The Dissertation Committee for Dimitrios Stylianos Parsinas Pylorof
certifies that this is the approved version of the following dissertation:

Optimization-based feedback control of nonlinear systems subject to input constraints

Committee:

Efstathios Bakolas, Supervisor

Maruthi R. Akella

Aristotle Arapostathis

Kevin S. Chan

Ryan P. Russell

Renato Zanetti

Optimization-based feedback control of nonlinear systems subject to input constraints

by

Dimitrios Stylianos Parsinas Pylorof

DISSERTATION

Presented to the Faculty of the Graduate School of

The University of Texas at Austin

in Partial Fulfillment

of the Requirements

for the Degree of

DOCTOR OF PHILOSOPHY

THE UNIVERSITY OF TEXAS AT AUSTIN

December 2018

Dedicated to my mom, Irina, and my uncle, Harry, for their unconditional love, encouragement, and support, which transcended continents and timezones,

as well as my classmates Jennifer and Natalie, for sharing parts of our graduate studies at the University of Texas and being by my side in good and bad times.

Acknowledgments

I am extremely grateful to my advisor, Professor Efstathios Bakolas for his guidance, his supervision, and the trust that he has bestowed on me since the beginning of our collaboration in 2013. Throughout my graduate studies, I have had the opportunity to think freely, as well as pose and address multiple questions, most of which form the backbone of this dissertation. I feel privileged for having been able to pursue a totally creative research plan, driven simply by curiosity about a few fundamental, open problems of importance in our field. Navigating through my studies has been both rewarding and challenging. Prof. Bakolas has helped me take good decisions at every turn of the way. As this chapter of my life comes to an end and I am getting ready for our future collaboration on new exciting research problems, I wish to all people embarking on a similar adventure to have the chance to meet and work with professors equally dedicated to their students' learning and success.

My formal, graduate-level education in control at the University of Texas began with Professor Maruthi Akella's classes, to whom I owe my introduction to fundamental topics in systems and control theory. During my years at the University of Texas, I also had the chance to be thoroughly introduced to the field of Orbital Mechanics, thanks to Professor Ryan P. Russell, to whom I am rather grateful for his teaching and our collaboration on applications of my research results to the orbital stabilization problem. I am also thankful to Professors Aristotle Arapostathis and Renato Zanetti, for being part of my dissertation committee, and Professor Srinivas Bettadpur, for his support and our collaboration on research topics involving spacecraft attitude control. My graduate studies at the University of Texas have been continuously supported though research and teaching assistanships provided by the Department of Aerospace Engineering and Engineering Mechanics and the Center for Space Research. This financial support made my studies possible in the first place and it is greatly appreciated.

Early on during my undergraduate years at the School of Mechanical Engineering of the National Technical University of Athens, I realized my desire to pursue research and advanced studies in the field of control. Professor Kostas J. Kyriakopoulos has been instrumental in introducing me to research at his lab, and guiding me through my undergraduate studies till I moved to the University of Texas.

Towards the end of my PhD studies, I joined the U.S. Army Research Laboratory (ARL) in Adelphi, MD, as a research fellow participating in ARL's Research Associateship Program managed by Oak Ridge Associated Universities. I am thankful to my ARL mentor, Dr. Kevin S. Chan, for his guidance, support, and trust throughout our collaboration. I am also thankful to staff members of ARL, in particular, Dr. Ananthram Swami, Dr. Tien Pham, Dr. Jemin George, and Mr. Gunjan Verma, for welcoming me and providing me with exciting research opportunities. I am glad to have been contributing to ARL's mission and I am looking forward to our future endeavors.

I am also grateful to Jhanani, Andrei, David, Chun-Yi, and many other students of the Control / Autonomy / Robotics and Orbital Mechanics groups at the University of Texas, for our collaboration and our stimulating discussions for the past five years. Last but not least, I am grateful to all other people and institutions that have contributed in one way or another to my learning and personal development. As memory fades and given the complexity of life nowadays, it is difficult to perform a fair and complete enumeration in the present context. We are all largely defined by our experiences and even seemingly small and unimportant things and interactions

can ultimately have a significant impact in our future. Maybe, I would have followed a different trajectory in my personal or academic life if it wasn't for my teachers in elementary school and high school, my friends and classmates, the anonymous reviewers who have handled my publications or the administrative staff at all places I have been at. I hope that I will have multiple opportunities to, likewise, positively influence the life and work of others in the years to come.

The research presented in this dissertation was sponsored in part by the Army Research Laboratory and was accomplished under Cooperative Agreements W911NF-18-2-0066 and W911NF-18-2-0251. The views and conclusions contained in this document are those of the author and should not be interpreted as representing the official policies, either expressed or implied, of the Army Research Laboratory or the U.S. Government. The U.S. Government is authorized to reproduce and distribute reprints for Government purposes notwithstanding any copyright notation herein.

Optimization-based feedback control of nonlinear systems subject to input constraints

by

Dimitrios Stylianos Parsinas Pylorof, Ph.D.
The University of Texas at Austin, 2018

Supervisor: Efstathios Bakolas

In this work, we are studying and solving feedback control problems for input constrained nonlinear systems under the influence of uncertainty. Our results are developed by fusing fundamental Lyapunov stability concepts with tools and techniques from the field of convex optimization that enable the derivation of computationally efficient control laws accompanied by robust stabilization guarantees.

When a nonlinear control system is subject to input constraints, a critical aspect of the stabilization problem with simple control laws based on a particular Control Lyapunov Function (CLF) is to characterize a subset of the state space starting from where stabilization to the origin is guaranteed. We consider polynomial systems which are affine in a control input constrained in a convex and compact polytope. We propose two alternative analysis methods that ultimately yield sufficient conditions for asymptotic stabilization under such input constraints and provide an estimate of the stabilization set for the system and the given CLF. Both methods relax the problem to the solution of Sum-of-Squares programs, which nominally can be cast as Semidefinite Programs that are solvable with interior point algorithms. Given a particular CLF, it is also possible to sequentially optimize over its coefficients to the end of reshaping

or enlarging the stabilization set, and thus, favorably altering the set of initial conditions from where the control objectives can be attained. A class of constrained control laws based on a particular CLF is shown to attain values equal to the minimizer of a Quadratic Program (QP), which is guaranteed to remain feasible along any closed loop trajectory emanating from the stabilization set. The input constraints are always respected and the closed loop system is rendered asymptotically stable. Additionally, such a QP is of a rather low dimension and can be solved efficiently, enabling the embedded implementation of the proposed control laws even on resource-constrained computational platforms.

For the case of systems subject to unknown, bounded uncertainties that enter the dynamics in an affine way, the aforedescribed results are extended to provide robust stabilization subject to input constraints. With the proposed methods, the min-max conditions typically encountered in Lyapunov methods with Robust CLFs (RCLFs) for such systems are handled in both the (R)CLF analysis and the feedback control problem. Therefore, one can estimate a subset of the robust stabilization set with SOS programming and, subsequently, calculate - online - the stabilizing control inputs using state feedback to render the system robustly practically stable.

An often encountered challenge in nonlinear control design and implementation is the large dimension of the underlying system, often resulting from the interconnection of multiple subsystems which interact with each other. The concept of Vector (Control) Lyapunov functions allows studying or warranting the applicable stability notion by focusing at the subsystem level and the respective subsystem-to-subsystem interactions. We are leveraging the premise of VCLF methods with our results on the robust stabilization problem to enable the solution of the input constrained robust stabilization problem for large scale systems, either in a distributed or a decentralized way (or in a combination of both), depending on whether state information is exchanged between interacting subsystems or not.

Lastly, we examine how uncertainty in the measurements of the system can affect the stabilization problem under input constraints. We propose a control framework with which one can steer a system to a neighborhood of the origin using only imperfect state feedback. The latter is achieved by enforcing a causality relationship between stabilizing the system from the point of view of an imperfect feedback control law and stabilizing the actual system. Ultimately, we use control laws based, again, on the minimizer of simple QPs, to provingly achieve the robust stabilization objective in a subset of the measurement space which is characterized by solving a sequence of SOS programming problems. For the case where only imperfect measurements either of a subset of the state vector of the system or of a linear combination of state vector components are available, we propose an extension of Lyapunov-based nonlinear observer design results from the literature to account for uncertainty in the dynamics and the measurement equation. The robust observer synthesis process takes place through SOS programming and produces observers with explicit performance guarantees with regards to the behavior of the state determination error.

The factors considered in this work are relevant to contemporary safety-critical control applications; nonlinearity, input constraints, uncertainty, and the need for embeddability and low footprint implementation are ubiquitous in control problems across fields ranging from robotics to industrial engineering, space exploration and cyber-physical systems. The proposed methods aim to collectively provide a theoretically sound, algorithmically implementable and practically useful framework to study and tackle challenging control problems.

Table of Contents

Chapter 1

Introduction

1.1 Contemporary challenges in control theory, algorithms and applications

Control theory and algorithms are nowadays at the backbone of most engineering systems, enabling (or hindering) their autonomous operation and decision making. As system components become smaller in size, more complex and fragmented in nature, and distributed over distant, heterogeneous and -possibly- adverse environments, questions related to a control solution's capabilities, robustness, and overall resilience emerge. Nonlinearity in the underlying dynamics, uncertainty, and constraints on the physically realizable control inputs make such questions critically important. Let us consider some of these factors in additional detail.

Nonlinearity and constraints: The Lyapunov stability theory provides us with a set of thorough tools, which one can use to address stability and control problems for nonlinear systems. However, there still exist multiple open issues with significant real-world implications. Here, we will examine one such fundamental problem pertaining to stabilization under input constraints. It will not be an exaggeration to claim that input constraints may be more relevant than even before in the controls field. First of all, system components such as actuators of any kind tend to become smaller in size and more capable, operating close to their physical limitations. Such limitations can impose constraints on the available control authority, which, in turn, can prevent the stabilization of the system from arbitrary initial conditions. Second, the widespread use of control systems everywhere around us with the goal to increase autonomy and improve human lives and activities (as, for instance, in autonomous cars, biomedical devices, safety-critical industrial control) means that a failure to stabilize a system, which can be caused by the lack of enough control authority, can have detrimental consequences.

Uncertainty: The aforedescribed issues become much more critical under the influence of uncertainty. Typically, robust control methods would attempt to attenuate the effect of uncertainty on the controlled system. However, this is not so trivial when the control inputs are subject to constraints, highlighting the need for the control designer to know, *a priori*, what the capabilities of a particular system are guaranteed to be, given the input constraints and the uncertainty characterization.

Scale: An even more challenging factor that affects contemporary control systems is the issue of scale. Systems nowadays can consist of tens, hundreds, thousands, or even more components that interact with each other. Solving such a control problem in a centralized way is typically impossible. One has to focus, instead, at the subsystem level and implement control laws locally in such a way that the entire system is stabilized, also. The large dimension and the complexity of such systems also highlight the need for any control design methodologies and the resulting solutions to be systematic and provably correct, since it may be impossible to *a posteriori* verify (say, with experiments or simulation techniques) the correctness of a control solution against all possible behaviors of the controlled system and its environment.

Embedded systems: Last but not least, contemporary control solutions are often deployed on embedded platforms in the real world. Except for the obvious requirements for robustness and resilience, it is often necessary for the control law to be computationally lightweight, so that it is easily implementable and embeddable on resource-constrained electronics. Embedded systems can be associated with additional complicating factors. For instance, as sensors get smaller, the quality of their measurements usually degrades. It may even be impossible to measure all quantities of interest for control purposes.

All the previously discussed complicating factors can make control problems rather hard to solve. Fortunately, contemporary progress in optimization methods and their use in stability and control problems has offered a path to designing highly capable control solutions while taking into consideration a multitude of aspects, something not possible if one was to solve such problems analytically. From fast-enough computational libraries and platforms that enable the solution of optimization problems online to the opposite end of the spectrum where one can design control laws offline with explicit guarantees about their safety and performance, computational methods in control offer exciting prospects and enable the development of gradually more robust, better performing and, overall, more useful control solutions.

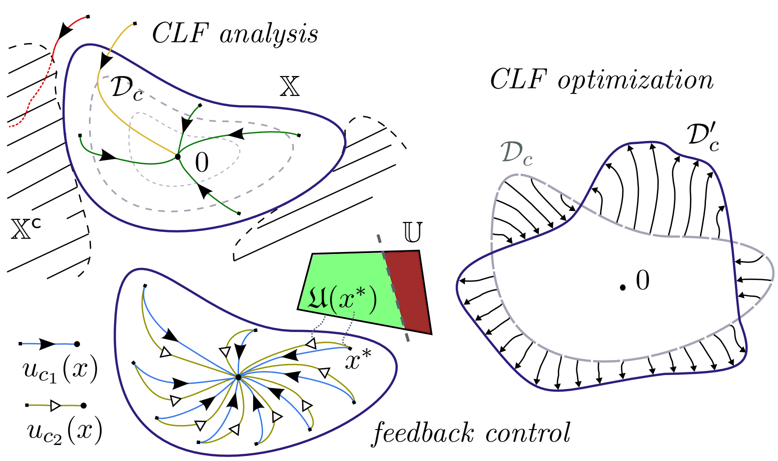


Figure 1.1: A conceptual illustration of the setting of the CLF-based stabilization problem under input constraints. *CLF analysis*: finding $\mathcal{D}_c$, a sublevel set of the CLF V contained within the non-shaded part $\mathbb{X}$ of the state space; everywhere in $\mathcal{D}_c$ one can find an input to render $\dot{V} < 0$ along the closed loop trajectories, which remain in $\mathcal{D}_c$; *CLF optimization*: enlarging or reshaping $\mathcal{D}_c$ by finding a new CLF; *feedback control*: closing the loop with a CLF-based, constrained control law.

1.2 Background and prior work

1.2.1 Nonlinear control

Despite the advent of Lyapunov stability concepts, Control Lyapunov Functions (CLFs) (Bacciotti and Rosier, 2005; Haddad and Chellaboina, 2008) and related methods such as backstepping (Krstić, Kanellakopoulos, and Kokotović, 1995; Freeman and Kokotović, 1996) as some of the main tools in nonlinear control, the stabilization of nonlinear systems remains a nontrivial feat. The ubiquitous presence of input constraints in the majority of physically meaningful systems can complicate such a control problem even further. One can use fundamental tools such as the Artstein theorem (Artstein, 1983), the Sontag formula (Sontag, 1989) and the pointwise min-norm approach (Freeman and Kokotović, 1996; Primbs, Nevistić, and Doyle, 2000) to design a feedback control law based on a particular CLF, say V, and achieve global or local asymptotic stabilization. However, if the control input is constrained to attain values in a particular set, such a control law may fail to stabilize the system. Even if it was possible to find a control input to render $\dot{V} < 0$ everywhere in the state space in the unconstrained case, this may not be possible under input constraints, ruling out global stabilization in the sense of Lyapunov (or altering the set from where local stabilization was originally attainable in the absence of input constraints). Note that there is more to the input constrained stabilization problem than merely finding the set where there exists an input vector that satisfies the constraints and renders $\dot{V} < 0$; a notion of invariance is also required, so that the controlled trajectories under an input constrained CLF-based control law will remain in such a set, which we hereafter call the stabilization set for a given system, input value set and CLF. The problem of bounded CLF-based feedback and backstepping has been studied in Lin and Sontag (1991); Freeman and Praly (1998). Note that the concepts of bounded and saturated control have important differences from the (polytopically) constrained case which we will ultimately consider here, as the latter necessitates an explicit, quantitative approach when dealing with the possibly coupled constraints on the components of the input vector.

Apart from input constraints, an often encountered difficulty in studying and controlling dynamical systems is that it is practically impossible to obtain a perfect model of the underlying dynamics. Even if one obtains a rather accurate, elaborate dynamical model, it may not be practical to use such a model for purposes of control design due to, for instance, its high dimension or the absence of a form amendable to control design techniques. Moreover, the parameters of many systems may naturally assume different values as time progresses in a way that significantly affects the underlying dynamics, or a system can also be affected by unknown, exogenous disturbances. For such systems, the typical Lyapunov control paradigm holds in a worst-case way with regards to the action of the unknown disturbances on the system, that is, by dominating over the unknown disturbances, no matter what their value may be, in which case we use the concept of a Robust CLF (RCLF) (Freeman and Kokotović, 1996; Krstić and Deng, 1998). The effect of uncertainty can be added on top of that of input constraints and can jeopardize the efficacy of an RCLF-based control scheme. In order to solve the robust stabilization problem in a provably correct (and thus, safe) way, it is necessary (a) to be able to robustly steer the system to the origin with an RCLF-based control law, under the influence of bounded uncertainty in the dynamics and in the presence of input constraints, and (b) to know from which initial conditions such a control law is guaranteed to be effective and achieve our stabilization objectives, under any possible action of the disturbance on the system. Similarly to the nominal case, it is critically important to recognize the necessity for a notion of invariance, in order for the controlled trajectory driven by an RCLF-based control law not to escape from the set where such a control law is well-defined and effective, while the system converges to the target equilibrium.

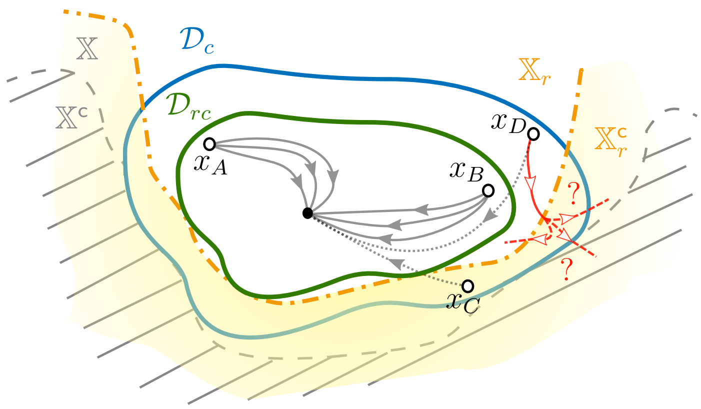


Figure 1.2: A conceptual illustration of the setting of the RCLF-based robust stabilization problem for uncertain systems under input constraints. All trajectories emanating from either x_A and x_B, both of which belong to the robust stabilization set $\mathcal{D}_{rc}$, are attracted to an arbitrarily small neighborhood of the origin, regardless of the particular action of w. Any such trajectories, driven under a simple min-max RCLF-based control law under input constraints, remain in the part of the state space $\mathbb{X}_r$ where a control law of this nature is feasible. Trajectories emanating from some point in $\mathbb{X}_r^{\mathsf{c}}$ may ultimately enter the robust stabilization set (as conceptually drawn for the case of x_C) or not (as shown for x_D). To provably solve the robust stabilization problem with an RCLF-based control law under constraints, it is critical to find a positively invariant set set $\mathcal{D}_{rc}$ that is contained within $\mathbb{X}_r$.

1.2.2 Convex optimization methods in nonlinear feedback control

Sum-of-squares (SOS) methods (Parrilo, 2000, 2013) have brought a small revolution in the field of control, by providing the means to numerically solve many problems of interest involving nonnegative polynomials. The original polynomial nonnegativity problem is rather hard to solve, however, one can use SOS techniques to relax it to an LMI (Boyd, Ghaoui, Feron, and Balakrishnan, 1994) in auxiliary variables, for which polynomial time algorithms exist (Vandenberghe and Boyd, 1996). Even if the resulting LMIs scale badly (in fact, factorially) with the dimension of the polynomial indeterminate, SOS methods have rendered the problem of polynomial nonnegativity practically decidable for many cases of interest. Multiple problems in stability and stabilization of nonlinear systems have benefited from such methods, especially for region of attraction calculations (Chesi, 2004a,b; Papachristodoulou and Prajna, 2002; Tan and Packard, 2008; Topcu, Packard, and Seiler, 2008; Kundu and Anghel, 2017). Feedback control synthesis with SOS for nonlinear systems without input constraints has been studied in Prajna, Parrilo, and Rantzer (2004b). The reader is referred, indicatively, to Vaidya, Mehta, and Shanbhag (2010); Manchester and Slotine (2017) for optimization-based synthesis methods that do not assume the knowledge of a particular CLF for the system. We also refer to Tedrake, Manchester, Tobenkin, and Roberts (2010); Majumdar and Tedrake (2017) for SOS-based control design in motion planning problems. For additional contemporary results on region of attraction calculation methods we refer to Henrion and Korda (2014); Wang, Lall, and West (2013), and the survey by Chesi (2010).

The fact that a CLF-based control law, such as the min-norm control law (Freeman and Kokotović, 1996), subject to polytopic input constraints can be associated with the solution of a Quadratic Program (QP) was first identified by Curtis (2003), to our knowledge. Such QPs are rather attractive, given their small dimension and the fact that they are rapidly solvable even on resource constrained embedded computational platforms. The recursive feasibility aspect of such a QP has not been studied, though. The latter implies that the QP associated with a control law could suddenly become unfeasible, if the controlled trajectory escapes to parts of the state space where no control input exists to render the time derivative of the CLF along the trajectory of the system negative definite. CLF techniques are fused with Control Barrier functions to study an automotive control system with state constraints and input saturation in Ames, Xu, Grizzle, and Tabuada (2017); Xu, Grizzle, Tabuada, and Ames (2017b).

The QP control scheme proposed therein essentially follows the min-norm paradigm except for a relaxation parameter that allows the violation of the asymptotic stabilization condition in the interest of prioritizing the satisfaction of state constraints.

1.2.3 Large-scale and networked systems

The tasks of designing, implementing and operating feedback control laws become increasingly challenging as the scale of the underlying system increases. The offline computational burden related to the derivation and verification of control laws can become prohibitive and even render such problems practically intractable. Moreover, using feedback to produce a stabilizing control input in real time poses additional challenges. For instance, any necessary data gathering, calculations, and actuation processes typically have to fit in a limited time frame and are performed on embedded computational platforms; the larger a system is, the more difficult it becomes to effectively close such a feedback loop. The various components of a large-scale system may not even be physically collocated, raising issues related to the communication between them and the possibility for them to collaborate (or not) in achieving the control objectives.

Centralized control paradigms treating a system as a whole throughout the life cycle of a control solution may, therefore, not be ideal for large-scale systems. It is often preferable to address control-related questions by considering subsystems of the original system and their interactions. Such control techniques fall under two paradigms: distributed and decentralized control. Distributed control laws operate at the subsystem level and also take into consideration information from any other subsystems which directly affect the local subsystem's dynamics. Conversely, subsystem-level decentralized control laws operate more independently, as they do not require any information from other subsystems.

Related techniques have been studied in the literature for more than half a century. The introduction of the fundamental notion of vector Lyapunov functions Bellman (1962); Matrosov (1962) helped extend Lyapunov stability concepts to large-scale systems. Instead of studying the stability of a system as a whole, vector Lyapunov function methods allow us to work with individual Lyapunov functions for each subsystem and consider the effect of their interactions via appropriate techniques. The reader is referred to Sandell, Varaiya, Athans, and Safonov (1978); Siljak (1978, 1991) and the more modern study in Haddad and Nersesov (2011). Sum-of-Squares (SOS) for calculating the region of attraction for a large-scale system have been presented in Kundu and Anghel (2017). The control problem has also been studied analytically in Karafyllis and Jiang (2013).

As any physically realizable control system can be expected to be subject to input constraints and uncertainty, the same is true for large-scale systems, also. The complexity of such systems emphasizes the need for a systematic framework to study and solve the stabilization problem in a correct and safe way.

1.2.4 Unavailability of perfect, full state feedback

Feedback control solutions are often designed assuming that perfect knowledge of the current state of the control system is available. This, however, is rarely the case in any practical implementation of a control law; measurement devices are universally affected by error sources and, thus, can only provide an imperfect measurement of the system's state vector, whereas in many systems only an incomplete state measurement may be possible. Moreover, in the more contemporary setting of distributed, networked, and cyberphysical systems, an emerging threat pertains to the possibility of measurements being maliciously manipulated by an adversarial entity, to the end of destabilizing or, in general, harming the control system. All these issues motivate the study of control problems with imperfect feedback and the design of observers, to the end of output feedback control.

Control under imperfect feedback

The presence of disturbances in the measurements of the state that results in imperfect state feedback is a known nontrivial problem in nonlinear stabilization (Freeman, 1995; Freeman and Kokotović, 1996; Ledyaev and Sontag, 1998); even a small, bounded measurement disturbance can have negative implications on the stability of a closed loop system or even render globally stabilizing a nonlinear system impossible. This is contrasted to the case of a (stabilized) linear system, where a bounded measurement disturbance will cause a bounded response. Such issues can hinder the straightforward application of common Lyapunov techniques for nonlinear stabilization in the case of imperfect state feedback, such as backstepping and CLF-based control laws (Bacciotti and Rosier, 2005; Krstić et al., 1995; Haddad and Chellaboina, 2008).

Observer design

The observer design problem is motivated by the often encountered difficulty to accurately measure the entire state vector of a dynamical system. Such information may be necessary in order to apply feedback control techniques to the system or to merely monitor the behavior of an otherwise controlled or autonomous system. Multiple issues can hinder direct and accurate measurements of the state of vector. First of all, any sensor device is inherently subject to errors of various kinds, which contribute to the overall uncertainty of the respective measurements. Secondly, some physical quantities can be rather difficult or even impractical to measure with applicable sensors. This is especially true in the context of modern embedded systems, where size, weight, cost and power requirements may inhibit the measurement of the entire state vector of the system. Similar situations can also arise in more contemporary settings, such as cyberphysical systems and large-scale, interconnected systems with mutually interacting agents.

Observers are dynamical systems which typically require knowledge of a system's measurements and the applied control inputs. The output of an observer is expected to converge, in some sense, to the current state of the observed system. The case where the system of interest is linear has been covered extensively in the fundamental controls literature (Luenberger, 1964). The observer design problem for the nonlinear case is significantly more complicated. We refer, indicatively, to extended Luenberger observers (Zeitz, 1987), high gain observers (Prasov and Khalil, 2013; Khalil, 2017; Farza, M'Saada, Triki, and Maatoug, 2011) and Lyapunov-based observers (Tsinias, 1989, 1990; Kazantzis and Kravaris, 1998). Extended Luenberger approaches essentially pursue a linearization-based approach and it should be mentioned that the knowledge of the input time derivatives is usually required. High gain observers offer good performance and have attracted significant interest in the contemporary literature. Nevertheless, such observers are only applicable to systems of a particular triangular structure and despite significant efforts to relax such restrictions (see, indicatively, Farza et al. (2011) and the additional references given therein, as well as Section 2.6 of Khalil (2017)), the results are still not directly applicable for the full state observation of systems containing arbitrary nonlinearities in each component of the dynamics (even a simple predator-prey system with dynamics $\dot{x}_1 = -x_2 - x_1 x_2 + u$, $\dot{x}_2 = x_1 + x_1 x_2$ cannot be brought to the necessary form in order to design a high-gain observer while measuring $y = x_1$). Lyapunov observers (Tsinias, 1989, 1990) offer an inherently nonlinear solution which is easy to design and handle, despite the non-surprising necessity to provide (or, inevitably, guess) appropriate Lyapunov-like functions. Also, such observers enable a rather easy characterization of the behavior of the state determination error, as the latter typically exhibits some Lyapunov-like stability notion.

1.3 Goals of this dissertation

In the present work, we consider a few fundamental stabilization problems for input constrained polynomial systems, along the lines of the contemporary challenges discussed in Section 1.1 and the topic areas presented in Section 1.2. We follow a bottom-up approach by gradually and systematically building on our results, which are ranging from a novel solution to the nominal stabilization problem for input constrained systems to decentralized control laws for uncertain networked systems and control laws that use imperfect feedback. In each case considered, we focus on bringing the problem formulation to forms amendable to convex optimization techniques, such as SOS programming for offline control law analysis and design calculations and quadratic programming for the online implementation of the various control laws. We use the power of the respective optimization methods to our advantage, to the end of developing solutions with previously unavailable traits especially with regards to stabilization guarantees under constraints for the applicable stabilization notion in each case considered. Ultimately, we hope to extend one's toolbox for tackling some challenging control problems, by providing dependable, safe, and provably correct stabilization results with relevance to difficulties often encountered in practical control applications.

1.4 Contributions of this dissertation

The particular contributions of this dissertation per topic area (and chapter) are described next.

1.4.1 Warranting asymptotic controllability under input constraints

In Chapter 3, we consider polynomial nonlinear control systems affine in the control input, which, in turn, is constrained to attain values in a convex, compact polytope. We assume the knowledge of a polynomial CLF for the system and we focus on the impact of the input constraints on the stabilization set (in the sense described in Section 1.2.1) of control laws stemming from the particular CLF. We exploit the geometry of the input value set and the input affine form of the dynamics to propose two alternative ways to efficiently characterize the subset of the state space where input constrained asymptotic controllability can be guaranteed with a continuous feedback control law based on the particular CLF. Sufficient conditions for the latter are given in terms of semialgebraic set containments, which allow for the use of Sum-of-Squares programming techniques that relax the problem to the solution of Semidefinite Programming feasibility problems. We also propose a method that allows for the (local) optimization over the CLF coefficients, to the end of reshaping or enlarging the stabilization set for a particular system. Figure 1.1 provides a conceptual illustration of the CLF analysis and optimization problems.

1.4.2 Feedback control through low-dimension Quadratic Programs

The set of admissible control laws is shown to consist of continuous selection functions for a certain set-valued map, the existence and nonemptyness of which are guaranteed by construction. In Chapter 4, such selection functions are shown to be pointwisely equal to the minimizer of a particular QP. This QP remains feasible along the trajectories of the closed loop system, for any initial point in the pre-calculated stabilization set. Moreover, the input vector can safely saturate on the boundary of the input value set, if that is necessary either for stabilization or for performance reasons. In the control approach proposed here, any performance considerations enter as a performance objective in the QP and not as a hard constraint (as is the case in the min-norm paradigm), effectively enlarging the stabilization set for the proposed approach further away from the origin. We refer to the dimension of such QPs as "low" with regards to alternative control methodologies also based on the online solution of optimization problems such as nonlinear Model Predictive Control, where one is typically faced with problems both of a much larger dimension and of a significantly more complex nature than QPs, such as nonlinear programming.

1.4.3 Robustness to bounded uncertainty

In Chapter 5, we consider the robust stabilization problem for uncertain nonlinear systems with polynomial dynamics which are affine in the control input and the unknown disturbance. The latter variables are assumed to be constrained in convex and compact polytopes. Given the dynamics and a Robust Control Lyapunov Function (RCLF), we calculate subsets of the state space starting from where robust stabilization is guaranteed by means of a simple Lyapunov-based controller, regardless of the possibly destabilizing action of the disturbance. This robust control analysis process extends our results on nominal (that is, without uncertainty) input constrained systems to the more general class of uncertain nonlinear systems with unknown, bounded disturbances. The proposed methods are based on solving sequences of Semidefinite Programs (SDPs) resulting from the parsing of SOS constraints. Ultimately, one obtains a sublevel set of the RCLF which we refer to as the robust stabilization set for the system; starting at any point in that set, any min-max, RCLF-based control law is guaranteed to be able to steer the system to a neighborhood of the origin, regardless of the value of the disturbance acting on the system along the closed loop trajectories. We next derive such a control law based on the minimizer of an appropriately formulated Quadratic Program (QP). Due to the perturbed dynamics, the underlying QP would nominally belong to the challenging class of robust optimization problems. By exploiting the structure of the control and disturbance value sets, we reduce the problem to a standard QP minimization problem which is easily embeddable given its low dimension and associated computational complexity. Figure 1.2 provides a conceptual illustration of the sets involved in this problem and the notion of invariance guaranteed by our results.

1.4.4 Distributed and decentralized control of large-scale systems

In Chapter 6, we leverage the premise of *vector* Lyapunov function methods for large-scale systems with our results on optimization-based robust control. Scalable solutions to the distributed and decentralized control problems for uncertain, input constrained large-scale nonlinear systems are developed, which aim to provide a notion of practically quantifiable robustness and a capacity for rather efficient implementation at the subsystem level. Input constraints and uncertainty are expected to significantly dominate the behavior of such a system and, accordingly, they form the theme of our work in this area. A set of SOS optimization problems, focusing at the subsystem level, can certify whether a large-scale system is collectively robustly stabilizable from a set of initial conditions, with either control technique. The proposed low-complexity QP-based control laws for each subsystem assume a worst-case scenario with regards to the uncertainty affecting the local dynamics, and either collaborate with the neighboring subsystems by accessing their current state information in the distributed case, or assume a worst-case scenario for their neighboring subsystems in the decentralized case. Both techniques isolate the potentially destabilizing effect of the uncertainty and the interconnections on the dynamics, and collectively achieve the robust stabilization objective.

1.4.5 Control with imperfect measurements

In Chapter 7, we develop technical results with which one can guarantee that an input constrained CLF-based control law operating on imperfect state feedback will provably stabilize the actual system. The operation of the proposed class of control laws relies on warranting a causality relationship between apparently (that is, from the point view of a control law having only imperfect knowledge of the state) and actually stabilizing the system, that holds in a subset of the state space which is explicitly quantifiable using the herein proposed SOS methods. The control law operating on the imperfect feedback is implemented in a simple and efficient way through a QP reminiscent of the one in our (perfect) state feedback case; nevertheless, thanks to the preceding analysis, such a QP-based control law adheres to the aforedescribed stabilization guarantees under imperfect feedback.

1.4.6 Robust nonlinear observer design

In Chapter 8, we are leveraging SOS methods and, in particular, our handling of the robust stabilization problem (Chapter 5), to extend results from the literature on Lyapunov-based observers for perfectly known systems with no uncertainty. We propose a methodology to design robust Lyapunov-based observers for affine polynomial systems with bounded uncertainty in the dynamics and the linear measurement equation.

1.5 Publications

The results presented in this dissertation have also appeared, in part, in the following publications: (given in chronological order)

- D. Pylorof and E. Bakolas, "Nonlinear control under polytopic input constraints with application to the attitude control problem," in Proceedings of the 2015 American Control Conference, pp. 4555-4560, 2015 *(the authors contributed equally).*
- D. Pylorof, E. Bakolas, and R. P. Russell, "A Nonlinear Controller for Low Thrust Stabilization of Spacecraft on CRTBP Orbits", in Proceedings of the 26$^{\text{th}}$ AAS/AIAA Spaceflight Mechanics Meeting, pp. 489-505, 2016 *(the authors contributed equally).*
- D. Pylorof and E. Bakolas, "Analysis and Synthesis of Nonlinear Controllers for Input Constrained Systems Using Semidefinite Programming Optimization," in Proceedings of the 2016 American Control Conference, pp. 6959-6964, 2016 *(the authors contributed equally).*
- D. Pylorof and E. Bakolas, "Robust Control of Input Constrained Nonlinear Systems Subject to Unknown Bounded Disturbances Based on Convex Optimization," in Proceedings of the 2017 American Control Conference, pp. 3700-3705, 2017 *(the authors contributed equally).*
- D. Pylorof and E. Bakolas, "Robust Distributed and Decentralized Control of Large-Scale Nonlinear Systems with Input Constraints Based on SOS Optimization," in Proceedings of the 2018 American Control Conference, pp. 4658-4663, 2018 *(the authors contributed equally).*

- D. Pylorof and E. Bakolas, "Stabilization of Input constrained nonlinear systems with imperfect state feedback using Sum-of-Squares Programming," in Proceedings of the 57$^{\text{th}}$ IEEE Conference on Decision and Control, pagination pending, 2018 *(the authors contributed equally)*.

- D. Pylorof and E. Bakolas, "Safe nonlinear control design for input constrained polynomial systems using sum-of-squares programming," journal submission, under revision, 2018 *(the authors contributed equally)*.

- D. Pylorof and E. Bakolas, "Robust nonlinear stabilization of uncertain polynomial systems subject to input constraints using Sum-of-Squares programming," journal submission, under review, 2018 *(the authors contributed equally)*.

1.6 Structure of this dissertation

Some fundamental results on optimization with Sum-of-Squares (SOS) polynomials are reviewed in Chapter 2. The rest of the chapters follow the enumeration of contributions given in Section 1.4. In Chapter 3 we consider the problems of analyzing the stabilization set of a control law based on a given CLF and optimizing over the CLF coefficients to reshape and / or enlarge the corresponding stabilization set. The feedback control problem is solved online with QPs in Chapter 4. The problem of robust stabilization of uncertain systems is studied in Chapter 5. The latter results are extended to the case of large-scale systems in Chapter 6. Finally, the problems of control with imperfect state feedback and robust observer design are considered in Chapters 7 and 8, respectively. The dissertation is concluded with Chapter 9. Each chapter contains numerical examples to illustrate the respective contributions and the efficacy of the proposed methods.

1.7 Notation

The sets of integers, real numbers and n-dimensional real vectors are denoted, respectively, by $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{R}^n$. Also, we use $\mathbb{R}_{\geq 0}$, $\mathbb{R}_{>0}$, $\mathbb{R}_{\leq 0}$, and $\mathbb{Z}_{>0}$ to denote nonnegative real, positive real, nonpositive real, and positive integer numbers, respectively. I_n is the $n \times n$ identity matrix. We denote element-wise inequalities between vectors $x, y \in \mathbb{R}^n$ by $x \preceq b$. Let $\mathbb{P} \subseteq \mathbb{R}^n$ be a convex, compact polytope with vertices $\eta_1, \eta_2, \ldots \eta_q \in \mathbb{P}$; we denote the index set of these vertices by $\mathcal{Q}_{\mathbb{P}} := \{1, 2, \ldots, q\} \subseteq \mathbb{Z}_{>0}$. Given any matrix $M = M^{\mathsf{T}} \in \mathbb{R}^{n \times n}$, its smallest eigenvalue is denoted by $\lambda_{\min}(M)$. $\overline{\mathcal{B}}^n_\epsilon$ and $\mathcal{B}^n_\epsilon$ denote the closed and open, respectively, n-ball of radius $\epsilon > 0$. The gradient of a continuously differentiable function $K : \mathbb{R}^n \to \mathbb{R}$ is denoted by $\nabla K(x)$ and is taken to be a row vector. Let $p : \mathbb{R}^n \to \mathbb{R}$ be a polynomial in $x \in \mathbb{R}^n$; $\deg(p(x))$ is the degree of $p(x)$, while $p(x) \in \Sigma[x]$ means that $p(x)$ belongs to a cone of nonnegative polynomials with indeterminate x. A set-valued map ρ with domain $\mathcal{A}$ and codomain $\mathcal{B}$ is denoted by $\rho : \mathcal{A} \rightrightarrows \mathcal{B}$. The interior of the set $\mathcal{A}$ is denoted by $\mathrm{Int}(\mathcal{A})$. The collection of nonempty sets $\{B_i \subseteq A, i \in \mathcal{I}\}$ is said to form a partition of the set A if $A = \cup_{i \in \mathcal{I}} B_i$ and $\mathrm{Int}(B_\ell) \cap \mathrm{Int}(B_\kappa) = \varnothing$ for $\ell, \kappa \in \mathcal{I}$, with $\ell \neq \kappa$. $\mathcal{A}^{\mathsf{c}}$ denotes the complement of a set $\mathcal{A}$. $\mathrm{Conv}\{x_1, \ldots, x_\kappa\}$ denotes the convex hull of the points $x_1, \ldots, x_\kappa \in \mathbb{R}^n$. Let $a \in \mathbb{R}$ and $b, c \in \mathbb{R}^n$; $|a|$, $\|b\|$ and $\|c\|_\infty$ denote the absolute value, 2-norm, and infinity norm, respectively.

Additional notation is introduced in each chapter, if necessary.

Chapter 2

Sum-of-Squares Programming preliminaries

The methods developed in this work involve optimization over nonnegative functions, to either directly parameterize optimization variables of interest, such as Lyapunov functions, or to appropriately parameterize auxiliary variables. To the end of obtaining finite dimensional and computationally amendable representations of nonnegative functions, we consider the approximation of the cone of nonnegative, multivariate polynomials with some finite degree bound given by the set of sum-of-squares (SOS) polynomials (Parrilo, 2000, 2013).

2.1 Sum-of-Squares polynomials

Definition 2.1.1. A multivariate polynomial $p : \mathbb{R}^n \to \mathbb{R}$ with $\deg(p(x)) \leq 2d$ for some $d \in \mathbb{Z}_{>0}$ is sum-of-squares (SOS) if it can be written as $p(x) = \sum_{i=1}^{n_i} q_i^2(x)$, where $q_i : \mathbb{R}^n \to \mathbb{R}$ are multivariate polynomials with $\deg(q_i(x)) \leq d$ for $i = 1, \dots, n_i$. For some implied (finite) upper degree bound $2d$, the set of all such $p(x)$ is denoted by $\Sigma[x]$.

For a given $p(x)$ with $\deg(p(x)) \leq 2d$, the existence of a matrix $Q = Q^\mathsf{T} \in \mathbb{R}^{n_d \times n_d}$ with $Q \succeq 0$ such that $p(x) = z_{n,d}^\mathsf{T}(x) Q z_{n,d}(x)$, where $n_d = (n + d)!/(n!d!)$ and the $z_{n,d}(x)$ contains all monomials of $x \in \mathbb{R}^n$ up to degree d, is necessary and sufficient for $p(x) \in \Sigma[x]$ to hold. In Parrilo (2000), the search for such a Q has been shown to be equivalent to solving a Linear Matrix Inequality (LMI) in auxiliary variables (that is, rather than in the polynomial indeterminate $x \in \mathbb{R}^n$), giving rise to a Semidefinite Programming (SDP) feasibility problem (Vandenberghe and Boyd, 1996). Contemporary results in Ahmadi and Majumdar (2018); Kamyar and Peet (2015) propose alternative parameterizations of subsets of the set of SOS polynomials (and even subsets of the set of nonnegative polynomials that may not be SOS) which result in linear and second order cone programs in auxiliary variables. These problems have the potential to be smaller in size compared to the SDP-based approach, allowing, thus, one to achieve a different balance between the size of the search space and the associated computational effort.

In this work, we denote the sets of nonnegative functions under consideration by $\Sigma[x]$, and we associate $\Sigma[x]$ with the set of SOS polynomials, the membership to which can be decided by solving an SDP. Nevertheless, the methods proposed here are agnostic to the particular parameterization employed for $\Sigma[x]$ and the results hold for any choice that ultimately associates $p(x) \in \Sigma[x]$ with a set of convex constraints on auxiliary decision variables.

2.2 The generalized S-procedure

The generalized S-procedure is a paradigm for showing the conditional satisfaction of polynomial inequalities. It builds on results on nonnegative polynomials, and can ultimately lead to a practically decidable sufficient condition for set containments to hold. In particular, let $f_i : \mathbb{R}^n \to \mathbb{R}$, $i = 0, \ldots, m$, describe the semialgebraic[1] sets

$$\{x \in \mathbb{R}^n : f_i(x) \geq 0\}$$

[1]The term *semialgebraic set* is used to refer to a subset of $\mathbb{R}^n$ which consists of all $x \in \mathbb{R}^n$ concurrently satisfying a finite number of equality and inequality constraints which are polynomials with indeterminate x.

and also let

$$F(x) := f_0(x) - \sum_{i=1}^{m} s_i(x) f_i(x),$$

where $s_i : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, for $i = 1, \ldots, m$. Then, $F(x) \geq 0$ for all $x \in \mathbb{R}^n$ implies

$$\cap_{i=1}^{m} \{x \in \mathbb{R}^n : f_i(x) \geq 0\} \subseteq \{x \in \mathbb{R}^n : f_0(x) \geq 0\}. \tag{2.1}$$

The nonnegativity of the $s_i(x)$ terms and $F(x)$ as a whole can be associated with the existence of the respective SOS decompositions. Typically, one formulates $f_0(x)$ and $f_i(x)$ for $i = 1, \ldots, m$ meaningfully for a particular problem (noting that the inequalities $f_i(x) \geq 0$ correspond to a set of conditions, whereas $f_0(x) \geq 0$ describes some useful implication for the particular x) and then looks for $s_i(x) \in \Sigma[x]$ such that

$$f_0(x) - \sum_{i=1}^{m} s_i(x) f_i(x) \in \Sigma[x]. \tag{2.2}$$

The feasibility of the SDP problem which contains the LMI constraints associated with the existence of all SOS decompositions involved in (2.2) implies, in turn, that the set containment (2.1) holds. The generalized S-procedure is frequently used in the SOS literature (see, indicatively, Parrilo (2000, 2013); Tan and Packard (2008); Topcu et al. (2008); Prajna, Papachristodoulou, and Wu (2004a); Kundu and Anghel (2017)) and can be regarded as an extension to the original S-procedure which applies to quadratic problems (see Boyd et al. (1994)).

2.3 Related symbolic libraries and numerical solvers

SOS programming typically requires the use of a symbolic parser, in order to automatically formulate the LMIs corresponding to the SOS constraints and transfer them to an applicable solver. Three common options that are often cited in SOS-related works in the literature are SOSTOOLS (Papachristodoulou, Anderson, Valmorbida, Prajna, Seiler, and Parrilo, 2013), YALMIP (Löfberg, 2009), and GloptiPoly 3 (Henrion, Lasserre, and Loefberg, 2009). Applicable interior point solvers for SDPs are MOSEK (MOSEK ApS, 2017), SeDuMi (Sturm, 1999), and SDPT3 (Tütüncü, Toh, and Todd, 2003).

In this work, we utilize YALMIP and MOSEK. Few examples were also solved with SeDuMi, in the interest of verifying, to a certain extent, the validity of the solutions produced by MOSEK. The default values for all solver parameters were used.

Chapter 3

Controllability under input constraints

We now begin our study of the asymptotic stabilization problem with CLF-based control laws under input constraints. The main question that we will be addressing first is how to find a subset of the state space, which we call the stabilization set for the system, where a simple CLF-based control law (that is, a control law satisfying the input constraints that can render the time derivative of the particular CLF negative definite everywhere in that set) can asymptotically stabilize the system. We will also see how we can optimize that set, that is, either reshape it or simply enlarge it, by optimizing over the coefficients of the CLF.

This chapter forms the foundation for the entire dissertation in two different ways. First, the methods proposed here yield the stabilization set for a particular system. Inside this set, there is a guarantee that one can stabilize the system with any continuous control law based on the particular CLF, in the aforedescribed sense. This enables the systematic development of rather simple control laws that are implemented by solving a computationally lightweight QP, as will be shown in Chapter 4. Second, the SOS-based methods proposed here to analyze the stabilization set for a given system and CLF will be further augmented in Chapters 5, 6, and 7 to account for input constrained control under uncertainty, the case of large-scale systems, and control under imperfect feedback, respectively.

3.1 System description

We consider nonlinear control systems of the form

$$\dot{x} = f(x) + g(x)u, \qquad x(0) = x_0, \tag{3.1}$$

where $x \in \mathbb{R}^n$ is the state vector at time $t \geq 0$ with initial value $x_0 \in \mathbb{R}^n$, and $f : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are known polynomial functions of x, with $f(0) = 0$. The control input is u and it is assumed that $u(t) \in \mathbb{U} \subseteq \mathbb{R}^m$, for all $t \geq 0$, where $\mathbb{U}$ is a convex, compact polytope with vertices $\{v_1, \dots, v_q\} \in \mathbb{U}$, $0 \in \operatorname{Int}(\mathbb{U})$, which is also parameterized by the intersection of halfspaces described by $Au \preceq b$ for appropriate $A \in \mathbb{R}^{p\times m}$, $b \in \mathbb{R}^p$. For notational convenience, we consider the index set $\mathcal{Q}_{\mathbb{U}} = \{1, \dots, q\}$ corresponding to the vertices $\{v_1, \dots, v_q\}$ of $\mathbb{U}$. Let $u_c : \mathcal{D}_c \to \mathbb{U}$ be a continuous feedback control law, where $\mathcal{D}_c \subseteq \mathbb{R}^n$ is a compact set with $0 \in \operatorname{Int}(\mathcal{D}_c)$; the solution of the closed loop system at time $t \geq 0$ is denoted by $\phi(t; x_0, u_c)$. For ϕ to be well defined, it must remain in $\mathcal{D}_c$, that is, $\phi(t; x_0, u_c) \in \mathcal{D}_c$ for all $t \geq 0$.

3.2 Lyapunov stabilization under input constraints

Next, we introduce the concept of Control Lyapunov Functions (CLFs) for systems of the form (3.1) and discuss stabilization in the presence of input constraints. First, consider the set $\mathcal{V}$ of *candidate* Control Lyapunov Functions (CLFs), consisting of polynomial, positive definite and radially unbounded functions $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$.

Definition 3.2.1. A function $V \in \mathcal{V}$ is a CLF for system (3.1) if there exists a set $\mathbb{X} \subseteq \mathbb{R}^n$ with $0 \in \operatorname{Int}(\mathbb{X})$ and a positive definite polynomial function $W : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, such that

$$\inf_{u\in\mathbb{U}} \; \psi(x, u) \leq -W(x), \tag{3.2}$$

for all $x \in \mathbb{X}$, where $\psi(x, u) := \nabla V(x)(f(x) + g(x)u)$.

In our work, $W(x)$ can be chosen to grow rather "slowly" with $\|x\|$, to merely provide that $\psi(x,u) < 0$ for $x \in \mathbb{X} \setminus \{0\}$. That allows us to include in $\mathbb{X}$ parts of the state space where convergence is slow, given the dynamics and the input constraints, whereas performance considerations enter our formulation explicitely at a later point in Chapter 4 (and not in the form of hard constraints). A significant difficulty arises from the fact that the set $\mathbb{X}$, which in view of Definition 3.2.1 can be defined as

$$\mathbb{X} := \left\{ x \in \mathbb{R}^n : \inf_{u \in \mathbb{U}} \psi(x,u) \leq -W(x) \right\}, \tag{3.3}$$

is not necessarily invariant under (3.1) and a control law conforming to $\psi(x, u_c(x)) \leq -W(x)$ for all $x \in \mathbb{X}$. Such a controlled trajectory of (3.1) emanating from some point in $\mathbb{X}$ may escape to $\mathbb{X}^{\mathsf{c}} = \mathbb{R}^n \setminus \mathbb{X}$, where no $u \in \mathbb{U}$ exists to render $\psi(x,u) < 0$ and V will start to grow, precluding stabilization in a Lyapunov sense. In pursuit of asymptotic stabilization results within invariant sets which are easy to handle analytically and computationally, we consider sublevel sets of the CLF, that is, $\Omega_{V,\gamma} := \{x \in \mathbb{R}^n : V(x) \leq \gamma\}$, which will be required to be contained in $\mathbb{X}$.

Lemma 3.2.1. *Let $V \in \mathcal{V}$ be a CLF for (3.1). If $\gamma > 0$ is such that*

$$\Omega_{V,\gamma} \subseteq \mathbb{X}, \tag{3.4}$$

there exists a continuous feedback control law $u_c : \Omega_{V,\gamma} \to \mathbb{U}$ such that the closed loop system resulting by setting $u = u_c(x)$ in (3.1), with $x_0 \in \Omega_{V,\gamma}$, is locally asymptotically stable and $\Omega_{V,\gamma}$ is a subset of the region of attraction.

Proof. By definition of $\mathbb{X}$, inequality (3.2) holds for all $x \in \Omega_{V,\gamma} \subseteq \mathbb{X}$. Let $\mathfrak{U} : \Omega_{V,\gamma} \rightrightarrows \mathbb{U}$ be the set-valued map that maps each $x \in \Omega_{V,\gamma}$ to $\mathfrak{U}(x) := \{u \in \mathbb{U} : \psi(x,u) \leq -W(x)\}$. Note that the set $\mathfrak{U}(x)$ is nonempty, since (3.2) holds for all $x \in \Omega_{V,\gamma}$. Let $u_c : \Omega_{V,\gamma} \to \mathbb{U}$ be any continuous selection function such that $u_c(x) \in \mathfrak{U}(x)$. The existence of such a selection is guaranteed following standard results on set-valued maps. In particular, $\mathfrak{U} : \Omega_{V,\gamma} \rightrightarrows \mathbb{U}$ is lower semicontinuous and it maps to the nonempty intersection of closed halfspaces so, Michael's theorem (Freeman and Kokotović, 1996, Theorem 2.18) applies. For all $x \in \Omega_{V,\gamma} \setminus \{0\}$ along the trajectories of (3.1), $\dot{V} = \psi(x, u_c(x)) < 0$ holds. By standard Lyapunov arguments, the asymptotic stabilization result follows. $\square$

The existence of a CLF V for (3.1) such that $\Omega_{V,\gamma} \subseteq \mathbb{X}$ for some $\gamma > 0$ results in the system (3.1) being called, in a Lyapunov sense, *asymptotically controllable* (Bacciotti and Rosier, 2005). Note that the requirement $0 \in \mathrm{Int}(\mathbb{X})$ serves not only as a form of a local controllability assumption near the origin using a continuous feedback control law, but also as a guarantee for the existence of some $\gamma > 0$ such that $\Omega_{V,\gamma} \subseteq \mathbb{X}$.

3.3 Problem statements

We now provide formal problem statements for the problems of *CLF analysis* and *CLF optimization* that we address in this chapter.

CLF analysis

Given a CLF $V \in \mathcal{V}$ for the system (3.1), the input value set $\mathbb{U}$ and some $\bar{\gamma} \in \mathbb{R}_{>0}$, find $\hat{\gamma}$ where

$$\hat{\gamma} := \sup \{\gamma \in (0, \bar{\gamma}] : (3.4) \text{ holds}\} . \tag{3.5}$$

Following Lemma 3.2.1, continuous feedback laws based on the particular CLF that asymptotically stabilize the system (3.1) for any $x_0 \in \mathcal{D}_c$, where $\mathcal{D}_c := \Omega_{V,\hat{\gamma}}$, are guaranteed to exist. We say that all such feedback control laws $u_c : \mathcal{D}_c \to \mathbb{U}$ (based on the same CLF) form a *control law family* parameterized by V, and define the respective set $\mathcal{U}$ as

$$\mathcal{U} := \{u_c : \psi(x, u_c(x)) \leq -W(x), \ \forall\, x \in \mathcal{D}_c\}, \tag{3.6}$$

while $\mathcal{D}_c$ is called the *stabilization set* of the particular family. $\mathcal{U}$ consists of continuous selection functions for the set-valued map $\widetilde{u} : \mathcal{D}_c \rightrightarrows \mathbb{U}$ which maps each $x \in \mathcal{D}_c$ to the intersection of halfspaces described by

$$\mathfrak{U}(x) := \{u \in \mathbb{U} : \psi(x, u) \leq -W(x)\}. \tag{3.7}$$

CLF optimization

The *CLF optimization* problem is concerned with adjusting a CLF via optimizing over its coefficients, to obtain control laws with a stabilization set that satisfies certain requirements which have been captured by forming a continuous shape function $P \in \mathcal{V}$.

Given the set of candidate CLFs $\mathcal{V}$ for the system (3.1), the shape function P, the input value set $\mathbb{U}$ and some $\bar{\sigma} \in \mathbb{R}_{>0}$, find $\hat{\sigma}$ and $\hat{V}$ which solve

$$\sup_{\sigma \in (0, \bar{\sigma}], V \in \mathcal{V}} \sigma, \text{ subject to } \Omega_{P,\sigma} \subseteq \Omega_{V,1} \subseteq \mathbb{X}. \tag{3.8}$$

Attempting to fit progressively larger sublevel sets of P inside the stabilization set corresponding to a control law family via optimizing over a set of CLFs can result in new control law families with stabilization sets conforming to the previously defined requirements. Note that the sublevel set $\Omega_{V,1}$ corresponding to the stabilization set of the control law family $\mathcal{U}$ has been normalized to $\hat{\gamma} = 1$ in order to avoid a redundancy in the problem's unknowns, which now include the CLF's coefficients.

In the remainder of the chapter, we focus on the set containment (3.4), which appears in both in the CLF analysis (3.5) and the CLF optimization (3.8) problems and which is a sufficient condition for the system to be asymptotically controllable with an input constrained CLF-based control law, for any $x_0 \in \Omega_{V,\gamma} \subseteq \mathbb{X}$.

3.4 Partitioning the state space based on the input value set geometry

The first method towards warranting (3.4) is based on a partition of the state space into a finite number of non-overlapping sets, which accounts for the interplay between the dynamics, the particular CLF and the shape of the input value set $\mathbb{U}$. In each such cell of the partition, the value of a particular vertex of $\mathbb{U}$ is expected to cause the largest possible decrease rate to V, at least near the origin. This allows us to characterize the stabilization action in that subset of the $\mathbb{R}^n$ using a numerical value of u, which in turn enables the formulation of appropriate semialgebraic set containments in $\mathbb{R}^n$. We first consider the following fundamental result from linear programming, which we use next to ultimately obtain a convenient characterization of the set $\mathbb{X}$.

Proposition 3.4.1. *(Bertsekas, 2009, Prop. 2.4.2) Let $\mathbb{P} \subseteq \mathbb{R}^m$ be a convex, compact polytope. Given $\alpha \in \mathbb{R}^m$, the linear function $u \mapsto \alpha^\mathsf{T} u$, where $u \in \mathbb{P}$, attains its minimum value at some vertex of $\mathbb{P}$.*

Lemma 3.4.2. *Given $i \in \mathcal{Q}_\mathbb{U}$, let $\psi_i(x) := \psi(x, v_i) = \nabla V(x)(f(x) + g(x)v_i)$ and $\mathbb{X}_i := \{x \in \mathbb{R}^n : \psi_i(x) \leq -W(x)\}$. Then, $\cup_{i \in \mathcal{Q}_\mathbb{U}} \mathbb{X}_i = \mathbb{X}$ holds.*

Proof. For a fixed $x \in \mathbb{R}^n$, the function $u \mapsto \psi(x, u)$ is linear in $u \in \mathbb{U}$, where $\mathbb{U}$ is a compact, convex polytope with vertices v_i, $i \in \mathcal{Q}_\mathbb{U}$. Following Proposition 3.4.1 and the definition of the set $\mathbb{X}$, $x \in \mathbb{X}$ holds if and only if $x \in \mathbb{X}_i$ holds for some index $i \in \mathcal{Q}_\mathbb{U}$ (that is, for either one of multiple indices i). Therefore, $\cup_{i \in \mathcal{Q}_\mathbb{U}} \mathbb{X}_i \supseteq \mathbb{X}$. The fact that $\cup_{i \in \mathcal{Q}_\mathbb{U}} \mathbb{X}_i \subseteq \mathbb{X}$ follows from the definitions of the sets $\mathbb{X}$ and $\mathbb{X}_i$, $i \in \mathcal{Q}_\mathbb{U}$, which, in connection with the previous result, yields $\cup_{i \in \mathcal{Q}_\mathbb{U}} \mathbb{X}_i = \mathbb{X}$. $\square$

Following Lemma 3.4.2, inequality (3.2) is equivalent to

$$\inf_{i \in \mathcal{Q}_\mathbb{U}} \psi_i(x) \leq -W(x),$$

allowing us to convert the disjunction over all elements of $\mathbb{U}$, introduced by the original definition of $\mathbb{X}$ in (3.3), to a disjunction over a finite number of values corresponding to the vertices v_i of $\mathbb{U}$, for $i \in \mathcal{Q}_\mathbb{U}$. Still, the disjunction over the vertex indices $i \in \mathcal{Q}_\mathbb{U}$ is not compatible with the S-procedure, which is based on a logical conjunction over semialgebraic sets. To overcome the hardship, we consider holding $i = \ell$ locally fixed and enforcing $\psi(x, v_\ell) \leq -W(x)$, for all x inside a certain subset of $\mathbb{R}^n$ corresponding to each ℓ. These sets are obtained as follows.

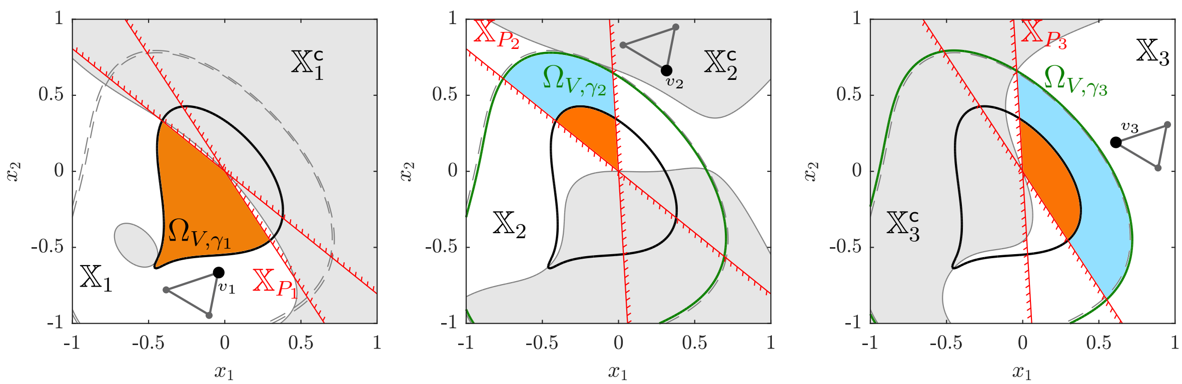


Figure 3.1: Illustration of the proposed partitioning scheme for each vertex v_i of the input value set $\mathbb{U}$, $i \in \mathcal{Q}_\mathbb{U}$, for the system considered in the example problem of Section 3.7.1. The gray-shaded parts of the state-space correspond to $x \in \mathbb{X}_i^\mathsf{c}$. The red hatched lines indicate the halfspaces determined by $(\eta_i - \eta_j)^\mathsf{T} x \leq 0$, for $j \in \mathcal{Q} \setminus \{i\}$. The color-filled sublevel sets of V correspond to the containment described by $\mathbb{X}_{P_i} \cap \Omega_{V,\gamma} \subseteq \mathbb{X}_i$, for $i = 1, 2, 3$. Since $\mathbb{X}_{P_i} \cap \Omega_{V,\gamma} \subseteq \mathbb{X}_i$ is required to hold concurrently for all $i \in \mathcal{Q}_\mathbb{U}$, the maximal $\Omega_{V,\gamma}$ corresponds to Ω_{V,γ_1}, where $\gamma_1 = 0.311$.

Proposition 3.4.3. *Assume that $g(0) \neq 0$ and let $\eta_i := \nabla^2 V(0) g(0) v_i$, $i \in \mathcal{Q}_\mathbb{U}$. The collection of sets*

$$\mathbb{X}_{P_i} := \cap_{j \in \mathcal{Q}_\mathbb{U} \setminus \{i\}} \left\{ x \in \mathbb{R}^n : (\eta_i - \eta_j)^\mathsf{T} x \leq 0 \right\}, \; i \in \mathcal{Q}_\mathbb{U},$$

forms a partition of $\mathbb{R}^n$.

Proof. Note that the η_i vectors are well defined, as $\nabla^2 V(x)$ exists and is a continuous function of x, since V is a polynomial and, therefore, smooth. First, we show that

$\cup_{i\in\mathcal{Q}_{\mathbb{U}}}\mathbb{X}_{P_i} = \mathbb{R}^n$. Consider any $x \in \mathbb{R}^n \setminus \{0\}$ and let $y = \alpha x$ with $\alpha \in (0,1]$. Also, consider the open ball $\mathcal{B}_r^n \subset \mathbb{X}$ for $r > 0$, the existence of which is provided by the fact that $0 \in \mathrm{Int}(\mathbb{X})$. For α such that $y = \alpha x \in \mathcal{B}_r^n$ and $\|y\|$ is sufficiently small, the Taylor expansion of $\psi_i(y)$ around the origin yields $\psi_i(y) = \eta_i^\mathsf{T} y + o(\|y\|)$, with $\lim_{y\to 0} o(\|y\|)/\|y\| = 0$. Since $\mathcal{B}_r^n \subset \mathbb{X}$, note that $y = \alpha x \in \mathbb{X}$ also holds. Let $\mathcal{I} \subseteq \mathcal{Q}_{\mathbb{U}}$ denote the index set such that $\psi_\ell(\alpha x) < 0$ for all $\ell \in \mathcal{I}$ and note that $\mathcal{I} \neq \varnothing$, in view of Lemma 3.4.2. Given our choice of α, $\|y\| = \alpha\|x\|$ is small so $\psi_i(y) < 0$ implies $\eta_i^\mathsf{T} x \leq 0$ for all $i \in \mathcal{I}$, while $\eta_i^\mathsf{T} x \geq 0$ for all $i \in \mathcal{Q}_{\mathbb{U}} \setminus \mathcal{I}$. Thus, there exists an $i^* \in \mathcal{I}$ such that $\eta_{i^*}^\mathsf{T} x \leq \eta_i^\mathsf{T} x$ for all $i \in \mathcal{Q}_{\mathbb{U}}$, which is equivalent to $x \in \mathbb{X}_{P_{i^*}}$. Consequently, $\cup_{i\in\mathcal{Q}_{\mathbb{U}}}\mathbb{X}_{P_i} = \mathbb{R}^n$.

To show that $\mathrm{Int}(\mathbb{X}_{P_i}) \cap \mathrm{Int}(\mathbb{X}_{P_j}) = \varnothing$ for any $i, j \in \mathcal{Q}_{\mathbb{U}}$ with $i \neq j$, let us assume, on the contrary, that $x^* \in \mathrm{Int}(\mathbb{X}_{P_\mu}) \cap \mathrm{Int}(\mathbb{X}_{P_\nu})$, for $\mu, \nu \in \mathcal{Q}_{\mathbb{U}}$ with $\mu \neq \nu$. By definition, $\eta_\mu^\mathsf{T} x^* < \eta_i^\mathsf{T} x^*$ for all $i \in \mathcal{Q}_{\mathbb{U}} \setminus \{\mu\}$ and $\eta_\nu^\mathsf{T} x^* < \eta_j^\mathsf{T} x^*$ for all $j \in \mathcal{Q}_{\mathbb{U}} \setminus \{\nu\}$. These inequalities imply that $\eta_\mu^\mathsf{T} x^* < \eta_\nu^\mathsf{T} x^*$ and $\eta_\nu^\mathsf{T} x^* < \eta_\mu^\mathsf{T} x^*$. At this point we have reached a contradiction; therefore, $\mathrm{Int}(\mathbb{X}_{P_\mu}) \cap \mathrm{Int}(\mathbb{X}_{P_\nu}) = \varnothing$ and the proof is complete. $\square$

Within each set $\mathbb{X}_{P_\ell}$ and at least near the origin, not only is the scalar quantity $\psi(x, v_i)$ negative for $i = \ell$, but also $\psi(x, v_\ell) \leq \psi(x, v_j) \leq \psi(x, u)$, for $j \in \mathcal{Q}_{\mathbb{U}}$ and $u \in \mathbb{U}$. This indication of stronger control authority forms the rationale of the proposed partition. Based on this partition, we now proceed to derive a semialgebraic set containment which is sufficient to show (3.4).

Proposition 3.4.4. *Suppose that*

$$\mathbb{X}_{P_i} \cap \Omega_{V,\gamma} \subseteq \mathbb{X}_i$$

holds for all $i \in \mathcal{Q}_{\mathbb{U}}$. Then $\Omega_{V,\gamma} \subseteq \mathbb{X}$.

Proof. By Lemma 3.4.2, we have $\mathbb{X}_i \subseteq \mathbb{X}$. According to Proposition 3.4.3, $\cup_{i\in\mathcal{Q}_\mathbb{U}}\mathbb{X}_{P_i} = \mathbb{R}^n$, while $\Omega_{V,\gamma} \subseteq \mathbb{R}^n$. Therefore,

$$(\cup_{i\in\mathcal{Q}_\mathbb{U}}\mathbb{X}_{P_i}) \cap \Omega_{V,\gamma} = \mathbb{R}^n \cap \Omega_{V,\gamma} = \Omega_{V,\gamma}$$

and, ultimately, one obtains $\Omega_{V,\gamma} \subseteq \mathbb{X}$. □

Following the S-procedure, for a particular $i \in \mathcal{Q}_\mathbb{U}$ (corresponding to the vertex $v_i \in \mathbb{U}$), consider

$$-(\psi_i(x) + W(x)) + c^{[i]}(x)(V(x) - \gamma) + \sum_{j\in\mathcal{Q}_\mathbb{U}\setminus\{i\}} s_j^{[i]}(x)(\eta_i - \eta_j)^\mathsf{T} x \in \Sigma[x], \tag{3.9}$$

where $c^{[i]}(x) \in \Sigma[x]$ and $s_j^{[i]}(x) \in \Sigma[x]$, for every $j \in \mathcal{Q}_\mathbb{U} \setminus \{i\}$, are unknown. When $\Sigma[x]$ is parameterized as SOS, the inclusion (3.9) can be expressed as an LMI which is convex in auxiliary variables that correspond to the unknown coefficients of all the nonnegative polynomials involved. The existence of $c^{[i]}(x)$ and all $q-1$ polynomials $s_j^{[i]}(x)$ such that (3.9) holds is a sufficient condition, in turn, for $\mathbb{X}_{P_i} \cap \Omega_{V,\gamma} \subseteq \mathbb{X}_i$ to hold for the particular $i \in \mathcal{Q}_\mathbb{U}$. Accordingly, the existence of all q polynomials $c^{[i]}(x)$ and all $q \times (q-1)$, in total, $s_j^{[i]}$, corresponding to all q instances of (3.9) for $i \in \mathcal{Q}_\mathbb{U}$, implies that $\mathbb{X}_{P_i} \cap \Omega_{V,\gamma} \subseteq \mathbb{X}_i$ holds for all $i \in \mathcal{Q}_\mathbb{U}$. Following Proposition 3.4.4, the set containment (3.4), which implies asymptotic controllability in the presence of input constraints, holds. For notational convenience, we associate the existence of all $c^{[i]}(x)$ and all $s_j^{[i]}(x)$ for all q instances of (3.9) for $i \in \mathcal{Q}_\mathbb{U}$ with the nonemptyness of the sets attained by $\mathcal{A}_\mathrm{I} : \mathbb{R}_{>0} \rightrightarrows \Sigma[x]^{q\times(q-1)} \times \Sigma[x]^q$, with

$$\begin{aligned}\mathcal{A}_\mathrm{I}(\gamma) := \big\{ & s_j^{[i]}(x) \in \Sigma[x] \;\; \forall i \in \mathcal{Q}_\mathbb{U}, j \in \mathcal{Q}_\mathbb{U} \setminus \{i\}, \\ & c^{[i]}(x) \in \Sigma[x] \; \forall i \in \mathcal{Q}_\mathbb{U} : (3.9) \text{ holds } \forall i \in \mathcal{Q}_\mathbb{U} \big\}.\end{aligned} \tag{3.10}$$

3.5 Employing a convex combination of the input value set vertices using polynomial coefficients

One can notice in Figure 3.1 that there exists a part of $\mathbb{R}^n$ which does not belong to $\mathbb{X}_1$ (as the partition-based scheme would anticipate to be case to determine that this part of the state space also belongs to $\mathbb{X}$), yet, it belongs to $\mathbb{X}_2$. This observation is not necessarily system- or CLF- specific. The input affine form of the dynamics and the polytopic form of $\mathbb{U}$ enable us to study and characterize the asymptotic controllabilty of the system by considering just the vertices of $\mathbb{U}$ in a lossless way (refer to Lemma 3.4.2). Yet, in our pursuit of sufficient conditions in terms of semialgebraic set containments, associating vertices v_i with each particular cell of the partition, based on local indications of stronger control authority, can introduce some conservatism to the solution as seen here. We now propose a method to show (3.4) that may result in larger SOS problems but avoids associating partition cells with vertices of $\mathbb{U}$ based on local information near the origin and, thus, cannot manifest the aforedescribed potential pitfalls for states away from the origin.

Proposition 3.5.1. *Let the polynomials $a_\kappa : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, $\kappa \in \mathcal{Q}_{\mathbb{U}}$, be such that $\zeta(x) := \sum_{\kappa \in \mathcal{Q}_{\mathbb{U}}} a_\kappa(x)$ satisfies $0 < \zeta(x) \leq 1$, for all $x \in \mathbb{R}^n$. Also, let $\hat{u}(x) := \sum_{\kappa \in \mathcal{Q}_{\mathbb{U}}} a_\kappa(x) v_\kappa$. Then,*

$$\{x \in \mathbb{R}^n : \psi(x, \hat{u}(x)) \leq -W(x)\} \subseteq \mathbb{X}. \tag{3.11}$$

Proof. One can write $\hat{u}(x) = \sum_{\kappa \in \mathcal{Q}_{\mathbb{U}}} \beta_\kappa(x)\zeta(x)v_k$, where $\beta_\kappa(x) := a_\kappa(x)/\zeta(x)$, $\kappa \in \mathcal{Q}_{\mathbb{U}}$. By the definition of $\zeta(x)$, $\beta_\kappa(x) \geq 0$ and $\sum_{\kappa \in \mathcal{Q}_{\mathbb{U}}} \beta_\kappa(x) = 1$, for all $x \in \mathbb{R}^n$. Therefore, $x \mapsto \hat{u}(x)$ maps each $x \in \mathbb{R}^n$ to $\mathbb{U}' \subseteq \mathbb{R}^m$, where $\mathbb{U}' := \mathrm{Conv}\{\zeta(x)v_\kappa, \kappa \in \mathcal{Q}_{\mathbb{U}}\}$. Note that $\mathbb{U}'$ results by uniformly shrinking $\mathbb{U}$ about $0 \in \mathrm{Int}(\mathbb{U})$, therefore, $\mathbb{U}' \subseteq \mathbb{U}$. Since $\hat{u} : \mathbb{R}^n \to \mathbb{U}' \subseteq \mathbb{U}$ and by the definition (3.3) of the set $\mathbb{X}$, if $x \in \mathbb{R}^n$ is such that $\psi(x, \hat{u}(x)) \leq -W(x)$ holds, then $x \in \mathbb{X}$ also holds, implying (3.11). □

The -yet undetermined- $a_\kappa(x)$ allow us to parameterize $\mathbb{X}$ while avoiding the disjunction found in the definition (3.3). By virtue of the S-procedure and Proposition 3.5.1, the existence of $c(x) \in \Sigma[x]$ and $a_\kappa(x) \in \Sigma[x]$, $\kappa \in \mathcal{Q}_{\mathbb{U}}$, such that

$$-\left(\psi\left(x, \sum_{\kappa\in\mathcal{Q}_{\mathbb{U}}} a_\kappa(x) v_\kappa\right) + W(x)\right) \in \Sigma[x], \tag{3.12a}$$

$$1 - \sum_{\kappa\in\mathcal{Q}_{\mathbb{U}}} a_\kappa(x) \in \Sigma[x], \tag{3.12b}$$

implies (3.4). In a similar vein to the partitioning approach, we associate the existence of the involved nonnegative polynomial decompositions with the nonemptyness of the set-valued map $\mathcal{A}_{\mathrm{II}} : \mathbb{R}_{>0} \rightrightarrows \Sigma[x]^q \times \Sigma[x]$, with

$$\mathcal{A}_{\mathrm{II}}(\gamma) := \{a_\kappa(x) \in \Sigma[x]\ \forall \kappa \in \mathcal{Q}_{\mathbb{U}}, c(x) \in \Sigma[x] : (3.12a), (3.12b) \text{ hold}\}. \tag{3.13}$$

Figure 3.2 illustrates the efficacy of the proposed method.

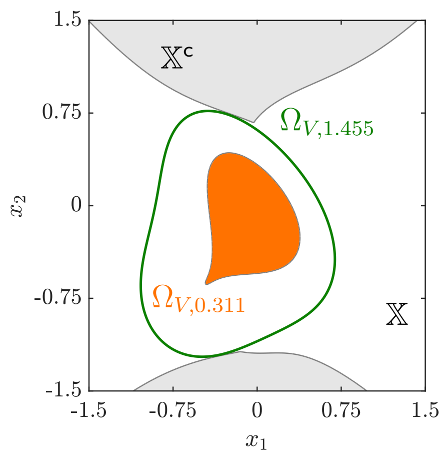


Figure 3.2: A -seemingly- non-conservative estimate of the maximal $\Omega_{V,\gamma} \subseteq \mathbb{X}$ for $\gamma = 1.455$, using the approach of Section 3.5 to warrant (3.4) for Example 1. The calculations required searching for degree 12 SOS polynomials (as opposed to degree 4 for the approach of Section 3.4, which, though, yields the much smaller sublevel corresponding to $\gamma = 0.311$).

3.6 SDP-based algorithms for CLF analysis and optimization

The analysis problem, is concerned with finding the largest positive number $\hat{\gamma} \in (0, \bar{\gamma}]$, such that (3.4) holds. Following the preceding discussion and given $\gamma > 0$, the nonemptyness of the sets attained by either $\mathcal{A}_{\mathrm{I}}(\gamma)$ or $\mathcal{A}_{\mathrm{II}}(\gamma)$, given, respectively by (3.10) or (3.13), is sufficient for (3.4) to hold for the particular γ. The nonemptyness of each of these sets corresponds to the existence of the respective nonnegative polynomial decompositions in (3.9) or (3.12). Accordingly, the latter is equivalent to the feasibility of the corresponding convex optimization problem, which is an SDP when nonnegative polynomials are parameterized as SOS. A simple bisection scheme, as outlined next in Algorithm 1, is used to asymptotically approximate $\hat{\gamma}$ up to some relative tolerance ϵ_{tol}. $\mathcal{A}_{\square}(\gamma)$ can refer to either $\mathcal{A}_{\mathrm{I}}(\gamma)$ or $\mathcal{A}_{\mathrm{II}}(\gamma)$, depending on the desired approach to express the input constrained controllability condition (3.4).

Algorithm 1 Bisection-based analysis algorithm

Require: $\{\gamma_l, \bar{\gamma} \in \mathbb{R}_{>0}\colon\ \gamma_l < \bar{\gamma}, \mathcal{A}_{\square}(\gamma_l) \neq \varnothing\}$, $\epsilon_{tol} > 0$
1: **if** $\mathcal{A}_{\square}(\bar{\gamma}) \neq \varnothing$ **then return** $\hat{\gamma} \leftarrow \bar{\gamma}$ **else** $\gamma_u \leftarrow \bar{\gamma}$
2: **repeat**
3:     **set** $\gamma_m \leftarrow (\gamma_l + \gamma_u)/2$
4:     **if** $\mathcal{A}_{\square}(\gamma_m) = \varnothing$ **then** $\gamma_u \leftarrow \gamma_m$ **else** $\gamma_l \leftarrow \gamma_m$
5: **until** $(\gamma_u - \gamma_l)/\gamma_l \leq \epsilon_{tol}$
6: **return** $\hat{\gamma} \leftarrow \gamma_l$

We now proceed to the solution of the CLF optimization problem, by assuming that the analysis problem has been solved, in the sense that a $\hat{\gamma} > 0$ has been found such that $\Omega_{V,\hat{\gamma}} \subseteq \mathbb{X}$. Without loss of generality, we scale V and $\hat{\gamma}$ in a way such that $\hat{\gamma} = 1$. Given a shape function $P(x)$, as described in Section 3.3, the CLF optimization problem can be formulated as the search for the largest $\sigma \in (0, \bar{\sigma}]$ as well as the corresponding CLF V and all other involved nonnegative polynomials, such that either

(3.9) or (3.12a)-(3.12b) hold, for $\gamma = 1$ and all $i \in \mathcal{Q}_{\mathbb{U}}$, and

$$(1 - V(x)) - s_P(x)(\sigma - P(x)) \in \Sigma[x], \tag{3.14a}$$

$$V(x) - \epsilon x^{\mathsf{T}} x \in \Sigma[x], \quad V(0) = 0, \tag{3.14b}$$

where $s_P(x) \in \Sigma[x]$. The existence of such an $s_P(x)$ so that (3.14a) holds implies $\Omega_{P,\sigma} \subseteq \Omega_{V,1}$; the former, coupled with either (3.9) or (3.12a)-(3.12b) holding for $\gamma = 1$ and all $i \in \mathcal{Q}_{\mathbb{U}}$, imply the double containment $\Omega_{P,\sigma} \subseteq \Omega_{V,1} \subseteq \mathbb{X}$ appearing in the definition of the CLF optimization problem (3.8). Finally, we use (3.14b), with $0 < \epsilon \ll 1$, to parameterize the set of candidate CLFs $\mathcal{V}$ as SOS polynomials. For the case where (3.4) is warranted following the partitioning approach of Section 3.4, let one consider the set-valued maps $\mathcal{S}_{A_{\mathrm{I}}} : \mathbb{R}_{>0} \times \mathcal{V} \rightrightarrows \Sigma[x]^{q\times(q-1)} \times \Sigma[x]^q \times \Sigma[x]$ and $\mathcal{S}_{B_{\mathrm{I}}} : \mathbb{R}_{>0} \times \Sigma[x]^q \times \Sigma[x]^{q\times(q-1)} \rightrightarrows \mathcal{V} \times \Sigma[x]$, with

$$\begin{aligned}\mathcal{S}_{A_{\mathrm{I}}}(\sigma, V) := \big\{ & s_j^{[i]}(x) \in \Sigma[x] \;\; \forall i \in \mathcal{Q}_{\mathbb{U}}, j \in \mathcal{Q}_{\mathbb{U}} \setminus \{i\}, \\ & c_i(x) \in \Sigma[x] \;\; \forall i \in \mathcal{Q}_{\mathbb{U}}, s_p(x) \in \Sigma[x] : \\ & (3.9) \text{ holds for } \gamma = 1, \forall i \in \mathcal{Q}_{\mathbb{U}}, (3.14a) \text{ holds} \big\},\end{aligned}$$

and

$$\begin{aligned}\mathcal{S}_{B_{\mathrm{I}}}(\sigma, \{c_i\}, \{s_j^{[i]}\}) := \{ & V \in \Sigma[x], s_P(x) \in \Sigma[x] : \\ & (3.14a) - (3.14b) \text{ hold}, (3.9) \text{ holds for } \gamma = 1, \forall i \in \mathcal{Q}_{\mathbb{U}} \}.\end{aligned}$$

The nonemptyness of the sets attained by $\mathcal{S}_{A_{\mathrm{I}}}$ and $\mathcal{S}_{B_{\mathrm{I}}}$ is equivalent to the feasibility of the convex optimization problem involving the respective nonnegative polynomial decompositions. Formulating set-valued maps similar to $\mathcal{S}_{A_{\mathrm{I}}}$ and $\mathcal{S}_{B_{\mathrm{I}}}$ that adhere to the convex combination approach of Section 3.5 to warrant (3.4) is straightforward and the prototyping is omitted for brevity reasons.

Algorithm 2 Sequential CLF optimization algorithm

Require: $\sigma_0 > 0, V_0 \in \mathcal{V}, P \in \mathcal{V} : \Omega_{P,\sigma_0} \subseteq \Omega_{V_0,1} \subseteq \mathbb{X}$

$k \leftarrow 0$, $\sigma \leftarrow \sigma_0$
repeat
    if $\mathcal{S}_{A_\mathrm{I}}(\sigma, V_k) \neq \varnothing$ **then**
        $k \leftarrow k + 1$
        $\sigma_k \leftarrow \sigma$, $V_k \leftarrow V_{k-1}$
        pick $\{s_j^{[i]}\}, \{c_i\} \in \mathcal{S}_{A_\mathrm{I}}(\sigma, V_k)$
        $\sigma \leftarrow \mathtt{step}_+$
    else if $\mathcal{S}_{B_\mathrm{I}}(\sigma, \{c_i\}, \{s_j^{[i]}\}) \neq \varnothing$ **then**
        $k \leftarrow k + 1$
        pick $V_k \in \mathcal{S}_{B_\mathrm{I}}(\sigma, \{c_i\}, \{s_j^{[i]}\})$.
        $\sigma_k \leftarrow \sigma$
        $\sigma \leftarrow \mathtt{step}_+$
    else
        $\sigma \leftarrow \mathtt{step}_\sim$
    end if
until $\mathtt{tf}$ = TRUE
$\hat{\sigma} \leftarrow \sigma_k$, $\hat{V} \leftarrow V_k$
return $\hat{\sigma}$, $\hat{V}$, $\{\sigma_i\}_{i=1}^k$, $\{V_i\}_{i=1}^k$

Starting with an initial $\sigma = \sigma_0 \in \mathbb{R}_{>0}$ and a CLF $V_0 \in \mathcal{V}$ such that (3.9) and (3.14a) hold, the CLF optimization process is driven by the expansion of the volume of the shape function sublevel set $\Omega_{P,\sigma}$ and evolves sequentially by alternating between the two different feasibility problems. A minimal working implementation of the proposed solution is described by Algorithm 2. At every step, identified by the index k, a pair (σ_k, V_k) satisfying $\Omega_{P,\sigma_k} \subseteq \Omega_{V_k,1} \subseteq \mathbb{X}$ is obtained. Depending on whether the problem of $\mathcal{S}_{B_\mathrm{I}}(\cdot,\cdot,\cdot)$ has been solved and optimization over the CLF space has taken place, V_k will be a new CLF. To influence the execution of Algorithm 2, one can use $\mathtt{step}_+$ to determine the increase strategy for σ (i.e. linear, geometric, etc.), $\mathtt{step}_\sim$ to decrease the σ increase rate if both problems are unfeasible, and $\mathtt{tf}$ to terminate the algorithm (i.e. when $\sigma_k = \bar{\sigma}$ or when an iteration or time limit has been reached). Figure 3.3 illustrates the evolution of the the sublevel sets $\Omega_{V_\ell,1}$ during a typical execution of

Algorithm 2 for one of the example problems considered at the end of this chapter.

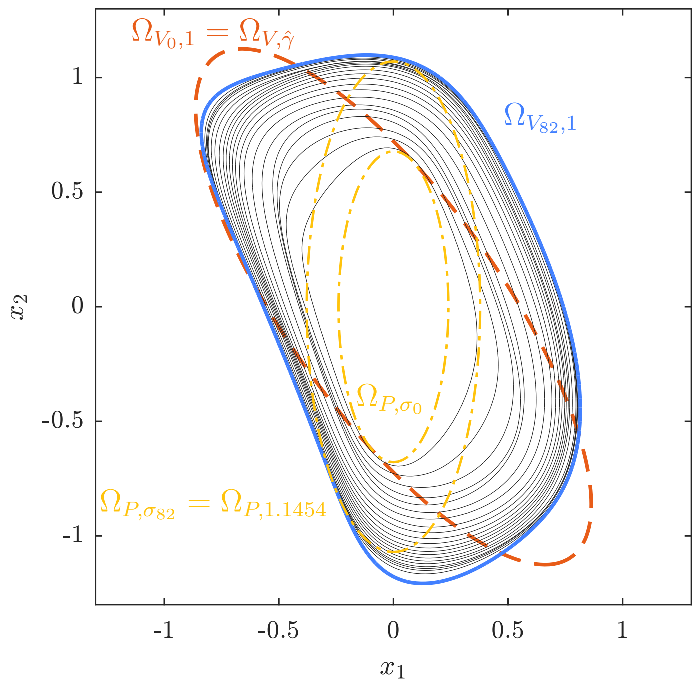


Figure 3.3: Different stabilization sets computed via application of Algorithm 2 to the system considered in the example of Section 3.7.2, to reshape and enlarge the stabilization set corresponding to the original CLF $V(x) = 1.7x_1^2 + 2x_1x_2 + x_2^2$. The (sub)level set illustrated with the dashed orange line corresponds to $\Omega_{V,\hat{\gamma}}$, where $\hat{\gamma} = 0.5217$ has been calculated using Algorithm 1. The CLF optimization is initialized by normalizing $\hat{\gamma}$, and proceeds by progressively expanding the sublevel sets of the shape function $P(x) = 8x_1^2 + x_2^2$, drawn with the yellow dashed line for the first and last iteration, which results in the generation of new CLFs V_ℓ for the system with stabilization sets $\mathcal{D}_c = \Omega_{V_\ell,1}$, for $\ell = 1, \ldots, 82$.

3.7 Numerical examples

The following examples help illustrate the contributions and the efficacy of methods proposed in this chapter. The necessary calculations have been performed using YALMIP (Löfberg, 2009), to parse the SOS constraints into SDPs and MOSEK (MOSEK ApS, 2017) to numerically solve the SDPs.

3.7.1 Planar nonlinear system with a triangular input value set

We consider the following system

$$\dot{x}_1 = x_2 + (1 - x_1^2 - x_2^2)x_1 + u_1,$$
$$\dot{x}_2 = -x_1 + (1 + x_1^2 + x_2^2)x_2 + u_2,$$

with $u = [u_1\ u_2]^\mathsf{T}$ constrained in the convex, compact polygon with vertices $v_1 = [2\ 3]^\mathsf{T}$, $v_2 = [1\ -2]^\mathsf{T}$, $v_3 = [-4\ 1]^\mathsf{T}$, and the CLF $V(x) = 2.36x_1^2 + 2.4x_1x_2 + 1.83x_2^2 + 2.52x_1^3 + 5.6001x_1^2x_2 + 5.7201x_1x_2^2 + 2.34x_2^3 + 2.05x_1^4 + 2.04x_1^3x_2 + 3.75x_1^2x_2^2 + 1.96x_1x_2^3 + 1.61x_2^4$. The reader should notice the coupling between u_1 and u_2 introduced by the triangular shape of $\mathbb{U}$.

Letting $\bar{\gamma} = 2$ and $W(x) = 10^{-6}x^\mathsf{T}x$, the CLF analysis algorithm, when we follow Section 3.4 to warrant the asymptotic controllability condition (3.4), yields $\hat{\gamma} = 0.311$. Searching for $\{c^{[i]}\}$ and $\{s_j^{[i]}\}$ SOS polynomials of degree no greater than 4, 6 and 8 yields similar results for $\hat{\gamma}$. Following Section 3.5 results in the non-conservative estimate $\hat{\gamma} = 1.455$, however, that required searching for degree 12 $\{c^{[i]}\}$ and $\{s_j^{[i]}\}$. Figures 3.1 and 3.2 illustrate the respective sublevel sets.

3.7.2 A system with multiple equilibria

The system described by

$$\dot{x}_1 = -x_1^3 - 0.5x_1^2 + 0.5x_1 + x_2,$$
$$\dot{x}_2 = x_1 - 2x_2 + u,$$

with $-1 \leq u \leq 1.2$, exhibits three equilibria: the origin is a saddle point, whereas the other two equilibria are stable nodes. Without applying control, trajectories emanating from the vicinity of the origin are attracted by either of the two nodes. Figure

3.3 illustrates the progression of the CLF optimization algorithm while searching for unknown SOS polynomials of degree no greater than 8.

3.7.3 A third order competing species system

We consider the third order system

$$
\begin{aligned}
\dot{x}_1 &= x_1 + x_1^2 - x_1 x_2 - x_1 x_3 + u_1, \\
\dot{x}_2 &= -0.1 x_2 + x_1 x_2 - 0.9 x_2 x_3, \\
\dot{x}_3 &= -x_1 x_3 + x_2 x_3 - x_3^2 + u_2,
\end{aligned}
$$

with $u = [u_1\ u_2]^\mathsf{T} \in \mathbb{U}$, where $\mathbb{U}$ is the convex, compact polygon with vertices $v_1 = [-2\ \ 2]^\mathsf{T}$, $v_2 = [-2\ \ -2]^\mathsf{T}$, $v_3 = [1.7\ \ -1.8]^\mathsf{T}$ and $v_4 = [1.7\ \ 1.8]^\mathsf{T}$. We choose the quadratic CLF $V(x) = x^\mathsf{T} x$, $W(x) = 10^{-6} V(x)$, and we let $\bar{\gamma} = 1$. Algorithm 1, following the approach of Section 3.4 and while searching for SOS multipliers of degree no larger than 4, yields $\hat{\gamma} = 0.7884$. The approach of Section 3.5 yields $\hat{\gamma} = 0.4569$ while searching for SOS multipliers of degree no greater than 10. Figure 3.4 illustrates the containment $\Omega_{V,\hat{\gamma}} \subseteq \mathbb{X}$.

3.7.4 Second order system with finite escape time in the control-free case

We consider the system with dynamics

$$
\begin{aligned}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= -0.5 x_1^2 - x_2 + u,
\end{aligned}
$$

with $-2 \leq u \leq 4$. If $u \equiv 0$, trajectories emanating from parts of the state space exhibit a finite escape time. Cancelling the nonlinearity would allow the global stabilization of the system, however, this is not an option under input constraints. The analysis algorithm for the CLF $V(x) = 1.7 x_1^2 + 2 x_1 x_2 + 1.7 x_2^2$ and $\bar{\gamma} = 20$, following Section

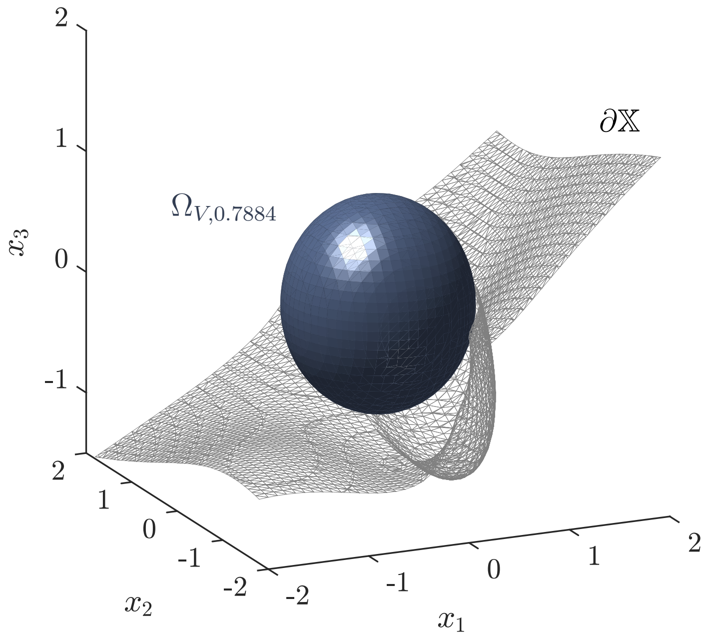


Figure 3.4: Illustration of the containment $\Omega_{V,\hat{\gamma}} \subseteq \mathbb{X}$ for the example of Section 3.7.3 for $\hat{\gamma} = 0.7884$. The containment appears to be tight, that is, a larger sublevel set of V would have crossed the boundary of $\mathbb{X}$, denoted here with $\partial\mathbb{X}$.

3.4 to warrant (3.4), yields the seemingly non-conservative estimate $\hat{\gamma} = 12.8961$ when searching for the involved SOS multipliers of degree no greater than 4. Algorithm 2 with $P(x) = 12x_1^2 + x_2^2$, $\bar{\sigma} = 25$ and degree 6 SOS yields $\hat{\sigma} = 21.6964$ and the corresponding new CLF (to the fourth significant digit)

$$
\begin{aligned}
\hat{V}(x) = {} & 2.337 \times 10^{-3} x_1^6 + 3.431 \times 10^{-3} x_1^5 x_2 + 7.139 \times 10^{-3} x_1^4 x_2^2 + 1.401 \times 10^{-3} x_1^3 x_2^3 \\
& + 0.001577 x_1^2 x_2^4 + 9.066 \times 10^{-5} x_1 x_2^5 + 2.752 \times 10^{-5} x_2^6 + 1.501 \times 10^{-3} x_1^5 \\
& - 2.528 \times 10^{-3} x_1^4 x_2 - 3.038 \times 10^{-3} x_1^3 x_2^2 - 1.114 \times 10^{-3} x_1^2 x_2^3 + 7.356 \times 10^{-5} x_1 x_2^4 \\
& + 6.621 \times 10^{-6} x_2^5 + 4.158 \times 10^{-3} x_1^4 + 5.275 \times 10^{-4} x_1^3 x_2 + 6.401 \times 10^{-4} x_1^2 x_2^2 \\
& + 2.063 \times 10^{-3} x_1 x_2^3 + 1.406 \times 10^{-3} x_2^4 - 2.991 \times 10^{-3} x_1^3 - 9.636 \times 10^{-5} x_1^2 x_2 \\
& + 9.195 \times 10^{-4} x_1 x_2^2 - 2.093 \times 10^{-4} x_2^3 + 7.215 \times 10^{-3} x_1^2 + 1.087 \times 10^{-3} x_1 x_2 \\
& + 2.833 \times 10^{-4} x_2^2.
\end{aligned}
$$

The involved (sub)level sets are illustrated in Figure 3.5.

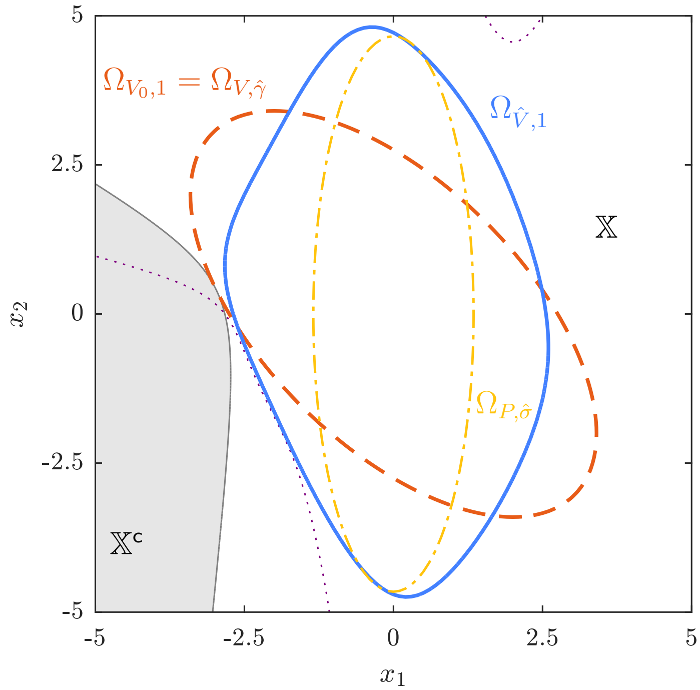


Figure 3.5: CLF analysis and optimization algorithms for the example of Section 3.7.4. The CLF analysis algorithm for the quadratic V yields the maximal sublevel set $\Omega_{V,\hat{\gamma}}$ contained in $\mathbb{X}$, while the CLF optimization algorithm yields a new degree 6 CLF $\hat{V}(x)$ with $\mathcal{D}_c = \Omega_{\hat{V},1}$, effectively reshaping and enlarging the stabilization set for the system towards larger values along the x_2 axis (the dotted line is used to draw the set $\mathbb{X}$, say $\hat{\mathbb{X}}$, corresponding to $\hat{V}$; observe that $\Omega_{\hat{V},1} \subseteq \Omega_{P,\hat{\sigma}} \subseteq \hat{\mathbb{X}}$ holds).

3.8 Summary

We have developed sufficient conditions for controllability using a CLF-based control law under input constraints. Using either of the two proposed methods, these sufficient conditions are numerically verifiable by solving SOS programs. We have also shown how to (locally) optimize over the coefficients of the given CLF in order to adjust the corresponding stabilization set. The results enable the efficient online implementation of constrained Lyapunov control laws, as well as the consideration of additionial complicating factors in the dynamics of the control system.

Chapter 4

Feedback control with Quadratic Programming

In view of the results of Chapter 3, the feedback control problem can be associated with obtaining a mapping $u_c \in \mathcal{U}$, where $\mathcal{U}$, as defined in (3.6), contains all asymptotically stabilizing control laws based on the CLF V, and can also be regarded as the set of all continuous selection functions for the set-valued map $\mathfrak{U}(x)$ given by (3.7). Here we proceed to develop a systematic way to design such a control law $u_c(x)$. The description of the class of systems under consideration is the same as in Chapter 3. We also assume knowledge of the stabilization set (which one can obtain by following the methods of Chapter 3) corresponding to the system under consideration, the input value set $\mathbb{U}$ and the particular CLF V.

4.1 Assembling the QP

At any $x \in \mathbb{R}^n$, let us consider the stabilization performance gap

$$\mathcal{H}(x,u) := (\psi(x,u) - \alpha(x))^2,$$

where $\alpha(x)$ is a continuous, negative definite function capturing the desired decrease rate of V along the closed loop trajectories. In pursuit of a pointwise feedback controller, we expand $\mathcal{H}$ and drop terms which do not contain u, obtaining a performance index quadratic in u where the state x appears as a parameter, that is,

$$\mathcal{J}(u;x) = u^\mathsf{T} Q(x) u + L(x) u,$$

where, for all $x \in \mathbb{R}^n$,

$$L(x) := 2\left[\nabla V(x) f(x) - \alpha(x)\right] \nabla V(x) g(x),$$
$$Q(x) := (\nabla V(x) g(x))^\mathsf{T} \nabla V(x) g(x).$$

By minimally shifting the spectrum of $Q(x)$, we obtain $\widetilde{\mathcal{J}}(u;x)$ which is a strictly convex function of u:

$$\widetilde{\mathcal{J}}(u;x) := u^\mathsf{T}[Q(x) + \mu(x) I_m]u + L(x)u, \tag{4.1}$$

where

$$\mu(x) := \max\Big(0, \epsilon - \lambda_{\min}\Big(g^\mathsf{T}(x)\left(\nabla V(x)\right)^\mathsf{T} \nabla V(x) g(x)\Big)\Big),$$

for some given small $\epsilon > 0$. Note that $x \mapsto \lambda_{\min}(Q(x))$ is continuous, given the continuity of the roots of the characteristic equation for $Q(x)$ (Henriksen and Isbell, 1953) and the fact that all roots are real, since $Q(x) = Q^\mathsf{T}(x)$. Consequently, $x \mapsto \mu(x)$ is also continuous.

Proposition 4.1.1. *For any given $x \in \Omega_{V,\hat{\gamma}}$, let $u^*(x)$ be the solution of the linearly constrained Quadratic Program (QP) of minimizing (4.1), subject to $u \in \mathfrak{U}(x)$, where $\mathfrak{U}(x)$ is given by (3.7). Then $u_c : \Omega_{V,\hat{\gamma}} \to \mathbb{U}$, where $u_c(x) = u^*(x)$ for all $x \in \Omega_{V,\hat{\gamma}}$, is an asymptotically stabilizing, continuous control law for (3.1).*

Proof. The domain $\mathfrak{U}(x)$ of the QP is nonempty for all $x \in \Omega_{V,\hat{\gamma}} \subseteq \mathbb{X}$. Therefore the minimizer $u^*(x)$ exists for all $x \in \Omega_{V,\hat{\gamma}}$. Moreover, given that $Q(x) + \mu(x) I_m$ is positive definite for all $x \in \Omega_{V,\hat{\gamma}}$, the QP problem is strictly convex and thus the minimizer $u^*(x)$ is unique. Also, the mapping $x \mapsto u^*(x)$ is continuous, since $x \mapsto \widetilde{\mathcal{J}}(u;x)$ and $x \mapsto \mathfrak{U}(x)$ are continuous and the matrix involved in the halfspace description of the polytopic domain $\mathfrak{U}(x)$ retains its rank, so, the results of Daniel (1973) hold. Thus,

$u_c : \Omega_{V,\hat{\gamma}} \to \mathbb{U}$ with $u_c(x) = u^*(x)$ constitutes a well-defined, continuous selection for $\mathfrak{U}(x)$ and, as is shown in the proof of Lemma 3.2.1, it is a continuous, asymptotically stabilizing control law for (3.1). $\square$

Remark 1. Calculating $u^*(x)$ requires solving a QP of dimension equal to the control input dimension m, where the current state x appears as a parameter. Such a QP is rapidly solvable on embedded computational platforms, enabling the use of the proposed control law in real-time embedded control applications.

Remark 2. The QP remains feasible along the trajectory of the system, for any $x_0 \in \Omega_{V,\hat{\gamma}}$, thanks to the preceding analysis.

Remark 3. For the special case where the approach of Section 3.5 of Chapter 3 has been used to warrant (3.4), one can identify that the polynomial $\hat{u}(x)$, as defined in Proposition 3.5.1, is an asymptotically stabilizing control law for (3.1). It is possible, therefore, to use $\hat{u}(x)$ to solve the control problem without considering the QP-based approach of the present chapter. Nevertheless, this method would essentially lack the tuning mechanism provided by the function $\alpha(x)$, introduced here to influence the convergence rate of the system. Incorporating such a feature in $\hat{u}(x)$ would entail considering additional constraints while looking for the polynomials $a_\kappa(x)$ appearing in $\hat{u}(x) := \sum_{\kappa \in \mathcal{Q}_{\mathbb{U}}} a_\kappa(x) v_\kappa$. Also, note that $\hat{u}(x)$ is smooth, and, as such, it may not as easily reach the boundary of $\mathbb{U}$ and saturate along it, compared to the QP-based control law which is only a continuous function of the state x, as mentioned before.

4.2 Numerical examples

For the following examples, we solved the QPs with the fast solver qpOASES (Ferreau, Kirches, Potschka, Bock, and Diehl, 2014).

4.2.1 Planar nonlinear system with a triangular input value set

We consider again the example system from Section 3.7.1, with dynamics

$$\dot{x}_1 = x_2 + (1 - x_1^2 - x_2^2)x_1 + u_1,$$
$$\dot{x}_2 = -x_1 + (1 + x_1^2 + x_2^2)x_2 + u_2,$$

where $u = [u_1\ u_2]^\mathsf{T}$ is constrained in the convex, compact polygon with vertices $v_1 = [2\ 3]^\mathsf{T}$, $v_2 = [1\ \ -2]^\mathsf{T}$, $v_3 = [-4\ 1]^\mathsf{T}$, and the quartic CLF $V(x)$ given by

$$V(x) = 2.36x_1^2 + 2.4x_1x_2 + 1.83x_2^2 + 2.52x_1^3 + 5.6001x_1^2x_2 + 5.7201x_1x_2^2 + 2.34x_2^3 + 2.05x_1^4 + 2.04x_1^3x_2 + 3.75x_1^2x_2^2 + 1.96x_1x_2^3 + 1.61x_2^4.$$

The state and input phase planes for four different initial conditions under the proposed control law with $\alpha(x) = -V(x)$ and $\varepsilon = 10^{-3}$ are illustrated in Figure 4.1.

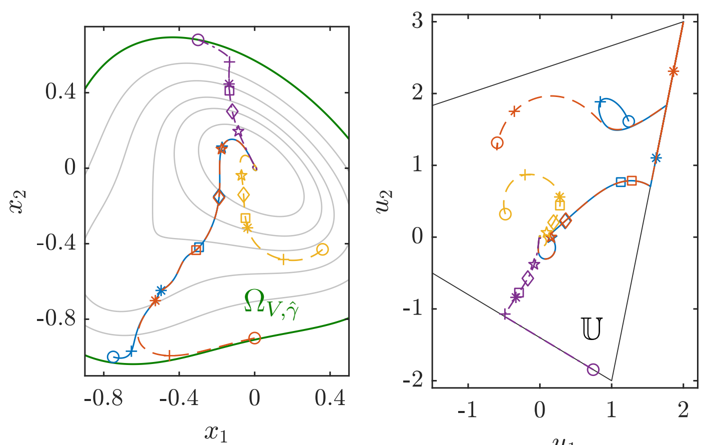


Figure 4.1: Phase-planes for the state and input variables of the system considered in Example 1, for four different initial conditions, timestamped according to $\circ(t = 0)$, $+(t = 0.3)$, $*(t = 0.97)$, $\square(t = 1.2)$, $\Diamond(t = 2)$, $\star(t = 3)$. Observe that the control input moves along the boundary of $\mathbb{U}$ without loss of asymptotic stability.

4.2.2 Second order system with finite escape time in the control-free case

Next, we consider again the example system from Section 3.7.4, with dynamics

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = -0.5x_1^2 - x_2 + u,$$

with $-2 \leq u \leq 4$ and the quadratic CLF $V(x) = 1.7x_1^2 + 2x_1x_2 + 1.7x_2^2$. We use the particular example to illustrate the operation of the proposed control law, as well as the differences between our general control methodology and the min-norm control paradigm (see Freeman and Kokotović (1996); Primbs et al. (2000); Curtis (2003) and also Ames et al. (2017); Xu et al. (2017b) for some contemporary applications). The min-norm control paradigm dictates controlling the system at every x using an input u that minimizes $u^\mathsf{T}u$ while satisfying a constraint of the form $\nabla V(x)(f(x)+g(x)u) \leq -W(x)$, where $W(x)$ is a positive definite function. It is clear that $-W(x)$ essentially determines, at every x, the rate of convergence of the system to the origin, by enforcing an upper bound on the rate of decrease of the CLF along the trajectories of the closed loop system system. A question that arises naturally at this point and in agreement with the theme of this dissertation is from where in the state space can a system be provingly stabilized to the origin under the min-norm control law, when the control input u is subject to constraints (say, of a polytopic form, that is, $u \in \mathbb{U} := \{u \in \mathbb{R}^M : Au \preceq b\}$).

Using the methods of Section 3 to estimate the stabilization set for solutions based on the min-norm control paradigm and, say, the quadratic CLF $V(x)$, would seemingly yield the same result, with some significant caveats, though, as explained next. The function $W : \mathbb{R}^n \to \mathbb{R}_{\geq 0})$ is used in our formulation to merely write the typical CLF asymptotic stabilization constraint (that is, $\inf_{u \in \mathbb{U}} \psi(x, u) < 0$, for $x \neq 0$)

as a non-strict inequality as in (3.2), and can be taken to grow rather slowly with $\|x\|$, for instance, $W(x) = cx^{\mathsf{T}}x$ or $W(x) = cV(x)$, with $0 < c \ll 1$. This has no practical effect on the stabilization performance of the closed loop system in terms of the actual or desired convergence rate, as the latter enters the formulation of the proposed control law through the function $\alpha : \mathbb{R}^n \to \mathbb{R}_{\leq 0}$ appearing in the objective (4.1) of the QP, and not as a hard constraint. One can choose $\alpha(x)$ with no effect on the size of the stabilization set for the system; if the input constraints or the dynamics prevent the system from attaining the desired performance indicated at every x by $\alpha(x)$, then the proposed control law will still render the system asymptotically stable while attempting, at every x, to minimize the performance gap. In min-norm control formulations, stabilization performance enters the problem as a hard constraint and if at some $x \in \mathbb{R}^n$ the convergence rate prescribed by $-W(x)$ is not attainable under the input constraints $u \in \mathbb{U}$, that point x will not be part of the stabilization set.

The above lead us to the following observations:

- The stabilization set for a given system, $\mathbb{U}$, CLF $V(x)$ and $W(x)$ may be the same sublevel set of the CLF for both the herein proposed control law and the min-norm paradigm, but in the latter case the convergence can be significantly slower. The proposed methodology can use more control authority as it becomes available, to match an originally unattainable convergence rate prescribed by $-W(x)$.

- Conversely, tuning a min-norm control law for faster convergence could significantly shrink the stabilization set in order to satisfy, everywhere in the latter set, the stricter stabilization constraint with the available control authority.

Figures 4.2 and 4.3 are used to illustrate these points on the present example.

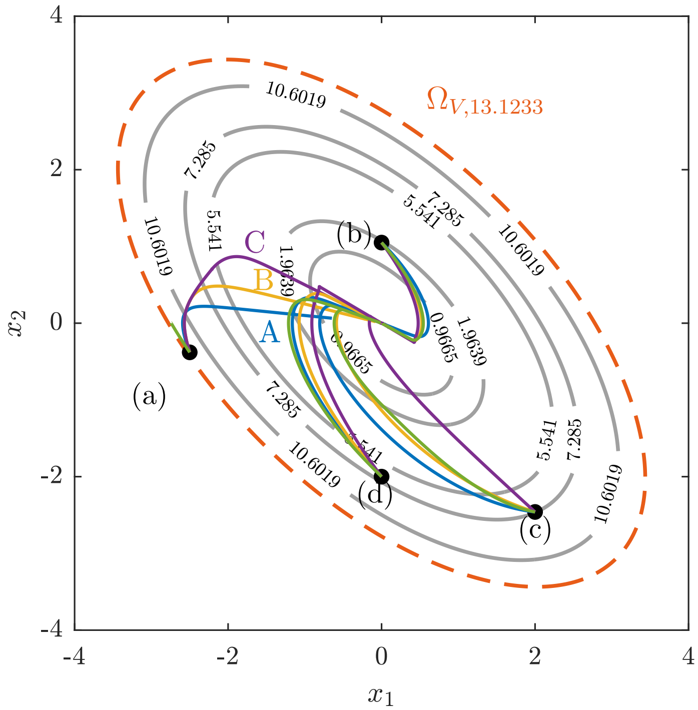
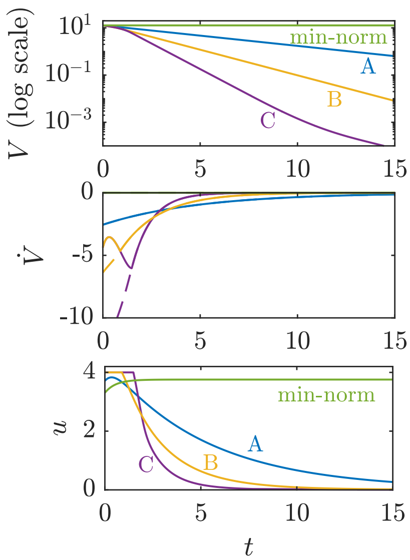


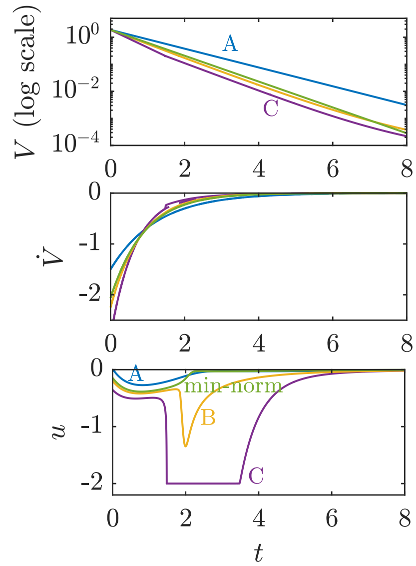
Figure 4.2: Stabilization sets for the system of Example 3 and the quadratic CLF V, for different $W(x) = cV(x)$, where $c \in \{10^{-6}, 0.3, 0.8, 1.1, 1.2, 1.3\}$. The outermost sublevel set $\Omega_{V,13.1233}$ corresponds to $c = 10^{-6}$. The feasibility and performance of the proposed solution are essentially decoupled and the control law can asymptotically stabilize the system from the maximal stabilization set, even if the convergence rate is initially rather slow.

4.3 Summary

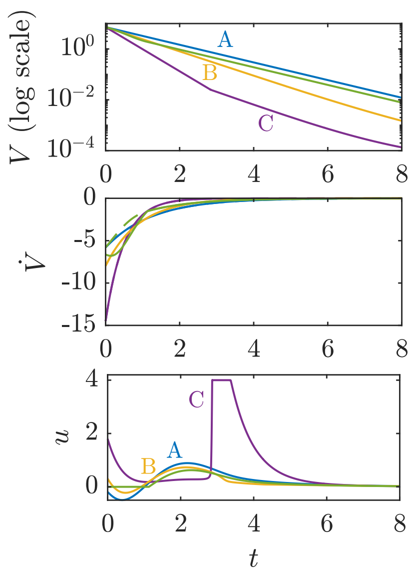
We presented a control law based on the minimizer of a QP that achieves the asymptotic stabilization objectives under input constraints and does not lose its feasibility along the closed loop trajectories of the system. An additional attractive feature of the proposed control law, in line with QP-based control laws presented before in the literature, are its low computational requirements which enable the embedded implementation of the proposed control solution.



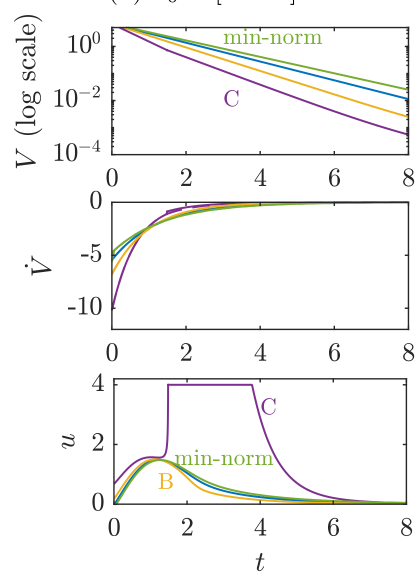
(a) $x_0 = [-2.5\ \ -0.38]^{\mathsf{T}}$



(b) $x_0 = [0\ 1.05]^{\mathsf{T}}$



(c) $x_0 = [2\ \ -2.46]^{\mathsf{T}}$



(d) $x_0 = [0\ \ -2]^{\mathsf{T}}$

Figure 4.3: Stabilization from 4 initial conditions (a), (b), (c), (d), under three cases (A,B,C) of the the proposed control law with the performance parameter $\alpha_{\square}(x) = -c_{\square}V(x)$, and the min-norm control law with the performance bound $W_{\mathrm{mn}}(x)$, for the system of Example 3 and the quadratic CLF V. The performance parameters per initial point are as follows $\{$(a): $c_{\mathrm{A}} = 0.2$, $c_{\mathrm{B}} = 0.5$, $c_{\mathrm{C}} = 1$, $W_{\mathrm{mn}}(x) = 10^{-6}V(x)$; (b): $c_{\mathrm{A}} = 0.8$, $c_{\mathrm{B}} = 1.2$, $c_{\mathrm{C}} = 1.5$, $W_{\mathrm{mn}}(x) = 1.1V(x)$; (c): $c_{\mathrm{A}} = 0.8$, $c_{\mathrm{B}} = 1.1$, $c_{\mathrm{C}} = 2$, $W_{\mathrm{mn}}(x) = 0.8V(x)$; (d): $c_{\mathrm{A}} = 0.8$, $c_{\mathrm{B}} = 1$, $c_{\mathrm{C}} = 1.5$, $W_{\mathrm{mn}}(x) = 0.7V(x)$.
All phase-plane trajectories are illustrated in Figure 4.2. The dashed lines, wherever visible in the $\dot{V}$ figures, correspond to the desired $\dot{V}$ for the proposed solution (that is, $\alpha(x)$) or the $\dot{V}$ bound for the min-norm control paradigm (that is, $W(x)$).

Chapter 5

Robust control of systems subject to bounded uncertainty

We now extend the methods of Chapters 3 and 4 to address the robust stabilization problem for uncertain polynomial systems.

5.1 System description

Let us consider uncertain nonlinear control systems with dynamics of the form

$$\dot{x} = f(x) + g_1(x)u + g_2(x)w, \qquad x(0) = x_0, \tag{5.1}$$

where $x \in \mathbb{R}^n$ is the state vector at time $t \geq 0$ with initial value $x_0 \in \mathbb{R}^n$, and $u \in \mathbb{U}$ is the control input. It is assumed that $u \in \mathbb{U} \subseteq \mathbb{R}^m$, for all $t \geq 0$, where $\mathbb{U}$ is a convex and compact polytope defined by the convex hull of its vertices $\{v_1, \ldots, v_q\} \in \mathbb{U}$, or, equivalently, by the intersection of halfspaces described by $Au \preceq b$ for appropriate $A \in \mathbb{R}^{p \times m}$, $b \in \mathbb{R}^p$. In addition, we assume that $0 \in \mathrm{Int}(\mathbb{U})$. The vector $w \in \mathbb{W} \subseteq \mathbb{R}^r$ is an unknown disturbance signal, which attains values in the convex and compact polytope $\mathbb{W}$ defined also by the convex hull of its vertices $\{z_1, \ldots, z_p\} \in \mathbb{W}$. One can use w to account for both exogenous disturbances and modeling uncertainty in the dynamics. For notational convenience, we consider the index sets $\mathcal{Q}_{\mathbb{U}} = \{1, \ldots, q\} \subset \mathbb{Z}_{>0}$ and $\mathcal{Q}_{\mathbb{W}} = \{1, \ldots, p\} \subset \mathbb{Z}_{>0}$, corresponding to the vertices $\{v_1, \ldots, v_q\}$ and $\{z_1, \ldots, z_p\}$ of $\mathbb{U}$ and $\mathbb{W}$, respectively. It is assumed that $f : \mathbb{R}^n \to \mathbb{R}^n$, $g_1 : \mathbb{R}^n \to \mathbb{R}^{n \times m}$ and $g_2 : \mathbb{R}^n \to \mathbb{R}^{n \times r}$ are known polynomial functions of x, with $f(0) = 0$.

We will consider control inputs that are piecewise continuous functions of time t. Under such a control input history $u(t)$ and a particular disturbance history $w(t)$, for $t \in [0, \tau)$, let $\phi(t; x_0, u, w)$ denote the solution[1] of (5.1) in the same time interval. Given the possible discontinuities in the right hand side of (5.1), the solution ϕ is regarded to exist in a non-classical sense (Bacciotti and Rosier, 2005; Clarke, Ledyaev, and Stern, 1998; Clarke, Ledyaev, Sontag, and Subbotin, 1997). The reader is referred to the relevant literature for more details and, in particular, to the fast-sampling-based interpretation of solutions to systems with discontinuous dynamics given by Clarke et al. (1997). We choose the latter interpretation because it is conceptually compatible with cases where the control law is implemented through the online solution of an optimization problem. In a similar vein to our work on nominal systems (that is, systems without uncertainty) in Chapters 3 and 4, we focus on developing ways to study from which states is a Lyapunov-based control law guaranteed to provide an applicable notion of robust stabilization and how to implement such a robust, input constrained control law through a computationally lightweight Quadratic Program.

5.2 Robust stabilization under disturbances and input constraints

For notational convenience and similarly to Chapter 3, we consider the set $\mathcal{V}$ of positive definite, radially unbounded polynomial functions $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$. Also, we use $\Omega_{V,\gamma} := \{x \in \mathbb{R}^n : V(x) \leq \gamma\}$ to refer to the γ-sublevel set of $V \in \mathcal{V}$.

Definition 5.2.1. A function $V \in \mathcal{V}$ is a Robust Control Lyapunov Function (RCLF) for system (5.1) if there exists a function $W \in \mathcal{V}$ and a set $\mathbb{X}_r \subseteq \mathbb{R}^n$, with $0 \in \mathrm{Int}(\mathbb{X}_r)$,

[1]Observe that here, in contrast to the nominal case of Chapter 3, the notation for the solution reflects the dependence of the latter on the time history of the disturbance w.

such that

$$\inf_{u\in\mathbb{U}}\sup_{w\in\mathbb{W}}\psi(x,u,w)\leq -W(x) \tag{5.2}$$

for all $x\in\mathbb{X}_r$, where $\psi:\mathbb{R}^n\times\mathbb{U}\times\mathbb{W}\to\mathbb{R}$ with $\psi(x,u,w):=\nabla V(x)(f(x)+g_1(x)u+g_2(x)w)$.

In the relevant literature (Bacciotti and Rosier, 2005; Clarke et al., 1998, 1997), such V and W are often called a *Control Lyapunov pair*. At this point, we emphasize that condition (5.2) does not suffice, in general, to guarantee stabilizability when the input u is subject to constraints. In fact, let us consider a trajectory starting from a point x_0 where (5.2) holds that is driven by a control law that guarantees $\psi(x,u,w)\leq -W(x)$ for all x along $\phi(t;x_0,u,w)$ and all $w\in\mathbb{W}$, where $t\in[0,t^*)$. There is nothing that prevents such a trajectory to escape $\mathbb{X}_r$ and enter the set $\mathbb{X}^{\mathsf{c}}=\mathbb{R}^n\setminus\mathbb{X}_r$ where (5.2) does not hold. At such a point $x^*\in\mathbb{X}_r^{\mathsf{c}}$ and depending on the particular action of the unknown disturbance w, the attainable rate of decrease of the RCLF V will be less than what prescribed by $W(x^*)$, while the value of V can even begin to increase, subsequently, as a function of time. The fate of such a trajectory cannot be easily determined, nor is it clear how to choose control inputs when the system's state is in $\mathbb{X}_r^{\mathsf{c}}$. Note that a different RCLF V would result in a different geometry for the problem in terms of the corresponding set $\mathbb{X}_r$, therefore, Lyapunov stabilization with *some* other RCLF cannot be definitively precluded. Focusing on our problem with the given RCLF, it is possible that the system returns to $\mathbb{X}_r$ under some particular control input and a favorable action of w. Alternatively, the system may remain indefinitely in $\mathbb{X}_r^{\mathsf{c}}$ or, even worse, escape to infinity in finite time. The various possibilities are conceptually illustrated in Figure 1.2, for the trajectory emanating from the point x_D. To provably achieve robust stabilization under input constraints, we proceed to warrant a sense of invariance within a subset of the set $\mathbb{X}_r$, given by

$$\mathbb{X}_r := \left\{ x \in \mathbb{R}^n : \inf_{u \in \mathbb{U}} \sup_{w \in \mathbb{W}} \psi(x, u, w) \leq -W(x) \right\}, \tag{5.3}$$

and where a control law based on the particular RCLF is well-defined and effective for any possible action of the disturbance w.

Next, we introduce the robust stabilization notion that we pursue in the present chapter.

Definition 5.2.2. The uncertain system (5.1) is called robustly practically stabilizable under input constraints, if for any $\epsilon > 0$ there exists a control law $u_{rc}(\cdot; \epsilon) : \mathcal{D}_{rc} \to \mathbb{U}$ such that the closed loop trajectories satisfy $\limsup_{t \to \infty} \phi(t; x_0, u_{rc}(\cdot; \epsilon), w) \leq \epsilon$, for and any $x_0 \in \mathcal{D}_{rc}$. The resulting closed loop system is called robustly practically stable, whereas we refer to the set $\mathcal{D}_{rc}$ as the robust stabilization set for (5.1).

The implication of Definition 5.2.2 is that such a control law can steer the system to an $\epsilon-$neighborhood of the origin, for any $\epsilon > 0$, starting at any $x_0 \in \mathcal{D}_{rc}$ and regardless of action of the disturbance w along the ensuing trajectory.

Lemma 5.2.1. *Let $V \in \mathcal{V}$ be an RCLF for (5.1) and $\epsilon > 0$ be given. If $\gamma > 0$ is such that $\Omega_{V,\gamma} \subseteq \mathbb{X}_r$, then the system is robustly practically stabilizable under input constraints, in the sense of Definition 5.2.2, with $\mathcal{D}_{rc} = \Omega_{V,\gamma}$.*

Proof. For any $x \in \Omega_{V,\gamma}$, consider the subset of the input value set $\mathbb{U}$ given by

$$\mathfrak{U}_r(x) := \left\{ u \in \mathbb{U} : \sup_{w \in \mathbb{W}} \psi(x, u, w) \leq -W(x) \right\} \tag{5.4}$$

and notice that it is nonempty, by the definition of the set $\mathbb{X}_r$ and given that $\Omega_{V,\gamma} \subseteq \mathbb{X}_r$. Consequently, no matter what the particular value of the disturbance $w \in \mathbb{W}$ may be, there exists $u^* \in \mathbb{U}$ such that $\psi(x, u^*, w) \leq -W(x)$, for all $(x, w) \in \Omega_{V,\gamma} \times \mathbb{W}$. Take $c \in$

$\mathbb{R}_{>0}$ such that $\Omega_{V,c} \subseteq \overline{\mathcal{B}}_\epsilon^n$ and consider a control law $u_{rc}(x;\epsilon)$ attaining values in $\mathfrak{U}_r(x)$, for all $x \in \Omega_{V,\gamma} \setminus \Omega_{V,c}$ and $u_{rc}(x;\epsilon) = 0$, otherwise. Since $\psi(x, u_{rc}(x;\epsilon), w) \leq -W(x)$ holds for all $(x, w) \in \Omega_{V,\gamma} \times \mathbb{W}$, it also holds along the trajectory $\phi(t; x_0, u_{rc})\cdot;\epsilon), w)$ emanating from any $x_0 \in \Omega_{V,\gamma}$, no matter what the action of w along ϕ is. Therefore, by typical Lyapunov arguments, $\phi(t; x_0, u_{rc}, w) \in \Omega_{V,\gamma}$ holds for all $t \in [0, \infty)$, so the control law is well-defined. Consider the function $\rho : \mathbb{R}^n \times \mathcal{V} \times \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$, with $\rho(x; V, c) := V(x) - c$, if $x \notin \Omega_{V,c}$, and $\rho(x; V, c) = 0$, otherwise. For any $x_0 \in \Omega_{V,\gamma} \setminus \Omega_{V,c}$, $\dot{\rho} = \psi(x, u_{rc}, w) \leq -W(x)$ holds for all $x \in \Omega_{V,\gamma} \setminus \Omega_{V,c}$ along the ensuing trajectory $\phi(t; x_0, u_{rc}, w)$, whereas $\min_x \rho(x; V, c) = 0$. Therefore, $\rho(\phi(t; x_0, u_{rc}, w); V, c) \to 0$ as $t \to \infty$ which concludes the proof. $\square$

Note that the function $\rho(x; V, c)$ can be regarded as a measure of the generalized distance of the point x from the sublevel set $\Omega_{V,c}$ of the RCLF V. It is easy to see that the results of Lemma 5.2.1 ensure that, regardless of the action of the disturbance w, the value of the RCLF V will decrease monotonically along the controlled trajectories, until the latter are ultimately confined in the sublevel set $\Omega_{V,c}$. Also, the controlled trajectories will never escape the sublevel set $\Omega_{V,\gamma} \subseteq \mathbb{X}_r$. This situation is conceptually illustrated in Fig. 1.2. It is noteworthy that the presence of uncertainty in the dynamics complicates the problem significantly. The geometry of the problem can be different compared to the nominal disturbance-free case which we studied in Chapter 3, in terms of the (simpler) set $\mathbb{X} = \{x \in \mathbb{R}^n : \inf_{u \in \mathbb{U}} \psi(x, u, 0) \leq -W(x)\}$ inside which a CLF-based input constrained control law is guaranteed to be effective and, consequently, in terms of the positive invariant (non-robust) stabilization set $\mathcal{D}_c$ corresponding to a CLF sublevel set contained in $\mathbb{X}$.

In the subsequent sections, we simplify the notation for the control laws of interest by dropping the ϵ argument. Of course, the latter is integral to the underlying stabilization notion as will be seen again in Section 5.4 and the numerical examples that follow in Section 5.5.

5.3 RCLF analysis with SOS

In this section, we develop two alternative methods that can be used to construct sufficient conditions for the set containment $\Omega_{V,\gamma} \subseteq \mathbb{X}_r$ to hold that are numerically verifiable. Both methods reduce the problem to the solution of convex feasibility problems - nominally Semidefinite Programs (SDPs) with Linear Matrix Inequality (LMI) constraints, which result from the parsing of sum-of-squares (SOS) constraints.

5.3.1 Using the shape of the input value set to induce a partition of the state space

At this point, we recall a fundamental result from linear programming.

Lemma 5.3.1. *Let $\mathbb{P} \subseteq \mathbb{R}^n$ be a convex, compact polytope. The function $h(y) := c^\mathsf{T} y + d$, where $c \in \mathbb{R}^n$, $d \in \mathbb{R}$ are constants and $y \in \mathbb{P}$, attains its minimum and maximum values at vertices of $\mathbb{P}$.*

Proof. Neglecting the constant term d, the case for the minimum corresponds to Proposition 2.4.2 in Bertsekas (2009). The same is true for the case of the maximum, by negating c and applying the same result. $\square$

It is possible to use Lemma 5.3.1 to the end of obtaining a more convenient parameterization of the set $\mathbb{X}_r$.

Lemma 5.3.2. *Let $\mathcal{C} \subseteq \mathbb{X}_r$ be a compact set. For any $x \in \mathcal{C}$,*

$$\inf_{u \in \mathbb{U}} \sup_{w \in \mathbb{W}} \psi(x, u, w) \leq -W(x)$$

is equivalent to

$$\min_{i\in\mathcal{Q}_{\mathbb{U}}}\max_{k\in\mathcal{Q}_{\mathbb{W}}}\psi(x,v_i,z_k)\leq -W(x).$$

Proof. Recall that v_i, z_k, where $i\in\mathcal{Q}_{\mathbb{U}}$, $k\in\mathcal{Q}_{\mathbb{W}}$, are the vertices of the input and disturbance value sets $\mathbb{U}$ and $\mathbb{W}$, respectively. Also, note that the quantity $\vartheta(x,u,w):=\psi(x,u,w)-W(x)$ is affine in u and w for a fixed x. The equivalence follows by applying Lemma 5.3.1 to $\vartheta(x,u,w)$ twice, that is, while regarding the latter to be a function of u with parameter w and vice-versa. Note that the infimum and the supremum are attained since $\vartheta(x,u,w)$ is jointly continuous in (x,u,w) and its arguments take values over compact sets. $\square$

Let $\mathcal{D}$ be any compact subset of $\mathbb{R}^n$; from Lemma 5.3.2, it readily follows that

$$\begin{aligned}\mathbb{X}_r\cap\mathcal{D}&=\left\{x\in\mathcal{D}:\min_{i\in\mathcal{Q}_{\mathbb{U}}}\max_{k\in\mathcal{Q}_{\mathbb{W}}}\psi(x,v_i,z_k)\leq -W(x)\right\}\\&=\bigcap_{k\in\mathcal{Q}_{\mathbb{W}}}\left\{x\in\mathcal{D}:\min_{i\in\mathcal{Q}_{\mathbb{U}}}\psi(x,v_i,z_k)\leq -W(x)\right\}.\end{aligned}\tag{5.5}$$

Expression (5.5) helps us in checking whether $x\in\mathbb{X}_r$ holds for a fixed point $x\in\mathcal{D}$, as it reduces the problem to studying the effect of the vertices of the respective value sets on $(u,w)\mapsto\psi(x,u,w)$. Note that we can account for the effect of the disturbance by requiring that $\min_i\psi(x,v_i,z_k)\leq -W(x)$ holds true for all $k\in\mathcal{Q}_{\mathbb{W}}$ for the particular x. Such a logical conjunction appears, at first sight, to be compatible with the S-procedure which we briefly described before. However, taking the minimum of $\psi(x,v_i,z_k)$ over all input value set vertices v_i cannot be readily accommodated in the same context and hinders the derivation of sufficient conditions for $\Omega_{V,\gamma}\subseteq\mathbb{X}_r$ to hold in terms of semialgebraic set containments.

The aforedescribed hardship is reminiscent of that encountered in the nominal case, that is, without uncertainty, in Chapter 3. We are first going to circumvent the problem by partitioning the state space into non-overlapping cells which are associated with each vertex v_i of the input value set $\mathbb{U}$. Essentially, we will be using the partition introduced in Section 3.4, by appropriately "robustifying" the respective results to make them amendable to the present uncertain context.

The pairing between cells of the partition and vertices of $\mathbb{U}$ is determined based on which vertex v_i tends to make $\psi(x, v_i, 0)$ decrease more for x near the origin $x = 0$. In the i^{th} cell of the partition, we hold u constant and equal to the corresponding vertex, and we check whether $\psi(x, v_i, z_k) \leq -W(x)$ holds, for all $k \in \mathcal{Q}_{\mathbb{W}}$ and for all x in the intersection of the cell with the sublevel set of interest $\Omega_{V,\gamma}$. The latter can be readily formulated as a semialgebraic set containment problem which one can solve with typical SOS techniques. If all the corresponding SOS problems are feasible for all $i \in \mathcal{Q}_{\mathbb{U}}$, then the set containment $\Omega_{V,\gamma} \subseteq \mathbb{X}_r$ holds.

Proposition 5.3.3. *Let the sets $\{\mathbb{X}_{P_i}, i \in \mathcal{Q}_{\mathbb{U}}\}$ be defined as in Proposition 3.4.3. Suppose that for each $i \in \mathcal{Q}_{\mathbb{U}}$,*

$$\mathbb{X}_{P_i} \cap \Omega_{V,\gamma} \subseteq \{x \in \mathbb{R}^n : \psi(x, v_i, z_k) \leq -W(x)\} \tag{5.6}$$

hold for all indices $k \in \mathcal{Q}_{\mathbb{W}}$. Then, $\Omega_{V,\gamma} \subseteq \mathbb{X}$.

Proof. The fact that for a particular $i \in \mathcal{Q}_{\mathbb{U}}$, (5.6) holds for all indices $k \in \mathcal{Q}_{\mathbb{W}}$ implies that it also holds for $k = \arg\max_{k \in \mathcal{Q}_{\mathbb{W}}} \psi(x, v_i, z_k)$; note that ψ is jointly continuous and defined over compact sets, so the supremum is attained. As a result,

$$\mathbb{X}_{P_i} \cap \Omega_{V,\gamma} \subseteq \left\{x \in \mathbb{R}^n : \sup_{w \in \mathbb{W}} \psi(x, v_i, w) \leq -W(x)\right\} \tag{5.7}$$

holds for all $i \in \mathcal{Q}_{\mathbb{U}}$. By definition of the set $\mathbb{X}_r$, for each $i \in \mathcal{Q}_{\mathbb{U}}$ we have

$$\left\{x \in \mathbb{R}^n : \sup_{w \in \mathbb{W}} \psi(x, v_i, w) \leq -W(x)\right\} \subseteq \mathbb{X}_r. \tag{5.8}$$

Using (5.7) and (5.8), the fact that (5.6) holds for all $i \in \mathcal{Q}_{\mathbb{U}}$ implies

$$\cup_{i \in \mathcal{Q}_{\mathbb{U}}} \{\mathbb{X}_{P_i} \cap \Omega_{V,\gamma}\} \subseteq \mathbb{X}_r. \tag{5.9}$$

Since the collection of sets $\{\mathbb{X}_{P_i}, i \in \mathcal{Q}_{\mathbb{U}}\}$ forms a partition of $\mathbb{R}^n$ from Proposition 3.4.3, we have $\cup_{i \in \mathcal{Q}_{\mathbb{U}}} \mathbb{X}_{P_i} = \mathbb{R}^n$. Note that, in addition, $\Omega_{V,\gamma} \subseteq \mathbb{R}^n$. The two latter facts imply $\cup_{i \in \mathcal{Q}_{\mathbb{U}}} \{\mathbb{X}_{P_i} \cap \Omega_{V,\gamma}\} = \Omega_{V,\gamma}$, which, in conjunction with (5.9), proves the Proposition. $\square$

We can now invoke the S-procedure in order to formulate sufficient conditions for Proposition 5.3.3 to hold that are practically decidable. In particular, assume there exist polynomials $c^{[i,k]}(x)$ and $s^{[i,j,k]}(x)$, for all $(i,j,k) \in \mathcal{Q}_{\mathbb{U}} \times \mathcal{Q}_{\mathbb{U}} \times \mathcal{Q}_{\mathbb{W}}$, such that

$$\begin{aligned} -\psi(x, v_i, z_k) - W(x) - c^{[i,k]}(x)\,(\gamma - V(x)) \\ - \sum_{j=1, j \neq i}^{q} s^{[i,j,k]}(x)(\eta_j - \eta_i)^{\mathsf{T}} x \in \Sigma[x], \quad \forall (i,k) \in \mathcal{Q}_{\mathbb{U}} \times \mathcal{Q}_{\mathbb{W}}, \end{aligned} \tag{5.10a}$$

$$c^{[i,k]}(x) \in \Sigma[x], \quad \forall (i,k) \in \mathcal{Q}_{\mathbb{U}} \times \mathcal{Q}_{\mathbb{W}}, \tag{5.10b}$$

$$s^{[i,j,k]}(x) \in \Sigma[x], \quad \forall (i,j,k) \in \mathcal{Q}_{\mathbb{U}} \times \mathcal{Q}_{\mathbb{U}} \times \mathcal{Q}_{\mathbb{W}}. \tag{5.10c}$$

When $\Sigma[x]$ is parameterized as SOS, (5.10a)-(5.10c) can be expressed as LMIs in terms of appropriate auxiliary variables, the feasibility of which can be determined in polynomial time by solving an SDP. A feasible solution implies that (5.6) holds for all $k \in \mathcal{Q}_{\mathbb{W}}$, and, by virtue of Proposition 5.3.3, $\Omega_{V,\gamma} \subseteq \mathbb{X}_r$ also holds.

5.3.2 Robust stabilizability via polynomial piecewise convex combinations

Our next approach towards showing $\Omega_{V,\gamma} \subseteq \mathbb{X}_r$ and, thus, guaranteeing that the system is robustly practically stabilizable in the set $\Omega_{V,\gamma}$, is based on parameterizing the action of the admissible, input constrained control laws with appropriate polynomial functions.

Proposition 5.3.4. *Let $\mathcal{I} \subset \mathbb{Z}_{>0}$ and consider the collection of semialgebraic sets $\{\mathcal{P}_\ell \subseteq \mathbb{R}^n, \ell \in \mathcal{I}\}$, where $\mathcal{P}_\ell := \left\{x \in \mathbb{R}^n : \pi_j^{[\ell]}(x) \geq 0\right\}$ and $\pi_j^{[\ell]} : \mathbb{R}^n \to \mathbb{R}$ are polynomial functions, forming a partition of $\mathbb{R}^n$. Also, for each $\ell \in \mathcal{I}$, let $\xi_i^{[\ell]} : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, $i \in \mathcal{Q}_{\mathbb{U}}$, be polynomial functions such that $0 < \sum_{i \in \mathcal{Q}_{\mathbb{U}}} \xi_i^{[\ell]}(x) \leq 1$ for all $x \in \mathbb{R}^n$, and take*

$$\tilde{u}_{rc}^{[\ell]}(x) := \sum_{i \in \mathcal{Q}_{\mathbb{U}}} \xi_i^{[\ell]}(x) v_i.$$

Then, if

$$\Omega_{V,\gamma} \cap \mathcal{P}_\ell \subseteq \left\{x \in \mathbb{R}^n : \psi\left(x, \tilde{u}_{rc}^{[\ell]}(x), z_k\right) \leq -W(x)\right\}, \quad \forall (k, \ell) \in \mathcal{Q}_{\mathbb{W}} \times \mathcal{I}, \tag{5.11}$$

the set containment $\Omega_{V,\gamma} \subseteq \mathbb{X}_r$ holds.

Proof. As $\{\mathcal{P}_\ell, \ell \in \mathcal{I}\}$ form a partition of $\mathbb{R}^n$, it is easy to see that

$$\cup_{\ell \in \mathcal{I}} \{\Omega_{V,\gamma} \cap \mathcal{P}_\ell\} = \Omega_{V,\gamma}. \tag{5.12}$$

Also,

$$\begin{aligned} &\cap_{k \in \mathcal{Q}_{\mathbb{W}}} \left\{x \in \mathbb{R}^n : \psi\left(x, \tilde{u}_{rc}^{[\ell]}(x), z_k\right) \leq -W(x)\right\} \\ &\qquad = \left\{x \in \mathbb{R}^n : \sup_{w \in \mathbb{W}} \psi\left(x, \tilde{u}_{rc}^{[\ell]}(x), w\right) \leq -W(x)\right\}. \end{aligned} \tag{5.13}$$

One can write $u_{rc}^{[\ell]}$ as

$$u_{rc}^{[\ell]}(x) = \sum_{i \in \mathcal{Q}_{\mathbb{U}}} \frac{\xi_i^{[\ell]}(x)}{\sum_{j \in \mathcal{Q}_{\mathbb{U}}} \xi_j^{[\ell]}(x)} \left(v_i \sum_{j \in \mathcal{Q}_{\mathbb{U}}} \xi_j^{[\ell]}(x)\right). \tag{5.14}$$

For any fixed $x \in \mathbb{R}^n$, note that $\xi_i^{[\ell]}(x) / \sum_{j \in \mathcal{Q}_{\mathbb{U}}} \xi_j^{[\ell]}(x) \geq 0, \forall i \in \mathcal{Q}_{\mathbb{U}}$. Moreover, recalling that $\sum_{i\ \in \mathcal{Q}_{\mathbb{U}}} \xi_i^{[\ell]}(x) \leq 1$, $0 \in \mathrm{Int}\{\mathbb{U}\}$ and that $\mathbb{U}$ is defined by the convex hull of its vertices $\{v_i, i \in \mathcal{Q}_{\mathbb{U}}\}$, observe that $v_i \sum_{j \in \mathcal{Q}_{\mathbb{U}}} \xi_j^{[\ell]}(x) \in \mathbb{U}$ holds, therefore, the polytope defined by the convex hull of the points $\left\{v_i \sum_{j \in \mathcal{Q}_{\mathbb{U}}} \xi_j^{[\ell]}(x), i \in \mathcal{Q}_{\mathbb{U}}\right\}$ is contained within $\mathbb{U}$ and the right hand side of (5.14) corresponds to a convex combination of the latter collection of points. Consequently, for each $\ell \in \mathcal{I}$ and any $x \in \mathbb{R}^n$, we have $x \mapsto u_{rc}^{[\ell]}(x) \in \mathbb{U}$. The latter fact, in conjunction with the definition of the set $\mathbb{X}_r$, implies

$$\left\{ x \in \mathbb{R}^n : \sup_{w \in \mathbb{W}} \psi \left(x, \tilde{u}_{rc}^{[\ell]}(x), w \right) \leq -W(x) \right\} \subseteq \mathbb{X}_r. \tag{5.15}$$

Using (5.12), (5.13) and (5.15), the set containment (5.11) implies $\Omega_{V,\gamma} \subseteq \mathbb{X}_r$, concluding the proof. □

With the help of the S-procedure and similarly to the case of Section 5.3.1, we can develop, in terms of semialgebraic set containments, sufficient conditions for the results of Proposition 5.3.4 to hold. Such semialgebraic set containments are, in turn, verifiable numerically by solving SDPs (or another class of convex problems, depending on the parameterization of $\Sigma[x]$). In the present case, we are looking for polynomials $\xi_i^{[\ell]}(x)$, $s_j^{[\ell]}(x)$ and $c^{[\ell]}(x)$, for all $(\ell, i, j, k) \in \mathcal{I} \times \mathcal{Q}_{\mathbb{U}} \times \mathcal{Q}_{\mathbb{U}} \times \mathcal{Q}_{\mathbb{W}}$, such that

$$\begin{aligned}
-\psi \left(x, \sum_{i \in \mathcal{Q}_{\mathbb{U}}} \xi_i^{[\ell]}(x) v_i, z_k \right) - W(x) \qquad & \\
-c^{[\ell]}(x)(\gamma - V(x)) - \sum_{j \in \mathcal{J}(\ell)} s_j^{[\ell]}(x) \pi_j^{[\ell]}(x) \in \Sigma[x], &\quad \forall (\ell, k) \in \mathcal{I} \times \mathcal{Q}_{\mathbb{W}}, &\text{(5.16a)}\\
1 - \sum_{i \in \mathcal{Q}_{\mathbb{U}}} \xi_i^{[\ell]}(x) \in \Sigma[x], &\quad \forall \ell \in \mathcal{I}, &\text{(5.16b)}\\
\xi_i^{[\ell]}(x) \in \Sigma[x], &\quad \forall (i, \ell) \in \mathcal{Q}_{\mathbb{U}} \times \mathcal{I}, &\text{(5.16c)}\\
s_j^{[\ell]}(x) \in \Sigma[x], &\quad \forall (j, \ell) \in \mathcal{Q}_{\mathbb{U}} \times \mathcal{I}. &\text{(5.16d)}
\end{aligned}$$

The reader should note that the partition of the state space employed in Proposition 5.3.4 has a different purpose compared to the partition of the state space used in the method described in Section 5.3.1. Here, the partition allows us to search for suitable nonnegative polynomials $\xi_i^{[\ell]}(x)$ such that the previous results hold only within the respective cell with index $\ell \in \mathcal{I}$ and not throughout the entire $\mathbb{R}^n$. We use the latter fact to our advantage, in order to make the proposed method of the present section more flexible.

5.4 Solving the control law min-max problem with simple QPs

If $\Omega_{V,\gamma} \subseteq \mathbb{X}_r$ holds, any $x \in \Omega_{V,\gamma}$ can be mapped to the set $\mathfrak{U}_r(x) := \{u \in \mathbb{U} : \sup_{w\in\mathbb{W}} \psi(x,u,w) \leq -W(x)\}$, which is guaranteed to be nonempty by the definition of $\mathbb{X}_r$. In turn, any $u \in \mathfrak{U}_r(x)$ is guaranteed to render $\psi(x,u,w) \leq -W(x)$, no matter what the value of the disturbance w happens to be. We now address the question of how to systematically choose such a $u \in \mathfrak{U}_r(x)$ as a function of x, to ultimately achieve our control objectives.

We proceed to formulate a Quadratic Program (QP), the solution of which will be equal to the value of the sought-after robust control law at the particular x. The domain of the QP will be given by $\mathfrak{U}_r(x)$; note that the current state x appears as a parameter in the formulation of the QP, the decision variable of which is the control input u to be applied to the system. We begin by considering, in a similar vein to our nominal control law of Chapter 4, a negative definite function $\alpha : \mathbb{R}^n \to \mathbb{R}_{\leq 0}$ which is assumed to correspond to the desired stabilization performance at every x, in terms of the rate of decrease of the RCLF V along the trajectories of the system. Next, we consider a measure of the stabilization performance gap $\mathcal{H} : \mathbb{R}^n \times \mathbb{U} \times \mathbb{W} \to \mathbb{R}$ at every x and given the value of the exerted control input u and the (unknown) value of $w \in \mathbb{W}$ which happens to act on the system at that time, with

$$\mathcal{H}(x,u,w) := (\psi(x,u,w) - \alpha(x))^2.$$

If terms which do not contain u are dropped from $\mathcal{H}$, one obtains the performance index

$$\mathcal{J}(u; x, w) = u^{\mathsf{T}} Q(x) u + L(x, w) u,$$

where $Q : \mathbb{R}^n \to \mathbb{R}^{m \times m}$ and $L : \mathbb{R}^n \times \mathbb{W} \to \mathbb{R}^m$ are given, respectively, by

$$Q(x) := g_1^{\mathsf{T}}(x) \left(\nabla V(x)\right)^{\mathsf{T}} \nabla V(x) g_1(x) + \mu(x) I_m,$$
$$L(x, w) := 2 \left[\nabla V(x) \left(f(x) + g_2(x) w\right) - \alpha(x)\right] \nabla V(x) g_1(x),$$

and $\mu : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is used to shift the spectrum of $g_1^{\mathsf{T}}(x) \left(\nabla V(x)\right)^{\mathsf{T}} \nabla V(x) g_1(x)$, if necessary, and render the resulting matrix *strictly* positive definite by some prespecified margin, that is,

$$\mu(x) := \max\left(0, \varepsilon - \lambda_{\min}\left(g_1^{\mathsf{T}}(x) \left(\nabla V(x)\right)^{\mathsf{T}} \nabla V(x) g_1(x)\right)\right),$$

for some given small $\varepsilon > 0$. Observe that the unknown quantity w appears in the performance index $\mathcal{J}(u; x, w)$, as well as in what we could regard to be our stabilization constraint embedded in the definition of $\mathfrak{U}_r(x)$, that is, $\nabla V(x) g_1(x) u \leq -\nabla V(x) f(x) - \nabla V(x) g_2(x) w - W(x)$. This is hindering our efforts if we were to try and pose the control law design problem as a simple QP. Taking the uncertainty into explicit consideration is a nontrivial feat in robust optimization. Often times, the resulting problems called robust counterparts (Ben-Tal, Nemirovski, and Roos, 2002; Ben-Tal, Ghaoui, and Nemirovski, 2009) assume forms which are more complicated and difficult to solve online, such as Semidefinite Programs (Vandenberghe and Boyd, 1996). We continue by formulating a control law based on a robustly safe solution of such an uncertain QP, which reduces to the solution of a simple and standard QP without any uncertainty.

Proposition 5.4.1. *Let* $\mathbb{U} := \{u \in \mathbb{R}^m : Au \preceq b\}$, *for appropriate* $A \in \mathbb{R}^{p \times m}$ *and*

$b \in \mathbb{R}^p$. *For any* $x \in \Omega_{V,\gamma}$ *and* $w \in \mathbb{W}$, *let* $u^*(x,w)$ *be the solution to the QP problem*

$$\begin{aligned} &\underset{u}{min} && \mathcal{J}\left(u; x, w\right), \\ &\textit{subject to} && \nabla V(x) g_1(x) u \leq -\nabla V(x) \left(f(x) + g_2(x) w\right) - W(x), \\ &&& Au \preceq b. \end{aligned}$$

The mapping $u_{rc} : \Omega_{V,\gamma} \to \mathbb{U}$, *with*

$$u_{rc}(x) := \begin{cases} u^* \left(x, z_{\overline{k}(x)}\right) & \textit{if } V(x) \geq \xi, \\ 0 & \text{otherwise}, \end{cases} \tag{5.17}$$

for any $\xi \in (0, \gamma)$, *where* $\overline{k}(x) := \arg\max_{k \in \mathcal{Q}_{\mathbb{W}}} \nabla V(x) g_2(x) z_k$, *is a robustly practically stabilizing control law for the system (5.1), in the sense of Definition 5.2.2.*

Proof. Let $\epsilon > 0$. Then, take $\xi \in (0, \gamma)$ such that $\Omega_{V,\xi} \subseteq \overline{\mathcal{B}}^n_\epsilon$. The QP is feasible for any $x \in \Omega_{V,\gamma}$ and $w \in \mathbb{W}$, since $\Omega_{V,\gamma} \subseteq \mathbb{X}_r$ and the set $\mathfrak{U}_r(x)$, which is the domain of the QP, is guaranteed by virtue of Lemma 5.2.1 to be nonempty. Additionally, since $Q(x) = Q^\mathsf{T}(x)$ is positive definite for any $x \in \mathbb{R}^n$, the solution of the QP is unique, and thus, $u^*\big(x, z_{\overline{k}(x)}\big)$ is well-defined. The control law under consideration is rendering $\psi(x, u_{rc}, w) \leq -W(x)$ for any $x \in \Omega_{V,\gamma}$ and any $w \in \mathbb{W}$. For any $x_0 \in \Omega_{V,\gamma}$ and following Lemma 5.2.1, the trajectory $\phi(t; x_0, u_{rc}, w)$ will remain in $\Omega_{V,\gamma}$ for all $t \geq 0$, which guarantees that the QP will remain feasible, and ultimately tends to $\Omega_{V,\xi}$, that is, $\rho(\phi(t; x_0, u_{rc}, w); V, \xi) \to 0$ as $t \to \infty$, for $\rho(\cdot; \cdot, \cdot)$ as given in the Proof of Lemma 5.2.1. This completes the proof. □

Remark 4. As we saw in Chapter 4, it is a known fact that the minimizer of a QP, whose objective and constraint functions depend continuously on a parameter, is a continuous function of the latter parameter under mild conditions (Daniel, 1973). In the present case, the QP which we have associated with the proposed control law depends continuously on the state x, but it also depends on $\overline{k}(x)$ which is defined in

Proposition 5.4.1. Observe that $x \mapsto \overline{k}(x) \in \mathcal{Q}_{\mathbb{W}}$ attains values from a discrete set that depend on the part of the state space where the system happens to be. It is possible to consider a partition of the state space (different than the one discussed in Section 5.3) based on $\overline{k}(x)$, such that $\overline{k}(x)$ attains a new value as the system's trajectory is crossing from one cell of the partition to another. As long as the system's trajectory remains in one such cell of the partition, $\overline{k}(x)$ will be constant and the control law QP will depend continuously only on the current state. Therefore, one can regard the resulting control input to be a piecewise continuous function of time t.

Remark 5. In accordance with the notion of practical stabilization, the herein proposed robust stabilization method is guaranteed to bring the controlled trajectory of the system from any $x_0 \in \Omega_{V,\gamma}$ for $\gamma > 0$ such that $\Omega_{V,\gamma} \subseteq \mathbb{X}_r$, to the prescribed neighborhood of the origin corresponding to the RCLF sublevel set $\Omega_{V,\xi}$. The control law will prevent the trajectory from escaping into $\mathcal{D}_{rc} \setminus \Omega_{V,\xi}$. This action of the control law can cause rapid "activations" and "deactivations" of the control input, reminiscent of chattering, which is typically undesirable. One can avoid this by taking an inner and an outer target sublevel set, $\Omega_{V,\underline{\xi}}$ and $\Omega_{V,\overline{\xi}}$, with $\overline{\xi} > \underline{\xi}$, and consider a variation of the control law operating as follows. Initially, the system is steered to $\Omega_{V,\underline{\xi}}$. No control is applied next, unless the trajectory reaches the boundary of $\Omega_{V,\overline{\xi}}$. When the latter happens, the control law brings the system back to $\Omega_{V,\underline{\xi}}$ and the process is repeated accordingly. Considering, also in the spirit of Clarke et al. (1997), that the feedback control law calculations are performed and applied to the system at time instants t_j, where $j \in \mathbb{Z}_{\geq 0}$ and $(t_j - t_{j-1})$ for $j \geq 1$ are rather small, we introduce the recursively-defined, binary-valued function

$$C_j = C\left(x\left(t_j\right), C_{j-1}\right)$$

with $C_0 = 1$, where $C : \mathbb{R}^n \times \{0,1\} \to \{0,1\}$ is given by

$$C(x,\eta) := \begin{cases} 1 & \text{if } \; V(x) \geq \overline{\xi} \; \text{ or } \; \left(\underline{\xi} \leq V(x(x) < \overline{\xi} \; \text{ and } \; \eta = 1\right), \\ 0 & \text{otherwise.} \end{cases} \tag{5.18}$$

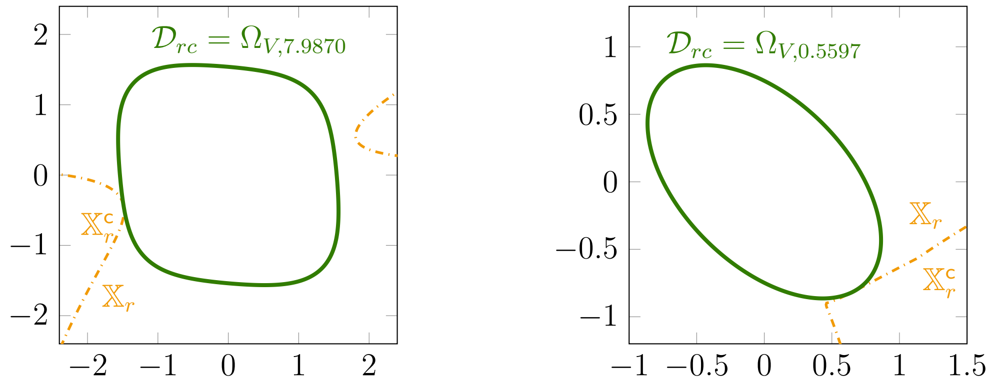
The control law defined in Proposition 5.4.1 assumes the form $u_{rc} : \Omega_{V,\gamma} \times \{0,1\} \to \mathbb{U}$, with

$$u_{rc}\left(x(t_j), C_j\right) := \begin{cases} u^*\left(x\left(t_j\right), z_{\overline{k}(x(t_j))}\right) & \text{if } C_j = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{5.19}$$

Remark 6. It is straightforward to see that the collection of polynomial functions $\left\{u_{rc}^{[\ell]}, \ell \in \mathcal{I}\right\}$, introduced in Section 5.3.2, can be used to assemble a robustly practically stabilizing control law for the system (that is, by taking $u_{rc}(x) = \tilde{u}_{rc}^{[\ell]}(x)$ for $\ell \in \mathcal{I}$ such that $x \in \mathcal{P}_\ell$. The QP-based control approach of the present section offers two advantages to using a control law based on $\left\{u_{rc}^{[\ell]}, \ell \in \mathcal{I}\right\}$. First, the QP-based control law can be used regardless of which method (that is, of either Section 5.3.1 or 5.3.2) has been used to show $\Omega_{V,\gamma} \subseteq \mathbb{X}_r$. Second, the QP-based control law is easily tunable online (via the function α), whereas performance considerations have not been factored into the derivations of Section 5.3.2, given that our focus there was on the controllability aspect of the input constrained robust control problem.

5.5 Numerical examples

Next, we illustrate the efficacy of the methods proposed in this chapter using three example problems. For each of these, we use the two methods presented in Section 5.3 to find an approximation of the the maximal γ such that $\Omega_{V,\gamma} \subseteq \mathbb{X}_r$, hereafter denoted by $\hat{\gamma}$, using a simple bisection scheme. For particular realizations of the unknown, bounded disturbance variable, we also propagate closed loop trajectories from x_0 in the respective robust stabilization set $\Omega_{V,\hat{\gamma}}$, driven by the proposed QP-based, min-max control law. The SOS constraints were handled with YALMIP (Löfberg, 2009), the resulting SDP problems were numerically solved primarily with MOSEK (MOSEK ApS, 2017) and also SeDuMi (Sturm, 1999), whereas the control law QPs were solved with qpOASES (Ferreau et al., 2014).



(a) Showing $\Omega_{V,\hat{\gamma}} \subseteq \mathbb{X}_r$ for the example of Section 5.5.1.

(b) Showing $\Omega_{V,\hat{\gamma}} \subseteq \mathbb{X}_r$ for the example of Section 5.5.2.

Figure 5.1: Illustrations of the set containment $\Omega_{V,\hat{\gamma}} \subseteq \mathbb{X}_r$ for the example problems of Sections 5.5.1 and 5.5.2, which is sufficient for robust practical stabilization under input constraints. The dashed-dotted line corresponds to the boundary of the set $\mathbb{X}_r^{\mathsf{c}}$. Observe that the containment happens to be tight for both cases, that is, slightly larger sublevel sets of the respective RCLF would violate $\Omega_{V,\hat{\gamma}} \subseteq \mathbb{X}_r$.

5.5.1 Uncertain system with finite escape time in the control-free case

We first consider the following uncertain nonlinear system with dynamics

$$\begin{aligned}\dot{x}_1 &= x_2,\\ \dot{x}_2 &= -x_1^2(0.6 + w_1) - x_2 + u + w_2,\end{aligned}$$

with $-0.01 \le w_1 \le 0.1$, $|w_2| \le 0.2$ and $-2 \le u \le 4$. The unknown disturbance w_1 can have a significant effect on the dynamics of the system, as the presence of the quadratic term results in trajectories with a finite escape time when no control is applied. We consider the RCLF $V(x) = x_1^4 + x_2^4 + x_1^2 + x_2^2 + x_1x_2$ and $W(x) = 10^{-6}V(x)$. Using the methods of Sections 5.3.1 and 5.3.2 within a bisection scheme in

$[10^{-3}, 10]$ to approximate $\hat{\gamma}$ yields 7.9868 and 7.9870, respectively, while searching for SOS polynomials appearing in the expressions (5.10a)-(5.10c) and (5.16a)-(5.16d) of degree no greater than 4. For the polynomial convex combination method of Section 5.3.2, we employed a partition of $\mathbb{R}^2$ into 8 cells, by equipartitioning each quadrant in 2 parts. We take the robust stabilization set for the system to be $\mathcal{D}_{rc} = \Omega_{V,\hat{\gamma}}$, for $\hat{\gamma} = 7.9870$. The set containment $\Omega_{V,\hat{\gamma}} \subseteq \mathbb{X}_r$ is illustrated in Figure 5.1a.

Starting from $x_0 = [-1 \quad -1]^\mathsf{T}$, we propagate a closed loop trajectory of the system under the proposed control law for $t \in [0, 15]$, with $\alpha(x) = -2.5V(x)$, $\varepsilon = 0.1$, $\underline{\xi} = 10^{-3}$, $\overline{\xi} = 5 \times 10^{-3}$. For our simulations, we take the disturbance to be time varying, attaining the values

$$w_1(t) = \begin{cases} 0.1 & \text{if } 0 \leq t \leq 3, \\ -0.01 \sin 10t & \text{otherwise,} \end{cases} \qquad w_2(t) = \begin{cases} 0.2 & \text{if } 0 \leq t \leq 4, \\ -0.2 & \text{if } 4 < t \leq 6, \\ -0.2\cos(t-6) & \text{if } t > 6. \end{cases}$$

The time histories of the control input u and the RCLF V are illustrated in Figure 5.2.

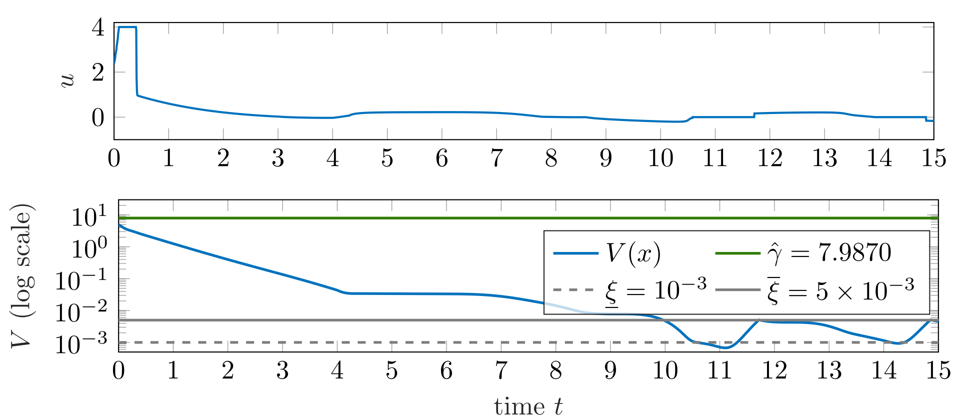


Figure 5.2: Numerical simulation results for the example of Section 5.5.1. Observe that the control input u initially saturates at the upper bound without loss of robust practical stability. The closed loop trajectory is ultimately confined in the sublevel set $\Omega_{V,\overline{\xi}}$, in accordance with the underlying robust practical stabilization notion. When no control is applied and the trajectory is about to exit $\Omega_{V,\overline{\xi}}$ near $t = 11.75$, the control law is engaged and brings the system back to $\Omega_{V,\underline{\xi}}$ at $t = 14.25$. The process is repeated accordingly.

5.5.2 Uncertain predator - prey system

Next, we consider the uncertain predator - prey system

$$\dot{x}_1 = -x_2 w_1 - x_1 x_2 + 0.1 x_1 w_2 + u + w_3,$$
$$\dot{x}_2 = x_1 w_4 + x_1 x_2,$$

with $-1.5 \leq u \leq 2.5$ and $w = [w_1\ w_2\ w_3\ w_4]^{\mathsf{T}} \in \mathbb{W} \subseteq \mathbb{R}^4$ such that $0.8 \leq w_1 \leq 1.1$, $0.9 \leq w_2 \leq 1.2$, $|w_3| \leq 0.1$ and $0.9 \leq w_4 \leq 1.1$. The nominal form of the control-free system, that is, with $w_2 = w_3 = 0$ and, say, $w_1 = w_4 = 1$, corresponds to the predator (x_2) - prey (x_1) dynamics where the equilibrium at the origin represents a coexistence condition between the two agents and is surrounded by periodic orbits. Of course, this may not necessarily be the case under the uncertainty and the control action introduced by u. One should notice here that the disturbance w_4 is unmatched, that is, it enters the system at a different equation than the control input u. We consider the RCLF $V(x) = x_1^2 + x_1 x_2 + x_2^2$ and $W(x) = 10^{-6} V(x)$. A simple bisection on γ in the interval $10^{-3}, 10]$ using either of the two methods to show $\Omega_{V,\gamma} \subseteq \mathbb{X}_r$ with degree 4 unknown SOS polynomials yields $\hat{\gamma} = 0.5597$. The respective set containment is illustrated in Figure 5.1b.

A closed loop trajectory from $x_0 = [0.5\ -0.6]^{\mathsf{T}}$ is propagated with the proposed control law, using the same parameters as in Example 1 and the disturbance values $w_1(t) = 0.81$, $w_2(t) = 1 + 0.1\cos 4t$, $w_3(t) = 0.1 \sin 5t$ and $w_4(t) = 1.05$, for all $t \geq 0$. The results are illustrated in Figure 5.3.

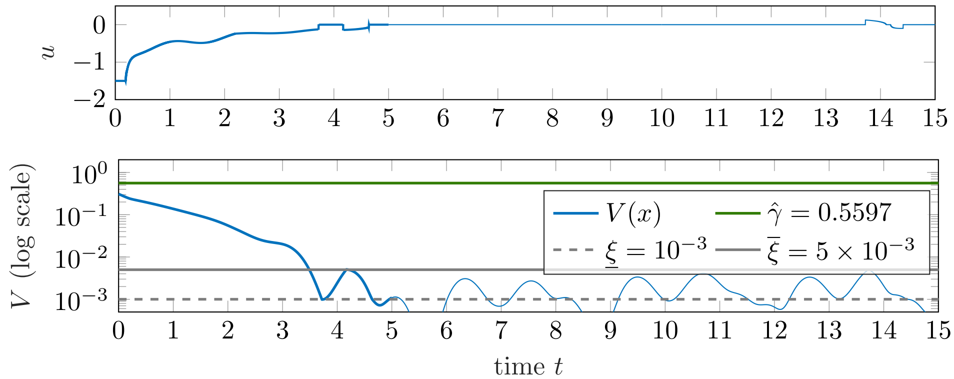


Figure 5.3: The control input u and the value of the RCLF V as functions of time for the example of Section 5.5.2.

5.5.3 Uncertain multi-species system

Finally, we consider the third order system

$$\dot{x}_1 = x_1 + x_1^2 - x_1x_2 - x_1x_3 + u_1 + w_1,$$

$$\dot{x}_2 = -0.7x_2 + x_1x_2 + 0.9x_2x_3,$$

$$\dot{x}_3 = -x_1x_3 + w_2x_3 - x_3^2 + u_2,$$

also encountered in a competing species context, with $u = [u_1\ u_2]^\mathsf{T}$ constrained in the compact polytope with vertices $z_1 = [4\ 4]^\mathsf{T}$, $z_2 = [4\ 0]^\mathsf{T}$, $z_3 = [1\ -2.2]^\mathsf{T}$, $z_4 = [-2\ -2]^\mathsf{T}$ and $z_5 = [-3\ 3]^\mathsf{T}$, $|w_1| \leq 0.1$ and $|w_2| \leq 0.2$. We consider the RCLF $V(x) = x^\mathsf{T}x$, $W(x) = 10^{-6}V(x)$, and then proceed to find the maximal γ using a bisection scheme in the interval $[10^{-3}, 10]$. The partition-based method described in Section 5.3.1 yields 1.2376, while the method of Section 5.3.2 yields 1.0871 and 1.1828, while searching for degree 4 and 6 unknown SOS polynomials, respectively, and considering a partition of $\mathbb{R}^3$ into its 8 octants.

The numerical results for the closed loop trajectory from $x_0 = [0.6\ -0.75\ 0.5]^\mathsf{T}$ under the proposed control law with the same parameters as before except for $\underline{\xi} = 10^{-4}$, $\overline{\xi} = 10^{-3}$, are illustrated in Figure 5.4. The disturbances are taken to be $w_1(t) =$

$0.1 \sin t$, for all $t \geq 0$, and

$$w_2(t) = w_2(x(t)) = \begin{cases} -0.2 & \text{if } x_2(t) \leq -0.2, \\ x_2(t) & \text{if } -0.2 \leq x_2(t) \leq 0.2, \\ 0.2 & \text{otherwise.} \end{cases}$$

Given the shape of the input value set $\mathbb{U}$, there exists a coupling between the admissible values of each input variable and it is not easy to tell from Figure 5.4 whether the particular input time history satisfies the input constraint. This is indeed the case, as one can see in Figure 5.5, where the time-parameterized input trajectory is drawn with respect to the set $\mathbb{U}$.

5.6 Summary

We developed a new solution to the Lyapunov-based robust stabilization problem while explicitly accounting for input constraints and uncertainty in the dynamics. The proposed methods guarantee that a control law based on a given RCLF and implemented online via a QP will robustly stabilize the system starting at any point in the robust stabilization set, which we explicitly characterize using SOS programming techniques.

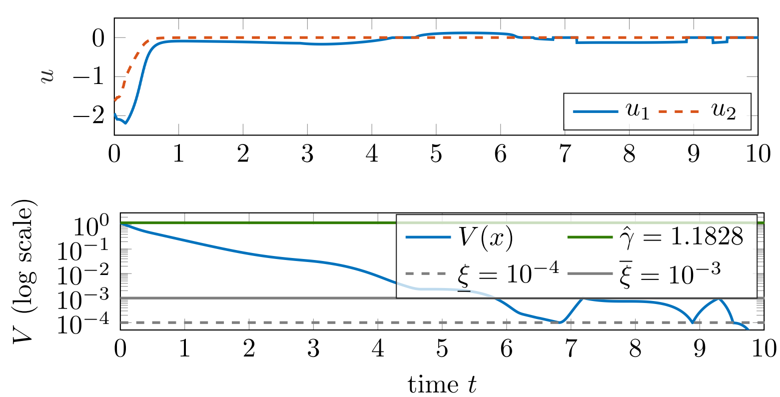


Figure 5.4: The control input $u = [u_1\ u_2]^\mathsf{T}$ and the value of the RCLF V as functions of time for the example of Section 5.5.3.

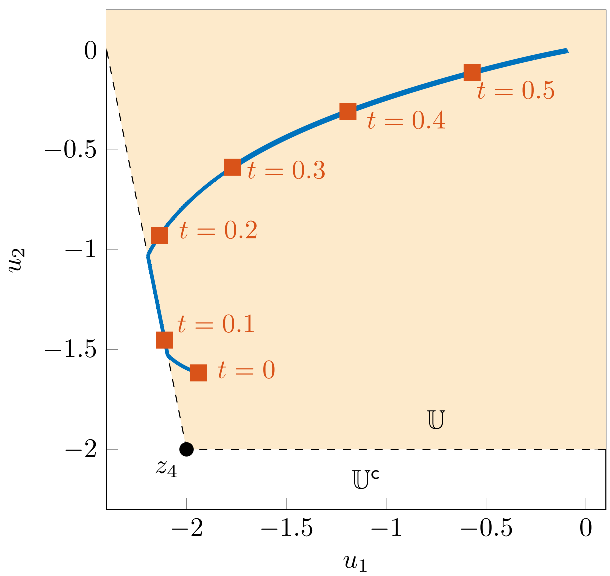


Figure 5.5: The input trajectory with respect to the input value set $\mathbb{U}$ for the example of Section 5.5.3. The dashed line denotes a part of the boundary of the set $\mathbb{U}$. As expected, the input constraints are satisfied, while the system is robustly practically stable.

Chapter 6

The case of large-scale, networked systems

Equipped with the technical results of the previous chapters, and, in particular, the robust control methodology of Chapter 5, we can now commence our study of large-scale or networked systems. As mentioned in Section 1.2.3, the typical characteristics of such systems are (i) the large dimension of the state vector and (ii) some underlying structure that can be used to decompose the original system into smaller, interacting subsystems.

In agreement with the approaches commonly encountered in the relevant literature, the main premise of the distributed and decentralized robust control methodologies proposed herein is that the control input calculations and the actuation process takes place locally, at the subsystem level. In the distributed case, the control laws utilize their knowledge of the state of their neighborhood, whereas in the decentralized case, the control laws are based on a worst-case assumption for the state of the subsystems affecting them. Thanks to the underlying Lyapunov stabilization concepts, which adhere to the Vector (Control) Lyapunov Function paradigm, the distributed and decentralized control laws can collectively achieve the robust stabilization objective for the entire large-scale system.

6.1 System description

We consider nonlinear control systems consisting of multiple individual subsystems which interact with each other. Systems of this nature are often referred to in the literature as *large-scale* or *network* systems. Let the dynamics of such a system $\mathbf{S}$ be given by

$$\dot{x} = f(x) + g(x)u + h(x)w, \qquad x(0) = x_0, \tag{6.1}$$

where $x \in \mathbb{R}^n$ is the state at time $t \geq 0$, with initial value x_0. The control input is u, with $u(t) \in \mathbb{U} \subseteq \mathbb{R}^m$ for all $t \geq 0$, whereas w is an unknown, bounded disturbance with $w(t) \in \mathbb{W} \subseteq \mathbb{R}^r$ for all $t \geq 0$. We can use w to account for modeling uncertainties and external disturbances. The sets $\mathbb{U}$ and $\mathbb{W}$ are convex, compact polytopes, with $0 \in \mathrm{Int}(\mathbb{U})$, defined by their vertices $\{v_\ell \in \mathbb{U}, \ell \in \mathcal{Q}\}$ and $\{z_k \in \mathbb{W}, k \in \mathcal{Q}_\mathbb{W}\}$, for $\mathcal{Q}_\mathbb{U}, \mathcal{Q}_\mathbb{W} \subseteq \mathbb{Z}_{>0}$. Also, $f : \mathbb{R}^n \to \mathbb{R}^n$, with $f(0) = 0$, $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$, and $h : \mathbb{R}^n \to \mathbb{R}^{n\times r}$ are polynomial functions of x.

Let $\mathbf{S}_i$ denote the i^{th} subsystem of $\mathbf{S}$, with $i \in \mathcal{S}$ where $\mathcal{S} = \{1, \ldots, s\}$. The state of $\mathbf{S}_i$ at time $t \geq 0$ is denoted by $x^{[i]} \in \mathbb{R}^{n_i}$, has initial value $x_0^{[i]}$ and is subject to the dynamics described by

$$\dot{x}^{[i]} = f_i\left(x^{[i]}\right) + g_i\left(x^{[i]}\right)u^{[i]} + \sum_{j\in\mathcal{S}\setminus\{i\}} \zeta_i^j\left(x^{[i]}, x^{[j]}\right) + h_i\left(x^{[i]}\right)w^{[i]}, \tag{6.2}$$

where the $u^{[i]}$, $\mathbb{U}^{[i]} \subseteq \mathbb{R}^{m_i}$, $w^{[i]}$, $\mathbb{W}^{[i]} \subseteq \mathbb{R}^{r_i}$, $\{v_\ell^{[i]} \in \mathbb{U}^{[i]}, \ell \in \mathcal{Q}_{\mathbb{U}^{[i]}}\}$, $\{z_k^{[i]} \in \mathbb{W}^{[i]}, k \in \mathcal{Q}_{\mathbb{W}^{[i]}}\}$ are defined in an analogous way for each subsystem $\mathbf{S}_i$. The mappings $f_i : \mathbb{R}^{n_i} \to \mathbb{R}^{n_i}$, $g_i : \mathbb{R}^{n_i} \to \mathbb{R}^{n_i\times m_i}$ and $h_i : \mathbb{R}^{n_i} \to \mathbb{R}^{n_i\times r_i}$ are polynomial functions of the subsystem's state $x^{[i]}$. The influence of subsystem $\mathbf{S}_j$ on $\mathbf{S}_i$ is captured through the polynomial function $\zeta_i^j : \mathbb{R}^{n_i} \times \mathbb{R}^{n_j} \to \mathbb{R}^{n_i}$; the case of single terms involving states from more than two subsystems is not covered here, in order to streamline the presentation, but it is a straightforward extension of the proposed formulation. We assume that the state, control input and disturbance variables of $\mathbf{S}$ are the concatenation of the subsystem components. The neighborhood $\mathcal{N}_i$ is a collection of the indices of subsystems with which $\mathbf{S}_i$ interacts, including i, that is,

$$\mathcal{N}_i := \{j \in \mathcal{S} : \zeta_i^j(\cdot,\cdot) \not\equiv 0\} \cup \{i\}.$$

We also consider $\widetilde{\mathcal{N}}_i := \mathcal{N}_i \setminus \{i\}$ (which excludes i). The collections of the state vectors of all subsystems in $\mathcal{N}_i$ and $\widetilde{\mathcal{N}}_i$ are denoted by $\chi^{[i]} \in \prod_{j\in\mathcal{N}_i} \mathbb{R}^{n_j}$ and $\widetilde{\chi}^{[i]} \in \prod_{j\in\widetilde{\mathcal{N}}_i} \mathbb{R}^{n_j}$, respectively.

A mapping $\theta : \mathcal{D}_c \to \mathbb{U}$, where $\mathcal{D}_c \subseteq \mathbb{R}^n$ is a compact set with $0 \in \mathrm{Int}(\mathcal{D}_c)$, is called an *admissible control law* for the large-scale system $\mathbf{S}$, if the solution of (6.1) at time $t \in [0,\infty)$, denoted by $\phi(t; x_0, \theta(\cdot), w(\cdot))$ or, simply, $\phi_c(t; x_0)$, exists and, also, $\phi_c(t; x_0) \in \mathcal{D}_c$ for $t \in [0,\infty)$. In a similar vein, $\{\theta^{[i]} : \mathcal{D}_c^{[i]} \times \prod_{j\in\mathcal{H}\subseteq\mathcal{S}\setminus\{i\}} \mathcal{D}_c^{[j]} \to \mathbb{U}^{[i]}, i \in \mathcal{S}\}$, where $\mathcal{D}_c^{[i]} \subseteq \mathbb{R}^{n_i}$, $i \in \mathcal{S}$, are compact sets with $0 \in \mathcal{D}_c^{[i]}$, is said to be a collection of *admissible subsystem-level control laws*, if all corresponding solutions for $t \in [0,\infty)$ of (6.2) for each $i \in \mathcal{S}$, denoted by $\phi^{[i]}(t; x_0^{[i]}, \theta^{[i]}(\cdot), \widetilde{\chi}^{[i]}(\cdot), w^{[i]}(\cdot))$ or, simply, $\phi_c^{[i]}(t; x_0^{[i]})$, exist and are contained within the respective $\mathcal{D}_c^{[i]}$.

Control schemes: centralized, decentralized & distributed

Our control objective for $\mathbf{S}$ is to robustly confine the closed loop trajectory of (6.1) to an $\epsilon-$neighborhood of $x = 0$, that is, $\limsup_{t\to\infty} \|\phi_c(t; x_0)\| \leq \epsilon$, for a given $\epsilon > 0$, regardless of the action of the disturbance, using control inputs from the input value set $\mathbb{U}$. Given the aforedescribed structure of $\mathbf{S}$, one can satisfy the control objective with admissible feedback control laws that take one of the following forms:

i) *centralized* control laws of the form $\theta : \mathcal{D}_c \to \mathbb{U}$, which are designed for the entire system (for instance, following the approach proposed in Chapter 5) and, accordingly, require knowledge of the entire state vector x;

ii) *decentralized* control laws of the form $\theta^{[i]} : \mathcal{D}_c^{[i]} \to \mathbb{U}^{[i]}$ which operate using knowledge of the subsystem's state only and render $\limsup_{t\to\infty} \|\phi_c^{[i]}(t; x_0^{[i]})\| \leq \epsilon_i$ for $\epsilon_i > 0$ such that $\limsup_{t\to\infty} \|\phi_c(t; x_0)\| \leq \epsilon$ for the given ϵ;

iii) *distributed* control laws of the form $\theta^{[i]} : \prod_{j\in\mathcal{N}_i} \mathcal{D}_c^{[j]} \to \mathbb{U}^{[i]}$, which collectively achieve the stabilization objective similarly to the decentralized control laws, but, in contrast to the latter, they do so by using knowledge of the state of all their neighboring subsystems.

For either the distributed or the decentralized case, we refer to each set $\mathcal{D}_c^{[i]}$ for the i^{th} subsystem $\mathbf{S}_i$ as the *domain* of the respective DiRCLF / DeRCLF -based, subsystem-level control law, and to $\mathcal{D}_c := \prod_{i\in\mathcal{S}} \mathcal{D}_c^{[i]}$ as the *robust stabilization set* for the large-scale system $\mathbf{S}$.

6.2 Distributed control

Proposition 6.2.1. *Let $V_i : \mathbb{R}^{n_i} \to \mathbb{R}_{\geq 0}$, for $i \in \mathcal{S}$, be polynomial, positive definite, radially unbounded functions and denote the time derivative of V_i along the trajectories of the i^{th} subsystem by $\psi_i : \mathbb{R}^{n_i} \times \mathbb{U}^{[i]} \times \prod_{j\in\widetilde{\mathcal{N}}_i} \mathbb{R}^{n_j} \times \mathbb{W}^{[i]} \to \mathbb{R}$, with*

$$\begin{aligned}\psi_i\left(x^{[i]}, u^{[i]}, \widetilde{\chi}^{[i]}, w^{[i]}\right) :=& \nabla V_i\left(x^{[i]}\right)\Big(f_i\left(x^{[i]}\right) + g_i\left(x^{[i]}\right)u^{[i]} \\ &+ \sum_{j\in\mathcal{S}\setminus\{i\}} \zeta_i^j(x^{[i]}, x^{[j]}) h_i(x^{[i]}) w^{[i]}\Big).\end{aligned} \tag{6.3}$$

Also, let $W_i : \mathbb{R}^{n_i} \to \mathbb{R}_{\geq 0}$, for $i \in \mathcal{S}$, be polynomial, positive definite, radially unbounded functions and $\mathbf{V} = [V_1(x^{[1]}) \ \ldots \ V_s(x^{[s]})]^\mathsf{T}$ and $\mathbf{W}(x) = [W_1(x^{[1]}) \ \ldots \ W_s(x^{[s]})]^\mathsf{T}$. Assume there exists a set $\mathcal{D}_c \subseteq \mathcal{D}$ with $\mathcal{D}_c := \prod_{i\in\mathcal{S}} \mathcal{D}_c^{[i]}$, where $\mathcal{D}_c^{[i]} := \Omega_{V_i,\gamma_i} \subseteq \mathcal{D}^{[i]}$, for some $\gamma_i > 0$, $i \in \mathcal{S}$, such that, for all $x \in \mathcal{D}_c$,

$$\min_{u\in\mathbb{U}} \max_{w\in\mathbb{W}} \ \dot{\mathbf{V}}(x, u, w) \leq -\mathbf{W}(x). \tag{6.4}$$

Then, there exist feedback control laws $\theta^{[i]} : \prod_{j\in\mathcal{N}_i} \mathcal{D}_c^{[j]} \to \mathbb{U}^{[i]}$ such that, for any $x_0 \in \mathcal{D}_c$, the trajectory $\phi_c(t;x_0)$ of the large-scale system $\mathbf{S}$ *when driven by the feedback law* $\theta(x) := [(\theta^{[1]}(\chi^{[1]}))^\mathsf{T} \dots (\theta^{[s]}(\chi^{[s]}))^\mathsf{T}]^\mathsf{T}$ *satisfies* $\limsup_{t\to\infty} \|\phi_c(t;x_0)\| \le \epsilon$, *for any given* $\epsilon > 0$.

Proof. Since (6.4) holds for all $x \in \mathcal{D}_c$, there exists $u_*^{[i]} \in \mathbb{U}^{[i]}$ such that

$$\psi_i(x^{[i]}, u_*^{[i]}, \widetilde{\chi}^{[i]}, w^{[i]}) \le -W_i(x^{[i]}),$$

for all $(x^{[i]}, \widetilde{\chi}^{[i]}, w^{[i]}) \in \mathcal{D}_c^{[i]} \times \prod_{j\in\widetilde{\mathcal{N}}_i} \mathcal{D}_c^{[j]} \times \mathbb{W}^{[i]}$. Let $\theta^{[i]} : \prod_{j\in\mathcal{N}_i} \mathcal{D}_C^{[j]} \to \mathbb{U}^{[i]}$ be a control law which, at every $\chi^{[i]}$, attains values equal to such a $u_*^{[i]}$. For any $x_0^{[i]} \in \mathcal{D}_c^{[i]}$, $\psi_i < 0$ holds along the ensuing trajectory $\phi_c^{[i]}(t;x_0^{[i]})$, for all $t \ge 0$ (as long as $\phi_c^{[i]}(t;x_0^{[i]}) \ne 0$). Therefore, $\mathcal{D}_c^{[i]}$ is positively invariant under the control law $\theta^{[i]}$.

Let $\epsilon > 0$ and let $\epsilon_i > 0$, $i \in \mathcal{S}$, be such that $\prod_{i\in\mathcal{S}} \overline{\mathcal{B}}_{\epsilon_i}^{n_i} \subseteq \overline{\mathcal{B}}_{\epsilon}^{n}$. Also, let $\xi_i := \max\{c > 0 : \Omega_{V_i,c} \subseteq \overline{\mathcal{B}}_{\epsilon_i}^{n_i}\}$ and consider $\rho = \rho(x^{[i]}; V_i, \xi_i)$, with ρ as defined in Chapter 5. Note that $\dot{\rho} = \psi_i(x^{[i]}, \theta^{[i]}(\cdot), \widetilde{\chi}^{[i]}(\cdot), w^{[i]}(\cdot)) < 0$ for all $x^{[i]} \in \mathcal{D}_c^{[i]} \setminus \{0\}$, where $\mathcal{D}_c^{[i]} \supseteq \Omega_{V_i,\xi_i}$, regardless of the values of $\chi^{[i]} \in \prod_{j\in\mathcal{N}_i^*} \mathcal{D}_c^{[j]}$ and $w^{[i]} \in \mathbb{W}^{[i]}$; also $\min_{x^{[i]}\in\mathcal{D}_c^{[i]}} \rho = 0$. Thus, $\rho \to 0$ as $t \to \infty$. Given the way the ξ_i and ϵ_i were chosen for $i \in \mathcal{S}$, the latter fact implies $\limsup_{t\to\infty} \|\phi_c^{[i]}(t;x_0^{[i]})\| \le \epsilon_i$, and, in turn, $\limsup_{t\to\infty} \|\phi_c(t;x_0)\| \le \epsilon$. □

If the conditions in Proposition 6.2.1 are satisfied, each V_i will be hereafter called a *Distributed* RCLF (DiRCLF) for the respective subsystem $\mathbf{S}_i$, while $\mathbf{V}$ will be called a *Vector* RCLF (VRCLF) for the large-scale system $\mathbf{S}$.

Proposition 6.2.2. *For each $i \in \mathcal{S}$ and all $\ell \in \mathcal{Q}_{\mathbb{U}^{[i]}}$, let*

$$\mathbb{X}_{P_\ell}^{[i]} \cap \mathcal{D}_c^{[i]} \subseteq \mathbb{X}_\ell^{[i]}, \tag{6.5}$$

where

$$\mathbb{X}_{P_\ell}^{[i]} := \cap_{\nu \in \mathcal{Q}_{\mathbb{U}^{[i]}} \setminus \{\ell\}} \{x^{[i]} \in \mathbb{R}^{n_i} : (\eta_\ell^{[i]} - \eta_\nu^{[i]})^\mathsf{T} x^{[i]} \leq 0\},$$

$\eta_\ell^{[i]} := \nabla^2 V_i(0) g_i(0) v_\ell^{[i]}$, $\mathcal{D}_c^{[i]} = \Omega_{V_i,\gamma_i}$, *and*

$\mathbb{X}_\ell^{[i]} := \left\{x^{[i]} \in \mathbb{R}^{n_i} : \sup_{k \in \mathcal{Q}_{\mathbb{W}^{[i]}}} \psi_i(x^{[i]}, v_\ell^{[i]}, \widetilde{\chi}^{[i]}, z_k^{[i]}) \leq -W_i(x^{[i]}),\ \textit{for all } \widetilde{\chi}^{[i]} \in \prod_{j \in \widetilde{\mathcal{N}}_i} \mathcal{D}_c^{[j]}\right\}$.

Then (6.4) holds.

Proof. By the fact that (6.5) holds for all $\ell \in \mathcal{Q}_{\mathbb{U}^{[i]}}$ and the properties of the partition of the state space $\mathbb{R}^n$ into the sets $\mathbb{X}_{P_i}$ for the centralized case as derived in Proposition 3.4.3 (which trivially carry over to each individual subsystem and its own state space $\mathbb{R}^{n_i}$), one obtains $\mathcal{D}_c^{[i]} \subseteq \cup_{\ell \in \mathcal{Q}_{\mathbb{U}^{[i]}}} \mathbb{X}_\ell^{[i]}$. Next, pick any $x^{[i]} \in \mathcal{D}_c^{[i]}$ and note that, by the definition of $\mathbb{X}_\ell^{[i]}$,

$$\min_{\ell \in \mathcal{Q}_{\mathbb{U}^{[i]}}} \max_{k \in \mathcal{Q}_{\mathbb{W}^{[i]}}} \psi_i\left(x^{[i]}, v_\ell^{[i]}, \widetilde{\chi}^{[i]}, z_k^{[i]}\right) \leq -W_i\left(x^{[i]}\right). \tag{6.6}$$

Since $\psi_i\left(x^{[i]}, u^{[i]}, \widetilde{\chi}^{[i]}, w^{[i]}\right)$ is affine in both $u^{[i]}$ and $w^{[i]}$, it can only attain its minima and maxima with respect to either of these two arguments on the vertices of the respective polytopic value set. Consequently, (6.6) is equivalent to

$$\min_{u^{[i]} \in \mathbb{U}^{[i]}} \max_{w^{[i]} \in \mathbb{W}^{[i]}} \psi_i\left(x^{[i]}, u^{[i]}, \widetilde{\chi}^{[i]}, w^{[i]}\right) \leq -W_i\left(x^{[i]}\right). \tag{6.7}$$

It is easy to see that (6.7) corresponds to the i^{th} component of (6.4); if (6.7) holds for all $i \in \mathcal{S}$, then (6.4) also holds. $\square$

The S-procedure with SOS polynomial multipliers can be used in order to express the containments in (6.5) via LMI constraints. In particular, for every subsystem index $i \in \mathcal{S}$ and all pairs $(\ell, k) \in \mathcal{Q}_{\mathbb{U}^{[i]}} \times \mathcal{Q}_{\mathbb{W}^{[i]}}$, consider the set containment

$$
\begin{aligned}
-\psi_i\left(x^{[i]}, v_\ell^{[i]}, \widetilde{\chi}^{[i]}, z_k^{[i]}\right) - W_i\left(x^{[i]}\right) - \sum_{j \in \mathcal{N}_i} c_j^{[i,\ell,k]}\left(\chi^{[i]}\right)\left(\gamma_j - V_j\left(x^{[j]}\right)\right) \\
+ \sum_{\nu \in \mathcal{Q}_{\mathbb{U}^{[i]}} \setminus \{\ell\}} s_\nu^{[i,\ell,k]}\left(\chi^{[i]}\right)\left(\eta_\ell^{[i]} - \eta_\nu^{[i]}\right)^{\mathsf{T}} x^{[i]} \in \Sigma[\chi^{[i]}],
\end{aligned}
\tag{6.8}
$$

where $c_j^{[i,\ell,k]} \in \Sigma[\chi^{[i]}]$ for $j \in \mathcal{N}_i$ and all $(\ell, k) \in \mathcal{Q}_{\mathbb{U}^{[i]}} \times \mathcal{Q}_{\mathbb{W}^{[i]}}$, and $s_\nu^{[i,\ell,k]} \in \Sigma[\chi^{[i]}]$ for $\nu \in \mathcal{Q}_{\mathbb{U}^{[i]}}$ and all $(\ell, k) \in \mathcal{Q}_{\mathbb{U}^{[i]}} \times \mathcal{Q}_{\mathbb{W}^{[i]}}$. If all the SOS polynomial decompositions involved in (6.8) for $i \in \mathcal{S}$ exist, then (6.5) holds for all $i \in \mathcal{S}$, and one can make use of Propositions 6.2.1 and 6.2.2, and thus take $\mathcal{D}_c^{[i]} = \Omega_{V_i,\gamma_i}$ as the domain of each $\theta^{[i]}$.

Subsystem-level distributed feedback control laws

Assume that $\mathbf{S}_i$ is at some state $x^{[i]} \in \mathcal{D}_c^{[i]}$ and is provided with the values of the states $\widetilde{\chi}^{[i]}$ of its neighboring subsystems. The stabilizing control inputs that can be attained by each control law $\theta^{[i]}$ are characterized through the affine inequalities described by $u^{[i]} \in \mathbb{U}^{[i]}$ and $\max_{w^{[i]} \in \mathbb{W}^{[i]}} \psi_i(x^{[i]}, u^{[i]}, \widetilde{\chi}^{[i]}, w^{[i]}) \leq -W_i(x^{[i]})$ (or, equivalently, in terms of the vertices of $\mathbb{W}^{[i]}$, $\max_{k \in \mathcal{Q}_{\mathbb{W}^{[i]}}} \psi_i(x^{[i]}, u^{[i]}, \widetilde{\chi}^{[i]}, z_k^{[i]}) \leq -W_i(x^{[i]})$). Note that the compact polytope in $\mathbb{R}^{m_i}$ described here is nonempty, thanks to the fact that (6.7) holds. We proceed to derive distributed control laws in a manner parallel to the approach pursued for the case of centralized control design presented in Chapter 5. To this end, let $\alpha_i : \mathcal{D}_c^{[i]} \to \mathbb{R}_{\leq 0}$ and consider the QP of finding $u^{[i]}$ that minimizes

$$
\mathcal{J}_i\left(u^{[i]}; \chi^{[i]}\right) := \left(u^{[i]}\right)^{\mathsf{T}} Q_i\left(x^{[i]}\right) u^{[i]} + L_i\left(\chi^{[i]}\right) u^{[i]},
$$

subject to $A^{[i]} u^{[i]} \preceq b^{[i]}$,

$$
\nabla V_i(x^{[i]}) g_i(x^{[i]}) u^{[i]} \leq -\nabla V_i(x^{[i]}) f(x^{[i]}) - \delta_i(x^{[i]}) - W_i(x^{[i]}) - \sum_{j \in \mathcal{S} \setminus \{i\}} \nabla V_i(x^{[i]}) \zeta_i^j(x^{[i]}, x^{[j]}),
$$

where

$$
\begin{aligned}
Q_i(x^{[i]}) &:= \left(\nabla V_i(x^{[i]}) g_i(x^{[i]})\right)^{\mathsf{T}} \nabla V_i(x^{[i]}) g_i(x^{[i]}), \\
L_i(x^{[i]}) &:= 2\Big[\nabla V_i(x^{[i]}) f(x^{[i]}) + \sum\nolimits_{j \in \mathcal{S} \setminus \{i\}} \nabla V_i(x^{[i]}) \zeta_i^j(x^{[i]}, x^{[j]}) \\
&\qquad\qquad\qquad\qquad + \delta_i(x^{[i]}) - \alpha_i(x^{[i]})\Big] \nabla V_i(x^{[i]}) g_i(x^{[i]}), \\
\delta_i(x^{[i]}) &:= \max_{k \in \mathcal{Q}_{\mathbb{W}^{[i]}}} \nabla V_i(x^{[i]}) h_i(x^{[i]}) z_k^{[i]}.
\end{aligned}
$$

The individual control laws $\theta^{[i]}$ which, at every $(x^{[i]}, \widetilde{\chi}^{[i]}) \in (\mathcal{D}_c^{[i]} \setminus \Omega_{V_i,\xi_i}) \times \prod_{j \in \widetilde{\mathcal{N}}_i} \mathcal{D}_c^{[j]}$, where $\xi_i := \max\{c > 0 : \Omega_{V_i,c} \subseteq \overline{\mathcal{B}}_{\epsilon_i}^{n_i}\}$ and $\epsilon_i > 0$ are such that $\prod_{i \in \mathcal{S}} \overline{\mathcal{B}}_{\epsilon_i}^{n_i} \subseteq \overline{\mathcal{B}}_{\epsilon}^{n}$, attain values equal to the minimizers of these QPs, for each $i \in \mathcal{S}$, robustly stabilize the large scale system $\mathbf{S}$ in the sense described by Proposition 6.2.1. The recursive feasibility of the QPs, that is, the guarantee that there exist solutions along any closed loop trajectories of the individual subsystems for any $x_0 \in \mathcal{D}_c$, is provided by the preceding analysis. The structure of the individual distributed control laws follows the discussion of Section 6.1: neighboring states appear as parameters in the formulation of the QP and their current numerical values are needed to calculate the control input.

6.3 Decentralized control

The decentralized control laws are designed under the assumption that they cannot access any state information pertaining to their corresponding neighboring subsystems. Any such state vector elements of neighboring subsystems which enter the dynamics of a particular subsystem have to be treated by a decentralized control law as unknown quantities, with specified uncertainty bounds. Pursuing a QP-based controller with even simple uncertainty characterizations either in the constraints or the objective of the underlying QP is not a trivial feat; the reader is referred to Ben-Tal et al. (2009) for a thorough treatment of the broader problem of robust conic optimization. For purposes of developing optimization-based control laws, it is reasonable to expect that solving such an uncertain optimization problem will be analytically complicated and more computationally demanding than what one can afford in a real time, embedded control context.

The aforedescribed hardship is overcome by drawing inspiration from the similar problem of robustly stabilizing (6.1) with a centralized control law with the techniques proposed in Chapter 5, which we have already utilized, to some extent, at the subsystem level for the distributed control case in Section 6.2. Assuming that any uncertainty or external influence which is bounded in a polytope enters the dynamics in an affine way allows for a rather efficient treatment both at the feedback control law level (since what would be a hard to solve uncertain optimization problem can be solved as a regular QP) and at the preceding analysis level (as any set containments are pursued in a space having dimension equal to the subsystem dimension n_i).

Sufficient conditions for decentralized stabilization with the proposed approach are developed next.

Proposition 6.3.1. *Let V_i, $\mathbf{V}$, $\mathcal{D}_c^{[i]}$, $\mathcal{D}_c$, W_i, $\mathbf{W}$ be defined as in Proposition 6.2.1 and assume that, for all $x \in \mathcal{D}_c$,*

$$\sum_{j \in \widetilde{\mathcal{N}}_i} \zeta_i^j(x^{[i]}, x^{[j]}) = \lambda_i(x^{[i]})\mu^{[i]}, \quad i \in \mathcal{S}, \tag{6.9}$$

where $\lambda_i : \mathbb{R}^{n_i} \to \mathbb{R}^{n_i \times \kappa_i}$ is a polynomial function of $x^{[i]}$, $\mu^{[i]} \in \mathbb{M}_i$ and $\mathbb{M}_i \subseteq \mathbb{R}^{\kappa_i}$ is a convex, compact polytope with vertices $\beta_\sigma^{[i]} \in \mathbb{M}_i$, where $\sigma \in \mathcal{Q}_{\mathbb{M}_i} \subseteq \mathbb{Z}_{>0}$. Finally, assume that

$$\min_{u \in \mathbb{U}} \max_{(w,\mu) \in \mathbb{W} \times \mathbb{M}} \dot{\hat{\mathbf{V}}}(x, u, \mu, w) \leq -\mathbf{W}(x), \tag{6.10}$$

for all $x \in \mathcal{D}_c$, where $\mu = [\mu_1^\mathsf{T} \ \ldots \mu_s^\mathsf{T}]^\mathsf{T}$, $\mathbb{M} := \prod_{i \in \mathcal{S}} \mathbb{M}_i$, and $\dot{\hat{\mathbf{V}}}$ is element-wise equal to the time derivative of the components of $\mathbf{V}$ along the trajectories of (6.2) subject to

([6.9](#)), as given by $\widehat{\psi}_i(x^{[i]}, u^{[i]}, \mu^{[i]}, w^{[i]}) := \nabla V_i(x^{[i]})(f_i(x^{[i]}) + g_i(x^{[i]})u^{[i]} + \lambda_i(x^{[i]})\mu^{[i]} + h_i(x^{[i]})w^{[i]})$*. Then, there exist control laws* $\widehat{\theta}^{[i]} : \mathcal{D}_c^{[i]} \to \mathbb{U}^{[i]}$*, for each subsystem* $\mathbf{S}_i$ *with* $i \in \mathcal{S}$*, such that the solution* $\phi_c(t; x_0)$ *to the dynamics (6.1) of* $\mathbf{S}$ *under the control law* $\widehat{\theta}(x) := [(\widehat{\theta}^{[1]}(x^{[1]}))^\mathsf{T} \ldots (\widehat{\theta}^{[s]}(x^{[s]}))^\mathsf{T}]^\mathsf{T}$ *satisfies* $\limsup_{t\to\infty} \|\phi_c(t; x_0)\| \leq \epsilon$ *for any* $x_0 \in \mathcal{D}_c$ *and any given* $\epsilon > 0$*.*

Proof. Given (6.10), it is possible to find $u_*^{[i]}$ such that $\widehat{\psi}_i(x^{[i]}, u_*^{[i]}, \mu^{[i]}, w^{[i]}) \leq -W_i(x^{[i]})$, regardless of the disturbance values $w^{[i]} \in \mathbb{W}^{[i]}$ and the cumulative contributions $\mu^{[i]} \in \mathbb{M}_i$ from the subsystems $\mathbf{S}_j$, $j \in \mathcal{N}_i$, with which $\mathbf{S}_i$ interacts. Let $\widehat{\theta}^{[i]} : \mathcal{D}_c^{[i]} \to \mathbb{U}^{[i]}$ be a control law attaining, at every $x^{[i]} \in \mathcal{D}_c^{[i]}$, values equal to such $u_*^{[i]}$. Note that the same control law $\widehat{\theta}$ also renders $\dot{V}_i = \psi_i(x^{[i]}, u_*^{[i]}, \widetilde{\chi}^{[i]}, w^{[i]}) \leq -W_i(x^{[i]})$, where the time derivatives are now computed along the actual trajectories of (6.2), which depend on the state vector values of the neighboring subsystems that $\widehat{\theta}^{[i]}$ does not have knowledge of. The proof with regards to the invariance of each $\mathcal{D}_c^{[i]}$ and the robust confinement of the closed-loop trajectories of $\mathbf{S}$ then proceeds similarly to the proof of Proposition 6.2.2. □

If the conditions in Proposition 6.3.1 are satisfied, each V_i will be hereafter called a *Decentralized* RCLF (DeRCLF) for the respective subsystem $\mathbf{S}_i$, while $\mathbf{V}$ will be called a *Vector* RCLF (VRCLF) for $\mathbf{S}$.

Proposition 6.3.2. *If, for every* $i \in \mathcal{S}$ *and for all* $\ell \in \mathcal{Q}_{\mathbb{U}^{[i]}}$*,*

$$\mathbb{X}_{P_\ell}^{[i]} \cap \mathcal{D}_c^{[i]} \subseteq \widehat{\mathbb{X}}_\ell^{[i]}, \tag{6.11}$$

holds, where the sets $\mathbb{X}_{P_\ell}^{[i]}$*,* $\mathcal{D}_c^{[i]}$ *are defined as in Proposition 6.2.2 and*

$$\widehat{\mathbb{X}}_\ell^{[i]} := \left\{ x^{[i]} \in \mathbb{R}^{n_i} : \max_{(k,\sigma)\in\mathcal{Q}_{\mathbb{W}^{[i]}}\times\mathcal{Q}_{\mathbb{M}_i}} \widehat{\psi}_i(x^{[i]}, v_\ell^{[i]}, \beta_\sigma^{[i]}, z_k^{[i]}) \leq -W_i(x^{[i]}) \right\}, \tag{6.12}$$

then (6.10) holds.

Proof. Similarly to the proof for Proposition 6.2.2, (6.11) implies $\mathcal{D}_c^{[i]} \subseteq \cup_{\ell \in \mathcal{Q}_{\mathbb{U}^{[i]}}} \widehat{\mathbb{X}}_\ell^{[i]}$. Consequently, for all $x^{[i]} \in \mathcal{D}_c^{[i]}$,

$$\min_{\ell \in \mathcal{Q}_{\mathbb{U}^{[i]}}} \max_{(k,\sigma) \in \mathcal{Q}_{\mathbb{W}^{[i]}} \times \mathcal{Q}_{\mathbb{M}_i}} \widehat{\psi}_i \left(x^{[i]}, v_\ell^{[i]}, \beta_\sigma^{[i]}, z_k^{[i]} \right) \leq -W_i \left(x^{[i]} \right). \tag{6.13}$$

Noting that $\widehat{\psi}_i \left(x^{[i]}, u^{[i]}, \mu^{[i]}, w^{[i]} \right)$ is affine in $u^{[i]}$, $\mu^{[i]}$ and $w^{[i]}$ and that (6.13) holds for all $i \in \mathcal{S}$, (6.10) also holds. □

The S-procedure with SOS polynomial multipliers can be used in order to express the containments in (6.11) via LMI constraints. In particular, for every subsystem index $i \in \mathcal{S}$ and all pairs $(\ell, k, \sigma) \in \mathcal{Q}_{\mathbb{U}^{[i]}} \times \mathcal{Q}_{\mathbb{W}^{[i]}} \times \mathcal{Q}_{\mathbb{M}_i}$ consider

$$-\widehat{\psi}_i \left(x^{[i]}, v_\ell^{[i]}, \beta_\sigma^{[i]}, z_k^{[i]} \right) - W_i \left(x^{[i]} \right) - c^{[i,\ell,k,\sigma]} \left(x^{[i]} \right) \left(\gamma_i - V_i \left(x^{[i]} \right) \right) \\ + \sum_{\nu \in \mathcal{Q}_{\mathbb{U}^{[i]}} \setminus \{\ell\}} s_\nu^{[i,\ell,k,\sigma]} \left(x^{[i]} \right) \left(\eta_\ell^{[i]} - \eta_\nu^{[i]} \right)^\mathsf{T} x^{[i]} \in \Sigma[x^{[i]}]$$

where $c^{[i,\ell,k,\sigma]} \in \Sigma[x^{[i]}]$ for all $(\ell, k, \sigma) \in \mathcal{Q}_{\mathbb{U}^{[i]}} \times \mathcal{Q}_{\mathbb{W}^{[i]}} \times \mathcal{Q}_{\mathbb{M}_i}$, and $s_\nu^{[i,\ell,k,\sigma]} \in \Sigma[x^{[i]}]$ for $\nu \in \mathcal{Q}_{\mathbb{U}^{[i]}}$ and all $(\ell, k, \sigma) \in \mathcal{Q}_{\mathbb{U}^{[i]}} \times \mathcal{Q}_{\mathbb{W}^{[i]}} \times \mathcal{Q}_{\mathbb{M}_i}$. If all the SOS polynomial decompositions involved in (6.8) for $i \in \mathcal{S}$ exist, then (6.5) holds for all $i \in \mathcal{S}$, and one can make use of Propositions 6.3.1 and 6.3.2 to ultimately take $\mathcal{D}_c^{[i]} = \Omega_{V_i, \gamma_i}$ as the domain of each $\widehat{\theta}^{[i]}$.

Subsystem-level decentralized feedback control laws

Defining α_i as in Section 6.2, we consider the QP of finding $u^{[i]}$ that minimizes

$$\mathcal{J}_i \left(u^{[i]}; x^{[i]} \right) := \left(u^{[i]} \right)^\mathsf{T} Q_i \left(x^{[i]} \right) u^{[i]} + L_i \left(x^{[i]} \right) u^{[i]},$$

$$\text{subject to} \quad A^{[i]} u^{[i]} \preceq b^{[i]},$$

$$\nabla V_i \left(x^{[i]} \right) g_i \left(x^{[i]} \right) u^{[i]} \leq -\nabla V_i \left(x^{[i]} \right) f_i \left(x^{[i]} \right) - \varepsilon_i \left(x^{[i]} \right) - W_i \left(x^{[i]} \right),$$

where

$$
\begin{aligned}
Q_i(x^{[i]}) &:= \left(\nabla V_i(x^{[i]}) g_i(x^{[i]})\right)^{\mathsf{T}} \nabla V_i(x^{[i]}) g_i(x^{[i]}), \\
L_i\left(x^{[i]}\right) &:= 2\left[\nabla V_i\left(x^{[i]}\right) f_i\left(x^{[i]}\right) + \varepsilon_i(x) - \alpha_i\left(x^{[i]}\right)\right] \nabla V_i(x^{[i]}) g_i(x^{[i]}), \\
\varepsilon_i\left(x^{[i]}\right) &:= \max_{(k,\sigma)\in \mathcal{Q}_{\mathbb{W}[i]} \times \mathcal{Q}_{\mathbb{M}_i}} \nabla V_i\left(x^{[i]}\right) h_i\left(x^{[i]}\right) z_k^{[i]} + \nabla V_i\left(x^{[i]}\right) \lambda_i\left(x^{[i]}\right) \beta_\sigma^{[i]}.
\end{aligned}
$$

Each decentralized control law $\widehat{\theta}^{[i]}$ attains values equal to the minimizer of this QP, at every $x^{[i]} \in \mathcal{D}_c \setminus \Omega_{V_i,\xi_i}$. These control laws collectively accomplish the same control objectives as the distributed control laws of Section 6.2, yet, they only require knowledge of the current state of their local subsystem only and assume the worst-case scenario with regards to any influence from other subsystems.

6.4 Remarks on the QP-based control laws

Remark 7. It is beneficial to ensure that the control law QPs are strictly convex, so that their minimizer is unique. To this end and similarly to our approach in Chapters 4 and 5, one can solve either QP for $\widetilde{Q}_i(x) = Q_i(x) + cI_{m_i}$ where $c \in \mathbb{R}_{\geq 0}$ with $0 < c \ll 1$ is such that no eigenvalue of $\widetilde{Q}_i(x)$ is smaller than a specified strictly positive number.

Remark 8. The robust stabilization methods proposed herein are guaranteed to bring the trajectories of all subsystems to the prescribed neighborhoods around the origin, which correspond to the DiRCLF / DeRCLF sublevel sets Ω_{V_i,ξ_i}. If some subsystem's trajectory is then about to exit this neighborhood, the respective control laws will prevent this at once. This can cause a behavior reminiscent of chattering, which is typically undesirable. Following the remedy proposed in Chapter 5 for the standalone system case, one can define an inner and an outer target sublevel set, $\Omega_{V_i,\underline{\xi}_i}$ and $\Omega_{V_i,\bar{\xi}_i}$ for each subsystem. Initially, each subsystem is brought to $\Omega_{V_i,\underline{\xi}_i}$, and then no control is applied unless the trajectory is about to exit $\Omega_{V_i,\bar{\xi}_i}$. Then, the control law brings the system back to $\Omega_{V_i,\bar{\xi}_i}$. The process is repeated accordingly. We illustrate this approach in practice in Section 6.6.

6.5 Computational considerations and scalability

Following the technical results presented for distributed and decentralized control, it is useful to reflect on the issue of *scalability*, one of the main motivating factors behind such control techniques. We approach the topic by estimating the computational requirements of the proposed methods, and looking at how these scale with the size of the large-scale system $\mathbf{S}$ and its subsystems $\mathbf{S}_i$.

The QP associated with the centralized control law has m unknowns, where m is the size of the input $u \in \mathbb{U} \subseteq \mathbb{R}^m$ of the large-scale system $\mathbf{S}$, and is subject to $1+q$ constraints, where q is the number of affine inequalities in $\mathbb{R}^m$ used to describe $\mathbb{U}$. Such a QP can be solved in $O(L^2(q)q^4)$ operations (Ye and Tse, 1989), where $q \mapsto L(q)$ is a positive definite polynomial function weighting the size of the input to the solution algorithm, in terms of bits. The underlying QPs for the distributed and the decentralized control laws for each subsystem $\mathbf{S}_i$, $i \in \mathcal{S}$, have m_i unknowns and $1+q_i$ constraints. Accordingly, each of these can be solved in $O(L^2(q_i)q_i^4)$ operations[1]. It is easy to see that the proposed solutions offer significant computational advantages as they do not scale with the size of the large-scale system $\mathbf{S}$. For example, using a loose interpretation of the asymptotic bounds, one can infer that for a system $\mathbf{S}$ consisting of 100 subsystems with 2 inputs each subject to box-like constraints the

[1]Note that these rough, worst case asymptotic estimates are ignoring any constant or linear time operations needed to assemble the QPs, since these are not expected to dominate in the complexity calculations, except, perhaps, for subsystems with dimensions $n_i \gg q_i$. Yet, even then, the subsystem-level control solutions would still be orders of magnitude faster than a centralized controller. Finally, applicable acceleration techniques (for instance, Nesterov and Nemirovski (2001)), as well as warm-starts are expected to significantly lower the actual number of operations required, in all cases.

relationship between $q_i = 4$ and $q = 400$ can yield a multiple-orders-of-magnitude difference in the respective computational costs, even if the calculations for all 100 controllers were performed on the same computational platform. Of course, the nature of the distributed and decentralized solutions allows one to implement and operate each subsystem-level controller locally, on relatively low-power and low-footprint embedded systems. In practice, solution times in the milliseconds range can be expected on modest embedded computers for some tens of unknowns in the QP.

An SDP with LMI constraints can be solved with polynomial time algorithms Vandenberghe and Boyd (1996). Yet, the size of the LMIs that result from the parsing of SOS constraints tends to grow exponentially as the dimension of either the underlying state-space or neighborhood state-space increases (for the decentralized or distributed case, respectively)[2]. SOS methods can be rendered practically intractable for systems with more than 10 states. Contemporary results in Ahmadi and Majumdar (2018) provide alternative SOS polynomial parameterizations based on affine and second order cone constraints, which can scale better than LMI-based SOS and possibly allow one to consider systems with some tens of states. For the methods developed herein, any practical upper bound on the size of the underlying (sub)problems, imposed by the particular SOS parameterization or hardware limitations, can be interpreted as follows: for the distributed case, such a bound is expected to interfere with the way the dimensions of the subsystem neighborhoods scale, while for the decentralized case, any such limitation will have a milder impact and mainly relate to the size of the individual subsystems. The total effort to analyze the large-scale system **S** for a given set of γ_i values only scales linearly with the individual effort required per subsystem,

[2]In fact, a problem with a single SOS polynomial constraint with indeterminate $x \in \mathbb{R}^n$ grows with the number of monomials of x, that is, $(n+d)!/(n!d!)$, where d is the degree of the polynomial.

and can also be (almost trivially) parallelized.

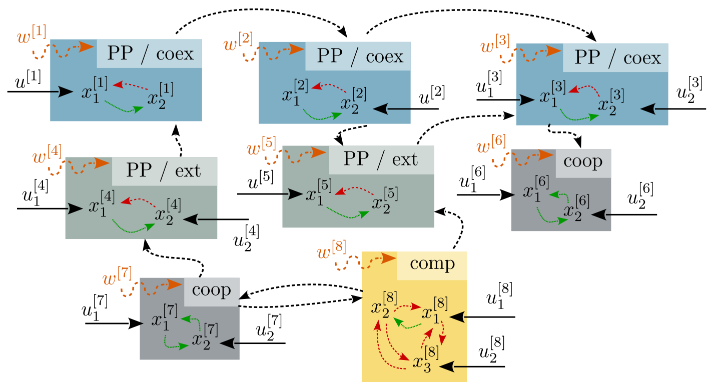


Figure 6.1: Large-scale system considered in Section 6.6. Dashed lines denote interactions between subsystems. Local interactions within subsystems are denoted with a dashed magenta arrow from agent A to B, if that interaction is nominally harmful to B's population (that is, the presence of A causes B to decrease), or a green arrow, if it is beneficial (that is, the presence of A causes B to increase).

6.6 Numerical example

We consider a large-scale system consisting of 8 interconnected subsystems with a total number of 17 agents, subject to variations of controlled, uncertain Lotka-Volterra type of dynamics, as conceptually illustrated in Figure 6.1. Such dynamics admit various interpretations in different contexts such as in biological, ecological, chemical and economic systems, as well as evolutionary game theory (Hofbauer and Sigmund, 2003; Samuelson, 1971; Nicolis and Portnow, 1973).

Subsystems $\mathbf{S}_1$, $\mathbf{S}_2$ and $\mathbf{S}_3$ each consist of two agents with predator/prey dynamics, where each nominal equilibrium at the origin (in the absence of uncertainty and interactions with other subsystems) corresponds to a *coexistence* point and is surrounded by multiple periodic orbits. The individual dynamics result from shifting the origin of the traditional predator/prey equations ($\dot{x} = ax - bxy$, $\dot{y} = cxy - dy$, where x is the prey population, y is the predator population and $a, b, c, d > 0$) so that it corresponds to the $(c/d, a/b)$ original equilibrium. This equilibrium is surrounded by periodic orbits. Interactions between agents from different subsystems, uncertainty in the coefficients (which both alter the nature of the dynamics) and control inputs are incorporated as shown in the equations. We have

$$\begin{aligned}
\dot{x}_1^{[1]} &= -x_2^{[1]}w_1^{[1]} - x_1^{[1]}x_2^{[1]} + 0.2x_1^{[1]}x_1^{[4]} + u^{[1]},\\
\dot{x}_2^{[1]} &= x_1^{[1]}w_2^{[1]} + x_1^{[1]}x_2^{[1]},\\
\dot{x}_1^{[2]} &= -x_2^{[2]}w_1^{[2]} - x_1^{[2]}x_2^{[2]},\\
\dot{x}_2^{[2]} &= x_1^{[2]}w_2^{[2]} + x_1^{[2]}x_2^{[2]} + x_2^{[2]}x_2^{[1]} + u^{[2]},\\
\dot{x}_1^{[3]} &= -x_2^{[3]}w_1^{[3]} - x_1^{[3]}x_2^{[3]} + x_2^{[3]}x_2^{[2]} + u_1^{[3]},\\
\dot{x}_2^{[3]} &= x_1^{[3]}w_2^{[3]} + x_1^{[3]}x_2^{[3]} + x_1^{[3]}x_1^{[2]} + x_1^{[3]}x_2^{[5]} + u_2^{[3]},
\end{aligned}$$

with $|u^{[1]}| \leq 1$, $|u^{[2]}| \leq 1.1$, $\|u^{[3]}\|_\infty \leq 1$, $0.9 \leq w_1^{[1]} \leq 1.1$, $0.9 \leq w_2^{[1]} \leq 1.1$, $0.8 \leq w_1^{[2]} \leq 1.1$, $0.9 \leq w_2^{[2]} \leq 1.1$, $0.9 \leq w_1^{[3]} \leq 1.1$, and $0.7 \leq w_2^{[3]} \leq 1.1$.

Each of $\mathbf{S}_4$ and $\mathbf{S}_5$ consists of two agents with predator/prey dynamics in which the origin would nominally correspond to an *extinction* saddle point:

$$\begin{aligned}
\dot{x}_1^{[4]} &= x_1^{[4]}w_1^{[4]} - x_1^{[4]}x_2^{[4]} - x_1^{[4]}x_1^{[7]} + u_1^{[4]},\\
\dot{x}_2^{[4]} &= x_1^{[4]}x_2^{[4]} - x_2^{[4]}w_2^{[4]} + x_2^{[4]}x_2^{[7]} + u_2^{[4]},\\
\dot{x}_1^{[5]} &= x_1^{[5]}w^{[5]} - x_1^{[5]}x_2^{[5]} + x_1^{[5]}x_1^{[2]} + 0.1x_1^{[5]}(x_3^{[8]})^3 + u^{[5]},\\
\dot{x}_2^{[5]} &= x_1^{[5]}x_2^{[5]} - x_2^{[5]},
\end{aligned}$$

with $\|u^{[4]}\|_\infty \leq 1$, $-1 \leq u^{[5]} \leq 2$, $w^{[4]} \in \mathrm{Conv}\{-(0.4, 0.4), (-0.5, 0.5), (0.4, 0.5), (0.4, -0.4)\}$, $-0.08 \leq w^{[5]} \leq 0.1$.

Subsystems $\mathbf{S}_6$ and $\mathbf{S}_7$ consist of two *cooperative*[3] agents each, subject to the

[3]Each agent's population is benefited (that is, tends to increase) from the presence of the other. The nominal equilibrium at the origin is unstable.

dynamics described by:

$$
\begin{aligned}
\dot{x}_1^{[6]} &= x_1^{[6]}(1 + x_1^{[6]} + x_2^{[6]}) + 0.2x_1^{[6]}x_2^{[3]} + u_1^{[6]} + w^{[6]} \\
\dot{x}_2^{[6]} &= x_2^{[6]}(1 + 0.2x_1^{[6]} + 0.6x_2^{[6]}) + 0.5x_1^{[6]}x_1^{[3]} + u_2^{[6]}, \\
\dot{x}_1^{[7]} &= x_1^{[7]}(1 + x_1^{[7]} + x_2^{[7]}) + 0.2x_1^{[7]}x_3^{[8]} + u_1^{[7]}, \\
\dot{x}_2^{[7]} &= x_2^{[7]}(1 + 0.2x_1^{[7]} + 0.6x_2^{[7]}) + 0.1x_1^{[7]}x_2^{[8]} + u_2^{[7]} + w^{[7]},
\end{aligned}
$$

with $\|u^{[6]}\|_\infty \leq 1$, $\|u^{[7]}\|_\infty \leq 1.2$, $-0.04 \leq w^{[6]} \leq 0.02$, and $-0.24 \leq w^{[7]} \leq 0.3$.

Finally, the subsystem $\mathbf{S}_8$ consists of three agents, which predominantly *compete*[4] with each other:

$$
\begin{aligned}
\dot{x}_1^{[8]} &= x_1^{[8]}(1 - x_1^{[8]} - x_2^{[8]} - x_3^{[8]}) + u_1^{[8]} + w^{[8]}, \\
\dot{x}_2^{[8]} &= x_2^{[8]}(-0.01 + x_1^{[8]} - 1.1x_3^{[8]}), \\
\dot{x}_3^{[8]} &= x_3^{[8]}(1 - x_1^{[8]} - x_2^{[8]} - x_3^{[8]}) - x_3^{[8]}x_2^{[7]} + u_2^{[8]},
\end{aligned}
$$

with $-0.4 \leq w^{[8]} \leq 0.5$ and $u^{[8]} \in \operatorname{Conv}\{-(2,2), (-2,2), (1.75, 1.8), (1.75, -1.8)\}$.

We choose $V_i(x^{[i]}) = \|x^{[i]}\|^2$, $W_i(x^{[i]}) = 10^{-7}V_i(x^{[i]})$, for all[5] $i \in \mathcal{S}$. Performing the proposed SOS-based calculations with YALMIP Löfberg (2009) and MOSEK MOSEK ApS (2017) shows that these V_i are DiRCLFs for the system with domain $\mathcal{D}_c^{[i]} = \Omega_{V_i,\gamma_i}$ for

$$\gamma_{\text{distr}} = [0.505\ \ 0.4\ \ 0.8\ \ 0.71\ \ 0.3\ \ 0.25\ \ 0.35\ \ 0.58]^\mathsf{T}$$

and also DeRCLFs with a slightly smaller domain corresponding to

$$\gamma_{\text{dec}} = [0.505\ \ 0.35\ \ 0.75\ \ 0.71\ \ 0.2\ \ 0.25\ \ 0.35\ \ 0.58]^\mathsf{T}.$$

[4]All interactions are harmful to the population of all agents, except for the effect of agent 1 on agent 2.

[5]In general, the proposed methods allow each V_i, $i \in \mathcal{S}$, to be different.

The fact that the robust stabilization set for the decentralized case ends up being smaller compared to the distributed case is not surprising, due to increased conservatism in the underlying uncertainty characterization as we saw in Section 6.3.

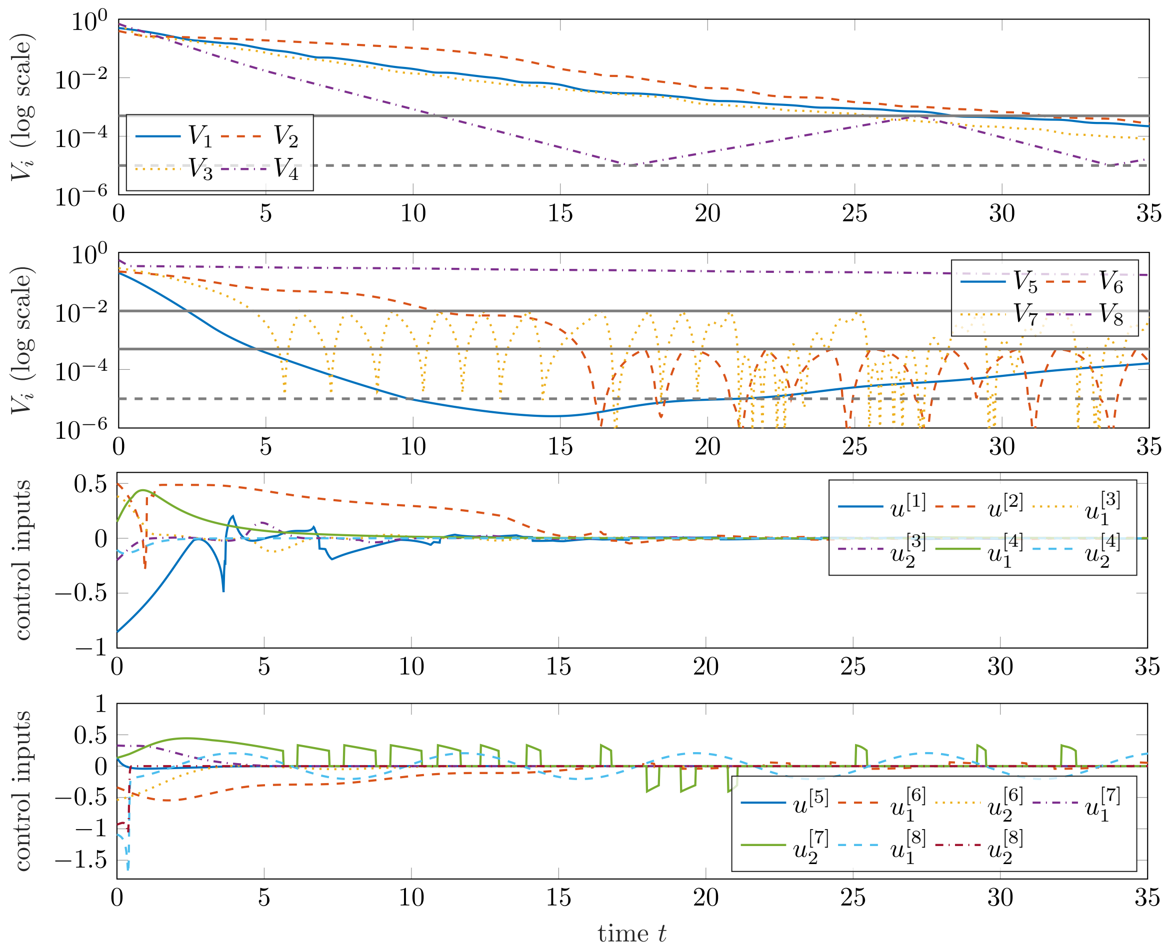


Figure 6.2: Numerical results for the distributed control case.

The performance parameter is set to $\alpha_i(x^{[i]}) = -0.2V_i(x^{[i]})$, for all $i \in \mathcal{S}$. The technique discussed in Section 6.4 is also employed, with $\underline{\xi}_i = 10^{-5}$ for $i \in \mathcal{S}$, $\bar{\xi}_i = 5 \times 10^{-4}$ for $i = 1, \ldots, 6$ and $\bar{\xi}_7 = 10^{-2}$. The closed loop system under both control schemes is simulated from an initial point x_0 close to the boundary of the decentralized control law robust stabilization set, with $x_0 = [0.3, -0.64, -0.38, 0.5, -0.75, 0.39, -0.68,$

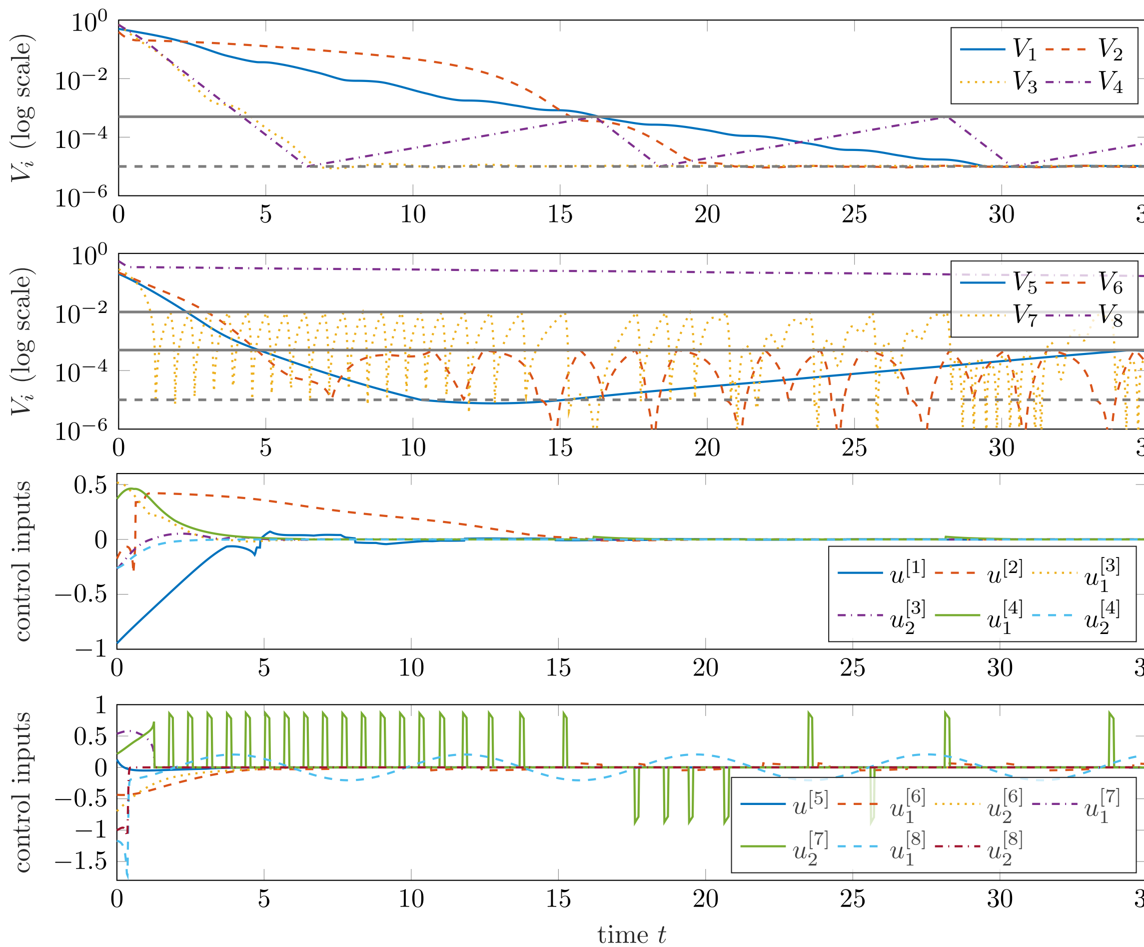


Figure 6.3: Numerical results for the decentralized control case. Observe that the control action is more aggressive compared to the distributed control case illustrated in Figure 6.2, as each decentralized control law is operating on a worst-case assumption with regards to the state of its neighborhood. The latter may not necessarily be the case, but the decentralized control laws have no way of learning about this, in contrast to the distributed case.

$0.48, 0.2, -0.4, 0.25, 0.4, -0.5, -0.2, 0.35, -0.58, 0.3]^{\mathsf{T}}$. The bounded uncertainty w is time varying. Figures 6.2 and 6.3 illustrate the results for the distributed and decentralized control cases, respectively. One can see that both control schemes achieve the robust stabilization control objective. All subsystems appear to converge faster under the decentralized solution, given the underlying worst-case scenario assumed by each individual control law with regards the current state of its neighborhood. The entire neighborhood is not necessarily in this worst-case state, yet, the decentralized robust control laws are based on such an assumption. The distributed control laws, to the contrary, are benefiting from each other's stabilization efforts and progress, since the respective information is communicated across each neighborhood.

6.7 Summary

In this chapter, we have addressed fundamental questions related to the stabilization problem for input constrained, uncertain large-scale nonlinear systems. Our results have been enabled by leveraging the power of vector Lyapunov functions with convex optimization methods and techniques such as SOS programming and rapidly solvable QPs. In particular, we address *from where* (that is, from which initial conditions) and *how* is robust stabilization guaranteed with subsystem-level control laws. The technical results are building directly on the contributions from Chapters 3, 4 and 5 and provide significant improvements in terms of the applicability of the proposed methods.

We have developed our results and the numerical example by assuming that the entire large scale system operates either in a distributed or a decentralized control mode. This was done in order to streamline the notation and the presentation. It is straightforward to also consider a more general case where each subsystem interacts with both cooperative and uncooperative neighbors.

Chapter 7

Control under imperfect state feedback

In this chapter, we turn our attention to the nominal case of the dynamics with no uncertainty, as in Chapters 3 and 4, and we focus on solving the input constrained stabilization problem using imperfect feedback. The solution is enabled by enforcing a causality relationship between apparently (that is, from the point of view of a control law with imperfect measurements) and actually stabilizing the system. The technical results lead to SOS programs that we use to calculate the sets where such a causality relationship holds, and an imperfect feedback control law based on a QP that can provingly stabilize the system in the aforedescribed sense.

7.1 System description

We consider nonlinear control systems of the form

$$\dot{x} = f(x) + g(x)u, \tag{7.1}$$

where $x \in \mathbb{R}^n$ is the state at time $t \geq 0$ with initial value $x(0) = x_0$, $u : [0, \infty) \to \mathbb{U} \subseteq \mathbb{R}^m$ is the control input, and $f : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^{n \times m}$ are known, polynomial functions of the state x with $f(0) = 0$. We assume that the input value set $\mathbb{U}$ is a convex, compact polytope, defined by its vertices $\{v_i \in \mathbb{U}, i \in \mathcal{Q}_{\mathbb{U}}\}$, where $\mathcal{Q}_{\mathbb{U}} \subseteq \mathbb{Z}_{>0}$, and, equivalently, by the halfspace description $Au \preceq b$, for appropriate $A \in \mathbb{R}^{p \times m}$, $b \in \mathbb{R}^p$. The solution of (7.1) for $t \in [0, \tau)$ is denoted by $\phi(t; x_0, u(\cdot))$. The output of the system is

$$y := x + \nu, \tag{7.2}$$

where $\nu : [0, \infty) \to \mathcal{N}_x$ is a continuous measurement disturbance attaining values in the compact set $\mathcal{N}_x \subseteq \mathbb{R}^n$, the description of which may depend continuously on the current state x. In particular, $\mathcal{N}_x$ is assumed to possess a semialgebraic description, which is discussed in detail in Section 7.3. Let $\mathcal{X} \subseteq \mathbb{R}^n$ and consider the set-valued map $\mathcal{Y} : \mathbb{R}^n \times \mathcal{N}_x \rightrightarrows \mathbb{R}^n$ given by

$$\mathcal{Y}(\mathcal{X}, \mathcal{N}_x) := \{x + \nu : x \in \mathcal{X}, \nu \in \mathcal{N}_x\};$$

also, consider its inverse $\mathcal{Y}^{-1} : \mathbb{R}^n \times \mathcal{N}_x \rightrightarrows \mathbb{R}^n$ with

$$\mathcal{Y}^{-1}(\mathcal{X}, \mathcal{N}_x) := \{z : z + \nu \in \mathcal{X}, \nu \in \mathcal{N}_x\}.$$

We use $\mathcal{Y}(\cdot, \mathcal{N}_x)$ to dilate its first argument, let it be a subset of $\mathbb{R}^n$ or a vector $x \in \mathbb{R}^n$, by the -possibly state dependent- uncertainty induced by (7.2). A conceptual illustration is provided by Figure 7.1. Note that in the special case where $\mathcal{N}_x$ does not depend on x, $\mathcal{Y}(\cdot, \mathcal{N}_x)$ and $\mathcal{Y}^{-1}(\cdot, \mathcal{N}_x)$ operate similarly to the Minkowski sum and difference, respectively.

Our control objective is introduced next.

Definition 7.1.1. The system (7.1) is *stabilizable under input constraints using imperfect feedback* if there exists a mapping $u_c(y)$, such that $y \mapsto u_c(y) \in \mathbb{U}$ for any $y \in \mathcal{D}_y$ where $\mathcal{D}_y \subset \mathbb{R}^n$ is a compact set, rendering the closed loop system stable, in the sense that all trajectories $\phi_c(t) = \phi(t; x_0, u_c)$ satisfy $\limsup_{t\to\infty} \|\phi_c(t)\| \leq \epsilon$, for some $\epsilon > 0$ and any $x_0 \in \mathcal{Y}^{-1}(\mathcal{D}_y, \mathcal{N}_x)$.

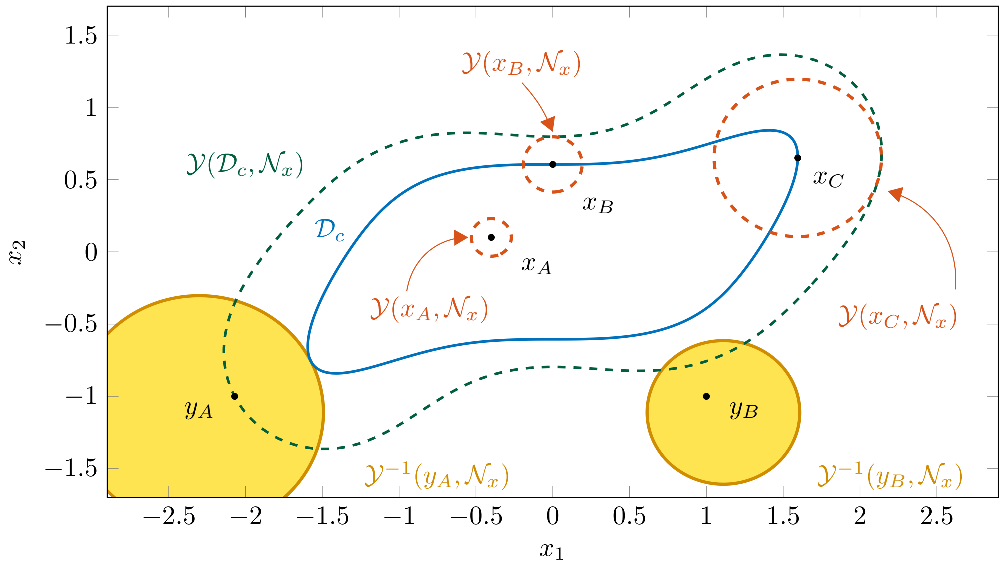


Figure 7.1: Illustration of the action of $\mathcal{Y}(\cdot, \mathcal{N}_x)$ and $\mathcal{Y}^{-1}(\cdot, \mathcal{N}_x)$ for $\mathcal{N}_x := \{\nu \in \mathbb{R}^2 : \|\nu\|^2 \leq 0.1\|x\|^2\}$, on the set $\mathcal{D}_c = \Omega_{V,0.5}$ for $V(x) = (0.3x_1^3 - x_2)^2 + x_2^4$, and the points $x_A = [-0.4 \;\; 0.1]^\mathsf{T}$, $x_B = [0 \;\; 0.605]^\mathsf{T}$, $x_C = [1.5948 \;\; 0.65]^\mathsf{T}$, $y_A = [-2.0704 \;\; -1]^\mathsf{T}$, and $y_B = [1 \;\; -1]^\mathsf{T}$. If the system is actually at x_A, x_B, x_C, or anywhere in the set $\mathcal{D}_c$, it may appear to be anywhere in the respective sets drawn with the dashed line. Conversely, if the system appears to be at either y_A or y_B, it may actually be anywhere in the respective shaded region.

7.2 A causality relationship between apparently and actually stabilizing the system

For notational convenience, we consider the set $\mathcal{V}$ of *candidate Control Lyapunov Functions* consisting of polynomial, positive definite, radially unbounded $V : \mathbb{R}^n \to \mathbb{R}$. We denote the sublevel sets of $V \in \mathcal{V}$ by $\Omega_{V,\gamma} := \{z \in \mathbb{R}^n : V(z) \leq \gamma\}$ and its time derivative along the trajectories of (7.1) given by $\psi(x, u) := \nabla V(x)(f(x) + g(x)u)$. Also, similarly to Chapters 3, 5, and 6, we consider a measure of the generalized distance $\rho(x; V, \gamma)$ of a point $x \in \mathbb{R}^n$ from the sublevel set $\Omega_{V,\gamma}$, as weighed by $V \in \mathcal{V}$, with $\rho(x; V, \gamma) = V(x) - \gamma$, if $x \notin \Omega_{V,\gamma}$, or $\rho(x; V, \gamma) = 0$, otherwise.

Definition 7.2.1. The function $V \in \mathcal{V}$ is an *Imperfect Feedback Control Lyapunov Function* (IF-CLF) for (7.1) if there exist polynomial, positive definite functions W_1, W_2 :

$\mathbb{R}^n \to \mathbb{R}_{\geq 0}$, and (nonempty) sets $\mathcal{X} \subseteq \mathbb{R}^n$ and

$$\mathbb{Y} := \{y \in \mathcal{Y}(\mathcal{X}, \mathcal{N}_x) : \inf_{u \in \mathbb{U}} \psi(y, u) \leq -W_1(y)\},$$

such that $\psi(x, u) \leq -W_2(x)$ holds for all $x \in \mathcal{X}$ and $u \in \overline{\mathbb{U}}(y) := \{u \in \mathbb{U} : \psi(y, u) \leq -W_1(y)\}$, where $y \in \mathcal{Y}(x, \mathcal{N}_x)$.

As we saw in the previous chapters, even under perfect state feedback, input constraints can complicate the stabilization problem significantly. Using a control law $x \mapsto u_c(x) \in \mathbb{U}$ that renders $\psi(x_0, u_c(x_0)) \leq -W(x)$ at some $x_0 \in \mathbb{R}^n$ (where $V \in \mathcal{V}$ is here a Control Lyapunov Function and $W : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is a positive definite polynomial function) and -at least initially- along the ensuing trajectory does not necessarily imply that this trajectory under $u_c(\cdot)$ will not reach parts of the state space where no $u \in \mathbb{U}$ exist to render $\psi(x, u) \leq -W(x)$, or even $\psi(x, u) < 0$. In other words, the set

$$\mathbb{X} := \{x \in \mathbb{R}^n : \inf_{u \in \mathbb{U}} \psi(x, u) \leq -W(x)\}$$

is not necessarily positively invariant under a control law u_c such that $\psi(x, u_c(x)) \leq -W(x)$ holds for all $x \in \mathbb{X}$. In our work on the perfect state feedback case, we address this issue by showing that the set containment $\Omega_{V,\gamma} \subseteq \mathbb{X}$ is sufficient to guarantee the input constrained controllability of the system, so that the origin is asymptotically stable in a Lyapunov sense under an input constrained control law $u_c : \Omega_{V,\gamma} \to \mathbb{U}$ based on the CLF V, that is, conforming to $\psi(x, u_c(x)) \leq -W(x)$ along all controlled trajectories emanating from $x_0 \in \Omega_{V,\gamma}$.

The IF-CLF concept introduced by Definition 7.2.1 is aligned with the aforementioned considerations related to input constrained stabilization and the intended architecture of the control scheme under development. There are two critical aspects that enable the proposed approach. First, according to Definition 7.2.1, V is an IF-CLF for the system if there exist y in the measurement space $\mathbb{Y}$ where using some $u^* \in \mathbb{U}$ to render $\psi(y, u^*) \leq -W_1(y)$ also renders $\psi(x, u^*) \leq -W_2(x)$, for any $x \in \mathcal{Y}^{-1}(y, \mathcal{N}_x)$ where the system may actually be. This causality relationship allows the development of a control law, based on the imperfect feedback y, with the objective of warranting stabilization in a way apparent to itself, that is, by rendering $\psi(y, u) \leq -W_1(y)$. The second critical aspect is related to a notion of invariance which is warranted by the following result.

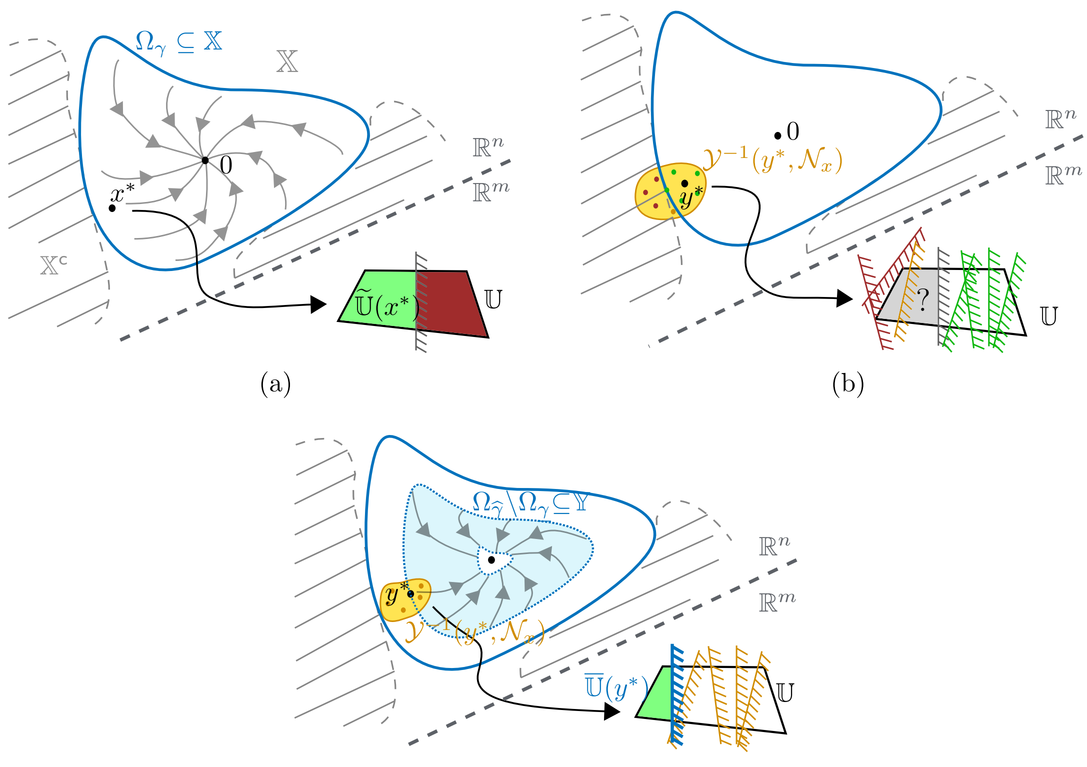
Proposition 7.2.1. *Let V be an IF-CLF for (7.1) and $\overline{\gamma}, \widehat{\gamma}, \underline{\gamma} \in \mathbb{R}_{>0}$, with $\overline{\gamma} \geq \widehat{\gamma} \geq \underline{\gamma} > 0$, be such that*

$$\Omega_{V,\overline{\gamma}} \setminus \Omega_{V,\underline{\gamma}} \subseteq \mathbb{Y}, \tag{7.3a}$$

$$\mathcal{Y}(\Omega_{V,\widehat{\gamma}}, \mathcal{N}_x) \subseteq \Omega_{V,\overline{\gamma}} \tag{7.3b}$$

hold. Then, there exists an imperfect feedback control law $u_c : \mathcal{D}_y \to \mathbb{U}$, where $\mathcal{D}_y := \Omega_{V,\widehat{\gamma}}$, such that the closed loop system is stable, in the sense of Definition 7.1.1.

Proof. Let $\widetilde{u} : \Omega_{V,\widehat{\gamma}} \setminus \Omega_{V,\underline{\gamma}} \to \mathbb{U}$ be any selection function for the set-valued map that maps every $y \in \Omega_{V,\widehat{\gamma}} \setminus \Omega_{V,\underline{\gamma}}$ to the set $\overline{\mathbb{U}}(y)$; note that $\overline{\mathbb{U}}(y)$ is nonempty, by virtue of (7.3a) and Definition 7.2.1. Then, take $u_c(y) := \widetilde{u}(y)$, if $y \in \Omega_{V,\widehat{\gamma}} \setminus \Omega_{V,\underline{\gamma}}$, or $u_c(y) := 0$, if $y \in \Omega_{V,\underline{\gamma}}$. Since V is an IF-CLF for (7.1) and (7.3a) holds, all $u \in \overline{\mathbb{U}}(y)$ render both $\psi(y, u) \leq -W_1(y)$, for all $y \in \Omega_{V,\widehat{\gamma}} \setminus \Omega_{V,\underline{\gamma}}$, and $\psi(x, u) \leq -W_2(x)$, for all $x \in \mathcal{Y}^{-1}(y, \mathcal{N}_x)$. Under the control law u_c and following typical Lyapunov arguments, these two facts along with (7.3b) imply that (i) any trajectory emanating from $x_0 \in \mathbb{R}^n$ such that $\mathcal{Y}(x_0, \mathcal{N}_x) \subseteq \Omega_{V,\widehat{\gamma}}$ will appear to remain in $\Omega_{V,\overline{\gamma}}$, and, additionally, (ii) $\dot{\rho}(x; V, \gamma^*) \leq -W_2(x) < 0$ holds for all $x \in \mathcal{Y}^{-1}(\Omega_{V,\overline{\gamma}} \setminus \Omega_{V,\underline{\gamma}}, \mathcal{N}_x) \setminus \{0\}$, where $\gamma^* := \sup\{\gamma : \Omega_{V,\gamma} \subseteq \mathcal{Y}^{-1}(\Omega_{V,\underline{\gamma}}, \mathcal{N}_x)\}$. Therefore, if the system at $t = 0$ appears to be at $y_0 \in \mathcal{D}_y$, implying that $x_0 \in \mathcal{Y}^{-1}(y_0, \mathcal{N}_x)$, the function $\rho(\phi(t; x_0, u_c); V, \gamma^*)$ is decrescent with regards to time t, and, ultimately, $\rho(\phi(t; x_0, u_c); V, \gamma^*) \to 0$, as $t \to \infty$. We then conclude that the system (7.1), when driven by the input constrained, imperfect feedback control law u_c, is stable, in the sense of Definition 7.1.1, with $\epsilon = \inf\{\sigma > 0 : \overline{\mathcal{B}}^n_\sigma \supseteq \Omega_{V,\gamma^*}\}$. $\square$



(a) (b) (c)

Figure 7.2: (a) In the perfect feedback case, considered in the previous chapters, every $x^* \in \Omega_{V,\gamma} \subseteq \mathbb{X}$ is mapped to the set $\widetilde{\mathbb{U}}(x^*) := \{u \in \mathbb{U} : \psi(x^*, u) \leq -W(x^*)\}$, which is guaranteed to be nonempty. Any $u^* \in \widetilde{\mathbb{U}}(x^*)$ renders $\psi(x^*, u^*) \leq -W(x^*)$. The set $\Omega_{V,\gamma}$ is positively invariant, by typical Lyapunov arguments. Continuous, input constrained feedback control laws can be found by considering any continuous selection function for the set-valued map that maps each x^* to $\widetilde{\mathbb{U}}(x^*)$. (b) Under imperfect feedback, choosing a u in order to render $\psi(y^*, u)$, where y^* is the apparent state of the system, may not necessarily render $\psi(x, u) < 0$, for the actual state $x \in \mathcal{Y}^{-1}(y^*, \mathcal{N}_x)$; in fact, there may not even be such a $u \in \mathbb{U}$. The system cannot be provably stabilized without explicitly considering the effect of the imperfect feedback. (c) The conditions in Proposition 7.2.1 enable the characterization of an invariant set where choosing any u that renders $\psi(y^*, u) \leq -W_1(y)$ guarantees that $\psi(x, u) < 0$ will hold for the actual state $x \in \mathcal{Y}^{-1}(y^*, \mathcal{N}_x)$.

Figure 7.1 illustrates the concepts discussed in this section. The set $\mathcal{D}_y$ considered in Proposition 7.2.1 will be hereafter called an *imperfect feedback stabilization* *set* for the particular system (7.1) and IF-CLF V. A computational approach based on SOS programming to certify whether the various conditions involved in Proposition 7.2.1 hold is presented next.

7.3 Analysis for imperfect state feedback stabilizability

According to Proposition 7.2.1, the set containments (7.3a) and (7.3b) are sufficient for the system to be stabilizable with an input constrained control law that uses imperfect feedback. Nevertheless, given the definition of the set $\mathbb{Y}$, showing these containments is nontrivial, even if the description of the measurement disturbance value set $\mathcal{N}_x$ happens to be relatively simple (as would be the case for an additive bias term, for instance). In this section, we develop techniques to algorithmically verify whether (7.3a) and (7.3b) hold using SOS programming and the S-procedure paradigm.

We first consider a semialgebraic parameterization of $\mathcal{N}_x$. Let $\mathcal{D}^{[j]} := \cap_{i=1}^{\kappa_j}\{x \in \mathbb{R}^n : p_i^{[j]}(x) \leq 0\} \subseteq \mathbb{R}^n$, where $p_i^{[j]} : \mathbb{R}^n \to \mathbb{R}$ are polynomial functions for $i = 1, \ldots, \kappa_j$ and each $j = 1, \ldots, J$, be semialgebraic sets such that $\cup_{j=1}^{J}\mathcal{D}^{[j]} = \mathbb{R}^n$ and $\mathrm{Int}(\mathcal{D}^{[\ell]}) \cap \mathrm{Int}(\mathcal{D}^{[\mu]}) = \varnothing$, for any $\ell, \mu \in \{1, \ldots, J\}$ with $\ell \neq \mu$. The collection of sets $\{\mathcal{D}^{[j]}\}_{i=1}^{J}$ constitutes a partition of the state space $\mathbb{R}^n$, which we employ in order to account for the possibility that the description of the dependence of the measurement disturbance value set $\mathcal{N}_x$ on the state of the system x, in turn, depends on the partition cell $\mathcal{D}^{[j]}$ of the state space $\mathbb{R}^n$ where the system is at time t. In particular, we assume that $\mathcal{N}_x$ is given by

$$\mathcal{N}_x := \begin{cases} \cap_{i=1}^{\lambda_1}\{\nu \in \mathbb{R}^n : \omega_i^{[1]}(\nu; x) \leq 0\}, & \text{if } x \in \mathcal{D}^{[1]}, \\ \vdots & \\ \cap_{i=1}^{\lambda_J}\{\nu \in \mathbb{R}^n : \omega_i^{[J]}(\nu; x) \leq 0\}, & \text{if } x \in \mathcal{D}^{[J]}, \end{cases}$$

where $\omega_i^{[j]} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, for $i = 1, \ldots, \lambda_j$ and each $j = 1, \ldots, J$, are polynomial functions. Using appropriate choices for $\{\mathcal{D}^{[j]}\}_{i=1}^{J}$ and $\mathcal{N}_x$, one can account, for instance, for measurement uncertainty that is proportional to the distance of the measured state from the origin but does not vanish, as well as other cases of practical significance.

Sufficient conditions to show (7.3a) and (7.3b) are now developed by means of a three-part strategy.

Part I: $\Omega_{V,\overline{\gamma}} \setminus \Omega_{V,\underline{\gamma}} \subseteq \mathbb{Y}$

The first part pertains to the existence of $u \in \mathbb{U}$ so that $\psi(y,u) \leq -W_1(y)$, for all $y \in \Omega_{V,\overline{\gamma}} \setminus \Omega_{V,\underline{\gamma}}$; in other words, we attempt to show the containment $\Omega_{V,\overline{\gamma}} \setminus \Omega_{V,\underline{\gamma}} \subseteq \mathbb{Y}$ without accounting, for now, for the requirements and conditions imposed on V and $\mathbb{Y}$ by the IF-CLF definition. To the end of formulating relevant sufficient conditions using the S-procedure, we first consider the following result.

Proposition 7.3.1. *Let* $\eta_i := \nabla^2 V(0) g(0) v_i$ *and, for* $i \in \mathcal{Q}_{\mathbb{U}}$*, consider the sets* $\mathbb{Y}_i := \{y \in \mathbb{R}^n : \psi(y, v_i) \leq -W_1(y)\}$ *and* $\mathbb{Y}_{P_i} := \cap_{j \in \mathcal{Q}_{\mathbb{U}} \setminus \{i\}} \{y \in \mathbb{R}^n : (\eta_i - \eta_j)^\mathsf{T} y \leq 0\}$*. If* $\mathbb{Y}_{P_i} \cap \Omega_{V,\gamma} \subseteq \mathbb{Y}_i$ *holds for all* $i \in \mathcal{Q}_{\mathbb{U}}$*, then* $\Omega_{V,\gamma} \subseteq \mathbb{Y}$*.*

We have proved this proposition, as Proposition 3.4.3, in the state space (that is, with x appearing instead of y). The proof, which employs the characterization of controllability under input constraints based on a partition of the state space induced by the vertices of the input value set $\mathbb{U}$, carries over to the present case in the measurement space. The peculiarities of the imperfect feedback problem are explicitly taken into consideration in Sections 7.1-7.2, and enter the analysis framework through parts II and III.

Following the S-procedure paradigm and by virtue of Proposition 7.3.1, the existence of $c^{[i]}(y) \in \Sigma[y]$, $i \in \mathcal{Q}_{\mathbb{U}}$, and $s_j^{[i]}(y) \in \Sigma[y]$, for $j \in \mathcal{Q}_{\mathbb{U}} \setminus \{i\}$, such that

$$-\psi(y, v_i) - W_1(y) - c^{[i]}(y)(\overline{\gamma} - V(y)) + \sum_{j \in \mathcal{Q}_{\mathbb{U}} \setminus \{i\}} s_j^{[i]}(y)(\eta_i - \eta_j)^\mathsf{T} y \in \Sigma[y] \tag{7.4}$$

holds for all $i \in \mathcal{Q}_{\mathbb{U}}$ implies $\Omega_{V,\overline{\gamma}} \subseteq \mathbb{Y}$.

Part II: additional requirements for $\mathbb{Y}$

We now enforce the causality relationship embedded in the IF-CLF definition. Let $z = [x^\mathsf{T}\ u^\mathsf{T}\ \nu^\mathsf{T}]^\mathsf{T} \in \mathbb{R}^{n+m+n}$. The existence, for each $j = 1, \ldots, J$ (indicating the respective partition cell involved in the parameterization of $\mathcal{N}_x$), of polynomials $s_A^{[j]}(z), s_{\overline{V}}^{[j]}(z), s_{\underline{V}}^{[j]}(z) \in \Sigma[z]$, and $s_{u,k}^{[j]}(z) \in \Sigma[z]$, for $k = 1, \ldots, p$, $s_{\omega,i}^{[j]}(z) \in \Sigma[z]$, for $i = 1, \ldots, \lambda_j$, and $s_{D,\mu}^{[j]}(z) \in \Sigma[z]$, for $\mu = 1, \ldots, \kappa_j$, such that

$$\begin{aligned} -\psi(x,u) - W_2(x) + \sum_{i=1}^{\lambda_j} s_{\omega,i}^{[j]}(z)\omega_i^{[j]}(\nu, x) - \sum_{k=1}^{p} s_{u,k}^{[j]}(z)\hat{e}_k^\mathsf{T}(b - Au) \\ -s_A^{[j]}(z)(-\psi(x+\nu, u) - W_1(x+\nu)) - s_{\underline{V}}^{[j]}(z)(V(x+\nu) - \underline{\gamma}) \\ -s_{\overline{V}}^{[j]}(z)(\overline{\gamma} - V(x+\nu)) + \sum_{\mu=1}^{\kappa_j} s_{D,\mu}^{[j]}(z)p_\mu(x) \in \Sigma[z] \end{aligned} \tag{7.5}$$

holds, implies, in turn, that $\psi(x,u) \leq -W_2(x)$, for all $x \in \mathcal{Y}^{-1}(y, \mathcal{N}_x)$, and $u \in \mathbb{U}$ such that $\psi(y,u) \leq -W_1(y)$.

Part III: $\mathcal{Y}(\Omega_{V,\widehat{\gamma}}, \mathcal{N}_x) \subseteq \Omega_{V,\overline{\gamma}}$

Finally, let $\xi = [x^\mathsf{T}\ \nu^\mathsf{T}]^\mathsf{T} \in \mathbb{R}^{n+n}$ and assume that, for each $j = 1, \ldots, J$, there exist $s_{\widetilde{V}}^{[j]}(\xi) \in \Sigma[\xi]$ and $s_{\widetilde{\omega},i}^{[j]}(\xi) \in \Sigma[\xi]$, for $i = 1, \ldots, \lambda_j$, and $s_{\widetilde{D},\mu}^{[j]}(\xi) \in \Sigma[\xi]$, for $\mu = 1, \ldots, \kappa_j$, such that

$$\bar{\gamma} - V(x+\nu) - s_{\widetilde{V}}^{[j]}(\xi)(\widehat{\gamma} - V(x)) + \sum_{i=1}^{\lambda_j} s_{\widetilde{\omega},i}^{[j]}(\xi)\omega_i^{[j]}(\nu, x) + \sum_{\mu=1}^{\kappa_j} s_{D,\mu}^{[j]}(\xi)p_\mu(x) \in \Sigma[\xi]. \tag{7.6}$$

This implies the set containment (7.3b).

Cumulative sufficient conditions

The existence of all nonnegative polynomials so that all instances of (7.4), (7.5) and (7.6) are satisfied is sufficient for the set containments (7.3a) and (7.3b) to hold for the particular V, W_1, W_2, $\overline{\gamma}$, $\widehat{\gamma}$ and $\underline{\gamma}$. When the set of nonnegative polynomials $\Sigma[\cdot]$ is parameterized as SOS, every instance of (7.4), (7.5) and (7.6) is equivalent, in turn, to LMIs in auxiliary variables, the satisfaction of which can be determined by the solution of the corresponding SDP feasibility problems.

Since $W_2(x)$, which determines the actual minimum rate of convergence of the closed loop system (in terms of the decrease rate of the IF-CLF along the actual trajectories), appears affinely in (7.5), one can parameterize it as an unknown, by letting $W_2(x) - \epsilon x^{\mathsf{T}} x \in \Sigma[x]$, for $0 < \epsilon \ll 1$. The rest of the parameters of interest, that is, V, W_1, $\overline{\gamma}$, $\widehat{\gamma}$ and $\underline{\gamma}$ have to be provided as constants, since they are multiplied with nonnegative polynomials which correspond to unknowns of the problem in the S-procedure paradigm, when the SOS constraints are parsed into LMIs. In absence of a more elaborate guess, a convenient parameterization of $W_1(y)$, such as $W_1(y) = cV(y)$ for $c \in \mathbb{R}_{>0}$, can help add more structure to the search space and the problem, in general, and one may be able to perform a bisection along either of c, $\overline{\gamma}$, $\widehat{\gamma}$, or $\underline{\gamma}$, with the rest remaining constant.

7.4 A QP-based control law operating on imperfect state feedback

At any point $y \in \Omega_{V,\widehat{\gamma}} \setminus \Omega_{V,\underline{\gamma}}$, for $\widehat{\gamma}$, $\underline{\gamma}$ such that the conditions of Proposition 7.2.1 are satisfied, the mapping $y \mapsto \overline{\mathbb{U}}(y) := \{u \in \mathbb{U} : \psi(y, u) \leq -W_1(y)\}$ provides a parameterization of the stabilizing control laws for a system with imperfect feedback stabilization set $\mathcal{D}_y = \Omega_{V,\widehat{\gamma}}$. Following Proposition 7.2.1, the conditions of which are verifiable using the computational analysis method developed in Section 7.3, the set $\overline{\mathbb{U}}(y)$ is nonempty. We next show a way to obtain a particular imperfect feedback control law, that is, a mapping from any $y \in \mathcal{D}_y$ to a $u \in \overline{\mathbb{U}}(y)$.

Consider, at every $y \in \mathcal{D}_y$, a measure of the apparent (that is, from the point of view of a control law having knowledge only of y) stabilization performance gap $\mathcal{H}$, in terms of the apparent decrease rate of the IF-CLF with regards to a reference value given by the negative definite, continuous function $\alpha : \mathbb{R}^n \to \mathbb{R}_{\leq 0}$, with $\mathcal{H}(u; y) := (\psi(y, u) - \alpha(y))^2$. Dropping terms which do not contain u leads to an expression quadratic in u with the current measured state y appearing as a parameter,

$$\mathcal{J}(u; y) = u^{\mathsf{T}}[Q(y) + \mu(y)I_m]u + L(y)u, \tag{7.7}$$

where $L(y) := 2\nabla V(y)g(y)[\nabla V(y)f(y) - \alpha(y)]$, $Q(y) := (\nabla V(y)g(y))^{\mathsf{T}}\nabla V(y)g(y)$, and $\mu(y)$ is used to minimally shift the spectrum of $Q(y)$, so that the smallest eigenvalue of the resulting matrix exceeds a given strictly positive threshold and, thus, $Q(y) + \mu(y)I_m \succ 0$ for any $y \in \mathbb{R}^n$.

Proposition 7.4.1. *Assume that V, $\widehat{\gamma}$ and $\underline{\gamma}$ are such that Proposition 7.2.1 holds. For any $y \in \Omega_{V,\widehat{\gamma}} \setminus \Omega_{V,\underline{\gamma}}$, consider the linearly constrained quadratic program of using u to minimize $\mathcal{J}(u; y)$, given by (7.7), subject to*

$$\nabla V(y)g(y)u \leq -\nabla V(y)f(y) - W_1(y), \tag{7.8a}$$

$$Au \preceq b, \tag{7.8b}$$

and denote its solution by $u^*(y)$. The control law $u_c : \Omega_{V,\widehat{\gamma}} \to \mathbb{U}$, with

$$u_c(y) := \begin{cases} u^*(y) & \text{if } y \in \Omega_{V,\widehat{\gamma}} \setminus \Omega_{V,\underline{\gamma}}, \\ 0 & \text{otherwise,} \end{cases}$$

renders the closed loop system stable, in the sense of Definition 7.1.1.

Proof. For any $y \in \Omega_{V,\widehat{\gamma}} \backslash \Omega_{V,\underline{\gamma}}$, the minimizer $u^*(y)$ of the QP exists and is unique; to see that, note that the domain of the QP, described by (7.8a) and (7.8b), or, equivalently, by $\overline{\mathbb{U}}(y)$, is nonempty, whereas $\mathcal{J}(u; y)$ is a strictly convex function of u. Therefore, the control law is well defined. Since the minimizer $u^*(y)$ satisfies $u^*(y) \in \mathbb{U}$ and $\psi(y, u^*) \leq -W_1(y)$, by virtue of Definition 7.2.1 and Proposition 7.2.1, $\psi(x, u^*(y)) \leq -W_2(x)$ holds, for any $x \in \mathcal{Y}^{-1}(\Omega_{V,\widehat{\gamma}} \setminus \Omega_{V,\underline{\gamma}}, \mathcal{N}_x)$. Therefore, the closed loop system under the proposed control law is stable. $\square$

The proposed control law is an adaptation of the control law we proposed for the perfect state feedback case in Chapter 4. At first sight, the control law assumes a similar form for the herein considered imperfect feedback, input constrained case; however, all stabilization guarantees in the present context stem from the fact that V is an IF-CLF and the conditions that W_1, $\widehat{\gamma}$ and $\underline{\gamma}$ provably satisfy, given the successful completion of the methods of Section 7.3.

When the (measured) trajectory enters $\Omega_{V,\underline{\gamma}}$, no control is applied, and the trajectory then appears to be about to exit $\Omega_{V,\underline{\gamma}}$, chattering may occur in order for the control law keep the trajectory confined in $\Omega_{V,\underline{\gamma}}$ and in agreement with the pursued stability notion. One can avoid chattering with simple techniques, similarly to our approach in Chapter 5.

7.5 Numerical example

We consider the nonlinear system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -0.5x_1^2 - x_2 + u,$$

with $-2 \leq u \leq 4$, and measurements $y = x + \nu$ with $\|\nu\| \leq 0.0548\|x\|$, if $\|x\| \geq 0.7071$, or $\|\nu\| \leq 0.0387$, if $\|x\| < 0.7071$. The perfect state feedback variant of this system has been one of the example cases used in the preceding chapters given its aggressive dynamics. Uncontrolled trajectories can exhibit a finite escape time, whereas the limited control authority that results from the constrained input precludes global stabilization even with state feedback. It is thus critical to explicitly characterize a subset of the measurement space from where input constrained stabilization is guaranteed with a IF-CLF-based control law.

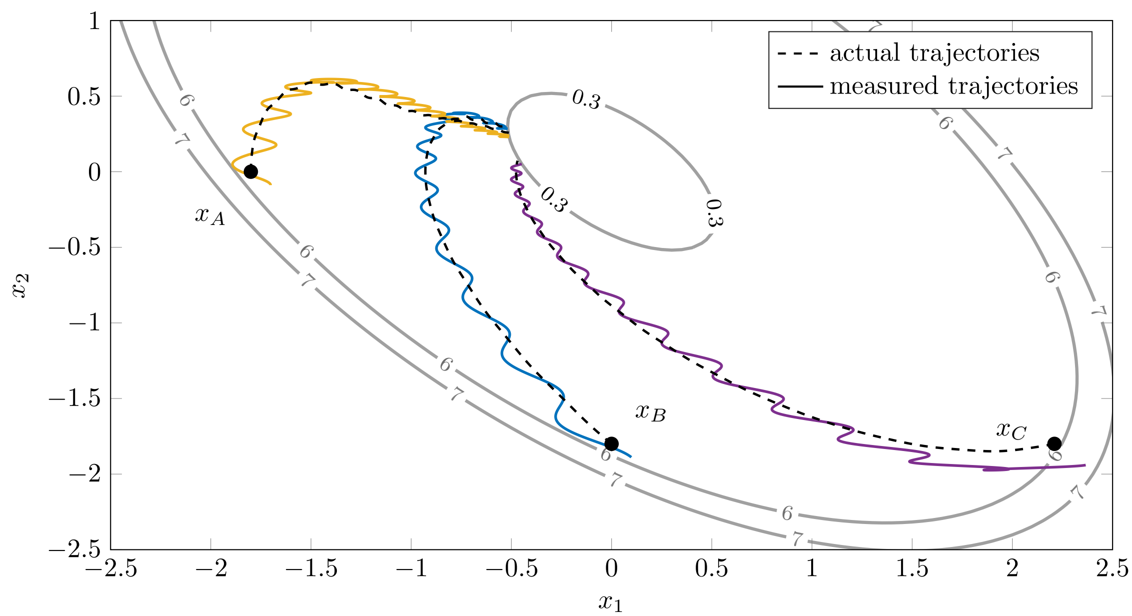


Figure 7.3: Closed loop trajectories on the phase plane, with regards to the sublevel sets of interest $\Omega_{V,\overline{\gamma}}$, $\Omega_{V,\widehat{\gamma}}$, and $\Omega_{V,\underline{\gamma}}$, for $\overline{\gamma} = 7$, $\widehat{\gamma} = 6$, and $\underline{\gamma} = 0.3$, respectively.

Assuming that the unknown nonnegative polynomial multipliers appearing in (7.4), (7.5) and (7.6) are SOS of degree no greater than 8, parsing the SOS constraints into SDP feasibility problems with YALMIP (Löfberg, 2009) and numerically solving the latter with MOSEK (MOSEK ApS, 2017) shows that $V(x) = 1.7x_1^2 + 2x_1x_2 + 1.7x_2^2$ is an IF-CLF for the system, with $\overline{\gamma} = 7$, $\widehat{\gamma} = 6$, $\underline{\gamma} = 0.3$ and $W_1(y) = 0.83V(y)$. Three closed loop trajectories are propagated until the measured state reaches $\Omega_{V,\underline{\gamma}}$, under a time-varying $\nu \in \mathcal{N}_x$ and using the proposed control law, from $x_A = [0 \ \ -1.8]^\mathsf{T}$,

$x_B = [-1.8 \;\; 0]^\mathsf{T}$ and $x_C = [2.21 \;\; -1.8]^\mathsf{T}$, for $\alpha_A(y) = -1.1V(y)$, $\alpha_B(y) = -0.9V(y)$ and $\alpha_C(y) = -V(y)$, respectively. The trajectories are illustrated on the phase-plane in Figure 7.3, while the time histories of the control inputs and the IF-CLF values are given, respectively, in Figures 7.4 and 7.5.

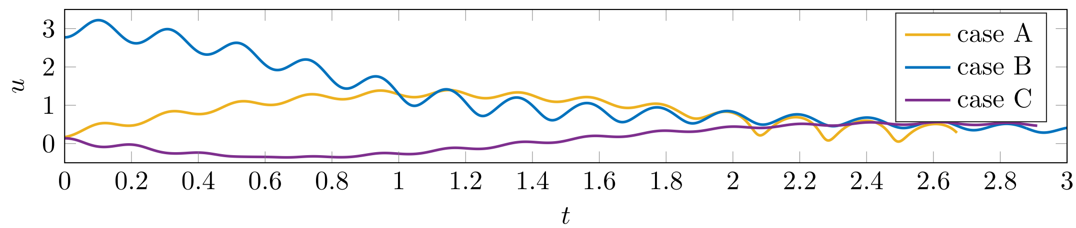


Figure 7.4: Control input u as a function of time t for each of the initial conditions.

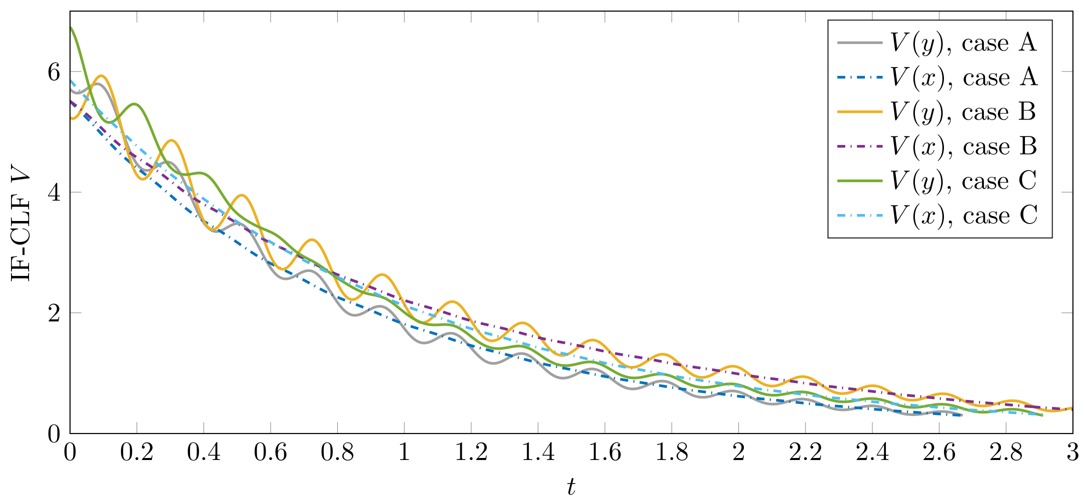


Figure 7.5: The apparent values of the IF-CLF, that is, $V(y)$, are oscillatory and not monotonically decreasing as functions of time under the effect of the measurement disturbances. The values of the IF-CLF V are monotonically decreasing along the actual trajectories of the system, as expected from the proposed imperfect feedback control law.

One can observe that the results are in agreement with the imperfect feedback stabilization notion provided by the proposed control solution.

7.6 Summary

Alongside input constraints, measurement disturbances that result in an imperfect state feedback signal are ubiquitous in control systems and can negatively affect a system's closed loop performance or, even worse, destabilize it. In this chapter, we presented a solution framework based on convex optimization and, in particular, sum-of-squares techniques, to analyze and implement imperfect feedback QP-based control laws that can provably stabilize nonlinear polynomial systems subject to input constraints.

Chapter 8

Robust observer design

Equipped with our technical results on the robust stabilization problem from Chapter 5, we now propose a simple method to design robust Lyapunov-based nonlinear observers. Our work extends earlier results from the literature (Tsinias, 1989, 1990) to the class of systems with uncertainty in the dynamics and the (linear) measurements. On a slightly different note compared to the previous chapters, we will use SOS methods to synthesize offline a robust dynamical observer. Our solution retains the attractive features of Lyapunov observers (easy design, straightforward performance characterization, possibly wider applicability than other nonlinear observers), while the use of SOS programming allows us to simplify the design even further by avoiding analytical calculations even in the much harder case of uncertain systems.

8.1 System description

We consider nonlinear, uncertain control systems with dynamics given by

$$\dot{x} = f(x) + g_1(x)u + g_2(x)w, \qquad x(0) = x_0, \tag{8.1}$$

and uncertain measurements of the form

$$y = Hx + \nu, \tag{8.2}$$

where $x \in \mathbb{R}^n$ is the state vector at time $t \geq 0$ with initial value $x_0 \in \mathbb{R}^n$, u is the control input attaining values in the compact, convex polytope $\mathbb{U} \subseteq \mathbb{R}^m$, w and

ν are unknown but bounded modeling and measurement, respectively, disturbances, attaining values in the compact, convex polytopes $\mathbb{W} \subseteq \mathbb{R}^r$ and $\mathcal{N} \subseteq \mathbb{R}^k$, described by their vertices $z_k \in \mathbb{W}$, for $k \in \mathcal{Q}_{\mathbb{W}}$, and $\beta_\ell \in \mathcal{N}$, for $\ell \in \mathcal{Q}_{\mathcal{N}}$. The mappings $f : \mathbb{R}^n \to \mathbb{R}^n$, $g_1 : \mathbb{R}^n \to \mathbb{R}^{n \times m}$ and $g_2 : \mathbb{R}^n \to \mathbb{R}^{n \times r}$ are known, polynomial functions of the state x, while $H \in \mathbb{R}^{k \times n}$ is a known, constant matrix, with $k \leq n$. One can use w to account for modeling uncertainty in the dynamics or exogenous disturbances and ν for measurement irregularities such as errors and bias. We assume that u, w and ν are, in general, functions of time t, attaining values in the respective compact sets, and can contain discontinuities. The solution of (8.1) at time $t \in [0, \tau)$ for some $\tau > 0$ is denoted by $\phi_x(t) = \phi_x(t; x_0, u, w)$ and is assumed to exist in the sense described in Chapter 5. For the observer design problem to be well posed, we assume that $\phi_x(t)$ exists for all $t \in [0, \infty)$ and, in fact, $\phi_x(t) \in \mathcal{D}_x$ for all $t \in [0, \infty)$, where $\mathcal{D}_x \subseteq \mathbb{R}^n$ is a compact set.

In the spirit of Tsinias (1989), let $\eta \in \mathbb{R}^n$ be the state of the sought-after observer at time $t \geq 0$, with dynamics given by

$$\dot{\eta} = f_o(\eta, u) + R(\eta, u)y, \qquad \eta(0) = \eta_0, \tag{8.3}$$

where u is the input of the observed system (8.1), y is the measured output given by (8.2), and $R : \mathbb{R}^n \times \mathbb{U} \to \mathbb{R}^{n \times k}$ is a matrix-valued polynomial function. Note that in contrast to Tsinias (1989), where the matrix R was only a function of u, here we allow R to depend on the state of the observer, also.

At any time $t \geq 0$, the state determination error is $e := x - \eta$. The goal of our robust observer design process is to find $f_o(\eta, u)$ and $R(\eta, u)$ so that e ultimately converges to a neighborhood of $e = 0$, regardless of the effect of the measurement disturbance ν and the action of the disturbance w in the system's dynamics. In absence of uncertainty (that is, when $w = 0$ and $\nu = 0$), the origin $e = 0$ is required to be an equilibrium of the error dynamics. This fact allows us to determine the form of the function f_o, that is,

$$f_o(\eta, u) := f(\eta) + g_1(\eta)u - R(\eta, u)H\eta \tag{8.4}$$

Using (8.1), (8.2), and (8.3), the state determination error dynamics can be written as

$$\dot{e} = f(e+\eta) - f(\eta) + (g_1(e+\eta) - g_1(\eta))\,u + g_2(e+\eta)w - R(\eta, u)He - R(\eta, u)\nu. \tag{8.5}$$

The solution of (8.5) at time $t \in [0, \tau)$ for some $\tau > 0$ is denoted by $\phi_e(t; e_0, \phi_x, \nu)$, where $e_0 = x_0 - \eta_0$, and is assumed to exist in the previously described sense, given the possibility for discontinuities in u, w and ν.

8.2 Lyapunov analysis for robust state observation

For notational convenience and similarly to the previous chapters, we consider the set $\mathcal{V}$ of all polynomial functions $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that $V(0) = 0$, $V(x) > 0$ for all $x \neq 0$, and $\lim_{\|x\| \to \infty} V(x) = \infty$.

Proposition 8.2.1. *Let $V_\epsilon \in \mathcal{V}$ and denote its time derivative along the trajectories of (8.5) by $\psi_e(e, \eta, u, w, \nu)$. Also, let $V_o, W_\epsilon \in \mathcal{V}$. Assume $R(\eta, u)$ is such that*

$$\sup_{(w,\nu) \in \mathbb{W} \times \mathcal{N}} \psi_e(e, \eta, u, w, \nu) \leq -W_\epsilon(e), \tag{8.6}$$

holds for all e such that $\underline{\epsilon} \leq V_\epsilon(e) \leq \overline{\epsilon}$ and all η such that $V_o(\eta) \leq \overline{\xi}$, for some $\overline{\epsilon}, \underline{\epsilon}, \overline{\xi} \in \mathbb{R}$ with $\overline{\epsilon} > \underline{\epsilon} \geq 0$ and $\overline{\xi} > 0$. Then, for any e_0 such that $\underline{\epsilon} \leq V_\epsilon(e_0) \leq \overline{\epsilon}$, the ensuing state determination error trajectories satisfy

$$\limsup_{t \to \infty} \|\phi_e(t; e_0, \phi_x, \nu)\| \leq \hat{\epsilon}, \tag{8.7}$$

for $\hat{\epsilon} = \{\min \sigma : \Omega_{V_\epsilon, \underline{\epsilon}} \subseteq \mathcal{B}^n_\sigma\}$.

Proof. It suffices to see that $V_\epsilon(e)$ and $W_\epsilon(e)$ serve as a Lyapunov pair for the state determination error dynamics (8.5) and that the latter system is robustly stable. In particular, for any e_0 such that $\underline{\epsilon} \leq V_\epsilon(e_0) \leq \bar{\epsilon}$, by typical Lyapunov arguments and similarly to the proof of Lemma 5.2.1, the function $\rho\left(\phi_e\left(t; e_0, \phi_x, \nu\right); V_\epsilon, \underline{\epsilon}\right)$ is decreasing monotonically and, also $\rho \to 0$ as $t \to \infty$, ultimately implying (8.7). $\square$

It is easy to notice that Proposition 8.2.1 exhibits some parallelism to our main result for the robust stabilization problem (that is, Lemma 5.2.1). This is not surprising, given the often documented connections between state observation and control. In both cases, we are guaranteeing an applicable stability notion for the underlying dynamics, which is the system's dynamics (5.1) in the robust stabilization case and the state determination error dynamics (8.5) in the observer case. Given the presence of persisting (that is, non-vanishing) disturbances in both the observed system's dynamics and its measurements, and the design philosophy of the proposed observer, the underlying stability notion is robust stability (as opposed to robust practical stability for the control case), that is, one can expect to converge to a neighborhood of $e = 0$, however, there can be a lower bound on the size of that neighborhood.

As we mentioned in Section 1.2.4 of the Introduction, one of the attractive features of Lyapunov observers is the ease with which one can characterize the behavior of the state determination error e. This is also true in our proposed robust observer for uncertain systems. Using typical comparison principle results (see, for instance, Theorem 4.16 in Haddad and Chellaboina (2008)), it is straightforward to derive a worst-case (with regards to the exact but unknown e_0 and the action of the disturbances w, ν) bound for e.

Lemma 8.2.2. *Let $W_\epsilon(e) = cV_\epsilon(e)$, where $c \in \mathbb{R}_{>0}$ and assume that the conditions of Proposition 8.2.1 hold. For any initial state determination error e_0 such that $\underline{\epsilon} \leq$*

$V_\epsilon(e_0) \le \bar{\epsilon}$, *the state determination error trajectories* $\phi_e(t; e_0, \phi_x, \nu)$ *satisfy*

$$V_\epsilon\left(\phi_e(t; e_0, \phi_x, \nu)\right) \le \begin{cases} \bar{\epsilon} e^{-ct}, & t \in [0, t^*), \\ \underline{\epsilon}, & t \in [t^*, \infty), \end{cases} \tag{8.8}$$

where

$$t^* = c^{-1} \log\left(\bar{\epsilon}/\underline{\epsilon}\right). \tag{8.9}$$

Proof. By virtue of Proposition 8.2.1, for any $e \in \mathbb{R}^n$ such that $\underline{\epsilon} \le V_\epsilon(e) \le \bar{\epsilon}$ and any η such that $V_o(\eta) \le \bar{\xi}$, $u \in \mathbb{U}$, $w \in \mathbb{W}$ and $\nu \in \mathcal{N}$,

$$\dot{V}_e = \psi(e, \eta, u, w, \nu) \le -cV_\epsilon(e).$$

Consequently, for all $t \in [0, t^*)$, for some $t^* > 0$, the state determination error trajectories satisfy

$$\begin{aligned} V_\epsilon\left(\phi_e(t; e_0, \phi_x, \nu)\right) &\le V(e_0) \times \exp(-ct) \\ &\le \bar{\epsilon} \times \exp(-ct). \end{aligned}$$

In a worst-case scenario with regards to the effect of the uncertainty on the system, an upper bound τ^* on the time t^* satisfies $\underline{\epsilon} = \bar{\epsilon} \times \exp(-c\tau^*)$, which yields (8.9). By virtue of Proposition 8.2.1, once the trajectory $\phi_e(t; e_0, \phi_x, \nu)$ enters the sublevel set $\Omega_{V_\epsilon, \underline{\epsilon}}$ it remains there for all subsequent t, implying (8.8). □

Remark 9. We will hereafter refer to the set $\Omega_{V_\epsilon, \bar{\epsilon}}$, for $\bar{\epsilon}$ satisfying the conditions given in Proposition 8.2.1, as the *region of attraction* for the proposed observer.

8.3 Robust Lyapunov observer synthesis with SOS

In the observer design problem, it is of obvious interest to find the matrix $R(\eta, u)$ which determines the observer's dynamics. While searching for $R(\eta, u)$, it can also be of interest to maximize $\overline{\epsilon}$ and $\overline{\xi}$, while minimizing $\underline{\epsilon}$, in order to enlarge the observer's region of attraction while shrinking the neighborhood of the origin where the state determination error is guaranteed to converge over time. An additional consideration is to accelerate the convergence rate of the observer. An intuitive way to achieve that is by tuning the function $W_\epsilon(e)$ determining the worst-case convergence rate, which is particularly straightforward especially in the case where we let $W_\epsilon(e) = cV_\epsilon(e)$ and we try to maximize $c \in \mathbb{R}_{>0}$. All these considerations can directly enter our proposed methodology, which is based the generalized S-procedure. First, we consider the following useful result.

Lemma 8.3.1. *For any given e such that $\underline{\epsilon} \leq V_\epsilon(e) \leq \overline{\epsilon}$, η such that $V_o(\eta) \leq \overline{\xi}$ and $u \in \mathbb{U}$,*

$$\max_{(\kappa,\ell)\in\mathcal{Q}_{\mathbb{W}}\times\mathcal{Q}_{\mathcal{N}}} \psi_e(e, \eta, u, z_\kappa, \beta_\ell) \leq -W_\epsilon(e) \tag{8.10}$$

is equivalent to (8.6).

Proof. The function $\psi_e(e, \eta, u, z_\kappa, \beta_\ell)$ is jointly continuous in its arguments, which take vales in compact sets. Therefore, its supremum is attained. Additionally, ψ_e is affine in w and ν, where the latter attain values in the convex, compact polytopes $\mathbb{W}$ and $\mathcal{N}$, respectively, so Lemma 5.3.1 holds and leads to the claimed equivalence. $\square$

Similarly to our approach in Section 5.3, we can guarantee that (8.10) holds for some given e, η, and u by requiring that $\psi_e(e, \eta, u, z_\kappa, \beta_\ell) \leq -W_\epsilon(e)$ holds for all disturbance value set vertices $(\kappa, \ell) \in \mathcal{Q}_{\mathbb{W}} \times \mathcal{Q}_{\mathcal{N}}$. The latter observation enables us to invoke the S-procedure to the end of deriving sufficient conditions for the results of Proposition 8.2.1 to hold. In particular, let $\zeta := [\eta^\mathsf{T} \quad e^\mathsf{T} \quad u^\mathsf{T}]^\mathsf{T} \in \mathbb{R}^{n+n+m}$. The sufficient conditions assume the form

$$
\begin{aligned}
-\psi_e(e,\eta,u,z_k,\beta_\ell) - W_\epsilon(e) - s_{\overline{z}}^{[\kappa,\ell]}(\zeta)(\overline{\xi} - V_o(\eta))& \\
-\sum_{i=1}^{p} s_u^{[\kappa,\ell,i]}(\zeta)\hat{e}_i^{\mathsf{T}}(b - Au)& \\
-s_{\overline{\epsilon}}^{[\kappa,\ell]}(\zeta)(\overline{\epsilon} - V_\epsilon(e))& \\
-s_{\underline{\epsilon}}^{[\kappa,\ell]}(\zeta)(V_\epsilon(e) - \underline{\epsilon}) &\in \Sigma[\zeta],\ \forall(\kappa,\ell) \in \mathcal{Q}_{\mathbb{W}} \times \mathcal{Q}_{\mathcal{N}}, \quad (8.11)\\
s_{\overline{z}}^{[\kappa,\ell]}(\zeta) &\in \Sigma[\zeta],\ \forall(\kappa,\ell) \in \mathcal{Q}_{\mathbb{W}} \times \mathcal{Q}_{\mathcal{N}}, \quad (8.12)\\
s_{\overline{\epsilon}}^{[\kappa,\ell]}(\zeta) &\in \Sigma[\zeta],\ \forall(\kappa,\ell) \in \mathcal{Q}_{\mathbb{W}} \times \mathcal{Q}_{\mathcal{N}}, \quad (8.13)\\
s_{\underline{\epsilon}}^{[\kappa,\ell]}(\zeta) &\in \Sigma[\zeta],\ \forall(\kappa,\ell) \in \mathcal{Q}_{\mathbb{W}} \times \mathcal{Q}_{\mathcal{N}}, \quad (8.14)\\
s_u^{[\kappa,\ell,i]}(\zeta) &\in \Sigma[\zeta],\ \forall(\kappa,\ell,i) \in \mathcal{Q}_{\mathbb{W}} \times \mathcal{Q}_{\mathcal{N}} \times [1,\ldots,p]. \quad (8.15)
\end{aligned}
$$

Remark 10. The satisfaction of the conditions appearing in Proposition 8.2.1, which is numerically verifiable by solving the corresponding SOS program, is sufficient for robust observability of the system in the herein described sense and with an observer that is synthesized via the same process. We do not provide separate observability conditions (as, for instance, is done in the nominal case presented in Tsinias (1989)) since our overall approach is less analytical and more numerical than the latter results, and its success, which implies the pursued observability notion, depends on the solution of the underlying SOS program.

Remark 11. The scenario which we will consider next involves taking $W_\epsilon(e) = cV_\epsilon(e)$ and attempting to maximize c, while keeping all other parameters constant. This would correspond to an attempt to synthesize an observer with the fastest convergence rate under the given uncertainty margins. Alternatively, one can also try to perform bisections to enlarge either of the sublevel set bounds appearing in Proposition 8.2.1.

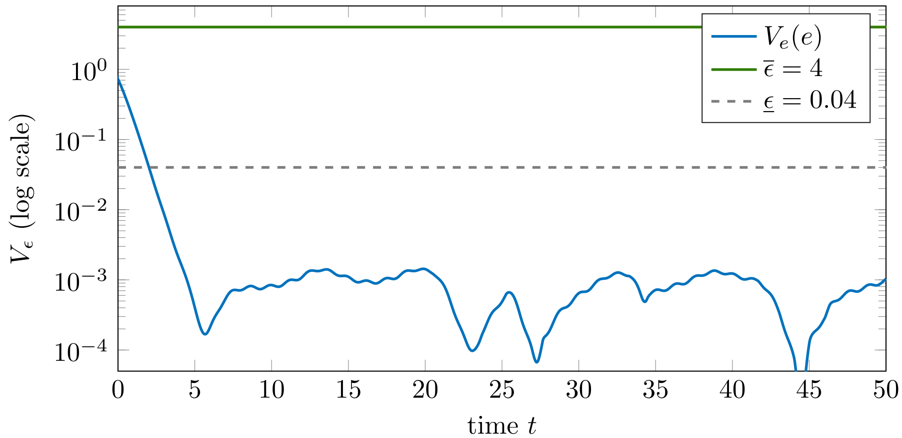


Figure 8.1: The state determination error e is initially within the sublevel set $\Omega_{V_\epsilon,\overline{\epsilon}}$ and it rapidly converges to $\Omega_{V_\epsilon,\underline{\epsilon}}$, where it subsequently remains.

8.4 Numerical example

We consider the following uncertain predator - prey dynamics, given by

$$\dot{x}_1 = -x_1 x_2 - x_2 + u + w,$$

$$\dot{x}_2 = x_1 + x_1 x_2,$$

where, in the absence of the control input u and the uncertainty w, the equilibrium at the origin corresponds to a coexistence condition between the prey (x_1) and the predator (x_2) populations. We let u and w be subject to $-1 \leq u \leq 1$ and $-0.02 \leq w \leq 0.01$. The measurements of the system are given by $y = x_1 + \nu$, where ν is a measurement disturbance satisfying $-0.01 \leq \nu \leq 0.02$. We take $V_\epsilon(e) = e^\mathsf{T} P e$ and $V_o = \eta^\mathsf{T} P \eta$, for

$$P = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

For $\overline{\epsilon} = 4$, $\underline{\epsilon} = 0.04$ and $\overline{z} = 0.9$, we parameterize the worst-case performance bound for the state determination error dynamics as $W_\epsilon(e) = cV_\epsilon(e)$ for $c > 0$, and we use the proposed SOS method in order to synthesize the robust Lyapunov observer while maximizing the performance parameter c. Searching for the involved unknown

SOS polynomials of degree no greater than 4 for the terms in $R(\eta, u)$ and no greater than 2 for the rest of the rest of the unknowns in the S-procedure expression with YALMIP (Löfberg, 2009) and MOSEK (MOSEK ApS, 2017) yields $c = 0.0087$ and

$$R(\eta, u) = \left[\frac{\begin{gathered} 8.02 \times 10^{-7}u^4 - 4.62 \times 10^{-5}u^3\eta_1 + 1.86 \times 10^{-6}u^3\eta_2 + 2.0710^{-5}u^3 \\ +0.0189u^2\eta_1^2 + 0.0257u^2\eta_1\eta_2 - 0.0145u^2\eta_1 + 0.0258u^2\eta_2^2 \\ +0.001u^2\eta_2 + 0.00526u^2 + 0.0408u\eta_1^3 \\ +0.231u\eta_1^2\eta_2 + 0.266u\eta_1^2 + 0.0724u\eta_1\eta_2^2 + 0.27u\eta_1\eta_2 \\ -0.248u\eta_1 - 0.252u\eta_2^3 + 0.557u\eta_2^2 + 0.16u\eta_2 \\ +0.0959u + 10.7\eta_1^4 + 3.63\eta_1^3\eta_2 - 5.88\eta_1^3 + 13.5\eta_1^2\eta_2^2 \\ +8.6\eta_1^2\eta_2 - 2.29\eta_1^2 - 3.73\eta_1\eta_2^3 + 1.78\eta_1\eta_2^2 \\ +2.71\eta_1\eta_2 - 6.49\eta_1 + 15.1\eta_2^4 - 5.53\eta_2^3 + 4.53\eta_2^2 \\ +3.2\eta_2 + 13.8 \end{gathered}}{\begin{gathered} -4.01 \times 10^{-7}u^4 + 2.31 \times 10^{-5}u^3\eta_1 - 9.29 \times 10^{-7}u^3\eta_2 - 1.03 \times 10^{-5}u^3 \\ -0.00943u^2\eta_1^2 - 0.0129u^2\eta_1\eta_2 + 0.00724u^2\eta_1 - 0.0129u^2\eta_2^2 \\ -5.07 \times 10^{-4}u^2\eta_2 - 0.00259u^2 - 0.0204u\eta_1^3 - 0.115u\eta_1^2\eta_2 \\ -0.131u\eta_1^2 - 0.0362u\eta_1\eta_2^2 - 0.133u\eta_1\eta_2 + 0.117u\eta_1 \\ +0.126u\eta_2^3 - 0.262u\eta_2^2 - 0.0812u\eta_2 - 0.0468u - 5.37\eta_1^4 \\ -1.81\eta_1^3\eta_2 + 2.92\eta_1^3 - 6.77\eta_1^2\eta_2^2 - 4.52\eta_1^2\eta_2 \\ +1.27\eta_1^2 + 1.87\eta_1\eta_2^3 - 0.271\eta_1\eta_2^2 - 1.39\eta_1\eta_2 \\ +2.24\eta_1 - 7.57\eta_2^4 + 2.41\eta_2^3 - 1.62\eta_2^2 - 1.36\eta_2 - 6.53 \end{gathered}}\right].$$

We now perform a numerical simulation to illustrate the efficacy of the proposed robust observer. The system is initially at $x_0 = [0\ 0.6]^\mathsf{T}$, whereas the observer is at $z_0 = [0\ 0]^\mathsf{T}$ (as if it has no "knowledge" of the presence of predators in the predator - prey "ecosystem"). As mentioned before, only imperfect measurements of the prey population are available. We take $w(t) = 0.005 \sin(5t + 0.5)$, for all $t \geq 0$ and we also assume that the control input is $u(t) = 0.01 \sin t$, for all $t \geq 0$ and is known to the observer. The trajectories of the system and the observer are illustrated in Figure 8.2, whereas the value of $V_\epsilon(e)$ as a function of time is illustrated in Figure 8.1.

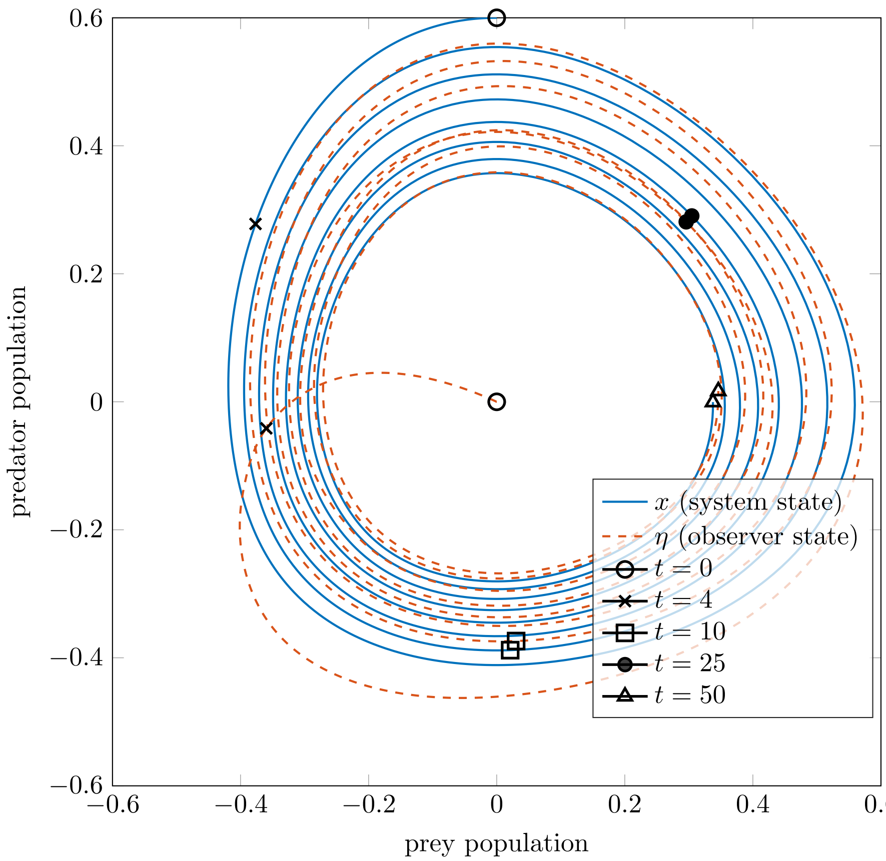


Figure 8.2: The oscillations around the origin are characteristic of the predator - prey dynamics. The observer quickly converges to the prescribed neighborhood of the true state of the system.

8.5 Summary

We proposed a design methodology for robust Lyapunov-based observers applicable to uncertain systems. The observer synthesis process takes place by means of solving an SOS program and is accompanied by explicitly quantified robustness and performance bounds. The literature on observers is significantly rich, given both the interest in the problem and the difficulties often encountered, few of which we briefly mentioned in Section 1.2.4 and in the present chapter. We do not regard our new results presented here to be an exhaustive treatment of the observer design problem, but, rather, a convenient application of the optimization-based design philosophy for

Lyapunov stability problems that we pursue in this work. Our results extend the capabilities of Lyapunov observers and, also, exhibit two interesting traits which we will be exploring in our future work:

- The output of the observer, η, as well as the error bound from Lemma 8.3.1, can be readily combined with our imperfect feedback, input constrained control law from Chapter 7 to solve an output feedback control problem.

- Using the concept of Vector Lyapunov Functions, the proposed design methodology for the observation of a standalone system can be extended to the case of networked systems, either in a distributed or a decentralized way, in parallel to our results in Chapter 6.

Chapter 9

Discussion, future work, and conclusion

9.1 Summary

In this dissertation, we have been motivated from the rather simple fact that *all practically meaningful control systems are subject to input constraints* and the obvious question of *how to design control laws* for such systems (in particular, for polynomial systems subject to input constraints and uncertainty). We focused on the importance of characterizing a set starting from where such a system is guaranteed to be asymptotically stabilizable with a simple Lyapunov-based control law. Such guarantees have been enabled by relaxing the respective problems to semialgebraic set containments that can be solved with SOS techniques. The proposed constrained control laws are implemented online by solving small QPs.

Our results on the nominal case were progressively extended to the cases of uncertain systems, large-scale systems, and systems with imperfect feedback. Finally, we used some of our tools for the robust control problem to design a robust Lyapunov-based observer.

9.2 Directions for future work

The work presented in this dissertation and the underlying optimization-based solution strategies can be used as building blocks to tackle various Lyapunov-based control problems. It may also be useful to consider additional techniques found in the

SOS literature (such as handling systems with rational or nonpolynomial dynamics), as well as the alternatives to computationally-heavy SDP-based SOS (which are fully compatible with our work, as all herein proposed methods are actually agnostic to the underlying convex parameterization of $\Sigma[x]$).

We briefly discuss two broader topics of particular interest: safe learning and large-scale networked control systems design.

The price to pay for robustness in a Lyapunov-based nonlinear control context under input constraints is increased conservatism with regards to the size of the robust stabilization set for the system. Since such a control law operates by making a worst-case assumption for the state of the system, the larger the uncertainty value set is, the smaller the robust stabilization set estimate will ultimately be. This is true for both standalone and networked systems. It is reasonable to wonder whether we can use the control input u to not only stabilize the system but to also infer any dynamics "hidden" in the uncertain term w. Such a feat can be connected to online system identification and relevant techniques that have been appearing in the literature for many decades. Contemporary results on learning theory (Smola and Schölkopf, 1998; Engel, Mannor, and Meir, 2004) provide us with additional tools to approximate any uncertainties in the dynamics. A critical aspect of such a problem is how to use the control input in such a way that (i) stability is maintained and (ii) useful information is extracted from the system's response. The former has been covered extensively in this dissertation, in the challenging case where the input is subject to constraints, also. It is envisioned that the latter part will be explored in our future work.

In a networked systems context, the capabilities of the entire system (for instance, in terms of its robust stabilization set, speed of convergence, etc.) can depend significantly on the individual subsystems. To be more specific and relevant to our work, the actuators and sensors used in one subsystem can directly affect its neighbors and vice-versa. A question that arises here naturally is how to distribute actuators and measurement devices across multiple subsystems, so that some specific objectives are met. This problem entails some binary decisions (that is, a particular actuator can be part of only one subsystem), for each of which the underlying convex problems we developed before determine whether the Lyapunov stabilization problem is feasible. Such a combination of optimization with integer and real variables, the latter of which are subject to convex constraints, has been recently considered in the Satisfiability Modulo Convex Programming framework (Shoukry, Nuzzo, Sangiovanni-Vincentelli, Seshia, Pappas, and Tabuada, 2018), which we plan to utilize in networked system design problems.

9.3 Conclusion

Not all contemporary problems in control and the broader fields of robotics, automation, and autonomy can be reduced to the nonlinear stabilization problem for a continuous time system. There is often more that needs to be done than just steering a system to a target equilibrium, (possibly) in the presence of various complicating factors such as the ones we considered in this dissertation. Path planning, trajectory generation and tracking, optimality, discrete or hybrid dynamics, control-oriented modeling of the underlying system, hierarchical design, collaboration between (sub)systems, high-level specifications, and many other aspects that it is impossible to completely enumerate, can also be of importance in many application areas. As one tries to focus on the big (and complicated) picture, it is natural to wonder where do robust control methods, such as the ones presented herein, fit in there.

We conclude this chapter and the dissertation with a three-part answer:

- Regardless of how complex a physical or cyberphysical system is and how many layers of abstraction it entails, it is often the case that *at least* some parts of the system interact with laws of physics or other phenomena best described, to some extent, by ordinary differential equations of forms amendable to nonlinear control techniques. This is the point, also, where a lot of uncertainty can enter the problem, in terms of mismodeled parameters or effects from the system's environment that cannot be precisely modeled or predicted. In the case of a robotic vehicle, for instance, the motion of the vehicle with regards to the ground is governed by well known rigid body dynamics equations. However, many of the involved physical parameters appearing therein can be uncertain. Significant actuation constraints can typically enter the problem formulation, also. Similar arguments are true for myriad other engineering systems, such as aircraft, spacecraft, robotic manipulators, process control systems, etc. Regardless of the high-level capabilities of any such system, an insufficiently capable and safe control law at the level where the system interfaces with the physical world can be detrimental to the entire system and its mission.

- The guaranteed robustness margins, which one can explicitly quantify with the proposed methods for the classes of systems we considered here, can be used as an interface with outer control loops focusing on progressively higher-level behavior and goals, as it typically the case in hierarchical control systems.

- Stabilization problems may ultimately be more common than what we, as control researchers and engineers, may tend to think. They can also show up in critically important areas, which do not seem to be related to "traditional" control problems, at least at first sight. We indicatively refer to four examples, from the fields of communications (Papachristodoulou and Peet, 2008), social networks (Proskurnikov and Tempo, 2017), applied mathematics (Karafyllis and Krstic, 2017) and cybersecurity (Xu, Lu, and Li, 2017a): following appropriate modeling, it has been shown that the flow of data over data links, the spread of opinions between elements of the society, the nonlinear programming problem, and the behavior of devices in a network can be cast as either nonlinear stability or stabilization problems.

Regardless of one's interest in one specific field versus another, or the choice of preferred mathematical tools, it is fair to argue that nonlinear feedback control, enabled by Lyapunov methods almost 130 years ago, developed further with analytical means in the past 30-40 years, and solidified with the power brought by optimization-based methods such as Sum-of-Squares and other techniques more recently, will be a integral part of control systems for the foreseeable future. Ongoing work on relevant problems will be beneficial to the end of promoting the development of safe and capable autonomous systems in a wide variety of application domains.

Bibliography

Ahmadi, A. A., Majumdar, A., 2018. DSOS and SDSOS Optimization: More Tractable Alternatives to Sum of Squares and Semidefinite Optimization. to appear at the SIAM Journal on Applied Algebra and Geometry.

Ames, A. D., Xu, X., Grizzle, J. W., Tabuada, P., 2017. Control barrier function based quadratic programs for safety critical systems. IEEE Transactions on Automatic Control 62 (8), 3861–3876.

Artstein, Z., 1983. Stabilization with relaxed controls. Nonlinear Analysis 7 (11), 1163–1173.

Bacciotti, A., Rosier, L., 2005. Liapunov Functions and Stability in Control Theory, 2nd Edition. Springer, Berlin.

Bellman, R., 1962. Vector Lyapunov functions. Journal of the Society for Industrial and Applied Mathematics Series A Control 1 (1), 32–34.

Ben-Tal, A., Ghaoui, L. E., Nemirovski, A., 2009. Robust Optimization. Princeton University Press, Princeton, NJ.

Ben-Tal, A., Nemirovski, A., Roos, C., 2002. Robust solutions of uncertain quadratic and conic-quadratic problems. SIAM Journal on Optimization 13 (2), 535–560.

Bertsekas, D. P., 2009. Convex Optimization Theory. Athena Scientific, Belmont, MA.

Boyd, S., Ghaoui, L. E., Feron, E., Balakrishnan, V., 1994. Linear Matrix Inequalities in System and Control Theory. SIAM, Philadelphia, PA.

Chesi, G., 2004a. Computing output feedback controllers to enlarge the domain of attraction in polynomial systems. IEEE Transactions on Automatic Control 49 (10), 1846–1850.

Chesi, G., 2004b. Estimating the domain of attraction for uncertain polynomial systems. Automatica 40, 1981–1986.

Chesi, G., 2010. LMI techniques for optimization over polynomials in control: A survey. IEEE Transactions on Automatic Control 55 (11), 2500–2510.

Clarke, F. H., Ledyaev, Y. S., Sontag, E. D., Subbotin, A. I., 1997. Asymptotic controllability implies feedback stabilization. IEEE Transactions on Automatic Control 42 (10), 1394–1407.

Clarke, F. H., Ledyaev, Y. S., Stern, R. J., 1998. Asymptotic stability and smooth Lyapunov functions. Journal of Differential Equations 149, 69–114.

Curtis, J. W., 2003. CLF-based nonlinear control with polytopic input constraints. In: Proceedings of the 42$^{\text{nd}}$ IEEE Conference on Decision and Control. Vol. 3. pp. 2228–2233.

Daniel, J. W., 1973. Stability of the solution of definite quadratic programs. Mathematical Programming 5 (1), 41–53.

Engel, Y., Mannor, S., Meir, R., 2004. The kernel recursive least-squares algorithm. IEEE Transactions on Signal Processing 52 (8), 2275–2285.

Farza, M., M'Saada, M., Triki, M., Maatoug, T., 2011. High gain observer for a class of non-triangular systems. Systems and Control Letters 60 (1), 27–35.

Ferreau, H., Kirches, C., Potschka, A., Bock, H., Diehl, M., 2014. qpOASES: A parametric active-set algorithm for quadratic programming. Mathematical Programming Computation 6 (4), 327–363.

Freeman, R. A., 1995. Global internal stabilizability does not imply global external stabilizability for small sensor disturbances. IEEE Transactions on Automatic Control 40 (12), 2119–2122.

Freeman, R. A., Kokotović, P. V., 1996. Robust Nonlinear Control Design, State-Space and Lyapunov Techniques. Birkhäuser, Boston, MA.

Freeman, R. A., Praly, L., 1998. Integrator backstepping for bounded controls and control rates. IEEE Transactions on Automatic Control 43 (2), 258–262.

Haddad, W. M., Chellaboina, V., 2008. Nonlinear Dynamical Systems and Control: A Lyapunov-Based Approach. Princeton University Press, Princeton, NJ.

Haddad, W. M., Nersesov, S. G., 2011. Stability and Control of Large-Scale Dynamical Systems: A Vector Dissipative Systems Approach. Princeton University Press, Princeton, NJ.

Henriksen, M., Isbell, J. R., 1953. On the continuity of the real roots of an algebraic equation. Proceedings of the American Mathematical Society 4 (3), 431–434.

Henrion, D., Korda, M., 2014. Convex computation of the region of attraction of polynomial control systems. IEEE Transactions on Automatic Control 59 (2), 297–312.

Henrion, D., Lasserre, J. B., Loefberg, J., 2009. Gloptipoly 3: moments, optimization and semidefinite programming. Optimization Methods and Software 24 (4-5), 761–779.

Hofbauer, J., Sigmund, K., 2003. Evolutionary game dynamics. Bulletin Of The American Mathematical Society 40 (4), 479–519.

Kamyar, R., Peet, M. M., 2015. Polynomial optimization with applications to stability analysis and control - alternatives to sum of squares. Discrete & Continuous Dynamical Systems - B 20 (8), 2383–2417.

Karafyllis, I., Jiang, Z.-P., 2013. Global stabilization of nonlinear systems based on vector control Lyapunov functions. IEEE Transactions on Automatic Control 58 (10), 2550–2562.

Karafyllis, I., Krstic, M., 2017. Global dynamical solvers for nonlinear programming problems. SIAM Journal on Control and Optimization 55 (2), 1302–1331.

Kazantzis, N., Kravaris, C., 1998. Nonlinear observer design using Lyapunov's auxiliary theorem. Systems and Control Letters 34 (5), 241–247.

Khalil, H. K., 2017. High-gain observers in nonlinear feedback control. SIAM, Philadelphia, PA.

Krstić, M., Deng, H., 1998. Stabilization of nonlinear uncertain systems. Springer, London.

Krstić, M., Kanellakopoulos, I., Kokotović, P., 1995. Nonlinear and Adaptive Control Design. Wiley-Interscience, New York, NY.

Kundu, S., Anghel, M., 2017. A multiple-comparison-systems method for distributed stability analysis of large-scale nonlinear systems. Automatica 78, 25–33.

Ledyaev, Y. S., Sontag, E. D., 1998. Stabilization under measurement noise: Lyapunov characterization. In: Proceedings of the 1998 American Control Conference. pp. 1653–1657.

Lin, Y., Sontag, E. D., 1991. A universal formula for stabilization with bounded controls. System and Control Letters 16 (6), 393–397.

Löfberg, J., 2009. Pre- and post-processing sum-of-squares programs in practice. IEEE Transactions on Automatic Control 54 (5), 1007–1011.

Luenberger, D. G., 1964. Observing the state of a linear system. IEEE Transactions on Military Electronics 8 (2), 74–80.

Majumdar, A., Tedrake, R., 2017. Funnel libraries for real-time robust feedback motion planning. International Journal of Robotics Research 36 (8), 947–982.

Manchester, I. R., Slotine, J.-J. E., 2017. Control contraction metrics: Convex and intrinsic criteria for nonlinear feedback design. IEEE Transactions on Automatic Control 62 (6), 3046–3053.

Matrosov, V. M., 1962. On the theory of stability of motion. Prikladnaia Matematika i Mekhanika (in Russian) 26 (6), 992–100.

MOSEK ApS, 2017. The MOSEK optimization toolbox for MATLAB, Version 8.1.

Nesterov, Y., Nemirovski, A., 2001. Interior-Point Polynomial Algorithms in Convex Programming. SIAM, Philadelphia, PA.

Nicolis, G., Portnow, J., 1973. Chemical oscillations. Chemical Reviews 73 (4), 365–384.

Papachristodoulou, A., Anderson, J., Valmorbida, G., Prajna, S., Seiler, P., Parrilo, P. A., 2013. SOSTOOLS: Sum of squares optimization toolbox for MATLAB.

Papachristodoulou, A., Peet, M. M., 2008. Global stability analysis of primal internet congestion control schemes with heterogeneous delays. IFAC Proceedings Volumes 41 (2), 2913–2918.

Papachristodoulou, A., Prajna, S., 2002. On the construction of lyapunov functions using the sum of squares decomposition. In: Proceedings of the 41^{st} IEEE Conference on Decision and Control. Vol. 3. pp. 3482–3487.

Parrilo, P. A., 2000. Structured semidefinite programs and semialgebraic geometry methods in robustness and optimization. Ph.D. thesis, California Institute of Technology.

Parrilo, P. A., 2013. Polynomial optimization, sums of squares, and applications. In: Blekherman, G., Parrilo, P. A., Thomas, R. R. (Eds.), Semidefinite Optimization and Convex Algebraic Geometry. SIAM and Mathematical Optimization Society, Philadelphia, PA, pp. 47–157.

Prajna, S., Papachristodoulou, A., Wu, F., 2004a. Nonlinear control synthesis by sum of squares optimization: a Lyapunov-based approach. In: 2004 5^{th} Asian Control Conference. Vol. 1. pp. 157–165.

Prajna, S., Parrilo, P. A., Rantzer, A., 2004b. Nonlinear control synthesis by convex optimization. IEEE Transactions on Automatic Control 49 (2), 310–314.

Prasov, A. A., Khalil, H. K., 2013. A nonlinear high-gain observer for systems with measurement noise in a feedback control framework. IEEE Transactions on Automatic Control 58 (3), 569–580.

Primbs, J. A., Nevistić, V., Doyle, J. C., 2000. A receding horizon generalization of pointwise min-norm controllers. IEEE Transactions on Automatic Control 45 (5), 898–909.

Proskurnikov, A. V., Tempo, R., 2017. A tutorial on modeling and analysis of dynamic social networks. part I. Annual Reviews in Control 43, 65 – 79.

Samuelson, P. A., 1971. Generalized predator-prey oscillations in ecological and economic equilibrium. Proceedings of the National Academy of Sciences of the United States of America 68 (5), 980–983.

Sandell, Jr., N. R., Varaiya, P., Athans, M., Safonov, M. G., 1978. Survey of decentralized control methods for large scale systems. IEEE Transactions on Automatic Control 23 (2), 108–128.

Shoukry, Y., Nuzzo, P., Sangiovanni-Vincentelli, A. L., Seshia, S. A., Pappas, G. J., Tabuada, P., 2018. SMC: Satisfiability Modulo Convex programming. Proceedings of the IEEE 106 (9), 1655–1679.

Siljak, D. D., 1978. Large-scale Dynamic Systems. Elsevier North Holland, New York.

Siljak, D. D., 1991. Decentralized Control of Complex Systems. Academic Press, San Diego, CA.

Smola, A. J., Schölkopf, B., 1998. Learning with kernels. MIT Press, Boston, MA.

Sontag, E. D., 1989. A 'universal' construction of Artstein's theorem on nonlinear stabilization. Systems and Control Letters 13 (2), 117 – 123.

Sturm, J., 1999. Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones. Optimization Methods and Software 11–12, 625–653.

Tan, W., Packard, A., 2008. Stability region analysis using polynomial and composite polynomial Lyapunov functions and sum-of-squares programming. IEEE Transactions on Automatic Control 53 (2), 565–571.

Tedrake, R., Manchester, I. R., Tobenkin, M., Roberts, J. W., 2010. LQR-trees: Feedback motion planning via sums-of-squares verification. International Journal of Robotics Research 29 (8), 1038–1052.

Topcu, U., Packard, A., Seiler, P., 2008. Local stability analysis using simulations and sum-of-squares programming. Automatica 44 (10), 2669–2675.

Tsinias, J., 1989. Observer design for nonlinear systems. Systems and Control Letters 13 (2), 135–142.

Tsinias, J., 1990. Further results on the observer design problem. Systems and Control Letters 14 (5), 411–418.

Tütüncü, R. H., Toh, K. C., Todd, M. J., 2003. Solving semidefinite-quadratic-linear programs using SDPT3. Mathematical Programming 95 (2), 189–217.

Vaidya, U., Mehta, P. G., Shanbhag, U. V., 2010. Nonlinear stabilization via control lyapunov measure. IEEE Transactions on Automatic Control 55 (6), 1314–1328.

Vandenberghe, L., Boyd, S., 1996. Semidefinite programming. SIAM Review 38 (1), 49–95.

Wang, T.-C., Lall, S., West, M., 2013. Polynomial level-set method for polynomial system reachable set estimation. IEEE Transactions on Automatic Control 58 (10), 2508–2521.

Xu, S., Lu, W., Li, H., 2017a. A stochastic model of active cyber defense dynamics. Internet Mathematics 11 (1), 23–61.

Xu, X., Grizzle, J. W., Tabuada, P., Ames, A. D., 2017b. Correctness guarantees for the composition of lane keeping and adaptive cruise control. IEEE Transactions on Automation Science and Engineering 62 (8), 1216–1229.

Ye, Y., Tse, E., 1989. An extension of Karmarkar's projective algorithm for convex quadratic programming. Mathematical Programming 44 (1-3), 157–179.

Zeitz, M., 1987. The extended luenberger observer for nonlinear systems. Systems and Control Letters 9 (2), 149–156.

Vita

Dimitrios Pylorof was born in 1989, in Thessaloniki, Greece. He received a Dipl.-Ing degree in Mechanical Engineering from the National Technical University of Athens in 2012 and an MS degree in Aerospace Engineering from the University of Texas at Austin in 2015. He is currently a PhD candidate at the University of Texas at Austin. During his graduate studies at UT, he had been a graduate research assistant at the Department of Aerospace Engineering and Engineering Mechanics and the Center for Space Research, as well as a teaching assistant at the Department of Aerospace Engineering and Engineering Mechanics. In May 2018 he joined the U.S. Army Research Laboratory in Adelphi, MD, as a research fellow participating in ARL's Research Associateship Program.

Email address: dpylorof@gmail.com

This dissertation was typed by the author.