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Adiabatic Limits of the Hermitian Yang-Mills Equations on Slicewise Stable Bundles

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Adiabatic Limits of the Hermitian Yang-Mills Equations on Slicewise Stable Bundles

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This is dedicated to my wife, Carla.

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would not have been possible.

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Adiabatic Limits of the Hermitian Yang-Mills Equations on Slicewise Stable Bundles

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A formal limit of the Hermitian Yang-Mills Equations on a SU(2) bundle over a product of two Riemann surfaces yields the Adiabatic Equations when the metric of the first surface is stretched ad infinitum. This thesis identifies the solutions of this new set of equations with holomorphic maps from the first surface into the moduli space of flat connections of the second one. Moreover, some advance is made in the study of the sort of bubbling phenomena that may occur when taking this limit. This dissertation is a step towards a rigorous proof of the relationship suggested by Bershadky, Johansen, Sadov and Vafa between Donaldson invariants and quantum cohomology, and relates to the program of Dostoglou and Salamon to prove the Atiyah-Floer conjecture.

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Chapter 1

Introduction

1.1 History

In Physics a time dependent process is called adiabatic if the rate of change is so slow that the solution can be approximately described by a time parameterized family of static solutions. This slow rate of change corresponds mathematically to stretching the time direction in the equations that govern the process, and the limit as time is infinitely stretched is referred to as the adiabatic limit.

Witten in [Wit85] introduced the idea of stretching the metric in some space directions instead of time, and this notion was subsequently employed in the study of η-invariants [Che87, BF86a, BF86b, BC89, Dai91]. This geometric adiabatic limit was used by Mazzeo and Melrose [MM90] to relate the spectrum of the Laplacian on forms to the Leray spectral sequence, and their results were generalized by Forman in [For95].

Atiyah [Ati88] used the idea of stretching the neck for the Heegaard splitting to study the Casson invariants and Floer homology. This was further explored by Dostoglou and Salamon [DS94] to prove a version of the Atiyah-Floer conjecture, relating the symplectic and instanton Floer homologies. In their proof they relate the adiabatic limit of anti-self-dual connections on the product of a cylinder and a Riemann surface to pseudo-holomorphic curves in the moduli space of flat connections on the surface.

When the base manifold is a product of two Riemann surfaces, physicists Bershadsky, Johansen, Sadov and Vafa [BJSV95] used the adiabatic limit to obtain a topological reduction of 4-dimensional supersymmetric Yang-Mills theory to 2-dimensional supersymmetric σ-models. Although their methods were not mathematically rigorous, an important consequence of such result would be a relation between the Donaldson invariants on the product manifold and the quantum cohomology of the flat connections on the fixed (non-stretched) surface. This is the BJSV conjecture.

Donaldson invariants are defined in terms of the topology of the moduli space of ASD equations, and quantum cohomology in terms of pseudo-holomorphic curves in the moduli space of flat connections. Such curves can be identified with solutions of the adiabatic equations, that are obtained as the formal adiabatic limit of the ASD equation. The problem then is how to relate the moduli space $\mathcal{M}_\lambda$ of ASD connections as one of the surfaces is stretched by a factor λ^{-1} and the moduli space $\mathcal{M}_{\mathrm{ad}}$ of adiabatic connections. One expects $\mathcal{M}_\lambda$ should converge in some sense to $\mathcal{M}_{\mathrm{ad}}$ as $\lambda \to 0$.

In his thesis [HI98], Handfield developed the first steps towards a rigorous proof the BJSV conjecture by showing how it is possible to construct a family of connections $[D_\lambda] \in \mathcal{M}_\lambda$ near an adiabatic connection $[D_{\mathrm{ad}}] \in \mathcal{M}_{\mathrm{ad}}$. There is however no guarantee that this construction will give a bijective correspondence between $\mathcal{M}_{\mathrm{ad}}$ and $\mathcal{M}_\lambda$, even for small values of λ.

1.2 The Product Structure

Let $\mathcal{E}$ be a SU(2) bundle over a product $\Sigma \times K$ of compact oriented Riemann surfaces.

If ρ^Σ and ρ^K are the Riemannian metrics of Σ and K, and ω^Σ and ω^K the corresponding Kähler forms, the product manifold $\Sigma \times K$ has a product metric and a product Kähler structure given by

$$\rho = \rho^\Sigma \oplus \rho^K, \tag{1.1}$$

$$\omega = \omega^\Sigma \oplus \omega^K. \tag{1.2}$$

The orthogonal splitting $T(\Sigma \times K) = T\Sigma \oplus TK$ induces in the obvious way similar splittings of forms:

$$\Omega^1(\Sigma \times K) = \Omega^\Sigma \oplus \Omega^K,$$
$$\Omega^2(\Sigma \times K) = \Omega^{\Sigma\Sigma} \oplus \Omega^{\Sigma K} \oplus \Omega^{KK}.$$

Here for example Ω^{KK} represents the 2-forms on $\Sigma \times K$ which are nonzero only when computed at a pair of vectors tangent to K.

Notation. If an object (vector, form, connection, etc.) has components along Σ or K, we use a single superscript to indicate the corresponding component (e.g. ω^Σ for the Σ component of the 2-form ω). For forms with mixed components we use multiple superscripts (e.g. $\eta^{\Sigma\Sigma\Sigma}$ or $\eta^{\Sigma KK}$ for components of a 3-form).

Hence a connection D on a bundle over $\Sigma \times K$ can be split according to its directional components

$$D = D^{\Sigma} + D^{K},$$

its holomorphic type

$$D = \partial_D + \bar{\partial}_D,$$

or both

$$D = \partial_D^{\Sigma} + \bar{\partial}_D^{\Sigma} + \partial_D^{K} + \bar{\partial}_D^{K}.$$

Its curvature splits as

$$F = F^{\Sigma\Sigma} + F^{\Sigma K} + F^{KK},$$

or, according to holomorphic type, as

$$F = F^{2,0} + F^{1,1} + F^{0,2}.$$

Definition. Let D, $\bar{\partial}$ and H be respectively a connection, a holomorphic structure and a Hermitian metric on $\mathcal{E}$. For any $z \in \Sigma$ denote by

- $\mathcal{E}_z$ the restriction of $\mathcal{E}$ over the slice $z \times K$,
- D_z, $\bar{\partial}_z$, H_z the restrictions of D^K, $\bar{\partial}^K$, H to $\mathcal{E}_z$.

As Σ is path-connected, all $\mathcal{E}_z$'s are isomorphic to some bundle $\mathcal{F} \to K$. Fix some Hermitian metric $H_{\mathcal{F}}$ on $\mathcal{F}$. As all Hermitian metrics in a bundle are related by bundle automorphisms, the isomorphism $\mathcal{E}_z \cong \mathcal{F}$ can be chosen such that $H_z = H_{\mathcal{F}}$ under this identification. Subject to this condition, the isomorphism is unique up to unitary gauge transformations of $(\mathcal{F}, H_{\mathcal{F}})$.

Remark. As $\mathcal{F}$ is a $SU(2)$ bundle over a Riemann surface, it is topologically trivial, even though $\mathcal{E}$ might be non-trivial.

Definition. A *trivialization* of $\mathcal{E}$ over a contractible open set $U \subset \Sigma$ is an isomorphism $\mathcal{E}|_U \cong U \times \mathcal{F}$.

1.3 The Adiabatic Rescaling

We define a parameterized family of metrics $\{\rho_\lambda\,;\lambda > 0\}$ on $\Sigma \times K$ by

$$\rho_\lambda \equiv \frac{\rho^\Sigma}{\lambda^2} \oplus \rho^K.$$

So as the parameter $\lambda \to 0$, the surface Σ is being indefinitely stretched, and we call this limiting process the *adiabatic limit.* The Kähler form ω_λ corresponding to the metric ρ_λ is

$$\omega_\lambda = \frac{\omega^\Sigma}{\lambda^2} \oplus \omega^K,$$

and the rescaled pointwise inner product (refer to Appendix E for notation) relates to the original one by

$$\begin{aligned} \big(\cdot\,,\,\cdot\big)_{\rho_\lambda} &= \lambda^2\big(\cdot\,,\,\cdot\big)^\Sigma_{\rho_1} + \big(\cdot\,,\,\cdot\big)^K_{\rho_1} && \text{on 1-forms,} \\ \big(\cdot\,,\,\cdot\big)_{\rho_\lambda} &= \lambda^4\big(\cdot\,,\,\cdot\big)^{\Sigma\Sigma}_{\rho_1} + \lambda^2\big(\cdot\,,\,\cdot\big)^{\Sigma K}_{\rho_1} + \big(\cdot\,,\,\cdot\big)^{KK}_{\rho_1} && \text{on 2-forms.} \end{aligned}$$

In particular, the pointwise norm for a 2-form $\eta = \eta^{\Sigma\Sigma} + \eta^{\Sigma K} + \eta^{KK}$ satisfies

$$|\eta|^2_{\rho_\lambda} \;=\; \lambda^4\,|\eta^{\Sigma\Sigma}|^2_{\rho_1} + \lambda^2\,|\eta^{\Sigma K}|^2_{\rho_1} + |\eta^{KK}|^2_{\rho_1}. \tag{1.3}$$

Define a 'trace' operator $\Lambda_\lambda : \Omega^{p,q}(\Sigma\times K) \to \Omega^{p-1,q-1}(\Sigma\times K)$, as in Appendix A, by $\big(\Lambda_\lambda\eta\,,\,\eta'\big)_{\rho_\lambda} = \big(\eta\,,\,\eta'\wedge\omega_\lambda\big)_{\rho_\lambda}$ for all $\eta \in \Omega^{p,q}$, $\eta' \in \Omega^{p-1,q-1}$. For a 2-form

$\eta \in \Omega^2(\Sigma \times K)$ this means

$$\begin{aligned}
\Lambda_\lambda \eta &= \big(\eta^{\Sigma\Sigma}\,,\,\omega_\lambda^{\Sigma}\big)_{\rho_\lambda} + \big(\eta^{KK}\,,\,\omega_\lambda^{K}\big)_{\rho_\lambda} \\
&= \lambda^4\big(\eta^{\Sigma\Sigma}\,,\,\frac{\omega^{\Sigma}}{\lambda^2}\big)_{\rho_1} + \big(\eta^{KK}\,,\,\omega^{K}\big)_{\rho_1} \\
&= \lambda^2\Lambda^{\Sigma}\eta^{\Sigma\Sigma} + \Lambda^K\eta^{KK},
\end{aligned}$$

where Λ^Σ and Λ^K are the trace operators on Σ and K corresponding to their original Kähler metrics.

We define the *rescaled Hermitian-Yang-Mills Equations*, or HYM_λ equations, as being the Hermitian-Yang-Mills equations for the metric ρ_λ, i.e.

$$\begin{cases} \bar{\partial}_\lambda^2 = 0 \\ \lambda^2\Lambda^\Sigma F_\lambda^{\Sigma\Sigma} + \Lambda^K F_\lambda^{KK} = 0, \end{cases} \tag{1.4}$$

where the subscript λ refers to any unitary connection D_λ satisfying these equations.

Lemma 1.1. *If D_λ satisfies the HYM_λ equations then*

$$|F_\lambda^{KK}|_{\rho_1} = |F_\lambda^{KK}|_{\rho_\lambda} = |F_\lambda^{\Sigma\Sigma}|_{\rho_\lambda} = \lambda^2|F_\lambda^{\Sigma\Sigma}|_{\rho_1}.$$

Proof. The first equation in (1.4) implies $F_\lambda \in \Omega^{1,1}(\Sigma\times K, \mathrm{ad}\,\mathcal{E})$ and therefore $F_\lambda^{\Sigma\Sigma} = \xi\omega^\Sigma$ and $F_\lambda^{KK} = \xi'\omega^K$ for some $\xi, \xi' \in \Omega^0(\Sigma \times K, \mathrm{ad}\,\mathcal{E})$. The second equation then means $\lambda^2\xi + \xi' = 0$ and the result follows from

$$|F_\lambda^{KK}|_{\rho_1} = |\xi'|\,|\omega^K|_{\rho_1} = |\lambda^2\xi| = \lambda^2|\xi|\,|\omega^\Sigma|_{\rho_1} = \lambda^2|F_\lambda^{\Sigma\Sigma}|_{\rho_1},$$

and from Equation (1.3). $\square$

The formal limit of the HYM_λ equations as $\lambda \to 0$ yields the *Adiabatic Limit Equations*

$$\begin{cases} \bar{\partial}_{\mathrm{ad}}^2 = 0 \\ F_{\mathrm{ad}}^{KK} = 0. \end{cases} \tag{1.5}$$

A solution D_{ad} of these equations is an *adiabatic connection.*

1.4 Statement of the Problem and Results

In this work, we attempt to improve the results obtained by Handfield in [HI98] in a program to establish a relation between moduli spaces of Yang-Mills connections and holomorphic curves in moduli spaces of flat connections via adiabatic connections.

On Chapter 2 we define some maps between connections on $\mathcal{E}$ and functions from Σ into $\mathcal{B}^*_{\mathcal{F}}$ (refer to Appendix F for notation). These are then used to prove a bijective correspondence between $\mathcal{M}^{\times}_{\mathrm{ad}}$ and $\mathrm{Hol}_{\mathcal{E}}(\Sigma, \mathcal{M}^*_{\mathcal{F}})$ (Theorem 2.4). Our approach differs from the one in [HI98] in that we work with SU(2) instead of SO(3) bundles, so we can not construct a connection on $\mathcal{E}$ from a function $\Sigma \to \mathcal{M}^*_{\mathcal{F}}$ by simply pulling back a universal connection defined over $\mathcal{M}^*_{\mathcal{F}}$.

On Chapter 3 we construct a family $[D_\lambda]$ corresponding to some $[D_{\mathrm{ad}}]$. But instead of doing so by using the Implicity Function Theorem as in [HI98], we impose a stability condition that allows us to obtain the family $[D_\lambda]$ in a very natural way. The idea is to look for this family along the orbit of $\mathcal{G}^{\mathbb{C}}$ passing through $[D_{\mathrm{ad}}]$, and use stability to guarantee the existence of a unique

$[D_\lambda]$ in such orbit for each small λ.

We however ran into problems in proving $[D_\lambda] \to [D_{\mathrm{ad}}]$, due to the possible occurrence of bubbling. Our approach was based in the techniques used by Dostoglou and Salamon in their proof of the Atiyah-Floer conjecture [DS94]. But after several attempts at adapting their methods to our case, we realized there is a serious gap in their argument about the formation of slow bubbles (which they call holomorphic spheres). A correction is attempted in [GS01], although the convergence is still not good enough to bridge this gap. So far we have been unable to obtain a better argument, but we conjecture that if the argument in [DS94] can somehow be corrected then the same should probably be true in our case.

Chapter 2

The Adiabatic Limit Equations

2.1 Connections and Maps Into the Moduli Space

2.1.1 Maps Induced by Connections

Given $D \in \mathcal{A}^\times$ and $z \in \Sigma$, the restriction D_z is an irreducible unitary connection on $(\mathcal{E}_z, H_z)$. The isomorphism between $(\mathcal{E}_z, H_z)$ and $(\mathcal{F}, H_\mathcal{F})$, which is defined up to unitary gauge transformations, determines a gauge equivalence class $[D_z] \in \mathcal{B}^*_\mathcal{F}$. This gives a map

Definition.
$$\begin{aligned} \phi_D : \Sigma &\to \mathcal{B}^*_\mathcal{F} \\ z &\mapsto [D_z]. \end{aligned}$$

Clearly ϕ_D depends only on the gauge equivalence class of D. Hence we obtain a well defined map:

Definition.
$$\begin{aligned} \Phi : \mathcal{B}^\times &\to C^\infty(\Sigma, \mathcal{B}^*_\mathcal{F}) \\ [D] &\mapsto \phi_D. \end{aligned}$$

Note that Φ is not injective, as we can have two connections on $\mathcal{E}$ whose restrictions along the slices $\{z\} \times K$ are gauge equivalent but not their restrictions along the Σ direction. Nor is it surjective. As $\mathcal{B}^\times$ is connected, all maps $\phi_D \in \operatorname{Im}\Phi$ are in the same homotopy class.

Definition. $C^\infty_{\mathcal{E}}(\Sigma, \mathcal{B}^*_{\mathcal{F}}) \equiv \{\text{maps in the homotopy class of } \operatorname{Im}\Phi\}$

$$= \{\phi \in C^\infty(\Sigma, \mathcal{B}^*_{\mathcal{F}}) : \phi \sim \phi_D \text{ for some } D \in \mathcal{A}^\times\}.$$

2.1.2 Connections Induced by Maps

Let $\phi \in C^\infty_{\mathcal{E}}(\Sigma, \mathcal{B}^*_{\mathcal{F}})$. We would like to construct a gauge equivalence class $[D_\phi]$ on $\mathcal{E}$ such that $\Phi([D_\phi]) = \phi$.

Given a cover $\{U_j\}$ of Σ by contractible open sets, ϕ can be lifted over U_j to a (non-unique) map $a_j : U_j \to \mathcal{A}^*_{\mathcal{F}}$. Depending on the context we write either $a_j(z)$ or $D_{a_j(z)}$ to represent the connection obtained from this map at z. For $z \in U_j \bigcap U_k$ there is a $g_{jk}(z) \in \mathcal{G}_{\mathcal{F}}$ such that

$$D_{a_j(z)} = g_{jk}(z) \cdot D_{a_k(z)}. \tag{2.1}$$

As these connections are irreducible and the center of $SU(2)$ is $\{\pm I\}$, $g_{jk}(z)$ is unique up to a $\pm$ sign. Also,

$$g_{jk}(z)g_{kj}(z) \cdot D_{a_j(z)} = D_{a_j(z)} \quad \Rightarrow \quad g_{jk}(z)g_{kj}(z) = \pm I,$$
$$g_{jk}(z)g_{kl}(z)g_{lj}(z) \cdot D_{a_j(z)} = D_{a_j(z)} \quad \Rightarrow \quad g_{jk}(z)g_{kl}(z)g_{lj}(z) = \pm I.$$

If the signs of the g_{jk}'s could be chosen in such a way to always have $+I$ above, they could be used as transition functions for "trivializations" $U_j \times \mathcal{F}$ of a bundle $\mathcal{E}' \to \Sigma \times K$. It is not possible in general to make such a choice [1]. However, as in this case $\phi \sim \phi_D$ for some $D \in \mathcal{A}^\times$, instead of lifting just ϕ we

[1]The g_{jk}'s modulo $\pm I$ define a $SO(3)$ bundle, and the choice of signs corresponds to lifting it to a $SU(2)$ bundle, what is only possible if the second Stiefel-Whitney class vanishes.

can lift the whole homotopy map, starting with "trivializations" $U_j \times \mathcal{F}$ of $\mathcal{E}$ and the obvious lifts $(D_z)_j$ of ϕ_D. In this way, not only we get a natural way of choosing the g_{jk}'s but they will also be homotopic to the transition functions of $\mathcal{E}$, so that $\mathcal{E}' \cong \mathcal{E}$. Note that choosing a different D, homotopy or lifts amounts only to obtaining another bundle isomorphic to $\mathcal{E}'$, or equivalently to a gauge transformation of $\mathcal{E}'$.

The partial connections D_{a_j} can be completed along the Σ directions in a natural gauge invariant way. Let $z = x + iy$ be local coordinates on U_j. For each z as $D^*_{a_j(z)} D_{a_j(z)} : \Omega^0(\mathrm{End}_0\,\mathcal{F}) \to \Omega^1(\mathrm{End}_0\,\mathcal{F})$ is elliptic there are $\eta_{xj}(z),\ \eta_{y_j}(z) \in \Omega^0(\mathrm{End}_0\,\mathcal{F})$ such that

$$D^*_{a_j}\left(D_{a_j}\eta_{xj} - \frac{\partial a_j}{\partial x}\right) = 0, \qquad D^*_{a_j}\left(D_{a_j}\eta_{y_j} - \frac{\partial a_j}{\partial y}\right) = 0, \tag{2.2}$$

and as $a_j(z)$ is stable Corollary J.3 implies these solutions are unique. Define a connection D_j on $U_j \times \mathcal{F}$ by

$$D_j = d + \eta_{xj}\,dx + \eta_{y_j}\,dy + D_{a_j}, \tag{2.3}$$

where d is the trivial connection along U_j. On $U_j \bigcap U_k$ an easy calculation using (2.1) shows that

$$g_{jk}^{-1} \cdot D_j = d + \left(g_{jk}^{-1}\eta_{xj}g_{jk} + g_{jk}^{-1}\frac{\partial g_{jk}}{\partial x}\right) dx + \left(g_{jk}^{-1}\eta_{y_j}g_{jk} + g_{jk}^{-1}\frac{\partial g_{jk}}{\partial y}\right) dy + D_{a_k},$$

and also

$$D^*_{a_k}\left(D_{a_k}\left(g_{jk}^{-1}\eta_{xj}g_{jk} + g_{jk}^{-1}\frac{\partial g_{jk}}{\partial x}\right) - \frac{\partial a_k}{\partial x}\right) = 0,$$

with a similar equation for y. Hence $D_j = g_{jk} \cdot D_k$ and these connections glue together to form a global connection D_ϕ on $\mathcal{E}'$. The isomorphism $\mathcal{E} \cong \mathcal{E}'$ now determines a gauge equivalence class $[D_\phi] \in \mathcal{B}^\times$, giving a well defined map

Definition. $\Theta : C^\infty_{\mathcal{E}}(\Sigma, \mathcal{B}^*_{\mathcal{F}}) \to \mathcal{B}^\times$

$$\phi \mapsto [D_\phi].$$

By construction, $\Phi \circ \Theta = \mathrm{id}$ and therefore $\mathrm{Im}\,\Phi = C^\infty_{\mathcal{E}}(\Sigma, \mathcal{B}^*_{\mathcal{F}})$.

Proposition 2.1. $\mathrm{Im}\,\Theta = \{[D] \in \mathcal{B}^\times : (D^K)^* F_D^{\Sigma K} = 0\}$.

Proof. Any $D \in \mathcal{A}^\times$ can be expressed locally as $D_j = d + \eta_{xj}\, dx + \eta_{yj}\, dy + (D_z)_j$ for some η_{xj}, $\eta_{yj} \in \Omega^0(\mathrm{End}_0 \mathcal{F})$, and writing $a_j \equiv (D_z)_j$ we have

$$F_{D_j}^{\Sigma K} = \left(D_{a_j} \eta_{xj} - \frac{\partial a_j}{\partial x} \right) \wedge dx + \left(D_{a_j} \eta_{yj} - \frac{\partial a_j}{\partial y} \right) \wedge dy.$$

If $[D] \in \mathrm{Im}\,\Theta$ then η_{xj} and η_{yj} can be chosen to satisfy (2.2), and in such case $(D^K)^* F_D^{\Sigma K} = 0$. Conversely, if $(D^K)^* F_D^{\Sigma K} = 0$ then η_{xj}, η_{yj} are the unique elements of $\Omega^0(\mathrm{End}_0 \mathcal{F})$ satisfying (2.2) and by construction $[D] = \Theta(\Phi(D))$. □

2.2 Adiabatic Connections and Holomorphic Maps

The moduli space $\mathcal{M}^*_{\mathcal{F}}$ of irreducible flat connections on $\mathcal{F}$ has a complex structure given by the Hodge start operator $*_K$ acting on its tangent space $T_A \mathcal{M}^*_{\mathcal{F}} = \{a \in \Omega^1(\mathrm{End}_0 \mathcal{F}) : D_A a = D_A^* a = 0\}$. In this section we show that the maps defined above establish a correspondence between the adiabatic connections on $\mathcal{E}$ and holomorphic maps from Σ into $\mathcal{M}^*_{\mathcal{F}}$.

Definition. $\mathrm{Hol}_{\mathcal{E}}(\Sigma, \mathcal{M}^*_{\mathcal{F}}) = \{\phi \in C^\infty_{\mathcal{E}}(\Sigma, \mathcal{B}^*_{\mathcal{F}}) :\ \phi$ is holomorphic, $\mathrm{Im}\,\phi \subset \mathcal{M}^*_{\mathcal{F}}\}$.

Proposition 2.2. $\mathcal{M}^{\times}_{ad} \subset \operatorname{Im}\Theta$.

Proof. Given $D = \partial^{\Sigma} + \bar\partial^{\Sigma} + \partial^K + \bar\partial^K$ we have[2]

$$F_D^{\Sigma K} = [\partial^{\Sigma}, \partial^K + \bar\partial^K] + [\bar\partial^{\Sigma}, \partial^K + \bar\partial^K], \tag{2.4}$$

$$\begin{aligned}(D^K)^* F_D^{\Sigma K} &= i\Lambda^K(\bar\partial^K - \partial^K) F_D^{\Sigma K} \\ &= i\Lambda^K \left[\bar\partial^K - \partial^K\, , [\partial^{\Sigma}, \partial^K + \bar\partial^K] + [\bar\partial^{\Sigma}, \partial^K + \bar\partial^K]\right] \\ &= 2i\Lambda^K \left[\bar\partial^K\, , [\partial^{\Sigma}, \partial^K]\right] + i\Lambda^K \left[\partial^{\Sigma}\, , [\partial^K, \bar\partial^K]\right] \\ &\quad - 2i\Lambda^K \left[\partial^K\, , [\bar\partial^{\Sigma}, \bar\partial^K]\right] - i\Lambda^K \left[\bar\partial^{\Sigma}\, , [\partial^K, \bar\partial^K]\right].\end{aligned} \tag{2.5}$$

If $D \in \mathcal{M}^{\times}_{\mathrm{ad}}$ the adiabatic equations $\bar\partial_D^2 = \partial_D^2 = 0$ and $F_D^{KK} = 0$ imply

$$[\bar\partial^{\Sigma}, \bar\partial^K] = 0, \qquad [\partial^{\Sigma}, \partial^K] = 0, \qquad [\partial^K, \bar\partial^K] = 0,$$

and therefore $(D^K)^* F_D^{\Sigma K} = 0$. The result follows from Proposition 2.1. □

Proposition 2.3. $\Phi(\mathcal{M}^{\times}_{ad}) = \operatorname{Hol}_{\mathcal{E}}(\Sigma, \mathcal{M}^*_{\mathcal{F}})$.

Proof. Let $[D] \in \mathcal{M}^{\times}_{\mathrm{ad}}$. By the previous Proposition $[D] = \Theta(\phi)$ for some $\phi \in C^{\infty}_{\mathcal{E}}(\Sigma, \mathcal{B}^*_{\mathcal{F}})$, and $\phi = \Phi([D])$ as $\Phi \circ \Theta = \mathrm{id}$. From the definition of Φ it is clear that $\operatorname{Im}\phi \subset \mathcal{M}^*_{\mathcal{F}}$ if and only if $F_D^{KK} = 0$. So all that remains to be shown is that $\bar\partial_D^2 = 0$ is equivalent to ϕ being holomorphic. As $[D] \in \operatorname{Im}\Theta$, D can be written locally as in (2.2) and (2.3) with $a_j \equiv (D_z)_j$, and so

$$\begin{aligned}\bar\partial_{D_j} &= \bar\partial + \frac{\eta_{x\,j} + i\,\eta_{y_j}}{2}\, d\bar z + \bar\partial_{a_j}, \\ \bar\partial^2_{D_j} &= -\left(\frac{\partial a_j^{0,1}}{\partial \bar z} - \bar\partial_{a_j}\left(\frac{\eta_{x\,j} + i\eta_{y_j}}{2}\right)\right) \wedge d\bar z.\end{aligned} \tag{2.6}$$

[2] We use the notation $[\partial_1, \partial_2] = \partial_1\partial_2 + \partial_2\partial_1$, and $[\partial_1, \partial_2\partial_3] = \partial_1\partial_2\partial_3 - \partial_2\partial_3\partial_1$.

The holomorphic condition for ϕ is

$$\frac{\partial\phi}{\partial x} + *_K \frac{\partial\phi}{\partial y} = 0.$$

The derivatives of ϕ at z can be computed by projecting the derivatives of its lift a_j onto $(\operatorname{Im} D_{a_j(z)})^\perp$. As $D_{a_j(z)}$ is flat it can be used to define an elliptic complex which on the other hand gives a Hodge-like decomposition $T_{a_j(z)}\mathcal{A}_\mathcal{F} = \operatorname{Im} D_{a_j(z)} \oplus \operatorname{Im} D^*_{a_j(z)} \oplus H^1_{a_j(z)} = \operatorname{Im} D_{a_j(z)} \oplus \ker D^*_{a_j(z)}$. Hence $\frac{\partial\phi}{\partial x}$ is given by

$$\frac{\partial\phi}{\partial x} = \frac{\partial a_j}{\partial x} - D_{a_j(z)}\eta,$$

for some η such that

$$D^*_{a_j}\Big(\frac{\partial a_j}{\partial x} - D_{a_j(z)}\eta\Big) \;=\; 0.$$

By the uniqueness of solutions of (2.2), $\eta = \eta_{xj}$. For $\frac{\partial\phi}{\partial y}$ there is a similar formula, and so

$$\begin{aligned}
\frac{\partial\phi}{\partial x} + *_K \frac{\partial\phi}{\partial y} \;&=\; \Big(\frac{\partial a_j}{\partial x} - D_{a_j}\eta_{xj}\Big) + *_K\Big(\frac{\partial a_j}{\partial y} - D_{a_j}\eta_{yj}\Big) \\
&= \frac{\partial a_j^{0,1}}{\partial x} + i\,\frac{\partial a_j^{0,1}}{\partial y} - \bar\partial_{a_j}\eta_{xj} - i\,\bar\partial_{a_j}\eta_{yj} \\
&\quad + \frac{\partial a_j^{1,0}}{\partial x} - i\,\frac{\partial a_j^{1,0}}{\partial y} - \partial_{a_j}\eta_{xj} + i\,\partial_{a_j}\eta_{yj} \\
&= 2\left(\frac{\partial a_j^{0,1}}{\partial \bar z} - \bar\partial_{a_j}\Big(\frac{\eta_{xj} + i\eta_{yj}}{2}\Big)\right) + 2\left(\frac{\partial a_j^{1,0}}{\partial \bar z} - \partial_{a_j}\Big(\frac{\eta_{xj} - i\eta_{yj}}{2}\Big)\right)
\end{aligned}$$

By (2.6) the vanishing of the $(0,1)$ component is equivalent to $\bar\partial^2_{D_j} = 0$, and similarly the vanishing of the $(1,0)$ component is equivalent to $\partial^2_{D_j} = 0$. For unitary connections $\bar\partial^2_{D_j} = 0$ if and only if $\partial^2_{D_j} = 0$, so ϕ is holomorphic if and only if $\bar\partial^2_D = 0$. $\square$

Theorem 2.4. *The maps* Φ *and* Θ *give a 1-1 correspondence between* $\mathcal{M}_{ad}^{\times}$ *and* $\mathrm{Hol}_{\mathcal{E}}(\Sigma, \mathcal{M}_{\mathcal{F}}^{*})$.

Proof. Follows from the two previous results and the fact that $\Phi \circ \Theta = \mathrm{id}$. $\square$

Theorem 2.5. *For each slicewise stable class* $\mathfrak{s} \in \mathcal{S}$ *there is a unique* $[D_{\mathfrak{s}}] \in \mathcal{M}_{ad}^{\times}$ *with* $\bar{\partial}_{\mathfrak{s}} \in \mathfrak{s}$.

Proof. For each $z \in \Sigma$, as $\mathfrak{s}_z \in \mathcal{S}_{\mathcal{F}}$ is stable by Corollary J.8 there is a unique $[D_{\mathfrak{s},z}] \in \mathcal{M}_{\mathcal{F}}^{*}$ such that $\bar{\partial}_{\mathfrak{s},z} \in \mathfrak{s}_z$. So $\mathfrak{s}$ determines a map

$$\begin{aligned} \phi_{\mathfrak{s}} : \Sigma &\to \mathcal{M}_{\mathcal{F}}^{*} \\ z &\mapsto [D_{\mathfrak{s},z}] \end{aligned}$$

and a gauge equivalence class $[D_{\mathfrak{s}}] = \Theta(\phi_{\mathfrak{s}})$. By construction $F_{\mathfrak{s}}^{KK} = 0$ (and therefore $[\partial_{\mathfrak{s}}^{K}, \bar{\partial}_{\mathfrak{s}}^{K}] = 0$) and $(\bar{\partial}_{\mathfrak{s}}^{K})_z \in \mathfrak{s}_z$ is stable for any $z \in \Sigma$. Proposition 2.1 implies $(D_{\mathfrak{s}}^{K})^{*} F_{\mathfrak{s}}^{\Sigma K} = 0$, and so by equation (2.5),

$$\left[\bar{\partial}_{\mathfrak{s}}^{K}, [\partial_{\mathfrak{s}}^{\Sigma}, \partial_{\mathfrak{s}}^{K}]\right] = 0, \qquad \left[\partial_{\mathfrak{s}}^{K}, [\bar{\partial}_{\mathfrak{s}}^{\Sigma}, \bar{\partial}_{\mathfrak{s}}^{K}]\right] = 0.$$

As over any slice $\bar{\partial}_{\mathfrak{s}}^{K}$ is stable, Lemma J.3 implies $[\partial_{\mathfrak{s}}^{\Sigma}, \partial_{\mathfrak{s}}^{K}] = 0$ and $[\bar{\partial}_{\mathfrak{s}}^{\Sigma}, \bar{\partial}_{\mathfrak{s}}^{K}] = 0$. Therefore $\bar{\partial}_{\mathfrak{s}}^{2} = 0$ and $[D_{\mathfrak{s}}] \in \mathcal{M}_{\mathrm{ad}}^{\times}$. Moreover, $\bar{\partial}_{\mathfrak{s}} \in \mathfrak{s}$ by Corollary J.6.

Now suppose D' is another adiabatic connection with $\bar{\partial}' \in \mathfrak{s}$. Then $D' = g(D_{\mathfrak{s}})$ for some $g \in \mathcal{G}^{\mathbb{C}}$. As D' is adiabatic each D'_z is flat, so as $D'_z = g(z)((D_{\mathfrak{s}})_z)$ Corollary J.8 implies $g(z) \in \mathcal{G}_{\mathcal{F}}$ for all z. Hence $g \in \mathcal{G}$. $\square$

Chapter 3

The Adiabatic Limit

3.1 The sequence of HYM_λ solutions

Fix a slicewise stable class $\mathfrak{s} \in \mathcal{S}$.

By Theorems J.7 and 2.5 there are a unique $[D_\lambda] \in \mathcal{M}_\lambda^\times$ for each $\lambda > 0$ and a unique $[D_{\mathrm{ad}}] \in \mathcal{M}_{\mathrm{ad}}^\times$ satisfying $\bar\partial_\lambda, \bar\partial_{\mathrm{ad}} \in \mathfrak{s}$. As $\bar\partial_\lambda$ and $\bar\partial_{\mathrm{ad}}$ are in the same class there is some $g_\lambda \in \mathcal{G}^{\mathbb{C}}$ such that $\bar\partial_\lambda = g_\lambda \cdot \bar\partial_{\mathrm{ad}}$, and as D_λ and D_{ad} are both unitary,

$$D_\lambda = g_\lambda(D_{\mathrm{ad}}).$$

The first step in proving convergence for a family of connections is to obtain some bounds on the curvature. Let $\{\lambda_n\}_{n\in\mathbb{N}}$ be any decreasing sequence with $\lambda_1 = 1$ and $\lambda_n \to 0$.

Notation. $D_n \equiv D_{\lambda_n},\ F_n \equiv F_{\lambda_n},\ \rho_n \equiv \rho_{\lambda_n},\ g_n \equiv g_{\lambda_n},$

Partition Σ as $\Sigma = \Sigma_1 \cup \ldots \cup \Sigma_4$, where

$$\Sigma_1 = C_1, \qquad \Sigma_2 = C_2 - C_1, \qquad \Sigma_3 = C_3 - C_2, \qquad \Sigma_4 = \Sigma - C_3,$$

$C_1 = \{z \in \Sigma :$ there are some $c > 0$ and a neighborhood of $z \times K$ in which $\sup |F_n|_{\rho_n} < c\lambda_n$ for all n large enough$\}$,

$$C_2 = \{z \in \Sigma : \text{for any } \delta > 0 \text{ there is a neighborhood of } z \times K \text{ in which}$$
$$\sup |F_n|_{\rho_n} < \delta \text{ for all } n \text{ large enough}\},$$

$$C_3 = \{z \in \Sigma : |F_n|_{\rho_n} \text{ is uniformly bounded on a neighborhood of } z \times K\}.$$

In the next sections we will deal with the Σ_i's one at a time, but first we give an idea of what should be expected from each case.

Σ_1 corresponds to the ideal situation for our purposes. Our aim is to prove that the other sets are empty and so $\Sigma = \Sigma_1$.

For $z \in \Sigma_2$ we will have a sequence of points converging to $z \times K$ with $|F_n|_{\rho_1} \to \infty$. Expanding the metric appropriately about these points we can keep F_n bounded without decaying to 0. As we are in C_2, expanding at the same rate as ρ_n would make $F_n \to 0$, so the appropriate expansion must be slower. The ρ_n-ASD equations then imply $F_n^{KK} \to 0$, and we end up with a non-flat adiabatic connection on $\mathbb{R}^2 \times K$.

In the case of Σ_3 we obtain a sequence $z_n \to z$ for which $|F_n|_{\rho_n}$ remains bounded about $z_n \times K$. Expanding the metric about these slices according to ρ_n we obtain a sequence of ASD connections with uniformly bounded curvature. As $z \notin C_2$ these will converge to a non-flat ASD connection on $\mathbb{R}^2 \times K$.

We prove Σ_2 and Σ_3 are empty by comparing what we obtained from D_n to what we get from D_{ad} by the same construction, and showing the existence of both objects contradicts our stability assumptions.

Finally, Σ_4 corresponds to points (or sequences) where even in the expanded metric ρ_n the curvature remains unbounded. As the connections are ASD with respect to ρ_n, there is a minimum amount of energy that must become concentrated at the point for the curvature to diverge. As the total energy is finite, there can be only a finite number of such points. Each of them will therefore be surrounded by Σ_1. This will provide a contradiction, as the energy of D_n inside a region can be controlled by the energy of D_{ad} (that is bounded) in the same region plus some contributions from the boundary.

3.2 Fiber Bubbles.

We now deal with Σ_3 as this case is simpler and allows us to introduce some techniques that will be used again in the next section, when dealing with Σ_2. Given $z \in \Sigma_3$ there is a $\delta > 0$ such that, after passing to a subsequence, it is possible to find a $k \in K$ and a sequence $(z_n, k_n) \to (z, k)$ such that $|F_n(z_n, k_n)|_{\rho_n} > \delta$ for all n.

In order to avoid the problem of having the metrics ρ_n diverge as $\lambda_n \to 0$ we pull shrinking neighborhoods of z_n back to $\mathbb{R}^2$.

Definition 3.1. Choosing orthonormal (with respect to ρ_1) basis $\{e_n^1, e_n^2\}$ for $T_{z_n}\Sigma$, define a sequence of maps $\varphi_n : \mathbb{R}^2 \times K \to \Sigma \times K$ by

$$\varphi_n(x, w) = (\exp_{z_n}(\lambda_n x_1 e_n^1 + \lambda_n x_2 e_n^2), w).$$

Define also sequences of metrics $\tilde{\rho}_n$ on $\mathbb{R}^2 \times K$, connections $\tilde{D}_n$ on $\tilde{\mathcal{E}} = \varphi_n^* \mathcal{E}$, and gauge transformations $\tilde{g}_n \in \mathcal{G}^{\mathbb{C}}(\tilde{\mathcal{E}})$ by

$$\tilde{\rho}_n = \varphi_n^* \rho_n, \qquad \tilde{D}_n = \varphi_n^* D_n, \qquad \tilde{g}_n = \varphi_n^* g_n.$$

Denote the curvature of $\tilde{D}_n$ by $\tilde{F}_n$.

Remark. Note that as $\mathbb{R}^2$ is contractible $\tilde{\mathcal{E}} = \varphi_n^*\mathcal{E}$ does not depend on n, and is isomorphic to $\mathbb{R}^2 \times \mathcal{F}$.

Lemma 3.2. *$\tilde{\rho}_n \to \rho \equiv \epsilon \times \rho^K$ in C^∞ over compact sets, where ϵ denotes the Euclidean metric on $\mathbb{R}^2$.*

Lemma 3.3. *$\{|\tilde{F}_n|_{\tilde{\rho}_n}\}$ is uniformly bounded over compact sets of $\mathbb{R}^2 \times K$.*

Proof. As $z \in C_3$ there is a $c > 0$ and a neighborhood U of z such that $|F_n|_{\rho_n} < c$ in $U \times K$ for all n. Given any compact set $\Omega \subset \mathbb{R}^2$, $\varphi_n(\Omega) \subset U$ for all n large enough. Hence $|\tilde{F}_n|_{\tilde{\rho}_n} = \varphi_n^*|F_n|_{\rho_n} < c$ in $\Omega \times K$ for n large. $\square$

As $\tilde{D}_n$ is ASD with respect to $\tilde{\rho}_n$, by Theorem I.6 we can assume, after passing to a subsequence and taking gauge transformations,

Lemma 3.4. *$\tilde{D}_n \to D$ in C^∞ over compact sets, where D is an ASD connection with respect to ρ.*

Lemma 3.5. *$0 < \|F\|_{L^2(\rho,\mathbb{R}^2\times K)} \leq 8\pi^2 c_2(\mathcal{E})$, where F is the curvature of D.*

Proof. As $|F(0,k)|_\rho = \lim_n |\tilde{F}_n(0,k_n)|_{\tilde{\rho}_n} = \lim_n |F_n(z_n,k_n)|_{\rho_n} > \delta$ and F is smooth (in an appropriate gauge), its L^2 norm is nonzero. Also, for any compact set $\Omega \in \mathbb{R}^2$, F satisfies

$$\begin{aligned}\int_{\Omega\times K} |F|_\rho^2 \; d\,\mathrm{vol}_\rho &= \lim_{n\to\infty} \int_{\Omega\times K} |\tilde{F}_n|_{\tilde{\rho}_n}^2 \; d\,\mathrm{vol}_{\tilde{\rho}_n} \\ &= \lim_{n\to\infty} \int_{\varphi_n(\Omega\times K)} |F_n|_{\rho_n}^2 \; d\,\mathrm{vol}_{\rho_n} \\ &\leq \lim_{n\to\infty} \int_{\Sigma\times K} |F_n|_{\rho_n}^2 \; d\,\mathrm{vol}_{\rho_n} \leq 8\pi^2 c_2(\mathcal{E}),\end{aligned}$$

what gives the upper bound. □

Corollary 3.6. $\sup_{z\times K}|F^{KK}|_{\rho^K}\to 0$ *as* $|z|\to\infty$.

Proof. As D is ASD the previous Lemma and Proposition I.2 imply that $\sup_{z\times K}|F|_\rho\to 0$ as $|z|\to\infty$. The result then follows from $|F^{KK}|_{\rho^K}=|F^{KK}|_\rho\le|F|_\rho$. □

Definition 3.7. Define a connection D_0 on $\mathbb{R}^2\times K$ by

$$D_0\equiv\lim_{n\to\infty}\varphi_n^* D_{\mathrm{ad}}=d^{\mathbb{R}^2}+(D_{\mathrm{ad}})_{z_0}.$$

Here $d^{\mathbb{R}^2}$ represents the trivial connection along $\mathbb{R}^2$ and $(D_{\mathrm{ad}})_{z_0}$ is the K component of D_{ad} along $z_0\times K$. The limit is uniform on compact sets for all derivatives.

Lemma 3.8. D_0 *is flat.*

Proof. As $D_0^{\mathbb{R}^2}$ is trivial and D_0^K is constant along $\mathbb{R}^2$, the result will be true if $(D_{\mathrm{ad}})_{z_0}$ is flat. But this is part of the definition of an adiabatic connection. □

Lemma 3.9. $D=g(D_0)$ *for some* $g\in\mathcal{G}^{\mathbb{C}}(\tilde{\mathcal{E}})$.

Proof. As $D_n=g_n(D_{\mathrm{ad}})$,

$$\tilde{D}_n=\varphi_n^*(D_n)=(\varphi_n^* g_n)(\varphi_n^* D_{\mathrm{ad}})=\tilde{g_n}(\varphi_n^* D_{\mathrm{ad}}),$$

and by Proposition J.12 $D=g(D_0)$ for some $g\in\mathcal{G}^{\mathbb{C}}(\tilde{\mathcal{E}})$. □

The curvatures of D and D_0 are related by

$$F = g\big(F_0 + \bar{\partial}_0(h^{-1}\partial_0 h)\big)g^{-1},$$

where $h = g^* g$. As D is ASD and D_0 is flat, this gives

$$\Lambda_\rho \bar{\partial}_0(h^{-1}\partial_0 h) = 0.$$

Setting $u \equiv \log h$ then $\triangle |u|^2 \leq 0$ by Lemma B.5, and therefore

$$\triangle^{\mathbb{R}^2} \int_{z\times K} |u|^2 \ d\,\mathrm{vol} = \int_{z\times K} (\triangle^{\mathbb{R}^2} + \triangle^K)\,|u|^2 \ d\,\mathrm{vol} \leq 0. \tag{3.1}$$

Lemma 3.10. *For any $\delta > 0$ there is $R > 0$ such that if $|z| > R$ then $\|D_z - g\cdot(D_0)_z\|_{C^1} < \delta$ for some $g \in \mathcal{G}_{\mathcal{F}}$.*

Proof. Using Corollaries 3.6 and J.10 we obtain a flat connection D_{flat} over $z \times K$ such that $\|D_z - D_{\mathrm{flat}}\|_{C^0} < \delta$ and $[\bar{\partial}_{\mathrm{flat}}] = [\bar{\partial}_{D_z}] = [(\bar{\partial}_0)_z]$. As both D_{flat} and $(D_0)_z$ are flat and in the same stable $\mathcal{G}^{\mathbb{C}}$-orbit, they differ by a unitary gauge transformation. □

Corollary 3.11. $\int_{z\times K} |u|^2 \ d\,\mathrm{vol} \to 0$ *as* $|z| \to \infty$.

Proof. Follows from the previous Lemma, Corollary J.15 and Lemma J.14. □

Equation (3.1) then implies $u = 0$ by the maximum principle, and therefore $D = g\cdot D_0$ with $g \in \mathcal{G}(\tilde{\mathcal{E}})$. This contradicts the fact that D is non-flat, proving

Proposition 3.12. $\Sigma_3 = \emptyset$.

3.3 Slow Bubbles.

Let $z_0 \in \Sigma_2$. As $z_0 \in C_2$ and $|F_n^{KK}|_{\rho_1} = |F_n^{KK}|_{\rho_n}$, for any $\delta > 0$ there is a neighborhood U of z_0 such that

$$\sup_{U \times K} |F_n^{KK}|_{\rho_1} < \delta. \tag{3.2}$$

for all n large enough. This means if we applied directly the techniques from the previous section we would end up with a flat connection D, which would not provide the desired contradiction. On the other hand, as $z_0 \notin C_1$ we can, after passing to a subsequence of the connections, find a sequence $z_n' \to z_0$ such that

$$\sup_{z_n' \times K} \lambda_n^{-1} |F_n|_{\rho_n} \to \infty.$$

By Lemma K.1 we can, after possibly changing to a different sequence $z_n' \to z_0$, assume

$$\lambda_n^{-1} \|F_n\|_{L^2(\rho_n, z_n' \times K)} \to \infty. \tag{3.3}$$

By expanding the metric on Σ at an appropriate rate we can get just enough control over the curvature to be able to use the Compactness Theorem K.3. The rate of expansion will be determined from the following function.

Definition. For any $z \in \Sigma$ and any $n \in \mathbb{N}$ let

$$\begin{aligned} e_n(z) &= \lambda_n^{-1} \|F_n\|_{L^2(\rho_n, z \times K)} \\ &= \sqrt{\lambda_n^2 \|F_n^{\Sigma\Sigma}\|^2_{L^2(\rho_1, z \times K)} + \|F_n^{\Sigma K}\|^2_{L^2(\rho_1, z \times K)} + \lambda_n^{-2} \|F_n^{KK}\|^2_{L^2(\rho_1, z \times K)}}\,. \end{aligned}$$

In order to obtain control over the curvature not only at z_n' but on a neighborhood we use the following Lemma due to Hofer (see [DS94, p. 631]).

Lemma 3.13. *Let M be a complete metric space and $f : M \to \mathbb{R}$ be continuous and nonnegative. Given $z' \in M$ and $r' > 0$ there exist $z \in M$ and $0 < r < r'$ such that*

$$d(z, z') \le r', \qquad \sup_{B_r(z)} f \le 2f(z), \qquad rf(z) \ge r'f(z')/2.$$

Here $B_r(z)$ is the geodesic ball of radius r centered at z.

Equation (3.3) implies $e_n(z_n') \to \infty$, so applying this Lemma with the metric ρ_1, $f = e_n$, $z' = z_n'$ and $r' = 1/\sqrt{e_n(z_n')}$ we obtain new sequences $z_n \to z_0$ and $r_n \to 0$ satisfying

$$\begin{gathered} \sup_{B_n} e_n \le 2e_n(z_n), \\ r_n e_n(z_n) \to \infty, \end{gathered}$$

where B_n is the geodesic ball centered at z_n of radius r_n with respect to ρ_1^Σ.

Let $\lambda_n' = e_n(z_n)^{-1}$ and $\alpha_n = \lambda_n/\lambda_n'$ and define a new sequence of metrics on $\Sigma \times K$ by

$$\rho_n' \equiv \frac{\rho^\Sigma}{{\lambda_n'}^2} \oplus \rho^K = (\alpha_n^2\, \rho_n^\Sigma) \oplus \rho_n^K.$$

With respect to these metrics,

$$\begin{aligned} \alpha_n^2 \|F_n^{\Sigma\Sigma}\|^2_{L^2(\rho_n', z_n \times K)} &+ \|F_n^{\Sigma K}\|^2_{L^2(\rho_n', z_n \times K)} \\ &+ \alpha_n^{-2} \|F_n^{KK}\|^2_{L^2(\rho_n', z_n \times K)} \;=\; {\lambda_n'}^2 e_n(z_n)^2 \;=\; 1, \end{aligned}$$

and

$$\alpha_n |F_n^{\Sigma\Sigma}|_{\rho_n'} = \alpha_n^{-1} |F_n^{\Sigma\Sigma}|_{\rho_n} = \alpha_n^{-1} |F_n^{KK}|_{\rho_n} = \alpha_n^{-1} |F_n^{KK}|_{\rho_n'}.$$

Also, by construction, for any $z \in B_n$,

$$\alpha_n^{-1} \|F_n\|_{L^2(\rho_n, z \times K)} = \frac{e_n(z)}{e_n(z_n)} \leq 2.$$

Let φ_n, $\tilde{\rho}_n$, $\tilde{D}_n$, $\tilde{F}_n$ and $\tilde{g}_n$ be as in Definition 3.1 with λ'_n instead of λ_n, and let $\tilde{\rho}'_n = \varphi_n^* \rho'_n$. Then $\tilde{\rho}'_n \to \epsilon \oplus \rho^K$, $\tilde{D}_n$ is ASD with respect to $\tilde{\rho}_n$,

$$\alpha_n^2 \|\tilde{F}_n^{\mathbb{R}\mathbb{R}}\|^2_{L^2(\tilde{\rho}'_n, 0 \times K)} + \|\tilde{F}_n^{\mathbb{R}K}\|^2_{L^2(\tilde{\rho}'_n, 0 \times K)} + \alpha_n^{-2} \|\tilde{F}_n^{KK}\|^2_{L^2(\tilde{\rho}'_n, 0 \times K)} \;=\; 1, \tag{3.4}$$

$$\alpha_n |\tilde{F}_n^{\mathbb{R}\mathbb{R}}|_{\tilde{\rho}'_n} = \alpha_n^{-1} |\tilde{F}_n^{KK}|_{\tilde{\rho}'_n}, \tag{3.5}$$

and

$$\sup_{z \in \tilde{B}_n} \|\tilde{F}_n\|_{L^2(\tilde{\rho}_n, z \times K)} \leq 2\alpha_n,$$

where $\tilde{B}_n$ is the ball in $\mathbb{R}^2$ of radius $r_n e_n(z_n)$ with respect to the metric $\tilde{\rho}_n$, so that $\tilde{B}_n \times K = \varphi_n^*(B_n \times K)$. Also, by Lemma K.1, for any bounded open set $U \subset \mathbb{R}^2$ there is a constant $c > 0$ such that

$$\|\tilde{F}_n\|_{L^\infty(\tilde{\rho}_n, U \times K)} \leq c\,\alpha_n.$$

Hence the conditions of Theorem K.3 are satisfied and there exists an adiabatic connection $\tilde{D}_0$ on $U \times K$ such that, after passing to a subsequence and taking gauge transformations,

$$\|\tilde{D}_n - \tilde{D}_0\|_{L^\infty(\Omega \times K)} \to 0, \qquad \sup_{z \in \Omega} \|\tilde{F}_n^{\mathbb{R}K} - \tilde{F}_0^{\mathbb{R}K}\|_{L^2(z \times K)} \to 0,$$

for every compact set $\Omega \subset U$. Moreover, by Lemma K.2, for any $p \geq 2$ there is a constant $c > 0$ such that

$$\|\tilde{F}_n^{KK}\|_{L^2(0 \times K)} \leq c\,\alpha_n^{2-2/p}.$$

Hence equations (3.4) and (3.5) imply

$$\|\tilde{F}_n^{\mathbb{R}K}\|_{L^2(\tilde{\rho}'_n,0\times K)} > 1/2, \tag{3.6}$$

for all n large enough, and therefore $\tilde{D}_0$ has nonzero curvature.

Defining D_0 as in Definition 3.7 we have

Lemma 3.14. *$\tilde{D}_0 = g \cdot D_0$ for some $g \in \mathcal{G}(\tilde{\mathcal{E}})$.*

Proof. By the same argument as in the proof of Lemma 3.9, $\tilde{D}_0 = g(D_0)$ for some $g \in \mathcal{G}^{\mathbb{C}}(\tilde{\mathcal{E}})$. Along each slice these two connections are both flat and in the same $\mathcal{G}^{\mathbb{C}}_{\mathcal{F}}$-orbit, so by Corollary J.8 we get $g \in \mathcal{G}$. □

This provides a contradiction, as D_0 is flat but $\tilde{D}_0$ is not, and therefore

Proposition 3.15. *$\Sigma_2 = \emptyset$.*

3.4 Fast Decay.

From the definition we have that for any open set $U \subset\subset \Sigma_1$ there is a constant $c > 0$ such that $|F_n|_{\rho_n} < c\lambda_n$ over $U \times K$ for all n. Therefore

$$|F_n^{\Sigma\Sigma}|_{\rho_1} < c\lambda_n^{-1},$$
$$|F_n^{\Sigma K}|_{\rho_1} < c,$$
$$|F_n^{KK}|_{\rho_1} < c\lambda_n,$$

on $U \times K$. These estimates will be useful in the next section when we deal with Σ_4.

3.5 Point Bubbles.

If $z \in \Sigma_4$ it is possible, after passing to a subsequence, to find a $k \in K$ and a sequence $(z_n, k_n) \to (z, k)$ such that $|F_n(z_n, k_n)|_{\rho_n} \to \infty$.

Pulling back to $\mathbb{R}^2 \times K$ as before we obtain a sequence of ASD connections with $|\tilde{F}_n(0, k_n)|_{\tilde{\rho}_n} \to \infty$. For any neighborhood V of z Corollary I.5 implies

$$\int_{V \times K} |F_n|^2_{\rho_n} \; d\,\mathrm{vol}_{\rho_n} = \int_{\varphi_n^*(V \times K)} |\tilde{F}_n|^2_{\tilde{\rho}_n} \; d\,\mathrm{vol}_{\tilde{\rho}_n} > \frac{\varepsilon^2}{2},$$

for all n large enough. Redefine C_3 and Σ_4 using this subsequence instead of the original sequence. If there is still another $z' \in \Sigma_3$, repeating the same argument (and therefore passing to a further subsequence) we obtain the same for any neighborhood V' of z'. Choosing V and V' to be disjoint we obtain

$$\int_{(V \cup V') \times K} |F_n|^2_{\rho_n} \; d\,\mathrm{vol}_{\rho_n} > 2 \cdot \frac{\varepsilon^2}{2},$$

for all n large enough. This process can be repeated inductively for as long as we can find new points in (the inductively redefined) Σ_4. But as these integrals are bounded by $8\pi^2 c_2(\mathcal{E})$, after a number of steps we should reach a subsequence for which Σ_4 is a finite set.

So without loss of generality we can assume $z \in \Sigma_4$ has a neighborhood V containing no other points of Σ_4. Any disc $D \subset\subset V$ centered at z satisfies

$$\int_{D \times K} |F_n|^2_{\rho_n} \; d\,\mathrm{vol}_{\rho_n} > \frac{\varepsilon^2}{2}, \tag{3.7}$$

for all n large enough. As $D_n = g_n(D_{\mathrm{ad}})$, Proposition D.7 gives

$$
\begin{aligned}
\int_{D\times K} |F_n|^2_{\rho_n} \; d\,\mathrm{vol}_{\rho_n} &= -\int_{D\times K} \big(F_n \,,\, F_n\big) \\
&= -\int_{D\times K} \big(F_{\mathrm{ad}} \,,\, F_{\mathrm{ad}}\big) \\
&\quad + \int_{\partial D\times K} \mathrm{tr}\Big(\big(F_{\mathrm{ad}} + g_n^{-1} F_n g_n\big) \wedge h_n^{-1} \partial_{\mathrm{ad}} h_n\Big),
\end{aligned}
$$

where $h_n = g_n^* g_n$. The first integral on the right hand side of the equation can be made as small as we want by choosing D small enough. The second integral decomposes into three pieces:

$$
\begin{aligned}
&\int_{\partial D\times K} \mathrm{tr}\Big(F_{\mathrm{ad}}^{\Sigma K} \wedge h_n^{-1} \partial_{\mathrm{ad}}^{K} h_n\Big), \\
&\int_{\partial D\times K} \mathrm{tr}\Big(g_n^{-1} F_n^{\Sigma K} g_n \wedge h_n^{-1} \partial_{\mathrm{ad}}^{K} h_n\Big), \\
&\int_{\partial D\times K} \mathrm{tr}\Big(g_n^{-1} F_n^{KK} g_n \wedge h_n^{-1} \partial_{\mathrm{ad}}^{\Sigma} h_n\Big).
\end{aligned}
$$

As $\Sigma_2 = \Sigma_3 = \emptyset$ and we assumed $V \cap \Sigma_4 = \emptyset$, then $\partial D \subset V \subset \Sigma_1$ and from the previous section we have

$$
\begin{aligned}
|F_n^{\Sigma K}|_{\rho_1} &< c, \\
|F_n^{KK}|_{\rho_1} &< c\lambda_n.
\end{aligned}
$$

The results of Appendix J then imply that $h_n^{-1}\partial_{\mathrm{ad}}^{K} h_n \to 0$ and $h_n^{-1}\partial_{\mathrm{ad}}^{\Sigma} h_n$ remains bounded. All these estimates are with respect to the fixed metric ρ_1, so the volume of $\partial D \times K$ does not change and therefore these three integrals decay to 0 as $n \to \infty$. This contradicts (3.7), what proves

Proposition 3.16. $\Sigma_4 = \emptyset$.

3.6 Conclusion

As the sequence $\{\lambda_n\}$ is arbitrary, showing $\Sigma = \Sigma_1$ proves

Theorem 3.17. *There is a constant $c > 0$ such that $|F_\lambda|_{\rho_\lambda} < c\lambda$, and therefore*

$$|F_\lambda^{\Sigma\Sigma}|_{\rho_\lambda} < c\lambda^{-1},$$
$$|F_\lambda^{\Sigma K}|_{\rho_\lambda} < c,$$
$$|F_\lambda^{KK}|_{\rho_\lambda} < c\lambda,$$

on $\Sigma \times K$ for all $\lambda > 0$.

Due to time constraints, we will not pursue here the natural development of this work, that would be to use these results to obtain some sort of convergence of $[D_\lambda]$ to $[D_{\mathrm{ad}}]$. However, on a subsequent work it should not be difficult to obtain from these estimates a result like the one in [HI98]:

Theorem 3.18. *For any $\epsilon > 0$ there is a $\lambda_0 > 0$ such that for any $0 < \lambda < \lambda_0$ there is a $g_\lambda \in \mathcal{G}$ such that*

$$\| g_\lambda(D_\lambda) - D_{ad} \|_{L_1^p(\rho_\lambda, \Sigma\times K)} < \epsilon.$$

Remark. Note that here the distance between $[D_\lambda]$ and $[D_{\mathrm{ad}}]$ in measured with respect to ρ_λ instead of a fixed metric.

Appendices

Appendix A

Kähler Manifolds and Riemann Surfaces

A Kähler manifold M is a complex manifold with a compatible symplectic form ω, called its *Kähler form.* By compatible we mean $\omega(Jv, Jv') = \omega(v, v')$ and $\omega(v, Jv) \geq 0$ for all $v, v' \in T_pM$, where $J : T_pM \to T_pM$, $J^2 = -1$, is the almost complex structure.

The condition $\omega(Jv, Jv') = \omega(v, v')$ implies that ω has holomorphic type $(1,1)$, and $\omega(v, Jv) \geq 0$ allows us to define a Riemannian metric on M by

$$\big(v\,,\,v'\big)_\omega \equiv \omega(v, Jv').$$

This extends to a hermitian inner product on $T_{\mathbb{C}}M$ by linearity in the first argument and anti-linearity in the second one.

Remark. Some authors use $\big(v\,,\,v'\big)_\omega \equiv 2\omega(v, Jv')$ because they adopt the convention that $dx \wedge dy(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}) = \frac{dx\otimes dy - dy\otimes dx}{2!}(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}) = \frac{1}{2}$. We instead use $dx \wedge dy = dx \otimes dy - dy \otimes dx$.

The relation between the Riemannian and complex structures on a Kähler manifold provides us with many useful properties, some of which we include here.

Lemma A.1. *Let* $n = \dim_{\mathbb{C}} M$. *Then* $\frac{\omega^n}{n!} = d\,\mathrm{vol}$.

Definition. Define a 'trace' operator $\Lambda : \Omega^{p,q}(M) \to \Omega^{p-1,q-1}(M)$ by

$$\big(\Lambda\eta\,,\,\eta'\big)_\omega = \big(\eta\,,\,\eta'\wedge\omega\big)_\omega\,,$$

for all $\eta \in \Omega^{p,q}$, $\eta' \in \Omega^{p-1,q-1}$.

Lemma A.2. *If η is a 2-form then*

$$\Lambda\eta = \big(\eta\,,\,\omega\big)_\omega = \big(\eta^{1,1}\,,\,\omega\big)_\omega\,,$$

where $\eta^{1,1}$ is the $(1,1)$ component of η. In particular,

$$\Lambda\omega = |\omega|^2_\omega = \dim_{\mathbb{C}} M$$

Lemma A.3. *For any $\eta, \eta' \in \Omega^{1,0}$,*

$$\Lambda(\eta\wedge\bar{\eta}') = -i\big(\eta\,,\,\eta'\big)_\omega.$$

Proof. This follows from the fact that ω can be expressed as

$$\omega = i\sum_j \theta_j\wedge\bar{\theta}_j,$$

with $\{\theta_j\}$ being an orthonormal basis for $(T^*_{\mathbb{C}}M)^{1,0}$. □

Proposition A.4. *Let D be an integrable connection on a bundle over a compact Kähler manifold. Then*

$$\Lambda\bar{\partial}_D - \bar{\partial}_D\Lambda = -i\,\partial^*_D, \tag{A.1}$$

$$\Lambda\partial_D - \partial_D\Lambda = +i\,\bar{\partial}^*_D. \tag{A.2}$$

Proof. See [Kob87, p. 65] for proof. □

On an oriented Riemann surface S the Hodge star operator takes 1-forms into 1-forms, i.e. $\star : \Omega^1 \to \Omega^1$, and $\star^2 = -1$ on 1-forms. So the complex valued 1-forms split orthogonally as $\Omega^1_{\mathbb{C}} = \Omega^{1,0} \oplus \Omega^{0,1}$ into the $-i$ and $+i$ (respect.) eigenspaces of $\star$. This gives the surface a complex manifold structure (which depends only on the conformal class of the metric and won't change under rescaling). Moreover, a Riemann surface is naturally a Kähler manifold, with its volume form being the Kähler form.

Appendix B

Hermitian Vector Bundles

Let $\mathcal{E} \to M$ be a complex vector bundle. A *hermitian structure* or *hermitian metric* H on $\mathcal{E}$ is a hermitian inner product on each fiber $\mathcal{E}_p$, varying smoothly with $p \in M$. We call $(\mathcal{E}, H)$ a *hermitian vector bundle.*

Definition. $\mathcal{G} = \{$bundle automorphisms of $\mathcal{E}$ preserving $H\}$.

Definition. $\mathcal{G}^{\mathbb{C}} = \{$bundle automorphisms of $\mathcal{E}\}$.

The elements of $\mathcal{G}$ are referred to as the *(unitary) gauge transformations* of $(\mathcal{E}, H)$, while the elements of $\mathcal{G}^{\mathbb{C}}$ are the *complex gauge transformations* of $\mathcal{E}$. The reason for this terminology is that $\mathcal{G}^{\mathbb{C}}$ is the complexification of the group $\mathcal{G}$.

Definition. A connection D on $(\mathcal{E}, H)$ is *unitary* or *H-compatible* if it satisfies

$$d\big(\xi\,,\,\eta\big) = \big(D\xi\,,\,\eta\big) + \big(\xi\,,\,D\eta\big),$$

for all $\xi, \eta \in \Gamma(\mathcal{E})$. Here $\big(\cdot\,,\,\cdot\big)$ is the hermitian inner product on the fibers given by H.

Definition. A *frame* for $\mathcal{E}$ over $U \subset M$ is a collection of local sections $\{\xi_j\}_{j=1,\ldots,\mathrm{rk}\,\mathcal{E}}$ defined over U and forming a basis $\{\xi_j(p)\}$ for each fiber $\mathcal{E}_p$, $p \in U$.

Definition. A *unitary frame* is a frame $\{\xi_j\}$ with $\big(\xi_j\,,\,\xi_k\big) = \delta_{ij}$.

Definition. The *connection matrix* of a connection D with respect to a frame $\{\xi_j\}$ is the matrix of 1-forms A_j^k defined by $D\,\xi_j = A_j^k\,\xi_k$. Its *curvature matrix* is the matrix of 2-forms F_j^k given by $F\,\xi_j = F_j^k\,\xi_k$.

Lemma B.1. *If D is a unitary connection then its connection and curvature matrices with respect to a unitary frame satisfy $A_j^k = -\bar{A}_k^j$, $F_j^k = -\bar{F}_k^j$.*

Proof. If D and the frame are unitary we have

$$0 = d\big(\xi_j\,,\,\xi_k\big) = \big(A_j^l\,\xi_l\,,\,\xi_k\big) + \big(\xi_j\,,\,A_k^m\,\xi_m\big) = A_j^k + \bar{A}_k^j. \tag{B.1}$$

The curvature matrix is given by $F_j^k\,\xi_k = D^2\,\xi_j = (dA_j^k - A_j^l \wedge A_l^k)\,\xi_k$. Thus

$$\bar{F}_k^j = d\bar{A}_k^j - \bar{A}_k^l \wedge \bar{A}_l^j = -dA_j^k - A_l^k \wedge A_j^l = -F_j^k. \tag{B.2}$$

$\square$

Lemma B.2. *Let D be a unitary connection over a complex manifold and A_j^k be its connection matrix with respect to a unitary frame $\{\xi_j\}$. Let $\{\xi_j^*\}$ denote the dual frame and $a_j^k \equiv (A^{1,0})_j^k$. Then*

$$\partial_D\,\xi_j = a_j^k\,\xi_k, \qquad \bar{\partial}_D\,\xi_j = -\bar{a}_k^j\,\xi_k, \tag{B.3}$$

$$\partial_D\,\xi_j^* = -a_k^j\,\xi_k^*, \qquad \bar{\partial}_D\,\xi_j^* = \bar{a}_j^k\,\xi_k^*. \tag{B.4}$$

Its curvature matrix satisfies

$$(F^{2,0})_j^k = -(\overline{F^{0,2}})_k^j, \qquad (F^{1,1})_j^k = -(\overline{F^{1,1}})_k^j. \tag{B.5}$$

Proof. Equations (B.3) and (B.5) are obtained by splitting equations (B.1) and (B.2) according to holomorphic type. By definition $(D\,\xi_j^*)(\xi_k) \equiv d\big(\xi_j^*(\xi_k)\big) - \xi_j^*(D\,\xi_k)$. As $\xi_j^*(\xi_k) = \delta_{ij}$, splitting $(D\,\xi_j^*)(\xi_k) = -\xi_j^*(D\,\xi_k)$ according to holomorphic type gives the remaining equations. □

Lemma B.3. *Let D be a unitary connection on a hermitian bundle $\mathcal{E}$ over a complex manifold. Then $(\bar\partial_D u)^* = \partial_D u^*$ for any $u \in \Omega^0(\operatorname{End}\mathcal{E})$.*

Proof. Write u locally with respect to a unitary frame as $u = u_k^j\,\xi_j \otimes \xi_k^*$. If a_j^k is as in Lemma B.2 then

$$\partial_D u = (\partial u_k^j + u_k^l a_l^j - u_l^j a_k^l)\,\xi_j \otimes \xi_k^*, \tag{B.6}$$

$$\bar\partial_D u = (\bar\partial u_j^k - u_j^l \bar a_k^l + u_l^k \bar a_l^j)\,\xi_k \otimes \xi_j^*. \tag{B.7}$$

Therefore

$$(\bar\partial_D u)^* = (\partial \bar u_j^k - \bar u_j^l a_k^l + \bar u_l^k a_l^j)\,\xi_j \otimes \xi_k^* = \partial_D u^*.$$

□

Lemma B.4. *Let D be a unitary connection on a hermitian bundle $\mathcal{E}$ over a complex manifold. If $u \in \Omega^0(\operatorname{End}\mathcal{E})$ is hermitian or skew-hermitian then*

$$|\partial_D u| = |\bar\partial_D u| = \frac{1}{\sqrt{2}}|Du|.$$

Proof. By Lemma B.3, $(\bar\partial_D u)^* = \partial_D u^*$. If $u^* = \pm u$ this implies $|\bar\partial_D u| = |\partial_D u|$, and as $|Du|^2 = |\partial_D u|^2 + |\bar\partial_D u|^2$ this proves the lemma. □

Lemma B.5. *Let D be a unitary connection on a hermitian bundle $\mathcal{E}$ over a Kähler manifold. If $h \in \Omega^0(\operatorname{Aut}\mathcal{E})$ is positive definite self-adjoint and $u = \log h$ then*

$$\frac{1}{4}\triangle\,|u|^2 \,\leq\, -i\big(u\,,\,\Lambda\bar{\partial}_D(h^{-1}\partial_D h)\big).$$

Proof. It is enough to prove it locally and for a dense subset of $C^\infty(\operatorname{Aut}\mathcal{E})$. Hence as h is positive definite self-adjoint we can assume it has distinct eigenvalues locally and can be diagonalized as $h = \sum_j e^{\lambda_j}\,\xi_j \otimes \xi_j^*$, with $\lambda_j \in \mathbb{R}$ and $\{\xi_j\}$ forming an unitary frame. Then $u = \sum_j \lambda_j\;\xi_j \otimes \xi_j^*$. Defining a_j^k as in Lemma B.2 we have

$$\partial_D u = \sum_j \partial\lambda_j\;\xi_j \otimes \xi_j^* + \sum_{j,k}(\lambda_j - \lambda_k)\,a_j^k\;\xi_k \otimes \xi_j^*, \tag{B.8}$$

$$h^{-1}\partial_D h = \sum_j \partial\lambda_j\;\xi_j \otimes \xi_j^* + \sum_{j,k}(e^{\lambda_j-\lambda_k} - 1)\,a_j^k\;\xi_k \otimes \xi_j^*, \tag{B.9}$$

$$\big(u\,,\,h^{-1}\partial_D h\big) = \sum_j \lambda_j\,\bar{\partial}\lambda_j = \frac{1}{2}\,\bar{\partial}\sum_j \lambda_j^2 = \frac{1}{2}\,\bar{\partial}\,|u|^2, \tag{B.10}$$

$$\big(\partial_D u\,,\,h^{-1}\partial_D h\big)_\rho = \sum_j |\partial\lambda_j|^2 + \sum_{j,k}(\lambda_j - \lambda_k)(e^{\lambda_j-\lambda_k} - 1)\,|a_j^k|^2. \tag{B.11}$$

Using Lemma A.3 and Proposition A.4 we obtain

$$\begin{aligned}
\big(u\,,\,\Lambda\bar{\partial}_D(h^{-1}\partial_D h)\big) &= \Lambda\partial\big(u\,,\,h^{-1}\partial_D h\big) - \Lambda\big(\partial_D u\,,\,h^{-1}\partial_D h\big) \\
&= \frac{i}{2}\,\bar{\partial}^*\bar{\partial}\,|u|^2 + i\big(\partial_D u\,,\,h^{-1}\partial_D h\big)_\rho \\
&= \frac{i}{4}\,\triangle\,|u|^2 + i\big(\partial_D u\,,\,h^{-1}\partial_D h\big)_\rho.
\end{aligned}$$

The result then follows from the fact that as $x(e^x - 1) \geq 0$ for any $x \in \mathbb{R}$, $\big(\partial_D u\,,\,h^{-1}\partial_D h\big)_\rho \geq 0$. $\square$

Appendix C

Holomorphic Vector Bundles

Let $\mathcal{E} \to M$ be a complex vector bundle over a complex manifold. A *holomorphic structure* on $\mathcal{E}$ is any of the following equivalent structures:

Proposition C.1. *Let $\mathcal{E} \to M$ be a complex vector bundle over a complex manifold. The following structures are equivalent (i.e., the existence of one induces in a unique way the other):*

1. *A maximal family of local trivializations $\varphi_U : \mathcal{E}|_U \to U \times \mathbb{C}^k$ covering $\mathcal{E}$ such that the transition functions are holomorphic functions.*

2. *Operators $\bar{\partial} : \Omega^{p,q}(M; \mathcal{E}) \to \Omega^{p,q+1}(M; \mathcal{E})$ which satisfy the integrability condition $\bar{\partial} \circ \bar{\partial} = 0$ and the Leibnitz rule*

$$\bar{\partial}(\sigma\theta) = (\bar{\partial}\sigma) \wedge \theta + \sigma\bar{\partial}\theta$$

for all $\sigma \in \Gamma(\mathcal{E})$, $\theta \in \Omega^{p,q}(M)$.

Proof. The Leibnitz condition implies $\bar{\partial}$ is a local operator, so it's enough to define it for local sections. Let $\sigma_1, \ldots, \sigma_k \in \Gamma(\mathcal{E}|_U)$ be local sections corresponding under the trivialization φ_U to a given frame of holomorphic functions $s_1, \ldots, s_k : U \to \mathbb{C}^k$. We set $\bar{\partial}\sigma_i \equiv 0$ and extend it to $\mathcal{E}$-valued forms over U

using the Leibnitz rule. As $\bar{\partial}^2 = 0$ on forms, this definition implies the same property for $\bar{\partial}$ on $\mathcal{E}$-valued forms.

Any other choice of frame would differ from $\{s_i\}$ by a holomorphic function $U \to GL(k, \mathbb{C})$ and would therefore define the same $\bar{\partial}$. In the same way, as the transition functions are holomorphic, the $\bar{\partial}$ operators defined on different trivializations agree at the intersections, giving a global operator.

Conversely, the condition $\bar{\partial}^2 = 0$ is a sufficient condition for the local existence of a basis of sections $\{\sigma_1, \ldots \sigma_k\}$ satisfying $\bar{\partial}\sigma_i = 0$ (see [DK90, p. 50] for a proof). Each such basis induces a local trivialization. The Leibnitz rule implies that the transition functions between any two trivializations constructed in this way will be holomorphic. □

Definition. $(\mathcal{E}, \bar{\partial})$ is a *holomorphic vector bundle.*

Definition. Two holomorphic structures $\bar{\partial}_1$, $\bar{\partial}_2$ on $\mathcal{E}$ are *equivalent* if they differ by a complex gauge transformation, i.e. if $\bar{\partial}_2 = g \circ \bar{\partial}_1 \circ g^{-1}$ for some $g \in \mathcal{G}^{\mathbb{C}}$.

Notation. $g \cdot \bar{\partial} \equiv g \circ \bar{\partial} \circ g^{-1} = \bar{\partial} + g\bar{\partial}g^{-1}$.

Notation. $[\bar{\partial}] \equiv \{g \cdot \bar{\partial} : g \in \mathcal{G}^{\mathbb{C}}\}$

Definition. A *holomorphic class* $\mathfrak{h}$ is a set of the form $\mathfrak{h} = [\bar{\partial}]$ for some holomorphic structure $\bar{\partial}$.

Definition. A *holomorphic section* of a holomorphic vector bundle $(\mathcal{E}, \bar{\partial})$ is a section σ satisfying $\bar{\partial}\sigma = 0$. A *holomorphic frame* is a frame consisting of holomorphic sections $\sigma_1, \ldots \sigma_{\mathrm{rk}\,\mathcal{E}}$.

Definition. A connection D on a complex vector bundle over a complex manifold is *integrable* if $F_D^{0,2} = \bar{\partial}_D^2 = 0$.

Definition. A connection D on a holomorphic vector bundle $(\mathcal{E}, \bar{\partial})$ is *compatible with the holomorphic structure* (or simply $\bar{\partial}$*-compatible*) if $\bar{\partial}_D = \bar{\partial}$.

Proposition C.2. *Let D be a connection on $\mathcal{E} \to M$. It is compatible with some holomorphic structure if and only if it is integrable.*

Proof. If D is compatible with a holomorphic structure $\bar{\partial}$ then $F_D^{0,2} = \bar{\partial}_D^2 = \bar{\partial}^2 = 0$. On the other hand, if $F_D^{0,2} = 0$ then $\bar{\partial} \equiv \bar{\partial}_D$ defines a holomorphic structure. $\square$

Appendix D

Hermitian Holomorphic Bundles

Lemma D.1. *An unitary connection D is integrable iff $F_D \in \Omega^{1,1}(\mathrm{End}\,\mathcal{E})$.*

Proof. If D is integrable then by Lemma B.1 we have $F_D^{2,0} = (-F_D^*)^{2,0} = -(F_D^{0,2})^* = 0$, hence $F_D = F_D^{1,1}$. The converse is imediate from the definition. $\square$

Proposition D.2. *Let $(\mathcal{E}, H, \bar\partial)$ be a holomorphic hermitian vector bundle. It has a unique $\bar\partial$-compatible unitary connection $D = D_{H,\bar\partial}$.*

Proof. If D is $\bar\partial$-compatible then by definition $\bar\partial_D = \bar\partial$. If it is also compatible with the hermitian metric, then its (1,0) component ∂_D is uniquely determined (at least locally) with respect to some holomorphic frame $\{\sigma_j\}$ by the relation

$$\big(\partial_D\,\sigma_j\,,\,\sigma_k\big) = \partial\big(\sigma_j\,,\,\sigma_k\big) - \big(\sigma_j\,,\,\bar\partial\,\sigma_k\big) = \partial\big(\sigma_j\,,\,\sigma_k\big)$$

and the Leibnitz rule.

Given another holomorphic frame $\{\sigma'_j\}$, the transition functions defined by $\sigma'_j = g_{jl}\,\sigma_l$ satisfy $\bar\partial g_{jl} = 0$ (hence $\partial \bar g_{jl} = 0$). Therefore

$$\begin{aligned}
\big(\partial_D\,\sigma'_j\,,\,\sigma'_k\big) &= \big((\partial g_{jl})\,\sigma_l + g_{jl}\,\partial_D\,\sigma_l\,,\,g_{km}\,\sigma_m\big)\\
&= (\partial g_{jl})\,\bar g_{km}\,\big(\sigma_l\,,\,\sigma_m\big) + g_{jl}\,\bar g_{km}\,\partial\big(\sigma_l\,,\,\sigma_m\big)\\
&= \partial(g_{jl}\,\bar g_{km}\big(\sigma_l\,,\,\sigma_m\big))\\
&= \partial\big(\sigma'_j\,,\,\sigma'_k\big).
\end{aligned}$$

Thus the relation defining ∂_D is not dependent on the choice of holomorphic trivialization, and gives a globally well defined operator. □

Notation. $F_{H,\bar\partial} \;\equiv\; F_{D_{H,\bar\partial}}$

Corollary D.3. $F_{H,\bar\partial} \in \Omega^{1,1}(\mathrm{End}\,\mathcal{E})$

Corollary D.4. *Let $(\mathcal{E}, H)$ be a hermitian vector bundle. The maps $\bar\partial \mapsto D_{H,\bar\partial}$ and $D \mapsto \bar\partial_D$ establish a 1-1 correspondence between holomorphic structures and integrable unitary connections on $\mathcal{E}$.*

Definition. $g \in \mathcal{G}^{\mathbb{C}}$ has three actions on connections on $(\mathcal{E}, H)$:

$$g\cdot D \;\equiv\; g\circ D\circ g^{-1} \;=\; D + g(Dg^{-1}), \tag{D.1}$$

$$\begin{aligned}
g(D)_H \;&\equiv\; (g^{*_H})^{-1}\circ\partial_D\circ g^{*_H} + g\circ\bar\partial_D\circ g^{-1} \\
&=\; D + (g^{*_H})^{-1}(\partial_D g^{*_H}) + g(\bar\partial_D g^{-1}),
\end{aligned}\tag{D.2}$$

$$\begin{aligned}
g\langle D\rangle_H \;&\equiv\; g^{-1}\cdot(g(D)_H) \\
&=\; h^{-1}\circ\partial_D\circ h + \bar\partial_D \;=\; D + h^{-1}(\partial_D h).
\end{aligned}\tag{D.3}$$

Here g^{*_H} means the dual of g with respect to H and $h \;\equiv\; g^{*_H}g$. The H subscripts will be omitted if no confusion is possible.

Remark. If $g \in \mathcal{G}$ then $g(D)_H = g \cdot D$ and $g\langle D \rangle_H = D$.

Proposition D.5. *On a Kähler manifold the following diagram is commutative:*

$$\begin{tikzcd} & D_{\bar\partial, H\circ g} \arrow[dr, "g\cdot"] & \\ D_{\bar\partial, H} \arrow[ur, "g\langle\ \rangle_H"] \arrow[rr, "g(\ )_H"'] & & D_{g\cdot\bar\partial, H} \end{tikzcd}$$

Here $H \circ g$ means of course $(H \circ g)(u, v) \equiv H(g(u), g(v))$.

Proof. See [Bra90] for proof. □

Definition. The *gauge equivalence class* of D is the set

$$[D] = \{D' : D' = g \cdot D \text{ for some } g \in \mathcal{G}\}.$$

Definition. Given an integrable unitary connection D or a holomorphic class $\mathfrak{h}$ on $(\mathcal{E}, H)$, their *($\mathcal{G}^{\mathbb{C}}$-)orbits* are defined as

$$\mathcal{O}(D) = \{D' : D' = g(D)_H \text{ for some } g \in \mathcal{G}^{\mathbb{C}}\},$$
$$\mathcal{O}(\mathfrak{h}) = \{D' \text{ integrable unitary} : \bar\partial_{D'} \in \mathfrak{h}\}.$$

Clearly $\mathcal{O}(D) = \mathcal{O}([\bar\partial_D])$.

Proposition D.6. $F_{g(D)} = g\big(F_D + \bar\partial_D(h^{-1}\partial_D h)\big)g^{-1}$.

Proposition D.7. *Let D_1, D_2 be unitary integrable connections on a hermitian bundle over a manifold of dimension 4. If $D_2 = g(D_1)$ for some $g \in \mathcal{G}^{\mathbb{C}}$ then*

$$\big(F_2\,,\,F_2\big) = \big(F_1\,,\,F_1\big) - d\,\mathrm{tr}\Big(\big(2F_1 + \bar\partial_1(h^{-1}\partial_1 h)\big) \wedge h^{-1}\partial_1 h\Big),$$

where $h = g^* g$.

Proof. By the previous Proposition $F_2 = g\big(F_1 + \bar\partial_1(h^{-1}\partial_1 h)\big)g^{-1}$, so

$$\begin{aligned}\big(F_2\,,\,F_2\big) &= -\operatorname{tr}\Big(\big(F_1 + \bar\partial_1(h^{-1}\partial_1 h)\big) \wedge \big(F_1 + \bar\partial_1(h^{-1}\partial_1 h)\big)\Big)\\ &= -\operatorname{tr}(F_1 \wedge F_1) - 2\operatorname{tr}\big(F_1 \wedge \bar\partial_1(h^{-1}\partial_1 h)\big) -\\ &\quad - \operatorname{tr}\big(\bar\partial_1(h^{-1}\partial_1 h) \wedge \bar\partial_1(h^{-1}\partial_1 h)\big).\end{aligned}$$

Using the hypothesis that $F_1^{2,0} = F_1^{0,2} = 0$ the Bianchi identity becomes $\bar\partial_1 F_1 = \partial_1 F_1 = 0$, and as the dimension is 4 we have

$$\begin{aligned}\operatorname{tr}\big(F_1 \wedge \bar\partial_1(h^{-1}\partial_1 h)\big) &= \bar\partial \operatorname{tr}(F_1 \wedge h^{-1}\partial_1 h) = \bar\partial \operatorname{tr}(F_1 \wedge h^{-1}\partial_1 h)\\ &= d \operatorname{tr}(F_1 \wedge h^{-1}\partial_1 h),\\ \operatorname{tr}\big(\bar\partial_1(h^{-1}\partial_1 h) \wedge \bar\partial_1(h^{-1}\partial_1 h)\big) &= \bar\partial \operatorname{tr}\big(\bar\partial_1(h^{-1}\partial_1 h) \wedge h^{-1}\partial_1 h\big) -\\ &\quad - \operatorname{tr}\big(\bar\partial_1\bar\partial_1(h^{-1}\partial_1 h) \wedge h^{-1}\partial_1 h\big)\\ &= d \operatorname{tr}\big(\bar\partial_1(h^{-1}\partial_1 h) \wedge h^{-1}\partial_1 h\big).\end{aligned}$$

□

Proposition D.8. *Let D, D' be integrable $SU(2)$ connections on a $SU(2)$ bundle $\mathcal{E}$ over a compact complex manifold. If $D' = g(D)$ for some $g \in \Omega^0(\operatorname{End}\mathcal{E})$ then, after a normalization, we can assume* $\det g = 1$ *and* $g \in \mathcal{G}^{\mathbb{C}}$.

Proof. As D and D' are $SU(2)$ connections they both induce the trivial connection on the associated bundle $\det\mathcal{E}$, and g induces the transformation $\det g$. The relation $\bar\partial_{D'} = \bar\partial_D - (\bar\partial_D g)g^{-1}$ becomes $\bar\partial = \bar\partial - (\bar\partial \det g)(\det g)^{-1}$ and so $\bar\partial \det g = 0$. As the base manifold is compact $\det g$ is a constant. The relation $D' = g(D)$ remains valid if g is replaced by a constant multiple of itself, therefore can normalize g in order to have $\det g = 1$. □

Appendix E

Notation for Inner Products and Norms

On a Riemannian manifold M, denote by $\left(\cdot\,,\,\cdot\right)_\rho$ the pointwise inner product of forms induced by the metric ρ, and by $\left\langle\cdot\,,\,\cdot\right\rangle_\rho$ the global inner product given by

$$\left\langle\cdot\,,\,\cdot\right\rangle_\rho \equiv \int_M \left(\cdot\,,\,\cdot\right)_\rho \;\; d\,\mathrm{vol}_\rho\,.$$

An inner product on the fibers of a vector bundle $\mathcal{E}$ over M will be represented as $\left(\cdot\,,\,\cdot\right)$. This product applied to a pair of forms with values on $\mathcal{E}$ denotes the wedge of the forms and the inner product of their fiber values. On the other hand, to take the products of both the fiber values and the forms, employ again the notation $\left(\cdot\,,\,\cdot\right)_\rho$. This same notation will also be used for an $\mathcal{E}$-valued form and a scalar valued one, in which case we take the inner product of the forms leaving the $\mathcal{E}$ values multiplied by the scalars.

The convention is that a hermitian inner product is $\mathbb{C}$-linear in the first argument and $\mathbb{C}$-antilinear in the second one.

In short, we can have:

$$
\begin{aligned}
\big(\cdot\,,\,\cdot\big)_\rho &: \Omega^{p,q} \times \Omega^{p,q} \;\to\; \mathrm{Maps}(M, \mathbb{C}),\\
\big(\cdot\,,\,\cdot\big)_\rho &: \Omega^{p,q} \times \Omega^{p,q}(\mathcal{E}) \;\to\; \Gamma(\mathcal{E}),\\
\big(\cdot\,,\,\cdot\big)_\rho &: \Omega^{p,q}(\mathcal{E}) \times \Omega^{p,q}(\mathcal{E}) \;\to\; \mathrm{Maps}(M, \mathbb{C}),\\
\big(\cdot\,,\,\cdot\big) \;\; &: \Omega^{p,q}(\mathcal{E}) \times \Omega^{p',q'}(\mathcal{E}) \;\to\; \Omega^{p+p',q+q'},\\
\big\langle\cdot\,,\,\cdot\big\rangle_\rho &: \Omega^{p,q} \times \Omega^{p,q} \;\to\; \mathbb{C},\\
\big\langle\cdot\,,\,\cdot\big\rangle_\rho &: \Omega^{p,q}(\mathcal{E}) \times \Omega^{p,q}(\mathcal{E}) \;\to\; \mathbb{C}.
\end{aligned}
$$

Denote by $|\cdot|_\rho$ the pointwise norm and by $\|\cdot\|_\rho$ the integrated norm given by

$$
\begin{aligned}
|\cdot|_\rho &= \sqrt{\big(\cdot\,,\,\cdot\big)_\rho}\\
\|\cdot\|_\rho &= \sqrt{\big\langle\cdot\,,\,\cdot\big\rangle_\rho}
\end{aligned}
$$

For example, for $F \in \Omega^2(M, \Omega^0(\operatorname{End}\mathcal{E}))$ we have $\big(F\,,\,F\big) = \operatorname{tr}(F \wedge F^*)$ and

$$
\|F\|_\rho^2 = \big\langle F\,,\,F\big\rangle_\rho = \int_M \big(F\,,\,*_g F\big) = \int_M \big(F\,,\,F\big)_\rho \; d\,\mathrm{vol}_\rho = \int_M |F|_\rho^2 \; d\,\mathrm{vol}_\rho\,.
$$

Remark. When there is no need to be explicit about which metric is being used we may omit the subscript ρ from $\big\langle\cdot\,,\,\cdot\big\rangle$, $|\cdot|$ and $\|\cdot\|$, but not from $\big(\cdot\,,\,\cdot\big)_\rho$.

L^p_k norms in the metric ρ (or simply $L^p_k(\rho)$ norms) over a subset $U \subset \Sigma \times K$ will be denoted by

$$
\|\cdot\|_{L^p_k(\rho,U)} = \Big(\sum_{|j|\le k} \int_U |\nabla^j_D \cdot|^p_\rho \; d\,\mathrm{vol}_\rho\Big)^{1/p},
$$

and similarly for L^∞, C^k, etc. We omit k if it is 0, ρ when it is clear from the context, and U if it is $\Sigma \times K$.

Appendix F

Notation

We fix some notation here for easy reference:

Σ, K = compact Riemann surfaces,

$(\mathcal{E}, H) = SU(2)$ bundle over $\Sigma \times K$ with metric H,

$(\mathcal{F}, H_{\mathcal{F}})$ = restriction of $(\mathcal{E}, H)$ to any slice $z \times K$,

$\operatorname{End}\mathcal{E}$ = endomorphism bundle of $\mathcal{E}$,

$\operatorname{End}_0 \mathcal{E} = \{g \in \operatorname{End}\mathcal{E} : \operatorname{tr} g = 0\}$,

$\operatorname{Aut}\mathcal{E} = \{g \in \operatorname{End}\mathcal{E} : g \text{ is invertible}\}$,

$\mathcal{G}^{\mathbb{C}}$ = group of complex gauge transformations of $\mathcal{E}$

$= \{g \in \Omega^0(\operatorname{Aut}\mathcal{E}) : \det g = 1\}$,

$\mathcal{G}$ = group of unitary gauge transformations of $(\mathcal{E}, H)$

$= \{g \in \Omega^0(\operatorname{Aut}\mathcal{E}) : g^{*_H} = g^{-1}, \det g = 1\}$,

$\mathcal{G}^{\mathbb{C}}_{\mathcal{F}}$ = group of complex gauge transformations of $\mathcal{F}$,

$\mathcal{G}_{\mathcal{F}}$ = group of unitary gauge transformations of $(\mathcal{F}, H_{\mathcal{F}})$,

$$\mathcal{H} \ = \{\text{holomorphic structures on } \mathcal{E}\}/\mathcal{G}^{\mathbb{C}},$$

$$\mathcal{H}_{\mathcal{F}} = \{\text{holomorphic structures on } \mathcal{F}\}/\mathcal{G}^{\mathbb{C}}_{\mathcal{F}}.$$

Given a holomorphic class $\mathfrak{h} = [\bar{\partial}] \in \mathcal{H}$ and $z \in \Sigma$, denote by $\mathfrak{h}_z$ the class $\mathfrak{h}_z = [\bar{\partial}_z] \in \mathcal{H}_{\mathcal{F}}$.

$$\begin{aligned}
\mathcal{S}_{\mathcal{F}} &= \{\text{stable holomorphic classes on } \mathcal{F}\} \\
&= \{\mathfrak{s} \in \mathcal{H}_{\mathcal{F}} : \mathfrak{s} \text{ is stable}\}, \\
\mathcal{S} &= \{\text{slicewise stable holomorphic classes on } \mathcal{E}\} \\
&= \{\mathfrak{s} \in \mathcal{H} : \mathfrak{s}_z \in \mathcal{S}_{\mathcal{F}} \quad \forall\, z \in \Sigma\}, \\
\mathcal{A} &= \{\text{unitary connections on } (\mathcal{E}, H)\}, \\
\mathcal{A}^{\times} &= \{D \in \mathcal{A} : D_z \text{ is irreducible } \forall\, z \in \Sigma\}, \\
\mathcal{A}^{*}_{\mathcal{F}} &= \{\text{irreducible unitary connections on } \mathcal{F}\}, \\
\mathcal{B}^{\times} &= \mathcal{A}^{\times}/\mathcal{G}, \\
\mathcal{B}^{*}_{\mathcal{F}} &= \mathcal{A}^{*}_{\mathcal{F}}/\mathcal{G}_{\mathcal{F}}, \\
\mathcal{M}^{\times}_{\lambda} &= \text{moduli space of slicewise stable } HYM_{\lambda} \text{ connections of } \mathcal{E} \\
&= \{[D] \in \mathcal{B}^{\times} : D \text{ is a solution of (1.4)}\}, \\
\mathcal{M}^{\times}_{\mathrm{ad}} &= \text{moduli space of slicewise stable adiabatic connections of } \mathcal{E} \\
&= \{[D] \in \mathcal{B}^{\times} : D \text{ is a solution of (1.5)}\}.
\end{aligned}$$

$$\begin{aligned}\mathcal{M}^*_{\mathcal{F}} &= \text{moduli space of irreducible flat connections of } \mathcal{F}\\ &= \{[D] \in \mathcal{B}^*_{\mathcal{F}} :\ F_D = 0\},\end{aligned}$$

Appendix G

Yang-Mills on Kähler Manifolds

The Yang-Mills functional for unitary connections on a hermitian vector bundle $\mathcal{E} \to M$ over a 4-dimensional[1] Kähler manifold can be rewritten as

$$\mathcal{YM}(D) = \|F_D\|^2 = \big\langle F_D \,,\, F_D \big\rangle = 2\big\langle \tfrac{F_D + *F_D}{2} \,,\, \tfrac{F_D + *F_D}{2} \big\rangle - \big\langle F_D \,,\, *F_D \big\rangle.$$

The term $\frac{F_D + *F_D}{2}$ is simply the self-dual component of F_D, which decomposes orthogonally as $F_D^{2,0} + F_D^{0,2} + \big(F_D \,,\, \frac{\omega}{\sqrt{2}}\big)\frac{\omega}{\sqrt{2}}$. For unitary connections $\|F_D^{2,0}\| = \|F_D^{0,2}\|$. And as $|\omega| = \sqrt{2}$ we have $\|\big(F_D \,,\, \frac{\omega}{\sqrt{2}}\big)\frac{\omega}{\sqrt{2}}\|^2 = \frac{1}{2}\|\big(F_D \,,\, \omega\big)\|^2 = \frac{1}{2}\|\Lambda F_D\|^2$. Hence

$$\mathcal{YM}(D) = 4\|F_D^{0,2}\|^2 + \|\Lambda F_D\|^2 + \int_M \operatorname{tr}(F_D \wedge F_D).$$

This integral is the topological invariant $-8\pi^2 ch_2(\mathcal{E})$. For an arbitrary constant μ, $\|\Lambda F_D\|^2$ can be rewritten as

$$\begin{aligned}
\|\Lambda F_D\|^2 &= \|i\Lambda F_D - \mu I\|^2 + 2i\mu\big\langle F_D \,,\, I\omega \big\rangle - \mu^2\|I\|^2 \\
&= \|i\Lambda F_D - \mu I\|^2 + 2i\mu \int_M \operatorname{tr}(F_D) \wedge \omega - \mu^2 \int_M \operatorname{tr}(I)\, d\,\mathrm{vol} \\
&= \|i\Lambda F_D - \mu I\|^2 + 4\pi\mu \int_M c_1(\mathcal{E}) \wedge \omega - \mu^2 \operatorname{rk}(\mathcal{E}) \operatorname{vol}(M).
\end{aligned}$$

[1] With slight modifications, this works in any dimension.

The last two terms will attain a maximum when

$$\mu = \frac{2\pi}{\operatorname{rk}(\mathcal{E})\operatorname{vol}(M)} \int_M c_1(\mathcal{E}) \wedge \omega. \tag{G.1}$$

The Yang-Mills functional can finally be expressed as

$$\mathcal{YM}(D) = 4\|F_D^{0,2}\|^2 + \|i\Lambda F_D - \mu I\|^2 + \mu^2 \operatorname{rk}(\mathcal{E})\operatorname{vol}(M) - 8\pi^2 ch_2(\mathcal{E}),$$

where μ is defined as in G.1. The last two terms depend only on the geometries of the bundle and the manifold, and the absolute minima of $\mathcal{YM}$ are attained by unitary connections satisfying

$$\begin{cases} F_D^{0,2} = 0 \\ i\Lambda F_D = \mu I. \end{cases}$$

The first equation means D is integrable. The second equation is the *Hermitian-Einstein condition* on D.

If the structure group is $SU(n)$ then $c_1(\mathcal{E}) = \frac{i}{2\pi}\operatorname{tr} F_D = 0$, therefore $\mu = 0$ and $ch_2(\mathcal{E}) = \frac{1}{2}(c_1^2 - 2c_2) = -c_2(E)$. In this case

$$\mathcal{YM}(D) = 4\|F_D^{0,2}\|^2 + \|\Lambda F_D\|^2 + 8\pi^2 c_2(\mathcal{E}),$$

with its minimum value $8\pi^2 c_2(\mathcal{E})$ attained when

$$\begin{cases} \bar{\partial}_D^2 = 0 \\ \Lambda F_D = 0. \end{cases} \tag{G.2}$$

Equations (G.2) are the *Hermitian-Yang-Mills (HYM) Equations*, and their solutions are *HYM-connections*. We note that in dimension 4 these equations are equivalent to the well known Anti-Self-Dual Equation $*F = -F$.

Appendix H

Sobolev Theorems.

We state here the Sobolev Theorems for L^p_k spaces of sections of a hermitian bundle $\mathcal{E}$ over a compact Riemannian manifold M^n (with or without boundary), and refer to [Pal68], [Sch95], [Heb99], [Mor98] and [FU84] for a more complete discussion.

Lemma H.1. *All Riemannian metrics on M and all unitary connections on $\mathcal{E}$ yield equivalent L^p_k norms.*

Let $0 \le l \le k, k'$ and $1 \le p, p', q < \infty$.

Theorem H.2 (Sobolev Embedding I). *If $k - \frac{n}{p} \ge l - \frac{n}{q}$ the embedding $L^p_k \hookrightarrow L^q_l$ is continuous and dense. If this inequality is strict and $k > l$ the embedding is actually compact.*

Theorem H.3 (Sobolev Embedding II). *If $k - \frac{n}{p} > l$ there is a dense compact embedding $L^p_k \hookrightarrow C^l$.*

Theorem H.4 (Sobolev Multiplication). *There is a continuous embedding $L^p_k(\mathcal{E}) \otimes L^{p'}_{k'}(\mathcal{E}) \hookrightarrow L^q_l(\mathcal{E} \otimes \mathcal{E})$ under either of these conditions:*

a) $\left(k - \frac{n}{p}\right) + \left(k' - \frac{n}{p'}\right) > \left(l - \frac{n}{q}\right)$,

b) $\left(k - \frac{n}{p}\right) + \left(k' - \frac{n}{p'}\right) = \left(l - \frac{n}{q}\right), \quad k - \frac{n}{p} < 0$ *and* $k' - \frac{n}{p'} < 0$.

These theorems are specially useful when used in connection with some basic facts from functional analysis:

(i) If $x_n \stackrel{w}{\rightharpoonup} x$ then $\{\|x_n\|\}$ is uniformly bounded and $\|x\| \leq \liminf \|x_n\|$;

(ii) In a reflexive Banach space (for example, L^p_k spaces on a compact manifold for $1 < p < \infty$) any bounded sequence has a weakly convergent subsequence;

(iii) A compact mapping sends weakly convergent sequences into strongly convergent ones (after passing to a subsequence).

Appendix I

Compactness Theorems

In this section we state a number of results concerning the convergence of sequences of connections having certain bounds on their curvatures. As their proofs are somewhat long, we give only references to where they can be found. For completeness, we first give a couple of results that are central to these proofs.

Lemma I.1. *There exist constants $c\,,\varepsilon > 0$ such that if D is any connection on a $SU(2)$ bundle over the 4-dimensional Euclidean unit ball B_1 with $\|F_D\|_{L^2} < \varepsilon$ then there exists an L^2_2 gauge (unique up to constant gauge transformations) in which $D = d + A$ with*

(i) $d^*A = 0$,

(ii) $*A|_{\partial B_1} = 0$,

(iii) $\|A\|_{L^2_1} < c\|F_D\|_{L^2}$.

Proof. See [Uhl82a] for proof. □

Remark. The constant ε that appears in the following propositions is the same as here.

Proposition I.2. *Let D be as in the previous Lemma, with the extra hypothesis that D be ASD with respect to a metric ρ on B_1. Then A is C^∞ in the half-sized ball $B_{1/2}$ and satisfies*

$$\|A\|_{C^k(B_{1/2})} \leq c(k)\, \|F_D\|_{L^2(B_1)},$$

for constants $c(k)$ independent of D. In particular,

$$\max_{B_{1/2}} |F_D| \leq c(1)\, \|F_D\|_{L^2(B_1)}.$$

These estimates are uniform over all metrics in a small C^k neighborhood of ρ.

Proof. Refer to [FU84, p. 119] for the proof. □

Proposition I.3. *Any ASD connection on a compact manifold is gauge equivalent to a smooth connection.*

Proof. See [FU84, p. 97]. □

These results are useful in proving the following compactness theorems.

Theorem I.4. *Let D_n be a sequence of connections on a 4-dimensional compact set Ω, ASD with respect to metrics ρ_n. Suppose $\rho_n \to \rho$ in $C^{k+1}(\Omega)$ and either*

(i) $\|F_{D_n}\|_{L^2(\Omega)} < \frac{\varepsilon}{2}$ *for all n,*

or

(ii) $\{|F_{D_n}|\}$ *is uniformly bounded over Ω.*

For any $\Omega' \Subset \Omega$ we then have, after passing to a subsequence and taking gauge *transformations, $D_n \to D$ in $C^k(\Omega')$, where D is an ASD connection with respect to ρ.*

Remark. As $\rho_n \to \rho$ and Ω is compact, the above norms can be taken with respect to either ρ_n or ρ.

Proof. See [FU84, p. 122] for proof. □

Corollary I.5. *Let D_n be a sequence of connections on a 4-dimensional compact set Ω, ASD with respect to metrics ρ_n. Suppose $\rho_n \to \rho$ in $C^2(\Omega)$ and $\|F_{D_n}\|_{L^2(\Omega)} < \frac{\varepsilon}{2}$ for all n. Then $\{|F_{D_n}|\}$ is uniformly bounded over any $\Omega' \Subset \Omega$.*

Proof. Suppose the conclusion is false. Then passing to a subsequence one can assume $\sup_{\Omega'} |F_{D_n}| \to \infty$. This would contradict the previous result, according to which there is a sub-subsequence for which $\sup_{\Omega'} |F_{D_n}| \to \sup_{\Omega'} |F_D|$. □

We still have compactness even if the curvature bounds are only local, as in the next Theorem.

Theorem I.6. *Let D_n be a sequence of unitary connections on a (possibly non-compact) manifold M^4, ASD with respect to metrics ρ_n. Suppose $\rho_n \to \rho$ in C^∞ over compact sets and each $x \in M$ has a neighborhood Ω such that either*

(i) $\|F_{D_n}\|_{L^2(\Omega)} < \frac{\varepsilon}{2}$ *for all n,*

or

(ii) $\{|F_{D_n}|\}$ *is uniformly bounded over Ω.*

Then, after passing to a subsequence and taking global gauge transformations, $D_n \to D$ in C^∞ over compact sets, and D is ASD with respect to ρ.

Proof. The proof is an easy adaptation of the proof in [DK90, p. 161] using Theorem I.4. $\square$

The next Theorem provides a similar, although weaker, result for other dimensions and when the connections are not ASD but have a common L^p bound on their curvatures.

Theorem I.7. *Let D_n be a sequence of L^p_1 connections on a $SU(2)$ bundle over a compact manifold M, with $2p > \dim M$. If $\|F_{D_n}\|_{L^p} \leq C$ for all n then, after passing to a subsequence and taking L^p_2 gauge transformations, $D_n \overset{w}{\rightharpoonup} D$ in L^p_1 and $\|F_D\|_{L^p} \leq C$.*

Proof. See [Uhl82a] for proof. $\square$

The proof of this Theorem applies readily to the following variation for non-compact manifolds.

Proposition I.8. *Let D_n be a sequence of L^p_1 connections on a $SU(2)$ bundle over an open set U, with $2p > \dim U$, and let $\Omega \subset U$ be compact. If $\|F_{D_n}\|_{L^p(U)} \leq C$ for all n then, after passing to a subsequence and taking L^p_2 gauge transformations, $D_n \overset{w}{\rightharpoonup} D$ in $L^p_1(\Omega)$ and $\|F_D\|_{L^p(\Omega)} \leq C$.*

Corollary I.9. *Let D_n be a sequence of smooth connections on a $SU(2)$ bundle over an open set U, and let $\Omega \subset U$ be compact. If $\{|F_{D_n}|\}$ is uniformly bounded* *over U then, after passing to a subsequence and taking L^p_2 gauge transformations, $D_n \stackrel{w}{\rightharpoonup} D$ in $L^p_1(\Omega)$ for $1 \leq p < \infty$ and $D_n \to D$ in $L^p(\Omega)$ for $1 \leq p \leq \infty$.*

Proof. Immediate from the previous proposition and the Sobolev Embedding Theorems. □

Remark. These propositions are still true if instead of a fixed metric one has a converging sequence $\rho_n \to \rho$.

From these results we can obtain some useful facts concerning the moduli space of flat connections. The next propositions refer to a flat $SU(2)$ bundle over a compact manifold M.

Proposition I.10. *Let $2p > \dim M$. For any $\delta > 0$ there is some $\epsilon > 0$ such that if $\|F_D\|_{L^p} < \epsilon$ there exists a flat connection D_0 with $\|D - D_0\|_{L^p} < \delta$.*

Proof. If there is no such ϵ, we can find a sequence $\{D_n\}$ of connections such that $\|F_{D_n}\|_{L^p} \to 0$ and each D_n is at a L^p distance of at least δ from any flat connection. But by Theorem I.7 $D_n \to D_0$ in L^p with $F_{D_0} = 0$, giving a contradiction. □

Proposition I.11. *For any $\delta > 0$ there is $\epsilon > 0$ such that if $\sup |F_D| < \epsilon$ there exists a flat connection D_0 with $\|D - D_0\|_{C^1} < \delta$.*

Proof. As before, assume it is not true and obtain a sequence $\{D_n\}$ away from any flat connection by at least a distance δ in C^1, with $\sup |F_{D_n}| \to 0$. Use Corollary I.9 and the compact embedding $L^p_1 \hookrightarrow C^0$ ($p > \dim M$) to obtain

$D_n \to D_0$ in C^0 with $F_{D_0} = 0$. Now as both $D_n \to D_0$ and $F_n \to F_{D_0}$ in C^0, the convergence can be improved to C^1, providing a contradiction. $\square$

Appendix J

Stability

In what follows $\mathcal{Q}$ denotes any $SU(2)$ bundle over a compact Kähler manifold (M, ω), $\mathcal{E}$ an arbitrary $SU(2)$ bundle over a product $(\Sigma \times K, \omega^{\Sigma} \oplus \omega^{K})$ of Riemann surfaces with volume forms ω^{Σ} and ω^{K}, and $\mathcal{F}$ a $SU(2)$ bundle over K. For each $z \in \Sigma$ denote by $\mathcal{E}_z$ the restriction of $\mathcal{E}$ to the slice $z \times K$.

For simplicity, we define stability for a holomorphic bundle $(\mathcal{Q}, \bar{\partial})$ only when $\dim_{\mathbb{C}} M = 1$ or 2. For the general case see [Kob87] or [UY86].

Definition. The *degree* of a line bundle $\mathcal{L}$ over (M, ω) is defined by

$$\deg(\mathcal{L}) = \deg_{\omega}(\mathcal{L}) = \int_M c_1(\mathcal{L}) \wedge \omega^{n-1}.$$

Definition. $(\mathcal{Q}, \bar{\partial})$ is *stable* (resp. *semi-stable*) if any holomorphic line bundle $\mathcal{L} \subset \mathcal{Q}$ has $\deg(\mathcal{L}) < 0$ (resp. $\deg(\mathcal{L}) \leq 0$). If we need to explicitly identify the Kähler form, we write ω-*stable* and ω-*semi-stable*.

Definition. A holomorphic structure $\bar{\partial}$ on $\mathcal{E}$ is *slicewise stable* if for all $z \in \Sigma$ its restriction $\bar{\partial}_z$ to $\mathcal{E}_z$ is stable.

Definition. An integrable connection D is *stable* (resp. *slicewise stable*) if $\bar{\partial}_D$ is stable (resp. slicewise stable). A $\mathcal{G}^{\mathbb{C}}$-orbit $\mathcal{O}$ is *stable* (resp. *slicewise stable*) if $\mathcal{O} = \mathcal{O}(D)$ for some stable (resp. slicewise stable) connection D.

Proposition J.1. *Let $\omega_\lambda = \frac{\omega^\Sigma}{\lambda^2} \oplus \omega^K$. Any slicewise stable holomorphic structure on $\mathcal{E}$ is ω_λ-stable for λ small enough.*

Proof. With respect to ω_λ the degree of a holomorphic line bundle $\mathcal{L} \subset \mathcal{E}$ becomes

$$\deg_{\omega_\lambda}(\mathcal{L}) = \int_{\Sigma\times K} c_1(\mathcal{L})^{\Sigma\Sigma} \wedge \omega^K + \frac{1}{\lambda^2} \int_{\Sigma\times K} c_1(\mathcal{L})^{KK} \wedge \omega^\Sigma. \tag{J.1}$$

Over any slice $z \times K$ we have $c_1(\mathcal{L})^{KK} = c_1(\mathcal{L}_z)$. As $\mathcal{E}$ is slicewise stable $\deg(\mathcal{L}_z) \leq -1$ and

$$\int_{\Sigma\times K} c_1(\mathcal{L})^{KK} \wedge \omega^\Sigma = \int_\Sigma \left(\int_{z\times K} c_1(\mathcal{L}_z) \right) \omega^\Sigma = \int_\Sigma \deg(\mathcal{L}_z)\, \omega^\Sigma \leq -\operatorname{vol}(\Sigma).$$

Also, the first integral in (J.1) has an upper bound independent of $\mathcal{L}$. This is a consequence of the principle that "the curvature decreases in holomorphic subbundles" (see [GH78, pp. 78–79] for details). Therefore there is an $\lambda_0 > 0$ such that $\deg_{\omega_\lambda}(\mathcal{L}) < 0$ for any $\lambda < \lambda_0$ and any holomorphic line bundle $\mathcal{L} \subset \mathcal{E}$. □

Proposition J.2. *Let $\bar\partial$, $\bar\partial'$ be semi-stable holomorphic structures on $\mathcal{Q}$, one of them being stable. If $g \in \Omega^0(\operatorname{End}\mathcal{Q})$ satisfies $\bar\partial \circ g = g \circ \bar\partial'$ then either $g = 0$ or g is invertible everywhere. Any other g' satisfying the same condition is a multiple of g.*

Proof. See [Kob87, pp. 172-173] for proof. □

Corollary J.3. *If $u \in \Omega^0(\operatorname{End}\mathcal{Q})$ is holomorphic with respect to a stable holomorphic structure then $u = cI$ for some $c \in \mathbb{C}$. If $u \in \Omega^0(\operatorname{End}_0\mathcal{Q})$ then $u = 0$.*

Lemma J.4. *For any stable holomorphic structure $\bar\partial$ on $\mathcal{Q}$ there is a constant $c > 0$ such that*

$$\|u\| \leq c\|\bar\partial u\|,$$

for any $u \in \Omega^0(\operatorname{End}_0 \mathcal{Q})$.

Proof. If $v \in \Omega^0(\operatorname{End}_0 \mathcal{Q})$ is such that $\bar\partial^*\bar\partial v = 0$ then $\bar\partial v = 0$ and by Corollary J.3 $v = 0$. Hence $\bar\partial^*\bar\partial$ has no 0 eigenvalue on $\Omega^0(\operatorname{End}_0 \mathcal{Q})$. If $\alpha > 0$ is its smallest eigenvalue then

$$\big\langle \bar\partial u\,,\, \bar\partial u\big\rangle = \big\langle u\,,\, \bar\partial^*\bar\partial u\big\rangle \geq \alpha\big\langle u\,,\, u\big\rangle,$$

for any $u \in \Omega^0(\operatorname{End}_0 \mathcal{Q})$. □

Lemma J.5. *Let $\bar\partial_1$, $\bar\partial_2$ be slicewise stable holomorphic structures on $\mathcal{E}$. If $\bar\partial_1^K = \bar\partial_2^K$ then $\bar\partial_1 = \bar\partial_2$.*

Proof. Let $a^\Sigma = \bar\partial_2^\Sigma - \bar\partial_1^\Sigma$. As $\bar\partial_2^\Sigma\bar\partial_2^K + \bar\partial_2^K\bar\partial_2^\Sigma = 0$ and $\bar\partial_1^K = \bar\partial_2^K$ we get

$$(\bar\partial_1^\Sigma + a^\Sigma)\circ\bar\partial_1^K + \bar\partial_1^K\circ(\bar\partial_1^\Sigma + a^\Sigma) = 0,$$

and as $\bar\partial_1^\Sigma\bar\partial_1^K + \bar\partial_1^K\bar\partial_1^\Sigma = 0$ this reduces to $\bar\partial_1^K a^\Sigma = 0$. Writing locally $a^\Sigma = a\,d\bar z^\Sigma$, this means $\bar\partial_1^K a = 0$, and so $a^\Sigma = 0$ by Corollary J.3. □

Proposition J.6. *Let $\bar\partial_1$, $\bar\partial_2$ be slicewise stable holomorphic structures on $\mathcal{E}$. Suppose for each $z \in \Sigma$ there is a $g(z) \in \mathcal{G}^{\mathbb{C}}_{\mathcal{E}_z}$ such that $\bar\partial_1^K = g(z)\cdot\bar\partial_2^K$. Then $g \in \mathcal{G}^{\mathbb{C}}$ and $\bar\partial_1 = g\cdot\bar\partial_2$.*

Proof. If we show that the $g(z)$'s fit together nicely along Σ so that $g \in \mathcal{G}^{\mathbb{C}}$ then the previous Lemma can be applied to $\bar{\partial}_2$ and $g^{-1} \cdot \bar{\partial}_1$. The structures $\bar{\partial}_1$ and $\bar{\partial}_2$ are related by

$$\bar{\partial}_1^K = g \circ \bar{\partial}_2^K \circ g^{-1},$$
$$\bar{\partial}_1 = \bar{\partial}_2 + a,$$

for some $a = a^\Sigma + a^K \in \Omega^{0,1}(\Sigma \times K, \operatorname{End}_0 \mathcal{E})$. Clearly

$$\bar{\partial}_2^K g + a^K g = 0.$$

As $\bar{\partial}_1^2 = \bar{\partial}_2^2 = 0$ we have

$$\begin{aligned}
0 = [\bar{\partial}_1^\Sigma, \bar{\partial}_1^K] &= [\bar{\partial}_2^\Sigma, g \circ \bar{\partial}_2^K \circ g^{-1}] + \bar{\partial}_1^K a^\Sigma \\
&= (\bar{\partial}_2^\Sigma g) \circ \bar{\partial}_2^K \circ g^{-1} + g \circ \bar{\partial}_2^\Sigma \circ \bar{\partial}_2^K \circ g^{-1} + g \circ \bar{\partial}_2^K \circ g^{-1} \circ \bar{\partial}_2^\Sigma + \bar{\partial}_1^K a^\Sigma \\
&= (\bar{\partial}_2^\Sigma g) \circ g^{-1} \circ \bar{\partial}_1^K - g \circ \bar{\partial}_2^K \circ \bar{\partial}_2^\Sigma \circ g^{-1} + g \circ \bar{\partial}_2^K \circ g^{-1} \circ \bar{\partial}_2^\Sigma + \bar{\partial}_1^K a^\Sigma \\
&= (\bar{\partial}_2^\Sigma g) \circ g^{-1} \circ \bar{\partial}_1^K - \bar{\partial}_1^K \circ g \circ \bar{\partial}_2^\Sigma \circ g^{-1} + \bar{\partial}_1^K \circ \bar{\partial}_2^\Sigma + \bar{\partial}_1^K a^\Sigma \\
&= (\bar{\partial}_2^\Sigma g) \circ g^{-1} \circ \bar{\partial}_1^K + \bar{\partial}_1^K \circ (\bar{\partial}_2^\Sigma g) \circ g^{-1} + \bar{\partial}_1^K a^\Sigma \\
&= \bar{\partial}_1^K \big((\bar{\partial}_2^\Sigma g) g^{-1} + a^\Sigma \big),
\end{aligned}$$

and the stability of $\bar{\partial}_1^K$ and Lemma J.4 now imply

$$\bar{\partial}_2^\Sigma g + a^\Sigma g = 0.$$

Hence $\bar{\partial}_2 g + a g = 0$, and as $\bar{\partial}_2$ is overdetermined elliptic the result follows from the regularity theorem for elliptic operators. □

Theorem J.7. *For each $\mathcal{G}^{\mathbb{C}}$-orbit $\mathcal{O}$ of $\mathcal{Q}$ there are three possibilities:*

1. *if $\mathcal{O}$ is stable it has a unique (up to unitary gauge transformation) HYM-connection, which is irreducible;*

2. *if $\mathcal{O}$ is semi-stable but not stable it can have at most one (up to unitary gauge transformation) HYM-connection. If it exists it is reducible;*

3. *otherwise it has no HYM-connections.*

Proof. See [Don85] for proof. $\square$

Corollary J.8. *Suppose $\mathcal{Q}$ is a flat bundle. Any flat connection on $\mathcal{Q}$ is semi-stable, and if it is irreducible then it is stable. Any two flat connections in the same $\mathcal{G}^{\mathbb{C}}$-orbit are gauge equivalent, and each stable $\mathcal{G}^{\mathbb{C}}$-orbit has a unique (up to gauge equivalence) flat connection.*

Proposition J.9. *Let $\mathcal{O}$ be a stable $\mathcal{G}^{\mathbb{C}}$-orbit on $\mathcal{Q}$, $\{D_n\}$ a sequence of connections in $\mathcal{O}$ and D_0 a flat connection. If $D_n \to D_0$ in C^0 then $D_0 \in \mathcal{O}$.*

Proof. The connections in the sequence are related to $D = D_1$ by $D_n = g_n(D)$ for some $g_n \in \mathcal{G}^{\mathbb{C}}$. This relation is invariant if g_n is multiplied by a constant, so choosing some $p > \dim M$ we can assume $\|g_n\|_{L^p} = 1$ for all n. If $a_n = \bar{\partial}_n - \bar{\partial}_D$ and $a = \bar{\partial}_0 - \bar{\partial}_D$ then $a_n \to a$ in C^0 and

$$\bar{\partial}_D g_n = -a_n g_n.$$

As on a Riemann surface $\bar{\partial}_D$ is elliptic this implies g_n is uniformly bounded in L^p_1. Passing to a subsequence we can assume $g_n \xrightarrow{w} g$ in L^p_1, where g satisfies

$\|g\|_{L^p} = 1$ and

$$\bar{\partial}_D g = -ag.$$

This equation means $\bar{\partial}_0 \circ g = g \circ \bar{\partial}_D$, so by Corollary J.8 and Proposition J.2 g is invertible and therefore $\bar{\partial}_0 = g \cdot \bar{\partial}_D$. As both connections are unitary $D_0 = g(D)$. By Proposition D.8 g can be normalized so that $\det g = 1$ and $g \in \mathcal{G}^{\mathbb{C}}$. □

Corollary J.10. *For each stable $\mathcal{G}^{\mathbb{C}}$-orbit $\mathcal{O}$ on $\mathcal{Q}$ and any $\delta > 0$ there is a $\epsilon > 0$ such that if $D \in \mathcal{O}$ and $\sup |F_D| < \epsilon$ there exists a flat connection $D_0 \in \mathcal{O}$ with $\|D - D_0\|_{C^1} < \delta$.*

Proof. Adapt Proposition I.11 by choosing the sequence of connections to be in $\mathcal{O}$, and use the previous Proposition to guarantee $D_0 \in \mathcal{O}$. □

Proposition J.11. *Let D_n, D'_n be integrable unitary connections on $\mathcal{F}$ such that $D'_n = g_n(D_n)$ for some $g_n \in \mathcal{G}^{\mathbb{C}}$. Suppose $D_n \to D$ and $D'_n \to D'$ in C^0, where D is semistable and D' stable. Then, after passing to a subsequence, $g_n \xrightarrow{w} g$ in L^p_1 ($p > 2$) for some $g \in \mathcal{G}^{\mathbb{C}}$ and $D' = g(D)$.*

Proof. The relation

$$D'_n = g_n(D_n) = D_n + (g_n^*)^{-1}(\partial_n g_n^*) - (\bar{\partial}_n g_n) g_n^{-1} \tag{J.2}$$

does not change if g_n is multiplied by $c_n = \|g_n\|_{C^0}^{-1}$, hence g_n can be normalized so that $\|g_n\|_{C^0} = 1$ for all n. Originally $\det g_n = 1$, so after this normalization $\det g_n = c_n^2$. Writing

$$a_n = \bar{\partial}_n - \bar{\partial}_D, \qquad a'_n = \bar{\partial}'_n - \bar{\partial}_D, \qquad a = \bar{\partial}_{D'} - \bar{\partial}_D,$$

we have $a_n \to 0$, $a'_n \to a$ in C^0, and equation (J.2) gives

$$\bar{\partial}_D g_n = g_n a_n - a'_n g_n. \tag{J.3}$$

The right side of this equation is bounded in L^p, so as on a Riemann surface $\bar{\partial}_D$ is elliptic we find that g_n is uniformly bounded in L^p_1. After passing to a subsequence we can assume $g_n \overset{w}{\rightharpoonup} g$ in L^p_1 for some $g \in \Omega^0(\operatorname{End}\mathcal{F})$. Equation (J.3) gives in the limit

$$\bar{\partial}_D g = -ag,$$

which is equivalent to

$$\bar{\partial}_{D'} \circ g = g \circ \bar{\partial}_D. \tag{J.4}$$

As the embedding $L^p_1 \hookrightarrow C^0$ is compact, $\|g\|_{C^0} = \lim \|g_n\|_{C^0} = 1$. Hence by Proposition J.2 g is invertible. Equation (J.4) then becomes $\bar{\partial}'_D = g \cdot \bar{\partial}_D$, and as both connections are unitary this means

$$D' = g(D).$$

Note that as $\det g_n = c_n^2$ converges to a finite nonzero value the original g_n's were already uniformly bounded and did not converge to 0 in C^0, and so the normalization is actually not necessary. □

Let $\mathcal{E}$ be a $SU(2)$ bundle over $U \times K$, where U is an open set of a Riemann surface or $\mathbb{R}^2$, and K is a compact Riemann surface. We then have:

Proposition J.12. *Let D_n, D'_n be integrable unitary connections on $\mathcal{E}$ such that $D'_n = g_n(D_n)$ for some $g_n \in \mathcal{G}^{\mathbb{C}}$. Suppose $D_n \to D$ and $D'_n \to D'$*

uniformly on compact sets and D*,* D' *are slicewise stable. Then* $D' = g(D)$ *for some* $g \in \mathcal{G}^{\mathbb{C}}$*.*

Proof. For each $z \in U$ the previous proposition gives $D'_z = g(z)(D_z)$ for some $g(z) \in \mathcal{G}^{\mathbb{C}}_{\mathcal{E}_z}$. Proposition J.6 then implies $\bar{\partial}_{D'} = g \cdot \bar{\partial}_D$ and the result follows from the fact that both connections are unitary. □

Proposition J.13. *Let* D_0 *be a stable flat connection on* $\mathcal{F}$*. For any* $\delta > 0$ *there is a* $\epsilon > 0$ *such that if* $\|D - D_0\|_{L^2_1} < \epsilon$ *there are another flat connection* D'_0 *and* $u \in \Omega^0(\operatorname{End}_0 \mathcal{F})$ *such that* $u^* = u$*,* $D = e^u(D'_0)$*,* $\|u\|_{L^2_2} < \delta$ *and* $\|D'_0 - D_0\|_{L^2_1} < \delta$*.*

Proof. For simplicity, we write $\operatorname{Herm}_0 \mathcal{F} = \{u \in \Omega^0(\operatorname{End}_0 \mathcal{F}) : u^* = u\}$ and use a subscript "k, p", as for example in $(\mathcal{A})_{k,p}$, $\Omega^1(\operatorname{ad} \mathcal{E})_{k,p}$, etc., to represent the completion of these spaces in the L^p_k norm. The space $(\mathcal{A}^*_{\text{flat}})_{1,2}$ of stable flat L^2_1 connections is an open Banach manifold, with tangent space $T_D(\mathcal{A}^*_{\text{flat}})_{1,2} = \{a \in \Omega^1(\operatorname{ad}\mathcal{F})_{1,2} : Da = 0\}$. The map $f : (\mathcal{A}^*_{\text{flat}})_{1,2} \times (\operatorname{Herm}_0 \mathcal{F})_{2,2} \to (\mathcal{A})_{1,2}$ defined as

$$f(D\,,\,u) = e^u(D) = D + e^{-u}\partial_D e^u + e^u \bar{\partial}_D e^{-u},$$

has derivative at $(D_0, 0)$ given by

$$df_{(D_0,0)}(a, v) = a + \partial_0 v - \bar{\partial}_0 v,$$

for any $a \in T_{D_0}(\mathcal{A}^*_{\text{flat}})_{1,2}$ and $v \in (\operatorname{Herm}_0 \mathcal{F})_{2,2}$. By the Surjective Mapping Theorem the Proposition will be proven if $df_{(D_0,0)}$ has a continuous right inverse. As $D_0 + D_0^*$ is elliptic any $b \in T_{f(D_0,0)}(\mathcal{A})_{1,2} = \Omega^1(\operatorname{ad}\mathcal{F})_{1,2}$ can be orthogonally decomposed as

$$b = D_0\alpha + D_0^*\beta + \gamma,$$

for some $\alpha \in \Omega^0(\mathrm{ad}\mathcal{F})_{2,2}$, $\beta \in \Omega^2(\mathrm{ad}\mathcal{F})_{2,2}$ and $\gamma \in \Omega^1(\mathrm{ad}\mathcal{F})_{1,2}$ such that $D_0\gamma = D_0^*\gamma = 0$. In general such decompositions do not yield a unique α and β, but for D_0 stable Lemma J.4 provides uniqueness. As D_0 is flat we have $D_0\alpha + \gamma \in T_{D_0}(\mathcal{A}^*_{\mathrm{flat}})_{1,2}$. Moreover $(i\beta)^* = i\beta$ and

$$D_0^*\beta = (\bar{\partial}_0^* + \partial_0^*)\beta = - * (\partial_0 + \bar{\partial}_0) * \beta = \partial_0(i * \beta) - \bar{\partial}_0(i * \beta),$$

so $b = df_{(D_0,0)}(D_0\alpha + \gamma, i * \beta)$. Note that the ellipticity of $D_0 + D_0^*$ and Lemma J.4 give

$$\|\beta\|_{L^2_2} \leq C\big(\|(D_0 + D_0^*)\beta\|_{L^2_1} + \|\beta\|_{L^2}\big) \leq C'\|D_0^*\beta\|_{L^2_1},$$

and therefore

$$\|D_0\alpha + \gamma\|_{L^2_1} + \|i * \beta\|_{L^2_2} \leq C''\big(\|D_0\alpha + \gamma\|_{L^2_1} + \|D_0^*\beta\|_{L^2_1}\big) = C''\|b\|_{L^2_1}.$$

Hence the linear map $L(b) = (D_0\alpha + \gamma, i * \beta)$ is a bounded right inverse of $df_{(D_0,0)}$. □

Lemma J.14. *If $g_1 \in \mathcal{G}^{\mathbb{C}}_{\mathcal{F}}$, $g \in \mathcal{G}_{\mathcal{F}}$, and $g_2 = g_1 g$ then $|\log(g_2^* g_2)| = |\log(g_1^* g_1)|$.*

Proof. $|\log(g_2^* g_2)| = |\log(g^{-1} g_1^* g_1 g)| = |g^{-1}\log(g_1^* g_1) g| = |\log(g_1^* g_1)|$. □

Corollary J.15. *Let D_0 be a stable flat connection on $\mathcal{F}$, $D = g(D_0)$ for some $g \in \mathcal{G}^{\mathbb{C}}_{\mathcal{F}}$, and $u = \log g^* g$. For any $\delta > 0$ there is a $\epsilon > 0$ such that if $\|D - D_0\|_{L^2_1} < \epsilon$ then $\|u\|_{L^2} < \delta$.*

Proof. By Proposition J.13 there is a $\epsilon > 0$ such that if $\|D - D_0\|_{L^2_1} < \epsilon$ then $D = e^{u'/2}(D_0')$ with D_0' flat, $u'^* = u'$ and $\|u'\|_{L^2} < \delta$. As D_0 and D_0' are both flat and in the same stable $\mathcal{G}^{\mathbb{C}}_{\mathcal{F}}$-orbit, they differ by a unitary gauge transformation. The previous Lemma then implies $\|u\|_{L^2} = \|u'\|_{L^2}$. $\square$

Appendix K

Extra Tools

Here we state without proof some results from [DS03]. Although they are proven for a different setting in that article, the proofs should adapt without difficulty to our case.

Lemma K.1. *Let $U \subset \Sigma$ be open and $\Omega \subset U$ a compact subset. Then for every constant $c_0 > 0$ there are constants c, $\lambda_0 > 0$ such that any HYM_λ $(0 < \lambda < \lambda_0)$ connection D satisfying*

$$\sup_{z \in U} \|F_D\|_{L^2(\rho_\lambda, z \times K)} \leq c_0 \, \lambda,$$

also satisfies

$$\|F_D\|_{L^\infty(\rho_\lambda, \Omega \times K)} \leq c \, \|F_D\|_{L^2(\rho_\lambda, U \times K)}.$$

Lemma K.2. *There is a constant $\delta > 0$ for which the following is true. Let $U \subset \mathbb{R}^2$ be open and $\Omega \subset U$ a compact subset. For every $c_0 > 0$ and $p \geq 2$ there are constants c, $\lambda_0 > 0$ such that any HYM_λ $(0 < \lambda < \lambda_0)$ connection D satisfying*

$$\|F_D^{UK}\|_{L^\infty(U \times K)} \leq c_0, \qquad \|F_D^{KK}\|_{L^\infty(U \times K)} \leq \delta,$$

also satisfies

$$\int_\Omega \Big(\|F_D^{KK}\|^p_{L^2(K)} + \lambda^p \|\nabla_x F_D^{KK}\|^p_{L^2(K)} + \lambda^p \|\nabla_y F_D^{KK}\|^p_{L^2(K)} \Big) \;\le\; c\,\lambda^{2p},$$
$$\sup_\Omega \big(\|F_D^{KK}\|_{L^2(K)} + \lambda \|\nabla_x F_D^{KK}\|_{L^2(K)} + \lambda \|\nabla_y F_D^{KK}\|_{L^2(K)} \big) \;\le\; c\,\lambda^{2-2/p}.$$

Theorem K.3. *Let ϵ_n be a sequence of metrics on an open set $U \subset \mathbb{R}^2$, converging in C^∞ to the Euclidean metric ϵ. Let D_n be a sequence of connections on $U \times \mathcal{F}$, ASD with respect to metrics $\rho_n = (\alpha_n^{-2}\epsilon_n) \oplus \rho^K$, where $\alpha_n \to 0$. Suppose there is some $c > 0$ such that for all n,*

$$\|F_n\|_{L^2(\rho_n, U\times K)} + \|F_n\|_{L^\infty(\rho_n, U\times K)} < c\,\alpha_n.$$

Then there exists an adiabatic connection D_0 on $U \times K$ such that, after passing to a subsequence and taking gauge transformations,

$$\|D_n - D_0\|_{L^\infty(\Omega\times K)} \to 0, \qquad \sup_{z\in\Omega} \|F_n^{UK} - F_0^{UK}\|_{L^2(z\times K)} \to 0,$$

for every compact set $\Omega \subset U$ (here the norms are taken with respect to either $\epsilon_n \oplus \rho^K$ or $\epsilon \oplus \rho^K$).

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