow & & \downarrow \\ \mathcal{F}(X \setminus K') & \longrightarrow & \mathcal{F}(X \setminus (K \cap K')) \end{array}$$

Applying a previous lemma, we know that $\{X \setminus K, X \setminus K'\}$ constitutes an admissible covering of $X \setminus (K \cap K')$, so the sheaf condition for $\mathcal{F}$ gives us the last of what we need to verify condition (ii), whence the proof is finished.

3.0.2 Verdier Duality Equivalence for Affinoids

Let us now proceed to demonstrate that, in the special case where we restrict to *overconvergent* sheaves and analogous cosheaves on an affinoid rigid analytic space, we can do better than the above: Lurie's Verdier duality functor furnishes us with a version of Verdier duality relevant to the rigid analytic setting. Let us recall the notion of overconvergence, and additionally consider an analogous notion to overconvergence for cosheaves. This section intimately uses Schneider's article *Points on Rigid Analytic Varieties,* and the basic notions, including many of those most relevant to us, are reviewed in the appendix on rigid geometry. This includes (and the reader is warned to be careful about this) the definition of wide opens (which should correspond to the notion of a partially proper open immersion from discussions of morphisms of rigid analytic spaces without boundary) relevant to this chapter, as opposed to those employed when specializing to curves and factorization theorems (which deal with a version used by Robert Coleman, among others, to discuss semistable coverings).

Definition 3.0.3. Let $\mathcal{F}$ be a sheaf on an affinoid rigid analytic space X valued in an ∞-category $\mathcal{C}$. We say that $\mathcal{F}$ is *overconvergent* if, for any compact admissible open $K \subset X$, the natural map $colim_{K \subset W} \mathcal{F}(W) \to \mathcal{F}(K)$ (where all W are wide opens) is an equivalence. Denote the overconvergent sheaves by $OverShv(X, \mathcal{C})$. Similarly, letting $\mathcal{F}$ be a $\mathcal{K}$-sheaf on X, we call it overconvergent as well, if the natural map $colim_{K \subset\subset K'} \mathcal{F}(K') \to \mathcal{F}(K)$ is an equivalence. Denote the overconvergent $\mathcal{K}$-sheaves by $OverShv_{\mathcal{K}}(X, \mathcal{C})$.

Definition 3.0.4. Similarly, define $\mathcal{G}$ a cosheaf on an affinoid rigid analytic space X as above to be *co-overconvergent* if, for any compact admissible open K of X, the natural map $\mathcal{G}(K) \to lim_{K \subset W} \mathcal{G}(W)$, again where W range over the wide opens containing K, is an equivalence. Denote the co-overconvergent cosheaves by $OverCShv(X, \mathcal{C})$. Similarly, we define a co-overconvergent $\mathcal{K}$-cosheaf to be a $\mathcal{K}$-cosheaf $\mathcal{G}$ so that the natural map $\mathcal{G}(K) \to lim_{K \subset\subset K'} \mathcal{G}(K')$ is an equivalence.

Here is the main theorem involving these two definitions:

Theorem 3.0.5. *Let X be an affinoid rigid analytic space. There is an equivalence of ∞-categories $OverShv(X, \mathcal{C}) \to OverCShv(X, \mathcal{C})$ furnished by the functor $\mathcal{F} \mapsto \mathcal{F}_c$ constructed earlier.*

PROOF: This proof will intimately use Schneider's characterization of the Berkovich space of an affinoid rigid analytic space, given by the assignment $M(-)$ discussed above. Note that $M(X)$ is a locally compact, Hausdorff space. Thus, Lurie's Verdier duality functor yields an equivalence of ∞-categories $Shv(M(X), \mathcal{C}) \cong CShv(M(X), \mathcal{C})$, and en route also an equivalence betwee the $\mathcal{K}$-sheaves and cosheaves.

The proof will consist of first demonstrating that the category of sheaves in the wide open G-topology on X, denoted $Shv_{wo}(X, \mathcal{C})$, is equivalent to the category of overconvergent $\mathcal{K}$-sheaves on X, and then checking that the category of sheaves for the wide open G-topology is equivalent with that of ordinary sheaves on $M(X)$, and that the category of overconvergent $\mathcal{K}$-sheaves is equivalent to that of overconvergent sheaves. This yields an equivalence $OverShv(X, \mathcal{C}) \cong Shv(M(X), \mathcal{C})$, and we will not give the details for the analogous equivalence $OverCShv(X, \mathcal{C}) \cong CShv(M(X), \mathcal{C})$, as the argument is clear by symmetry. We will then furnish our final equivalence by the string $OverShv(X, \mathcal{C}) \cong OverShv_{\mathcal{K}}(X, \mathcal{C}) \cong Shv_{wo}(X, \mathcal{C}) \cong Shv(M(X), \mathcal{C}) \cong CShv(M(X), \mathcal{C}) \cong OverCShv(M(X), \mathcal{C})$. We touch on why this is given by the Verdier duality functor constructed earlier.

Let us now prove all that we need in a sequence of lemmas. Denote by $\mathcal{W}(X)$ the poset of wide opens of X. First note that the posets $\mathcal{K}(X)$ and that $M(\mathcal{K}(X))$ can be identified, due to the behavior of $M(-)$ with respect to inclusions. We can also identify $\mathcal{W}(X)$ and $M(\mathcal{W}(X))$. Thus, we can identify functors out of $N(\mathcal{W}(X))^{op}$ with those out of $N(M(\mathcal{W}(X)))^{op}$, and analogous remarks apply to $\mathcal{K}(X)$.

Lemma 3.0.6. *Let F be a functor $N(\mathcal{K}(X) \cup \mathcal{W}(X))^{op} \to \mathcal{C}$. Then, the following conditions are equivalent: (1) The restriction $F|(N(\mathcal{K}(X))^{op}$ is an* *overconvergent $\mathcal{K}$-sheaf, and F is a right Kan extension of this restriction. (2) The restriction $F|(N(\mathcal{W}(X))^{op}$ is a sheaf for the wide open G-topology, and F is a left Kan extension of this restriction.*

PROOF: First suppose (2). We show that the restriction of F to $N(\mathcal{K}(X))^{op}$ is an overconvergent $\mathcal{K}$-sheaf as follows. By our aforementioned identification, we can identify F with a functor $N(M(\mathcal{K}(X))\cup M(\mathcal{W}(X)))^{op} \to \mathcal{C}$. To show the appropriate restriction is a $\mathcal{K}$-sheaf first of all, let us note what needs to be shown is that the empty object is sent to 0, and the usual gluing condition must be satisfied for unions of two compacts K, K'. Note that, under our identification, we can think of $F|N(M(\mathcal{W}(X))^{op})$ equivalently as a sheaf, in the topological sense, with respect to the opens $M(\mathcal{W}(X))$. To see this, we must show that presheaves (under our identification) satisfy the gluing condition in the topological sense precisely if they do so in the wide open G-topology sense. First, note that, as Schneider shows, a collection in $\mathcal{W}(X)$ $\{U_i\}_{i\in I}$ is an admissible covering of U if and only if $\{M(U_i)\}_{i\in I}$ covers $M(U)$. Further, either applying or undoing M preserves finite intersections, and this completes the identification of the topological sheaf condition with the wide open G-topology one.

This is used as follows. Note that F is a left Kan extension of the restriction to $N(M(\mathcal{W}(X))^{op}$ under our identification precisely if it is a left Kan extension of the restriction to $N(\mathcal{W}(X))^{op}$ before the identification. This means that our F is a left Kan extension of a topological sheaf on images of wide opens under M. From now on, let us call such sheaves $\mathcal{W}$-sheaves. First, we take for granted that this means the restriction to $N(M(\mathcal{K}(X))^{op}$ of F must satisfy that $M(\emptyset)$ is sent to 0, and the usual $\mathcal{K}$-sheaf gluing condition for $M(K), M(K')$ unioning to $M(K \cup K') = M(K) \cup M(K')$ and intersecting at $M(K) \cap M(K') = M(K \cap K')$. However, under our identification of compacts with their images under M, this is exactly the $\mathcal{K}$-sheaf gluing condition (on X) for the restriction of $F|N(\mathcal{K}(X))^{op}$. This leaves justifying what we took for granted above. Again viewing the restriction of F to the wide opens as a $\mathcal{W}$-sheaf on $M(X)$, note such a sheaf can be uniquely extended to a sheaf on $M(X)$. When we left Kan extend this latter sheaf to the compacts, we claim subsequently restricting to $N(M(\mathcal{K}(X)))^{op}$ yields a functor equivalent to the restriction of F to $N(M(\mathcal{K}(X)))^{op}$. This follows by noting that, for any given $K \subset X$ compact, the poset $\{M(W) : W \in \mathcal{W}(X), M(K) \subset M(W)\}$ includes cofinally into the poset of opens of $M(X)$ containing $M(K)$. This uses that the $M(W)$ constitute a fundamental system of open neighborhoods of $M(K)$. Basically, this shows that F restricted to $N(M(\mathcal{K}(X)))^{op}$ is produced by left Kan extending a sheaf on $M(X)$ to the $M(K)$ with $K \subset X$ compact, whence by *Higher Topos Theory* 7.3.4.9, it must of course satisfy the $\mathcal{K}$-sheaf condition for the $M(K), M(K')$ as above.

So we've verified that the restriction of F to the compacts yields a $\mathcal{K}$-sheaf. Now we must show that it is overconvergent. This follows quite easily by identifying the colimit condition involved in left Kan extending F from its restriction to the wide opens with the one involved in overconvergence (it uses that the $M(K')$ so that $K \subset\subset K'$ constitute a fundamental system of compact neighborhoods of $M(K)$). Let K be some compact subspace of X. Let $\mathcal{V}$ be the union of the posets of wide opens containing K and the compacts K' so that $K \subset\subset K'$. Let $\mathcal{V}'$ be the further union adding the element K itself. It is clear that F restricted to both $N(\mathcal{V})^{op}$ and $N(\mathcal{V}')^{op}$ are left Kan extensions of their restrictions to the wide opens. Thus, the restriction to $N(\mathcal{V}')^{op}$ is a left Kan extension of that to $N(\mathcal{V})^{op}$. Denote the poset of compacts K' so $K \subset\subset K'$ by $\mathcal{K}_{K\subset\subset}(X)$. It is now enough to demonstrate that the inclusion

$$N(\mathcal{K}_{K\subset\subset}(X))^{op} \subset N(\mathcal{K}_{K\subset\subset}(X)) \cup \{U \in \mathcal{W}(X) : K \subset U\})^{op}$$

is cofinal. However, this follows from the aforementioned fact about fundamental systems of compact neighborhoods, using the good behavior of M along inclusions. In particular, to demonstrate that the usual contractibility criterion is satisfied, let us note that for any $K \subset W \in \mathcal{W}(X)$, we can find some $K \subset\subset K' \subset W$. This is because we can certainly find $M(K) \subset M(K') \subset M(W)$ where $M(K')$ is a compact neighborhood of $M(K)$. But this implies $K \subset W' \subset K' \subset W$ for some W' wide open. We must show $K \subset\subset K'$. To do this, pick some x an analytic point of X so that K is a neighborhood of it. Then, we must show x to be inner in K'. But notice that $x \in M(W')$, since $x \in M(K) \subset M(W')$, and by definition of wide open, this implies that x is inner in W'. Now it follows from the definitions that x is also inner in K', as desired (the desired affinoid subdomain of K' we must find to prove this can be taken to be the open affinoid of W' guaranteed by virtue of x being inner in W'). Also, such K' are clearly closed under finite intersection, whence the proof is finished.

So we have shown that the restriction of F to the compacts yields an overconvergent $\mathcal{K}$-sheaf. Now, we must demonstrate, further, that F is actually a right Kan extension of the restriction to the compacts. To see this, we will produce a new functor as follows: restrict F to the wide opens, and extend this to a sheaf on $M(X)$. Then, produce by left Kan extension a functor on the union of the posets of compacts in $M(X)$ and opens in $M(X)$. Notice that this is a functor of the sort dealt with in *Higher Topos Theory,* 7.3.4.9. Also, by construction, its restriction to the wide opens coincides with that of F, and we can also see the restriction to $N(\mathcal{K}(X))^{op}$ coincides with that of F. Thus, it will be enough to show that this functor is a right Kan extension of its restriction to $N(\mathcal{K}(X))^{op}$ at every element of $N(\mathcal{W}(X))^{op}$. But, this now directly follows because it is a right Kan extension of its restriction to the category of compact subspaces of $M(X)$, and, using that for a given $M(W)$ for W wide open, the category of $M(K)$ with $K \subset X$ compact such that $M(K) \subset M(W)$ includes into that of compacts contained in $M(W)$, and this is cofinal, meaning that the right Kan extension condition we desire for F follows from the one F' satisfies.

We have now completed the proof that (2) implies (1). It remains to show that (1) implies (2). To this end, suppose that F restricted to the compacts yields an overconvergent $\mathcal{K}$-sheaf, and that it is a right Kan extension of this restriction. We must demonstrate, first of all, that its restriction to the wide opens is a sheaf for the wide open G-topology, and secondly that it is a left Kan extension of the restriction to the wide opens. This can be done as follows. First of all, that the restriction to the wide opens is a sheaf for the wide open G-topology follows very easily, since left Kan extending from the compacts to $N(E(X))^{op}$, where $E(X)$ denotes the admissible opens poset, yields a sheaf with respect to all admissible coverings, so certainly with respect to the wide open G-topology.

It remains to demonstrate that F is a left Kan extension of the restriction to the wide opens. This follows by considering a given compact K, and simply noting the inclusions $N(\mathcal{W}_{K\subset}(X))^{op}, N(\mathcal{K}_{K\subset\subset}(X))^{op} \subset N(\mathcal{W}_{K\subset}(X) \cup \mathcal{K}_{K\subset\subset}(X))^{op}$ are cofinal. Here $\mathcal{W}_{K\subset}$ denotes the wide opens containing K. The idea is that this lets us use the colimit condition guaranteed by the overconvergent $\mathcal{K}$-sheaf property to deduce the analogous one with respect to wide opens. This completes the proof that (1) and (2) are equivalent.

It is now clear that our proof actually demonstrates an equivalence between $OverShv_{\mathcal{K}}(X, \mathcal{C})$ and $Shv_{wo}(X, \mathcal{C})$, since the category $\mathcal{E}$ satisfying the equivalent conditions of the lemma maps to both of these by the appropriate restriction functors, both of which yield trivial Kan fibrations, due to our characterization of $\mathcal{E}$ using Kan extensions. Thus, we are now ready for the next lemma:

Lemma 3.0.7. *There is a functor $OverShv_{\mathcal{K}}(X, \mathcal{C}) \to OverShv(X, \mathcal{C})$ furnished by right Kan extension, and it is an equivalence of ∞-categories.*

PROOF: The functor is simply given by restricting the functor $Shv_{\mathcal{K}}(X, \mathcal{C}) \to Shv(X, \mathcal{C})$ to $OverShv_{\mathcal{K}}(X, \mathcal{C})$. That this has image in $OverShv(X, \mathcal{C})$ is verified, since the proof of the lemma above shows that any $\mathcal{F}$ in the image, if restricted to $N(\mathcal{K}(X) \cup \mathcal{W}(X))^{op}$, is a left Kan extension of its restriction to the wide opens. Note that, letting $\mathcal{E}$ denote the full subcategory of functors $N(E(X))^{op} \to \mathcal{C}$ that are right Kan extensions of their restriction to the compacts, we have a trivial Kan fibration $\mathcal{E} \to Fun(N(\mathcal{K}(X))^{op}, \mathcal{C})$ given by restriction. This implies that there is also a trivial Kan fibration $\mathcal{E}' \to OverShv_{\mathcal{K}}(X, \mathcal{C})$, where $\mathcal{E}'$ is the full subcategory of functors $N(E(X))^{op} \to \mathcal{C}$ whose restriction to the compacts yields an overconvergent $\mathcal{K}$-sheaf, and are right Kan extensions of these restrictions. We have seen any object of $\mathcal{E}'$ is an overconvergent sheaf. Composing the inverse to $\mathcal{E}' \to OverShv_{\mathcal{K}}(X, \mathcal{C}))$ with the inclusion of $\mathcal{E}'$ into the overconvergent sheaves, we have a fully faithful functor. $OverShv_{\mathcal{K}}(X, \mathcal{C})$. It remains to see that it is essentially surjective. This now follows, because any overconvergent sheaf is the right Kan extension of its restriction to compacts (this is true for any sheaf), but in addition, this restriction defines an overconvergent $\mathcal{K}$-sheaf. This again follows from examining our work for the previous lemma, and the proof is finished.

Let us finally note the lemma:

Lemma 3.0.8. *The ∞-categories $Shv_{wo}(X, \mathcal{C})$ and $Shv(M(X), \mathcal{C})$ are equivalent.*

PROOF: The functor will be given as follows: first, we can identify sheaves for the wide open G-topology with sheaves for the basis elements $M(W)$ of the topology on $M(X)$. This was covered earlier. Then, simply appeal to the equivalence between sheaves for these basis elements and all sheaves on $M(X)$, since this is a basis closed under finite intersections.

It is now time to note why the equivalence we have built between overconvergent sheaves and co-overconvergent cosheaves is actually given by the rigid Verdier duality functor we defined earlier. To see this, let us note that the application of Lurie's Verdier duality equivalence between sheaves on $M(X)$ and cosheaves on it is ultimately determined by composing the restriction functor from sheaves to cosheaves with the analogous equivalence between $\mathcal{K}$-sheaves and cosheaves on $M(X)$. However, this composition is given by sending a sheaf on $M(X)$, call it $\mathcal{F}$, to the $\mathcal{K}$-cosheaf given by $C \mapsto \Gamma_C(X, \mathcal{F})$. In the particular case where C is of form $M(K)$, this is given by $fib(\mathcal{F}(M(X)) \to \mathcal{F}(M(X) \setminus M(K)))$. However, by identifying $M(X) \setminus M(K)$ with $M(X \setminus K)$, it is now clear (and left to the reader) that given our descriptions above that our equivalence $OverShv(X, \mathcal{C}) \cong OverCShv_{\mathcal{K}}(X, \mathcal{C})$ is given by the usual $\mathcal{F} \mapsto (K \mapsto fib(\mathcal{F}(X) \to \mathcal{F}(X \setminus K)))$.

We now note that our Verdier duality functor from sheaves to cosheaves is, in a sense, most meaningful (because it is given by genuine compact supports) when we consider what the cosheaves do on wide opens (that is, for partially proper morphisms $U \hookrightarrow X$, where X is separated). Let W be an admissible open of X containing a compact admissible open K. Note the diagram

$$\begin{array}{ccc} fib & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ \mathcal{F}(X) & \longrightarrow & \mathcal{F}(X \setminus K) \\ \downarrow & & \downarrow \\ \mathcal{F}(W) & \longrightarrow & \mathcal{F}(W \setminus K) \end{array}$$

where the top square is a pullback (that is, fib is $\Gamma_K(X, \mathcal{F})$). We would like the outer square to be a pullback as well, so as to identify fib with $\Gamma_K(W, \mathcal{F})$. However, as $\{W, X \setminus K\}$ may not be an admissible covering of X, the bottom square may not always be a pullback, which would be the natural way to get the desired identification. It turns out this condition holds naturally with further conditions (related to the idea of being wide open) on W.

Lemma 3.0.9. *Let $W \subset X$ be an admissible open, where X is a separated rigid analytic space, and suppose $K \subset W$ is a compact admissible open of X. Suppose that there is an admissible covering of X by open affinoids $\{X_i\}_{i \in I}$ so that the intersection of W with each X_i is a wide open of it. Then, $\{W, X \setminus K\}$ is an admissible covering of X.*

PROOF: First, let us demonstrate the result for X affinoid. Note that in this case, W defines a wide open in the sense of *Points on Rigid Analytic Varieties* . Note that we can verify the lemma for this case by showing $\{M(W), M(X \setminus K)\}$ is a topological covering of $M(X)$. However, this follows because $M(X \setminus K) = M(X) \setminus M(K)$, and $M(K) \subset M(W)$, as $M(-)$ preserves inclusions.

We now suppose X is a general separated space. Let $\{X_i\}_{i \in I}$ be the given admissible covering of X by open affinoids. Now, note that we can check our covering $\{W, X \setminus K\}$ is admissible by checking it by intersecting with every X_i. An arbitrary such intersection looks like $\{W \cap X_i, X_i \setminus (X_i \cap K)\}$. As X is separated, $X_i \cap K$ is a compact open of X_i. Thus, as $W \cap X_i$ is a wide open of X_i, containing $K \cap X_i$, we can apply the lemma using this wide open of X_i and the compact $K \cap X_i$ of X_i to conclude the intersection covering is an admissible covering of X_i, and the proof is finished.

We will now give a somewhat more detailed presentation of the functor passing from sheaves to cosheaves, and also comment on what appears to work best about extending Lurie's proof to the specialized situation considered above (as opposed to for more general rigid analytic sheaves).

3.0.3 Further Details of Duality Functor

Proceeding much as Lurie does in *Higher Algebra,* let us consider a poset M, consisting of pairs (i, S) with $i = 0, 1, 2$ and $S \subset X$, where X is a separated rigid analytic space. Further, require that S is compact if $i = 0$ and $X \setminus S$ is compact if $i = 2$. We declare $(i_1, S_1) \leq (i_2, S_2)$ if $i_1 \leq i_2$ and $S_1 \subset S_2$ or if $i_1 = 0, i_2 = 2$. We can, in addition, consider a larger poset M' consisting of pairs (i, S) with i as above, and S as above for $i = 1, 2$, but with the lax requirement that $X \setminus S$ is an admissible open of X for $i = 2$. We give here an argument for Verdier duality more directly building a version of the ideas of Lurie, rather than by more exclusive appeal to Lurie's theorem in context to the locally compact Hausdorff Berkovich spaces attached to certain rigid analytic spaces.

Theorem 3.0.10. *Let X be an affinoid rigid analytic space. Then the Verdier duality functor yields an equivalence between overconvergent sheaf and co-overconvergent cosheaf categories.*

Proposition 3.0.11. *Let X, $\mathcal{C}$ be as above. Then, the following are equivalent conditions for a functor $F : N(M) \to \mathcal{C}$ to satisfy: (i) The restriction to $N(M_0)$ determines an overconvergent $\mathcal{K}$-cosheaf, the restriction to $N(M_1)$ is zero, and F is a left Kan extension of the restriction to $N(M_0 \cup M_1)$. (ii) The restriction to $N(M_2)$ determines an overconvergent $\mathcal{K}$-sheaf, the restriction to $N(M_1)$ is zero, and F is a right Kan extension of the restriction to $N(M_1 \cup M_2)$.*

Assuming the proposition, the proof of the theorem proceeds by defining $\mathcal{E}(\mathcal{C})$ as the full subcategory of $Fun(N(M), \mathcal{C})$ satisfying the equivalent conditions of the proposition. There are restriction functors to the overconvergent $\mathcal{K}$-sheaf and cosheaf categories. These are trivial Kan fibrations by application for *Higher Topos Theory* 4.3.2.15.

PROOF OF PROPOSITION: We prove just one direction, noting the other follows by symmetry. Suppose the truth of (ii) and consider a functor $F : N(M) \to \mathcal{C}$ satisfying the relevant hypotheses. Extend this to a functor $F' : N(M') \to \mathcal{C}$, and consider the restriction of this extended functor to a rigid analytic sheaf $\mathcal{F}$ (using that M_2' can be identified with the opposite of the poset category of admissible opens of X). One part of (i) essentially follows once we demonstrate that the restriction of F' to M_0 sends a pair $(0, K) \mapsto \Gamma_K(X, \mathcal{F})$, since we have already demonstrated this is a $\mathcal{K}$-cosheaf, so we will only need to verify the co-overconvergence condition (that the restriction has the desired interpretation in terms of supports at compacts is established towards the end of our discussion). For this, we must consider the following diagram, which is a map of fiber sequences

$$\begin{array}{ccc} \mathcal{G}(K) & \longrightarrow & lim_{\leftarrow, K \subset\subset K'} \mathcal{G}(K') \\ \downarrow & & \downarrow \\ \mathcal{F}(X) & \longrightarrow & lim_{\leftarrow, K \subset\subset K'} \mathcal{F}(X) \\ \downarrow & & \downarrow \\ \mathcal{F}(X \setminus K) & \longrightarrow & lim_{\leftarrow K \subset\subset K'} \mathcal{F}(X \setminus K') \end{array}$$

The middle horizontal arrow is an equivalence since its right-hand limit is taken with respect to a filtered poset, and the lower horizontal arrow is an equivalence because it is precisely asking for the sheaf condition for $M(\mathcal{F})$ for the covering of $M(X) \setminus M(K)$ by the sets $M(X) \setminus M(K')$ (that this is a covering amounts to the fact that the $M(K')$ form a fundamental system of compact neighborhoods of $M(K)$, which is all we need, since the set of all complements in $M(X)$ of compact neighborhoods of $M(K)$ covers $M(X) \setminus M(K)$, and any such is contained in some $M(X) \setminus M(K')$). Also, note that this covering is closed under finite intersections, as $(X \setminus K_1) \cap (X \setminus K_2) = X \setminus (K_1 \cup K_2)$, and in particular, if $K \subset\subset K_1, K \subset\subset K_2$, $K \subset\subset K_1 \cup K_2$ is clear.

At this point, we just need to establish the following: the original functor $N(M) \to \mathcal{C}$ is a left Kan extension of the restriction to $M_1 \cup M_2$. To get this, put M'' to consist of those objects of form (i, S) where $i = 0, 1$ and $S \subset X$ is compact. We note the restriction of F to $M_0 \cup M_1$ is a left Kan extension of the restriction to M'', so all we need to do is show that F is a left Kan extension of $F|N(M'')$ at every object $(2, X \setminus K)$ with K compact. To demonstrate this, we will use an auxiliary poset B, which is defined to consist of the pairs $(2, X \setminus U)$ with U wide open of form $X \setminus C$ with C compact. It now suffices to demonstrate that $F'|N(M'' \cup B \cup M_2)$ is a left Kan extension of the restriction to $N(M'')$.

To demonstrate this last assertion, we will proceed in two steps. The second step will be to show that $F'|N(M'' \cup B)$ is a left Kan extension of the restriction to $N(M'')$. The first will be to demonstrate that $F'|N(M'' \cup M_2 \cup B)$ is a left Kan extension of the restriction to $N(M'' \cup B)$.

The first step's proof proceeds by first noting that we know $F'|N(M_2 \cup B)$ is a left Kan extension of the restriction to $N(B)$. This follows from noting that the proof of Proposition 4 in *Points on Rigid Analytic Varieties* demonstrates that, given $C \subset M(X)$ any compact subset, given any compact V with ever analytic point occurring in C inner in V, we can find some $M(\Omega)$ with Ω one of the wide opens occurring in B, with $C \subset M(\Omega) \subset M(V)$. In particular, this means that, for any $(2, X \setminus K) \in M_2$, the colimits appearing in our left Kan extension condition can be identified with $colim_{K \subset\subset K'} \mathcal{F}(K)$, where $\mathcal{F} = F|N(M_2)$. We can now finish the proof by noting that for every object

$(2, X \setminus K) \in M_2$, the inclusion $N(B)_{/(2,X\setminus K)} \subset N(B \cup M'')_{(2,X\setminus K)}$ is cofinal; this is seen by noting that, for any $(i, S) \in M''$ satisfying $(i, S) \leq (2, X \setminus K)$, the poset $\{(2, X \setminus U) \in B : (i, S) \leq (2, X \setminus U) \leq (2, X \setminus K)\}$ is nonempty and stable under finite unions. To expand on this, the nonemptiness claim can be demonstrated by showing that, given $(i, S) \leq (2, X \setminus K)$ as above, there is always U as above with $K \subset U$ but with U having empty intersection with S. Note that $M(S)$ and $M(K)$ are compact in $M(X)$, and we can certainly find an open neighborhood of $M(K)$ that does not overlap $M(S)$ based on elementary properties of our locally compact Hausdorff spaces. However, we can certainly find some $M(\Omega)$, where Ω occurs in some element of B, between the open neighborhood of $M(K)$ mentioned and $M(K)$ itself. This means $M(\Omega)$ does not overlap $M(S)$. Hence, neither does Ω overlap with S, yet Ω is a wide open containing K of the desired form. The closure under finite unions follows from the fact that $X \setminus U_1$ and $X \setminus U_2$ union to $X \setminus (U_1 \cap U_2)$, and we know that wide opens in B are closed under finite intersection, since for C_i compact, $X \setminus C_1$ and $X \setminus C_2$ intersect at $X \setminus (C_1 \cup C_2)$.

We are now down to demonstrating the second of the two desired steps. Here, we will consider a particular $(2, X \setminus U) \in B$, with $U = X \setminus C$, and in particular will note the diagram $(0, X) \leftarrow (0, C) \rightarrow (1, C)$ is left cofinal in $N(M'')_{/(2,X\setminus U)}$. Thus, we are reduced to proving that the diagram

$$\begin{tikzcd}
F'(0,C) \arrow[r] \arrow[d] & F'(1,C) \arrow[d] \\
F'(0,X) \arrow[r] & F'(2,X\setminus U)
\end{tikzcd}$$

is a pushout square. We will demonstrate this by considering the following diagram (of which the above is a part):

$$\begin{tikzcd}
F'(0,C) \arrow[r] \arrow[d] & F'(1,C) \arrow[d] & \\
F'(0,X) \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & F'(1,X) \arrow[d] \\
F'(2,\emptyset) \arrow[r] & F'(2,X\setminus U) \arrow[r] & F'(2,X)
\end{tikzcd}$$

The lower right square is assumed to be a pullback. Notice that the outer lower square is a pullback simply by basic properties of F' we will establish at the end (basically why it is given on $(0,K)$ by $\Gamma_K(X,\mathcal{F})$ where $\mathcal{F} = F'|N(M_2'))$. So, the left lower square is also a pullback square. Also, noting $X\setminus(X\setminus U) = C$, the same holds true of the left larger/outer square. Noting the left outer and left lower squares are pullback squares, the same is true of the left upper square. Now, to finish the proof, all we need is to know that the map $Z \to F'(2,X\setminus U)$ is an equivalence, since the left-upper square being a pullback (which we have established) is equivalent to its being a pushout because $\mathcal{C}$ is a stable ∞-category. However, noting that $F'(1,X)$ and $F'(2,X)$ are both zero, and noting the lower right square is a pullback square, this is immediate, and the proof is finished.

It remains to give an explicit description of our purported functor $OverShv_{\mathcal{K}}(X, \mathcal{C}) \to OverCShv_{\mathcal{K}}(X, \mathcal{C})$. We will do this by constructing a full subcategory $\mathcal{D} \subset Fun(N(M'), \mathcal{C})$ closely related to $\mathcal{E}(\mathcal{C})$ from earlier, again following Lurie. We let $\mathcal{D}$ consist of the functors so that (i) The restriction to $N(M_2)$ is an overconvergent $\mathcal{K}$-sheaf; (ii) the restriction to $N(M_2')$ is a right Kan extension of that to $N(M_2)$; (iii) the restriction to $N(M_1'$ is zero; and finally, (iv) the restriction to $N(M')$ is a right Kan extension of that to $N(M_1' \cup M_2')$.

Let us note that we can reconstrue condition (ii) as asking that the restriction to $N(M_1' \cup M_2')$ is a right Kan extension of that to $N(M_1 \cup M_2)$. That is, let $(2, X \setminus U)$ be in M_2'. The limit condition involving $N(M_2)_{(2,X\setminus U)/}$ is the same as that involving $N(M_1 \cup M_2)_{(2,X\setminus U)/}$, since it is impossible for any $(1, S)$ to satisfy $(2, X \setminus U) \leq (1, S)$.

We can thus also reconstrue condition (iv) as asking that $F'|N(M')$ is a right Kan extension of the restriction to $N(M_1 \cup M_2)$. This in particular implies that restricting a functor in $\mathcal{D}$ to $N(M)$ will produce something in $\mathcal{E}(\mathcal{C})$: the zero condition at M_1 is satisfied, as M_1 coincides with M_1', the overconvergent $\mathcal{K}$-sheaf condition on $N(M_2)$ is built in, and the final condition is taken care of by what we have just discussed.

We now note that the goal is to show that the composition $OverShv(X, \mathcal{C}) \to OverShv_{\mathcal{K}}(X, \mathcal{C}) \to OverCShv_{\mathcal{K}}(X, \mathcal{C})$ is given by sending $\mathcal{F}$ to the assignment given on K by $\Gamma_K(X, \mathcal{F})$. This composition is given as follows. Note the restriction functor $\mathcal{D} \to OverShv(X, \mathcal{C})$ is a trivial Kan fibration. Composing its inverse with the restriction to $\mathcal{E}(\mathcal{C})$ and subsequently to $OverCShv_{\mathcal{K}}(X, \mathcal{C})$ is the desired composition we are trying to make explicit. We thus realize that it suffices to know the behavior of F' on elements $(0, K)$ where K is compact. However, this is easily done: the diagram $(2, \emptyset) \to (2, K) \leftarrow (1, K)$ is left cofinal in $N(M')_{(0,K)/} \times_{N(M')} N(M'_1 \cup M'_2)$. Thus, we know that the square

$$\begin{array}{ccc} F'(0,K) & \longrightarrow & F'(1,K) \\ \downarrow & & \downarrow \\ F'(0,\emptyset) & \longrightarrow & F'(2,K) \end{array}$$

is a pullback. We know that $F'(1, K) = 0$, $F'(0, \emptyset) = \mathcal{F}(X)$, and $F'(2, K) = \mathcal{F}(X \setminus K)$. The bottom arrow is given by restriction. That this square is a pullback shows that $F'(0, K)$ is in fact given by $\Gamma_K(X, \mathcal{F})$, and our recharacterization of the significance of F' being a right Kan extension of its restriction to $N(M'_1 \cup M'_2)$ now shows that $F'|N(M_0)$ is indeed the functor on compacts given by the assignment $K \mapsto \Gamma_K(X, \mathcal{F})$, as desired.

Chapter 4

Construction of Examples of Nonarchimedean Factorization Algebras

This section will construct several of the analogues of factorization algebras associated to Lie algebras familiar from the world of locally constant factorization algebras on topological manifolds.

The most basic examples that we produce are the factorization algebras arising from a symmetric algebra construction on (homotopy) cosheaves. Since we have shown that homotopy cosheaves can be constructed from sheaves by taking compactly supported sections, we will have one easy source of examples, albeit basic. The arguments here owe debt to the work of Costello and Gwilliam on factorization algebras, although the reader should be warned that the technical details differ considerably, even if the flavor is the same.

We will appeal to some results first about external tensors of cosheaves, which will consider the same situation from infinity and model categorical points of view. First, we need a definition.

Definition 4.0.1. Consider $\mathcal{F} : \mathcal{C} \to \mathcal{V}$ some ∞-cosheaf, where $\mathcal{C}$ arises from a Grothendieck topology on a rigid analytic space X as in *Higher Topos Theory Remark 6.2.2.3 (that is,* $\mathcal{C}$ *is endowed with a Grothendieck topology on an* *infinity-category given by the nerve of the ordinary category of admissible opens of X, and the reader can see the appendix on homotopical notions for more details), and $\mathcal{V}$ is stable, symmetric monoidal ∞-category where $\otimes_{\mathcal{V}}$ commutes with colimits. Write $\mathcal{F}^{\boxtimes k}$ for the functor on k-fold products of admissible opens of X given by $U_1 \times \cdots \times U_k \mapsto \mathcal{F}(U_1) \otimes \cdots \otimes \mathcal{F}(U_k)$.*

Proposition 4.0.1. *With notation as in the above definition, for any positive integer k, $\mathcal{F}^{\boxtimes k}$ satisfies the ∞-codescent condition with respect to p-admissible covers stable under finite intersection.*

PROOF: Denote the collection of all products of affinoids contained in some element of a given collection of products of admissible opens $\mathcal{W}$ in a given product admissible open $U = U_1 \times \cdots \times U_n$ by $\mathcal{PK}_{\mathcal{W}}(U)$. For the special case of the cover equal to U itself, we use the notation $\mathcal{PK}(U)$. The major fact we use again and again in this proof is that the gluing condition for products of admissible coverings of individual factors follows from the cosheaf condition for $\mathcal{F}$ along with the commutativity of the monoidal product with colimits.

First, let us consider $\mathcal{W}$ to be a p-admissible covering of some product of admissible opens as above. We want that $\mathcal{F}^{\boxtimes k}|N(\mathcal{PK}(U))$ is a left Kan extension of the restriction to $N(\mathcal{PK}_{\mathcal{W}}(U)$ first of all (this will help immensely in the proof later). Notice that this assertion needs to be demonstrated for every product of affinoids in U, so in fact, it is enough to assume that U itself is a product of affinoids. Then, notice $\mathcal{W}$ has a refinement that is given by taking a product of admissible coverings (closed under finite intersection as well) by affinoids. We call this $\mathcal{W}'$. We now have a tower

$$N(\mathcal{PK}_{\mathcal{W}'}(U)) \subset N(\mathcal{PK}_{\mathcal{W}}(U)) \subset N(\mathcal{PK}(U)).$$

If we get that the restriction of $\mathcal{F}^{\boxtimes k}$ to the center is a left Kan extension of that to the left, and similarly for the right and left, we will have the result for p-admissible covers of products of affinoids. To see that both of these facts hold, note that the collection of all products of affinoids, each of which is contained in some element of $\mathcal{W}'$, is given as follows: consider every admissible open of X occurring in a kth factor of $\mathcal{W}'$. Consider all affinoids contained in some such admissible open. These affinoids constitute an admissible covering of U_k, since they are built by considering the union of admissible coverings of each admissible open occurring in a kth factor of $\mathcal{W}'$. Further, these affinoids are closed under finite intersection. This being said, notice that the collection of all products of affinoids contained in some element of $\mathcal{W}'$and contained in some given product of affinoids can be assembled by taking all mutual products of affinoids built as above, letting k vary, and replacing $\mathcal{W}'$ with its interesection with the given product of affinoids. Thus, it is a product of admissible covers, and $\mathcal{F}^{\boxtimes k}$ satisfies codescent with respect to these, and we are done showing the restriction of $\mathcal{F}^{\boxtimes k}$ to $N(\mathcal{PK}(U))$ is a left Kan extension of that to $N(\mathcal{PK}_{\mathcal{W}}(U))$.

Now, consider an arbitrary p-admissible covering $\mathcal{W}$, closed under finite intersections, of arbitrary U. We can reduce, via cofinality of $N(\mathcal{W}) \subset N * \mathcal{W} \cup \mathcal{PK}_{\mathcal{W}}(U))$, to showing that the restriction of $\mathcal{F}^{\boxtimes k}$ to $N(\mathcal{W} \cup \mathcal{PK}_{\mathcal{W}}(U) \cup \{U\})$ is a left Kan extension of that to $N(\mathcal{W} \cup \mathcal{PK}_{\mathcal{W}}(U))$. Considering the analogous fact for $N(\mathcal{PK}_{\mathcal{W}}(U))$ and $N(\mathcal{W} \cup \mathcal{PK}_{\mathcal{W}}(U))$ is just asking for the colimit gluing condition for all products of affinoids contained in a given element of $\mathcal{W}$, we can in fact reduce to showing the restriction of $\mathcal{F}^{\boxtimes k}$ to $N(\mathcal{PK}_{\mathcal{W}}(U) \cup \{U\})$ is a left Kan extension of that to $N(\mathcal{PK}_{\mathcal{W}}(U))$.

To show this, we can consider the tower

$$N(\mathcal{PK}_{\mathcal{W}}(U)) \subset N(\mathcal{PK}(U)) \subset N(\mathcal{PK}(U) \cup \{U\})$$

because the Kan extension condition from left to right entails the one we need. To show this, we can simply appeal to showing the result from left to center and center to right. Center to right is obvious. Left to right follows from our earlier work on p-admissible covers of products of affinoids.

Theorem 4.0.2. *Let $\mathcal{F}$ be an ∞-cosheaf valued in a stable ∞-category with small limits and colimits. Then the assignment $U \mapsto Sym(\mathcal{F}(U))$ is an ∞-categorical admissible Weiss cosheaf (it satisfies codescent for admissible Weiss covers).*

PROOF: We are asked to show that, for $\{U_i\}_{i \in I}$ an admissible Weiss cover of U, the natural map $colim_{U_i} Sym\mathcal{F}(U_i) \to Sym\mathcal{F}(U)$ is an equivalence. Notice that this is simply a direct sum of the maps $colim_{U_i} Sym^k\mathcal{F}(U_i) \to Sym^k\mathcal{F}(U)$ for $k = 0, 1, ...$, and it suffices to show each such map is an equivalence. However, we know from our earlier work that the maps $colim_{U_i}\mathcal{F}^{\boxtimes k}(U_i^k) \to \mathcal{F}^{\boxtimes k}(U)$ are equivalences. Passing to S_k-coinvariants now yields the result for the k-fold symmetric powers, and thus the theorem as well.

Corollary 4.0.3. *Let $\mathcal{F}$ be a homotopy cosheaf in $dgVect_K$ endowed with the projective model structure, and suppose K to contain $\mathbf{Q}$. Also, let us suppose that the natural map $\mathcal{F}(U) \oplus \mathcal{F}(V) \to \mathcal{F}(U \coprod V)$ is an isomorphism (stricter than weak equivalence), where U, V are admissible opens admissibly covering their disjoint union. Then, the assignment $U \mapsto Sym(\mathcal{F}(U))$ is a factorization algebra.*

PROOF: This has two parts: first, we should note this assignment naturally defines a multiplicative, unital prefactorization algebra. The details of this are exactly as in Costello-Gwilliam's presentation for ordinary topological spaces. The structure maps are determined, for disjoint U, V admissibly covering their union and contained in some admissible W, for instance, by the precosheaf maps $Sym\mathcal{F}(U \coprod V) \to Sym\mathcal{F}(W)$ by using the isomorphism $Sym(\mathcal{F}(U \coprod V)) \cong Sym(\mathcal{F}(U)) \otimes Sym(\mathcal{F}(V))$ (which also gives multiplicativity). Unitality arises from noting that $Sym(\mathcal{F}(\emptyset)) \cong K$, using the identity map $1_{dgVect_K} = K \to Sym(\mathcal{F}(\emptyset))$. The only thing left to check is admissible Weiss locality: this follows from considering the associated ∞-functor, and noting that we can identify the homotopy admissible Weiss codescent condition with the analogous ∞-categorical one. Thus, it will suffice to know that the ∞-functor associated to our assignment is simply given by the composition of the ∞-functor associated to $\mathcal{F}$ with the ∞-categorical total symmetric powers functor. For this, we just need to observe that the naive Sym construction presents the ∞-categorical one, since our hypotheses guarantee that $dgVect_K$ is freely powered (that is, for any positive integer n and any cofibrant object

X, the S_n-shaped diagram given by endowing $X^{\otimes n}$ with the natural S_n action permuting factors is projectively cofibrant), and since any object is automatically cofibrant in $dgVect_K$. Thus, the proof is finished.

The above result will come in handy for the main type of factorization algebras we are concerned with, namely ones associated to Lie algebras. This is because, by applying a PBW type result in addition to a spectral sequence argument, we can reduce the locality condition for these enveloping factorization algebras to that for the above symmetric factorization algebras. The usual type of (precosheaf of) dg Lie algebras relevant to universal enveloping factorization algebra constructions looks like the tensor of an ordinary Lie algebra with the compactly supported sections of a resolution of a structure sheaf. Costello and Gwilliam, for instance, consider compactly supported Dolbeaut forms, where the Dolbeaut resolution is one of a sheaf of holomorphic functions. This is analogous to considering a de Rham resolution of the locally constant sheaf valued at $\mathbf{R}$, which is used to construct locally constant factorization algebras corresponding to universal enveloping algebras of (ordinary) Lie algebras.

We now proceed with defining (universal) enveloping factorization algebras; a stepping stone is recalling a notion used by Costello and Gwilliam, which we of course are considering in a rigid analytic setting.

Definition 4.0.2. A *Lie-structured cosheaf* is a precosheaf of dg Lie algebras that is a homotopy cosheaf at the level of underlying dg vector spaces (using the projective stable symmetric monoidal model structure, as usual).

This leads immediately to the enveloping factorization algebra:

Definition 4.0.3. Let $\mathcal{L}$ be a Lie-structured cosheaf. Its associated *enveloping prefactorization algebra* denoted $\mathcal{U}(\mathcal{L})$ is defined by sending $U \mapsto C_*(\mathcal{L}(U))$, the Chevalley-Eilenberg chains of $\mathcal{L}(U)$.

Theorem 4.0.4. *In the above definition, once again assuming K contains $\mathbf{Q}$, and the underlying homotopy cosheaf of $\mathfrak{g}[1]$, satisfies the strictness condition from the earlier result on the symmetric factorization algebra, the enveloping prefactorization algebra is actually a factorization algebra valued in the model category of dg vector spaces (with the projective model structure).*

PROOF: The details of the multiplicative, unital prefactorization algebra structure are left to the reader and are analogous to the details for the symmetric factorization algebra. So we focus on demonstrating admissible Weiss locality. There is a filtration on the Chevalley-Eilenberg complex attached to a given admissible open by expressions of degree $\leq k$, and similarly on the Čech complex of the locality condition (for concreteness, use the total complex realization, which can transparently be filtered). In both cases, the filtration is ascending, and we have a filtered map from the Čech side to the other. At every step of the filtration, we have a filtered quasi-isomorphism associated to finite filtrations (this follows from the *Sym* functor yielding a factorization algebra), hence a quasi-isomorphism outright (see for instance the Stacks Project). This actually gives that the original filtered map from the Čech complex to sections of the enveloping factorization algebra is itself a quasi-isomorphism. The reason is that we can view each filtration as defining a sequential diagram, and the filtered map yields a map of sequential diagrams, inducing a map on colimits that we want to be a quasi-isomorphism. Note that each sequential diagram is a diagram consisting of cofibrant objects (since everything is cofibrant in $dgVect_K$), with maps between the cofibrant objects all cofibrations (since these maps are monomorphisms, and projective cofibrations of dg vector spaces coincide with injective cofibrations). Such a diagram is automatically projectively cofibrant. So, we have a map of projectively cofibrant diagrams that is a weak equivalence. The colimit functor preserves weak equivalences between cofibrant diagrams, so the induced map on colimits is a weak equivalence as well, which completes the proof of locality. That our prefactorization algebra is multiplicative follows from elementary facts about the Chevalley-Eilenberg chains functor.

The following is an example of an enveloping factorization algebra, the analogue of the vertex algebra associated to a loop algebra.

Let us assume K contains the rational numbers for this example. Let $\mathcal{L}$ be a homotopy sheaf of dg vector spaces that is a presheaf of dg Lie algebras. Then, its associated ∞-functor is an ∞-categorical sheaf in $dgVect_K$ which is a presheaf of dg Lie algebras. The assignment $U \mapsto \mathcal{L}_c(U)$ now produces a precosheaf of dg Lie algebras that is a cosheaf (all in the ∞-sense) of dg vector spaces. The latter point follows because sifted ∞-colimits commute with the forgetful functor from dg Lie algebras to dg vector spaces. Last, we can produce a Lie-structured cosheaf whose associated ∞-functor is $\mathcal{L}_c$. We will justify this claim in the first appendix.

A special case of this construction can be performed for $\mathfrak{g}$ a Lie algebra over K, where $\mathcal{L}$ as an ∞-categorical presheaf of dg Lie algebras that is a sheaf of dg vector spaces is given by $\mathfrak{g} \otimes \mathcal{O}_X$. Here, to give the idea without full rigor, we can regard $\mathcal{O}_X$ as an object of presheaves of *dg*-algebras modulo weak equivalences, where the weak equivalences are given by those maps which, on applying the forgetful functor to dg vector spaces, are those inducing isomorphisms on cohomology sheaves (viewing a presheaf of dg vector spaces as a complex of presheaves).

In any case, applying Chevalley-Eilenberg cochains to a Lie-structured cosheaf yields a factorization algebra, so we have a good source of examples. In the special case outlined above, we call the example the *factorization algebra associated to* $\mathfrak{g}$. This also corresponds roughly to the vertex algebra attached to an affine Lie algebra at level 0.

We will now give a somewhat more nuanced, homotopical treatment of the ideas involved in establishing admissible Weiss locality of the factorization algebra associated to a Lie-structured cosheaf. We begin with a definition.

Definition 4.0.4. A *filtered prefactorization algebra* is a prefactorization algebra $\mathcal{F}$ whose underlying precosheaf is equipped with map from a sequence of precosheaves $\mathcal{F}_0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \cdots$, where all maps are monomorphisms when evaluating on any admissible open U, and induce filtrations of $\mathcal{F}(U)$ for every U. A *filtered factorization algebra* is just a factorization algebra that is filtered as a prefactorization algebra.

Remark 4.0.1. For any admissible open U, the sequence

$$\mathcal{F}_0(U) \to \mathcal{F}_1(U) \to \cdots$$

is automatically a projectively cofibrant diagram, since all the arrows are monomorphisms, and the objects $\mathcal{F}_i(U)$ are automatically cofibrant. Further, notice that the natural map $colim_i \mathcal{F}_i(U) \to \mathcal{F}(U)$ represents the natural map $hocolim_i \mathcal{F}_i(U) \to \mathcal{F}(U)$. That the former is an isomorphism shows that the latter is a weak equivalence.

We make an additional definition before we state the main proposition about filtered factorization algebras.

Definition 4.0.5. A filtered prefactorization algebra satisfies *graded admissible Weiss-locality* if, for any given admissible Weiss cover $\{U_k\}_{k \in I}$, and for any i, we have that the natural map $hocolim_{U_k}(\mathcal{F}_i(U_k)/\mathcal{F}_{i-1}(U_k)) \to \mathcal{F}_i(U)/\mathcal{F}_{i-1}(U)$ is a weak equivalence.

Proposition 4.0.5. *Suppose $\mathcal{F}$ is a filtered prefactorization algebra satisfying graded admissible Weiss-locality. Then, it is a factorization algebra.*

PROOF: Let $\{U_k\}_{k \in I}$ be an admissible Weiss cover of admissible open U. To show that $hocolim_{U_k}\mathcal{F}(U_k) \to \mathcal{F}(U)$ is a weak equivalence, we note that this is asking $hocolim_{U_k}(colim_i \mathcal{F}_i(U_k)) \to colim_i \mathcal{F}_i(U)$ to be a weak equivalence. However, since the filtrations define projectively cofibrant diagrams, these colimits represent homotopy colimits, and we can just show that $hocolim_{U_k}(hocolim_i \mathcal{F}_i(U_k)) \to hocolim_i \mathcal{F}_i(U)$ is a weak equivalence. But, since all maps $\mathcal{F}(U_k) \to \mathcal{F}(U)$ respect the degrees of the filtration, this amounts to showing that $hocolim_i(hocolim_{U_k}\mathcal{F}_i(U_k)) \to hocolim_i(\mathcal{F}_i(U))$ is a weak equivalence. This can be demonstrated after assuming that we have projectively cofibrant representatives for the sequential diagrams whose homotopy colimits are being taken (that is, referred to by the $hocolim_i$), so let us assume this. We are left with demonstrating that the natural maps

$$hocolim_{U_k}\mathcal{F}_i(U_k)/hocolim_{U_k}\mathcal{F}_{i-1}(U_k) \to \mathcal{F}_i(U)/\mathcal{F}_{i-1}(U)$$

are weak equivalences. Note that all these cofibers are homotopy cofibers (due to the projective cofibrance assumption). Thus, it is enough to show the natural

$$hocolim_{U_k}(\mathcal{F}_i(U_k)/\mathcal{F}_{i-1}(U_k)) \to \mathcal{F}_i(U_k)/\mathcal{F}_{i-1}(U)$$

are weak equivalences. However, this is just graded admissible Weiss locality, so we are done.

Remark 4.0.2. The relation to the more classical proof for factorization algebras associated to Lie-structured cosheaves is that we found a projectively cofibrant representative there for the sequence given by the $hocolim_{U_k}(\mathcal{F}_{i-1}(U_k)) \to$

$hocolim_{U_k} \mathcal{F}_i(U_k)$ by simply filtering the total complex of the Čech complex by degrees (all objects are cofibrant, and the degree-wise inclusions are of course cofibrations).

Chapter 5

Nonarchimedean Factorization Theorems

This section is concerned with providing a version of constructions due to Dwyer-Stolz-Teichner relating functorial field theories and factorization algebras on manifolds with very general geometric structure. They prove a version of the excision theorem often proved for locally constant factorization algebras by associating dg-categories to factorization algebras and slightly thickened compact (d-1)-manifolds and bimodules over these dg-categories to d-manifolds with boundary.

Let us fix some helpful terminology for convenience.

Remark 5.0.1. When not otherwise specified, in this section, a *closed sub-annulus* of an annulus defined informally by inequalities $r \leq |T| \leq s$, (where T is a coordinate - for instance, if talking of subannuli of the unit disk, T would correspond to the variable of the one-variable Tate algebra) refers to an admissible open defined by $r \leq r' \leq |T| \leq s' \leq s$. An open sub-annulus and semiopen sub-annulus are defined similarly, and we can also talk of sub-annuli of other annuli, like wide open/semiopen annuli. Such a sub-annulus will be called a boundary annulus if either $s' = s$ or $r' = r$. Further, if either the inner or outer r or s is marked left or right (see below), we can talk correspondingly of left and right (boundary) subannuli in the obvious way (for instance, if the inner one is marked left, say for a wide open annulus of inner radius r and outer radius s, a left boundary semiopen annulus would be one of form given by $r < |T| \leq r' < s$). An *interior circle* of either a closed or (semi)open annulus of inner radius r and outer radius s is one defined by inequalities $r < |T| = c < s$. Further, we will denote a closed annulus of inner radius r and outer radius s by $A[r,s]$, an open annulus of the analogous radii by $A(r,s)$, and a semiopen annulus of form $r < |T| \leq s$ by $A(r,s]$, and one of form $r \leq T < s$ by $A[r,s)$. We will also often refer to the analogous notions for annuli that are isomorphic to any of the ones given above, and the analogous notions (subannuli, boundary annuli, and so on) are defined using the isomorphism. When a marked structure (as below) is involved, we will fix a specific isomorphism.

Definition 5.0.1. Define a *marked open annulus* to be a rigid analytic space *Ann* with a fixed isomorphism to some $A(r,s)$ as above, with one of r,s marked left, and the other marked right.

Definition 5.0.2. Similarly, define a *marked semiopen annulus* to be a rigid space S with a fixed isomorphism to either some $A(r,s]$ or some $A[r,s)$, again with a marking of left or right for each of r,s.

We will frequently take it for granted that we can refer to things like the inner/outer radii or the left and right markings of a marked annulus (and analogously with other structures with markings) via the appropriate isomorphisms defining the marked structure.

Using our notion of subannuli of annuli described above, let us make the following definition, which plays a key role in our constructions.

Definition 5.0.3. Let Ann be a marked open annulus of form $A(r, s)$, and assume r is marked left and s marked right. A *marked left semiopen subannulus* S is a marked semiopen subannulus of form $r < |T| \leq s' < s$ (in the terminology established earlier, a left boundary subannulus), with r marked left and s' marked right. We could make the analogous definition for the case where s is marked left in $A(r, s)$, and here, the marked semiopen subannulus would be of form $r' \leq |T| < s$, and here, s would be marked left and r' would be marked right. Also, let us call the circle corresponding to the right-marked circle of a marked left semiopen subannulus the *right boundary* and denote it by $\partial_R S$. In this definition, we are considering Ann to be endowed with a fixed isomorphism (giving its marked structure) to $A(r, s)$, so all of the above is to be understood in terms of this isomorphism.

We will now attach a dg-category to a marked open annulus.

Definition 5.0.4. Let Ann be a marked open annulus as above, and suppose $\mathcal{F}$ is a unital, multiplicative prefactorization algebra. We construct/define the dg-category attached to these data, called $\mathcal{A}_{\mathcal{F}}(Ann)$ or $\mathcal{A}(Ann)$ for short (if the prefactorization algebra is understood) as follows. The objects are marked left semiopen subannuli of Ann. The maps attached to two such objects S_1, S_2 are given by $\mathcal{F}(S_2 \setminus \{S_1 \cup \partial_R S_2\})$ whenever $S_1 \subset S_2$, and zero otherwise. That is, we are evaluating our prefactorization algebra on the space between the right boundaries. The composition for $S_1 \subset S_2 \subset S_3$, namely when it is not forced trivially to be the zero map, as happens if $S_i \subset S_{i+1}$ is false, is a map $\mathcal{A}(Ann)(S_1, S_2) \otimes \mathcal{A}(Ann)(S_2, S_3) \to \mathcal{A}(Ann)(S_1, S_3)$ that is derived from noting that $S_2 \backslash \{S_1 \cup \partial_R S_2\}$, $S_3 \backslash \{S_2 \cup \partial_R S_3\}$ constitute an admissible covering of the disjoint union (by applying a standard fact from BGR regarding finite unions involving certain strict inequalities). Thus, we can simply define this composition map to arise from the structure maps of the factorization algebra $\mathcal{F}(S_2 \setminus \{S_1 \cup \partial_R S_2\}) \otimes \mathcal{F}(S_3 \setminus \{S_2 \cup \partial_R S_3\}) \to \mathcal{F}(S_3 \setminus \{S_1 \cup \partial_R S_3\})$. Last, we have a unital structure $1 \to \mathcal{A}(Ann)(S, S)$ arising from the unital structure $1 \to \mathcal{F}(\emptyset)$ of our prefactorization algebra. That this defines a dg-category (in other words, the natural compatibilities that composition must satisfy, and so on) follows from the multiplicative, unital prefactorization algebra structure of $\mathcal{F}$.

We now construct bimodule categories over the above dg-categories associated to a suitable notion of rigid analytic curve with boundary annuli.

Definition 5.0.5. Let $\mathcal{A}, \mathcal{B}$ be dg-categories. An $\mathcal{A} - \mathcal{B}$-bimodule is a dg-functor $\mathcal{M} : \mathcal{A}^{op} \otimes \mathcal{B} \to dgVect$. This is given by maps $\mathcal{A}^{op}(a, a') \otimes \mathcal{B}(b, b') \to [\mathcal{M}(a, b), \mathcal{M}(a', b')]$, or equivalently, using adjunctions, action maps $\mathcal{A}(a', a) \otimes \mathcal{M}(a, b) \otimes \mathcal{B}(b, b') \to \mathcal{M}(a', b')$ satisfying the natural compatibilities.

Remark 5.0.2. The basic example to keep in mind is that, given a dg-category $\mathcal{A}$, we can think of $\mathcal{A}(-, -) : \mathcal{A}^{op} \otimes \mathcal{A} \to dgVect$, the bimodule given by the hom-spaces. This lets us think of any dg-category $\mathcal{A}$ as a $\mathcal{A} - \mathcal{A}$-bimodule.

Here, the action maps from above are just given by composition in the dg-category.

Definition 5.0.6. Let W be a smooth, one-dimensional rigid analytic curve, and let $X \subset W$ be an open affinoid. We define an isomorphism of pairs of form (W, X) (say $(W_1, X_1), (W_2, X_2)$) to be an isomorphism $W_1 \to W_2$ such that X_1 is sent isomorphically onto X_2.

Definition 5.0.7. Consider a pair $\Sigma = (W, X)$ given by, for C a smooth, complete rigid analytic curve, $(C \setminus \{X_1 \cup \cdots \cup X_k\}, C \setminus \{D_1 \cup \cdots \cup D_k\})$, where the D_i are the wide open interiors of non-overlapping affinoid disks D_i' in C, and X_i is an affinoid disk contained in D_i. We call a wide open pair in Coleman's sense *simple* if it is endowed with an isomorphism of pairs to some (W, X) as above. We will refer to the complements $D_i' \setminus D_i$ as boundary circles of X. We note that each $X_i \subset D_i$ defines a wide open annulus in C given by $D_i \setminus X_i$, and we can call these boundary annuli of our wide open pair. *We will assume for the rest of this chapter that all wide open pairs considered are simple.* We use the term simple, because these simply arise via data of disks in a smooth, complete curve.

Definition 5.0.8. Now, define a *marked wide open pair* to be a wide open pair endowed with a fixed isomorphism to one of the above standard simple ones of form $(C \setminus \cup_i X_i, C \setminus \cup_i D_i)$ with some collection of D_i, X_i (and thus the wide open annuli in C that they define) marked as left, with the rest marked as right. We endow the corresponding wide open annuli (called boundary annuli for obvious reasons) with marked structures: the left-marked ones get their inner radii marked left via an isomorphism to an annulus of form $A(r, s)$ with r marked left, s marked right, and the right-marked ones analogously get their inner radii marked right, each via an isomorphism to an annulus of form $A(r, s)$ with r marked right, and s marked left.

Remark 5.0.3. We will refer to a (marked) basic wide open pair in this chapter: this simply means a (marked) wide open pair as above which is in addition basic in the sense of Coleman/etc, given in the appendix.

Definition 5.0.9. Consider a marked wide open pair $\Sigma = (W, X)$ with one left boundary annulus and one right boundary annulus (call the first Y_1 and the second Y_2). Given a factorization algebra $\mathcal{F}$ valued in dg vector spaces, we produce $\mathcal{M}(\Sigma)$, a $\mathcal{A}(Y_1)-\mathcal{A}(Y_2)$-bimodule. This is a functor $\mathcal{A}(Y_1)^{op} \otimes \mathcal{A}(Y_2) \to dgVect$ given by sending a pair of marked left semiopen subannuli of the Y_i, denoted (S_1, S_2), to the dg vector space given by evaluating our factorization algebra on the space between the right boundaries of S_i. More precisely, setting W' equal to the wide open annulus given by deleting the right boundary of S_2, we evaluate $\mathcal{F}$ on the the following space: $W \setminus (S_1 \cup (Y_2 \setminus W'))$. There are action maps given as follows: for $S_0 \subset S_1$ in Y_1, we define the action $\mathcal{A}(Y_1)(S_0, S_1) \otimes \mathcal{M}(\Sigma)(S_1, S_2) \to \mathcal{M}(\Sigma)(S_0, S_2)$ by the structure maps of the factorization algebra $\mathcal{F}(S_1 \setminus \{S_0 \cup \partial_R S_1\}) \otimes \mathcal{F}(W \setminus (S_1 \cup (Y_2 \setminus W'))) \to \mathcal{F}(W \setminus (S_0 \cup (Y_2 \setminus W')))$. The right action of $\mathcal{A}(Y_2)$ is constructed exactly analogously.

Note in our definition that we only need to evaluate our factorization algebra (to construct the mapping spaces) on admissible opens. One way to see this is to note that our wide open W can be gotten by deleting two affinoid disks from a proper curve, and the space we evaluate our factorization algebra can thus be gotten by deleting two larger affinoid disks (here, we appeal to the fact that deleting a compact from a quasi-separated rigid space yields an admissible open). The disjoint unions considered to define the action maps yield admissible coverings of the given union, and this is not hard to see, but we record an explanation in a lemma briefly.

Lemma 5.0.1. *The disjoint unions involved in defining the action maps above are all admissibly covered by the involved disjoint admissible opens.*

PROOF: In the end, this reduces to the BGR fact we considered with regard to the analogous disjoint unions involved in defining the dg category attached to a wide open annulus. We will give the details, for illustration, regarding the disjoint union corresponding to an action map $\mathcal{A}(a_3, a_2) \otimes \mathcal{A}(a_2, a_1) \otimes \mathcal{M}(a_1, b_1) \otimes \mathcal{B}(b_1, b_2) \to \mathcal{M}(a_3, b_2)$. This involves considering three semiopen annuli corresponding to objects of $\mathcal{A}(Y_1)$, call these $S_0 \subset S_1 \subset S_2$ where the containment is proper, and considering some semiopens $T \subset T' \subset Y_2$ corresponding to objects of $\mathcal{A}(Y_2)$. We will show that the union of $S_2 \setminus (S_1 \cup \partial_R S_2)$, $S_1 \setminus (S_0 \cup \partial_R S_1)$, $T' \setminus (T \cup \partial_R T')$ and the space in between the right boundaries of S_2 and T is an admissible disjoint union. To do so, we will proceed by producing an admissible covering by open affinoids of the space between the right boundaries of S_0 and T' and intersecting it with our disjoint union. Then, we will show that the intersection of each element of our disjoint union with each of these open affinoids yields an admissible covering of the corresponding disjoint union contained in the given affinoid.

Here is how we produce the affinoid covering described above. Pick an affinoid subannulus of Y_1 having $\partial_R S_1$ as an interior circle, but contained between the right boundaries of S_0 and S_2. Pick another such affinoid subannulus of Y_1 having $\partial_R S_2$ as an interior circle, not overlapping the earlier subannulus, and contained in $Y_1 \setminus S_1$. Finally, pick an affinoid subannulus of Y_2 having $\partial_R T$ as an interior circle and contained in $T' \setminus \partial_R T'$. Now, let us suppose that Y_1 has marked structure given by an isomorphism to $A(r, s)$, and Y_2 has marked structure given by an isomorphism to $A(r', s')$. We will suppose the right boundaries of S_0, S_1, S_2 correspond to radii $r < r_0 < r_1 < r_2 < s$, and suppose also that the right boundaries of T, T' correspond to radii of form $s' > s_T > s_{T'} > r'$. We will include in our affinoid covering the underlying affinoid $X \subset W$, and also several sequences of subannuli of Y_1, Y_2 that will now be specified. Let us suppose the first subannulus we chose corresponds to $A[c_0, c_1] \subset A(r, s)$. Then, we include in our desired affinoid covering a sequence of affinoid subannuli of Y_1 of form $A[c_{0,1}, c_0], A[c_{0,2}, c_{0,1}], ...$, where the $c_{0,k}$ approach r_0 but remain strictly larger. Further, we include the subannuli of Y_1 of form $A[c_1, c_{1,0}], A[c_{1,0}, c_{1,1}], A[c_{1,1}, c_{1,2}], ...$, where the $c_{1,k} \to r_1$ while remaining strictly smaller. Also, if the affinoid subannulus that we picked out containing $\partial_R S_2$ as an interior circle is of form $A[d_1, d_2]$, let us consider subannuli of Y_1 of form $A[d_2, d_{2,0}], A[d_{2,0}, d_{2,1}], ...$ where $d_{2,k} \to s$ while remaining strictly smaller. Similarly denoting the affinoid subannulus of Y_2 we chose containing $\partial_R T$ as an interior circle, supposing this subannulus is of form $A[e_1, e_2]$, we can consider sequences of affinoid subannuli of Y_2 of form $A[e_2, e_{2,0}], A[e_{2,0}, e_{2,1}], ...$ where $e_{2,k} \to s'$ while remaining strictly smaller, as well as a sequence $A[e_{1,0}, e_1], A[e_{1,1}, e_{1,0}], ..$ where $e_{1,k}$ approaches q, where q is the radius of the boundary circle of T' under the isomorphism $Y_2 \cong A(r', s')$. Here, $e_{1,k}$ remains strictly bigger than q.

It now follows from elementary facts about admissible coverings of annuli in rigid geometry that the union of X and all the affinoid subannuli we considered above actually is an admissible covering of the space between the right boundaries of S_0 and T'.

We now check that intersection of our original disjoint union, namely the space between the right boundaries of S_0, S_1, the analogous space for S_1, S_2, and the ones for S_2, T and T, T', with each element of our affinoid covering above yields an admissible covering of the given union. For the affinoid containing the right boundary of S_1 as an interior circle, the intersection of our finite disjoint union with this affinoid simply yields a covering of form $A[c_0, a), A(a, c_1]$. This is admissible, because of a standard BGR fact. Similar remarks apply to all the other affinoid subannuli we chose containing given interior circles. Notice that each of the other affinoid subannuli we consider is contained in some element of our finite disjoint union, so the intersection of that finite disjoint union with it simply yields the same affinoid subannulus, whence the resulting covering is clearly admissible. The underlying affinoid $X \subset W$ is also contained completely between the right boundaries of S_2 and

T, whence the intersection of our finite disjoint union with X simply yields the one-element covering of X. This completes our proof. The reader should note that the analogous argument works for all the possible disjoint unions we ever consider in our definition of $\mathcal{M}(\Sigma)$.

Definition 5.0.10. Given a $\mathcal{A}-\mathcal{B}$-bimodule $\mathcal{M}$ and a $\mathcal{B}-\mathcal{C}$-bimodule $\mathcal{N}$, we now recall the definition of the bar construction, which will compute their derived tensor product (and be denoted accordingly). This can be formed as follows. Consider the simplicial object in $\mathcal{A}-\mathcal{C}$-bimodules given by $[n] \mapsto \oplus_{b_0,\dots,b_n}(\mathcal{M}(-,b_0)\otimes\mathcal{B}(b_0,b_1)\otimes\cdots\otimes\mathcal{B}(b_{n-1},b_n)\otimes\mathcal{N}(b_n,-))$, with the simplicial face and degeneracies given by composition and action maps. This is the simplicial bar construction associated to the two bimodules. The bar construction, denoted $\mathcal{M}\otimes^{\mathbb{L}}_{\mathcal{B}}\mathcal{N}$, is given by taking the geometric realization of the simplicial bar construction. (The mapping space associated to a pair $(a,c)\in\mathcal{A}^{op}\otimes\mathcal{C}$ is given by the geometric realization of the simplicial diagram determined by the above by evaluating at the pair.)

To continue on to the nonarchimedean gluing factorization theorem that is the main attraction of the section, we will now need to spell out a version of the notion of semistable covering that takes into account the marked structure that our constructions appeal to.

Definition 5.0.11. A *marked semistable covering* of a marked wide open pair (W,X) (for simplicity assumed to have just two boundary annuli A,B, one marked left, and the other marked right) is a semistable covering involving precisely two basic wide open pairs $\Sigma_1 = (W_1, X_1)$,$\Sigma_2 = (W_2, X_2)$, each also with precisely two boundary annuli, overlapping at precisely one wide open annulus $Ann \cong A(1, s)$. The boundary annuli of the first will be denoted A and, of course, Ann, and those of the second are Ann, as well as another called B. In addition, each basic wide open pair has a marked structure, and the overlap is between the right boundary annulus of W_1 and the left one of W_2. There is also a requirement of compatibility of the marked structures on the annulus of overlap: the left and right markings must coincide.

Now, we consider a marked semistable covering exactly as defined above. Denote the underlying affinoids of Σ_i by X_i and the corresponding larger affinoids containing X_i in the definition of wide open by Y_i, and define X and Y similarly for the case of Σ. Now, let $\mathcal{F}$ be a rigid factorization algebra on W, and regard its restrictions to the admissible opens of W_1 and W_2 as factorization algebras $\mathcal{F}_1, \mathcal{F}_2$. Denote by $\mathcal{M}(\Sigma_1)$ the $\mathcal{A}(A) - \mathcal{A}(Ann)$-bimodule associated to the factorization algebra $\mathcal{F}_1$, and similarly denote by $\mathcal{M}(\Sigma_2)$ the $\mathcal{A}(Ann) - \mathcal{A}(B)$-bimodule associated to the factorization algebra $\mathcal{F}_2$. Finally, denote by $\mathcal{M}(\Sigma)$ the $\mathcal{A}(A) - \mathcal{A}(B)$-bimodule associated to $\mathcal{F}$.

Theorem 5.0.2. *Notation as above, the natural map $\mathcal{M}(\Sigma_1) \otimes^{\mathbb{L}}_{\mathcal{A}(Ann)} \mathcal{M}(\Sigma_2) \to \mathcal{M}(\Sigma)$ is a weak equivalence.*

PROOF: The natural map from the derived tensor product to $\mathcal{M}(\Sigma)$ is induced from a map from the simplicial bar construction to the constant simplicial $\mathcal{A}(A)-\mathcal{A}(B)$-bimodule $\mathcal{M}(\Sigma)$. This map can be defined by evaluating on pairs of objects of $\mathcal{A}(A)$ and $\mathcal{A}(B)$ by realizing the associated simplicial object in $dgVect$ is weakly equivalent to the simplicial Čech complex associated to a certain covering of the space in Σ between the right boundaries of the objects in the given pair, and considering the composition of this weak equivalence with the natural map from this simplicial Čech complex to $\mathcal{F}$ evaluated on this space between boundaries (as in the definition of locality for factorization algebras).

To get that the map $\mathcal{M}(\Sigma_1) \otimes^{\mathbb{L}}_{\mathcal{A}(Ann)} \mathcal{M}(\Sigma_2) \to \mathcal{M}(\Sigma)$ is a weak equivalence, we can thus simply show that the factorization algebra $\mathcal{F}$ satisfies locality with respect to the coverings mentioned above. That is, we simply must have that each covering referenced above is admissible Weiss.

Let us now demonstrate this. Let U be the admissible open given informally (see earlier for the precise definition) by the space in between the right boundaries of S_1, S_2, corresponding to objects of $\mathcal{A}(A)$ and $\mathcal{A}(B)$. This can be gotten by deleting closed affinoid disks from a proper curve arising by gluing disks onto the boundary annuli of Σ, just as Σ itself is gotten by deleting (smaller) such affinoid disks from the same proper curve. Hence, U is admissible open in Σ. An element of the covering of U is given by taking a semiopen annulus corresponding to an object of Ann, and examining the complement of the right boundary in U. Note that U is separated, since it corresponds to an open subspace of a proper curve. Therefore, taking the complement of something compact (like a circle, as we are doing) yields an admissible open, so the elements of our covering are all admissible open. Doing this for all the semiopen annuli corresponding to objects of $\mathcal{A}(Ann)$ yields all of the covering (that it covers is obvious). Denote our constructed cover by $\{U_i\}_{i\in I}$. We now describe somewhat more explicitly the maps $\mathcal{M}(\Sigma_1)\otimes^{\mathbb{L}}_{\mathcal{A}(Ann)} \mathcal{M}(\Sigma_2)(S_1,S_2)\to\mathcal{M}(\Sigma)(S_1,S_2)$, once again induced from a map of simplicial $\mathcal{A}(A){-}\mathcal{A}(B)$-bimodules. Denote the component of U_i containing A by U_i^L, and likewise denote the component of U_i containing B by U_i^R. In the lowest degree, the simplicial diagram in dg vector spaces that the left side is the geometric realization of is given by $\oplus_i\mathcal{F}(U_i^L)\otimes\mathcal{F}(U_i^R)$. This naturally maps to $\oplus_i F(U_i)$, yielding a weak equivalence due to multiplicativity. We then have a map $\oplus_i\mathcal{F}(U_i)\to\mathcal{F}(U)$, and the composition $\oplus_i\mathcal{F}(U_i^L)\otimes\mathcal{F}(U_i^R)\to\oplus_i\mathcal{F}(U_i)\to\mathcal{F}(U)$ now yields the lowest degree part of the map of simplicial diagrams inducing our desired map $\mathcal{M}(\Sigma_1)\otimes^{\mathbb{L}}_{\mathcal{A}(Ann)}\mathcal{M}(\Sigma_2)(S_1,S_2)\to\mathcal{M}(\Sigma)(S_1,S_2)$. All the other degrees are constructed analogously, using multiplicativity and the usual maps involved in checking locality.

We now note that $\{U_i\}_{i\in I}$ is a Weiss cover (any finite collection of points is contained in some element), so the only question is if it is an admissible Weiss cover. To conclude this is admissible Weiss, we can simply use the criterion established earlier involving n-Weiss coverings of compact subspaces of U by other such compact subspaces. In particular, we can demonstrate what we need by showing that our covering is admissible and that, for each positive integer $n\geq 2$, for a given compact subspace $K\subset U$, there exists a finite refinement of the original covering by the U_i, call it $\{K_j\}_{j\in J}$, so that the intersection of $\left\{K_j^n\right\}_{j\in J}$ with K^n constitutes a covering of K^n. In fact, we will just give the argument for the latter point, as the fact that our cover is admissible follows by an analogous argument (since it just corresponds to demonstrating an analogous claim when $n = 1$).

To produce this refinement, we proceed in a couple steps. First, we note that $\left\{U_i^k\right\}_{i\in I}$ does indeed cover U^k, but actually has a subcover given by finitely many U_i. This is seen by noting that the U_i form a k-Weiss covering (any collection of k points is contained in some element of the cover) because at worst, each point in a given collection of k points lies in a distinct circle deleted to produce a given U_i. This shows that there is a subcover $\{U_j\}_{j\in J}$ of $\{U_i\}_{i\in I}$ with J a finite set of size $k+1$, so that $\left\{U_j^k\right\}_{j\in J}$ covers U^k.

We now define $K_j \subset U_j$ compact as follows. It is given by considering the circle C_j deleted from Ann to produce U_j and instead considering Ann_j some open subannulus of Ann containing C_j as an interior circle. In addition, the Ann_j for all j are required to be pairwise disjoint. Further, let us pick a right semiopen subannulus $A' \subset A$ and a left semiopen subannulus $B' \subset B$ so that $K \cap A \subset A', K \cap B \subset B'$. Then, $X_1 \cup X_2 \cup Ann \setminus Ann_j \cup A' \cup B' := K_j$ produces the desired refinement of $\{U_j\}_{j\in J}$ of compacts.

Remark 5.0.4. Let C be a smooth, complete rigid analytic curve with a semistable model $\mathcal{C}$ over R_K having reduction with just two irreducible components meeting at a single double point singularity (with no other singularities). Denoting these components Γ_1, Γ_2, and putting $\Sigma_i = Red^{-1}(\Gamma_i)$ for $i = 1, 2$, we have that these are part of a semistable covering consisting of two basic wide opens overlapping in a single wide open annulus Ann. If we have a marked semistable covering arising in this fashion, we can think of the result above as related to/a nonarchimedean version of the factorization theorems traditionally proved in an algebro-geometric context.

Chapter 6

Sketch of Relation to Vertex Algebras Theory

We sketch here, without promise of completeness or rigor, an expected analogue of the result of Costello-Gwilliam stating that the pushforward of the cohomology of their Kac-Moody factorization algebra on $\mathbf{C}$ to $\mathbf{R}_{\geq 0}$ under $z \mapsto |z|$ admits a dense approximation by a suitable locally constant prefactorization algebra, roughly given by the Kac-Moody vertex algebra vector space on open disks and by $\mathcal{U}(\mathfrak{g}[t, t^{-1}])$ on open annuli.

Consider a locally constant prefactorization algebra $\mathcal{V}$ associated to the (we only consider level 0 here) Kac-Moody factorization algebra for a Lie algebra $\mathfrak{g}$, situated on $\mathbf{R}_{\geq 0}$, as follows: send opens of form $[0, a)$ to $\mathcal{U}(t^{-1}\mathfrak{g}[t^{-1}])$. Send $(a, b) \mapsto \mathcal{U}(\mathfrak{g}[t, t^{-1}])$. Send disjoint unions to tensor products (for example, send $[0, a) \coprod (x, y)$ where $a < x$ to $\mathcal{U}(t^{-1}\mathfrak{g}[t^{-1}]) \otimes \mathcal{U}(\mathfrak{g}[t, t^{-1}])$. The structure maps are given by the action maps of $\mathcal{U}(\mathfrak{g}[t, t^{-1}])$ on $\mathcal{U}(t^{-1}\mathfrak{g}[t^{-1}])$ and the multiplication of the former (since it has the structure of an associative algebra).

Similarly, there should be a prefactorization algebra valued in vector spaces given as follows. Let $\mathcal{F}$ denote the factorization algebra on rigid analytic $\mathbf{A}^1$ associated to $\mathfrak{g}$. Send $[0, a)$ to $H^*(\mathcal{F}(D(0, a)))$, where $D(0, a)$ denotes the wide open disk of radius a centered at zero. Send (a,b) to $H^*(\mathcal{F}(A(a,b))$. Call this the *cohomology prefactorization algebra* on rigid analytic $\mathbf{A}^1$ associated to $\mathfrak{g}$, and denote it by $H^*(\mathcal{F})$.

There should be a map from $\mathcal{V}$ to $H^*(\mathcal{F})$ as prefactorization algebras valued in vector spaces, which can be built from an explicit understanding of the latter's values on opens of form $[0,a)$ and (a,b) that should yield a dense inclusion of subspaces on such opens when the sections of $H^*(\mathcal{F})$ are appropriately topologized and/or bornologized. There is a nice Serre duality for rigid analytic spaces that justifies why we might expect such a dense inclusion (basically, due to sections supported at a point densely approximating compactly supported sections).

Remark 6.0.1. To justify the title of the section briefly, note that $\mathcal{U}(t^{-1}(\mathfrak{g}[t^{-1}])$ is precisely the vector space of the level zero vacuum module for the Kac-Moody Lie algebra, which identifies it as the vector space of the corresponding vertex algebra.

Appendices

Appendix A

Appendix on Homotopical Matters

A.0.1 Homotopy (Co)limits

We review the relation between homotopy (co)limits and ∞-(co)limits, showing they correspond to each other nicely in the settings of fibrant simplicial categories and combinatorial model categories.

The first is *Higher Topos Theory* Proposition 4.2.4.4.

Proposition A.0.1. *Let S be a small simplicial set, $\mathcal{C}$ a small simplicial category, and $u : \mathfrak{C}[S] \to \mathcal{C}$ an equivalence. Suppose $\mathbf{A}$ is a combinatorial simplicial model category. Then we have a categorical equivalence of simplicial sets*

$$N(\mathbf{A}^{\mathcal{C}^\circ}) \cong Fun(S, N(\mathbf{A}^\circ))$$

This leads to *Higher Topos Theory* Corollary 4.2.4.7, which tells us how to get a pre(co)sheaf of simplicial categories with an associated functor of ∞-categories of choice.

Corollary A.0.2. *Let $\mathcal{J}$ be a fibrant simplicial category, S a simplicial set, and $p : N(\mathcal{J}) \to S$ be a map. We can then find the following:*

(1) A fibrant simplicial category $\mathcal{C}$

(2) A simplicial functor $P : \mathcal{J} \to \mathcal{C}$

(3) A categorical equivalence of simplicial sets $j : S \to N(\mathcal{C})$

(4) An equivalence beween $j \circ p$ and $N(P)$ as objects of the ∞-category $Fun(N(J), N(\mathcal{C}))$.

This then naturally leads to the following result, relating homotopy colimits (and the same for homotopy limits) and ∞-(co)limits.

Theorem A.0.3. *Let $\mathcal{C}$ and $\mathcal{J}$ be fibrant simplicial categories, and $F : \mathcal{J} \to \mathcal{C}$ be a simplicial functor. Suppose we are given $C \in \mathcal{C}$ and a compatible family of maps $\{\eta_I : F(I) \to C\}_{I \in \mathcal{J}}$. The following conditions are then equivalent:*

(1) The map η_I exhibits C as the homotopy colimit of the diagram F.

(2) Consider the extension of $N(F)$ as a functor $N(\mathcal{J})^{\triangleright} \to N(\mathcal{C})$ determined by η_I. This extension is a colimit of $N(F)$.

This now yields the following result for combinatorial model categories [HA] Proposition 1.3.4.23, and similarly for colimits:

Proposition A.0.4. *Let $\mathbf{A}$ be a cominatorial model category, and let $\mathcal{J}$ be a small category. Let $F : \mathcal{J} \to \mathbf{A}^c$ be a functor, and let $\alpha : X \to lim_{\leftarrow, i \in \mathcal{J}} F(i)$ be a map in $\mathbf{A}^c$. The following are equivalent: (1) The map α exhibits X as the homotopy limit of F; (2) The induced map $N(\mathcal{J})^{\triangleleft} \to N(\mathbf{A}^c) \to N(\mathbf{A}^c)[W^{-1}]$ is a limit diagram.*

For our purposes, the most important application of the above concerns the relation between homotopy (co)sheaves and ∞-categorical (co)sheaves. We will give a review of the theory of (co)sheaves first in the specific context we are concerned. We focus on cosheaves, and leave the sheaf case to the reader.

Definition A.0.1. Let X be a (separated, as always) rigid analytic space, and denote by $\mathcal{U}(X)$ the poset of admissible opens in X. The covering sieves in Lurie's sense associated to the G-topology on X are given for a specific admissible open U by full subcategories of $N(\mathcal{U}(X))_{/U}$ of the following form: the objects must consist of all admissible opens contained in a given admissible open occurring in an admissible covering $\mathcal{W}$ of U. (Note that this automatically means the admissible opens occurring in this full subcategory must be, as a collection, closed under finite intersection.) We say that a functor $\mathcal{G} : N(\mathcal{U}(X)) \to \mathcal{C}$ is a *cosheaf* if, for any covering sieve $\mathcal{W}'$, the natural map $colim_{U_i \in \mathcal{W}'} \mathcal{G}(U_i) \to \mathcal{G}(U)$ is an equivalence.

We will record a helpful, basic proposition that we did not know where to find to cite, about how we can see the above codescent condition.

Proposition A.0.5. *A functor* $\mathcal{G} : N(\mathcal{U}(X)) \to \mathcal{C}$ *is a cosheaf if and only if, for any* $\mathcal{W}$ *an admissible covering of* U *closed under finite intersection, the natural map* $colim_{U_i \in \mathcal{W}} \mathcal{G}(U_i) \to \mathcal{G}(U)$ *is an equivalence.*

PROOF: Clearly, any functor $\mathcal{G}$ satisfying the codescent condition with respect to all $\mathcal{W}$ as above must in particular satisfy the analogous one with respect to covering sieves. So, it remains to see the other direction. Let us assume $\mathcal{G}$ is a cosheaf, and suppose $\mathcal{W}$ is an admissible covering of admissible open U) closed under finite intersection. We form a covering sieve $\mathcal{W}'$ given by all admissible opens contained in some element of $\mathcal{W}'$. To see codescent is satisfied with respect to $\mathcal{W}'$, let us note that $N(\mathcal{W}) \subset N(\mathcal{W}')$ is a cofinal inclusion. This is by the standard application of *Higher Topos Theory,* 4.1.3.1. Notice that any element of $\mathcal{W}'$ is contained in one of $\mathcal{W}$, and by assumption, the elements of $\mathcal{W}$ containing a given one of $\mathcal{W}'$ are closed under finite intersection.

The proof is now finished. We are now ready to spell out the relation between homotopy cosheaves and cosheaves in the above ∞-categorical sense. Suppose that $\mathcal{F}$ is a functor $\mathcal{U}(X) \to dgVect_K$. We can associate to it an ∞-categorical functor $N(\mathcal{U}(X)) \to N(dgVect_K)[W^{-1}]$.

Proposition A.0.6. *$\mathcal{F}$ is a homotopy cosheaf if and only if the associated ∞-functor defines a cosheaf.*

PROOF: Let us note that to be a homotopy cosheaf is precisely to satisfiy that, for any admissible cover $\mathcal{W}$ of arbitrary admissible open U, the natural map $hocolim_{U_i \in \mathcal{W}} \mathcal{F}(U_i) \to \mathcal{F}(U)$ is a weak equivalence. However, this is true if and only if the induced map

$$N(\mathcal{W})^{\triangleright} \to N(dgVect_K) \to N(dgVect_K)[W^{-1}]$$

defines a colimit diagram. Notice that the first map here is given by the family of maps from the $\mathcal{F}(U_i)$ into $\mathcal{F}(U)$, and the second is localization. Note that this composition can be rewritten as

$$N(\mathcal{W})^{\triangleright} \to N(\mathcal{U}(X)) \to N(dgVect_K) \to N(dgVect_K)[W^{-1}].$$

For such diagrams to always be colimit diagrams is exactly the definition of the ∞-functor associated to $\mathcal{F}$ being a cosheaf.

Now, let us additionally note that, by the equivalence of ∞-categories

$$N(dgVect_K^{\mathcal{U}(X)}) \cong Fun(N(\mathcal{U}(X)), N(dgVect_K)[W^{-1}]),$$

we can always, by invoking the above discussion, produce a homotopy cosheaf whose associated ∞-functor is equivalent to the functor underlying a given ∞-cosheaf. Also, the associated ∞-functor of some homotopy cosheaf is automatically an ∞-cosheaf.

We now justify a remark made in the main body of the paper that, given an ∞-categorical precosheaf of dg Lie algebras $\mathcal{L}$ that is a cosheaf at the level of underlying dg vector spaces, there is a Lie-structured cosheaf whose associated ∞-functor yields the stated ∞-categorical precosheaf. Assume, as we were at the relevant time, that our base K contains the rationals and appeal to the model categorical structure on dg Lie algebras over K considered by Wallbridge's work referenced in the bibliography.

First, we can certainly produce a precosheaf of cofibrant dg Lie algebras (call this $L : \mathcal{U}(X) \to dgLie^c_K$ with $\mathcal{U}(X)$ denoting the poset category of admissible opens) that has associated infinity-categorical functor the above $\mathcal{L}$. We just claim this is the desired Lie-structured cosheaf. To see this, we need precisely that, after composing L with the forgetful functor to dg vector spaces, we get a homotopy cosheaf. However, we know from the above that this can be verified by verifying that the associated infinity-categorical functor valued in dg vector spaces is itself a cosheaf. Since $dgLie^c_K \to dgVect^c_K$ preserves weak equivalences, because in particular, a weak equivalence of dg Lie algebras is precisely one that is so at the level of underlying dg vector spaces, the associated infinity-functor for $\mathcal{U}(X) \to dgLie^c_K \to dgVect^c_K$ (the first arrow is given by L) is equivalent to $N(\mathcal{U}(X)) \to N(dgLie^c_K) \to N(dgLie^c_K)[W^{-1}] \to N(dgVect_K)[W^{-1}]$. But this is the ∞-functor gotten by composing $\mathcal{L}$ with the forgetful functor to dg vector spaces, and is a cosheaf by assumption.

A.0.2 Functor Tensor Products and Bar Constructions

In this subsection, we review the theory of bar constructions and derived tensor products of functors to the extent needed for our discussion of the nonarchimedean factorization formulas, which utilize the theory of dg-categories, bimodules over them, and tensor products of these bimodules. The main references for this are Shulman's work on bar constructions/colimits in enriched homotopy theory (see bibliography) and Emily Riehl's book on categorical homotopy theory (and some shorter articles on homotopy colimits and weighted colimits).

The basic idea is (denoting by $\mathcal{V}$ the category of dg vector spaces) that we consider tensor products of bimodules $\mathcal{A} \otimes \mathcal{B}^{op} \to \mathcal{V}$ with ones $\mathcal{B} \otimes \mathcal{C}^{op} \to \mathcal{V}$ over the dg-category $\mathcal{B}$, and think of these in terms of bar constructions. This subsection collects a little background relevant to such discussion.

Recall that the bar constructions in such situations are defined using geometric realizations of simplicial bar constructions. Recall that, for a $\mathcal{V}$-enriched category $\mathcal{U}$, tensored and cotensored over $\mathcal{V}$, we can make sense of geometric realizations of simplicial objects in $\mathcal{U}$ if $\mathcal{V}$ is equipped with a a functor $\Delta^{\cdot} : \Delta \to \mathcal{V}$ used to define geometric realizations of simplicial objects $X : \Delta \to \mathcal{V}$ of $\mathcal{V}$. We recall this definition:

Definition A.0.2. Let $\mathcal{U}$ be $\mathcal{V}$-enriched and tensored over $\mathcal{V}$, and suppose $\Delta^{\cdot} : \Delta \to \mathcal{V}$ is as above. Let $X : \Delta \to \mathcal{U}$ be a simplicial object. Then, define the geometric realization by $|X| := X \otimes_{\Delta} \Delta^{\cdot}$.

Remark A.0.1. Recall that the tensor structure on a $\mathcal{V}$-enriched functor category $[\mathcal{J}, \mathcal{V}]$, where $\mathcal{V}$ is enriched over itself, is given pointwise.

This justifies the notion of geometric realization used earlier in discussion of nonarchimedean factorization rules. We note that there are technicalities in the construction of *derived* enriched functor tensor products. We refer the reader to Shulman's detailed article for details, and will present a high-level, vaguer summary of this aspect of his work.

Definition A.0.3. Let us define a small $\mathcal{V}$-enriched category $\mathcal{D}$ to be *good* if the bar construction preserves weak equivalences in both factors between objectwise cofibrant diagrams, and if it preserves appropriate cofibrance conditions.

For us, at least, goodness is just a way of knowing that bar constructions correctly compute the derived tensor product of left and right modules over some dg-category. A main result on goodness to keep in mind is that, if the Homs of $\mathcal{D}$ are cofibrant, and the maps $1_{\mathcal{V}} \to Hom_{\mathcal{D}}(d_1, d_2)$ are all cofibrations, then $\mathcal{D}$ is good.

Appendix B

Appendix on Rigid Geometry

B.0.1 Fundamentals

Here, we review the basic definitions of rigid analytic geometry. A relatively detailed survey is Brian Conrad's *Several Approaches to Nonarchimedean Geometry.* A standard textbook presentation of the foundations is in Fresnel and van der Put's *Rigid Geometry and its Applications.* Last, there is the encyclopedic preprint called *Foundations of Rigid Geometry* by Fujiwara and Kato, which takes the Raynaud formal models approach to nonarchimedean geometry as fundamental.

Here, we mainly review the classical theory of rigid geometry, since it really is all we need. We begin by noting the basic building blocks of rigid geometry called affinoids.

Definition B.0.1. Let n be a natural number satisfying $n \geq 1$. The *n-variable Tate algebra over K* is given by

$$T_n = \left\{ \sum a_j X^j : |a_j| \to 0 \right\},$$

where $a_j \in K$.

We then define affinoid algebras as quotients of these Tate algebras. These are topologized in a natural way, induced from a norm on the Tate algebras (we will not recall these details here), and maps of affinoid algebras are maps continuous with respect to the topologies.

Definition B.0.2. A K-affinoid algebra A is a K-algebra admitting an isomorphism $T_n/I \cong A$ for some ideal $I \subset T_n$. The set $MaxSpec(A)$ of maximal ideals of A is denoted by $M(A)$.

We now build up the basic ingredients of a Grothendieck topology (a simple sort called a G-topology) associated to $M(A)$.

Definition B.0.3. Let A be an affinoid algebra over K. A subset $U \subset M(A)$ is said to be an *affinoid subdomain* if there exists a map $i : A \to A'$ of K-affinoids, so that $M(i) : M(A') \to M(A)$ lands in U, and is universal for this condition in the following sense: a map $\phi : A \to B$ factors through A' precisely if $M(\phi)$ carries $M(B)$ into U, in which case the factorization is unique.

We now define the Tate G-topology on $M(A)$.

Definition B.0.4. A subset $U \subset M(A)$ is *admissible open* if there exists a covering by $\{U_i\}$ where each U_i is an affinoid subdomain, so that for any affinoid B, with $\phi : A \to B$ a map, the pullback of the U_i under $M(\phi)$ admits a refinement by a covering via finitely many affinoid subdomains.

We say that a covering $\{U_i\}$ of its union U is itself an admissible covering, if for $\phi : A \to B$ as above, the pullback of the cover under $M(\phi)$ has the property given above. This forces U to be admissible open.

Remark B.0.1. A covering of an admissible open is an admissible covering if and only if it has an admissible refinement.

Definition B.0.5. The Tate G-topology has as objects the admissible opens and coverings the admissible coverings.

Remark B.0.2. A result called the Tate acyclicity theorem allows us to construct a structure sheaf $\mathcal{O}_A$ with respect to the G-topology on $M(A)$.

Definition B.0.6. We define an affinoid space to be the locally ringed G-topologized space $(M(A), \mathcal{O}_A)$. This is denoted $Sp(A)$.

We can now globalize.

Definition B.0.7. A rigid analytic space over K is a locally ringed G-topologized space $(X, \mathcal{O}_X)$, along with an admissible (with respect to the G-topology) covering $\{U_i\}$ of X and isomorphisms $(U_i, \mathcal{O}_{X U_i}) \cong Sp(A_i)$ where each A_i is affinoid over K.

B.0.2 Wide Opens

For the purposes of this section, we say that K as in the introduction satisfies Hypothesis B if R_K contains a bald subring with the same residue field. This hypothesis is stated for completeness, but the interested reader should examine, for example, Bosch's *Lectures on Formal Geometry* for a precise definition of baldness. For our purposes, we note (as Coleman-McMurdy do) that K satisfies Hypothesis B if, for instance, it is discretely valued with perfect residue field. These cases suffice for our purposes.

Definition B.0.8. Let K be a complete discretely valued field with nontrivial valuation. A wide open rigid curve over K is a smooth rigid K-curve containing affinoid subdomains X and Y so that $W \setminus X$ is a disjoint union of open annuli, X is relatively compact in Y, and for every component V of $W \setminus X$, $Y \cap V$ is a semiopen annulus. We call X an *underlying affinoid* of W.

For X a rigid space over K, and $f \in \mathcal{O}_X(X) = A(X)$, let $|f|_{sup}$ denote the supremum of $|f(x)|$ over all $x \in X(\mathbf{C})$, where $\mathbf{C}$ is the completion of an algebraic closure of K. Denote by $\mathbb{F}_K$ the residue field of K. We make the definition $A^\circ(X) := \{f \in A(X) : |f|_{sup} \leq 1\}$.

Definition B.0.9. A *basic wide open pair* is a pair (W, X) where W is a connected wide open, and X is an underlying affinoid, so that $W \setminus X$ is a disjoint union of wide open annuli of form $A(1, s)$. In addition, we require that X has reduction with at worst double points as singularities, and that $A^\circ(X) \otimes_{R_K} \mathbb{F}_K$ is reduced.

Definition B.0.10. A *semistable covering of a rigid curve X consists of a finite admissible covering $\{U\}_{U \in \mathfrak{U}}$ where each U comes from a basic wide open pair (U, U^u). The intersection of any two distinct U, V must consist of a disjoint union of connected components of $U \setminus U^u$, by definition annuli of form $A(1, s)$. Last, triple intersections are required to be empty.*

The following are two important results proved in Coleman and McMurdy's work. Here is *Stable Reduction of $X_0(p^3)$,* Theorem 2.18:

Theorem B.0.1. *Let W be a wide open over K with underlying affinoid X. Then W can be completed into a proper, algebraic curve C over K by gluing open disks onto the connected components of $W \setminus X$.*

Here is *Stable Reduction of $X_0(p^3)$,* Theorem 2.36(i), stated slightly differently for some additional specificity.

Theorem B.0.2. *Let C be a smooth, complete curve over a stable field K, satisfying Hypothesis B. If C has a semistable model R_K whose reduction has at least two components, then C has an associated semistable covering over K given as follows. Let $\mathscr{C}$ be the given semistable model. Let $\mathscr{I}_{\mathscr{C}}$ be the set of irreducible components in the reduction of $\mathscr{C}$. For every $\Gamma \in \mathscr{I}_{\mathscr{C}}$, let $\Gamma^\circ = \Gamma \setminus \cup_{\Gamma' \in \mathscr{I}_{\mathscr{C}}, \Gamma' \neq \Gamma} \Gamma'$. If $\Gamma \in \mathscr{I}_{\mathscr{C}}$, put $W_\Gamma = Red^{-1}\Gamma$ and $X_\Gamma = Red^{-1}\Gamma^\circ$. Then, $\{(W_\Gamma, X_\Gamma) : \Gamma \in \mathscr{I}_{\mathscr{C}}\}$ is a semistable covering.*

B.0.3 Analytic Points on Rigid Analytic Varieties

This subsection recalls the basic notions of Peter Schneider's article *Points on Rigid Analytic Varieties,* which builds a version of the underlying topological space of a Berkovich space using the notion of analytic points of a rigid analytic variety. The notation K is used as it has always been in the body of the paper.

Definition B.0.11. A *complete extension field F of K is an extension field of it equipped with an absolute value $|-|_F$ that restricts to the one on K, so that F is complete with respect to it.*

Throughout this section, $X = Sp(A)$ is an affinoid over K.

Definition B.0.12. An *analytic point* x of X is a continuous K-algebra map $A \to F$, where F is a complete extension field of K. Here, F is called the *field of values* of K, and x is said to be F-valued.

Remark B.0.3. For any maximal ideal m_x of A, A/m_x is a finite extension of K and defines a complete extension field of it in a natural way. The map $A \to A/m_x$ defines an analytic point corresponding to one of the ordinary points of X.

We now define the notion of a neighborhood of an analytic point, which is central to defining the topology on $M(X)$, Schneider's version of the underlying topological space of the Berkovich space associated to X.

Definition B.0.13. Let $Sp(B) \subset X$ be an affinoid subdomain. It is said to be an *affinoid neighborhood* of analytic point $x : A \to F$ of X if there is a continuous K-algebra map $B \to F$ so that there is a factorization $A \to F = A \to B \to F$. An admissible open $U \subset X$ is said to be a *neighborhood* of x if it contains an affinoid subdomain which is a neighborhood of x.

Remark B.0.4. The map $B \to F$ above is unique, and will be denoted x by abuse of notation.

Definition B.0.14. Let $U \subset X$ be a neighborhood of x. We call x *inner* in U and say that U is a *wide neighborhood of* x if U contains an affinoid $Sp(B)$ so that there is an affinoid generating system $f_1, ..., f_n$ of B over A so that $|x(f_i)|_F < 1$ for each $i = 1, ..., n$.

Definition B.0.15. Let $V \subset U \subset X$ be admissible opens. We call V inner in U, or U a *wide neighborhood of* V *if any analytic point of* X *of which* V *is a neighborhood is inner in* U.

Definition B.0.16. We say two analytic points $x, x' : A \to F$ are *congruent* if $|x(-)_F| = |x'(-)|_F$. Define $M(X)$ to consist of the set of all congruence classes of analytic points of X, and endow it with the coarsest topology so that all maps $M(X) \to \mathbf{R}$ given by $(x : A \to F) \mapsto |x(f)|_F$ for $f \in A$ are continuous.

We can regard subsets of X as subsets of $M(X)$ using a natural map $X \to M(X)$ which induces a homeomorphism if X is endowed with the canonical topology. Further, there are certain distinguished subsets of $M(X)$ attached to admissible opens of X. To make sense of the definition below, note that two congruent analytic points share the same system of neighborhoods, so it makes sense to talk of the neighborhood of a point of $M(X)$ without distinguishing between analytic points and their congruence classes.

Definition B.0.17. Let $U \subset X$ be admissible open. Define $M(U)$ to consist of the set of $x \in M(X)$ so that U is a neighborhood of x.

We now define the important notion of a wide open of X.

Definition B.0.18. An admissible open $U \subset X$ is a *wide open* of X if it is a wide neighborhood of any of its analytic points.

There are lots of nice properties of the assignment $M(-)$ which we will record below.

(1) Let $U \subset X$ be admissible open. Then, it is wide open in X if and only if $M(U)$ is an open of X.

(2) The assignment $M(-)$ preserves finite unions and intersections of compact subspaces of X. Further, for any compact subspace $K \subset X$, $X \setminus K$ is wide open in X, and $M(X \setminus K) = M(X) \setminus M(K)$.

(3) M preserves inclusions of all admissible opens of X. If $U \subset X$ is admissible open, $M(U) \cap X = U$.

(4) Let $\{\Omega_i\}_{i \in I}$ be a covering by wide opens of some wide open Ω. It is an admissible covering if and only if the $M(\Omega_i)$ cover $M(\Omega)$. Also, intersections of finitely many wide opens are wide open, and the collection of all $M(\Omega)$ with Ω wide open in X form a basis for the topology on $M(X)$.

(5) Let $C \subset M(X)$ be compact. The collection of all $M(K) \subset M(X)$ so $K \subset X$ is compact with it being a wide neighborhood of every analytic point of C is a fundamental system of compact neighborhoods of C. For fixed compact K, the collection of all $M(K')$ so that $K \subset\subset K'$ are compact wide neighborhoods of K is a fundamental system of compact neighborhoods of $M(K)$.

(6) Let $C \subset M(X)$ be compact. The collection of all $M(W)$ so that W is a wide open of X and $C \subset M(W)$ is a fundamental system of neighborhoods of C in $M(X)$.

B.0.4 Serre Duality in Rigid Geometry

There is a nice Serre Duality theory for rigid geometry of smooth Stein spaces and smooth, projective curves (basically, smooth rigid geometric objects without boundary), which we recall here, as it plays a key role in our examples. As with the other appendices, unless stated otherwise, there is no claim to originality in this section. The main reference for this section is *Serre Duality for Rigid Analytic Spaces* by van der Put, but also Peter Beyer's *On Serre Duality for Coherent Sheaves on Rigid Analytic Spaces.* We are not extensively detailed in this section, and only give the flavor, and refer the reader seeking specifics to these two works.

Definition B.0.19. Let X be a separated rigid analytic space, and $K \subset X$ compact. For $\mathcal{F}$ an abelian sheaf on X, define $H^0_K(X, \mathcal{F}) = ker(H^0(X, \mathcal{F}) \to H^0(X \setminus K, \mathcal{F}))$. This has derived functors H^i_K, being left-exact.

Proposition B.0.3. *The above fit into a long exact sequence of form*

$$0 \to H^0_K(X, \mathcal{F}) \to H^0(X, \mathcal{F}) \to H^0(X \backslash K, \mathcal{F}) \to H^1_K(X, \mathcal{F}) \to H^1(X, \mathcal{F}) \to H^1(X \backslash K, \mathcal{F})$$

This will be useful in characterizing what certain compactly supported cohomology groups look like. There is some subtlety in thinking about how to define compactly supported cohomology, however: as discussed earlier in our Verdier duality chapter, extension by zero is problematic for our G-topology, so authors such as van der Put do not define compactly supported cohomology in terms of all compact subspaces contained in the given rigid space, but roughly only consider ones contained in wide opens.

Definition B.0.20. We now make the definition of cohomology with compact support: let c be the family of compact $K \subset X$. The cohomology with compact support is given by $H^i_c(U, \mathcal{F}) := colim_{K \in c} H^i_K(X, \mathcal{F})$.

Remark B.0.5. Notice that compactly supported cohomology coincides with ordinary cohomology for proper curves and affinoids. There will be no useful Serre duality theory for affinoids, but there will be for proper curves, and also for smooth Stein domains.

Remark B.0.6. We did not place restrictions on the compact subspaces considered, unlike van der Put, because in the end, he only considers special spaces without boundary like Stein spaces and proper spaces. For the former, any affinoid is relatively compact in some other affinoid. For the latter, compactly supported cohomology corresponds to ordinary cohomology.

Remark B.0.7. There is a useful explicit characterization of compactly supported cohomology of a wide open disk of radius one over K. Every cohomology class is represented by a Laurent series $\Sigma_{\alpha<0} a_\alpha X^\alpha$, so that there exists $0 < \epsilon < 1$ in $|K^*|$ so that $lim_\alpha |a_\alpha|/\epsilon^\alpha = 0$.

Let us now summarize van der Put's discussion on the topologies on cohomology groups. First, finitely generated $O(Z)$-modules for Z affinoid have a canonical Banach space structure. An arbitrary $O(Y)$-module is a strict limit of Banach spaces given by the finitely generated submodules, hence has a locally convex structure.

Now, let $\mathcal{F}$ be a coherent sheaf on a rigid space of countable type. Its global sections can be given a Fréchet space structure using the sheaf axiom for an admissible, countable affinoid covering, by noting that the product of sections over affinoids has such a structure, plus that the global sections are a closed subspace of that product. Compactly supported cohomology has a direct limit locally convex topology induced from the Fréchet space structure on sections with support.

This summary underway, let us now recollect the definition of Stein space in rigid geometry.

Definition B.0.21. A Stein space X/K is a separated rigid analytic K-space such that there is an admissible affinoid covering $\{U_i\}$ so that there are topological generators of $O(U_n)/K$ call them $h_1(n), ..., h_{r_n}(n)$ and constants $a_n \in \sqrt{|K^*|}$ with $0 < a_n < 1$ so $U_{n-1} = (u \in U_n | h_i(n)(u) \leq a_n \forall i = 1, ..., r(n))$

There is always a closed immersion of a Stein space into some affine space.

We now formulate Serre duality. Fix the following notation. Let X/K be a smooth, separated rigid analtic space of dimension n. Put ω for the nth exterior power of $\Omega^1_{X/K}$. If X is either Stein or proper, there is a residue map $Res_X : H^n_c(\omega) \to K$. This is continuous and K-linear. Let $\mathcal{F}$ denote a coherent sheaf on X. There are two maps induced from considering Yoneda pairings: (i)

$Ext_c^{n-i}(\mathcal{F}, \omega) \to Hom_K(H^i(\mathcal{F}), K)$ and (ii) $Ext^{n-i}(\mathcal{F}, \omega) \to Hom_K(H_c^i(\mathcal{F}), K)$. The right-hand sides refer to K-linear maps. The results of Serre duality are that, for a smooth Stein space, we have the following:

(a) The map (i) induces an isomorphism $Ext_c^{n-i}(\mathcal{F}, \omega) \to Hom_K - cont(H^i(\mathcal{F}), K)$, where the right side consists of continuous K-linear maps. (b) The map (ii) induces an isomorphism similarly $Ext^{n-i}(\mathcal{F}, \omega) \to Hom_K - cont(H_c^i(\mathcal{F}), K)$.

For X smooth and proper, we get for $i = n$ that $Ext^{n-i}(\mathcal{F}, \omega) \to Hom_K(H^i(\mathcal{F}), K)$ is an isomorphism, with the cases of the other i following if all $Ext^j(H^i(\mathcal{F}), K)$ vanish for all i and nonzero j.

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Vita

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