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Dimer Models and Hochschild Cohomology

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Dimer Models and Hochschild Cohomology

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Dimer Models and Hochschild Cohomology

by

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Dimer models have appeared in the context of noncommutative crepant resolutions and homological mirror symmetry for punctured Riemann surfaces. For a zigzag consistent dimer embedded in a torus, we explicitly describe the Hochschild cohomology of its Jacobi algebra in terms of dimer combinatorics. We then compute the compactly supported Hochschild cohomology of the category of matrix factorizations for the Jacobi algebra with its canonical potential.

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Chapter 1: Introduction

A dimer model is a type of directed graph that cellularly decomposes a Riemann surface. It comes with a canonical superpotential whose derivatives determine the relations of an associative, generally noncommutative algebra, called the Jacobi algebra. Under certain consistency conditions on the dimer, the Jacobi algebra is Calabi-Yau 3 [13]. Furthermore, when the ambient surface is a torus, the center is a three dimensional toric Gorenstein singularity, of which the Jacobi algebra is a noncommutative crepant resolution [9].

As Bocklandt showed in [8], the role of dimers in noncommutative geometry extends to homological mirror symmetry of punctured Riemann surfaces. On the A-side, given such a space X, one embeds a dimer, say $\mathcal{Q}^\vee$, into the closure of X in such a way that the vertices align with the punctures. The arrows of $\mathcal{Q}^\vee$ are exact Lagrangian submanifolds of X between the punctures, and the full subcategory $fuk(\mathcal{Q}^\vee)$ of these objects in the $\mathbb{Z}/2\mathbb{Z}$-graded wrapped Fukaya category $wFuk(X)$ (see [1] for a definition) generate the whole category.

On the B-side, a dimer $\mathcal{Q}$ is obtained from $\mathcal{Q}^\vee$ by an involution called dimer duality. Essentially, dimer duality preserves the arrow set but exchanges vertices and zigzag cycles: i.e., closed paths that alternate between clockwise and anti-clockwise faces. The Jacobi algebra $J(\mathcal{Q})$ has a special central element ℓ, called the potential, given by the sum of the boundary cycles in the cellular decomposition. The pair $(J(\mathcal{Q}), \ell)$ constitutes a noncommutative $\mathbb{Z}/2\mathbb{Z}$-graded Landau-Ginzburg (LG) model, and a matrix factorization of $(J(\mathcal{Q}), \ell)$

$$\mathcal{Q} = \begin{tikzcd} 1 \arrow[r, "x"] & 1 \arrow[dl, "z"] \\ 1 \arrow[u, "y"] \arrow[r, "x"'] & 1 \arrow[u, "y"'] \end{tikzcd} \qquad \mathcal{Q}^\vee = \begin{tikzcd} 2 \arrow[r, "y"] & 3 \arrow[dl, "z"] \\ 1 \arrow[u, "x"] \arrow[r, "x"'] & 2 \arrow[u, "y"'] \end{tikzcd}$$

Figure 1.1: Noncommutative mirror symmetry for the three-punctured sphere [8]. The dimer $\mathcal{Q}^\vee$ is embedded in the punctured sphere, with vertices corresponding to the punctures. Its dimer dual $\mathcal{Q}$ is embedded in the torus.

is a curved complex of projective $J(\mathcal{Q})$-modules. This can be represented as a diagram of the following form:

$$P_0 \underset{d_1}{\overset{d_0}{\rightleftarrows}} P_1 \quad d_1 d_0 = \ell \cdot Id_{P_0},\ d_0 d_1 = \ell \cdot Id_{P_1}.$$

Each arrow in $\mathcal{Q}$ determines a matrix factorization (see §2.7), and the collection of such objects forms a full subcategory $mf(\mathcal{Q})$ of the differential $\mathbb{Z}/2\mathbb{Z}$-graded (DG) category of matrix factorizations $MF(J(\mathcal{Q}), \ell)$.

The statement of mirror symmetry in [8] is an equivalence between $fuk(\mathcal{Q}^\vee)$ and $mf(\mathcal{Q})$.

Theorem 1.0.1 ([8] Corollary 8.4)**.** *Suppose $\mathcal{Q}$ is a zigzag consistent dimer in a surface of nonpositive Euler characteristic. Then there exists an A_∞-quasi-isomorphism*

$$mf(\mathcal{Q}) \cong fuk(\mathcal{Q}^\vee).$$

Here, zigzag consistency is a condition on the intersection properties of zigzag cycles (see §2.3.2). Commutative versions of this mirror equivalence were proved in [28] and [25].

A natural question is if the Hochschild cohomology HH^* of the A_∞-categories in Theorem 1.0.1 can be computed. Like its classical counterpart for associative algebras, Hochschild cohomology of categories governs their deformations. Additionally, Hochshchild cohomology of the Fukaya category of an exact symplectic manifold is conjecturally equivalent to its symplectic cohomology as Gerstenhaber algebras [31]. This equivalence was proved in [20] for punctured surfaces satisfying a certain nondegeneracy condition.

For matrix factorizations of commutative LG models, a general computation of Hochschild cohomology was provided by Lin–Pomerleano.

Theorem 1.0.2 ([26] Theorem 3.1)**.** *Suppose X is a smooth variety over $\mathbb{C}$ and $W : X \to \mathbb{C}$ a function whose only critical value is 0. Let $MF(X,W)$ be the category of matrix factorizations of the Landau-Ginzburg model (X,W). Then*

$$HH^*\big(MF(X,W)\big) = \mathbb{R}\Gamma(\wedge T_X, [W,-]) \;\; mod \;\; 2$$

where $\wedge T_X$ is the sheaf of polyvector fields on X and $[-,-]$ is the Schouten-Nijenhuis bracket.

A dual description of homology in terms of differential forms follows when X is Calabi-Yau (done generally in [16]). These results generalize the theorem of Dyckerhoff [15], which gives the cohomology of a local LG model with an isolated singularity. The method in common among these computations is to identify a compact generator of $MF(X,W)$ from its equivalence with the derived singularities category. Then derived Morita theory [33] can be applied to compute Hochschild cohomology as the derived endomorphism algebra of the generator.

Our goal is to compute the Hochschild cohomology of the matrix factorization category of the noncommutative LG model $(J(\mathcal{Q}), \ell)$. However, commutative methods do not readily transfer. If $\mathcal{Q}$ is embedded in a hyperbolic surface, for example, then $J(\mathcal{Q})$ is not Noetherian, and it is unclear if $MF(J(\mathcal{Q}), \ell)$ has a compact generator.

An alternative is to compute the so-called Hochschild cohomology of the second kind of the matrix factorization category. This is an example of a derived functor of the second kind, the foundations of which were established in [30] and [29]. The essential difference is that, whereas ordinary Hochschild cohomology of a graded category is defined as a direct product totalization of the Hochschild complex, cohomology of the second kind is defined as a direct sum totalization. In analogy with the similarly defined topological invariants, Hochschild (co)homology of the second kind is also called compactly supported (Borel-Moore) Hochschild (co)homology, denoted HH_c^* (HH_*^{BM}).

Polischuk–Positselski [29] show that the two kinds of Hochschild cohomology for the commutative LG models in [26] and [15] coincide. Furthermore, they identify the compactly supported Hochschild cohomology of matrix factorizations with that of the LG model treated as a curved algebra. More precisely, if A is an associative algebra and h is a central element, then the LG model (A, h) is equivalently the data of a curved A_∞-structure on A for which all multiplication maps $\{m_k\}_{k=0}^{\infty}$ are trivial except the associative product m_2 and $m_0 = h$. We label this curved algebra A_h. The category of matrix factorizations of (A, h) can be reinterpreted as the category of curved $\mathbb{Z}/2\mathbb{Z}$-graded modules over A_h, projective as A-modules.

Theorem 1.0.3 ([29] §2.6)**.** *For a Landau-Ginzburg model (A, h), there are isomorphisms of $\mathbb{Z}/2\mathbb{Z}$-graded vector spaces*

$$HH_c^*(A_h) \cong HH_c^*(MF(A, h)), \quad HH_*^{BM}(A_h) \cong HH_*^{BM}(MF(A, h))$$

We are unsure if the two kinds of Hochschild cohomology of $MF(J(\mathcal{Q}), \ell)$ are equivalent. Nonetheless, support for the affirmative may come from computing $HH_c^*(J(\mathcal{Q})_\ell)$. This can be accomplished by a spectral sequence as in [12], where it is done for a commutative local LG model with an isolated hypersurface singularity. The result, which we prove in Chapter 5, is analogous to Theorem 1.0.2.

Proposition 1.0.4. *Suppose $\mathcal{Q}$ is a zigzag consistent dimer in a surface of nonpositive Euler* *characteristic. Suppose further that $\mathcal{Q}$ admits a perfect matching. Then*

$$HH^*_c(MF(J(\mathcal{Q}),\ell)) \cong H_*\big(HH^*(J(\mathcal{Q})),\,\{\ell,-\}\big) \;\; mod \;\; 2$$
$$HH^{BM}_*(MF(J(\mathcal{Q}),\ell)) \cong H_*\big(HH_*(J(\mathcal{Q})),\,\mathcal{L}_\ell\big) \;\; mod \;\; 2$$

where $\{-,-\}$ is the Gerstenhaber bracket and $\mathcal{L}_-$ is the Lie derivative.

Here, a perfect matching is a subset of $\mathcal{Q}$ containing exactly one arrow from every boundary cycle. It can be used to define a $\mathbb{Z}$-grading on $J(\mathcal{Q})$, which features prominently in the proof of the proposition.

When $\mathcal{Q}$ is a zigzag consistent dimer embedded in a torus Σ, we describe $HH^*(J(\mathcal{Q}))$ explicitly in terms of the underlying toric data of $J(\mathcal{Q})$. The perfect matchings generate a lattice N^{out} of outer derivations of $J(\mathcal{Q})$. A difference of perfect matchings can be identified as an element of $H_1(\Sigma)$, the integer homology of Σ. Translating the perfect matchings by a fixed reference matching and taking the convex hull produces a lattice polygon in $H_1(\Sigma)\otimes_{\mathbb{Z}}\mathbb{R} \cong \mathbb{R}^2$. The toric variety from the cone on the polygon has coordinate ring isomorphic to the center $\mathcal{Z}$ of $J(\mathcal{Q})$. The rays of the dual cone are generated by the opposites to the homology classes of the zigzag cycles. Moreover, in the facet of the dual cone orthogonal to a corner perfect matching, the interior lattice points determine outer derivations of $J(\mathcal{Q})$ that have degree -1 with respect to the perfect matching. Let $N^{out}_{\mathbb{R}} = N^{out} \otimes_{\mathbb{Z}} \mathbb{R}$, $\{\nu_1, \ldots, \nu_k\} \subset H_1(\Sigma)$ be the opposite homology classes of the zigzags, $\Gamma = \bigcup_{i=1}^k \mathbb{Z}_{>0} \cdot \nu_i$, and $H_1(\Sigma)^* = H_1(\Sigma) \setminus \{0\}$. In Chapter 4, we prove the following.

Theorem 1.0.5. *Suppose $\mathcal{Q}$ is a zigzag consistent dimer in a torus Σ. As a vector space,*

$$HH^*(J(\mathcal{Q})) \cong \begin{cases} \mathcal{Z} & \text{if } * = 0 \\ \mathcal{Z} \otimes_{\mathbb{R}} N_{\mathbb{R}}^{out} \oplus \mathbb{C} \cdot H_1(\Sigma)^* \setminus \Gamma & \text{if } * = 1 \\ \mathcal{Z} \otimes_{\mathbb{R}} N_{\mathbb{R}}^{out} \wedge N_{\mathbb{R}}^{out} \oplus \big(\mathbb{C} \cdot H_1(\Sigma)^* \setminus \Gamma\big)^2 \oplus \mathbb{C} \cdot \Gamma \oplus tor_\ell^+ & \text{if } * = 2 \\ \mathcal{Z} \otimes_{\mathbb{R}} N_{\mathbb{R}}^{out} \wedge N_{\mathbb{R}}^{out} \wedge N_{\mathbb{R}}^{out} \oplus \mathbb{C} \cdot H_1(\Sigma) \oplus tor_\ell & \text{if } * = 3 \\ 0 & \text{otherwise.} \end{cases}$$

Here, tor_ℓ is the subspace of $HH_0(J(\mathcal{Q}))$ of torsional elements under the action of ℓ, and tor_ℓ^+ consists of all such elements with positive degree in some perfect matching. A more geometric description in terms of paths in the dimer is provided in §4.3.

In Chapter 5, the above description of $HH^*(J(\mathcal{Q}))$ allows us to compute $HH_c^*(MF(J(\mathcal{Q}), \ell))$. Let x_{ν_i} be the central element of $J(\mathcal{Q})$ corresponding to the homology class ν_i.

Theorem 1.0.6. *Suppose $\mathcal{Q}$ is a zigzag consistent dimer in a torus Σ. Then*

$$\begin{aligned} HH_c^{even}\big(MF(J(\mathcal{Q}), \ell)\big) &\cong tor_\ell^+ \oplus \mathbb{C}[x_{\nu_1}, \ldots, x_{\nu_k}]/\big(x_{\nu_i} x_{\nu_j} \,|\, i \neq j\big) \\ HH_c^{odd}\big(MF(J(\mathcal{Q}), \ell)\big) &\cong tor_\ell \oplus \mathbb{C}[x_{\nu_1}, \ldots, x_{\nu_k}]/\big(x_{\nu_i} x_{\nu_j} \,|\, i \neq j\big) \\ &\oplus \mathbb{C} \end{aligned}$$

In examples, this computation gives the answer expected from considerations of mirror symmetry.

We give here an outline of the paper. In Chapter 2, we briefly review the prerequisites on dimer models, Calabi-Yau algebras, matrix factorizations, and Hochschild cohomology. In Chapter 3, we characterize the Batalin-Vilkovisky (BV) structure of $HH^*(J(\mathcal{Q}))$ induced by the Calabi-Yau structure of $J(\mathcal{Q})$. The idea is to relate Hochschild cohomology of $J(\mathcal{Q})$

to that of its localization with respect to ℓ, $J(\mathcal{Q})[\ell^{-1}]$, which is Morita equivalent to the fundamental group algebra of a circle bundle over Σ. This work helps us with the explicit computation of the Hochschild cohomology of $J(\mathcal{Q})$ in Chapter 4. Finally, in Chapter 5, the compactly supported Hochschild cohomology of $J(\mathcal{Q})$ is addressed.

1.1 Basic notation and conventions

We work generally over the complex numbers $\mathbb{C}$. The following notation will be common throughout the text.

- $\mathcal{Q}$ is a finite quiver (or directed graph) with vertex set $\mathcal{Q}_0$ and arrow set $\mathcal{Q}_1$.
- $t, h : \mathcal{Q}_1 \to \mathcal{Q}_0$ are the tail and head functions, respectively.
- $\mathbb{C}\mathcal{Q}$ is the path algebra of $\mathcal{Q}$.
- For every vertex $v \in \mathcal{Q}_0$ and arrow $a \in \mathcal{Q}_1$, the same symbols v and a will also denote the corresponding elements of $\mathbb{C}\mathcal{Q}$ and $J(\mathcal{Q})$.
- $\Bbbk := \mathbb{C}\mathcal{Q}_0$, the semisimple subalgebra of $\mathbb{C}\mathcal{Q}$ spanned by the idempotents $\{v \in \mathcal{Q}_0\}$.
- $\bar{\mathcal{Q}}$ is the double of $\mathcal{Q}$.
- The unadorned tensor product $\otimes$ stands for $\otimes_{\mathbb{C}}$.

The convention of forward concatenation of paths for multiplication in $\mathbb{C}\mathcal{Q}$ will be followed. That is, for arrows $a_1, \dots, a_n \in \mathcal{Q}_1$,

$$a_1 a_2 \dots a_n \neq 0 \in \mathbb{C}\mathcal{Q} \Longleftrightarrow h(a_i) = t(a_{i+1}) \; \forall \, i = 1, \dots, n.$$

A symbol such as $p : v \to w$ will indicate a path p such that $t(p) = v$ and $h(p) = w$, either in $\mathbb{C}\mathcal{Q}$ or in $J(\mathcal{Q})$.

Chapter 2: Preliminaries

This chapter lays the groundwork for the rest of the paper. The exposition about dimer models in §2.1 - 2.4 is adapted largely from the works of Bocklandt and Broomhead, [5, 8, 7, 9]. In §2.5, we discuss Ginzburg's notion of Calabi-Yau algebras [22]. In §2.6-2.7, we briefly review the essentials from Polischuk–Positselski [29] about curved differential graded categories and Hochschild cohomology.

2.1 Dimer models

Conventionally, a dimer model is defined as a tiling of a Riemann surface by a bipartite graph. The edges of the dual cellular decomposition can be oriented to give a quiver, from which the Jacobi algebra is constructed. Since the algebra is our focus, we will simplify the exposition by defining dimers from the quiver perspective.

Let Σ be a compact Riemann surface of genus g. We say a quiver $\mathcal{Q}$ embeds into Σ if

1. $\mathcal{Q}_0$ is identified with a finite subset of Σ,
2. each arrow $a \in \mathcal{Q}_1$ has a smooth embedding $\phi_a : [0,1] \to \Sigma$ such that $\phi_a(0) = t(a)$ and $\phi_a(1) = h(a)$, and
3. the images of distinct arrows intersect only at the vertices.

We also impose the condition that no arrow is a contractible loop. Such a quiver is said to split Σ if $\Sigma \setminus \mathcal{Q}$ is a disjoint union of open disks. The closure of such a disk is called a face of $\mathcal{Q}$, which we denote generically by F.

Definition 2.1.1. A *dimer model* (or simply dimer) is a quiver $\mathcal{Q}$ splitting a Riemann surface Σ such that every face is bounded by a path of length at least 3: that is, an element $a_1 a_2 \ldots a_m \neq 0 \in \mathbb{C}\mathcal{Q}$ where $a_1, \ldots, a_m \in \mathcal{Q}_1$ and $m \geq 3$. We call such a closed path, considered up to cyclic permutation of the arrows, a boundary cycle, and label it ∂F.

Let $\mathcal{Q}_2$ be the set of faces of a dimer model. It can be divided into two subsets: the collection $\mathcal{Q}_2^+$ of faces whose boundary cycles are oriented anti-clockwise and the collection $\mathcal{Q}_2^-$ of faces whose boundary cycles are oriented clockwise. We describe faces and their boundary cycles as positive and negative accordingly. Every arrow is contained in the boundary cycle of exactly one positive face and one negative face.

We see that a dimer model provides a cellular decomposition of the ambient Riemann surface, and we write the associated chain complex with integer coefficients as

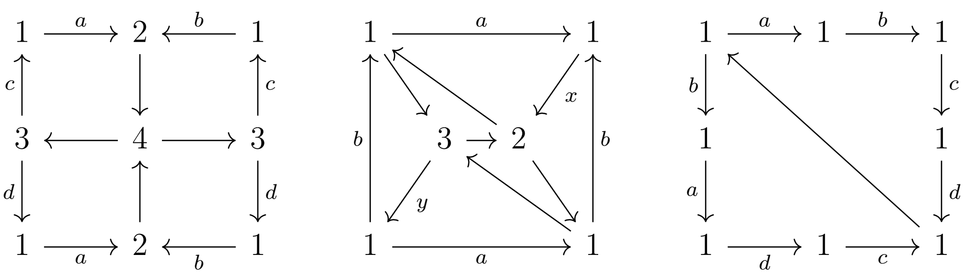
$$\mathbb{Z}\mathcal{Q}_2 \xrightarrow{d} \mathbb{Z}\mathcal{Q}_1 \xrightarrow{d} \mathbb{Z}\mathcal{Q}_0. \tag{2.1}$$

Dually, we write the cellular cochain complex as

$$\mathbb{Z}^{\mathcal{Q}_0} \xrightarrow{\partial} \mathbb{Z}^{\mathcal{Q}_1} \xrightarrow{\partial} \mathbb{Z}^{\mathcal{Q}_2}. \tag{2.2}$$

The Euler characteristic of Σ can be computed in the standard way from these complexes. Hence, the Euler characteristic can be ascribed to the dimer itself, and we denote it $\chi(\mathcal{Q})$.

Example 2.1.2 ([8]).



The first two dimers have genus 1, while the third dimer has genus 2.

2.2 Jacobi algebras

Let $\mathcal{Q}$ be an arbitrary quiver. A superpotential of $\mathcal{Q}$ is an element $\Phi \in \mathbb{C}\mathcal{Q}/[\mathbb{C}\mathcal{Q}, \mathbb{C}\mathcal{Q}]$, the vector space with basis given by the cyclic words in $\mathcal{Q}$. For each arrow $x \in \mathcal{Q}_1$, Ginzburg [22] defines an operator

$$\partial_x : \mathbb{C}\mathcal{Q}/[\mathbb{C}\mathcal{Q}, \mathbb{C}\mathcal{Q}] \to \mathbb{C}\mathcal{Q}$$

called the cyclic derivative with respect to x. It evaluates the equivalence class of a monomial $a_1 a_2 \ldots a_m \in \mathbb{C}\mathcal{Q}$ where $a_i \in \mathcal{Q}_1$ as

$$\partial_x[a_1 a_2 \ldots a_m] = \sum_{i \,|\, a_i = x} a_{i+1} \ldots a_m a_1 \ldots a_{i-1}.$$

Then the Jacobi algebra of the pair $(\mathcal{Q}, \Phi)$ is defined to be the quotient of the path algebra by the ideal generated by the cyclic derivatives of Φ,

$$J(\mathcal{Q}, \Phi) = \mathbb{C}\mathcal{Q}/(\partial_a \Phi \,|\, a \in \mathcal{Q}_1).$$

If $\mathcal{Q}$ is a dimer model, the boundary of a face, ∂F, can be viewed as an element in

$\mathbb{C}\mathcal{Q}/[\mathbb{C}\mathcal{Q}, \mathbb{C}\mathcal{Q}]$. Then the dimer is equipped with the canonical superpotential

$$\Phi_0 = \sum_{F \in \mathcal{Q}_2^+} \partial F - \sum_{F \in \mathcal{Q}_2^-} \partial F.$$

Definition 2.2.1. The Jacobi algebra of a dimer model $\mathcal{Q}$ is the algebra $J(\mathcal{Q}) := J(\mathcal{Q}, \Phi_0)$.

To write the relations explicitly, let $x \in \mathcal{Q}_1$ and $R_x^{\pm}$ be the paths in $\mathbb{C}\mathcal{Q}$ completing x to positive and negative boundary paths, respectively. We see that

$$\partial_x \Phi_0 = R_x^+ - R_x^-. \tag{2.3}$$

Since the boundary paths of a dimer have path length at least 3, the terms $R_x^{\pm}$ have length at least 2. Hence, the quotient map $\mathbb{C}\mathcal{Q} \twoheadrightarrow J(\mathcal{Q})$ preserves $\mathcal{Q}_0$ and $\mathcal{Q}_1$. In general, path length induces only an increasing filtration on $J(\mathcal{Q})$, as the relations need not be homogeneous.

We denote by $\mathcal{Z}$ the center of the algebra $J(\mathcal{Q})$, the underlying dimer being implicit. Choosing path that bounds a face at each vertex v and letting c_v be its image in $J(\mathcal{Q})$, we define

$$\ell = \sum_{v \in \mathcal{Q}_0} c_v \in J(\mathcal{Q}).$$

From the definition of the Jacobi algebra, it is straightforward to check that ℓ is independent of the choice of boundary paths and, moreover, is in $\mathcal{Z}$. This special central element is called the potential of $\mathcal{Q}$, and it pairs with $J(\mathcal{Q})$ to form a Landau-Ginzburg model.

2.3 Consistency conditions

Several related notions of consistency of a dimer are defined in the literature. We discuss a couple versions that will be most relevant to our interests. Throughout, it is assumed that $\mathcal{Q}$ is a dimer model in a surface Σ.

2.3.1 Cancellation

Consider the set $\{1, \ell, \ell^2, \dots \ell^n, \dots\}$ consisting of all nonnegative powers of ℓ in $J(\mathcal{Q})$. We denote by $J(\mathcal{Q})[\ell^{-1}]$ the central Ore localization of $J(\mathcal{Q})$ with respect to this multiplicative set and call it the localized Jacobi algebra. Geometrically, if $J(\mathcal{Q})$ is viewed as the coordinate ring of a hypothetical noncommutative affine variety, then $J(\mathcal{Q})[\ell^{-1}]$ is the coordinate ring of the complement to the zero locus of ℓ. Letting $\mathcal{Z}[\ell^{-1}] = \mathcal{Z} \otimes_{\mathbb{C}[\ell]} \mathbb{C}[\ell, \ell^{-1}]$, we can realize it as

$$J(\mathcal{Q}) \otimes_{\mathcal{Z}} \mathcal{Z}[\ell^{-1}].$$

It can also be constructed from the path algebra of the double quiver $\overline{\mathcal{Q}}$ by imposing the relations

$$aa^{-1} = t(a), \; a^{-1}a = h(a) \;\; \forall\, a \in \mathcal{Q}_1 \tag{2.4}$$

in addition to those in (2.3). Consequently, the localized algebra has an important cancellation property: for any arrow a and paths $p, q \in J(\mathcal{Q})[\ell^{-1}]$ such that $h(p) = h(q) = t(a)$, then

$$pa = qa \Longrightarrow p = q,$$

and similarly for products in the reverse direction.

Definition 2.3.1. A dimer model $\mathcal{Q}$ is said to be *cancellation* if $J(\mathcal{Q})$ also has the cancellation property, or equivalently, if the natural map $L : J(\mathcal{Q}) \to J(\mathcal{Q})[\ell^{-1}]$ is injective.

The kernel of L consists of all torsion elements under the action of ℓ, so a dimer is cancellation if and only if $J(\mathcal{Q})$ is torsion-free. Generally, cancellation can be difficult to check directly, but in nonpositive Euler characteristic, there is an equivalent geometric condition to which we now turn.

2.3.2 Zigzag consistency

Let $\widetilde{\Sigma}$ be the universal cover of Σ. The dimer can be lifted to a quiver $\widetilde{\mathcal{Q}}$ embedded in $\widetilde{\Sigma}$ that locally exhibits the same properties as $\mathcal{Q}$.

Definition 2.3.2. A *zigzag flow* is an infinite path in $\widetilde{\mathcal{Q}}$

$$\tilde{Z} := \ldots \tilde{Z}[-2]\tilde{Z}[-1]\tilde{Z}[0]\tilde{Z}[1]\tilde{Z}[2]\ldots, \quad \tilde{Z}[i] \in \widetilde{\mathcal{Q}}_1 \;\; \forall\, i \in \mathbb{Z}$$

such that $\tilde{Z}[i]\tilde{Z}[i+1]$ is contained in a positive boundary cycle when i is even and a negative boundary cycle when i is odd, or vice versa. Two zigzag flows are considered to be equivalent if one is obtained from the other by an integer shift in parametrization. An arrow $\tilde{Z}[i]$ is called a *zig* if $\tilde{Z}[i]\tilde{Z}[i+1]$ is contained in a positive boundary cycle and a *zag* if $\tilde{Z}[i]\tilde{Z}[i+1]$ is contained in a negative boundary cycle.

The projection of the zigzag flow $\tilde{Z}$ to $\mathcal{Q}$ is an infinite periodic path. We call a single period of this path at a given vertex a *zigzag path* and denote it by Z; when considered up to cyclic permutation of the arrows, we call it a *zigzag cycle*. An important related construction is the path that runs opposite to a zigzag along the positive or negative boundary cycles it meets.

Definition 2.3.3. Let Z be a zigzag path. The *left opposite path* to Z, denoted $\mathcal{O}_L$, is the path in $\mathcal{Q}$ at $t(Z) = h(Z)$ consisting of the arrows in the positive boundary cycles meeting Z but not in Z. The *right opposite path* to Z, denoted $\mathcal{O}_R$, is defined similarly but with negative boundary cycles. When considered up to cyclic permutation of the arrows, we call them the *left and right opposite cycles* to Z.

Zigzag and opposite cycles can be identified as 1-cycles in the cellular chain complex (2.1). Let $\{\nu_1, \ldots, \nu_k\} \subset H_1(\Sigma)$ be the homology classes of the opposite cycles, so $\{-\nu_1, \ldots, -\nu_k\}$

are the homology classes of the zigzag cycles. In general, there are multiple opposites and zigzags for each class.

Any arrow $a \in \widetilde{\mathcal{Q}}_1$ is contained in exactly two zigzag flows: one for which a is a zig and one for which a is a zag. Let

$$\tilde{Z}_a^+ = \tilde{Z}_a^+[0]\tilde{Z}_a^+[1]\tilde{Z}_a^+[2]\dots$$
$$\tilde{Z}_a^- = \tilde{Z}_a^-[0]\tilde{Z}_a^-[1]\tilde{Z}_a^-[2]\dots$$

be the respective seminfinite subpaths emanating from $\tilde{Z}_a^+[0] = \tilde{Z}_a^-[0] = a$, called the *zig* and *zag* rays of a.

Definition 2.3.4. A dimer model $\mathcal{Q}$ is *zigzag consistent* if for all arrows $a \in \widetilde{\mathcal{Q}}_1$, $\tilde{Z}_a^+$ and $\tilde{Z}_a^-$ intersect only in a:

$$\tilde{Z}_a^+[i] = \tilde{Z}_a^-[j] \Longrightarrow i = j = 0.$$

Note that a dimer model in the sphere can never be zigzag consistent because $\widetilde{\mathcal{Q}} = \mathcal{Q}$ is finite. Hence, whenever we assume a dimer is zigzag consistent, it will be implicit that $\chi(\mathcal{Q}) \leq 0$, in which case zigzag consistency is actually equivalent to cancellation.

Theorem 2.3.5 ([5] Theorem 5.5)**.** *Suppose $\mathcal{Q}$ is a dimer with $\chi(\mathcal{Q}) \leq 0$. Then $\mathcal{Q}$ is cancellation if and only if $\mathcal{Q}$ is zigzag consistent.*

It is straightforward to check that the first and third examples in 2.1.2 are zigzag consistent. However, the second example is not, as the zigzag rays emanating from a lift of the arrow x intersect in a lift of y. Thus, the dimer is not cancellation.

When $\mathcal{Q}$ is a zigzag consistent dimer embedded in a torus, zigzag flows behave similarly to lines in the Euclidean plane. As observed in Remark 5.6 [5], a zigzag path in this setting cannot intersect itself in an arrow, so the ν_i are nonzero and are primitive elements of $H_1(\Sigma)$. For a given zigzag flow $\tilde{Z}$, the homology $-\nu_i$ of the cycle to which it projects is the gradient of the line in $\mathbb{R}^2$ drawn through a vertex and its periodic shifts in $\tilde{Z}$. Hence, ν_i can be thought of as the direction of $\tilde{Z}$. We may assume, then, that the homology classes $\{\nu_1, \dots, \nu_k\}$ are ordered cyclically in anti-clockwise fashion. Distinct zigzag flows (paths, cycles) are said to be *parallel* if they have the same direction, as justified by the following.

Proposition 2.3.6 ([5], [9]). *Suppose $\mathcal{Q}$ is a zigzag consistent dimer model in a torus.*

1. *If two zigzag flows have the same homology, they do not intersect in an arrow.*
2. *If two zigzag paths have linearly independent homology, then they intersect in at least one arrow.*

Notation 2.3.7. Suppose $\mathcal{Q}$ is a zigzag consistent dimer in a torus. For each $i \in \mathbb{Z}/k\mathbb{Z}$, let $\gamma_i = \mathbb{Z}_{>0} \cdot \nu_i$, and let $\sigma_i = \mathrm{IntCone}(\nu_i, \nu_{i+1}) \cap \mathbb{Z}^2$, the set of lattice points in the interior of the cone in $\mathbb{R}^2$ spanned by ν_i and ν_{i+1}.

2.4 Perfect matchings

Let $\mathcal{Q}$ be a dimer model. In the cellular cochain complex (2.2), the image of a vertex $v \in \mathbb{Z}^{\mathcal{Q}_0}$ under the differential ∂ is the function

$$\partial(v) : a \mapsto \delta_{vh(a)} - \delta_{vt(a)}, \quad \forall a \in \mathcal{Q}_1$$

where δ_{vw} is the Kronecker delta on $\mathcal{Q}_0$. The kernel of ∂ is precisely the sublattice generated by $\sum_{v \in \mathcal{Q}_0} v$. Modifying the notation in [9], write N^{in} for $\partial(\mathbb{Z}^{\mathcal{Q}_0})$. We thus have an exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Z}^{\mathcal{Q}_0} \to N^{in} \to 0.$$

If $\alpha \in \mathbb{Z}^{\mathcal{Q}_1}$, then

$$\partial\alpha(F) = \sum_{a \in \partial F} \alpha(a), \ \ \forall\, F \in \mathcal{Q}_2.$$

Let $\underline{1} \in \mathbb{Z}^{\mathcal{Q}_2}$ be the constant function with value 1, $N = \partial^{-1}(\mathbb{Z} \cdot \underline{1})$, and $N^{out} = N/N^{in}$. An element $\alpha \in N$ has homogeneous summation on the boundary cycles of $\mathcal{Q}$: there exists a constant $m \in \mathbb{Z}$ such that

$$\sum_{a \in \partial F} \alpha(a) = m, \;\; \forall\, F \in \mathcal{Q}_2.$$

Since the relations of the Jacobi algebra (2.3) are homogeneous with respect to such α, it determines an $\mathbb{Z}$-grading on $J(\mathcal{Q})$. By the rule $\alpha(a^{-1}) = -\alpha(a)$, it extends to a grading on $J(\mathcal{Q})[\ell^{-1}]$.

Definition 2.4.1. A *perfect matching* $\mathcal{P}$ is a subset of $\mathcal{Q}_1$ containing exactly one arrow from every boundary cycle. Such a set can be identified with the element of $\partial^{-1}(\underline{1})$ sending an arrow a to 1 if $a \in \mathcal{P}$ and 0 otherwise. We write $PM(\mathcal{Q})$ for the set of perfect matchings of $\mathcal{Q}$ and $\deg_{\mathcal{P}}(p)$ for the degree of a path p in $J(\mathcal{Q})$ or $J(\mathcal{Q})[\ell^{-1}]$ with respect to $\mathcal{P}$.

Not every dimer model admits a perfect matching. Broomhead gives a necessary and sufficient condition for its existence ([9] Lemma 2.8). He also proves that $N^+ := \partial^{-1}(\mathbb{N} \cdot \underline{1})$ is generated integrally by $PM(\mathcal{Q})$. If every arrow of $\mathcal{Q}$ is contained in a perfect matching, then the sum

$$\sum_{\mathcal{P} \in PM(\mathcal{Q})} \mathcal{P}$$

is a strictly positive element of N^+. In this case, the perfect matchings generate the lattice N.

Proposition 2.4.2 ([9] Lemma 2.11; [7] Lemma 1.39)**.** *Suppose $\mathcal{Q}$ is a dimer admitting a strictly positive element of N^+. Then N is integrally generated by $PM(\mathcal{Q})$.*

The difference of two perfect matchings is a cocycle (2.2) and so determines a class in $H^1(\Sigma)$. Fixing a reference perfect matching $\mathcal{P}'$, we obtain a lattice polytope from the convex hull of $\{\mathcal{P} - \mathcal{P}' \,|\, \mathcal{P} \in PM(\mathcal{Q})\}$ in $H^1(\Sigma) \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^{2g+1}$, unique to $\mathcal{Q}$ up to affine integral transformation. We call this polytope the *matching polytope* and denote it $MP(\mathcal{Q})$.

When $\mathcal{Q}$ is a zigzag consistent dimer in a torus, the combinatorics are especially well-understood. Every lattice point in the matching polygon is the image of some perfect matching, which can then be classified as

- an *internal matching* if its image lies in the interior of $MP(\mathcal{Q})$,
- a *boundary matching* if its image lies on the boundary of $MP(\mathcal{Q})$, and
- *corner matching* if its image lies at the intersection of two boundary components.

Generally, multiple perfect matchings can map to the same lattice point. However, the corner matchings are unique, and they can be constructed geometrically from an isoradial embedding of $\widetilde{\mathcal{Q}}$ into $\mathbb{R}^2$ [5]. In fact, every arrow is contained in some corner matching, so by Proposition 2.4.2, the lattice N is generated by $PM(\mathcal{Q})$.

The homology classes of the zigzag cycles $\{-\nu_i \mid i \in \mathbb{Z}/k\mathbb{Z}\}$ are precisely the outward pointing normals to $MP(\mathcal{Q})$ [23]. Hence, we can cyclically order the corner matchings $\{\mathcal{P}_i \mid i \in \mathbb{Z}/k\mathbb{Z}\}$ so that $-\nu_i$ is the normal to the boundary component between $\mathcal{P}_i$ and $\mathcal{P}_{i+1}$. The detailed relationship between perfect matchings and zigzag cycles can be summarized as follows.

Theorem 2.4.3 ([23] §3; see also [7] Theorem 1.47)**.** *Suppose $\mathcal{Q}$ is a zigzag consistent dimer in a torus.*

1. *The corner matchings $\mathcal{P}_i$ and $\mathcal{P}_{i+1}$ contain the zigs and zags, respectively, of all zigzag cycles of homology $-\nu_i$. In each boundary cycle that does not meet a zigzag cycle of homology $-\nu_i$, $\mathcal{P}_i$ and $\mathcal{P}_{i+1}$ coincide.*
2. *The number of zigzag cycles n_i of homology $-\nu_i$ is one less than the number of lattice points on the boundary between $\mathcal{P}_i$ and $\mathcal{P}_{i+1}$. A boundary matching of length d away*

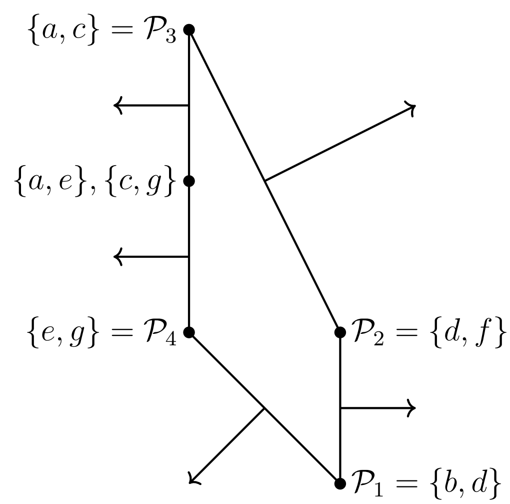
from $\mathcal{P}_i$ is the union of $\mathcal{P}_i \cap \mathcal{P}_{i+1}$, all arrows in $\mathcal{P}_i$ from d chosen zigzag cycles of homology $-\nu_i$, and all arrows in $\mathcal{P}_{i+1}$ from the remaining $n_i - d$ zigzag cycles of homology $-\nu_i$.

3. *The internal matchings meet every nontrivial closed path of $\mathcal{Q}$.*

As a corollary, the opposite paths $\mathcal{O}_R$ and $\mathcal{O}_L$ to a zigzag path of homology $-\nu_i$ have degree 0 in $\mathcal{P}_i$, $\mathcal{P}_{i+1}$, and all boundary matchings between them. Since the potential ℓ has degree 1 in all perfect matchings, this implies that the opposite paths are *minimal paths* in $J(\mathcal{Q})$: namely, they are not a multiple of ℓ. Note, however, that $\mathcal{O}_R$ and $\mathcal{O}_L$ have positive degree in all other perfect matchings, as can be deduced from Proposition 2.3.6.

Example 2.4.4 ([7] Example 1.5)**.** The suspended pinchpoint can be modeled by a zigzag consistent dimer in a torus.

$$\begin{tikzcd}
1 \arrow[r, "d"] & 1 \arrow[dl, "e"'] \\
3 \arrow[u, "c"] \arrow[d, "b"'] & 3 \arrow[u, "c"'] \arrow[d, "b"] \\
2 \arrow[ur, "f"] & 2 \arrow[dl, "g"] \\
1 \arrow[u, "a"] \arrow[r, "d"'] & 1 \arrow[u, "a"']
\end{tikzcd}$$



Observe that the homology classes of the paths d and $afcec$ generate $H_1(\Sigma)$. The matching polygon $MP(\mathcal{Q})$ is represented with respect to this basis. There are 4 corner matchings, 2 boundary matchings, and no internal matchings. We list the the zigzag cycles and represent them as normal vectors in the diagram.

zigzag cycle	homology class
ag	$(-1,0)$
ce	$(-1,0)$
$dafc$	$(2,1)$
fb	$(1,0)$
$debg$	$(-1,-1)$

2.5 Calabi-Yau algebras

Let A be an associative algebra and $A^e = A \otimes A^{op}$, the enveloping algebra of A.

Definition 2.5.1. An algebra A is *(homologically) smooth* if it has a bounded resolution by finitely generated projective A^e-modules. A smooth algebra A is *Calabi-Yau of dimension n* (CY-n) if there exists an A-bimodule quasi-isomorphism

$$A[n] \to \mathbb{R}\mathrm{Hom}_{A^e}(A, A \otimes A)$$

where $[-]$ denotes the shift in homological degree and $\mathbb{R}\mathrm{Hom}_{A^e}(A, A \otimes A)$ has A-bimodule structure from the inner bimodule action on $A \otimes A$.

The definition implies that, if A is CY-n, the Serre functor on the derived category of finitely generated A-modules is translation by n,

$$\mathbb{R}\mathrm{Hom}_A(M, N) \cong \mathbb{R}\mathrm{Hom}_A(N, M[n])^*.$$

In this sense, it is analogous to the geometric notion of a Calabi-Yau variety.

The quasi-isomorphism in the definition is equivalently an A-bimodule isomorphism

$$A \to \mathrm{Ext}^n_{A^e}(A, A \otimes A), \tag{2.5}$$

which is determined by the image of $1 \in A$. The image, which is a central element in $\mathrm{Ext}^n_{A^e}(A, A \otimes A)$, is called a *volume* of A, and the set of all volumes $Vol(A)$ is a torsor over the ring of central units, $\mathcal{Z}(A)^\times$ [22].

Not every dimer model yields a Jacobi algebra that is Calabi-Yau 3. As it turns out, the Jacobi algebra of a dimer in the sphere can never be so [5]. However, a sufficient condition when the Euler characteristic is nonpositive is that $\mathcal{Q}$ is cancellation or, equivalently, zigzag consistent.

Theorem 2.5.2 ([13]). *If $\mathcal{Q}$ is a zigzag consistent dimer model, then $J(\mathcal{Q})$ is Calabi-Yau* 3.

2.5.1 Resolutions of the Jacobi algebra

Let $\mathcal{Q}$ be a general quiver and Φ a superpotential of $\mathcal{Q}$. As for any associative algebra, $A = J(\mathcal{Q}, \Phi)$ can be resolved as a bimodule by the bar complex,

$$\mathrm{Bar}(A) = A \otimes A^{\otimes *} \otimes A,$$
$$d_{\mathrm{Bar}}(p_1 \otimes \cdots \otimes p_n) = \sum_{i=1}^{n} (-1)^{i-1} p_1 \otimes \cdots \otimes p_i p_{i+1} \otimes \cdots \otimes p_n.$$

A somewhat smaller resolution is obtained by normalizing with respect to the semisimple subalgebra $\Bbbk$. Letting $\overline{A} = A/\Bbbk$, we have

$$\overline{\mathrm{Bar}}(A) = A \otimes_{\Bbbk} \overline{A}^{\otimes_{\Bbbk} *} \otimes_{\Bbbk} A, \tag{2.6}$$

a quotient of $\mathrm{Bar}(A)$ by an acyclic subcomplex, with differential induced from d_{Bar}.

The Calabi-Yau 3 property of $A := J(\mathcal{Q}, \Phi)$ is equivalent to the exactness of a certain bimodule complex. To define it, let

- $V_0 = \Bbbk$,
- $V_1 = \mathbb{C}\mathcal{Q}_1$, the vector space with basis given by the arrows of $\mathcal{Q}$,
- $V_2 = \mathbb{C}\{\partial_a \Phi \,|\, a \in \mathcal{Q}_1\}$, the vector space with basis given by the cyclic derivatives of Φ, and
- $V_3 = \mathbb{C}\{\Phi^v \,|\, v \in \mathcal{Q}_0\}$, the vector space with basis given by the syzygies

$$\Phi^v = \sum_{a \,|\, t(a)=v} a \, \partial_a \Phi = \sum_{a \,|\, h(a)=v} \partial_a \Phi \, a.$$

These vector spaces have obvious $\Bbbk$-bimodule structures and thus generate projective A-bimodules

$$\mathbb{P}_i := A \otimes_{\Bbbk} V_i \otimes_{\Bbbk} A.$$

Then define maps $\mu_i : \mathbb{P}_i \to \mathbb{P}_{i-1}$ for $i = 1, 2, 3$ and $\mu_0 : \mathbb{P}_0 \to A$ by

$$\begin{aligned}
\mu_3 \quad &: p \otimes \Phi^v \otimes q \mapsto \sum_{a \,|\, t(a)=v} pa \otimes \partial_a \Phi \otimes q - \sum_{a \,|\, h(a)=v} p \otimes \partial_a \Phi \otimes aq, \\
\mu_2 \quad &: p \otimes \partial_a \Phi \otimes q \mapsto \sum_{b \in \mathcal{Q}_1} (\partial_b \, \partial_a \Phi)' \otimes b \otimes (\partial_b \, \partial_a \Phi)'', \\
\mu_1 \quad &: p \otimes a \otimes q \mapsto pa \otimes q - p \otimes aq, \\
\mu_0 \quad &: p \otimes q \mapsto pq,
\end{aligned}$$

where, for a path $Y \in \mathbb{C}\mathcal{Q}$, the element $(\partial_b Y)' \otimes (\partial_b Y)'' \in \mathbb{C}\mathcal{Q} \otimes \mathbb{C}\mathcal{Q}$ is the sum over all elements $X \otimes Z$ such that $XbZ = Y$. It is straightforward to check that $\mu_i \mu_{i-1} = 0$, so we have a finitely generated A-bimodule complex

$$\mathbb{P}_3 \xrightarrow{\mu_3} \mathbb{P}_2 \xrightarrow{\mu_2} \mathbb{P}_1 \xrightarrow{\mu_1} \mathbb{P}_0 \xrightarrow{\mu_0} A. \tag{2.7}$$

Theorem 2.5.3 ([22] Corollary 5.3.3)**.** *The algebra A is CY-3 if and only if the complex $(\mathbb{P}_*, \mu_*)$ is a projective resolution of A.*

In fact, the resolution $\mathbb{P}_*$ is self-dual in the derived category of A-bimodules: there is an isomorphism of complexes

$$\mathrm{Hom}_{A^e}(\mathbb{P}_*, A \otimes A) \cong \mathbb{P}_{3-*}. \tag{2.8}$$

Generally, if an algebra has a self-dual resolution of length n, then it is Calabi-Yau n [6].

As a consequence of Theorems 2.5.2 and 2.5.3, a zigzag consistent dimer $\mathcal{Q}$ has a resolution of the form (2.7). Moreover, since $J(\mathcal{Q})[\ell^{-1}]$ is a flat $J(\mathcal{Q})$-module, the complex $J(\mathcal{Q})[\ell^{-1}] \otimes_{J(\mathcal{Q})} \mathbb{P} \otimes_{J(\mathcal{Q})} J(\mathcal{Q})[\ell^{-1}]$ is a self-dual resolution of $J(\mathcal{Q})[\ell^{-1}]$, which is therefore $J(\mathcal{Q})[\ell^{-1}]$ CY-3 as well.

2.6 Curved algebras and matrix factorizations

Curved differential graded categories provide a unified framework to discuss matrix factorizations, curved algebras, and Hochschild cohomology. We follow the exposition in [29], but we restrict our attention to small $\mathbb{C}$-linear categories and grading group Γ equal to $\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z}$.

Definition 2.6.1. A curved differential Γ-graded ($\Gamma - \mathrm{CDG}$) category is the data $(\mathcal{C}, \delta, h)$ where

1. $\mathcal{C}$ is a small Γ-graded $\mathbb{C}$-linear category,

2. δ (the differential) is a collection of degree 1 endomorphisms $\delta_{XY} : \mathcal{C}(X,Y) \to \mathcal{C}(X,Y)$ for all $X, Y \in \mathbb{C}$, and

3. h (the curvature) is a collection of degree 2 morphisms $h_X \in \mathcal{C}(X,X)$ for each $X \in \mathbb{C}$,

satisfying the equations

1. $\delta_{XZ}(gf) = \delta_{YZ}(g)f + (-1)^{|g|} g\delta_{XY}(f)$ for all morphisms $f : X \to Y$ and $g : Y \to Z$,

2. $\delta^2_{XY}(f) = h_Y f - f h_X$ for all $f \in \mathcal{C}(X,Y)$, and

3. $\delta_{XX}(h_X) = 0$ for all $X \in \mathcal{C}$.

We will omit the subscripts from δ and h when they are clear from context.

A CDG category is a generalization of a differential graded (DG) category. Since the square of the differential is generally nonzero, however, there is no natural definition of homology or quasi-isomorphism. The correct notion of equivalence comes from derived categories of the second kind ([30], [29]), but since the examples of CDG categories we encounter are well-established in other contexts, we will not explore this topic.

A curved algebra A_h is a $\Gamma-$CDG category with one object and trivial differential. From the definition, it consists of a Γ-graded associative algebra A and a central element h in degree 2. A curved differential graded module over A_h is a Γ-graded left A-module with an A-linear endomorphism d_M of degree 1 satisfying

$$d_M^2 = h \cdot Id_M.$$

We define $A_h - Mod^{\Gamma}_{\text{CDG}}$ to be the category of such objects with morphism spaces given by internal Hom of Γ-graded A-modules. In fact, there is a natural differential

$$\delta(f) = d_N f - (-1)^{|f|} f d_M \quad \forall f : M \to N,$$

and one easily checks $\delta^2 = 0$. Hence, $A_h - Mod_{\mathrm{CDG}}^{\Gamma}$ is actually DG- category.

Definition 2.6.2. Let (A, h) be a $\mathbb{Z}/2\mathbb{Z}$-graded Landau-Ginzburg model. The category of matrix factorizations $MF(A, h)$ is the full DG-subcategory of $A_h - Mod_{\mathrm{CDG}}^{\mathbb{Z}/2\mathbb{Z}}$ consisting of curved differential graded modules that are projective and finitely generated as A-modules.

In detail, a matrix factorization of (A, h) is a diagram

$$\begin{tikzcd} P_0 \arrow[r, bend left, "d_P^0"] & P_1 \arrow[l, bend left, "d_P^1"] \end{tikzcd}$$

where P_0 and P_1 are finitely generated projective A-modules and

$$d_P^1 d_P^0 = h \cdot Id_{P_0}, \;\; d_P^0 d_P^1 = h \cdot Id_{P_1}.$$

For the Landau-Ginzburg model $(J(\mathcal{Q}), \ell)$ associated to a dimer model, every arrow $a \in \mathcal{Q}_1$ defines a matrix factorization

$$M_a = \begin{tikzcd} J(\mathcal{Q})t(a) \arrow[r, bend left, "a"] & J(\mathcal{Q})h(a) \arrow[l, bend left, "r_a"] \end{tikzcd}$$

where r_a is the equivalence class of $R_a^{\pm}$ (2.3) in $J(\mathcal{Q})$. We consider the action of a and r_a to be on the right. The full DG subcategory of $MF(J(\mathcal{Q}), \ell)$ consisting of these matrix factorizations is the category $mf(\mathcal{Q})$ appearing in Theorem 1.0.1. For two arrows $a, b \in \mathcal{Q}_1$, a pair of paths

$$\big(p : t(a) \to t(b), q : h(a) \to h(b)\big)$$

defines a degree 0 morphism $M_a \to M_b$, while a pair

$$\big(p : t(a) \to h(b), q : h(a) \to t(b)\big)$$

defines a degree 1 morphism.

2.7 Hochschild cohomology

Let $(\mathcal{C}, \delta, h)$ be a $\Gamma-$ CDG category. For any objects $X_0, \ldots, X_n \in \mathcal{C}$, the vector space

$$\mathcal{C}(X_0, X_1, \ldots, X_n) := \mathcal{C}(X_n, X_0^{op}) \otimes \mathcal{C}(X_0, X_1) \otimes \cdots \otimes \mathcal{C}(X_{n-1}, X_n)$$

has tensor degree n, taken modulo 2 if $\Gamma = \mathbb{Z}/2\mathbb{Z}$. Moreover, it has the induced Γ-grading of tensor products, and we denote by $\mathcal{C}(X_0, X_1, \ldots, X_n)_m$ the homogeneous degree m component. We define the Hochschild chain complex $C_*(\mathcal{C})$ (Hochschild chains of the first kind) as the Γ-graded complex whose homogeneous degree k component is the direct sum totalization

$$\bigoplus_{m+n=k} \mathcal{C}(X_0, X_1, \ldots, X_n)_m$$

with differential given by the sum of three terms:

$$\begin{aligned} d_C(c_0 \otimes \cdots \otimes c_n) &= \sum_{i=0}^{n-1} (-1)^i c_0 \otimes \cdots \otimes c_i c_{i+1} \cdots \otimes c_n \\ &\quad + (-1)^{n+|c_n|(|c_0|+\cdots+|c_{n-1}|)} c_n c_0 \otimes c_1 \otimes \cdots \otimes c_{n-1} \\ d_\delta(c_0 \otimes \cdots \otimes c_n) &= \sum_{i=0}^{n} (-1)^{n+|c_0|+\cdots+|c_{i-1}|} c_0 \otimes \cdots \otimes \delta(c_i) \otimes \cdots \otimes c_n \\ d_h(c_0 \otimes \cdots \otimes c_n) &= \sum_{i=0}^{n} (-1)^i c_0 \otimes \cdots \otimes c_i \otimes h \otimes c_{i+1} \otimes \cdots \otimes c_n. \end{aligned} \tag{2.9}$$

Alternatively, the Borel-Moore Hochschild chain complex $C_*^{BM}(\mathcal{C})$ (Hochschild chains of the second kind) is defined to be the Γ-graded complex whose homogeneous degree k component is the direct product totalization

$$\prod_{m+n=k} \mathcal{C}(X_0, X_1, \ldots, X_n)_m$$

with differential given by the same formula. Denote the homologies of these complexes by $HH_*(\mathcal{C})$ and $HH_*^{BM}(\mathcal{C})$, respectively.

Dually, for any objects $X_0, \ldots, X_n \in \mathcal{C}$, the internal Hom space of Γ-graded vector spaces

$$\mathrm{Hom}\left(\mathcal{C}(X_0, X_1) \otimes \cdots \otimes \mathcal{C}(X_{n-1}, X_n), \, \mathcal{C}(X_0, X_n^{op})\right)$$

has tensor degree n, taken modulo 2 if $\Gamma = \mathbb{Z}/2\mathbb{Z}$. Let

$$\mathrm{Hom}^m\left(\mathcal{C}(X_0, X_1) \otimes \cdots \otimes \mathcal{C}(X_{n-1}, X_n), \, \mathcal{C}(X_0, X_n^{op})\right)$$

be the homogeneous Γ-degree m component. We define the Hochschild cochain complex $C^*(\mathcal{C})$ (Hochschild cochains of the first kind) as the Γ-graded complex whose homogeneous degree k component is the direct product totalization

$$\prod_{m+n=k} \mathrm{Hom}^m\left(\mathcal{C}(X_0, X_1) \otimes \cdots \otimes \mathcal{C}(X_{n-1}, X_n), \, \mathcal{C}(X_0, X_n^{op})\right)$$

with differential given by the sum of three terms:

$$\begin{aligned}
d_C f(c_1 \otimes \cdots \otimes c_{n+1}) &= (-1)^{|f||c_1|} c_1 f(c_2 \otimes \cdots \otimes c_{n+1}) \\
&\quad + \sum_{i=1}^{n} (-1)^i f(c_1 \otimes \cdots \otimes c_i c_{i+1} \otimes \cdots \otimes c_{n+1}) \\
&\quad + (-1)^{n+1} f(c_1 \otimes \cdots \otimes c_n) c_{n+1} \\
d_\delta f(c_1 \otimes \cdots \otimes c_n) &= (-1)^n \delta(f(c_1 \otimes \cdots \otimes c_n)) \\
&\quad - \sum_{i=1}^{n} (-1)^{n+|f|+|c_1|+\cdots+|c_{i-1}|} f(c_1 \otimes \cdots \otimes \delta(c_i) \otimes \cdots \otimes c_n) \\
d_h f(c_1 \otimes \cdots \otimes c_{n-1}) &= \sum_{i=0}^{n-1} (-1)^{i+1} f(c_1 \otimes \cdots \otimes c_i \otimes h \otimes c_{i+1} \otimes \cdots \otimes c_{n-1}).
\end{aligned} \tag{2.10}$$

Alternatively, the compactly supported Hochschild cochain complex $C^*_c(\mathcal{C})$ (Hochschild cochains of the second kind) is defined to be the Γ-graded complex whose homogeneous degree k component is the direct sum totalization

$$\bigoplus_{m+n=k} \mathrm{Hom}^m \big(\mathcal{C}(X_0, X_1) \otimes \cdots \otimes \mathcal{C}(X_{n-1}, X_n), \, \mathcal{C}(X_0, X_n^{op})\big)$$

with differential given by the same formula. The homologies of these complexes are denoted $HH^*(\mathcal{C})$ and $HH^*_c(\mathcal{C})$, respectively.

If the curvature h is trivial, then $C_*(\mathcal{C})$ and $C^*(\mathcal{C})$ recover the usual definition of Hochschild (co)homology of a $\Gamma - \mathrm{DG}$ category. For an associative algebra A concentrated in degree 0, the Γ-grading of the two kinds of Hochschild complexes equals the tensor grading. If $\Gamma = \mathbb{Z}$, there is precisely one homogeneous component in each degree, so the direct product totalization equals the direct sum totalization. Consequently, the two kinds of Hochschild cohomology coincide. On the other hand, for $\Gamma = \mathbb{Z}/2\mathbb{Z}$, each degree has infinitely many homogeneous components:

$$C_*(A) = \bigoplus_{n\geq 0} A \otimes A^{\otimes n}, \quad C^*(A) = \prod\nolimits_{n\geq 0} \mathrm{Hom}_{\mathbb{C}}(A^{\otimes n}, A),$$
$$C_*^{BM}(A) = \prod_{n\geq 0} A \otimes A^{\otimes n} \quad C_c^*(A) = \bigoplus\nolimits_{n\geq 0} \mathrm{Hom}_{\mathbb{C}}(A^{\otimes n}, A).$$

In particular, $C_*^{BM}(A)$ is the completion of $C_*(A)$ with respect to the tensor degree, and similarly for $C^*(A)$ is the completion of $C_c^*(A)$.

For a $\mathbb{Z}/2\mathbb{Z}$-graded Landau-Ginzburg model (A, h) with nonzero potential, ordinary Hochschild (co)homology is trivial.

Theorem 2.7.1 ([12] Theorem 4.2)**.** *Let A_h be a $\mathbb{Z}/2\mathbb{Z}$-graded curved algebra such that $h \neq 0$. Then*

$$HH_*(A_h) = HH^*(A_h) = 0.$$

Consequently, the classical Hochschild invariants provide no information about $\mathbb{Z}/2\mathbb{Z}$-graded curved algebras. Caldararu–Tu also show in [12] that if A is a smooth affine variety of dimension n and h is a regular function with isolated singularity, then

$$HH_c^*(A_h) \cong Jac(h), \quad HH_*^{BM}(A_h) \cong \omega(h)[n] \mod 2$$

where $Jac(h)$ is the ring of regular functions and $\omega(h)$ is the relative dualizing sheaf of the critical locus. This agrees with the Hochschild cohomology of $MF(A, h)$ computed in [15].

More generally, drawing an analogy with derived Morita theory [33], one might hope from Definition 2.6.2 that there is at least a relationship between the compactly supported invariants of a $\mathbb{Z}/2\mathbb{Z}$-graded curved algebra A_h and those of the matrix factorization category. This is indeed part of a broader theorem of [29] relating a CDG category to its DG category of modules.

Theorem 2.7.2 ([29] §2.6). *For a $\mathbb{Z}/2\mathbb{Z}$-graded Landau-Ginzburg model (A, h), there are natural isomorphisms*

$$\begin{aligned} HH_c^*(A_h) &\cong HH_c^*(MF(A,h)) \\ HH_*^{BM}(A_h) &\cong HH_*^{BM}(MF(A,h)). \end{aligned}$$

The subtlety is that, unlike for ordinary algebras, A_h is not naturally a left or right curved module over itself (but is a curved bimodule over itself). The theorem is proved by embedding both A_h and $MF(A, h)$ into the larger category of so called QDG-modules and establishing an isomorphism there.

The relationship between the two kinds of Hochschild invariants for the DG category $MF(A, h)$ is more complicated. The inclusion of direct sum into direct product totalizations provides maps

$$HH_*(MF(A,h)) \to HH_*^{BM}(MF(A,h)), \quad HH_c^*(MF(A,h)) \to HH^*(MF(A,h)). \tag{2.11}$$

By [29] Corollary 4.7B, a sufficient condition for these maps to be isomorphisms is the existence of a kind of resolution of A_h as an A_h-bimodule. This is satisfied, for example, for smooth commutative algebras with potential having critical value only 0 [26].

For the Landau-Ginzburg model $(J(\mathcal{Q}), \ell)$ of a dimer, it is unknown whether the comparison maps (2.11) are isomorphisms. The main issue is that the Jacobi algebra is generally not Noetherian. However, in the case of a zigzag consistent dimer in a torus, $J(\mathcal{Q})$ is Noetherian and is a noncommutative crepant resolution of $\mathbb{Z}/2\mathbb{Z}$ [9]. This leads us to make a conjecture.

Conjecture 2.7.3. *If $\mathcal{Q}$ is a zigzag consistent dimer model, then*

$$\begin{aligned} HH_c^*(MF(J(\mathcal{Q}),\ell)) &\cong HH^*(MF(J(\mathcal{Q}),\ell)) \\ HH_*^{BM}(MF(J(\mathcal{Q}),\ell))) &\cong HH_*^{BM}(MF(J(\mathcal{Q}),\ell)) \end{aligned}$$

Failure of the conjecture would provide a geometrically interesting example of the disagreement between the two kinds of Hochschild cohomology. In any case, it provides some motivation for computing the compact-type invairants of $MF(J(\mathcal{Q}),\ell)$.

2.7.1 Noncommutative calculus

Let $\Gamma = \mathbb{Z}$ and A be an associative algebra concentrated in degree 0. As explained previously, the two kinds of Hochschild cohomology agree in this setup, so there is no need to distinguish between them. The Hochschild homology and cohomology of A form a noncommuative calculus [32],

$$\big(HH^*(A),\, \cup,\, \{-,-\},\, HH_*(A),\, i_-,\, B\big),$$

which we now review.

The cup product $\cup$ and interior (or cap) product i_- are well-known, but it will be useful to have formulas for resolutions other than the bar resolution. For a projective A-bimodule resolution P_* of A, there is a diagonal map (unique up to homotopy equivalence)

$$D : P_* \to P_* \otimes_A P_*$$

lifting the identity of A. If $P_* = \mathrm{Bar}(A)$, for example, the diagonal map is

$$D : a_1 \otimes \cdots \otimes a_n \mapsto \sum_{i=0}^{n} (a_1 \otimes \cdots \otimes a_i \otimes 1) \otimes (1 \otimes a_{i+1} \otimes \cdots \otimes a_n). \tag{2.12}$$

Then the cup product is defined as

$$\alpha \cup \beta = \mu \circ (\alpha \otimes \beta) \circ D, \quad \forall\, \alpha, \beta \in \mathrm{Hom}_{A^e}(P_*, A)$$

where $\mu : A \otimes A \to A$ is multiplication. Similarly, the cap product is defined as

$$\alpha \cap \eta = (\mu \otimes Id_P) \circ (Id_A \otimes \alpha \otimes Id_P) \circ (Id_A \otimes D)\, \eta, \quad \forall\, \alpha \in \mathrm{Hom}_{A^e}(P_*, A), \ \eta \in A \otimes_{A^e} P_*.$$

One can check that these operations descend to the usual cup and cap operations on Hochschild (co)homology [2].

On cochains $\alpha, \beta \in C^*(A)$, the Gerstenhaber bracket has formula

$$\begin{aligned} \{\alpha, \beta\}(a_1, \ldots, a_{d+e-1}) &= \sum_{j \geq 0} (-1)^{j(|\beta|+1)} \alpha(a_1, \ldots, a_j, \beta(a_{j+1}, \ldots, a_{j+e}), \ldots, a_{d+e-1}) \\ &- (-1)^{(|\alpha|+1)(|\beta|+1)} \sum_{j \geq 0} (-1)^{j(|\alpha|+1)} \beta(a_1, \ldots, a_j, \alpha(a_{j+1}, \ldots, a_{j+d}), \ldots, a_{d+e-1}). \end{aligned} \tag{2.13}$$

The cup product and Gerstenhaber bracket make $HH^*(A)$ into a Gerstenhaber algebra [21]. In particular, the Leibniz identity is satisfied,

$$\{\alpha, \beta \cup \gamma\} = \{\alpha, \beta\} \cup \gamma + (-1)^{(|\alpha|-1)|\beta|} \beta \{\alpha, \gamma\} \quad \forall \alpha, \beta, \gamma \in HH^*(A). \tag{2.14}$$

Note that, for a central element h, the differential d_h of (2.10) is the adjoint action of h, $d_h(\alpha) = -\{\alpha, h\}$.

The map B is the Connes differential, which on $C_*(A)$ has the formula

$$\begin{aligned} B(a_0 \otimes a_1 \otimes \cdots \otimes a_n) &= \sum_{i=0}^{n} (-1)^{ni} 1 \otimes a_i \otimes \cdots \otimes a_n \otimes a_0 \otimes \cdots \otimes a_{i-1} \\ &+ \sum_{i=0}^{n} (-1)^{n(i+1)} a_{i-1} \otimes 1 \otimes a_i \otimes \cdots \otimes a_n \otimes a_0 \otimes \cdots \otimes a_{i-2}. \end{aligned} \tag{2.15}$$

The action of $C^*(A)$ on $C_*(A)$ as a Lie algebra is defined by the Lie derivative, $\mathcal{L}_-$. It satisfies the Cartan identity on $HH_*(A)$,

$$\mathcal{L}_\alpha = [B, i_\alpha] \quad \forall \alpha \in HH^*(A). \tag{2.16}$$

For a central element h, $\mathcal{L}_h$ coincides with the differential d_h in (2.9).

Van den Bergh showed that if A is Calabi-Yau n, there is a Poincaré-type duality isomorphism between Hochschild cohomology and homology [35]. A volume π (2.5) determines a quasi-isomorphism of bimodules

$$\pi^+ : \mathbb{R}\mathrm{Hom}_{A^e}(A, A \otimes A) \to A[n], \quad \pi \mapsto 1.$$

Furthermore, since A is smooth, there is a quasi-isomorphism

$$\mathbb{R}\mathrm{Hom}_{A^e}(\mathbb{R}\mathrm{Hom}_{A^e}(A, A \otimes A), A) \cong A \otimes^{\mathbb{L}}_{A^e} A,$$

sending π^+ to a Hochschild cycle of degree n. In general, an element of $HH_n(A)$ which is the image of a quasi-isomorphism under this identification is called a nondegenerate element. By abuse of notation, we write π for the nondegenerate element corresponding to π^+. Then we have a quasi-isomorphism

$$\mathbb{R}\mathrm{Hom}_{A^e}(A[n], A) \xrightarrow{\circ \pi^+} \mathbb{R}\mathrm{Hom}_{A^e}(\mathbb{R}\mathrm{Hom}_{A^e}(A, A \otimes A), A) \xrightarrow{\cong} A \otimes^{\mathbb{L}}_{A^e} A, \tag{2.17}$$

and the induced isomorphism on homology is [14]

$$\mathbb{D}_\pi : HH^*(A) \to HH_{n-*}(A), \quad \alpha \mapsto \alpha \cap \pi. \tag{2.18}$$

One easily sees that $\mathbb{D}_\pi$ exchanges $\alpha \cup -$ with $\alpha \cap \mathbb{D}_\pi(-)$. Moreover, the Connes differential B is sent under $\mathbb{D}_\pi$ to a Batalin-Vilkovisky (BV) operator Δ_π compatible with the Gerstenhaber structure, making $HH^*(A)$ into a BV algebra [22]. The precise relationship is given by

$$\Delta_\pi(\alpha \cup \beta) = \Delta_\pi(\alpha) \cup \beta + (-1)^{|\alpha|}\alpha\Delta_\pi(\beta) + (-1)^{|\alpha|}\{\alpha, \beta\} \quad \forall\, \alpha, \beta \in HH^*(A). \tag{2.19}$$

Chapter 3: Batalin-Vilkovisky structure of the Jacobi algebra

Throughout, we will assume that $\mathcal{Q}$ is a dimer model in a surface Σ that admits a perfect matching. In §3.1, we recount the fact that the localized algebra $J(\mathcal{Q})[\ell^{-1}]$ is isomorphic to a matrix algebra with coefficients in the fundamental group algebra of an $\mathbb{S}^1$-bundle over Σ [6]. Then, in a general setting, we relate the BV structure of a Calabi-Yau algebra to that of a central localization of the algebra. Combined, these results allow us in §3.4 to describe the BV structure of $J(\mathcal{Q})$ in terms of the calculus of Laurent polynomials and the group algebra $\mathbb{C}[\pi_1(\Sigma)]$.

3.1 The localized algebra as a matrix algebra

In addition to the $\mathbb{Z}$-grading by a perfect matching, we shall consider a grading on $J(\mathcal{Q})[\ell^{-1}]$ afforded by the homotopical structure [13]. Let $\pi_1(\Sigma, \mathcal{Q}_0)$ be the full subcategory of the fundamental groupoid of Σ whose objects are the vertices of $\mathcal{Q}$. The embedding of a $\mathcal{Q}$ into Σ can be extended to the double $\bar{\mathcal{Q}}$ by letting the image of a^{-1} be the inverse path of the image of $a \in \mathcal{Q}_1$. Since the paths in each relation of (2.3) and (2.4) are homotopic, a path in $J(\mathcal{Q})[\ell^{-1}]$ represents a morphism in $\pi_1(\Sigma, \mathcal{Q}_0)$ between its endpoints. Hence, the algebra $J(\mathcal{Q})[\ell^{-1}]$ is graded by $\pi_1(\Sigma, \mathcal{Q}_0)$, and $J(\mathcal{Q})$ inherits the grading via the localization map $L : J(\mathcal{Q}) \to J(\mathcal{Q})[\ell^{-1}]$.

Upon choosing a basepoint, the $\pi(\Sigma, \mathcal{Q}_0)$-grading of $J(\mathcal{Q})[\ell^{-1}]$ can be transformed into a grading by the fundamental group. Fix a vertex v_0, and for every $v \in \mathcal{Q}_0$, fix a path $p_v : v_0 \to v$ in $J(\mathcal{Q})[\ell^{-1}]$, taking p_{v_0} to be the idempotent v_0. By multiplying by the appropriate power of ℓ, we can ensure that $\deg_{\mathcal{P}}(p_v) = 0$ for a chosen $\mathcal{P} \in PM(\mathcal{Q})$. Then define the $\pi_1(\Sigma, v_0)$-degree of a path $p \in J(\mathcal{Q})[\ell^{-1}]$, denoted $|p|$, to be the $\pi_1(\Sigma, \mathcal{Q}_0)$-degree of $p_{t(p)}\, p\, p_{h(p)}^{-1}$. For convenience, we suppress the basepoint and simply write $\pi_1(\Sigma)$. Passing to the abelianization of $\pi_1(\Sigma)$ gives a grading of $J(\mathcal{Q})[\ell^{-1}]$ by the homology $H_1(\Sigma)$, independent of the choice of basepoint and connecting paths p_v.

The gradings can be leveraged to describe the Jacobi algebra in more familiar terms. For any perfect matching $\mathcal{P}$, consider the $\pi_1(\Sigma) \times \mathbb{Z}$-bigrading on $J(\mathcal{Q})[\ell^{-1}]$ in which the bidegree of a path p is $(|p|, \deg_{\mathcal{P}}(p))$. As the next lemma states, the homogeneous subspace of paths between given vertices is one-dimensional.

Lemma 3.1.1 ([6] Lemma 7.2)**.** *Let $\mathcal{Q}$ be a dimer model admitting a perfect matching. Two paths $p, q : v \to w \in J(\mathcal{Q})[\ell^{-1}]$ are equal if and only if $(|p|, \deg_{\mathcal{P}}(p)) = (|q|, \deg_{\mathcal{P}}(q))$ for any $\mathcal{P} \in PM(\mathcal{Q})$.*

Consequently, keeping track of the head and tail data as well as the gradings, we can write an isomorphism from $J(\mathcal{Q})[\ell^{-1}]$ to a matrix algebra.

Theorem 3.1.2 ([6] Theorem 7.4)**.** *Let $\mathcal{Q}$ be a dimer model admitting a perfect matching. For any $\mathcal{P} \in PM(\mathcal{Q})$, the map*

$$\Psi_{\mathcal{P}} : J(\mathcal{Q})[\ell^{-1}] \to Mat_{\#\mathcal{Q}_0}(\mathbb{C}[\pi_1(\Sigma)] \otimes \mathbb{C}[z^{\pm 1}]).$$

sending a path $p : v \to w$ to

$$\big(|p| \otimes z^{\deg_{\mathcal{P}}(p)}\big) e_{vw},$$

where e_{vw} is the (v, w)-elementary matrix, is an isomorphism of algebras.

Thus, when $\chi(\mathcal{Q}) = 0$, $J(\mathcal{Q})[\ell^{-1}]$ is Morita equivalent to the algebra of Laurent polynomials $\mathbb{C}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}]$. When $\chi(\mathcal{Q}) < 0$, the algebra $\mathbb{C}[\pi_1(\Sigma)]$ is noncommutative, but its Hochschild (co)homology is still well-understood. In §3.4, this Morita equivalence will be used to describe the BV structure of the Jacobi algebra explicitly.

Remark 3.1.3. We note here that the Hochshild homologies of $J(\mathcal{Q})$ and $J(\mathcal{Q})[\ell^{-1}]$ inherit the $H_1(\Sigma) \times \mathbb{Z}$-bigrading with respect to any perfect matching. Since in each degree the resolution $\mathbb{P}_*$ is finitely generated by homogeneous elements, the Hochschild cohomologies also inherit the $H_1(\Sigma) \times \mathbb{Z}$-bigrading. This auxiliary data will allow for easy deductions about the structure of Hochschild (co)homology.

3.2 Hochschild cohomology of a central localization

Let A be an associative algebra, $\mathcal{Z}$ be the center of A, and $S \subset \mathcal{Z}$ a multiplicative subset containing 1 and excluding 0. We denote by $\widehat{\mathcal{Z}}$ the localization of the center with respect to S. Then the Ore localization of A with respect to S can be defined as

$$\widehat{A} := A \otimes_{\mathcal{Z}} \widehat{\mathcal{Z}}.$$

Moreover, for any A-module M, its Ore localization is the $\widehat{A}$-module

$$\widehat{M} := M \otimes_{\mathcal{Z}} \widehat{\mathcal{Z}}.$$

The natural map $L : M \to \widehat{M}$ has kernel equal to the S-torsion of M,

$$tor_S(M) = \{m \in M \,|\, sm = 0 \text{ for some } s \in S\}.$$

See for example [36] for a detailed account about localization.

Let $\mathcal{D}(A^e)$ and $\mathcal{D}(\widehat{A}^e)$ be the derived categories of A and $\widehat{A}$-bimodules, respectively. The algebra $\widehat{A}$ is flat as a left and as a right A-module, and the functors

$$\begin{aligned} F &= \widehat{A} \otimes_A - \otimes_A \widehat{A} : \mathcal{D}(A^e) \to \mathcal{D}(\widehat{A}^e), \\ G &= \widehat{A} \otimes_{\widehat{A}} - \otimes_{\widehat{A}} \widehat{A} : \mathcal{D}(\widehat{A}^e) \to \mathcal{D}(A^e) \end{aligned}$$

form an adjoint pair. They yield canonical maps

$$\begin{aligned} I_* &: A \otimes^{\mathbb{L}}_{A^e} A \to A \otimes^{\mathbb{L}}_{A^e} GF(A) \cong \widehat{A} \otimes^{\mathbb{L}}_{\widehat{A}^e} \widehat{A}, \\ I^* &: \mathrm{Hom}_{\mathcal{D}(A^e)}(A, A[i]) \to \mathrm{Hom}_{\mathcal{D}(\widehat{A}^e)}(F(A), F(A)[i]) \cong \mathrm{Hom}_{\mathcal{D}(\widehat{A}^e)}(\widehat{A}, \widehat{A}[i]), \quad \forall i \in \mathbb{Z}. \end{aligned}$$

Letting

$$\widehat{\mathrm{Bar}}(A) := F(A) = \widehat{A} \otimes_A \mathrm{Bar}(A) \otimes_A \widehat{A} \cong \bigoplus_{n \in \mathbb{N}} \widehat{A} \otimes A^{\otimes n} \otimes \widehat{A},$$

we can write the maps explicitly on (co)chains:

$$I_* : A \otimes_{A^e} \mathrm{Bar}(A) \to \widehat{A} \otimes_{\widehat{A}^e} \widehat{\mathrm{Bar}}(A),$$

$$I_*(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = L(a_0) \otimes L(a_1) \otimes a_2 \cdots \otimes a_{n-1} \otimes L(a_n)$$

$$I^* : \mathrm{Hom}_{A^e}(\mathrm{Bar}(A), A) \to \mathrm{Hom}_{\widehat{A}^e}(\widehat{\mathrm{Bar}}(A), \widehat{A}),$$

$$I^*(\alpha)(\hat{a}_1 \otimes a_2 \otimes \cdots \otimes a_{n-1} \otimes \hat{a}_n) = \hat{a}_1 L\alpha(1 \otimes a_2 \otimes \cdots \otimes a_{n-1} \otimes 1)\hat{a}_n.$$

To arrive at maps on Hochschild (co)chains, consider the comparison map

$$\widehat{\mathrm{Bar}}(A) \to \mathrm{Bar}(\widehat{A}), \quad \hat{a}_1 \otimes a_2 \otimes \cdots \otimes a_{n-1} \otimes \hat{a}_n \mapsto \hat{a}_1 \otimes L(a_2) \otimes \cdots \otimes L(a_{n-1}) \otimes \hat{a}_n.$$

It lifts the identity of $\widehat{A}$ and so is a homotopy equivalence. Let

$$\phi : \widehat{A} \otimes_{\widehat{A}^e} \widehat{\mathrm{Bar}}(A) \to \widehat{A} \otimes_{\widehat{A}^e} \mathrm{Bar}(\widehat{A})$$

be the induced map on chains and

$$\phi^\vee : \mathrm{Hom}_{\widehat{A}^e}(\widehat{\mathrm{Bar}}(A), \widehat{A}) \to \mathrm{Hom}_{\widehat{A}^e}(\mathrm{Bar}(\widehat{A}), \widehat{A})$$

be the map on cochains given by precomposition with the homotopy inverse. Then define

$$\begin{aligned} L_* &:= \phi \circ I_* : C_*(A) \to C_*(\widehat{A}), \\ L_* &:= \phi^\vee \circ I^* : C^*(A) \to C^*(\widehat{A}) \end{aligned}$$

as the maps on Hochschild (co)chains induced by localization. We see in particular that L_* is given simply by

$$L_* : a_0 \otimes a_1 \otimes a_2 \otimes \cdots \otimes a_n \mapsto L(a_0) \otimes L(a_1) \otimes L(a_2) \otimes \cdots \otimes L(a_n). \tag{3.1}$$

Throughout, we denote the Connes differentials on the Hochschild complexes of A and $\widehat{A}$ as B_A and $B_{\widehat{A}}$, respectively. As shown in [10], the functor HH_* commutes with central localization. This result can be slightly enhanced to include the Connes differentials.

Proposition 3.2.1. *Let A be an associative algebra and $S \subset \mathcal{Z}$ a multiplicative subset. The map*

$$\widehat{L}_* : \widehat{HH}_*(A) = HH_*(A) \otimes_{\mathcal{Z}} \widehat{\mathcal{Z}} \longrightarrow HH_*(\widehat{A}), \;\; \eta \otimes \hat{z} \mapsto L_*(\eta) \cap \hat{z}$$

is an isomorphism of $\widehat{\mathcal{Z}}$-modules. Moreover, $\widehat{L}_*$ intertwines the Connes differential $B_{\widehat{A}}$ with *the differential*

$$\widehat{B}(\eta \otimes s^{-1}) := B_A(\eta) \otimes s^{-1} - \mathcal{L}_s(\eta) \otimes s^{-2}, \ \ \forall s \in S.$$

Proof. It is clear from the formulas for the Connes differential (2.15) and L_* (3.1) that L_* intertwines B_A and $B_{\widehat{A}}$. Then the formula for the differential $\widehat{B}$ is obtained from the calculus identities. Observe

$$\begin{aligned} B_{\widehat{A}}\,\widehat{L}_*(\eta \otimes s^{-1}) &= B_{\widehat{A}}\, i_{s^{-1}} L_*(\eta) \\ &= i_{s^{-1}} B_{\widehat{A}}\, L_*(\eta) + \mathcal{L}_{s^{-1}}\, L_*(\eta) \\ &= i_{s^{-1}} L_*\, B_A(\eta) + \mathcal{L}_{s^{-1}}\, L_*(\eta). \end{aligned}$$

We also have the identity $\mathcal{L}_{s^{-1}} = -i_{s^{-2}} \mathcal{L}_s$ so the last expression equals

$$i_{s^{-1}} L_*\, B_A(\eta) - i_{s^{-2}}\, \mathcal{L}_s\, L_*(\eta) = i_{s^{-1}}\, L_*\, B_A(\eta) - i_{s^{-2}}\, L_*\, \mathcal{L}_s(\eta).$$

Under the isomorphism $\widehat{L}$, this is precisely the image of

$$B_A(\eta) \otimes s^{-1} - \mathcal{L}_s(\eta) \otimes s^{-2}. \qquad \square$$

We would like to prove analogously that HH^* commutes with central localization in a way that preserves the algebraic structure. To do so, the cup and cap products can be defined for the resolution $\widehat{\mathrm{Bar}}(A)$. Let D_A be the diagonal map for $\mathrm{Bar}(A)$ (2.12) and $\widehat{D}$ be the diagonal map for $\widehat{\mathrm{Bar}}(A)$, which has the form

$$\widehat{D} : \widehat{\mathrm{Bar}}(A) \to \widehat{\mathrm{Bar}}(A) \otimes_{\widehat{A}} \widehat{\mathrm{Bar}}(A)$$

$$\hat{a}_1 \otimes a_2 \otimes \cdots \otimes a_{n-1} \otimes \hat{a}_n \mapsto \sum_{i=0}^{n} (\hat{a}_1 \otimes a_2 \otimes \cdots \otimes a_i \otimes 1) \otimes (1 \otimes a_{i+1} \otimes \cdots \otimes a_{n-1} \otimes \hat{a}_n).$$

Lemma 3.2.2. *The map $L^* : HH^*(A) \to HH^*(\widehat{A})$ is a morphism of algebras with respect to the cup products.*

Proof. Consider the diagram of cochain complexes

$$\begin{tikzcd}
\mathrm{Hom}_{A^e}(\mathrm{Bar}(A), A) \arrow[r, "I^*"] \arrow[d, "\alpha\cup -"] & \mathrm{Hom}_{\widehat{A}^e}(\widehat{\mathrm{Bar}}(A), \widehat{A}) \arrow[r, "\phi^\vee"] \arrow[d, "I^*(\alpha)\cup -"] & \mathrm{Hom}_{\widehat{A}^e}(\mathrm{Bar}(\widehat{A}), \widehat{A}) \arrow[d, "L^*(\alpha)\cup -"] \\
\mathrm{Hom}_{A^e}(\mathrm{Bar}(A), A) \arrow[r, "I^*"] & \mathrm{Hom}_{\widehat{A}^e}(\widehat{\mathrm{Bar}}(A), \widehat{A}) \arrow[r, "\phi^\vee"] & \mathrm{Hom}_{\widehat{A}^e}(\mathrm{Bar}(\widehat{A}), \widehat{A}).
\end{tikzcd}$$

The horizontal composition is L^*. If we take homology, commutativity of the second square follows from the independence of the cup product from the choice of resolution and diagonal map. So to prove L^* is an algebra morphism, it suffices to prove commutativity of the first square. Observe

$$\begin{aligned}
I^*(\alpha\cup\beta)(\hat{a}_1\otimes\cdots\otimes\hat{a}_n) &= \hat{a}_1 L(\alpha\cup\beta)(1\otimes a_2\otimes\cdots\otimes a_{n-1}\otimes 1)\hat{a}_n \\
&= \hat{a}_1 L\circ\mu(\alpha\otimes\beta)D_A(1\otimes a_2\otimes\cdots\otimes a_{n-1}\otimes 1)\hat{a}_n \\
&= \hat{a}_1 L\alpha(1\otimes a_2\otimes\cdots\otimes a_i\otimes 1)L\beta(1\otimes a_{i+1}\otimes\cdots\otimes a_{n-1}\otimes 1)\hat{a}_n
\end{aligned}$$

and

$$\begin{aligned}
I^*(\alpha)\cup I^*(\beta)(\hat{a}_1\otimes\cdots\otimes\hat{a}_n) &= \mu(I^*(\alpha)\cup I^*(\beta))\widehat{D}(\hat{a}_1\otimes\cdots\otimes\hat{a}_1) \\
&= I^*(\alpha)(\hat{a}_1\otimes a_2\otimes a_i\otimes 1)I^*(\beta)(1\otimes a_{i+1}\otimes\cdots\otimes\hat{a}_n) \\
&= \hat{a}_1 L\alpha(1\otimes a_2\otimes\cdots\otimes a_i\otimes 1)L\beta(1\otimes a_{i+1}\otimes\cdots\otimes a_{n-1}\otimes 1)\hat{a}_n,
\end{aligned}$$

so indeed the first square commutes. $\square$

Lemma 3.2.3. *For all* $\alpha \in HH^*(A)$ *and* $\eta \in HH_*(A)$, *the diagram*

$$\begin{tikzcd} HH^*(A) \arrow[r, "L^*"] \arrow[d, "-\cap\eta"] & HH^*(\widehat{A}) \arrow[d, "-\cap L_*(\eta)"] \\ HH_*(A) \arrow[r, "L_*"] & HH_*(\widehat{A}) \end{tikzcd}$$

commutes.

Proof. We use a similar argument as for the previous lemma. Consider the diagram of complexes

$$\begin{tikzcd} \mathrm{Hom}_{A^e}(\mathrm{Bar}(A), A) \arrow[r, "I^*"] \arrow[d, "-\cap\eta"] & \mathrm{Hom}_{\widehat{A}^e}(\widehat{\mathrm{Bar}}(A), \widehat{A}) \arrow[r, "\phi^\vee"] \arrow[d, "-\cap I_*(\eta)"] & \mathrm{Hom}_{\widehat{A}^e}(\mathrm{Bar}(\widehat{A}), \widehat{A}) \arrow[d, "-\cap L_*(\eta)"] \\ A \otimes_{A^e} \mathrm{Bar}(A) \arrow[r, "I_*"] & \widehat{A} \otimes_{\widehat{A}^e} \widehat{\mathrm{Bar}}(A) \arrow[r, "\phi"] & \widehat{A} \otimes_{\widehat{A}^e} \mathrm{Bar}(\widehat{A}) \end{tikzcd}$$

The top horizontal composition is the map L^*, while the bottom horizontal composition is the mapL_*. If we take homology, commutativity of the second square follows from the independence of the cap product from choice of resolution and diagonal map. So to prove the result, it suffices to prove commutativity of the first square.

Without loss of generality, suppose

$$\eta = a_0 \otimes a_1 \otimes \cdots \otimes a_n \in A \otimes_{A^e} A^{\otimes n}.$$

Observe

$$\begin{aligned} I_*(\alpha \cap \eta) &= I_*\big(a_0 \otimes (\alpha \otimes Id) D_A(a_1 \otimes \dots a_n)\big) \\ &= I_*\big(a_0 a_1 \alpha(1 \otimes a_2 \otimes \cdots \otimes a_i \otimes 1) \otimes (1 \otimes a_{i+1} \otimes \cdots \otimes a_n)\big) \\ &= L(a_0) L(a_1) L\alpha(1 \otimes a_2 \otimes \cdots \otimes a_i \otimes 1) \otimes (1 \otimes a_{i+1} \otimes \cdots \otimes a_{n-1} \otimes L(a_n)), \end{aligned}$$

and

$$\begin{aligned}
I^*(\alpha)\cap I_*(\eta) &= L(a_0)\otimes (I^*(\alpha)\otimes Id)\widehat{D}(L(a_1)\otimes a_2\otimes\cdots\otimes a_{n-1}\otimes L(a_n))\\
&= L(a_0)I^*(\alpha)(L(a_1)\otimes a_2\otimes\cdots\otimes a_i\otimes 1)\otimes(1\otimes a_{i+1}\otimes\cdots\otimes a_{n-1}\otimes L(a_n))\\
&= L(a_0)L(a_1)L\alpha(1\otimes a_2\otimes\cdots\otimes a_i\otimes 1)\otimes(1\otimes a_{i+1}\otimes\cdots\otimes a_{n-1}\otimes L(a_n)),
\end{aligned}$$

so indeed the first square commutes. □

If A is Calabi-Yau n, then its central localization $\widehat{A}$ is also Calabi-Yau n [19]. Combined with the preceding lemmas, the fact that Van den Berg duality (2.18) is the interior product with a nondegenerate element shows that L^* commutes with the BV operator.

Proposition 3.2.4. *If A is CY-n with nondegenerate element π, then $\widehat{A}$ is CY-n with nondegenerate element $L_*(\pi)$, and the map L^* is a morphism of BV-algebras.*

Proof. Since A is smooth, we have a commutative diagram

$$\begin{tikzcd}
\mathbb{R}\mathrm{Hom}_{A^e}(\mathbb{R}\mathrm{Hom}_{A^e}(A, A\otimes A), A) \arrow[r, "\cong"] \arrow[d] & A\otimes^{\mathbb{L}}_{A^e} A \arrow[d, "I_*"] \\
\mathbb{R}\mathrm{Hom}_{A^e}(\mathbb{R}\mathrm{Hom}_{A^e}(A, A\otimes A), GF(A)) \arrow[r, "\cong"] & A\otimes^{\mathbb{L}}_{A^e} GF(A) \cong \widehat{A}\otimes^{\mathbb{L}}_{\widehat{A}^e}\widehat{A}
\end{tikzcd}$$

By the adjunction $F\dashv G$ and again by the fact that A is smooth, we have

$$\begin{aligned}
\mathbb{R}\mathrm{Hom}_{A^e}(\mathbb{R}\mathrm{Hom}_{A^e}(A, A\otimes A), GF(A)) &\cong \mathbb{R}\mathrm{Hom}_{\widehat{A}^e}(\widehat{A}\otimes_A \mathbb{R}\mathrm{Hom}_{A^e}(A, A\otimes A)\otimes_A\widehat{A}, \widehat{A})\\
&\cong \mathbb{R}\mathrm{Hom}_{\widehat{A}^e}(\mathbb{R}\mathrm{Hom}_{\widehat{A}^e}(\widehat{A}, \widehat{A}\otimes\widehat{A}), \widehat{A})
\end{aligned}$$

Consequently, a quasi-isomorphism in

$$\mathbb{R}\mathrm{Hom}_{A^e}(\mathbb{R}\mathrm{Hom}_{A^e}(A, A\otimes A), A)$$

corresponding to the nondegenerate element $\pi \in HH_*(A)$ maps to a quasi-isomorphism in

$$\mathbb{R}\mathrm{Hom}_{\widehat{A}^e}(\mathbb{R}\mathrm{Hom}_{\widehat{A}^e}(\widehat{A}, \widehat{A} \otimes \widehat{A}), \widehat{A})$$

corresponding to $L_*(\pi)$.

By Lemma 3.2.3, we then have a commutative diagram

$$\begin{tikzcd} HH^*(A) \arrow[r, "L^*"] \arrow[d, "\mathbb{D}_\pi"] & HH^*(\widehat{A}) \arrow[d, "\mathbb{D}_{L_*(\pi)}"] \\ HH_*(A) \arrow[r, "L_*"] & HH_*(\widehat{A}). \end{tikzcd}$$

Since L_* intertwines the Connes differentials by Lemma **??**, L^* must intertwine the BV operators Δ_π and $\Delta_{L_*(\pi)}$. □

The dual statement to Proposition 3.2.1 can now be formulated.

Theorem 3.2.5. *Let A be an associative algebra and $S \subset \mathcal{Z}$ a multiplicative subset.*

1. *The map*

$$\widehat{L}^* : \widehat{HH^*}(A) = HH^*(A) \otimes_{\mathcal{Z}} \widehat{\mathcal{Z}} \longrightarrow HH^*(\widehat{A}), \ \ \alpha \otimes \hat{z} \mapsto L^*(\alpha) \cup \hat{z}.$$

   is a morphism of graded $\widehat{\mathcal{Z}}$-algebras.

2. *If A has a bimodule resolution by finitely generated projectives, then $\widehat{L}^*$ is an isomorphism.*

3. *If A is CY-n with nondegenerate element π, then map $\widehat{L}^*$ is an isomorphism of BV algebras, intertwining $\Delta_{L_*(\pi)}$ with the differential*

$$\widehat{\Delta}_\pi(\alpha \otimes s^{-1}) := \Delta_\pi(\alpha) \otimes s^{-1} - \{s, \alpha\} \otimes s^{-2} \ \ \forall s \in S.$$

Proof.

(1) This is clear from Lemma 3.2.2.

(2) Let P_* be a resolution of A by finitely generated projectives, and let

$$\widehat{P}_* := F(P_*) = \widehat{A} \otimes_A P_* \otimes_A \widehat{A}.$$

By a comparison between P_* and $\mathrm{Bar}(A)$, the map $I_* : \mathrm{Hom}_{A^e}(\mathrm{Bar}(A), A) \to \mathrm{Hom}_{\widehat{A}^e}(\widehat{\mathrm{Bar}}(A), \widehat{A})$ is equivalent to the horizontal arrow of the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{A^e}(P_*, A) \arrow[r] \arrow[dr] & \mathrm{Hom}_{\widehat{A}^e}(\widehat{P}_*, \widehat{A}) \\
& \mathrm{Hom}_{A^e}(P_*, A) \otimes_{\mathcal{Z}} \widehat{\mathcal{Z}} \arrow[u]
\end{tikzcd}$$

By the action of $\widehat{\mathcal{Z}}$ on the codomain $\widehat{A}$, the map factors as shown. Since P_* is finitely generated in each degree,

$$\mathrm{Hom}_{A^e}(P_*, A) \otimes_{\mathcal{Z}} \widehat{\mathcal{Z}} \cong \mathrm{Hom}_{A^e}(P_*, \widehat{A}).$$

The vertical map in the diagram is then identified as the isomorphism given by $\widehat{A}^e$-linearly extending a morphism $P_* \to \widehat{A}$ to $\widehat{P}_* \to \widehat{A}$.

(3) We have a commutative diagram of isomorphisms

$$\begin{tikzcd}
HH^*(A) \otimes_{\mathcal{Z}} \widehat{\mathcal{Z}} \arrow[r, "\widehat{L}^*"] \arrow[d, "\mathbb{D}_\pi \otimes Id"] & HH^*(\widehat{A}) \arrow[d, "\mathbb{D}_{L_*(\pi)}"] \\
HH_*(A) \otimes_{\mathcal{Z}} \widehat{\mathcal{Z}} \arrow[r, "\widehat{L}_*"] & HH_*(\widehat{A}).
\end{tikzcd}$$

By Proposition 3.2.1, $\widehat{L}_*$ is a chain map where the left side is given differential

$$\widehat{B}(\eta \otimes s^{-1}) = B(\eta) \otimes s^{-1} - \mathcal{L}_s(\eta) \otimes s^{-2}, \ \ \forall s \in S$$

The dual to this operator under $\mathbb{D}_\pi \otimes Id$ has precisely the stated formula. $\square$

An immediate consequence of Proposition 3.2.1 and Theorem 3.2.5 is that the kernels of L_* and L^* consist of the S-torsion.

Corollary 3.2.6. *Let A be an associative algebra and $S \subset \mathcal{Z}$ a multiplicative subset. Then*

$$Ker(L_*) = tor_S(HH_*(A)).$$

If furthermore A has a bimodule resolution by finitely generated projectives, then

$$Ker(L^*) = tor_S(HH^*(A)).$$

In general, injectivity of the localization map $L : A \to \widehat{A}$ does not prevent the existence of torsion in Hochschild (co)homology, a fact that will play a crucial role in our computation of $HH^*(J(\mathcal{Q}))$.

3.3 Morita invariance

Let A be an associative algebra and $Mat_r(A)$ be the algebra of $r \times r$-matrices with coefficients in A. It is well-known that Morita equivalence induces isomorphisms on Hochschild (co)homology. In the instance at hand, the isomorphisms are

$$HH_*(A) \underset{tr_*}{\overset{inc_*}{\rightleftarrows}} HH_*(Mat_r(A)) \qquad HH^*(A) \underset{inc^*}{\overset{cotr^*}{\rightleftarrows}} HH_*(Mat_r(A))$$

where

- inc_* is induced by the inclusion of A into the $(1,1)$-entry;

- tr_* is the generalized trace,

$$tr_* : m_0 \otimes m_1 \otimes \cdots \otimes m_n \mapsto \sum_{(i_0,\ldots,i_n)} m_0^{i_0,i_1} \otimes m_1^{i_1,i_2} \otimes \cdots \otimes m_n^{i_n,i_0}, \quad m_i \in Mat_r(A), \tag{3.2}$$

  the sum being over all indices $(i_0, \ldots, i_n) \in \{1, 2, \ldots, r\}^{n+1}$;

- $cotr_*$ is the cotrace;

- inc^* is the co-inclusion,

$$inc^*(\alpha)(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = proj_{11}\, \alpha(a_0 e_{11} \otimes a_1 e_{11} \otimes \cdots \otimes a_n e_{11}), \tag{3.3}$$

  the map $proj_{11}$ being the projection onto the $(1,1)$-coordinate.

If A is Calabi-Yau n, then $Mat_n(A)$ is also Calabi-Yau n (see e.g.[38]), and the analogous statement to Theorem 3.2.5 holds.

Proposition 3.3.1. *If A is CY-n with nondegenerate element π, then the maps*

$$\big(HH^*(A), \Delta_\pi\big) \underset{inc^*}{\overset{cotr^*}{\rightleftarrows}} \big(HH^*(Mat_r(A)), \Delta_{inc_*(\pi)}\big)$$

are isomorphisms of BV algebras.

Proof. It is known from general theory that the Morita isomorphisms on Hochschild (co)homology preserve the cup and cap products [3]. Hence, just as in Proposition 3.2.4, it remains to show that tr_* or inc_* commutes with the Connes differential. But this is clear from the formulas for B (2.15) and tr_* (3.2). □

3.4 Batalin-Vilkovisky structure of $HH^*(J(\mathcal{Q}))$

Given any $\mathcal{P} \in PM(\mathcal{Q})$, let $\Psi := \Psi_{\mathcal{P}}$ be the isomorphism in Theorem 3.1.2, and let Ψ^* and Ψ_* be the induced isomorphisms on Hochschild (co)homology. Then we have isomorphisms

$$\begin{aligned} cotr^* \circ \Psi^* : HH^*(J(\mathcal{Q})[\ell^{-1}]) &\rightarrow HH^*(\mathbb{C}[\pi_1(\Sigma)] \otimes \mathbb{C}[z^{\pm 1}]) \\ tr_* \circ \Psi_* : HH_*(J(\mathcal{Q})[\ell^{-1}]) &\rightarrow HH_*(\mathbb{C}[\pi_1(\Sigma)] \otimes \mathbb{C}[z^{\pm 1}]). \end{aligned}$$

If $\chi(\mathcal{Q}) = 0$ so $\pi_1(\Sigma) \cong H_1(\Sigma) \cong \mathbb{Z}^2$, then

$$\mathbb{C}[H_1(\Sigma)] \otimes \mathbb{C}[z^{\pm 1}] \cong \mathbb{C}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}]$$

with x and y corresponding to generators of $H_1(\Sigma)$. The Hochschild-Kostant-Rosenberg isomorphism [24] gives an identification of calculus structures

$$\begin{aligned} HH^*(\mathbb{C}[H_1(\Sigma)] \otimes \mathbb{C}[z^{\pm 1}]) &\cong \mathbb{C}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}][\partial_x, \partial_y, \partial_z] \\ HH_*(\mathbb{C}[H_1(\Sigma)] \otimes \mathbb{C}[z^{\pm 1}]) &\cong \mathbb{C}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}][dx, dy, dz] \end{aligned} \tag{3.4}$$

where $\partial_x, \partial_y, \partial_z$ are the coordinate vector fields in cohomological degree 1 and dx, dy, dz are the dual Kahler forms. In particular, the BV differential is the usual divergence operator on polyvector fields, depending on a choice of 3-form. Explicitly, if ξ_x, ξ_y, and ξ_z are the $\mathbb{C}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}]$-linear vector fields for the coordinates ∂_x, ∂_y, and ∂_z and if $\pi = x^r y^s z^t dx\, dy\, dz$, then the associated divergence operator is

$$\mathrm{div}_\pi := x^{-r}\partial_x x^r \xi_x + y^{-s}\partial_y y^s \xi_y + z^{-t}\partial_z z^t \xi_z.$$

In the case that $\chi(\mathcal{Q}) < 0$, the algebra $\mathbb{C}[\pi_1(\Sigma)]$ is noncommutative. It is, by a result of

Kontsevich, Calabi-Yau 2 (see [22] Corollary 6.1.4), so its Hochschild cohomology has a BV structure under Van den Bergh duality. Vaintrob [34] explicitly described the BV structure in terms of the Chas-Sullivan string topology of Σ. Namely, let $L\Sigma$ be the space of free loops of Σ and $\mathbb{H}_*(\Sigma) := H_*(L\Sigma, \mathbb{C})$, the *loop homology of* Σ. The latter is endowed with an associative multiplication, called the *loop product*, that is defined in terms of the intersection product on Σ, as well as a differential $\rho : \mathbb{H}_*(\Sigma) \to \mathbb{H}_{*+1}(\Sigma)$ induced by the natural $\mathbb{S}^1$-action on $L\Sigma$ [11]. Together, these operations make $\mathbb{H}_*(\Sigma)$ a BV algebra.

Theorem 3.4.1 ([34] Theorem 3.2)**.** *Suppose Σ is a Riemann surface of genus $g > 1$. There is an isomorphism of BV algebras*

$$HH^*(\mathbb{C}[\pi_1(\Sigma)]) \cong \mathbb{H}_{2-*}(\Sigma) \cong \begin{cases} \mathbb{C} & \text{if } * = 0 \\ H_1(\Sigma, \mathbb{C}) \oplus H_0(L\Sigma, \mathbb{C})/\mathbb{C}e & \text{if } * = 1 \\ H_0(L\Sigma, \mathbb{C}) & \text{if } * = 2 \end{cases}$$

where e is the class of the trivial loop. The only nontrivial product is of elements in $HH^1(\mathbb{C}[\pi_1(\Sigma)])$ and has formula

$$(\alpha_1, \beta_1) \cdot (\alpha_2, \beta_2) = \langle \alpha_1, \alpha_2 \rangle e + \langle \alpha_2, \beta_1 \rangle \beta_1 + \langle \alpha_1, \beta_2 \rangle \beta_2 + [\beta_1, \beta_2]_{Gold}$$

for all $\alpha_i \in H_1(\Sigma, \mathbb{C})$ and $\beta_i \in H_0(L\Sigma, \mathbb{C})/\mathbb{C}e$, where $\langle -, - \rangle$ is the intersection pairing on $H_1(\Sigma, \mathbb{C})$ and $[-, -]_{Gold}$ is the Goldmann bracket. The BV differential ρ is trivial except on $HH^2(\mathbb{C}[\pi_1(\Sigma)])$, where it is the projection $H_0(L\Sigma, \mathbb{C}) \to H_0(L\Sigma, \mathbb{C})/\mathbb{C}e$.

Notice that, as the center of $\mathbb{C}[\pi_1(\Sigma)]$ is trivial, there is a unique nondegenerate element (or volume (2.5)) for the Calabi-Yau structure, up to scaling. Consequently, the BV structure from Van den Bergh duality is unique. Letting π_s be the nondegenerate element, we have $\rho = \Delta_{\pi_s}$.

To relate the BV structure of $\mathbb{C}[\pi_1(\Sigma)] \otimes \mathbb{C}[z^{\pm 1}]$ to the BV structures of its constituents, we can apply the Kunneth isomorphism. The isomorphism for Hochschild homology holds generally ([27] Theorem 4.2.5), so if π is a nondegenerate element for $\mathbb{C}[\pi_1(\Sigma)] \otimes \mathbb{C}[z^{\pm 1}]$, it corresponds to $\pi_s \otimes z^t dz$ for some $t \in \mathbb{Z}$. As stated in the next lemma, the necessary finiteness conditions hold for the Kunneth isomorphism on Hochschild cohomology to respect BV structures.

Lemma 3.4.2. *Suppose Σ is a Riemann surface of genus $g > 1$. There is an isomorphism of BV algebras*

$$\left(HH^*(\mathbb{C}[\pi_1(\Sigma)] \otimes \mathbb{C}[z^{\pm 1}]),\, \Delta_\pi\right) \cong \left(\mathbb{H}_*(\Sigma) \otimes \mathbb{C}[z^{\pm 1}][\partial_z],\, \rho \otimes id + id \otimes z^{-t}\partial_z z^t \xi_z\right)$$

Proof. Let $A = \mathbb{C}[\pi_1(\Sigma)]$ and $B = \mathbb{C}[z^{\pm 1}]$. Since A is smooth, it has a resolution by finitely generated projective biomdules, $P_*(A)$. By Theorem 3.13 of [2], we have only to show that there exists a bimodule resolution $P_*(B)$ of B such that

$$\mathrm{Hom}_{(A\otimes B)^e}(P_*(A) \otimes P_*(B), A \otimes B) \cong \mathrm{Hom}_{A^e}(P_*(A), A) \otimes \mathrm{Hom}_{B^e}(P_*(B), B).$$

But this is clear if we choose $P_*(B)$ to be the Koszul bimodule resolution of B. $\square$

Now let $\mathcal{Q}$ be a zigzag consistent dimer model. We would like to use the results of the previous sections and the above characterizations to relate the BV structure on $HH^*(J(\mathcal{Q}))$ to string topology and calculus of Laurent polynomials. To do so, we first characterize the set of volumes for the Calabi-Yau structure of $J(\mathcal{Q})$. Recall the definition of the bimodule resolution $\mathbb{P}_*$ for $J(\mathcal{Q})$ (2.7) and the grading of Remark 3.1.3.

Lemma 3.4.3. *Suppose $\mathcal{Q}$ is a zigzag consistent dimer admitting a perfect matching.*

1. *Up to scaling, the unique volume of $J(\mathcal{Q})$ is the class in* $\mathrm{Ext}^3_{J(\mathcal{Q})^e}(J(\mathcal{Q}), J(\mathcal{Q}) \otimes J(\mathcal{Q}))$

of the map

$$\pi_0 : \mathbb{P}_3 \to J(\mathcal{Q}) \otimes J(\mathcal{Q}), \;\; p \otimes \Phi_0^v \otimes q \mapsto pv \otimes vq$$

Hence, Δ_{π_0} is the unique BV differential induced from the Calabi-Yau structure of $J(\mathcal{Q})$.

2. *For any $\mathcal{P} \in PM(\mathcal{Q})$, the Van den Bergh isomorphism*

$$\mathbb{D}_{\pi_0} : HH^*(J(\mathcal{Q})) \to HH_{3-*}(J(\mathcal{Q})), \;\; \alpha \mapsto \alpha \cap \pi_0$$

has homogeneous $H_1(\Sigma) \times \mathbb{Z}$-bidegree $(0, 1)$.

Proof. By self-duality (2.8), there are $J(\mathcal{Q})$-bimodule isomorphisms

$$\mathrm{Ext}^3_{J(\mathcal{Q})^e}(J(\mathcal{Q}), J(\mathcal{Q}) \otimes J(\mathcal{Q})) \cong H_0(\mathbb{P}_*) \cong J(\mathcal{Q}).$$

The latter is given, for example, by $[1 \otimes_{\Bbbk} 1] \mapsto 1$, where $[-]$ denotes the class in $H_0(\mathbb{P}_*)$. Tracing this element back through the first isomorphism gets the class of

$$\pi_0 : p \otimes \Phi_0^v \otimes q \mapsto pv \otimes vq.$$

Any other volume element is in the $\mathcal{Z}^\times$-orbit of π_0. However, the only units in $J(\mathcal{Q})$ are of the form $\sum_{v \in \mathcal{Q}_0} \epsilon(v)v$ where $\epsilon \in (\mathbb{C}^*)^{\mathcal{Q}_0}$, and among these, the only central units are those with $\epsilon \equiv \lambda$ for some $\lambda \in \mathbb{C}^*$. We conclude Δ_{π_0} is the unique BV differential.

The volume π_0 is homogeneous of bidegree $(0, -1)$ with respect to any perfect matching (Remark 3.1.3). This implies that the quasi-isomorphism (2.17) that descends to $\mathbb{D}_{\pi_0}$ has bidegree $(0, 1)$, as desired. $\square$

With the lemma, degree considerations are enough to deduce which BV structure corresponds to that of $J(\mathcal{Q})$.

Theorem 3.4.4. *Suppose $\mathcal{Q}$ is a zigzag consistent dimer admitting a perfect matching.*

1. *If $\chi(\mathcal{Q}) = 0$, then there is an isomorphism of BV algebras*

$$\big(\widehat{HH^*}(J(\mathcal{Q})),\, \widehat{\Delta}_{\pi_0}\big) \cong \big(\mathbb{C}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}][\partial_x, \partial_y, \partial_z],\, \mathrm{div}\,\big)$$

   where div *is the divergence operator*

$$\mathrm{div} := x\partial_x x^{-1}\xi_x + y\partial_y y^{-1}\xi_y + \partial_z\xi_z.$$

2. *If $\chi(\mathcal{Q}) < 0$, then there is an isomorphism of BV algebras*

$$\big(\widehat{HH^*}(J(\mathcal{Q})),\, \widehat{\Delta}_{\pi_0}\big) \cong \big(\mathbb{H}_*(\Sigma) \otimes \mathbb{C}[z^{\pm 1}][\partial_z],\, \rho \otimes id + id \otimes \partial_z\xi_z\big)$$

   where ρ is the string topology BV operator.

Proof. By Theorem 3.2.5 and Proposition 3.3.1, the map

$$inc^* \circ \Psi^* \circ \widehat{L}^* : \widehat{HH^*}(J(\mathcal{Q})) \longrightarrow HH^*(\mathbb{C}[\pi_1] \otimes \mathbb{C}[z^{\pm 1}])$$

is an isomorphism of BV algebras when the BV structures are induced by the nondegenerate elements π_0 and $\pi_0' := tr_*\Psi_*L_*(\pi_0)$. Clearly, each map tr_*, Ψ_*, and L_* preserves the $H_1(\Sigma) \times \mathbb{Z}$-bigrading with respect to any perfect matching. So by Lemma 3.4.3, π_0' must have bidegree $(0, 1)$.

If $\chi(\mathcal{Q}) < 0$, then under the Hochschild-Kostant-Rosenberg isomorphism (3.4), the only nondegenrate element with bidegree $(0, 1)$ is, up to to scaling, the 3-form $x^{-1}y^{-1}dxdydz$. The resulting BV differential is precisely the divergence operator div with the stated formula.

If $\chi(\mathcal{Q}) < 0$, then under the Kunneth isomorphism, the only nondegenerate element of bidegree $(0,1)$ is, up to scaling, $\pi_s \otimes dz$. By Lemma 3.4.2, the the resulting BV differential on the tensor product is

$$\rho \otimes id + id \otimes \partial_z \xi_z.$$

$\square$

Chapter 4: Hochschild cohomology of the Jacobi algebra

Hereafter, we assume that $\mathcal{Q}$ is a zigzag consistent dimer in a surface Σ and that $\mathcal{Q}$ admits a perfect matching. We analyze in more detail the Hochschild (co)homology groups of the Jacobi algebra. For a dimer embedded in a torus, the Hochschild cohomology is computed explicitly in terms of perfect matchings and zigzag cycles.

In §4.1, we review the relevant facts regarding the center of $J(\mathcal{Q})$. This material appears in many sources, e.g. [9, 5, 7]. For a dimer in a torus, the center is isomorphic to the coordinate ring of the toric variety associated to the matching polygon. However, we opt for a more intrinsic description that elucidates the bigrading by the homology of Σ and any perfect matching (Remark 3.1.3). Subsequently, in §4.2 - 4.3, we describe $HH^1(J(\mathcal{Q}))$ and $HH_0(J(\mathcal{Q}))$, the latter of which is generally found to have ℓ-torsion. Under Van den Bergh duality, $HH_0(J(\mathcal{Q}))$ is isomorphic to $HH^3(J(\mathcal{Q}))$, allowing us to use the BV structure to compute $HH^2(J(\mathcal{Q}))$ in §4.4. Throughout, the Hochschild (co)homology class of an element will be denoted in brackets $[-]$.

4.1 Zeroth Hochschild cohomology

The center is a Morita invariant, so from Theorem 3.1.2, we immediately deduce

$$\mathcal{Z}[\ell^{-1}] \cong \mathcal{Z}(\mathbb{C}[\pi_1(\Sigma)]) \otimes \mathbb{C}[z^{\pm 1}].$$

For a hyperbolic surface, the center of the fundamental group algebra $\mathbb{C}[\pi_1(\Sigma)]$ is trivial, implying that the subalgebra $\mathcal{Z}$ is simply the polynomial algebra in ℓ.

Proposition 4.1.1. *Suppose $\mathcal{Q}$ is a zigzag consistent dimer with $\chi(\mathcal{Q}) < 0$, and suppose $\mathcal{Q}$ admits a perfect matching. Then $HH^0(J(\mathcal{Q})) = \mathbb{C}[\ell]$.*

When $\chi(\mathcal{Q}) = 0$, then the center of $J(\mathcal{Q})[\ell^{-1}]$ is isomorphic to the algebra of Laurent polynomials in three variables,

$$\mathcal{Z}[\ell^{-1}] \cong \mathbb{C}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}].$$

If x and y correspond to generators X and Y of $H_1(\Sigma)$ and $\mathcal{P} \in PM(\mathcal{Q})$ is used to define Ψ in Theorem 3.1.2, the monomial $x^r y^s z^t$ corresponds to a sum of closed paths, one for each vertex, with homology $mX + nY$ and degree t with respect to $\mathcal{P}$. Because in fact any perfect matching can be used to construct Ψ, the central element is homogeneous in all perfect matchings. By Lemma 3.1.1, a closed path $p \in J(\mathcal{Q})[\ell^{-1}]$ is determined uniquely by its homology and degree in any perfect matching, so if f is the homogeneous central element with the same bidegree, then $fh(p) = p$.

Lemma 4.1.2. *Suppose $\mathcal{Q}$ is a zigzag consistent dimer in a torus, and let p be a closed path in $J(\mathcal{Q})[\ell^{-1}]$ at a vertex v. Then there exists a unique $f \in \mathcal{Z}[\ell^{-1}]$ such that $f v = p$. Moreover, f is homogeneous with the same homology class and degree as p in all perfect matchings.*

To describe the subalgebra $\mathcal{Z}$, we use the following important result.

Proposition 4.1.3 ([9] Proposition 6.2; c.f. [7] Lemma 3.18)**.** *Suppose $\mathcal{Q}$ is a zigzag consistent dimer in a torus. For any two vertices $v, w \in \mathcal{Q}_0$ and any homotopy class, there exists a path $v \to w$ in $\mathcal{Q}$ of that homotopy class having degree 0 in some corner matching $\mathcal{P}_i$.*

In other words, for every pair of vertices and every homotopy class, there exists a minimal path (§2.4) of that homotopy class running between the vertices. As an immediate consequence, the paths in $J(\mathcal{Q})$ can be recognized as those in $J(\mathcal{Q})[\ell^{-1}]$ that have nonnegative degree in all perfect matchings.

Corollary 4.1.4. *Suppose $\mathcal{Q}$ is a zigzag consistent dimer in a torus. A path $p \in J(\mathcal{Q})[\ell^{-1}]$ lies in the subalgebra $J(\mathcal{Q})$ if and only if $\deg_{\mathcal{P}}(p) \geq 0$ for all $\mathcal{P} \in PM(\mathcal{Q})$.*

Proof. The forward direction is clear. For the converse, by Proposition 4.1.3, there is path $q : t(p) \to h(p) \in J(\mathcal{Q})$ homotopic to p such that $\deg_{\mathcal{P}}(q) = 0$ for some $\mathcal{P} \in PM(\mathcal{Q})$. Then by Lemma 3.1.1, $p = q\ell^{\deg_{\mathcal{P}}(p)}$. □

The idea of the proof of Proposition 4.1.3 is to construct the minimal path (up to homotopy) from pieces of opposite cycles with consecutive homologies ν_i and ν_{i+1}, for some $i \in \mathbb{Z}/k\mathbb{Z}$. Since by Theorem 2.4.3 these opposite cycles have degree 0 in the corner matching $\mathcal{P}_{i+1}$, the resulting path has degree 0 in $\mathcal{P}_{i+1}$ as well. In particular, it follows from the proof that, if p is a minimal closed path with homology $\eta \in \sigma_i$ (Notation 2.3.7), then $\mathcal{P}_{i+1}$ is the unique perfect matching for which p has degree 0.

Therefore, for a given $\eta \in H_1(\Sigma)$, the sum of the minimal closed paths of homology η is a central element, which we denote as x_η. Just as in the proof of Corollary 4.1.4, it is deduced that every element of $\mathcal{Z}$ with homology η equals $x_\eta \ell^m$ for some $m \in \mathbb{Z}_{\geq 0}$. We summarize these facts in the following.

Proposition 4.1.5. *Suppose $\mathcal{Q}$ is a zigzag consistent dimer in a torus.*

1. *If $\eta \in \gamma_i$, then x_η has degree* 0 *in $\mathcal{P}_i$, $\mathcal{P}_{i+1}$, and all boundary matchings in between them.*

2. *If $\eta \in \sigma_i$, then $\mathcal{P}_{i+1}$ is the unique perfect matching in which x_η has degree* 0.

3. *The center $\mathcal{Z}$ is generated over $\mathbb{C}[\ell]$ by $\{x_\eta \,|\, \eta \in H_1(\Sigma)\}$. As a vector space,*

$$HH^0(J(\mathcal{Q})) \cong \bigoplus_{\substack{\eta\in H_1(\Sigma)\\ m\in\mathbb{Z}_{\geq 0}}} \mathbb{C}\cdot x_\eta\,\ell^m \,\cong\, \mathbb{C}\cdot H_1(\Sigma)\times\mathbb{Z}_{\geq 0}$$

In this light, the relations of $\mathcal{Z}$ are of the form

$$x_\eta \cdot x_\mu = x_{\eta+\mu}\ell^m$$

for some $m \geq 0$. They can be characterized more precisely by realizing $\mathcal{Z}$ as the coordinate ring of the toric variety associated to $MP(\mathcal{Q})$ (see [9, 7]).

4.2 First Hochschild cohomology

Let $\mathrm{Der}_{\Bbbk}(J(\mathcal{Q}))$ be the space of derivations of $J(\mathcal{Q})$ that evaluate trivially on $\Bbbk$. Furthermore, let $\mathrm{Inner}_{\Bbbk}(J(\mathcal{Q}))$ be the subspace of $\mathrm{Der}_{\Bbbk}(J(\mathcal{Q}))$ of *inner derivations*: namely, those of the form

$$ad_p : q \mapsto [p,q] = pq - qp, \quad \forall q \in J(\mathcal{Q})$$

where $p \in \bigoplus_{v\in\mathcal{Q}_0} v\, J(\mathcal{Q})\, v$. We define $\mathrm{Der}_{\Bbbk}(J(\mathcal{Q})[\ell^{-1}])$ and $\mathrm{Inner}_{\Bbbk}(J(\mathcal{Q})[\ell^{-1}])$ similarly for the localized algebra. From the normalized relative bar resolution (2.6), the first Hochschild cohomology can be computed as

$$HH^1(J(\mathcal{Q})) \cong \operatorname{Der}_{\Bbbk}(J(\mathcal{Q}))/\operatorname{Inner}_{\Bbbk}(J(\mathcal{Q})),$$
$$HH^1(J(\mathcal{Q})[\ell^{-1}]) \cong \operatorname{Der}_{\Bbbk}(J(\mathcal{Q})[\ell^{-1}])/\operatorname{Inner}_{\Bbbk}(J(\mathcal{Q})[\ell^{-1}]).$$

An element of $\operatorname{Der}_{\Bbbk}(J(\mathcal{Q}))$ or $\operatorname{Der}_{\Bbbk}(J(\mathcal{Q})[\ell^{-1}])$ is specified by its values on arrows. Hence, a derivation $D \in \operatorname{Der}_{\Bbbk}(J(\mathcal{Q}))$ uniquely extends to an element of $\operatorname{Der}_{\Bbbk}(J(\mathcal{Q})[\ell^{-1}])$ by prescribing

$$D(a^{-1}) = -a^{-1}D(a)a^{-1}, \quad \forall\, a \in \mathcal{Q}_1.$$

Conversely, any derivation of $J(\mathcal{Q})[\ell^{-1}]$ is uniquely determined by its values on $\mathcal{Q}_1$. Thus, we have an injection $\operatorname{Der}_{\Bbbk}(J(\mathcal{Q})) \hookrightarrow \operatorname{Der}_{\Bbbk}(J(\mathcal{Q})[\ell^{-1}])$ respecting the $\mathcal{Z}$-module structures.

The relations of the Jacobi algebra require that a derivation $D \in \operatorname{Der}_{\Bbbk}(J(\mathcal{Q})[\ell^{-1}])$ is homogeneous on boundary cycles: if $a_1 \ldots a_m$ and $b_1 \ldots b_n$ are the positive and negative boundary cycles containing the arrow $a_1 = b_1 \in \mathcal{Q}_1$, then

$$\sum_{i=1}^{m} a_1 \ldots a_{i-1} D(a_i) a_{i+1} \ldots a_m = \sum_{j=1}^{n} b_1 \ldots b_{j-1} D(b_j) b_{j+1} \ldots b_n. \tag{4.1}$$

This constraint suggests a description of $\operatorname{Der}_{\Bbbk}(J(\mathcal{Q})[\ell^{-1}])$ in terms of the lattice N (§2.4). Let $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$, $N_{\mathbb{R}}^{in} = N^{in} \otimes_{\mathbb{Z}} \mathbb{R}$, and $N_{\mathbb{R}}^{out} = N^{out} \otimes_{\mathbb{Z}} \mathbb{R}$.

Lemma 4.2.1. *Suppose $\mathcal{Q}$ is a zigzag consistent dimer. There is an injection of $\mathcal{Z}[\ell^{-1}]$-modules*

$$\mathcal{Z}[\ell^{-1}] \otimes_{\mathbb{R}} N_{\mathbb{R}} \hookrightarrow \operatorname{Der}_{\Bbbk}(J(\mathcal{Q})[\ell^{-1}]),$$

under which $\mathcal{Z}[\ell^{-1}] \otimes_{\mathbb{R}} N_{\mathbb{R}}^{in}$ maps into $\operatorname{Inner}_{\Bbbk}(J(\mathcal{Q})[\ell^{-1}])$. *If $\chi(\mathcal{Q}) = 0$, the map is an isomorphism, under which $\mathcal{Z}[\ell^{-1}] \otimes_{\mathbb{R}} N_{\mathbb{R}}^{out}$ maps onto* $\operatorname{Inner}_{\Bbbk}(J(\mathcal{Q})[\ell^{-1}])$.

Proof. Letting $f \in \mathcal{Z}[\ell^{-1}]$ and $\beta \in N$, define a map

$$D_{f,\beta} : \mathcal{Q}_1 \to J(\mathcal{Q})[\ell^{-1}], \quad a \mapsto f\beta(a)a.$$

For any boundary cycle $a_1 a_2 \dots a_m$,

$$\sum_{i=1}^{m} a_1 \dots a_{i-1} D_{f,\beta}(a_i) a_{i+1} \dots a_m = f\partial(\beta) a_1 a_2 \dots a_m,$$

so $D_{f,\beta}$ satisfies condition (4.1) and defines an element of $\mathrm{Der}_{\Bbbk}(J(\mathcal{Q})[\ell^{-1}])$. We therefore have a map

$$\mathcal{Z}[\ell^{-1}] \otimes_{\mathbb{R}} N_{\mathbb{R}} \to \mathrm{Der}_{\Bbbk}(J(\mathcal{Q})[\ell^{-1}]), \quad f \otimes \beta \mapsto D_{f,\beta}$$

that obviously respects the $\mathcal{Z}[\ell^{-1}]$-module structures and is injective. Furthermore, the coboundary of a vertex

$$\partial(v) : a \mapsto \delta_{v\,h(a)} - \delta_{v\,t(a)}$$

corresponds to ad_v, so $\mathcal{Z}[\ell^{-1}] \otimes_{\mathbb{R}} N_{\mathbb{R}}^{in}$ maps into $\mathrm{Inner}_{\Bbbk}(J(\mathcal{Q})[\ell^{-1}])$.

Now suppose that $\chi(\mathcal{Q}) = 0$ and let $D \in \mathrm{Der}_{\Bbbk}(J(\mathcal{Q})[\ell^{-1}])$. For each arrow $a \in \mathcal{Q}_1$, $D(a)$ is an element of $t(a)J(\mathcal{Q})[\ell^{-1}]h(a)$. Therefore, $D(a)a^{-1}$ is a linear combination of closed paths at $t(a)$, implying $D(a) = f_a \cdot a$ for some $f_a \in \mathcal{Z}[\ell^{-1}]$ by Lemma 4.1.2. The assignment $a \mapsto f_a$ must satisfy condition (4.1),

$$\sum_{a \in \partial F_1} f_a = \sum_{a \in \partial F_2} f_a, \;\; \forall\, F_1, F_2 \in \mathcal{Q}_2,$$

and thus is an element of $\mathcal{Z}[\ell^{-1}] \otimes_{\mathbb{R}} N_{\mathbb{R}}$ mapping to D. Therefore, $\mathcal{Z}[\ell^{-1}] \otimes_{\mathbb{R}} N_{\mathbb{R}} \cong \mathrm{Der}_{\Bbbk}(J(\mathcal{Q})[\ell^{-1}])$.

Finally, if p is a closed path at vertex v, then $p = fv$ for a unique element $f \in \mathcal{Z}[\ell^{-1}]$ by Lemma 4.1.2. Therefore, $ad_p = f ad_v$, implying $\mathcal{Z}[\ell^{-1}] \otimes_{\mathbb{R}} N_{\mathbb{R}}^{in}$ maps onto $\mathrm{Inner}_{\Bbbk}(J(\mathcal{Q})[\ell^{-1}])$.

$\square$

An immediate consequence of the lemma is a confirmation of what is already deduced from the Morita equivalence of Theorem 3.1.2.

Corollary 4.2.2. *Suppose $\mathcal{Q}$ is a zigzag consistent dimer with $\chi(\mathcal{Q}) < 0$. There is an injection of $\mathcal{Z}[\ell^{-1}]$-modules*

$$\mathcal{Z}[\ell^{-1}] \otimes_{\mathbb{R}} N_{\mathbb{R}}^{out} \hookrightarrow HH^1(J(\mathcal{Q})[\ell^{-1}]).$$

If $\chi(\mathcal{Q}) = 0$, then the map is an isomorphism.

The image of N under the map of Lemma 4.2.1 is a lattice of derivations that preserve the $H_1(\Sigma) \times \mathbb{Z}$-bidegree with respect to all perfect matchings. In particular, the image of a perfect matching $\mathcal{P}$ is the derivation

$$E_P : J(\mathcal{Q})[\ell^{-1}] \to J(\mathcal{Q})[\ell^{-1}], \quad E_{\mathcal{P}}(p) = \deg_{\mathcal{P}}(p)\, p.$$

By Lemma 2.4.2, when $\chi(\mathcal{Q}) = 0$, such derivations generate $\mathrm{Der}_{\Bbbk}(J(\mathcal{Q})[\ell^{-1}])$ over $\mathcal{Z}[\ell^{-1}]$, which can be decomposed into $H_1(\Sigma) \times \mathbb{Z}$-homogeneous subspaces,

$$\mathrm{Der}_{\Bbbk}(J(\mathcal{Q})[\ell^{-1}]) \cong \bigoplus_{\substack{\eta \in H_1(\Sigma) \\ m \in \mathbb{Z}}} \mathbb{C} \cdot x_\eta \ell^m \otimes_{\mathbb{R}} N_{\mathbb{R}}. \tag{4.2}$$

The $\mathcal{Z}$-submodule $\mathrm{Der}_{\Bbbk}(J(\mathcal{Q}))$ contains the image of $\mathcal{Z} \otimes_{\mathbb{R}} N_{\mathbb{R}}$ but is generally larger.

Lemma 4.2.3. *Suppose $\mathcal{Q}$ is a zigzag consistent dimer in a torus. As a $\mathcal{Z}$-module,* $\mathrm{Der}_{\Bbbk}(J(\mathcal{Q}))$ *is generated by*

$$\{E_{\mathcal{P}} \,|\, \mathcal{P} \in PM(\mathcal{Q})\} \cup \{x_\eta \ell^{-1} E_{\mathcal{P}_{i+1}} \,| i \in \mathbb{Z}/k\mathbb{Z},\, \eta \in \sigma_i\}.$$

Proof. We determine which elements of $\mathrm{Der}_{\Bbbk}(J(\mathcal{Q})[\ell^{-1}])$ preserve the subalgebra $J(\mathcal{Q})$. A derivation preserves $J(\mathcal{Q})$ if and only if each component in (4.2) preserves $J(\mathcal{Q})$. So without loss of generality, suppose

$$x_\eta \, \ell^m D \in \mathrm{Der}_{\Bbbk}(J(\mathcal{Q}))$$

for some $\eta \in H_1(\Sigma)$, $m \in \mathbb{Z}$, and $D \in N_{\mathbb{R}}$. Since x_η is minimal, it has degree 0 in some corner matching $\mathcal{P}_i$. Then for all $a \in \mathcal{Q}_1$,

$$\deg_{\mathcal{P}_i}(x_\eta \, \ell^m a) = m + \deg_{\mathcal{P}_i}(a).$$

By Corollary 4.1.4, this must be nonnegative to land in $J(\mathcal{Q})$. Thus, we must have $m \geq -1$. Obviously, if m is nonnegative, then the derivation preserves $J(\mathcal{Q})$, but if $m = -1$ and $\eta = 0$, then it does not. So it remains to analyze the case $\eta \neq 0$ and $m \geq -1$.

First, suppose $\eta \in \sigma_i$ for some $i \in \mathbb{Z}/k\mathbb{Z}$, so the corner matching $\mathcal{P}_{i+1}$ is the unique perfect matching evaluating x_η to 0 (Proposition 4.1.5). For all $a \in \mathcal{Q}_1$,

$$\deg_{\mathcal{P}_{i+1}}(x_\eta \, \ell^{-1} a) = -1 + \deg_{\mathcal{P}_i}(a).$$

Hence, $D(a)$ is nonzero only if $a \in \mathcal{P}_{i+1}$, implying $D = E_{\mathcal{P}_{i+1}}$ up to scaling.

Next, suppose $\eta \in \gamma_i$, so x_η has degree 0 in the corner matchings $\mathcal{P}_i$ and $\mathcal{P}_{i+1}$ (Proposition 4.1.5). For all $a \in \mathcal{Q}_1$,

$$\begin{aligned} \deg_{\mathcal{P}_i}(x_\eta \ell^{-1} a) &= -1 + \deg_{\mathcal{P}_i}(a) \\ \deg_{\mathcal{P}_{i+1}}(x_\eta \ell^{-1} a) &= -1 + \deg_{\mathcal{P}_{i+1}}(a). \end{aligned}$$

Consequently, $D(a)$ is nonzero only if $a \in \mathcal{P}_i \cap \mathcal{P}_{i+1}$. However, by Proposition 2.4.3, in any boundary cycle meeting a zigzag cycle of homology class ν_i, there is no arrow in the intersection. Hence, D must evaluate trivially on all boundary cycles, constraining $D(a)$ to be 0 for all $a \in \mathcal{P}_i \cap \mathcal{P}_{i+1}$. Therefore, D is the trivial derivation. $\square$

Recall from Corollary 3.2.6 that the kernel of the localization map $L^* : HH^*(J(\mathcal{Q})) \to HH^*(J(\mathcal{Q})[\ell^{-1}])$ is the ℓ-torsion of $HH^*(J(\mathcal{Q}))$. As we now prove, the first Hochschild cohomology is torsion free. Thus, it is generated over $\mathcal{Z}$ by the rank 3 lattice N^{out} along with the additional derivations of Lemma 4.2.3.

Theorem 4.2.4. *Suppose $\mathcal{Q}$ is a zigzag consistent dimer in a torus. Then $HH^1(J(\mathcal{Q}))$ is the $\mathcal{Z}$-lattice in $HH^1(J(\mathcal{Q})[\ell^{-1}])$ generated by*

$$\{[E_{\mathcal{P}}] \,|\, \mathcal{P} \in PM(\mathcal{Q})\} \cup \{[x_\eta \ell^{-1} E_{\mathcal{P}_{i+1}}] \,| i \in \mathbb{Z}/k\mathbb{Z},\, \eta \in \sigma_i\}.$$

As a vector space,

$$HH^1(J(\mathcal{Q})) = \mathcal{Z} \otimes_{\mathbb{R}} N^{out}_{\mathbb{R}} \;\oplus \bigoplus_{\substack{i \in \mathbb{Z}/k\mathbb{Z} \\ \eta \in \sigma_i}} \mathbb{C} \cdot [x_\eta \, \ell^{-1} E_{\mathcal{P}_{i+1}}]$$

Proof. To prove the localization map $L^* : HH^1(J(\mathcal{Q})) \to HH^1(J(\mathcal{Q})[\ell^{-1}])$ is injective, it suffices to show that no element D of

$$\mathrm{Inner}_{\Bbbk}(J(\mathcal{Q})[\ell^{-1}]) \setminus \mathrm{Inner}_{\Bbbk}(J(\mathcal{Q})) \cong \mathcal{Z}[\ell^{-1}] \otimes_{\mathbb{R}} N^{in}_{\mathbb{R}} \setminus \mathcal{Z} \otimes_{\mathbb{R}} N^{in}_{\mathbb{R}}$$

preserves $J(\mathcal{Q})$. Without loss of generality, we may assume that D is homogeneous in the decomposition (4.2),

$$D = x_\eta \ell^m D'$$

for some $\eta \in H_1(\Sigma)$, $m < 0$, and $D' \in N^{in}_{\mathbb{R}}$. By Lemma 4.2.3, in order for D to preserve $J(\mathcal{Q})$, m must be -1, $\eta \in \sigma_i$ for some $i \in \mathbb{Z}/k\mathbb{Z}$, and D must be (up to scaling) $x_\eta \ell^{-1} E_{\mathcal{P}_{i+1}}$.

However, this derivation is not inner.

□

The derivations $E_{\mathcal{P}}$ provide convenient generators for working with the calculus structure. Under the isomorphism of Theorem 3.4.4, they correspond to weighted Euler vector fields of $\mathbb{C}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}]$.

Proposition 4.2.5. *Suppose $\mathcal{Q}$ is a zigzag consistent dimer in a torus. Then*

1. $\Delta_{\pi_0}(f[E_{\mathcal{P}}]) = (1 + \deg_{\mathcal{P}}(f))f$ *for all homgeneous* $f \in \mathcal{Z}$ *and* $\mathcal{P} \in PM(\mathcal{Q})$*;*
2. $\Delta_{\pi_0}([x_\eta \ell^{-1} E_{\mathcal{P}_{i+1}}]) = 0$ *for all* $i \in \mathbb{Z}/k\mathbb{Z}$ *and* $\eta \in \sigma_i$*.*
3. $\{[E_{\mathcal{P}}], [E_{\mathcal{P}'}]\} = 0$ *for all* $\mathcal{P}, \mathcal{P}' \in PM(\mathcal{Q})$*.*

Proof. By Theorem 3.4.4, the composition

$$\zeta := inc^* \Psi^* L^* : \left(HH^*(J(\mathcal{Q})), \Delta_{\pi_0}\right) \to \left(\mathbb{C}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}][\partial_x, \partial_y, \partial_z], \operatorname{div}\right)$$

is a morphism of BV algebras. Recall that the map Ψ of Theorem 3.1.2 was defined for a choice of basepoint v_0 for the fundamental group and a perfect matching $\mathcal{P}'$. Let p_x and p_y be closed paths in $v_0 J(\mathcal{Q})[\ell^{-1}]v_0$ whose homology classes correspond to generators x and y that, moreover, have degree 0 in $\mathcal{P}'$. Then we see that $[E_{\mathcal{P}}]$ is sent to the Euler vector field weighted by the cohomology class $(n_x, n_y) = (\deg_{\mathcal{P}}(p_x), \deg_{\mathcal{P}}(p_y))$ of $\mathcal{P} - \mathcal{P}'$,

$$\zeta([E_{\mathcal{P}}]) = n_x \, x\partial_x + n_y \, y\partial_y + z\partial_z.$$

Hence,

$$\zeta(\Delta_{\pi_0}([E_{\mathcal{P}}])) = \operatorname{div}(n_x x\partial_x + n_y y\partial_y + z\partial_z) = 1,$$

implying $\Delta_{\pi_0}([E_{\mathcal{P}}]) = 1$. The formula for $\Delta(f[E_{\mathcal{P}}])$ for homogeneous $f \in \mathcal{Z}$ follows from identity (2.19) and the fact that

$$\{f, E_{\mathcal{P}}\} = E_{\mathcal{P}}(f) = \deg_{\mathcal{P}}(f)\, f.$$

For $i \in \mathbb{Z}/k\mathbb{Z}$ and $\eta \in \sigma_i$, a similar computation shows

$$\Delta_{L_*(\pi_0)}\big(L^*([x_\eta \ell^{-1} E_{\mathcal{P}_{i+1}}])\big) = \Delta_{L_*(\pi_0)}\big(x_\eta \ell^{-1} L^*([E_{\mathcal{P}_{i+1}}])\big) = 0.$$

Since $HH^0(J(\mathcal{Q})) \to HH^0(J(\mathcal{Q})[\ell^{-1}])$ is injective, it must be that

$$\Delta_{\pi_0}([x_\eta \ell^{-1} E_{\mathcal{P}_{i+1}}]) = 0.$$

Finally, using the definition (2.13), observe

$$\{E_{\mathcal{P}}, E_{\mathcal{P}'}\}(p) = E_{\mathcal{P}}(E_{\mathcal{P}'}(p)) - E_{\mathcal{P}'}(E_{\mathcal{P}})(p) = \deg_{\mathcal{P}}(p)\deg_{\mathcal{P}'}(p)\, p - \deg_{\mathcal{P}}(p)\deg_{\mathcal{P}'}(p)\, p = 0$$

for all $\mathcal{P}, \mathcal{P}' \in PM(\mathcal{Q})$.

$\square$

4.3 Zeroth Hochschild homology

To describe $HH^0(J(\mathcal{Q}))$, we follow a similar strategy as in the preceding section. Let $\mathcal{R}$ be the vector subspace of $J(\mathcal{Q})$ generated by elements $[p, q] = pq - qp$ where

1. p and q are paths in $J(\mathcal{Q})$,
2. $h(p) = t(q)$ and $h(q) = t(p)$, and
3. $q \notin \Bbbk$.

Define $\mathcal{R}[\ell^{-1}]$ to be the analogous subspace of $J(\mathcal{Q})[\ell^{-1}]$ where p, q are allowed to be paths in the localized algebra. Using the normalized relative bar resolution (2.6), we can compute zeroth Hochschild homology as

$$\begin{aligned} HH_0(J(\mathcal{Q})) &\cong \bigoplus_{v \in \mathcal{Q}_0} vJ(\mathcal{Q})v \,/\, \mathcal{R}, \\ HH_0(J(\mathcal{Q})[\ell^{-1}]) &\cong \bigoplus_{v \in \mathcal{Q}_0} vJ(\mathcal{Q})[\ell^{-1}]v \,/\, \mathcal{R}[\ell^{-1}]. \end{aligned} \tag{4.3}$$

Thus, it is spanned by equivalence classes of closed paths. In particular, if $\chi(\mathcal{Q}) = 0$, then Lemma 4.1.2 implies that $HH_0(J(\mathcal{Q}))$ and $HH_0(J(\mathcal{Q})[\ell^{-1}])$ are generated over the respective centers by the classes of the vertices, $\{[v] \mid v \in \mathcal{Q}_0\}$.

Under the Morita equivalence of Theorem 3.1.2, $HH_0(J(\mathcal{Q}))$ is isomorphic to the zeroth Hochschild homology of $\mathbb{C}[\pi_1(\Sigma)] \otimes \mathbb{C}[z^{\pm 1}]$.

Lemma 4.3.1. *Suppose $\mathcal{Q}$ is a zigzag consistent dimer that admits a perfect matching. Then*

$$HH_0(J(\mathcal{Q})[\ell^{-1}]) \cong \bigoplus_{\gamma \in Conj(\pi_1(\Sigma))} \mathbb{C} \cdot \gamma \otimes \mathbb{C}[z^{\pm 1}]$$

where $Conj(\pi_1(\Sigma))$ is the set of conjugacy classes of the fundamental group. If $\chi(\mathcal{Q}) = 0$, then as $\mathcal{Z}[\ell^{-1}]$-modules,

$$HH_0(J(\mathcal{Q})[\ell^{-1}]) \cong \mathcal{Z}[\ell^{-1}].$$

Consequently, if c and c' are closed paths in $J(\mathcal{Q})[\ell^{-1}]$, then the following are equivalent:

1. $[c] = [c']$ in $HH_0(J(\mathcal{Q})[\ell^{-1}])$;
2. c and c' are homotopic free loops in Σ and have the same degree in a perfect matching;
3. $|c|$ and $|c'|$ are conjugate in $\pi_1(\Sigma)$ and have the same degree in all perfect matchings.

In the third statement, the conjugating loop can be taken as a path in $J(\mathcal{Q})[\ell^{-1}]$ since $\mathcal{Q}$ splits Σ. Hence, $c' = pcp^{-1}$ for some $p : t(c) \to t(c')$, and

$$c' - c = pcp^{-1} - cp^{-1}p = [p, cp^{-1}] \in \mathcal{R}[\ell^{-1}].$$

This affirms $[c] = [c']$ in the description of $HH_0(J(\mathcal{Q})[\ell^{-1}])$ in (4.3).

However, the same statements are not equivalent for $HH_0(J(\mathcal{Q}))$ with c and c' closed paths in $J(\mathcal{Q})$. Indeed, if p is a path in $J(\mathcal{Q})$, then p^{-1} lies in $J(\mathcal{Q})$ if and only if $p = p^{-1}$ is a vertex. Thus, while distinct vertices v and w are equivalent in $HH_0(J(\mathcal{Q})[\ell^{-1}])$, they are not equivalent in $HH_0(J(\mathcal{Q}))$. The difference $[v] - [w]$ lies in the kernel of the localization map $L_* : HH_0(J(\mathcal{Q})) \to HH_0(J(\mathcal{Q})[\ell^{-1}])$.

If $\chi(\mathcal{Q}) = 0$, then $HH_0(J(\mathcal{Q}))$ is generated over $\mathcal{Z}$ by $\{[v] \mid v \in \mathcal{Q}_0\}$. The relations in $\mathcal{R}$ can be recast as equivalence relations among the vertices.

Definition 4.3.2. Let f be an element of $\mathcal{Z}$ with homogeneous $H_1(\Sigma) \times \mathbb{Z}$-bidegree. We say that two vertices $v, v' \in \mathcal{Q}_0$ are *connected through* f if there exist vertices $\{w_j \mid j = 1, \ldots, n\} \subset \mathcal{Q}_0$ and closed paths $\{C_j \mid 1 \leq j < n\} \subset \mathbb{C}\mathcal{Q}$ such that

1. $w_1 = v$ and $w_n = v'$,
2. the image of C_j in $J(\mathcal{Q})$ is $f w_j$, and
3. C_j contains vertex v_{j+1} in addition to v_j for all $1 \leq j < n-1$.

The notion of a path in $J(\mathcal{Q})$ containing a vertex or intersecting another path in $J(\mathcal{Q})$ is not generally well-defined. Hence, the definition uses lifts of paths to the path algebra $\mathbb{C}\mathcal{Q}$.

Lemma 4.3.3. *Suppose $\mathcal{Q}$ is a zigzag consistent dimer in a torus. Let v and v' be vertices in $\mathcal{Q}_0$, and let $f \in \mathcal{Z}$ be a homogeneous element. Then $f\,[v] = f\,[v']$ in $HH_0(J(\mathcal{Q}))$ if and only if v and v' are connected through f.*

Proof. The vector space $\mathcal{R}$ is generated by elements of the form

$$[p, q] = pq - qp = f\, t(p) - f\, h(p)$$

where $f \in \mathcal{Z}$ has the same homology class and degree as pq and qp in all perfect matchings. Taking $P, \mathcal{Q} \in \mathbb{C}\mathcal{Q}$ to be respective representatives of p and q, the closed path PQ shows that $t(p)$ is connected to $h(p)$ through f. Thus, for any two vertices $v, v' \in \mathcal{Q}_0$, if $f\,[v] = f\,[v']$ in $HH_0(J(\mathcal{Q}))$, then v and v' are connected through f.

Conversely, suppose v and v' are connected through f. Let $\{w_j \,|\, j = 1, \ldots, n\}$ and $\{C_j \,|\, 1 \leq j < n\}$ be vertices and closed paths as in the definition. For each $1 \leq j < n$, we may write $C_j = P_j Q_j$ where $P_j \in \mathbb{C}\mathcal{Q}$ is a path from w_j to w_{j+1} and $Q_j \in \mathbb{C}\mathcal{Q}$ is a path from w_{j+1} to w_j. Let p_j and q_j be their images in $J(\mathcal{Q})$. Then

$$f\, v - f\, v' = \sum_{j=1}^{n-1} f(w_j - w_{j+1}) = \sum_{j=1}^{n-1} p_j q_j - q_j p_j = \sum_{j=1}^{n-1} [p_j, q_j] \in \mathcal{R}$$

Hence, $f\,[v] = f\,[v']$. $\square$

As a result, the f-homogeneous subspace of $HH_0(J(\mathcal{Q}))$ has dimension equal to the number of equivalence classes of vertices connected through f. Clearly, neighboring vertices are connected through ℓ, and thus any two vertices are connected through any multiple of ℓ. The ℓ-torsion, then, is concentrated in subspaces corresponding to the minimal elements of $\mathcal{Z}$. For each $\eta \in H_1(\Sigma)$, let r_η be the number of equivalence classes of vertices connected through x_η, and let $v_1^\eta, v_2^\eta, \ldots, v_{r_\eta}^\eta$ be a full list of representative vertices. Note that $r_0 = \#\mathcal{Q}_0$ by the discussion following Lemma 4.3.1.

Proposition 4.3.4. *Suppose $\mathcal{Q}$ is a zigzag consistent dimer in a torus.*

1. *For all $i \in \mathbb{Z}/k\mathbb{Z}$ and $\eta \in \gamma_i$, r_η is at least the number of zigzag cycles of homology ν_i.*

2. The ℓ-torsion of $HH_0(J(\mathcal{Q}))$ is

$$tor_\ell(HH_0(J(\mathcal{Q}))) = \bigoplus_{\substack{\eta \in H_1(\Sigma) \\ 1 < j \leq r_\eta}} \mathbb{C} \cdot x_\eta([v_1^\eta] - [v_j^\eta])$$

3. As a vector space,

$$HH_0(J(\mathcal{Q})) \cong \mathcal{Z} \cdot \ell \ \oplus \bigoplus_{\substack{\eta \in H_1(\Sigma) \\ 1 \leq j \leq r_\eta}} \mathbb{C} \cdot x_\eta[v_j^\eta].$$

Proof. For the first statement, suppose $\mathcal{Q}$ has two zigzag cycles Z_1 and Z_2 of homology $-\nu_i$. Fix a fundamental domain $\mathbb{U}$ in the universal cover, and let $\tilde{Z}_1$ and $\tilde{Z}_2$ be lifts of Z_1 and Z_2 to zigzag flows incident to $\mathbb{U}$. By Proposition 2.3.6, they are parallel. Let v be a vertex whose lift $\tilde{v}$ in $\mathbb{U}$ lies between $\tilde{Z}_1$ and $\tilde{Z}_2$, and let w be a vertex whose lift $\tilde{w}$ in $\mathbb{U}$ lies outside the region between $\tilde{Z}_1$ and $\tilde{Z}_2$. If $\eta \in \gamma_i$ and v, w are connected through x_η, then a path $p : v \to w$ can be constructed from arrows contained in representatives of x_η at various vertices. In particular, p has degree 0 in the corner matchings $\mathcal{P}_i$ and $\mathcal{P}_{i+1}$. But a lift of $\tilde{p}$ to $\mathbb{U}$ must intersect either $\tilde{Z}_1$ or $\tilde{Z}_2$ in an arrow, implying p intersects either Z_1 or Z_2 in an arrow. This contradicts Theorem 2.4.3. Thus, v and w are not connected through x_η. The argument can be generalized to prove the statement for any number of zigzag cycles.

The second and third statements are immediate from the fact that any two vertices are connected through ℓ. $\square$

4.4 Second and third Hochschild cohomology

Suppose $\mathcal{Q}$ is a zigzag consistent dimer for which every arrow is contained in a perfect matching. Then the sum

$$\sum_{\mathcal{P}\in PM(\mathcal{Q})} \mathcal{P}$$

is a strictly positive element of N^+, implying N is generated by $PM(\mathcal{Q})$. With respect to the grading imparted by the sum, $J(\mathcal{Q})$ is a nonnegatively graded and connected $\Bbbk$-algebra. Consequently, by Lemma 3.6.1 of [17], the rows of the following commutative diagram are exact:

$$\begin{tikzcd} \Bbbk \arrow[r] & HH^3(J(\mathcal{Q})) \arrow[r, "\Delta_{\pi_0}"] \arrow[d, "\mathbb{D}"] & HH^2(J(\mathcal{Q}))) \arrow[r, "\Delta_{\pi_0}"] \arrow[d, "\mathbb{D}"] & HH^1(J(\mathcal{Q})) \arrow[r, "\Delta_{\pi_0}"] \arrow[d, "\mathbb{D}"] & HH^0(J(\mathcal{Q})) \arrow[r, "\Delta_{\pi_0}"] \arrow[d, "\mathbb{D}"] & 0 \\ \Bbbk \arrow[r] & HH_0(J(\mathcal{Q})) \arrow[r, "B"] & HH_1(J(\mathcal{Q})) \arrow[r, "B"] & HH_2(J(\mathcal{Q})) \arrow[r, "B"] & HH_3(J(\mathcal{Q})) \arrow[r, "B"] & 0. \end{tikzcd} \tag{4.4}$$

The map $\Bbbk \to HH_0(J(\mathcal{Q}))$ is the inclusion sending $v \in \mathcal{Q}_0$ to $[v]$. The second cohomology decomposes as a vector space as

$$\begin{aligned} HH^2(J(\mathcal{Q})) \;&\cong\; Im\big(\Delta_{\pi_0} : HH^3(J(\mathcal{Q})) \to HH^2(J(\mathcal{Q}))\big) \\ &\oplus\; Ker\big(\Delta_{\pi_0} : HH^1(J(\mathcal{Q})) \to HH^0(J(\mathcal{Q}))\big). \end{aligned} \tag{4.5}$$

When $\chi(\mathcal{Q}) = 0$, we can therefore use the description of $HH^1(J(\mathcal{Q}))$ and $HH^3(J(\mathcal{Q})) \cong HH_0(J(\mathcal{Q}))$ to deduce the structure of $HH^2(J(\mathcal{Q}))$. First, we identify elements of $HH^2(J(\mathcal{Q}))$ corresponding to the rays $\{\gamma_i \,|\, i \in \mathbb{Z}/k\mathbb{Z}\}$.

Lemma 4.4.1. *Suppose $\mathcal{Q}$ is a zigzag consistent dimer in the torus. For all $i \in \mathbb{Z}/k\mathbb{Z}$ and $\eta \in \gamma_i$, the element $x_\eta \ell^{-1} E_{\mathcal{P}_i} \cup E_{\mathcal{P}_{i+1}}$ is a Hochschild 2-cocycle of $J(\mathcal{Q})$.*

Proof. As the cup product of derivations, the element $x_\eta \ell^{-1} E_{\mathcal{P}_i} \cup E_{\mathcal{P}_{i+1}}$ is a 2-cocycle of

$J(\mathcal{Q})[\ell^{-1}]$. Thus, it suffices to show the map restricts to a map $J(\mathcal{Q}) \otimes_{\Bbbk} J(\mathcal{Q}) \to J(\mathcal{Q})$. Let p and q be paths in $J(\mathcal{Q})$ such that $h(p) = t(q)$. Observe

$$x_\eta \ell^{-1} E_{\mathcal{P}_i} \cup E_{\mathcal{P}_{i+1}}(p,q) = -\deg_{\mathcal{P}_i}(p)\deg_{\mathcal{P}_{i+1}}(q) x_\eta \ell^{-1} pq,$$

which is nonzero if and only if $\deg_{\mathcal{P}_i}(p)$ and $\deg_{\mathcal{P}_{i+1}}(q)$ are positive. If either p or q is represented by a path containing an arrow in $\mathcal{P}_i \cap \mathcal{P}_{i+1}$, then by Theorem 2.4.3, $\deg_{\mathcal{P}}(pq) > 0$ in all boundary matchings $\mathcal{P}$ on the component of $MP(\mathcal{Q})$ between $\mathcal{P}_i$ and $\mathcal{P}_{i+1}$. Therefore,

$$\deg_{\mathcal{P}}(x_\eta \ell^{-1} pq) \geq 0 \quad \forall \mathcal{P} \in PM(\mathcal{Q}),$$

implying by Corollary 4.1.4 that $x_\eta \ell^{-1} pq \in J(\mathcal{Q})$. Otherwise, p must be represented by a path containing a zig a of a zigzag cycle Z_1, and q must be represented by a path containing a zag b of a zigzag cycle Z_2, both of homology ν_i. The part of pq running from a to b must also contain a zag of Z_1 or a zig of Z_2. Then pq contains a zig and a zag of a single zigzag cycle of homology ν_i, so again $\deg_{\mathcal{P}}(pq) > 0$ in all boundary matchings $\mathcal{P}$ between $\mathcal{P}_i$ and $\mathcal{P}_{i+1}$. As before, by Corollary 4.1.4, $x_\eta \ell^{-1} pq \in J(\mathcal{Q})$. $\square$

Taking products and comparing with Proposition 4.3.4, we arrive at a description of third cohomology. In the notation of §4.3, let $\tau = \mathbb{D}^{-1}([v_0^0])$.

Proposition 4.4.2. *Suppose $\mathcal{Q}$ is a zigzag consistent dimer in a torus. As a vector space,*

$$\begin{aligned} HH^3(J(\mathcal{Q})) &\cong \mathcal{Z} \otimes_{\mathbb{R}} N_{\mathbb{R}}^{out} \wedge N_{\mathbb{R}}^{out} \wedge N_{\mathbb{R}}^{out} \\ &\oplus \bigoplus_{\substack{i \in \mathbb{Z}/k\mathbb{Z} \\ \eta \in \sigma_i}} \mathbb{C} \cdot [x_\eta \ell^{-1} E_{\mathcal{P}_{i+1}} \cup E_{\mathcal{P}_{i+2}} \cup E_{\mathcal{P}_{i+3}}] \\ &\oplus \bigoplus_{\substack{i \in \mathbb{Z}/k\mathbb{Z} \\ \eta \in \gamma_i}} \mathbb{C} \cdot [x_\eta \ell^{-1} E_{\mathcal{P}_i} \cup E_{\mathcal{P}_{i+1}} \cup E_{\mathcal{P}_{i+2}}] \\ &\oplus \ \mathbb{C}\tau \ \oplus \ tor_\ell(HH_0(J(\mathcal{Q}))). \end{aligned}$$

Proof. By Theorem 4.2.4, the subalgebra generated under cup products by $HH^1(J(\mathcal{Q}))$ over $HH^0(J(\mathcal{Q}))$ is torsion free. In degree 3, we thereby obtain $\mathcal{Z} \otimes_{\mathbb{R}} N_{\mathbb{R}}^{out} \wedge N_{\mathbb{R}}^{out} \wedge N_{\mathbb{R}}^{out}$ and $[x_\eta \ell^{-1} E_{\mathcal{P}_{i+1}} \cup E_{\mathcal{P}_{i+2}} \cup E_{\mathcal{P}_{i+3}}]$ for each $i \in \mathbb{Z}/k\mathbb{Z}$ and $\eta \in \sigma_i$. From Lemma 4.4.1, we also obtain, for each $i \in \mathbb{Z}/k\mathbb{Z}$ and $\eta \in \gamma_i$, the element

$$[x_\eta \ell^{-1} E_{\mathcal{P}_i} \cup E_{\mathcal{P}_{i+1}}] \cup [E_{\mathcal{P}_{i+2}}] = [x_\eta \ell^{-1} E_{\mathcal{P}_i} \cup E_{\mathcal{P}_{i+1}} \cup E_{\mathcal{P}_{i+2}}].$$

Multiplying it by ℓ lands in $\mathcal{Z} \otimes_{\mathbb{R}} N_{\mathbb{R}}^{out} \wedge N_{\mathbb{R}}^{out} \wedge N_{\mathbb{R}}^{out}$, so it is not torsional. By Lemma 3.4.3, Van den Bergh duality $\mathbb{D}$ has $H_1(\Sigma) \times \mathbb{Z}$-bidegree $(0,1)$ with respect to all perfect matchings. So for homogeneous $f \in \mathcal{Z}$, the image of the homogeneous $f\ell^{-1}$-subspace of $HH^3(J(\mathcal{Q}))$ is the homogeneous f-subspace of $HH_0(J(\mathcal{Q}))$. Along with, say, $[v_0^0] \in \Bbbk \hookrightarrow HH_0(J(\mathcal{Q}))$, the images under $\mathbb{D}$ of the elements of $HH^3(J(\mathcal{Q}))$ thus identified span a subspace complementary to $tor_\ell(HH_0(J(\mathcal{Q})))$. $\square$

To describe second cohomology, the decomposition (4.5) can now be used for a dimension count. Let

$$tor_\ell^+(HH_0(J(\mathcal{Q}))) = tor_\ell(HH_0(J(\mathcal{Q}))) \cap HH_0(J(\mathcal{Q})) \setminus \Bbbk,$$

the subspace spanned by torsional elements that have positive degree in some perfect matching.

Theorem 4.4.3. *Suppose $\mathcal{Q}$ is a zigzag consistent dimer in a torus. As a vector space,*

$$\begin{aligned} HH^2(J(\mathcal{Q})) &\cong \mathcal{Z} \otimes_{\mathbb{R}} N_{\mathbb{R}}^{out} \wedge N_{\mathbb{R}}^{out} \\ &\oplus \bigoplus_{\substack{i \in \mathbb{Z}/k\mathbb{Z} \\ \eta \in \sigma_i}} \mathbb{C} \cdot \{[x_\eta \ell^{-1} E_{\mathcal{P}_{i+1}} \cup E_{\mathcal{P}_{i+2}}],\, [x_\eta \ell^{-1} E_{\mathcal{P}_{i+1}} \cup E_{\mathcal{P}_{i+3}}]\} \\ &\oplus \bigoplus_{\substack{i \in \mathbb{Z}/k\mathbb{Z} \\ \eta \in \gamma_i}} \mathbb{C} \cdot [x_\eta \ell^{-1} E_{\mathcal{P}_i} \cup E_{\mathcal{P}_{i+1}}] \\ &\oplus tor_\ell^+\big(HH_0(J(\mathcal{Q}))\big) \end{aligned}$$

Proof. Once again, by Theorem 4.2.4, the degree 2 part of the subalgebra generated by $HH^1(J(\mathcal{Q}))$ over $HH^0(J(\mathcal{Q}))$ is torsion free. This accounts for $\mathcal{Z} \otimes_{\mathbb{R}} N_{\mathbb{R}}^{out} \wedge N_{\mathbb{R}}^{out}$ and the elements $[x_\eta \ell^{-1} E_{\mathcal{P}_{i+1}} \cup E_{\mathcal{P}_{i+2}}]$ and $[x_\eta \ell^{-1} E_{\mathcal{P}_{i+1}} \cup E_{\mathcal{P}_{i+3}}]$ for $i \in \mathbb{Z}/k\mathbb{Z}$ and $\eta \in \sigma_i$. By Lemma 4.4.1, for each $i \in \mathbb{Z}/k\mathbb{Z}$ and $\eta \in \gamma_i$, we also have the element $[x_\eta \ell^{-1} E_{\mathcal{P}_i} \cup E_{\mathcal{P}_{i+1}}]$. Upon multiplying by ℓ, it lands in $\mathcal{Z} \otimes_{\mathbb{R}} N_{\mathbb{R}}^{out} \wedge N_{\mathbb{R}}^{out}$ and so is not a torsional element.

The torsion of $HH^*(J(\mathcal{Q}))$ is the kernel of the localization map $L^* : HH^*(J(\mathcal{Q})) \to HH^*(J(\mathcal{Q})[\ell^{-1}])$, which is a morphism of BV algebras by Theorem 3.2.5. Hence, the torsion of $HH^2(J(\mathcal{Q}))$ is the image under Δ_{π_0} of the torsion of $HH^3(J(\mathcal{Q}))$. The latter is isomorphic to $tor_\ell(HH_0(J(\mathcal{Q})))$ under $\mathbb{D}$, and the kernel of $B : HH_0(J(\mathcal{Q})) \to HH_1(J(\mathcal{Q}))$ is precisely $\Bbbk$. Therefore, the torsion of $HH^2(J(\mathcal{Q}))$ is isomorphic to $tor_\ell^+(HH_0(J(\mathcal{Q})))$.

Let f be a homogeneous element of $\mathcal{Z}$. The BV operator Δ_{π_0} preserves the $H_1(\Sigma) \times \mathbb{Z}$-bigrading with respect to all perfect matchings. Hence, by the exactness of (4.4), the dimension of the $f\ell^{-1}$-homogeneous subspace of $HH^2(J(\mathcal{Q}))$, modulo torsion, is

- 3 if $f \in \mathcal{Z} \cdot \ell$;
- 2 if $f = x_\eta$ with $\eta \in \sigma_i$, for all $i \in \mathbb{Z}/k\mathbb{Z}$;
- 1 if $f = x_\eta$ with $\eta \in \gamma_i$, for all $i \in \mathbb{Z}/k\mathbb{Z}$;
- 0 if $f = 1$.

Consequently, the elements of $HH^2(J(\mathcal{Q}))$ we have identified, along with torsion, span all of $HH^2(J(\mathcal{Q}))$. □

As is well-known, the second Hochschild cohomology of an associative algebra classifies its infinitesimal deformations up to gauge equivalence (see e.g. [4]). The kernel of

$$\Delta_{\pi_0} : HH^2(J(\mathcal{Q})) \to HH^1(J(\mathcal{Q}))$$

consists of the Hochschild classes of deformations of $J(\mathcal{Q})$ within the class of Calabi-Yau algebras. By exactness, this space equals the image of $\Delta_{\pi_0} : HH^3(J(\mathcal{Q})) \to HH^2(J(\mathcal{Q}))$, which corresponds to first-order deformations of the superpotential of $J(\mathcal{Q})$ ([18] Proposition 2.1.5). More precisely, if $u \in HH_0(J(\mathcal{Q}))$ and $U \in \mathbb{C}\mathcal{Q}/[\mathbb{C}\mathcal{Q}, \mathbb{C}\mathcal{Q}]$ is a lift of u, then the element $\Delta_{\pi_0} \mathbb{D}^{-1}(u) \in HH^2(J(\mathcal{Q}))$ is the Hochshild class of the deformation

$$J(\mathcal{Q}, \Phi_0 + \hbar u) := \frac{\mathbb{C}\mathcal{Q}[\hbar]/(\hbar^2)}{\big(\partial_a(\Phi_0 + \hbar U) \,|\, a \in \mathcal{Q}_1\big).}$$

Lifting the elements of $HH_0(J(\mathcal{Q}))$ in Proposition 4.3.4 to $\mathbb{C}\mathcal{Q}/[\mathbb{C}\mathcal{Q}, \mathbb{C}\mathcal{Q}]$, we can thus describe all Calabi-Yau deformations of $J(\mathcal{Q})$. In particular, the second cohomology classes of the Calabi-Yau deformations corresponding to the non-torsional elements of Proposition 4.4.2 can be computed explicitly by (2.19), (2.14), and Proposition 4.2.5. We obtain

$$\begin{aligned}\Delta_{\pi_0}([fE_{\mathcal{P}_0} \cup E_{\mathcal{P}_1} \cup E_{\mathcal{P}_2}]) &= ((1 + \deg_{\mathcal{P}_0}(f))[fE_{\mathcal{P}_1} \cup E_{\mathcal{P}_2}] - (1 + \deg_{\mathcal{P}_1}(f))[fE_{\mathcal{P}_0} \cup E_{\mathcal{P}_2}] \\ &+ (1 + \deg_{\mathcal{P}_2}(f))[E_{\mathcal{P}_0} \cup E_{\mathcal{P}_1}]\end{aligned}$$

for all homogeneous $f \in \mathcal{Z}$,

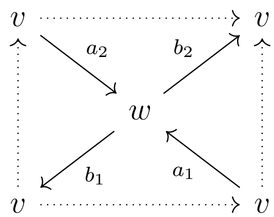
$$\begin{aligned}\Delta_{\pi_0}([x_\eta \ell^{-1} E_{\mathcal{P}_{i+1}} \cup E_{\mathcal{P}_{i+2}} \cup E_{\mathcal{P}_{i+3}}]) &= -\deg_{\mathcal{P}_{i+2}}(x_\eta)[x_\eta \ell^{-1} E_{\mathcal{P}_{i+1}} \cup E_{\mathcal{P}_{i+3}}] \\ &+ \deg_{\mathcal{P}_{i+3}}(x_\eta)[x_\eta \ell^{-1} E_{\mathcal{P}_{i+1}} \cup E_{\mathcal{P}_{i+2}}]\end{aligned}$$

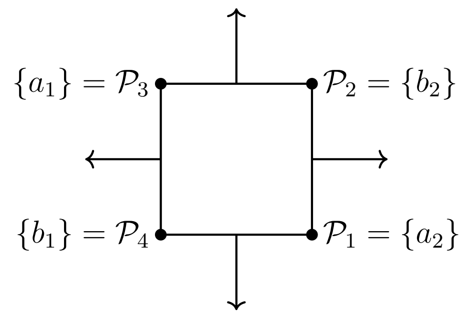
for all $i \in \mathbb{Z}/k\mathbb{Z}$ and $\eta \in \sigma_i$, and

$$\Delta_{\pi_0}([x_\eta \ell^{-1} E_{\mathcal{P}_i} \cup E_{\mathcal{P}_{i+1}} \cup E_{\mathcal{P}_{i+2}}]) = \deg_{\mathcal{P}_{i+2}}(x_\eta)[x_\eta \ell^{-1} E_{\mathcal{P}_i} \cup E_{\mathcal{P}_{i+1}}]$$

for all $i \in \mathbb{Z}/k\mathbb{Z}$ and $\eta \in \gamma_i$.

4.5 Example: mirror to 4-punctured sphere





Consider the zigzag consistent dimer in a torus illustrated above. There are four zigzag cycles, which coincide with the opposite cycles, and four perfect matchings, one for each arrow. The dimer dual has genus 0 and four vertices, determining the sphere with 4-punctures.

The minimal central elements corresponding to the opposite cycles are

$$\begin{aligned} x_1 &:= a_1 b_1 + b_1 a_1 \\ x_2 &:= a_2 b_1 + b_1 a_2 \\ x_3 &:= a_2 b_2 + b_2 a_2 \\ x_4 &:= a_1 b_2 + b_2 a_1. \end{aligned}$$

In this case, they generate the center $\mathcal{Z}$ as algebra, with the single relation $\ell = x_1 x_3 = x_2 x_4$, so

$$HH^0(J(\mathcal{Q})) \cong \mathcal{Z} \cong \mathbb{C}[x_1, x_2, x_3, x_4]/(x_1 x_3 - x_2 x_4).$$

By Theorem 4.2.4, the first Hochschild cohomology is given by

$$HH^1(J(\mathcal{Q})) = \mathcal{Z} \otimes_{\mathbb{R}} N_{\mathbb{R}}^{out} \;\oplus \bigoplus_{\substack{i \in \mathbb{Z}/4\mathbb{Z} \\ m,n>0}} \mathbb{C} \cdot [x_i^m x_{i+1}^n \ell^{-1} E_{\mathcal{P}_{i+1}}].$$

Every closed path in $\mathcal{Q}$ contains both vertices v and w, and so the torsion of $HH_0(J(\mathcal{Q}))$ is simply

$$tor_\ell(HH_0(J(\mathcal{Q}))) = \mathbb{C} \cdot \{[v] - [w]\} \subset \Bbbk,$$

Therefore, according to Proposition 4.3.4,

$$HH_0(J(\mathcal{Q})) \cong \mathcal{Z} \cdot [v] \oplus \mathbb{C} \cdot [w].$$

which is isomorphic under Van den Bergh duality $\mathbb{D}$ to $HH^3(J(\mathcal{Q}))$. Writing $\tau = \mathbb{D}^{-1}([v])$, we can present third cohomology as in Proposition 4.4.2,

$$\begin{aligned} HH^3(J(\mathcal{Q})) \ \cong\ & \mathcal{Z} \otimes_{\mathbb{R}} N_{\mathbb{R}}^{out} \wedge N_{\mathbb{R}}^{out} \wedge N_{\mathbb{R}}^{out} \\ \oplus & \bigoplus_{\substack{i \in \mathbb{Z}/4\mathbb{Z} \\ m,n>0}} \mathbb{C} \cdot [x_i^m x_{i+1}^n \ell^{-1} E_{\mathcal{P}_{i+1}} \cup E_{\mathcal{P}_{i+2}} \cup E_{\mathcal{P}_{i+3}}] \\ \oplus & \bigoplus_{\substack{i \in \mathbb{Z}/4\mathbb{Z} \\ m>0}} \mathbb{C} \cdot [x_i^m E_{\mathcal{P}_i} \cup E_{\mathcal{P}_{i+1}} \cup E_{\mathcal{P}_{i+2}}] \\ \oplus & \ \mathbb{C}\tau \oplus tor_\ell(HH_0(J(\mathcal{Q}))). \end{aligned}$$

Since the torsion of $HH_0(J(\mathcal{Q}))$ is concentrated in $\Bbbk$, the second cohomology is torsion free. Then by Theorem 4.4.3,

$$\begin{aligned} HH^2(J(\mathcal{Q})) \ =\ & \mathcal{Z} \otimes_{\mathbb{R}} N_{\mathbb{R}}^{out} \wedge N_{\mathbb{R}}^{out} \\ \oplus & \bigoplus_{\substack{i \in \mathbb{Z}/4\mathbb{Z} \\ m,n>0}} \mathbb{C} \cdot \{[x_i^m x_{i+1}^n \ell^{-1} E_{\mathcal{P}_{i+1}} \cup E_{\mathcal{P}_{i+2}}], [x_i^m x_{i+1}^n \ell^{-1} E_{\mathcal{P}_{i+1}} \cup E_{\mathcal{P}_{i+3}}]\} \\ \oplus & \bigoplus_{\substack{i \in \mathbb{Z}/4\mathbb{Z} \\ m>0}} \mathbb{C} \cdot [x_i^m \ell^{-1} E_{\mathcal{P}_i} \cup E_{\mathcal{P}_{i+1}}]. \end{aligned}$$

Chapter 5: Hochschild cohomology of matrix factorizations

According to Theorem 2.7.2, the compactly supported Hochschild cohomology of the matrix factorization category $MF(J(\mathcal{Q}), \ell)$ is isomorphic to that of the curved algebra $J(\mathcal{Q})_\ell$. In §5.1, the latter is computed by a spectral sequence of a mixed double complex associated to the compactly supported Hochschild cochain complex. We follow [12], where the computation is done for a Landau-Ginzburg model of a commutative local algebra with isolated hypersurface singularity. The result is a complex given by the ordinary cohomology $HH^*(J(\mathcal{Q}))$ equipped with differential $\{\ell, -\}$. In §5.2, the homology of this complex is computed explicitly in the case $\chi(\mathcal{Q}) = 0$.

5.1 The spectral sequence

We set up the computation for Borel–Moore homology as in [12]; the computation for compactly supported cohomology will be adapted easily from it. Let A be an associative algebra, $\mathcal{Z}(A)$ be the center of A, and h be an element of $\mathcal{Z}(A)$. In the notation of §2.6, A_h is a curved associative algebra. A mixed double complex supported above the diagonal is obtained by letting

$$C_{i,j} = \begin{cases} A \otimes A^{\otimes(j-i)} & \text{if } j \geq i \\ 0 & \text{otherwise,} \end{cases}$$

equipped with d_A (2.9) as the vertical differential (of homological degree -1) and $\mathcal{L}_h = d_h$ (2.9) as the horizontal differential (of homological degree $+1$).

$$
\begin{tikzcd}
\cdot & \cdots \arrow[d] & \cdots \arrow[d] & \cdots \arrow[d] & & \\
\cdots & A \otimes A^{\otimes 3} \arrow[l] \arrow[d] & A \otimes A^{\otimes 2} \arrow[l] \arrow[d] & A \otimes A \arrow[l] \arrow[d] & A \arrow[l] \arrow[d] & \cdots \arrow[l] \\
\cdots & A \otimes A^{\otimes 2} \arrow[l] \arrow[d] & A \otimes A \arrow[l] \arrow[d] & A \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & \cdots \arrow[l] \\
\cdots & A \otimes A \arrow[l] \arrow[d] & A \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & \cdots \arrow[l] & \\
\cdots & A \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & & \\
\cdots & \cdots & \cdots & \cdots & &
\end{tikzcd}
$$

Note that $C_{*,*}$ is 2-periodic along the main diagonal. The even and odd degrees of the direct product totalization $\mathrm{Tot}^{\Pi}(C_{*,*})$ coincide respectively with the even and odd degrees of the Borel–Moore chain complex $C_*^{BM}(A_h)$. Therefore,

$$H_*(\mathrm{Tot}^{\Pi}(C_{*,*})) = H_*^{BM}(A_h) \bmod 2.$$

The periodicity can be leveraged to reduce the computation to the first quadrant. Let $C_{*,*}^{+}$ be the subcomplex

$$C_{*,*}^{+} = \begin{cases} A \otimes A^{\otimes (j-i)} & \text{if } j \geq i \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

For $r \in \mathbb{N}$, let $C_{*,*}^{+}[2r]$ denote the complex shifted by $2r$ along the diagonal in the direction of the third quadrant. If $s > r$, then $C_{*,*}^{+}[2r]$ is a quotient of $C_{*,*}^{+}[2s]$ by the subcomplex consisting of terms $C_{i,j}$ for which $-2s \leq i < -2r$ or $-2s \leq j < -2r$. Thus, there are quotient maps

$$\mathrm{Tot}(C^+_{*,*})[2s] \to \mathrm{Tot}(C^+_{*,*})[2r], \quad s > r \geq 0.$$

Here, we ignore the distinction between direct product and direct sum totalizations since they coincide for these complexes. The inverse system $\{\mathrm{Tot}(C^+_{*,*})[2r] \mid r \in \mathbb{N}\}$ has limit $\mathrm{Tot}^{\Pi}(C_{*,*})$, and because it satisfies the Mittag-Leffler condition ([37] Theorem 3.5.8), there is an exact sequence

$$0 \to \varprojlim{}^1 H_{i+1}(\mathrm{Tot}(C^+_{*,*})[2r]) \to H_i(\mathrm{Tot}^{\Pi}(C_{*,*})) \to \varprojlim H_i(\mathrm{Tot}(C^+_{*,*})[2r]) \to 0. \tag{5.1}$$

for all $i \in \mathbb{Z}$. The symbol $\varprojlim^1$ denotes the first derived functor of the inverse limit. Since $H_i(\mathrm{Tot}(C^+_{*,*})[2r]) = H_{i+2r}(\mathrm{Tot}(C^+_{*,*}))$, the Borel–Moore Hochschild homology is determined from (5.1) by the homology of $\mathrm{Tot}(C^+_{*,*})$.

Now, suppose $\mathcal{Q}$ is a zigzag consistent dimer admitting a perfect matching, and take $A = J(\mathcal{Q})$ and $h = \ell$. To compute the homology of $\mathrm{Tot}(C^+_{*,*})$, we first need a lemma characterizing some terms of the spectral sequence.

Lemma 5.1.1. *Suppose $\mathcal{Q}$ is a zigzag consistent dimer that admits a perfect matching.*

1. *If $\chi(\mathcal{Q}) < 0$, then*

$$HH^0(J(\mathcal{Q}))/\{\ell, HH^1(J(\mathcal{Q}))\} \cong \mathbb{C}.$$

2. *If $\chi(\mathcal{Q}) = 0$, then*

$$HH^0(J(\mathcal{Q}))/\{\ell, HH^1(J(\mathcal{Q}))\} \cong \mathbb{C}[x_{\nu_1}, \ldots, x_{\nu_k}]/\big(x_{\nu_i}x_{\nu_j} \mid i \neq j\big)$$

Proof. For any $\mathcal{P} \in PM(\mathcal{Q})$ and $f \in \mathcal{Z}[\ell^{-1}]$ such that $[fE_{\mathcal{P}}] \in HH^1(J(\mathcal{Q}))$, observe

$$\{\ell, [fE_{\mathcal{P}}]\} = fE_{\mathcal{P}}(\ell) = f\ell.$$

When $\chi(\mathcal{Q}) < 0$, the center $\mathcal{Z}$ is $\mathbb{C}[\ell]$ by Proposition 4.1.1. Letting f range in $\mathbb{C}[\ell]$, we see that $\ell \cdot \mathbb{C}[\ell] \subset \{\ell, HH^1(J(\mathcal{Q}))\}$. On the other hand, the image of ℓ under any derivation of $J(\mathcal{Q})$ must have degree at least 2 in the filtration by path length. Therefore, the reverse inclusion holds, so

$$HH^0(J(\mathcal{Q}))\,/\{\ell, HH^1(J(\mathcal{Q}))\} = \mathbb{C}[\ell]/(\ell) \cong \mathbb{C}$$

When $\chi(\mathcal{Q}) = 0$, Theorem 4.2.4 implies that $\{\ell, HH^1(J(\mathcal{Q}))\}$ is the ideal generated by ℓ and the minimal elements $\{x_\eta \,|\, i \in \mathbb{Z}/k\mathbb{Z},\ \eta \in \sigma_i\}$. Hence, the quotient is the algebra generated by the x_{ν_i} for all $i \in \mathbb{Z}/k\mathbb{Z}$. $\square$

Proposition 5.1.2. *Suppose $\mathcal{Q}$ is a zigzag consistent dimer model admitting a perfect matching. Then*

$$HH_*^{BM}\big(MF(J(\mathcal{Q}),\ell)\big) \cong H_*\big(HH_*(J(\mathcal{Q})),\, \mathcal{L}_\ell\big)\ mod\, 2$$
$$HH_c^*\big(MF(J(\mathcal{Q}),\ell)\big) \cong H_*\big(HH^*(J(\mathcal{Q})),\, \{\ell,-\}\big)\ mod\, 2.$$

Proof. Let $E^*_{*,*}$ be the homological spectral sequence for the first-quadrant double complex $C^+_{*,*}$. With respect to the vertical filtration, the first page is the Hochschild homology of $J(\mathcal{Q})$ with horizontal differential $\mathcal{L}_\ell$.

$$\begin{tikzcd}
0 & HH_3(J(\mathcal{Q})) \arrow[l] & HH_2(J(\mathcal{Q})) \arrow[l] & HH_1(J(\mathcal{Q})) \arrow[l] & HH_0(J(\mathcal{Q})) \arrow[l] & 0 \arrow[l] \\
0 & HH_2(J(\mathcal{Q})) \arrow[l] & HH_1(J(\mathcal{Q})) \arrow[l] & HH_0(J(\mathcal{Q})) \arrow[l] & 0 \arrow[l] & {} \arrow[l] \\
0 & HH_1(J(\mathcal{Q})) \arrow[l] & HH_0(J(\mathcal{Q})) \arrow[l] & 0 \arrow[l] & {} \arrow[l] & \\
0 & HH_0(J(\mathcal{Q})) \arrow[l] & 0 \arrow[l] & {} \arrow[l] & &
\end{tikzcd}$$

Evidently, the only possible nonzero differential of the second page is

$$d^2_{i,i} : Ker\big(\mathcal{L}_\ell : HH_0(J(\mathcal{Q})) \to HH_1(J(\mathcal{Q}))\big) \to Coker\big(\mathcal{L}_\ell : HH_2(J(\mathcal{Q})) \to HH_3(J(\mathcal{Q}))\big).$$

The differentials d_A and $\mathcal{L}_\ell$ have $H_1(\Sigma) \times \mathbb{Z}$-bidegree $(0,0)$ and $(0,1)$ with respect to all perfect matchings, so the differential $d^2_{*,*}$ has bidegree $(0,2)$. Consequently, the image of $d^2_{*,*}$ must be concentrated in degrees greater than or equal to 2 in all perfect matchings. However, under the Van den Bergh isomorphism $\mathbb{D}$,

$$HH^0(J(\mathcal{Q}))/\{\ell, HH^1(J(\mathcal{Q}))\} \cong HH_3(J(\mathcal{Q}))/\mathcal{L}_\ell\big((HH_2(J(\mathcal{Q}))\big),$$

and by Lemma 3.4.3, $\mathbb{D}$ has bidegree $(0,1)$. It is then deduced from Lemma 5.1.1 that the subspace of $Coker\big(\mathcal{L}_\ell : HH_2(J(\mathcal{Q})) \to HH_3(J(\mathcal{Q}))\big)$ lying in degrees greater than or equal to 2 in all perfect matchings is trivial. Consequently, the differential $d^2_{*,*}$ is 0, and $E^*_{*,*}$ collapses at the second page.

For $r \geq 2$,

$$H_{i+2r}(\mathrm{Tot}(C^+_{*,*})) \cong \begin{cases} H_{\mathrm{even}}\big(HH_*(J(\mathcal{Q})), \mathcal{L}_\ell\big) & \text{if } i \equiv 0 \bmod 2 \\ H_{\mathrm{odd}}\big(HH_*(J(\mathcal{Q})), \mathcal{L}_\ell\big) & \text{if } i \equiv 1 \bmod 2, \end{cases}$$

and so

$$\varprojlim H_*(\mathrm{Tot}(C^+_{*,*}[2r])) = H_*\big(HH_*(J(\mathcal{Q})), \mathcal{L}_w\big) \bmod 2.$$

Because the projections

$$H_*(\mathrm{Tot}(C^+_{*,*}[2s])) \to H_*(\mathrm{Tot}(C^+_{*,*}[2r]))$$

are isomorphisms for $s > r \geq 2$, the inverse system $\{H_*(\mathrm{Tot}(C^+_{*,*}[2r]) \,|\, r \in \mathbb{N}\}$ satisfies the

Mittag-Leffler condition ([37] Theorem 3.5.8), ensuring that

$$\varprojlim^1 H_*(\mathrm{Tot}(C^+_{*,*})[2r]) = 0$$

in all homological degrees. As a result, (5.1) the gives the desired description of the Borel–Moore Hochschild homology of $MF(J(\mathcal{Q}), \ell)$.

The computation of compactly supported Hochschild cohomology follows similarly. The relevant mixed double complex is $C^{i,j} = \mathrm{Hom}(J(\mathcal{Q})^{\otimes j-i}, J(\mathcal{Q}))$ equipped with vertical differential d_A (2.10) and horizontal differential $\{\ell, -\}$. Since compactly supported cohomology is a direct sum totalization, there is no need for truncating to the first quadrant and taking inverse limits. The same argument as above shows that the spectral sequence with respect to the vertical filtration collapses at the second page. □

5.2 Compactly supported cohomology of $MF(J(\mathcal{Q}), \ell)$

After Proposition 5.1.2, it remains to compute the homology of the complex $HH^*(J(\mathcal{Q}))$ equipped with differential $\{\ell, -\}$. We begin with a lemma about the kernel of $\{\ell, -\}$ in degree 3 that holds generally in nonpositive Euler characteristic. Subsequently, we specialize to $\chi(\mathcal{Q}) = 0$ and complete the computation.

Lemma 5.2.1. *Suppose $\mathcal{Q}$ is a zigzag consistent dimer model admitting a perfect matching. Then*

$$Ker\big(\{\ell, -\} : HH^3(J(\mathcal{Q})) \to HH^2(J(\mathcal{Q}))\big) \subset tor_\ell(HH^3(J(\mathcal{Q}))).$$

Proof. Under the Van den Bergh isomorphism $\mathbb{D}$, the Cartan identity (2.16) dualizes to

$$\{\ell, -\} = [\Delta_{\pi_0}, \ell].$$

Recall from §4.4 that, for $\alpha \in HH^3(J(\mathcal{Q}))$, the element $\Delta_{\pi_0}(\alpha)$ is the Hochschild class of the deformation of the superpotential determined by $\mathbb{D}(\alpha) \in HH_0(J(\mathcal{Q}))$. We will use this interpretation to show that $\{\ell, \alpha\}$ is the equivalence class of the trivial deformation only if α is torsion.

Let $\pi : \mathbb{C}\mathcal{Q} \to J(\mathcal{Q})$ be the quotient map and $s : J(\mathcal{Q}) \to \mathbb{C}\mathcal{Q}$ be a section. If $u \in J(\mathcal{Q})$, then $[s(u)] \in \mathbb{C}Q/[\mathbb{C}Q, \mathbb{C}Q]$ projects to $[u] \in HH_0(J(\mathcal{Q})) = J(\mathcal{Q})/[J(\mathcal{Q}), J(\mathcal{Q})]$, and the infinitesimal deformation of $J(\mathcal{Q})$ with Hochschild class $\Delta_{\pi_0}\mathbb{D}^{-1}([u])$ is, up to gauge equivalence ([18] Proposition 2.1.5),

$$J(\Phi_0 + \hbar[s(u)]) := \frac{\mathbb{C}Q[\hbar]/(\hbar^2)}{\big(\partial_a(\Phi_0 + \hbar[s(u)]) \,|\, a \in \mathcal{Q}_1\big)}.$$

We would like to relate this to a first-order star product on $J(\mathcal{Q})[\hbar]/(\hbar^2)$. Since the cyclic derivatives of Φ_0 (2.3) have path length at least 2, a gauge transformation can be applied if necessary to obtain an $\hbar$-linear isomorphism

$$F : J(\Phi_0 + \hbar[s(u)]) \to J(\mathcal{Q})[\hbar]/(\hbar^2)$$

that preserves vertices and arrows. Then the $*$-product endowed on $J(\mathcal{Q})[\hbar]/(\hbar^2)$ via F has, for each $a \in \mathcal{Q}_1$, the relation

$$x_1 * \cdots * x_m - y_1 * \cdots * y_n = -\hbar\, \pi(\partial_a[s(u)]) \tag{5.2}$$

where $ax_1 \dots x_m$ is the positive boundary cycle starting at a and $ay_1 \dots y_n$ is the negative boundary cycle starting at a.

Without loss of generality, we may assume that $s(\ell) \in \mathbb{C}\mathcal{Q}$ is a sum of only positive boundary cycles. Applying (5.2) to $[u]$ and $\ell[u]$ separately, observe that the $*$-product with

Hochschild class

$$\{\ell, \mathbb{D}^{-1}([u])\} = \Delta_{\pi_0}(\mathbb{D}^{-1}(\ell[u])) - \ell\Delta_{\pi_0}(\mathbb{D}^{-1}[u])$$

has relation

$$\begin{aligned} x_1 * \cdots * x_m - y_1 * \cdots * y_n &= -\hbar\,\pi(\partial_a[s(\ell u)]) + \hbar\,\ell\,\pi(\partial_a([s(u)])) \\ &= \hbar \sum_{i=1}^{m+1} \epsilon_i x_1 \ldots x_{i-1} u x_i \ldots x_m \end{aligned} \tag{5.3}$$

where

$$\epsilon_i = \begin{cases} -1 & \text{if } x_i \ldots x_m a x_1 \ldots x_{i-1} \text{ is a summand of } s(\ell) \\ 0 & \text{otherwise.} \end{cases}$$

We claim that, unless $[u]$ is torsion, this relation cannot be made 0 for all $a \in \mathcal{Q}_1$ simultaneously by a gauge transformation. Let $\psi \in C^1(J(\mathcal{Q})) = \mathrm{Hom}(J(\mathcal{Q}), J(\mathcal{Q}))$ and extend $\hbar$-linearly. The gauge equivalent $*$-product under ψ has relation

$$\begin{aligned} x_1 * \cdots * x_m - y_1 * \cdots * y_n &= \hbar \sum_{i=1}^{m+1} \epsilon_i x_1 \ldots x_{i-1} u x_i \ldots x_m \\ &- \hbar \sum_{i=1}^{m} x_1 \ldots \psi(x_i) \ldots x_m + \hbar \sum_{i=1}^{n} y_1 \ldots \psi(y_i) \ldots y_n. \end{aligned}$$

Fixing $\mathcal{P} \in PM(\mathcal{Q})$, we $*$-multiply both sides by $\deg_{\mathcal{P}}(a)\,a$ and sum over all $a \in \mathcal{Q}_1$. The result on the right side is

$$\sum_{a \in \mathcal{Q}_1} \deg_{\mathcal{P}}(a) \Big\{ \sum_{i=1}^{m+1} \epsilon_i a x_1 \ldots x_{i-1} u x_i \ldots x_m - \sum_{i=1}^{m} a x_1 \ldots \psi(x_i) \ldots x_m + \sum_{i=1}^{n} a y_1 \ldots \psi(y_i) \ldots y_n \Big\}.$$

Projected to $HH_0(J(\mathcal{Q}))$, the expression is the sum of two terms:

$$\lambda \ell[u], \quad \lambda := \sum_{a \in \mathcal{Q}_1} \deg_{\mathcal{P}}(a) \sum_{i=1}^{m+1} \epsilon_i$$

and

$$\sum_{a\in\mathcal{Q}_1}\deg_{\mathcal{P}}(a)\Big\{-\sum_{i=1}^{m}[\psi(x_i)x_{i+1}\dots x_m a x_1\dots x_{i-1}]+\sum_{i=1}^{n}[\psi(y_i)y_{i+1}\dots y_n a y_1\dots y_{i-1}]\Big\} \quad (5.4)$$

The coefficient λ is nonzero, since an arrow a can be chosen in $\mathcal{P}$ that also is contained in a summand of $s(\ell)$. On the other hand, for any $b\in\mathcal{Q}_1$, the terms of (5.4) having a factor of $\psi(b)$ sum to 0,

$$\begin{aligned}-\sum_{a\in R^b_+}\deg_{\mathcal{P}}(a)[\psi(b)\pi(R^b_+)]+\sum_{a\in R^b_-}\deg_{\mathcal{P}}(a)[\psi(b)\pi(R^b_-)]&\\ =(1-\deg_{\mathcal{P}}(b))[\psi(b)(\pi(R^b_+)-\pi(R^b_-))]&=0.\end{aligned}$$

Consequently, in order for the $*$-product to be the trivial deformation, $\ell[u]$ must be 0. □

With the explicit description of $HH^*(J(\mathcal{Q}))$ when $\chi(\mathcal{Q})=0$, we can finish the computation.

Theorem 5.2.2. *Suppose $\mathcal{Q}$ is a zigzag consistent dimer a torus. Then*

$$\begin{aligned}HH_c^{even}\big(MF(J(\mathcal{Q}),\ell)\big)&\cong tor^+_\ell(HH_0(J(\mathcal{Q})))\oplus\mathbb{C}[x_{\nu_1},\dots,x_{\nu_k}]/\big(x_{\nu_i}x_{\nu_j}\,|\,i\neq j\big)\\ HH_c^{odd}\big(MF(J(\mathcal{Q}),\ell)\big)&\cong tor_\ell(HH_0(J(\mathcal{Q})))\oplus\mathbb{C}[x_{\nu_1},\dots,x_{\nu_k}]/\big(x_{\nu_i}x_{\nu_j}\,|\,i\neq j\big)\\ &\oplus\ \mathbb{C}\end{aligned}$$

Proof. First, we claim that $\{\ell,-\}$ evaluates the ℓ-torsion of $HH^*(J(\mathcal{Q}))$ to 0. By Lemma 5.2.1, the kernel of $\{\ell,-\}:HH^3(J(\mathcal{Q}))\to HH^2(J(\mathcal{Q}))$ is contained in $tor_\ell(HH^3(J(\mathcal{Q})))\cong tor_\ell(HH_0(J(\mathcal{Q})))$. On the other hand, by Proposition 4.3.4, $tor_\ell(HH_0(J(\mathcal{Q})))$ is generated as a $\mathcal{Z}$-module by $\{[v]-[w]\,|\,v\neq w\in\mathcal{Q}_0\}$. Observe

$$\mathcal{L}_\ell([v]-[w]) = [B,\ell]([v]-[w]) = \ell B([v]-[w]) = 0,$$

the last equality following from the fact that the kernel of B is $\Bbbk \hookrightarrow HH_0(J(\mathcal{Q}))$ (4.4). Therefore,

$$Ker(\{\ell,-\} : HH^3(J(\mathcal{Q})) \to HH^2(J(\mathcal{Q}))) \cong tor_\ell(HH_0(J(\mathcal{Q}))). \tag{5.5}$$

Since the localization map $L^* : HH^*(J(\mathcal{Q})) \to HH^*(J(\mathcal{Q})[\ell^{-1}])$ is a morphism of BV algebras (Theorem 3.2.5), the map $\{\ell,-\}$ must send $tor_\ell(HH^2(J(\mathcal{Q}))) \cong tor_\ell^+(HH_0(J(\mathcal{Q}))$ into the torsion of $HH^1(J(\mathcal{Q}))$. But $HH^1(J(\mathcal{Q}))$ is torsion free by Theorem 4.2.4, so $\{\ell,-\}$ is trivial on the torsion of $HH^2(J(\mathcal{Q}))$.

Next, we compute the map $\{\ell,-\}$ on the subspace of $HH^*(J(\mathcal{Q}))$ complementary to the torsion. Recall the presentation of $HH^2(J(\mathcal{Q}))$ from Theorem 4.4.3,

$$\begin{aligned} HH^2(J(\mathcal{Q})) \;&\cong\; \mathcal{Z}\otimes_{\mathbb{R}} N^{out}_{\mathbb{R}} \wedge N^{out}_{\mathbb{R}} \\ &\oplus \bigoplus_{\substack{i\in\mathbb{Z}/k\mathbb{Z}\\ \eta\in\sigma_i}} \mathbb{C}\cdot\{[x_\eta\ell^{-1}E_{\mathcal{P}_{i+1}}\cup E_{\mathcal{P}_{i+2}}],\,[x_\eta\ell^{-1}E_{\mathcal{P}_{i+1}}\cup E_{\mathcal{P}_{i+3}}]\} \\ &\oplus \bigoplus_{\substack{i\in\mathbb{Z}/k\mathbb{Z}\\ \eta\in\gamma_i}} \mathbb{C}\cdot[x_\eta\ell^{-1}E_{\mathcal{P}_i}\cup E_{\mathcal{P}_{i+1}}] \\ &\oplus\; tor_\ell^+\big(HH_0(J(\mathcal{Q}))\big). \end{aligned}$$

Let f be a homogeneous element of $\mathcal{Z}$ and $\mathcal{P}_i \neq \mathcal{P}_j$ be any corner matchings such that

$$[f\ell^{-1}E_{\mathcal{P}_i}\cup E_{\mathcal{P}_j}] \in HH^2(J(\mathcal{Q})).$$

From the definition of the bracket (2.13), we compute

$$\{\ell, [f\ell^{-1}E_{\mathcal{P}_i} \cup E_{\mathcal{P}_j}]\} = f([E_{\mathcal{P}_j}] - [E_{\mathcal{P}_i}]). \tag{5.6}$$

Consequently, modulo torsion, the kernel of $\{\ell, -\} : HH^2(J(\mathcal{Q})) \to HH^1(J(\mathcal{Q}))$ is the $\mathcal{Z}$-submodule of $\mathcal{Z} \otimes_{\mathbb{R}} N_{\mathbb{R}}^{out} \wedge N_{\mathbb{R}}^{out}$ generated by the cocycle

$$\alpha := [E_{\mathcal{P}_2}] \cup [E_{\mathcal{P}_3}] - [E_{\mathcal{P}_1}] \cup [E_{\mathcal{P}_3}] + [E_{\mathcal{P}_1}] \cup [E_{\mathcal{P}_2}].$$

By (5.5), $\{\ell, -\} : HH^3(J(\mathcal{Q})) \to HH^2(J(\mathcal{Q}))$ is injective on the subspace complementary to the torsion, and for degree reasons, the image lies in $\mathcal{Z} \cdot \alpha$. But then it must be onto, so we conclude that the homology of $\{\ell, -\}$ in degree 2 is $tor_\ell^+(HH_0(J(\mathcal{Q})))$.

In the presentation of Theorem 4.2.4,

$$HH^1(J(\mathcal{Q})) \cong \mathcal{Z} \otimes_{\mathbb{R}} N_{\mathbb{R}}^{out} \oplus \bigoplus_{\substack{i \in \mathbb{Z}/k\mathbb{Z} \\ \eta \in \sigma_i}} [x_\eta \ell^{-1} E_{\mathcal{P}_{i+1}}].$$

As we saw in Lemma 5.1.1,

$$Coker(\{\ell, -\} : HH^1(J(\mathcal{Q})) \to HH^0(J(\mathcal{Q}))) \cong \mathbb{C}[x_{\nu_1}, \ldots, x_{\nu_k}]/\big(x_{\nu_i} x_{\nu_j} \,|\, i \neq j\big).$$

The kernel of $\{\ell, -\} : HH^1(J(\mathcal{Q})) \to HH^0(J(\mathcal{Q}))$ is generated over $\mathcal{Z}$ by differences of perfect matchings,

$$Ker(\{\ell, -\} : HH^1(J(\mathcal{Q})) \to HH^0(J(\mathcal{Q}))) = \mathcal{Z} \otimes_{\mathbb{R}} H^1(\Sigma, \mathbb{R}).$$

From (5.6), it is seen that the image of $\{\ell, -\} : HH^2(J(\mathcal{Q})) \to HH^1(J(\mathcal{Q}))$ is the span of

- $\mathcal{Z} \cdot \ell \otimes_{\mathbb{R}} H^1(\Sigma, \mathbb{R})$

- $\mathcal{Z} \cdot x_\eta \otimes_{\mathbb{R}} H^1(\Sigma, \mathbb{R})$ for all $i \in \mathbb{Z}/k\mathbb{Z}$ and $\eta \in \sigma_i$, and

- $\mathcal{Z} \cdot x_\eta([E_{\mathcal{P}_{i+1}}] - [E_{\mathcal{P}_i}])$ for all $i \in \mathbb{Z}/k\mathbb{Z}$ and $\eta \in \gamma_i$.

Thus, the homology of $\{\ell, -\}$ in degree 1 is isomorphic to

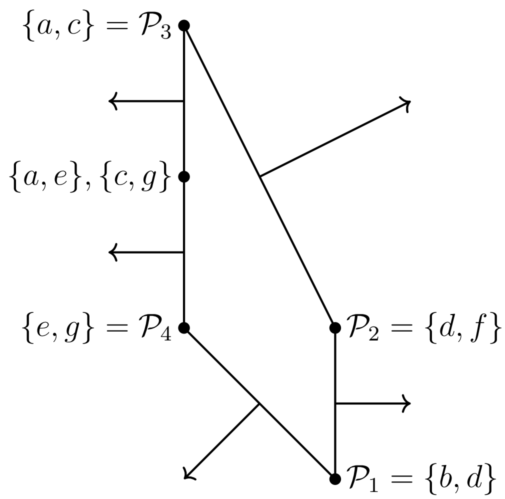
$$\mathbb{C}[x_{\nu_1}, \ldots, x_{\nu_k}]/\big(x_{\nu_i}x_{\nu_j} \mid i \neq j\big) \oplus \mathbb{C}.$$

$\square$

5.3 Example: suspended pinchpoint

Recall from Example 2.4.4 the dimer of the suspended pinchpoint.

$$\begin{tikzcd}
v_1 \arrow[r, "d"] & v_1 \arrow[dl, "e"'] \\
v_3 \arrow[u, "c"] \arrow[d, "b"'] & v_3 \arrow[u, "c"'] \arrow[d, "b"] \\
v_2 \arrow[ur, "f"] & v_2 \arrow[dl, "g"'] \\
v_1 \arrow[u, "a"] \arrow[r, "d"'] & v_1 \arrow[u, "a"']
\end{tikzcd}$$



The minimal central elements corresponding to the opposite cycles are

$$\begin{aligned}
x_1 &= ag + ga + ce \\
x_2 &= ebg + bge + geb \\
x_3 &= bf + fb + d \\
x_4 &= afc + fca + caf.
\end{aligned}$$

The vertices v_2 and v_3 are connected through x_3, but by Proposition 4.3.4, neither are connected to vertex v_1 through x_3. Otherwise, it can be seen directly that, for all homogeneous $f \neq x_3^m$ in $\mathcal{Z}$, all vertices are connected through f. Thus,

$$tor_\ell(HH_0(J(\mathcal{Q}))) = \mathbb{C} \cdot \{[v_1] - [v_2], [v_1] - [v_3]\} \oplus \bigoplus_{m>0} \mathbb{C} \cdot x_3^m([v_1] - [v_2]).$$

Then the compactly supported Hochschild cohomology of $MF(J(\mathcal{Q}), \ell)$ is

$$\begin{aligned} HH_c^{\text{even}}(MF(J(\mathcal{Q}), \ell)) &= \bigoplus_{m>0} \mathbb{C} \cdot x_3^m([v_1] - [v_2]) \oplus \mathbb{C}[x_1, x_2, x_3, x_4]/(x_i x_j \,|\, i \neq j) \\ HH_c^{\text{odd}}(MF(J(\mathcal{Q}), \ell)) &= \mathbb{C} \cdot \{[v_1] - [v_2], [v_1] - [v_3]\} \oplus \bigoplus_{m>0} \mathbb{C} \cdot x_3^m([v_1] - [v_2]) \\ &\oplus \mathbb{C}[x_1, x_2, x_3, x_4]/(x_i x_j \,|\, i \neq j) \oplus \mathbb{C}. \end{aligned}$$

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