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**Functional Approximation Methods for Solving Stochastic Control Problems in Finance**

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# Functional Approximation Methods for Solving Stochastic Control Problems in Finance

**by**

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Dedicated to my family and everyone I love.

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**Functional Approximation Methods for Solving Stochastic Control Problems in Finance**

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I develop a numerical method that combines functional approximations and dynamic programming to solve high-dimensional discrete-time stochastic control problems under general constraints. The method relies on three building blocks: first, a quasi-random grid and the radial basis function method are used to discretize and interpolate the high-dimensional state space; second, to incorporate constraints, the method of Lagrange multipliers is applied to obtain the first order optimality conditions; third, the conditional expectation of the value function is approximated by a second order polynomial basis, estimated using ordinary least squares regressions. To reduce the approximation error, I introduce the test region iterative contraction (TRIC) method to shrink the approximation region around the optimal solution. I apply the method to two Finance applications: a) dynamic portfolio choice with constraints, a continuous control problem; b) dynamic portfolio choice with capital gain taxation, a high-dimensional singular control problem.

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# Chapter 1

# Introduction

Many problems in Engineering, Economics, Finance, and other fields can be formulated as stochastic control problems, where a series of decisions must be made along time in the face of uncertainty and constraints. Solving high-dimensional stochastic control problems naively, using dynamic programming techniques and a regular discretization of the state space, is challenging due to the exponentially increasing requirements in computational resources. Recently, a series of papers have proposed solving such problems with a simulation-based approach, combining Monte Carlo simulation and dynamic programming - see Tsitsiklis and Van Roy (2001), Longstaff and Schwartz (2001), Brandt, Goyal, Santa-Clara, and Stroud (2005), and Koijen, Nijman, and Werker (2009). Although the simulation-based approach has been successfully applied to several applications in Finance, this approach has been largely limited to stochastic control problems with exogenous state variables due to the dependency of the discretization of the state space on a set of forward-simulated paths of the exogenous state variables.

In this dissertation I develop a numerical methodology that combines functional approximations and dynamic programming to solve high-dimensional discrete-time stochastic control problems under general constraints. The method can handle problems with both a large number of endogenous and exogenous state variables. To incorporate constraints, I apply the method of Lagrange multipliers to obtain the optimality conditions that characterize the optimal control. There are two difficulties in solving the resulting system of equations: a) the high cost associated with repeated evaluations of conditional expectations in the optimality conditions; and, b) the lack of efficient schemes to discretize and interpolate a high-dimensional state space including both endogenous and exogenous state variables. To overcome the first difficulty, I extend the functional approximation approach and approximate the conditional expectation of the value function using a second order polynomial basis, where the coefficients are estimated using regression-like methods. The benefit is that the resulting optimality conditions become a system of linear equations, which can be solved efficiently. To improve the accuracy of the second order approximation I introduce the test region iterative contraction (TRIC) method and estimate the conditional expectation in a region getting smaller and smaller around the estimated value of the optimal control. To overcome the second difficulty, two meshfree approximation methods are employed: first, a quasi-random grid is used to discretize a high-dimensional state space uniformly with a smaller number of grid points than a regular grid; second, the value function on the quasi-random grid is interpolated using a set of radial basis functions whose parameters are chosen by applying machine learning techniques, such as cross-validation and iterative center selection.

There is an extensive literature on numerical methods for solving stochastic control problems in Finance. Tsitsiklis and Van Roy (2001) and Longstaff and Schwartz (2001) solve an optimal stopping problem, pricing American-style options, using the simulation/regression method. They approximate the continuation value of an American option; i.e., an expectation conditioning on the current stock price, as a linear combination of basis functions, where the weights of the basis functions are estimated using ordinary least squares regression. Inspired by the work of pricing American options, Brandt et al. (2005) apply the idea of simulation and regression to solve a continuous control problem, dynamic portfolio choice. They approximate conditional expectations using a Taylor series expansion in the control variables and estimate the coefficients of the expansion using the simulation/regression method. Koijen et al. (2009) apply the simulation-based approach to solve a portfolio choice problem with constraints on portfolio weights by solving the first order conditions of optimality.

The method I propose is different from the simulation-based approach in three ways: first, my method achieves an accurate approximation of the conditional expectation using a different approach, the test region iterative contraction (TRIC) method with quadratic basis functions; second, my method can solve problems with both endogenous state variables and exogenous state variables; third, my method explicitly incorporates a general class of constraints on control variables.

A closely related method is developed by Garlappi and Skoulakis (2008). To find an efficient and accurate approximation of the conditional expectation, Garlappi and Skoulakis (2008) extend the Taylor series expansion idea of Brandt et al. (2005) by choosing a different center of expansion and separating the terms involving the control variables from the terms involving the random shocks. As a result of this separation, they are able to focus on evaluating the conditional expectations of the terms involving the random shocks, where analytical results are available under certain distributional assumptions. Their method relies on a Taylor series expansion of the value function in the control variables, which becomes a bottleneck for problems with a complicated dependency of the value function on the control variables. Instead of using a high order Taylor series approximation to achieve accurate approximation within the entire control space, the method I propose focuses on approximating the conditional expectation within a small region around the optimal value of the control variables using a second order polynomial.

I demonstrate the potential of the general methodology based on functional approximations by solving two applications in Finance. The first application considers a continuous control problem, where an investor chooses her lifetime portfolio and consumption plan to maximize her expected utility while facing financial constraints and receiving an income stream. The difficulty of this problem is to find the optimal portfolio weights that satisfy the margin requirements on the long positions (positive portfolio weights) and the short positions (negative portfolio weights). The method I propose is general enough to handle this type of constraints by applying the method of Lagrange multipliers and solving the first order optimality conditions, where the expected future utility is approximated by the TRIC method with a second order polynomial. The main finding is that severely constrained younger investors often optimally hold a under-diversified portfolio compared to the portfolios held by older investors, which is consistent with the empirical literature.

The second application of the TRIC method considers a state-dependent transaction cost that affects the investor's investment and consumption decision, taxation on capital gains. Unlike the rest of the literature, who assumes that capital losses can be immediately realized for a tax refund, I make the more realistic assumption that capital losses can only be used to offset current capital gains or be carried forward to offset future capital gains. This application is challenging because it is a high-dimensional problem with singular controls. To model capital gain taxation for each risky asset one needs to keep track of both the average purchasing price and the number of shares held. Modeling the limited use of capital losses requires an additional state variable. The resulting high-dimensional state space is discretized and interpolated using the mesh-free techniques. The problem belongs to the class of singular control problems because the relationship between capital gain taxes and portfolio choice is non-differentiable at certain points. To handle singularity, I partition the control space into a set of disjoint regions at the non-differentiable points, and select the optimal control by comparing all the local optima obtained within each region. The results show that limiting the use of losses to only offset gains results in significant reductions in the overall position of risky assets held by the investor.

The rest of the dissertation is organized as follows. Chapter 2 describes the general framework of the method. Chapter 3 and Chapter 4 illustrate the two applications in Finance. Chapter 5 concludes and discusses plans for future work.

# Chapter 2

# General Framework

## 2.1 Model Setup

I consider a finite-horizon discrete-time stochastic control problem with a given terminal condition $V_T(X_T)$ at time $T$ and a general form of recursion characterized by the Bellman equation at time $t = 0, 1, \ldots, T-1$[1]:

$$\begin{aligned} V_t(X_t) = \quad & \max_{x_t} \mathcal{H}_t \left\{ u_t(X_t, x_t), E_t\left[ V_{t+1}(X_{t+1}) \middle| X_t \right] \right\} \\ \text{s.t.} \quad & X_{t+1} = \mathcal{G}_t(X_t, x_t, \varepsilon_{t+1}) \\ & \mathcal{F}_t(X_t, x_t) \geq 0 \end{aligned} \tag{2.1}$$

where $X_t$ is the vector of state variables at time $t$ with dimension $d_X$; $V_t(\cdot)$ is the value function at time $t$; $x_t$ is the vector of control variables at time $t$ with dimension $d_x$; $u_t(\cdot,\cdot)$ is the utility function at time $t$ when taking action $x_t$ at state $X_t$; $\mathcal{H}_t(\cdot,\cdot)$ is the function that aggregates current utility and expected future utilities at time $t$; $\varepsilon_{t+1}$ is the vector of random noise over period $[t, t+1]$ with dimension $d_\varepsilon$; $\mathcal{G}_t(\cdot,\cdot,\cdot)$ is the evolution rule of state variables; $\mathcal{F}_t(\cdot,\cdot)$ is the vector of constraints on control variables $x_t$ with dimension $d_F$. The general setup (2.1) accommodates a large class of stochastic control problems by choosing the functions $u_t$, $\mathcal{H}_t$, $\mathcal{G}_t$, and $\mathcal{F}_t$. For example, time separable utilities can be specified by choosing $\mathcal{H}_t(a,b) = a + \beta b$, $\beta \in (0,1]$, and recursive preferences can be specified by choosing $\mathcal{H}_t(a,b) = \left[ (1-\beta) a^{\frac{1}{\theta}} + \beta b^{\frac{1}{\theta}} \right]^{\theta}$,

[1]Problem (2.1) has the same form of the general case discussed in Garlappi and Skoulakis (2008), except that I explicitly specify the constraints on the control variables $\mathcal{F}_t(X_t, x_t) \geq 0$, which is a vector of $d_F$ constraints on the control variables $x_t$; i.e., $\left\{ \mathcal{F}_t^i(X_t, x_t) \geq 0 \right\}_{i=1}^{d_F}$.

$\theta \neq 0$. The constraint functions, $\left\{\mathcal{F}_t^k\left(X_t, x_t\right)\right\}_{k=1}^{d_F}$, can be either state-dependent or state-independent, linear or non-linear.

The state variable vector $X_t$ contains both endogenous state variables $X_t^{en}$ and exogenous state variables $X_t^{ex}$; i.e., $X_t = \left(X_t^{en}, X_t^{ex}\right)$. By definition, the exogenous state variables at time $t+1$ do not depend on the control variables and endogenous state variables at time $t$, while the endogenous state variables do; i.e., $X_{t+1}^{en} = \mathcal{G}_t^{en}\left(X_t^{en}, X_t^{ex}, x_t, \varepsilon_{t+1}\right)$ and $X_{t+1}^{ex} = \mathcal{G}_t^{ex}\left(X_t^{ex}, \varepsilon_{t+1}\right)$. Unlike the simulation-based methods that focus on exogenous state variables, I provide a method that can handle both endogenous and exogenous state variables.

## 2.2 Algorithm

The solution methodology is based on dynamic programming that solves the Bellman equation (2.1) backward along time. The difficulties of finding the optimal solution to the Bellman equation arise from the following: first, the number of dimensions of the state variable space can be very high which makes discretization and interpolation using regular grids infeasible because the number of grid points increases exponentially with the number of dimensions of the state variable space; second, the objective function involves a conditional expectation which is costly to evaluate and needs to be repeatedly evaluated for different values of state variables and control variables during the process of searching for the optimal solution; third, a systematic way to incorporate constraints is needed. To overcome the first difficulty I use meshfree techniques: the quasi-random grid for discretization and the radial basis function method for interpolation. To overcome the second difficulty I focus on the functional approximation approach to approximate the conditional expectations using a set of basis functions. The third problem is addressed by the method of Lagrange multipliers. I summarize the solution methodology using the algorithm below and further discuss every step in the following sections.

**Algorithm: Main Steps**

**Step 1:** Set the terminal condition at time $T : V_T\left(S_T\right)$

**Step 2:** Find the optimal policy backwards at time $t = T-1, T-2, \cdots, 0:$

- **Step 2.1:** Discretize the space of state variables $X_t$ with a quasi-random grid $\left\{X_t^{(i)}\right\}_{i=1}^{n_g}$.
- **Step 2.2:** Find the optimal control at each state variable grid point $X_t^{(i)}$ using the TRIC method:
  - **Step 2.2.1:** Approximate conditional expectations and their derivatives using functional approximations
  - **Step 2.2.2:** Solve the optimality conditions with the approximations obtained in Step 2.2.1
  - **Step 2.2.3:** Repeat Step 2.2.1 and 2.2.2 in a smaller approximation region until a convergence condition is satisfied
- **Step 2.3:** Approximate the value functions on the quasi-random grid with a set of radial basis functions

## 2.3 Optimality Conditions

One challenge of finding the optimal solution to problem (2.1) is to incorporate constraints on control variables, $\left\{\mathcal{F}_t^k\left(S_t, x_t\right) \geq 0\right\}_{k=1}^{d_F}$, which can be state-dependent or state-independent, linear or non-linear. To handle constraints, I apply the method of Lagrange multipliers to the Bellman equation by defining a set of Lagrange multipliers $\left\{\lambda_t^k\right\}_{k=1}^{d_F}$. The Lagrangian and the corresponding Karush–Kuhn–Tucker conditions are given below:

**Lagrangian**

$$\mathcal{L}\left(x_t, \lambda_t | X_t\right) = \mathcal{H}_t\left\{u_t\left(X_t, x_t\right), E_t\left[V_{t+1}\left(X_{t+1}\right) | X_t\right]\right\} + \sum_{k=1}^{d_F} \lambda_t^k \mathcal{F}_t^k\left(X_t, x_t\right) \tag{2.2}$$

**Karush–Kuhn–Tucker Conditions**

$$
\begin{aligned}
&0 = \tfrac{\partial \mathcal{L}}{\partial x_t^i} = \tfrac{\partial}{\partial x_t^i}\mathcal{H}_t + \sum_{k=1}^{d_F} \lambda_t^k \tfrac{\partial}{\partial x_t^i}\left[\mathcal{F}_t^k\left(X_t, x_t\right)\right], i = 1, \cdots, d_x && \text{First Order Conditions} \\
&0 = \lambda_t^k \mathcal{F}_t^k\left(X_t, x_t\right), k = 1, \cdots, d_F && \text{Complementary Conditions} \\
&0 \leq \lambda_t^k, \mathcal{F}_t^k\left(X_t, x_t\right), k = 1, \cdots, d_F && \text{Feasibility Conditions}
\end{aligned} \tag{2.3}
$$

where the first order conditions can be further specified as

$$
\begin{aligned}
0 &= \tfrac{\partial}{\partial x_t^i}\mathcal{H}_t\left\{u_t\left(X_t, x_t\right), E_t\left[V_{t+1}\left(X_{t+1}\right)\middle| X_t\right]\right\} + \sum_{k=1}^{d_F}\lambda_t^k \tfrac{\partial}{\partial x_t^i}\left[\mathcal{F}_t^k\left(X_t, x_t\right)\right] \\
&= \partial\mathcal{H}_{t,1}\tfrac{\partial}{\partial x_t^i}\left[u_t\left(X_t, x_t\right)\right] + \partial\mathcal{H}_{t,2}\tfrac{\partial}{\partial x_t^i}E_t\left[V_{t+1}\left(X_{t+1}\right)\middle| X_t\right] \\
&\quad + \sum_{k=1}^{d_F}\lambda_t^k \tfrac{\partial}{\partial x_t^i}\left[\mathcal{F}_t^k\left(X_t, x_t\right)\right], i = 1, \cdots, d_x
\end{aligned}
$$

where $\partial\mathcal{H}_{t,1}$ is the partial derivative of $\mathcal{H}_t$ with respect to its first argument.

The Karush–Kuhn–Tucker conditions (2.3) at a given grid point, $X_t^{(i)}, i = 1, \ldots, n_g$, form a system of equations that characterize the optimal solution $x_t^*$. In general the Karush–Kuhn–Tucker conditions are necessary conditions for optimality but not sufficient. One needs to find all the candidate solutions that satisfy the Karush–Kuhn–Tucker conditions by enumerating all the possible specifications of the complementary conditions. The optimal solution is chosen among the candidates by comparing the value function achieved by each candidate solution.[2]

The optimality of the Karush–Kuhn–Tucker condition requires the differentiability of functions, $u_t$, $\mathcal{H}_t$, $\mathcal{G}_t$, and $\mathcal{F}_t$ with respect to the control variables $x_t$. However, the method developed in this work can also accommodate situations where there are non-differentiable kinks in those functions. The idea is to partition the control space

[2]For some special cases, the Karush–Kuhn–Tucker conditions are both necessary and sufficient for optimality. In those cases once a candidate solution is found, the remaining specifications can be skipped.

at the kinks and find the optimal solution locally within each subset of the partition, and pick the globally optimal solution by comparing the objective functions across all locally optimal solutions.

## 2.4 Approximation of Conditional Expectations

Solving the optimality conditions (2.3) requires evaluations of the conditional expectation

$$E_t\left[V_{t+1}\left(X_{t+1}\right)\middle| X_t\right] = E_t\left[V_{t+1}\left(\mathcal{G}_t\left(X_t, x_t, \varepsilon_{t+1}\right)\right)\middle| X_t\right] \tag{2.4}$$

and its derivatives with respect to the control variables $x_t^i, i = 1, \ldots, d_x$. As a function of the state variables $X_t$ and the control variables $x_t$, the conditional expectation (2.4) needs to be evaluated repeatedly during the process of solving the optimality conditions. Thus it is important to find an efficient approximation.

Under the functional approximation approach, the conditional expectation (2.4) is approximated by a linear combination of basis functions[3]

$$E_t\left[V_{t+1}\left(X_{t+1}\right)\middle| X_t\right] = E_t\left[V_{t+1}\left(\mathcal{G}_t\left(X_t, x_t, \varepsilon_{t+1}\right)\right)\middle| X_t\right] \approx \sum_{j=1}^{n_f} \omega_j\left(X_t\right) f_j\left(x_t\right) \tag{2.5}$$

where $n_f$ is the number of basis functions and $\left\{f_j\left(\cdot\right)\right\}_{j=1}^{n_f}$ are the basis functions on control variables $x_t$. The state-dependent coefficients $\left\{\omega_j\left(X_t\right)\right\}_{j=1}^{n_f}$ are estimated using cross-test-solution regression

$$\begin{bmatrix} E_t\left[V_{t+1}\left(\mathcal{G}_t\left(X_t, x_t^{(1)}, \varepsilon_{t+1}\right)\right)\middle| X_t\right] \\ \vdots \\ E_t\left[V_{t+1}\left(\mathcal{G}_t\left(X_t, x_t^{(n_t)}, \varepsilon_{t+1}\right)\right)\middle| X_t\right] \end{bmatrix} = \begin{bmatrix} f_1\left(x_t^{(1)}\right) & \cdots & f_{n_f}\left(x_t^{(1)}\right) \\ \vdots & & \vdots \\ f_1\left(x_t^{(n_t)}\right) & \cdots & f_{n_f}\left(x_t^{(n_t)}\right) \end{bmatrix} \begin{bmatrix} \omega_1 \\ \vdots \\ \omega_{n_f} \end{bmatrix} \tag{2.6}$$

[3]I am abusing terminology by calling the set of functions $\left\{f_j\left(\cdot\right)\right\}_{j=1}^{n_f}$ a basis since they do not necessarily span the entire function space.

where $\left\{x_t^{(k)}\right\}_{k=1}^{n_t}$ is a set of trial values of the control variables, called test solutions, that satisfy all the constraints. The conditional expectation (2.4) is approximated using the following steps:

**Algorithm: Approximation of Conditional Expectations**

**Step 1:** Generate test solutions $\left\{x_t^{(k)}\right\}_{k=1}^{n_t}$ quasi-randomly within a set $\theta$, called the test region. To guarantee that all the test solutions are feasible, the test region $\theta$ is chosen to be a subset of the feasible region $\bar{\theta}(X_t) = \left\{x_t \in \mathbb{R}^{d_x} : \mathcal{F}_t(X_t, x_t) \geq 0\right\}$; i.e., $x_t^{(k)} \in \theta \subseteq \bar{\theta}(X_t), k = 1, \ldots, n_t$.

**Step 2:** Evaluate the left-hand-side of (2.6) by evaluating the conditional expectation for the different test solutions $x_t = x_t^{(k)}, k = 1, \ldots, n_t$.

**Step 3:** Evaluate the right-hand-side of (2.6) by evaluating the basis functions at different test solutions.

**Step 4:** Estimate the coefficients $\{\omega_j(X_t)\}_{j=1}^{n_f}$ using ordinary least squares regression.

The algorithm expands Step 2.2.1 of the main algorithm described in Section 2.2. In Step 2 there are many possible ways to evaluate the conditional expectation under different test solutions such as quadrature methods (as in Judd (1998)), simulation/regression methods (as in Tsitsiklis and Van Roy (2001), Longstaff and Schwartz (2001), and Brandt et al. (2005)), decomposition methods (as in Garlappi and Skoulakis (2008)), and multinomial discretization methods (as in He (1990)).

Given approximation (2.5) the derivatives of the conditional expectation are given by

$$\frac{\partial}{\partial x_t^i} E_t\left[V_{t+1}(X_{t+1}) \middle| X_t\right] = \sum_{j=1}^{n_f} \omega_j(X_t) \frac{\partial f_j(x_t)}{\partial x_t^i}, i = 1, \ldots, d_x \tag{2.7}$$

where $\{\omega_j(X_t)\}_{j=1}^{n_f}$ are the set of coefficients in (2.5) and the derivatives of the basis functions $\{f_j(\cdot)\}_{j=1}^{n_f}$ with respect to the control variables $\{x_t^i\}_{i=1}^{d_x}$ are easy to evaluate since the basis functions are known.

## 2.5 Reducing Approximation Error

Once the conditional expectation and its derivatives are approximated through (2.5) and (2.7), the optimality conditions are approximated by a system of deterministic equations that can be solved by many numerical packages.[4]

**Approximate Karush–Kuhn–Tucker Conditions**

$$
\begin{aligned}
&0 = \partial\mathcal{H}_1 \tfrac{\partial}{\partial x_t^i}\left[u_t\left(X_t, x_t\right)\right] + \partial\mathcal{H}_2 \sum_{j=1}^{n_f} \omega_j\left(X_t\right) \tfrac{\partial f_j(x_t)}{\partial x_t^i} + \sum_{k=1}^{d_F} \lambda_t^k \tfrac{\partial}{\partial x_t^i}\left[\mathcal{F}_t^k\left(X_t, x_t\right)\right], i = 1, \cdots, d_x \\
&0 = \lambda_t^k \mathcal{F}_t^k\left(X_t, x_t\right), k = 1, \cdots, d_F \\
&0 \leq \lambda_t^k, \mathcal{F}_t^k\left(X_t, x_t\right), k = 1, \cdots, d_F
\end{aligned}
\tag{2.8}
$$

The solution to the approximate optimality conditions (2.8) is an approximation to the true optimal solution $x_t^*$ that solves the original optimality conditions (2.3). To reduce the approximation error, one needs to improve the quality of the functional approximation defined in equation (2.5). There are three ways to improve this approximation: reducing the sampling error, reducing the specification error, and reducing the size of the approximation region; i.e., the test region $\theta$.

The sampling error can be reduced by increasing the number of test solutions $n_t$. However, additional computational cost is needed to evaluate conditional expectations with respect to more test solutions.

The specification error can be reduced by carefully choosing the type of basis functions and using more basis functions; i.e. increasing $n_f$. For a particular model, it is possible to choose appropriate types of basis functions given problem-specific information about the functional form of the conditional expectation to be approximated. For the purpose of developing a general methodology, a common choice of basis functions is the polynomial basis, where the number of basis functions is specified by the

[4] In the application section, I use the multidimensional root-finding solver of the GSL library to solve the two numerical examples.

order of the polynomial and the number of control variables. Choosing the order of the polynomial basis is a balance between the specification error and the complexity of the resulting approximate optimality conditions. Using a high order polynomial basis reduces the specification error, but leads to a high order system of equations (the approximate optimality conditions), which requires more effort to solve and the solution to the system of equations is more sensitive with respect to the initial guess. Moreover, estimating the coefficients of a high order polynomial basis requires a larger sample of test solutions.

To reduce the specification error while still keeping the benefit of a low order polynomial basis, I consider ways to reduce the size of the test region. First I use second order polynomial basis functions so that the approximate optimality conditions become a system of linear equations that can be solved efficiently without an initial guess. Then I reduce the size of the test region $\theta$; i.e., approximate the conditional expectation in a smaller region where the quadratic function becomes a better fit. This motivates the idea of contracting the test region iteratively.

## 2.6 Test Region Iterative Contraction (TRIC) Method

Test region iterative contraction (TRIC) is a method that I introduce to improve the accuracy of the functional-approximation-based approach of solving stochastic control problems. When the conditional expectation is approximated through the functional approximation in equation (2.5), reducing the size of the test region $\theta$ reduces the approximation error. However, without knowing the optimal solution $x_t^*$, reducing the size of the test region blindly could lead to a test region that does not contain the optimal solution. I demonstrate the basic idea of the TRIC method using an example of optimal portfolio choice over two assets in Figure 2.1.

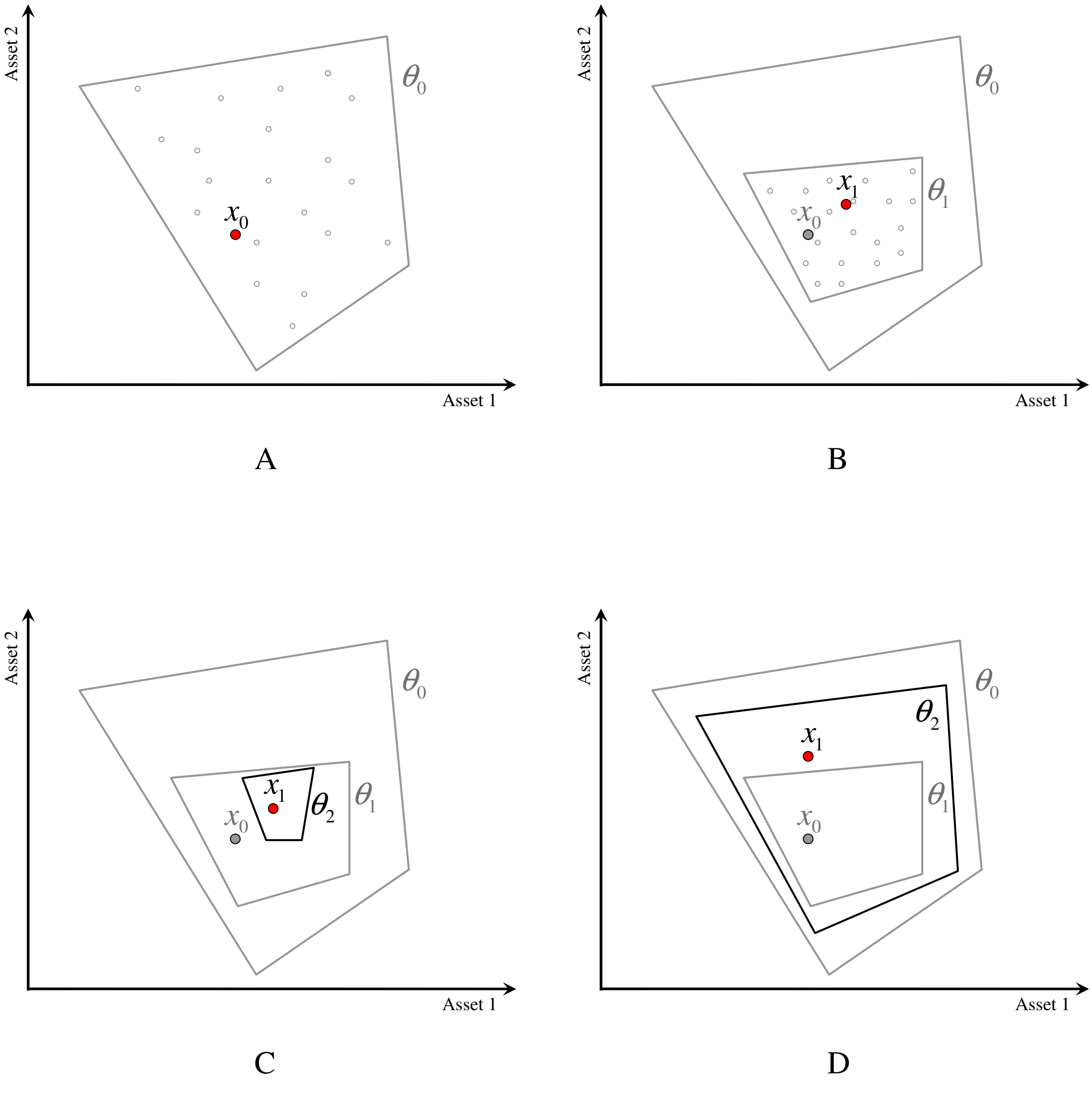


Figure 2.1: An Illustration of the TRIC Method

As shown in Panel A of Figure 2.1, at Iteration 0 a set of test solutions is generated randomly within the initial test region $\theta_0$. Using a cross-test-solution regression over these test solutions I can estimate a functional approximation of the conditional expectation and obtain the approximate optimality conditions. Solving the approximate optimality conditions provides an approximate optimal solution $x_0$. Iteration 1 starts with a smaller test region $\theta_1$ around $x_0$, as shown in Panel B of Figure 2.1, and finds a new approximate optimal solution $x_1$ by repeating the procedures of generating test solutions, estimating the cross-test-solution regression, and solving the approximate optimality conditions. If $x_1$ falls in the current test region $\theta_1$, Iteration 2 keeps shrinking the test region from $\theta_1$ to $\theta_2$ around $x_1$, as shown in Panel C of Figure 2.1. Otherwise, Iteration 2 uses a slightly larger test region $\theta_2$ as shown in Panel D of Figure 2.1. The iteration stops after some convergence criterion is met.

The following algorithm summarizes the TRIC method. It expands Step 2.2 of the main algorithm described in Section 2.2 and requires the following inputs: the error cutoff $\varepsilon$, the number of test solutions $n_t$, the initial test region $\theta^{[0]}$, and the test region update rule, $\Theta(\cdot, \cdot)$. If no further information is available one can set $\theta^{[0]} = \bar{\theta}(X_t)$, where $\bar{\theta}(X_t) = \left\{x_t \in \mathbb{R}^{d_x} : \mathcal{F}_t(X_t, x_t) \geq 0\right\}$ is the feasible region at state $X_t$. However, it is possible to obtain a smaller $\theta^{[0]}$ if the optimal solution at a similar state, called a reference solution, is available. Finding a reference solution requires problem-specific knowledge. Usually the optimal solution at a similar state of the next period or the current period is available as a reference solution. Sometimes analytical solutions are available at certain states.

**Algorithm: the TRIC Method**

**Step 1:** Initialization

- Set initial iteration number: $i = 0$
- Set initial test region: $\theta^{[0]}$
- Set accuracy level: $\varepsilon$

**Step 2:** Update the optimal solution

- Generate a set of test solutions, $\left\{x_t^{[i,k]}\right\}_{k=1}^{n_t}$, within the test region $\theta^{[i]}$.
- Update the coefficients of the basis functions, $\left\{\omega_j^{[i]}\right\}_{j=1}^{n_f}$, using the cross-test-solution regression (2.6) over the test solutions $\left\{x_t^{[i,k]}\right\}_{k=1}^{n_t}$.
- Update the approximate optimality conditions (2.8) using $\left\{\omega_j^{[i]}\right\}_{j=1}^{n_f}$.
- Find the approximate optimal solution $x^{[i]}$ by solving the approximate optimality conditions.

**Step 3:** Update the test region using

$$\theta^{[i+1]} = \Theta\left(x^{[i]}, \theta^{[i]}\right)$$

where the test region update rule, $\Theta\left(x^{[i]}, \theta^{[i]}\right)$, satisfies

- If $x^{[i]} \in \theta^{[i]}$, shrink the test region to $\theta^{[i+1]} \subset \theta^{[i]}$.
- If $x^{[i]} \notin \theta^{[i]}$, enlarge the test region to $\theta^{[i]} \subset \theta^{[i+1]} \subset \theta^{[i-1]}$.

**Step 4:** Test convergence

- If $\left\|x^{[i]} - x^{[i-1]}\right\| < \varepsilon$, stop.
- If $\left\|x^{[i]} - x^{[i-1]}\right\| \geq \varepsilon$, update the iteration number $i = i+1$ and go to Step 2.

**Comparison Between the TRIC Method and the STRONG Method**

Independent of this work, Chang, Hong, and Wan (2010) introduced a method similar to TRIC, the Stochastic Trust-Region Response-Surface Method (STRONG). STRONG combines the response surface method and the trust region method to iteratively solve unconstrained stochastic optimization problems with continuous control variables. Within each iteration, STRONG estimates the value, the gradient, and the curvature of the objective function at the current solution using the design of experiment method and the ordinary least squares regression. Based on the information at the current solution, a local model is constructed to approximate the objective function locally around the current solution. Associated with the local model is a region around the current solution, the trust region, where the local model is believed to be trustworthy. Depending on the size of the trust region, either a linear or quadratic local model is built. Solving the local model exactly or finding a nearly optimal solution of the local model, such as a Cauchy point, within the trust region provides a new solution that potentially improves the objective value. Based on the observed improvement and the model-predicted improvement in the objective value, STRONG automatically determines whether the new solution should be accepted and updates the size of the trust region for the next iteration.

TRIC and STRONG share some common features. They are both iterative algorithm to solve stochastic optimization problems with continuous control variables. They both use low order polynomials to approximate the objective function within some approximation regions (test regions in TRIC and trust regions in STRONG), whose sizes are adjusted adaptively at each iteration. However, TRIC and STRONG are different in many aspects. First, TRIC is for constrained optimization, but STRONG is for unconstrained optimization. Second, STRONG finds new solutions that potentially improve the objective value by searching along the improving direction. TRIC finds new solutions by solving the first order optimality conditions, the Karush–Kuhn–Tucker Conditions. Third, STRONG uses the information (the value of the objective function together with its gradient and curvature) at the center of the trust region to construct the local model, while TRIC uses the information of a random sample, drawn from the entire test region, to construct the local model. Fourth, new solutions of STRONG must be inside of the trust region, while new solutions of TRIC can be either inside or outside of the test region. Fifth, STRONG evaluates the reliability of the local model and updates the size of the trust region by comparing the observed improvement and the model-predicted improvement in the objective value. TRIC updates the size of the test region by checking whether the new solution is inside of the current test region. Sixth, if a new solution is found to be satisfactory, TRIC contracts the test region of the next iteration, but STRONG expands the trust region of the next iteration.

## 2.7 Meshfree Discretization and Interpolation of the State Space

How to discretize the state space and interpolate the results on the discretized state space is a problem faced by all numerical methods of solving stochastic control problems. To overcome the problem of exponential growth of a regular mesh, I employ two meshfree techniques: the quasi-random grid for discretization and the radial basis function method for interpolation. The benefit of using a quasi-random grid is that it can fill a high-dimensional state space uniformly with a relatively small number of grid points.[5] However, interpolating a function evaluated on a quasi-random grid is difficult because of the lack of ready-to-use geometry structures across the data points. The remedy is to use the radial basis function method.[6]

Radial basis functions $\Phi(X) : \mathbb{R}^n \to \mathbb{R}$ are a family of functions whose values

[5]Problem-specific knowledge may help to place the grid points in a more efficient manner. For example, additional grid points may be needed in regions of the state space where optimal controls are more sensitive with respect to changes in the state variables.

[6]More information about the radial basis function method and other meshfree approximation methods can be found in Fasshauer (2007).

only depend on the "distance" (measured by some norm) from the center; i.e.,

$$\Phi(X) = \varphi(\|X - C\|)$$

where $C \in \mathbb{R}^n$ is the center; $\|\cdot\|$ is a norm on $\mathbb{R}^n$; $\varphi : [0, \infty) \to \mathbb{R}$ is a univariate function. A well known radial basis function family is the Gaussian family

$$\Phi(X) = e^{-\varepsilon^{-2}\|X-C\|_2^2}$$

where $\varepsilon \in \mathbb{R}^+$ is the width of the radial basis function; $C \in \mathbb{R}^n$ is the center of the radial basis function; $\|\cdot\|_2$ is the Euclidean norm. A Gaussian radial basis function is closely related to the density function of the Normal distribution with $\varepsilon$ related to the variance $\sigma^2$ by $\varepsilon^2 = 2\sigma^2$. A larger $\varepsilon$; i.e., a larger $\sigma^2$, corresponds to a "flatter" radial basis function that leads to more averaging across the surrounding radial basis functions. A smaller $\varepsilon$ causes faster decay of the radial basis function from its center and, thus, a more localized effect.

I use a set of Gaussian radial basis functions together with low order polynomials to interpolate the value function on the quasi-random grid. To capture the overall pattern, the value function on the quasi-random grid is projected onto a set of low order polynomials of the state variables. The local fluctuations left in the residuals are approximated by a linear combination of a set of Gaussian radial basis functions:

$$\mathcal{V}(X) = \sum_{i=1}^{n_r} \beta_i e^{-\varepsilon_i^{-2}\|X-C_i\|_2^2} \tag{2.9}$$

where $n_r$ is the number of radial basis functions; $\{\beta_i\}_{i=1}^{n_r}$, $\{\varepsilon_i\}_{i=1}^{n_r}$, and $\{C_i\}_{i=1}^{n_r}$ are the weights, widths, and centers of the radial basis functions.

The radial basis function approximation (2.9) is very flexible in terms of the large number of parameters: the number of radial basis functions, the location of the center of each radial basis function, the width around each center, and the weights on each radial basis function. How to choose these parameters to fit the given data points and, more importantly, to provide good out-of-sample performance for interpolation is a challenge. I use trial-and-error to find a set of parameters that can provide both good in-sample and out-of-sample performance. First, all data points are randomly grouped into two sets: a training set and a validation set. Then different sets of radial basis functions, characterized by different sets of parameters, are fitted to the training data through ordinary least squares regression. The best set of radial basis functions is chosen based on out-of-sample performance, measured by the mean square error over the validation set.

Instead of choosing all the parameters simultaneously, the following algorithm is used to choose the best set of parameters in a hierarchical manner. The algorithm expands Step 2.3 of the main algorithm described in Section 2.2. There are three levels of decisions in the algorithm. The optimal combination of $n_r$ and $\varepsilon_0$ is selected by trying different combinations through Level 1 and Level 2 and comparing their out-of-sample mean square errors computed by Level 3. Given $n_r$ and $\varepsilon_0$, Level 3 fits a set of radial basis functions to the training set using iterative center selections. In general the radial basis function method can place centers at locations that are not data points. For simplicity, I only place centers on data points. The chosen centers in each iteration are the training points with the largest absolute residuals; i.e., the new centers are placed at the points where the current set of radial basis functions cannot fit well. To prevent singularity of the ordinary least squares regression, all the chosen centers are located at different points. The width of each radial basis function is set by $\varepsilon_0 h$, where the standard width $\varepsilon_0$ determines the average width across all centers, and the filling distance $h$ measures the density of surrounding centers. A larger $h$ indicates a lower density of centers in the area around this center and, thus, suggests a larger radial basis function width to cover the space between this center to the surrounding centers. A smaller $h$ indicates an area filled by more centers, where a radial basis function with smaller width and a more localized effect is more appropriate.

**Algorithm: Radial Basis Function Approximation of the Value Function**

**Level 1:** Choose the number of radial basis functions $n_r$ from a set of trial values.

**Level 2:** Given $n_r$, choose the standard width $\varepsilon_0$ from a set of trial values.

**Level 3:** Given $n_r$ and $\varepsilon_0$, fit a set of radial basis functions to the training set iteratively and report the out-of-sample mean square error.

**Step 1:** Update residuals.

**Step 2:** Sort all training points based on their absolute residuals.

**Step 3:** Expand the set of centers by choosing $n_{inc}$ non-center points from the training set with the largest absolute residuals.[7]

**Step 4:** Update the width of each radial basis function $i$ using $\varepsilon_i = \varepsilon_0 h_i$, where $h_i$ is the filling distance of center $i$.[8]

**Step 5:** Fit the current radial basis functions to the training set using ordinary least squares regression.

**Step 6:** If the total number of centers selected is less than $n_r$, go to Step 1 and select $n_{inc}$ more centers; otherwise, stop and report the out-of-sample mean square error over the validation set.

**End Level 3**

**End Level 2**

**End Level 1**

[7] In the first iteration $n_{ini}$ centers are chosen from the training set.

[8] The filling distance of center $i$, $h_i$, is defined as the minimum distance between center $i$ and all the other centers.

# Chapter 3

# Finance Application I: Portfolio Choice with Financial Constraints and Income

To demonstrate the capabilities of the methodology I solve two dynamic portfolio choice problems in Finance. The first problem considers the effects of financial constraints on an investor's optimal portfolio and consumption choices. The second problem studies the effects of an important type of friction in financial markets, capital gain taxation. Each problem is discussed in depth in Chapter 3 and Chapter 4 with the emphasis on the algorithm, the related numerical challenges, and main results.

In the paper of Roche, Tompaidis, and Yang (2009), we consider a finite-horizon discrete-time optimal portfolio and consumption choice problem for an investor who has constant relative risk aversion (CRRA) preferences, receives a stochastic income stream, has access to five risky assets, and faces financial and margin constraints: the investor cannot borrow against future income and faces a margin requirement to invest in risky assets. This model provides a rational explanation for the under-diversification of household portfolios documented in the empirical literature.

The literature on dynamic portfolio choice, pioneered by Merton (1971), suggests that under certain conditions all investors prefer the same optimal mix of risky assets (the market portfolio) regardless of their specific risk aversion or initial wealth. Individual risk aversion only affects the proportion of wealth that the investors invest in the market portfolio and the riskless asset. This is the so-called two-fund separation theorem or the mutual fund separation theorem. However under the margin requirements and a non-tradable income stream the two-fund separation theorem may not hold. Instead, investors may deviate from the market portfolio and shift their portfolios towards under-diversified portfolios which concentrate on a few assets with higher expected returns. Intuitively this asset substitution behavior can be explained by the notion of effective wealth which includes not only the investor's current financial wealth but also the discounted future income stream. Since the investor is not allowed to trade her future income, she can only make the optimal investment and consumption decisions based on her current wealth complying with the margin requirement. Comparing to an unconstrained investor who can allocate an optimal proportion of the total effective wealth into the fully diversified risky portfolio, the constrained investor's overall exposure to risky assets is restricted by the current wealth and the margin requirement. When the current wealth is only a small fraction of the effective wealth; i.e., the investor has a large amount of future income with respect to the current wealth, the constrained investor's overall exposure to risky assets may be well below the optimal amount that the unconstrained investor can achieve. Thus the constrained investor has to balance between her diversification motive and her motive for higher returns. As the constraint becomes more binding (the ratio of current wealth over the effective wealth becomes smaller) the motive for higher returns becomes stronger, which induces the investor to remove assets with lower expected returns from her portfolio.

## 3.1 Model Setup

### 3.1.1 Financial Market and Income Stream

The investor can choose from one risk-free asset and $n_a$ risky assets. The investor also receives a stochastic income $Y_t$. Both the risky assets' prices $\{S_t^i\}_{i=1}^{n_a}$ and the income stream $Y_t$ follow geometric Brownian motions:

$$\frac{dS_t^1}{S_t^1} = \mu_1 dt + \sigma_1 dZ_t^1$$
$$\vdots$$
$$\frac{dS_t^{n_a}}{S_t^{n_a}} = \mu_{n_a} dt + \sigma_{n_a} dZ_t^{n_a}$$
$$\frac{dY_t}{Y_t} = \mu_Y dt + \sigma_Y dZ_t^Y$$

where $Z_t = \left(Z_t^1, \cdots, Z_t^{n_a}, Z_t^Y\right) \in \mathbb{R}^{n_a+1}$ is a vector of Brownian motion with correlation matrix $\rho_{(n_a+1)\times(n_a+1)}$. We approximate the continuous-time dynamics by a discrete-time Markov chain using the discretization described in He (1990). In this discretization an $N$ dimensional multivariate normal distribution is described by $N+1$ nodes. Discretizing returns in this fashion preserves market completeness in discrete time.

### 3.1.2 Optimization Problem

We consider an investor who starts working at time 0 and retires at time $T$. At each intermediate time $t = 0, \cdots, T-1$, the investor's problem is described as:

**Bellman Equation**[1]

$$\begin{aligned} V_t\left(W_t\right) = \quad & \max_{x_t, c_t} u\left(c_t\right) + \beta E_t\left[g_t^{1-\gamma} V_{t+1}\left(W_{t+1}\right)\right] \\ \text{s.t.} \quad & W_{t+1} = g_t^{-1}\left(W_t - c_t + 1\right)\left(\sum_{i=1}^{n_a} x_t^i R_t^{e,i} + R^f\right) \quad \text{Wealth Evolution} \qquad (3.1) \\ & m^+ \sum_{i=1}^{n_a} x_t^{i+} + m^- \sum_{i=1}^{n_a} x_t^{i-} \leq 1 \quad \text{Margin Constraint} \end{aligned}$$

with the terminal condition

$$V_T\left(W_T\right) = \phi \frac{\left(W_T - 1\right)^{1-\gamma}}{1-\gamma}$$

[1]The Bellman equation is homogenous in terms of wealth, income, and consumption. For simplicity, we choose to scale by income. As a result, the investor receives one unit of income at each period. Wealth and consumption are interpreted as wealth over income ratio and consumption over income ratio.

where $V(\cdot)$ is the value function; $W$ is the wealth (scaled by income); $x$ is the vector of portfolio weights; $c$ is the consumption (scaled by income); $u(\cdot) = (\cdot)^{1-\gamma} / (1-\gamma)$ is the power utility with risk aversion coefficient $\gamma$; $\beta$ is the time discount factor; $g$ is the income growth rate; $R^e$ is the vector of excess return of risky assets; $R^f$ is the return of the risk-free asset; $m^+$ and $m^-$ are the margin requirements on long positions and short positions; $n_a$ is the number of risky assets; $\phi$ is the bequest factor.

The bequest factor $\phi$ determines the relative importance of the terminal wealth $W_T$ versus the intermediate consumption $c_t$. If the investor has an expected remaining life of $\tau$ years after retirement and the opportunity set remains constant, then the factor $\phi$ is given by

$$
\begin{aligned}
\phi &= \left[\frac{1-(\beta\alpha)^{1/\gamma}}{1-(\beta\alpha)^{(\tau+1)/\gamma}}\right]^{-\gamma} \\
\alpha &= E\left[\left(\sum_{i=1}^{n_a} x^* R_t^{e,i} + R^f\right)^{1-\gamma}\right]
\end{aligned}
$$

where $x^*$ is the vector of optimal portfolio weights after retirement — see Ingersoll (1987).

### 3.1.3 Model Simplification

Model (3.1) can be further simplified by defining the total investment $I_t = W_t - c_t + 1$ as suggested by Carroll (2006). There is a one-to-one correspondence between wealth $W_t$ and total investment $I_t$:

$$
\begin{aligned}
I_t &\rightarrow W_t = I_t + c_t^*(I_t) - 1 \\
W_t &\rightarrow I_t = W_t - c_t^*(W_t) + 1
\end{aligned}
$$

Therefore we can specify a particular grid $G$ either through wealth $W_t(G)$ or equivalently through investment $I_t(G)$. However, discretizing the total investment $I_t$, instead of the wealth $W_t$, allows the separation of the portfolio choice and the consumption choice:

**Portfolio Optimization**[2]

$$
\begin{aligned}
J_t\left(I_t\right) = \quad & \max_{x_t^+, x_t^-} \beta E_t\left[g_t^{1-\gamma} V_{t+1}\left(W_{t+1}\right)\right] \\
\text{s.t.} \quad & W_{t+1} = g_t^{-1} I_t\left[\sum_{i=1}^{n_a}\left(x_t^{i+} - x_t^{i-}\right) R_t^{e,i} + R^f\right] \\
& m^+ \sum_{i=1}^{n_a} x_t^{i+} + m^- \sum_{i=1}^{n_a} x_t^{i-} \leq 1 \\
& x_t^{i+}, x_t^{i-} \geq 0, i = 1, \cdots, n_a
\end{aligned}
\tag{3.2}
$$

**Consumption Optimization**

$$
V_t\left(W_t\right) = \max_{c_t} u\left(c_t\right) + J_t\left(I_t\right) = \max_{c_t} u\left(c_t\right) + J_t\left(W_t - c_t + 1\right) \tag{3.3}
$$

where $J(\cdot)$ is the value function of the portfolio optimization step; $x^+$ and $x^-$ are vectors of long positions and short positions on the risky assets. We can solve problem (3.1) by solving the two subproblems in a sequential order; i.e., solve the portfolio optimization first then solve the consumption optimization.

## 3.2 Solution Methodology

As a special case of the general recursion (2.1), problem (3.1) can be solved by the general algorithm described in Chapter 2 with further improvements considering its special features. First, the one-dimensional state space can be efficiently discretized using a grid with more grid points placed at low wealth levels in an exponential manner as suggested by Carroll (2006). Second, given the separation of portfolio optimization and consumption optimization, finding the optimal control in a sequential order (portfolio optimization followed by consumption optimization) is easier than finding the optimal values of all the controls simultaneously. Third, the Karush–Kuhn–Tucker conditions

[2] To maintain equivalence we also need the constraints $x_t^{i+} x_t^{i-} = 0, i = 1, \cdots, n_a$. However, one can show that dropping these constraints will expand the feasible region but will not introduce new optimal solutions which are non-trivially different.

of portfolio optimization are both necessary and sufficient,[3] which makes enumerating all specifications of the complementary conditions unnecessary.

The consumption optimization problem (3.3) is an one-dimensional unconstrained optimization problem, which can be solved efficiently by many existing techniques. The portfolio optimization problem (3.2) is a constrained stochastic optimization problem, which can be solved by applying the TRIC algorithm to approximate the conditional expectations in the following optimality conditions:

**Karush–Kuhn–Tucker Conditions of Portfolio Optimization**

$$
\begin{array}{ll}
0 = \beta I_t E_t \left\{ g_t^{-\gamma} \frac{\partial V_{t+1}(W_{t+1})}{\partial W_{t+1}} R_t^{e,i} \right\} + \mu_t^{i+} - \lambda_t m^+, i = 1, \cdots, n_a & \text{FOCs} \\
0 = -\beta I_t E_t \left\{ g_t^{-\gamma} \frac{\partial V_{t+1}(W_{t+1})}{\partial W_{t+1}} R_t^{e,i} \right\} + \mu_t^{i-} - \lambda_t m^-, i = 1, \cdots, n_a & \text{FOCs} \\
0 = \mu_t^{i+} x_t^{i+}, i = 1, \cdots, n_a & \text{Complementarity} \\
0 = \mu_t^{i-} x_t^{i-}, i = 1, \cdots, n_a & \text{Complementarity} \\
0 = \lambda_t \left( 1 - m^+ \sum_{i=1}^{n_a} x_t^{i+} - m^- \sum_{i=1}^{n_a} x_t^{i-} \right) & \text{Complementarity} \\
1 \geq m^+ \sum_{i=1}^{n_a} x_t^{i+} + m^- \sum_{i=1}^{n_a} x_t^{i-} & \text{Feasibility} \\
0 \leq x_t^{i+}, x_t^{i-}, \mu_t^{i+}, \mu_t^{i-}, \lambda_t, i = 1, \cdots, n_a & \text{Feasibility}
\end{array}
$$

where $\left\{\mu_t^{i+}\right\}_{i=1}^{n_a}$ and $\left\{\mu_t^{i-}\right\}_{i=1}^{n_a}$ are Lagrange multipliers of the non-negative constraints on long positions and short positions; $\lambda_t$ is the Lagrange multiplier of the margin constraint.

The resulting algorithm that combines the general algorithm and the special features can solve problem (3.1) efficiently. On a computer with a 2.66GHz CPU and 1.92GB memory, it takes approximately 20 hours to solve a numerical example with 45 periods (age 20 to age 65), five risky assets, a stochastic income stream, 1,000 grid points, 300 test solutions, and no-short-sale-no-borrowing constraint.

[3] The sufficiency is justified by the concave objective function and the linear constraints.

## 3.3 Numerical Results

### 3.3.1 Calibration

To apply the numerical algorithm, we consider the case of an investor that receives income from age 20 to age 65, at which point she retires. After retirement the investor has an expected lifetime of 20 years, which matches the data for a 65 year old female in the 2004 Mortality Table — see Social Security Administration (2004). For the base case we assume that income grows deterministically at a constant growth rate of 3% per year, in line with the assumptions in Viceira (2001). We assume that the investor is not able to either borrow or short any of the assets. For comparative statics we also consider stochastic income stream, margin constraints on long and short positions, and non-negative wealth constraints. The base case parameters are listed in Table 3.1.

The opportunity set available to the investor includes five risky assets corresponding to the indices of five industries: Consumer, Manufacturing, High Tech, Health, and Other. To calculate the covariance matrix for the five industries we constructed real returns for each industry using the inflation data provided in Robert Shiller's website, see Shiller (2003), to deflate the annual returns of the five industry portfolios between 1927 and 2004, provided in Kenneth French's website, see French (2008). The expected returns for each industry were computed using the methodology proposed by Black and Litterman (1992), by matching the market capitalization weights for each industry in July 2008, provided in Kenneth French's website, to the relative weights that a CRRA investor who receives no income would allocate to each industry within her equity portfolio. The risk free interest rate was computed from the data in Robert Shiller's website to match the realized one year real interest rate between 1927 and 2004.

Table 3.1: Application I - Parameter Values of the Base Case

| | | | | | |
|---|---|---|---|---|---|
| Number of periods | 45 years (age 20 to age 65) | | | | |
| Risk aversion | | | | | 3 |
| Long margin | | | | | 1 |
| Short margin | | | | | $\infty$ |
| Time discount factor (annually) | | | | | 0.98 |
| Interest rate (annually) | | | | | 1.4% |
| Income growth rate (annually) | | | | | 3% |
| | | | | | |
| | Cnsmr | Manuf | HiTec | Hlth | Other |
| Asset drift (annually) | 8.51% | 7.83% | 9.51% | 6.97% | 8.87% |
| Asset volatility (annually) | 28.9% | 25.8% | 33.3% | 26.6% | 29.7% |
| Correlations between assets | 1.000 | 0.898 | 0.832 | 0.732 | 0.932 |
| | | 1.000 | 0.848 | 0.698 | 0.930 |
| | | | 1.000 | 0.772 | 0.856 |
| | | | | 1.000 | 0.727 |
| | | | | | 1.000 |

### 3.3.2 Diversification Measures

Calvet, Campbell, and Sodini (2008) present an empirical analysis of diversification of household portfolios in Sweden, and describe several measures that quantify the degree that investors deviate from mean-variance optimal portfolios. We use the same measures in order to determine the potential magnitude of the impact of the financial constraints on diversification. We present the measures below, following the description in Calvet et al. (2008).

Denoting by $r_{h,t}, r_{B,t}$ the returns of the risky asset portfolios of the constrained and unconstrained investors, respectively, we have the following variance decomposition

$$r_{h,t} = \alpha_h + \beta_h r_{B,t} + \epsilon_{h,t}$$

and, if we denote by $\sigma_B, \sigma_h$ the standard deviation of the returns of the portfolio of the unconstrained and constrained investors respectively, we have

$$\sigma_h^2 = \beta_h^2 \sigma_B^2 + \sigma_{i,h}^2$$

The interpretation of this decomposition is that the portfolio of the constrained investor has *systematic risk* $|\beta_h|\sigma_B$ and *idiosyncratic risk* $\sigma_{i,h}$. The *idiosyncratic variance share* is given by

$$\frac{\sigma_{i,h}^2}{\sigma_h^2} = \frac{\sigma_{i,h}^2}{\beta_h^2 \sigma_B^2 + \sigma_{i,h}^2}$$

Another measure of portfolio diversification is the Sharpe ratio of the risky portion of the portfolio. We denote the Sharpe ratio of the portfolio of an investor that does not face financial constraints $S_B$, and the Sharpe ratio of a constrained investor $S_h$. These ratios are defined by the ratio of the excess return of the respective portfolio to the standard deviation of excess returns

$$S_h = \frac{\mu_h}{\sigma_h}$$

where $\mu_h, \sigma_h$, are the excess return and standard deviation of excess return for the portfolio of the constrained investor. The *relative Sharpe ratio loss* is defined by

$$RSRL_h = 1 - \frac{S_h}{S_B}$$

While the relative Sharpe ratio loss is a measure of the diversification loss in the risky asset portion of the portfolio, it does not necessarily reflect the overall efficiency loss in the portfolio. To capture this loss, we define the *return loss* as the average return loss by the investor by choosing a suboptimal portfolio

$$RL_h = w_h(S_B\sigma_h - \mu_h)$$

where $w_h$ is the portion of the portfolio invested in risky assets.

Finally, we define a measure associated with utility losses for the constrained investor, compared to the unconstrained one. It is defined as the increase in the risk-free rate that would make the constrained investor indifferent between being constrained with the higher risk-free rate and being unconstrained. In the case of a risk-averse investor with CRRA preferences with risk aversion coefficient $\gamma$, Calvet et al. (2008) calculate the utility loss from the relationship

$$UL_h = \frac{S_B^2 - S_h^2}{2\gamma}$$

### 3.3.3 Base Case

The optimal asset allocations for the base case parameters are presented in Figure 3.1 for investors 30 and 60 years old over a range of wealth to income ratios. From Figure 3.1 we notice that as the financial wealth of the investor decreases compared to her income, the investor allocates a larger proportion of her wealth to the risky assets. For a 30 year old investor the margin constraint binds if the investor's financial wealth is smaller than 12.9 times her annual income. While the proportion in which each risky asset is held within the equity portfolio does not change when the margin constraint is not binding, once the constraint binds the investor shifts her portfolio to increase the portfolio's expected return, sacrificing diversification. When the financial wealth reaches a level of 8.2, 6.5, 3.7, and 0.93 times the investor's annual income, the investor drops the Health, Manufacturing, Consumer, and Other industry stocks from her portfolio, respectively. For financial wealth levels below 93% of the investor's annual income, the investor's equity portfolio consists only of the stock of the High Tech industry. A similar pattern is observed for an investor of age 60. In that case, since the remaining income spans a smaller number of years; i.e., the discounted value of future earnings is smaller than the 30 year old investor, the constraint binds at a lower level of the financial wealth equal to 2.4 times annual income. For lower levels of the financial wealth to income ratio the 60 year old investor also shifts her equity portfolio, dropping the Health, Manufacturing, Consumer, and Other industry stocks at ratios of 1.8, 1.6, 1.1, and 0.4 respectively.

Table 3.2 presents further details of the optimal allocations for different levels of the financial wealth to annual income ratio, as well as values for the various diversification measures and the investor's lifetime relative risk aversion. From the table, we notice that when the margin constraint is not binding and the ratio of financial wealth to income decreases, the investor increases the portfolio's expected return by increasing the percentage of her wealth invested in risky assets while maintaining a diversified portfolio. Once the constraint binds, further reductions in the financial wealth to annual income ratio result in a deterioration of the portfolio diversification measures. As an example, a 30 year old investor whose financial wealth is equal to one year of her labor income holds a portfolio that has 11.1% idiosyncratic volatility — which corresponds to 11.3% of the portfolio's variance — Sharpe ratio of 25.9% compared to 27.3% achieved when the portfolio is diversified, and a return loss of 48 basis points per year. The beta of the investor's equity portfolio is 14% greater than the beta of the equity part of the diversified portfolio, while the lifetime relative risk aversion of the investor is 0.23, close to that of a risk-neutral investor. Panel B of

Table 3.2 presents allocations and diversification measures for a 60 year old investor. The results are qualitatively similar to the results in Panel A, with the main difference being that the margin constraint binds at lower levels of the financial wealth to annual income ratio.

Table 3.3 presents results obtained by simulating the evolution of the portfolio of an investor starting at age 20. From Panel A we notice that the investor whose financial wealth at age 20 was twice her annual income holds, at age 30, a portfolio that almost always consists of one or two risky assets. At the same time the investor consumes slightly more than her annual labor income. At age 45 the investor starts saving for retirement and consumes less than her annual income. her portfolio is still mostly constrained by the margin requirements. At age 60 the investor has accelerated her saving behavior and is mostly unconstrained in her financial portfolio.[4]

Panel B of Table 3.3 presents the simulation results for an investor whose financial wealth at age 20 is equal to ten times her annual income. Even though this investor is relatively richer than the investor in Panel A, the margin constraint still largely binds at age 30, leading to the investor holding an under-diversified equity portfolio. Given her large financial wealth, this investor postpones saving much longer than the investor in Panel A. Overall, the results in both panels indicate that younger investors, even if they have significant amounts of financial wealth, are holding portfolios far from those held by older, unconstrained, investors.

[4]Since consumption is measured with respect to current annual labor income and since in this example income increases 3% annually, the reduction in consumption relative to income observed in the table does not necessarily imply a reduction in the actual amount consumed by the investor. Nevertheless, consumption to income ratios above 1.0 imply that the investor consumes part of her financial wealth while ratios below 1.0 imply that the investor saves part of her labor income for retirement.

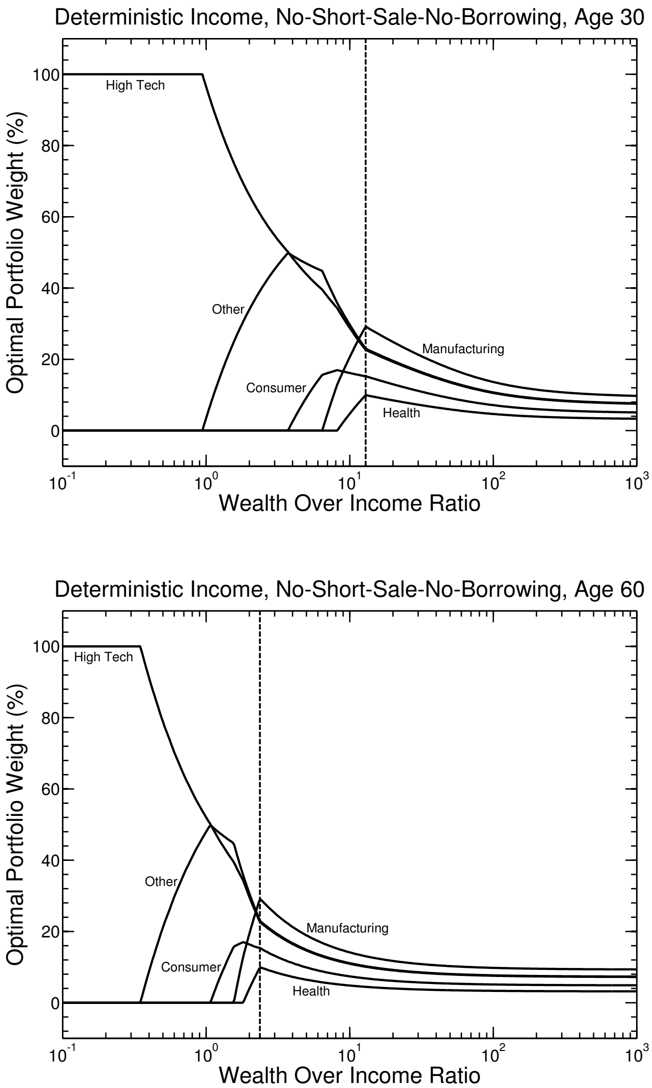


Figure 3.1: Application I - Asset Allocations of the Base Case

This figure presents the asset allocations for different levels of the financial wealth to annual income ratio. The investor receives deterministic income. The investor's opportunity set consists of a riskless asset and five risky assets calibrated to the returns of stock industry indices for the industries High Tech, Consumer, Manufacturing, Health, and Other. The parameter values for the processes followed by the risky and riskless assets are given in Table 3.1. The investor is not allowed to borrow or short an asset and is required to pay 100% of an asset's value. The top panel corresponds to a 30 year old investor and the bottom panel to a 60 year old investor.

Table 3.2: Application I - Asset Allocations and Diversification Measures of the Base Case

This table presents the optimal asset allocations and diversification measures for the base case : deterministic income with no-short-sale-no-borrowing constraint. W/Y is the current wealth to income ratio. Cnsmr, Manuf, HiTec, Hlth, and Other are the portfolio weights (as a percentage of current wealth) of the five industry indices: Consumer, Manufacturing, High Tech, Health, and Other. Margin is the total usage of the margin account in percentage. $\mu_h$ and $\sigma_h$ are the expected value and standard deviation of the excess return of the risky part of the portfolio. $\sigma_{i,h}$ is the idiosyncratic standard deviation. IVarS is the idiosyncratic variance share. $S_h$ is the Sharpe ratio of the risky part of the portfolio. RSRL$_h$ is the relative Sharpe ratio loss. RL$_h$ is the return loss of the total portfolio. UL$_h$ is the utility loss. $\beta_h$ is the $\beta$ of the constrained portfolio with respect to the unconstrained portfolio. LRRA is the lifetime relative risk aversion.

| Panel A: Age 30 | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| W/Y | Cnsmr (%) | Manuf (%) | HiTec (%) | Hlth (%) | Other (%) | Margin (%) | $\mu_h$ (%) | $\sigma_h$ (%) | $\sigma_{i,h}$ (%) | IVarS (%) | $S_h$ (%) | RSRL$_h$ (%) | RL$_h$ (%) | UL$_h$ (%) | $\beta_h$ | LRRA |
| $\infty$ | 5 | 9 | 7 | 3 | 7 | 32 | 7.43 | 27.2 | 0.0 | 0.0 | 27.3 | 0.0 | 0.00 | 0.00 | 1.00 | 3.00 |
| 1000.0 | 5 | 10 | 8 | 3 | 8 | 33 | 7.43 | 27.2 | 0.0 | 0.0 | 27.3 | 0.0 | 0.00 | 0.00 | 1.00 | 2.87 |
| 100.0 | 7 | 14 | 11 | 5 | 11 | 47 | 7.43 | 27.2 | 0.0 | 0.0 | 27.3 | 0.0 | 0.00 | 0.00 | 1.00 | 2.05 |
| 20.0 | 13 | 25 | 19 | 8 | 20 | 85 | 7.43 | 27.2 | 0.0 | 0.0 | 27.3 | 0.0 | 0.00 | 0.00 | 1.00 | 1.13 |
| 15.0 | 14 | 28 | 21 | 9 | 22 | 95 | 7.43 | 27.2 | 0.0 | 0.0 | 27.3 | 0.0 | 0.00 | 0.00 | 1.00 | 1.02 |
| 12.0 | 15 | 27 | 24 | 9 | 25 | 100 | 7.49 | 27.4 | 0.3 | 0.0 | 27.3 | 0.0 | 0.00 | 0.00 | 1.01 | 0.92 |
| 10.0 | 16 | 21 | 29 | 5 | 30 | 100 | 7.67 | 28.1 | 1.3 | 0.2 | 27.3 | 0.0 | 0.00 | 0.00 | 1.03 | 0.87 |
| 9.0 | 17 | 17 | 32 | 2 | 33 | 100 | 7.78 | 28.5 | 1.8 | 0.4 | 27.3 | 0.1 | 0.00 | 0.00 | 1.05 | 0.82 |
| 8.0 | 17 | 12 | 35 | 0 | 37 | 100 | 7.91 | 29.0 | 2.5 | 0.7 | 27.3 | 0.2 | 0.01 | 0.00 | 1.06 | 0.80 |
| 7.0 | 16 | 5 | 38 | 0 | 42 | 100 | 8.01 | 29.4 | 3.1 | 1.1 | 27.2 | 0.3 | 0.02 | 0.01 | 1.08 | 0.77 |
| 6.0 | 14 | 0 | 41 | 0 | 45 | 100 | 8.10 | 29.8 | 3.6 | 1.5 | 27.2 | 0.5 | 0.04 | 0.01 | 1.09 | 0.73 |
| 5.0 | 9 | 0 | 44 | 0 | 47 | 100 | 8.14 | 29.9 | 3.9 | 1.7 | 27.2 | 0.6 | 0.05 | 0.01 | 1.09 | 0.63 |
| 4.0 | 2 | 0 | 48 | 0 | 49 | 100 | 8.20 | 30.2 | 4.6 | 2.3 | 27.1 | 0.8 | 0.07 | 0.02 | 1.10 | 0.56 |
| 3.0 | 0 | 0 | 55 | 0 | 45 | 100 | 8.25 | 30.5 | 5.2 | 2.9 | 27.0 | 1.1 | 0.09 | 0.03 | 1.11 | 0.49 |
| 2.5 | 0 | 0 | 59 | 0 | 41 | 100 | 8.28 | 30.7 | 5.7 | 3.5 | 26.9 | 1.4 | 0.12 | 0.03 | 1.11 | 0.44 |
| 2.0 | 0 | 0 | 66 | 0 | 34 | 100 | 8.33 | 31.1 | 6.6 | 4.5 | 26.8 | 1.9 | 0.16 | 0.05 | 1.12 | 0.41 |
| 1.7 | 0 | 0 | 72 | 0 | 28 | 100 | 8.37 | 31.4 | 7.4 | 5.5 | 26.7 | 2.4 | 0.21 | 0.06 | 1.12 | 0.37 |
| 1.4 | 0 | 0 | 80 | 0 | 20 | 100 | 8.42 | 31.9 | 8.5 | 7.1 | 26.4 | 3.2 | 0.28 | 0.08 | 1.13 | 0.29 |
| 1.2 | 0 | 0 | 87 | 0 | 13 | 100 | 8.47 | 32.3 | 9.6 | 8.8 | 26.2 | 4.1 | 0.36 | 0.10 | 1.14 | 0.28 |
| 1.0 | 0 | 0 | 96 | 0 | 4 | 100 | 8.54 | 33.0 | 11.1 | 11.3 | 25.9 | 5.3 | 0.48 | 0.13 | 1.14 | 0.23 |
| 0.7 | 0 | 0 | 100 | 0 | 0 | 100 | 8.57 | 33.3 | 11.6 | 12.2 | 25.7 | 5.8 | 0.53 | 0.14 | 1.15 | 0.19 |
| 0.4 | 0 | 0 | 100 | 0 | 0 | 100 | 8.57 | 33.3 | 11.6 | 12.2 | 25.7 | 5.8 | 0.53 | 0.14 | 1.15 | 0.09 |
| 0.0 | 0 | 0 | 100 | 0 | 0 | 100 | 8.57 | 33.3 | 11.6 | 12.2 | 25.7 | 5.8 | 0.53 | 0.14 | 1.15 | 0.00 |

Table 3.2, cont.

| Panel B: Age 60 | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| W/Y | Cnsmr (%) | Manuf (%) | HiTec (%) | Hlth (%) | Other (%) | Margin (%) | $\mu_h$ (%) | $\sigma_h$ (%) | $\sigma_{i,h}$ (%) | IVarS (%) | $S_h$ (%) | RSRL$_h$ (%) | RL$_h$ (%) | UL$_h$ (%) | $\beta_h$ | LRRA |
| $\infty$ | 5 | 9 | 7 | 3 | 7 | 32 | 7.43 | 27.2 | 0.0 | 0.0 | 27.3 | 0.0 | 0.00 | 0.00 | 1.00 | 3.00 |
| 1000.0 | 5 | 9 | 7 | 3 | 7 | 32 | 7.43 | 27.2 | 0.0 | 0.0 | 27.3 | 0.0 | 0.00 | 0.00 | 1.00 | 2.98 |
| 100.0 | 5 | 10 | 8 | 3 | 8 | 33 | 7.43 | 27.2 | 0.0 | 0.0 | 27.3 | 0.0 | 0.00 | 0.00 | 1.00 | 2.85 |
| 20.0 | 6 | 12 | 9 | 4 | 9 | 40 | 7.43 | 27.2 | 0.0 | 0.0 | 27.3 | 0.0 | 0.00 | 0.00 | 1.00 | 2.35 |
| 15.0 | 7 | 13 | 10 | 4 | 10 | 43 | 7.43 | 27.2 | 0.0 | 0.0 | 27.3 | 0.0 | 0.00 | 0.00 | 1.00 | 2.22 |
| 12.0 | 7 | 13 | 10 | 5 | 10 | 46 | 7.43 | 27.2 | 0.0 | 0.0 | 27.3 | 0.0 | 0.00 | 0.00 | 1.00 | 2.07 |
| 10.0 | 7 | 14 | 11 | 5 | 11 | 48 | 7.43 | 27.2 | 0.0 | 0.0 | 27.3 | 0.0 | 0.00 | 0.00 | 1.00 | 1.97 |
| 9.0 | 8 | 15 | 11 | 5 | 12 | 50 | 7.43 | 27.2 | 0.0 | 0.0 | 27.3 | 0.0 | 0.00 | 0.00 | 1.00 | 1.87 |
| 8.0 | 8 | 15 | 12 | 5 | 12 | 52 | 7.43 | 27.2 | 0.0 | 0.0 | 27.3 | 0.0 | 0.00 | 0.00 | 1.00 | 1.81 |
| 7.0 | 8 | 16 | 13 | 6 | 13 | 55 | 7.43 | 27.2 | 0.0 | 0.0 | 27.3 | 0.0 | 0.00 | 0.00 | 1.00 | 1.71 |
| 6.0 | 9 | 17 | 13 | 6 | 14 | 59 | 7.43 | 27.2 | 0.0 | 0.0 | 27.3 | 0.0 | 0.00 | 0.00 | 1.00 | 1.62 |
| 5.0 | 10 | 19 | 15 | 6 | 15 | 65 | 7.43 | 27.2 | 0.0 | 0.0 | 27.3 | 0.0 | 0.00 | 0.00 | 1.00 | 1.48 |
| 4.0 | 11 | 21 | 16 | 7 | 17 | 73 | 7.43 | 27.2 | 0.0 | 0.0 | 27.3 | 0.0 | 0.00 | 0.00 | 1.00 | 1.29 |
| 3.0 | 13 | 25 | 20 | 9 | 20 | 86 | 7.43 | 27.2 | 0.0 | 0.0 | 27.3 | 0.0 | 0.00 | 0.00 | 1.00 | 1.08 |
| 2.5 | 15 | 28 | 22 | 10 | 22 | 97 | 7.43 | 27.2 | 0.0 | 0.0 | 27.3 | 0.0 | 0.00 | 0.00 | 1.00 | 0.98 |
| 2.0 | 16 | 19 | 30 | 4 | 31 | 100 | 7.71 | 28.2 | 1.5 | 0.3 | 27.3 | 0.0 | 0.00 | 0.00 | 1.04 | 0.83 |
| 1.7 | 16 | 8 | 36 | 0 | 39 | 100 | 7.97 | 29.2 | 2.8 | 0.9 | 27.3 | 0.2 | 0.02 | 0.01 | 1.07 | 0.74 |
| 1.4 | 12 | 0 | 42 | 0 | 46 | 100 | 8.12 | 29.8 | 3.8 | 1.6 | 27.2 | 0.5 | 0.04 | 0.01 | 1.09 | 0.67 |
| 1.2 | 5 | 0 | 47 | 0 | 48 | 100 | 8.17 | 30.1 | 4.3 | 2.0 | 27.1 | 0.7 | 0.06 | 0.02 | 1.10 | 0.57 |
| 1.0 | 0 | 0 | 52 | 0 | 48 | 100 | 8.23 | 30.4 | 5.0 | 2.7 | 27.0 | 1.0 | 0.08 | 0.02 | 1.10 | 0.56 |
| 0.7 | 0 | 0 | 64 | 0 | 36 | 100 | 8.31 | 31.0 | 6.3 | 4.2 | 26.9 | 1.7 | 0.15 | 0.04 | 1.11 | 0.44 |
| 0.4 | 0 | 0 | 91 | 0 | 9 | 100 | 8.51 | 32.6 | 10.3 | 9.9 | 26.1 | 4.6 | 0.41 | 0.11 | 1.14 | 0.22 |
| 0.0 | 0 | 0 | 100 | 0 | 0 | 100 | 8.57 | 33.3 | 11.6 | 12.2 | 25.7 | 5.8 | 0.53 | 0.14 | 1.15 | 0.00 |

Table 3.3: Application I - Base Case Simulations

This table presents summary statistics of the simulated wealth as well as the portfolio and consumption choices that an individual investor faces starting from a given initial wealth and following the optimal strategy. The results are based on 10,000 simulation paths. $W/Y$ and $C/Y$ are the realized wealth to income ratio and consumption to income ratio. Cnsmr, Manuf, HiTec, Hlth, and Other are the portfolio weights (as a percentage of current wealth) of the five industry indices: Consumer, Manufacturing, High Tech, Health, and Other. Margin is the total usage of the margin account in percentage. $Q_{25}$, $Q_{50}$, and $Q_{75}$ are the 25% quantile, the 50% quantile (median), and the 75% quantile. SD is the standard deviation.

| Panel A: Initial wealth equal to two years of income | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | $W/Y$ | $C/Y$ | Cnsmr (%) | Manuf (%) | HiTec (%) | Hlth (%) | Other (%) | Margin (%) |
| Age 30 | $Q_{25}$ | 0.3 | 1.0 | 0 | 0 | 78 | 0 | 0 | 100 |
| | $Q_{50}$ | 0.7 | 1.1 | 0 | 0 | 100 | 0 | 0 | 100 |
| | $Q_{75}$ | 1.4 | 1.1 | 0 | 0 | 100 | 0 | 22 | 100 |
| | Mean | 1.2 | 1.1 | 1 | 0 | 88 | 0 | 11 | 100 |
| | SD | 1.7 | 0.1 | 3 | 2 | 19 | 1 | 16 | 1 |
| Age 45 | $Q_{25}$ | 1.0 | 0.8 | 0 | 0 | 59 | 0 | 0 | 100 |
| | $Q_{50}$ | 1.5 | 0.9 | 0 | 0 | 78 | 0 | 20 | 100 |
| | $Q_{75}$ | 2.5 | 0.9 | 0 | 0 | 100 | 0 | 37 | 100 |
| | Mean | 2.3 | 0.9 | 2 | 1 | 76 | 0 | 21 | 100 |
| | SD | 2.9 | 0.1 | 4 | 5 | 22 | 2 | 17 | 4 |
| Age 60 | $Q_{25}$ | 5.2 | 0.6 | 7 | 14 | 11 | 5 | 11 | 49 |
| | $Q_{50}$ | 7.1 | 0.7 | 8 | 16 | 12 | 5 | 13 | 55 |
| | $Q_{75}$ | 9.9 | 0.8 | 10 | 19 | 14 | 6 | 15 | 64 |
| | Mean | 8.1 | 0.7 | 9 | 17 | 13 | 6 | 13 | 58 |
| | SD | 4.4 | 0.2 | 2 | 4 | 3 | 1 | 4 | 13 |
| Panel B: Initial wealth equal to ten years of income | | | | | | | | | |
| | | $W/Y$ | $C/Y$ | Cnsmr (%) | Manuf (%) | HiTec (%) | Hlth (%) | Other (%) | Margin (%) |
| Age 30 | $Q_{25}$ | 2.7 | 1.2 | 0 | 0 | 26 | 0 | 21 | 100 |
| | $Q_{50}$ | 5.7 | 1.4 | 11 | 0 | 42 | 0 | 33 | 100 |
| | $Q_{75}$ | 11.3 | 1.7 | 15 | 21 | 57 | 7 | 45 | 100 |
| | Mean | 8.4 | 1.5 | 8 | 9 | 45 | 3 | 32 | 97 |
| | SD | 8.2 | 0.4 | 7 | 11 | 23 | 4 | 14 | 8 |
| Age 45 | $Q_{25}$ | 1.9 | 0.9 | 0 | 0 | 22 | 0 | 15 | 95 |
| | $Q_{50}$ | 4.0 | 1.0 | 6 | 0 | 46 | 0 | 26 | 100 |
| | $Q_{75}$ | 10.6 | 1.3 | 12 | 18 | 69 | 6 | 42 | 100 |
| | Mean | 8.4 | 1.2 | 6 | 8 | 49 | 2 | 27 | 93 |
| | SD | 10.5 | 0.4 | 6 | 10 | 28 | 3 | 15 | 15 |
| Age 60 | $Q_{25}$ | 6.3 | 0.6 | 7 | 13 | 10 | 4 | 10 | 43 |
| | $Q_{50}$ | 9.4 | 0.8 | 8 | 14 | 11 | 5 | 11 | 50 |
| | $Q_{75}$ | 14.7 | 1.1 | 9 | 17 | 13 | 6 | 13 | 58 |
| | Mean | 11.9 | 0.9 | 8 | 15 | 12 | 5 | 12 | 52 |
| | SD | 8.9 | 0.5 | 2 | 4 | 3 | 1 | 3 | 13 |

### 3.3.4 Comparative Statics

#### Regulation T Margin Requirements

Figure 3.2 presents the optimal asset allocations for an investor facing a margin constraint in line with the requirements in Regulation T of 50% for long positions and 150% for short positions. For the calibrated parameter values from Table 3.1, the investor never shorts any of the risky assets. Compared to Figure 3.1, the investor is unconstrained for a greater range of her financial wealth to income ratio, with the allocations being identical when the margin constraint does not bind for either investor. The margin constraint for the investor that faces the Regulation T margin requirements binds at a level of financial wealth equal to 2.3 times her annual income at age 30 and 91% of her annual income at age 60. The order that assets drop out is the same as in the base case, and the last asset held in the portfolio is the stock corresponding to the High Tech industry, which is exclusively held at levels of financial wealth below 18% of annual income at age 30 and below 12% of annual income at age 60.

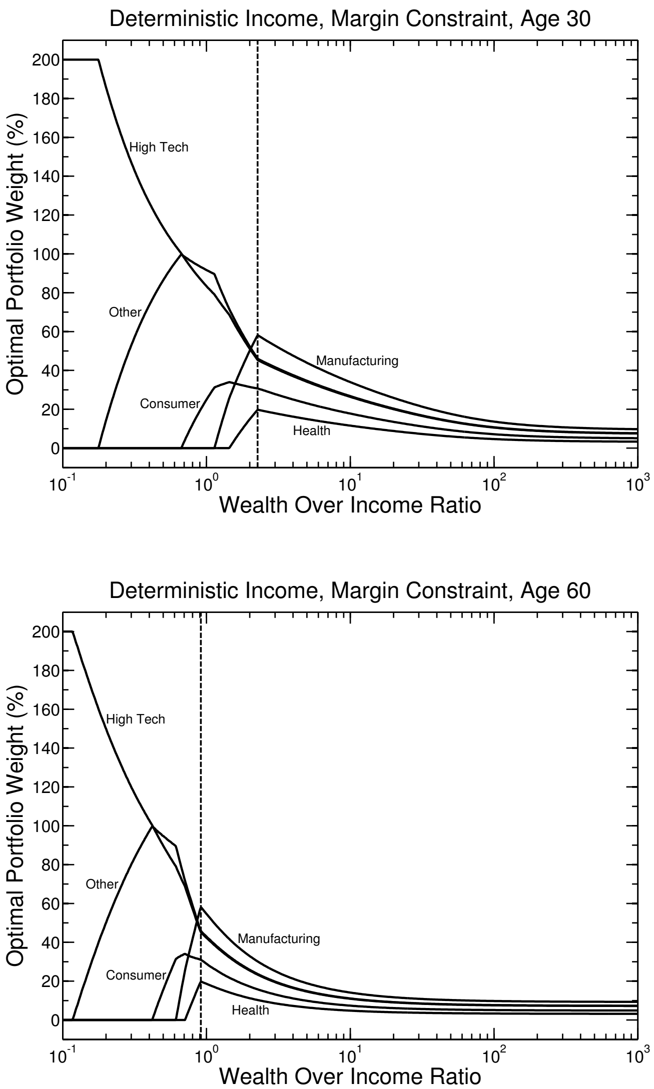


Figure 3.2: Application I - Asset Allocations with Margin Requirements

This figure presents the asset allocations for different levels of the financial wealth to annual income ratio. The investor receives deterministic income. The investor's opportunity set consists of a riskless asset and five risky assets calibrated to the returns of stock industry indices for the industries High Tech, Consumer, Manufacturing, Health, and Other. The parameter values for the processes followed by the risky and riskless assets are given in Table 3.1. The investor is allowed purchase a risky asset with 50% margin. The top panel corresponds to a 30 year old investor and the bottom panel to a 60 year old investor.

### Stochastic Income

Figure 3.3 presents the optimal asset allocations when the annual standard deviation of income growth is 10%, in line with the value used by Viceira (2001). The remaining parameters are the same as in the base case, given in Table 3.1. From the figure we notice that stochastic income has an effect in asset allocations: both for age 30 and age 60 investors, allocations in the industry stocks are reduced compared to the base case of deterministic income, an effect intuitively expected due to the higher risk implied by the stochastic nature of income growth. While the order in which assets are dropped from the equity portfolio when the ratio of financial wealth to income decreases is the same as in Figure 3.1, the threshold when the margin binds is lower. For a 30 year old investor who receives income with deterministic growth the margin binds at a financial wealth level equal to 12.9 times her annual income, while for the investor who receives income with stochastic growth the margin constraint binds at a level of financial wealth equal to 10.4 times her annual income.

Overall, Figure 3.3 illustrates that even in the case of stochastic income the intuition developed in the base case by assuming deterministic income remains valid; i.e., an investor with low levels of financial wealth compared to labor income holds under-diversified portfolios consisting of only a few out of many possible risky assets.

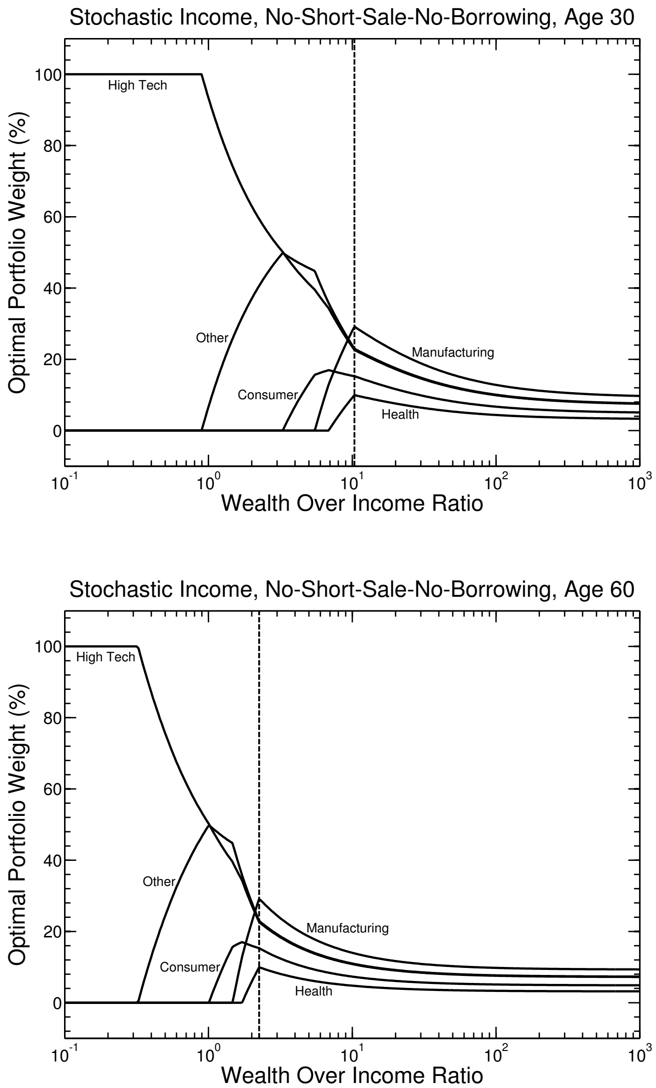


Figure 3.3: Application I - Asset Allocations with Stochastic Income

This figure presents the asset allocations for different levels of the financial wealth to annual income ratio. The investor receives stochastic income with annualized volatility 10%. The investor's opportunity set consists of a riskless asset and five risky assets calibrated to the returns of stock industry indices for the industries High Tech, Consumer, Manufacturing, Health, and Other. The parameter values for the processes followed by the risky and riskless assets are given in Table 3.1. The investor is not allowed to borrow or short an asset and is required to pay 100% of an asset's value. The top panel corresponds to a 30 year old investor and the bottom panel to a 60 year old investor.

**Non-negative Wealth Constraints**

A case of constrained choice previously studied in the literature is the case when the investor's wealth is required to remain greater or equal to zero but where the investor does not face a margin requirement, see He and Pagés (1993), El Karoui and Jeanblanc-Picqué (1998), and Duffie, Fleming, Soner, and Zariphopoulou (1997). The margin requirement is a stricter constraint, since it automatically guarantees non-negative wealth. To quantify the difference in asset allocations, Figure 3.4 presents the optimal asset allocation for an investor facing a non-negative wealth constraint, but who is otherwise identical to our base-case investor. From the figure we notice that in both the cases of a non-negative wealth constraint and of a margin requirement, investment in risky assets increases as the wealth to income ratio decreases. On the other hand, there are significant differences: unlike the case of a margin requirement, an investor that faces a non-negative wealth constraint maintains a diversified portfolio, even when her income is much greater than her wealth; in addition, the size of the risky asset portfolio for the investor that faces a non-negative wealth constraint is much larger than for an investor that faces a margin requirement. In order to finance this larger investment in risky assets, the results in the figure indicate that the investor that is constrained to maintain wealth non-negative borrows amounts up to 10 times her wealth or more, using her income as collateral.

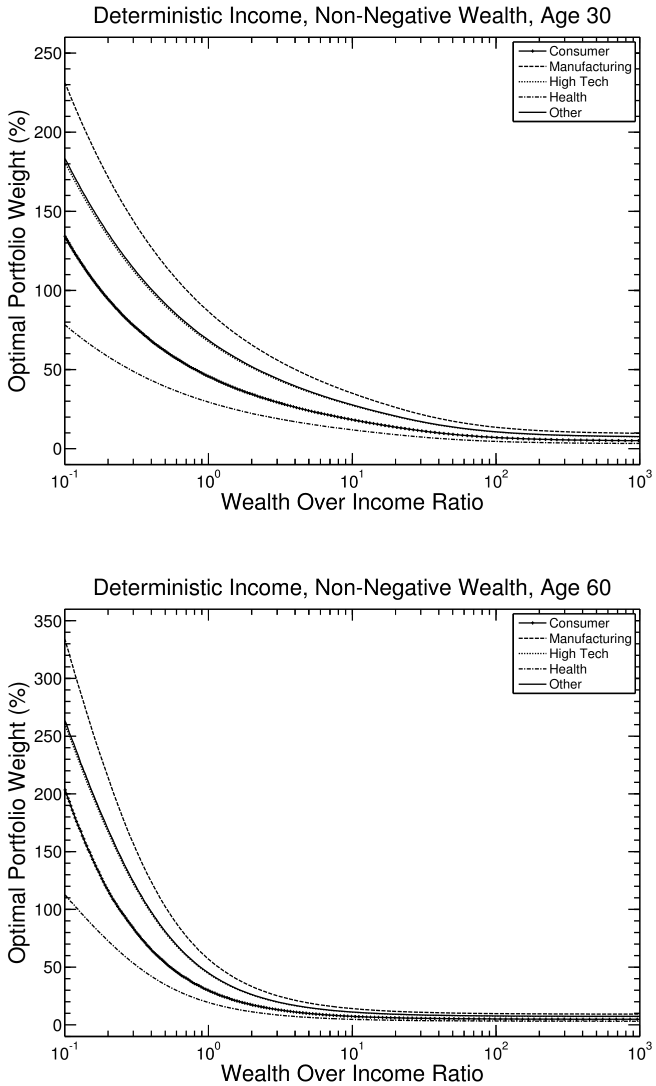


Figure 3.4: Application I - Asset Allocations with a Non-Negative Wealth Constraint

This figure presents the asset allocations for different levels of the financial wealth to annual income ratio for an investor that faces a non-negative wealth constraint but no borrowing or margin constraints. The investor receives deterministic income and has access to a riskless asset and five risky assets calibrated to the returns of stock industry indices for the industries High Tech, Consumer, Manufacturing, Health, and Other. The parameter values for the processes followed by income and the risky and riskless assets are given in Table 3.1. The top panel corresponds to a 30 year old investor and the bottom panel to a 60 year old investor.

# Chapter 4

# Finance Application II: Portfolio Choice with Capital Gain Taxation and Constraints

Capital gain taxation is an important friction faced by taxable investors when they make asset allocation and consumption decisions. However, integrating capital gain taxation into a portfolio choice setting is notoriously difficult. First, capital gain taxes are realization-based and the portfolio choice problem faced is essentially one with a state-dependent transaction cost. Second, in most countries, the portion of the tax code involving capital gain taxation is full of a myriad of details such as the treatment of long-term versus short-term capital gains, the computation of the tax basis, the taxation of different risky securities, especially derivatives, and how capital gain taxes are computed at an investor's death.

Due to the complexity of modeling the capital gain taxation, academic work studying portfolio choice with capital gain taxation often assumes the Full Use of Capital Losses (FUL); i.e., capital losses can be immediately realized for a tax rebate — see Constantinides (1983), Dybvig and Koo (1996), and Dammon, Spatt, and Zhang (2001). The tax rebate cushions the downside of holding equity by providing a positive cashflow from tax rebates whenever realized capital losses exceed realized capital gains. While the FUL assumption simplifies the analysis, it is inconsistent with the US tax code. In Ehling, Gallmeyer, Srivastava, Tompaidis, and Yang (2009) we investigate a more realistic assumption, which is consistent with the US tax code, the Limited Use of Capital Losses (LUL); i.e., realized capital losses can only be used to offset current realized capital gains or be carried forward to offset future capital gains.

Our main finding is that assuming the limited use of capital losses significantly reduces the attractiveness of risky assets and leads to a lower overall allocation to risky assets than in the FUL case. Another finding is that the tax rebate feature under the FUL assumption results in a counterfactual welfare effect, where an untaxed investor would actually prefer to pay a capital gain tax.

## 4.1 Model Setup

The investor chooses an optimal consumption and investment policy in the presence of realized capital gain taxation at trading dates $t = 0, \ldots, T$. Our assumptions concerning the exogenous price system, taxation, and the investor's portfolio problem are outlined below. The notation and model structure are based on the setting in Gallmeyer, Tompaidis, and Kaniel (2006) with an extension to accommodate for the limited use of capital losses.

### 4.1.1 Financial Market

The set of financial assets available to the investor consists of $n_a$ dividend-paying stocks and a riskless money market that pays a continuously-compounded pre-tax rate of return $r$. The stocks pay dividends with constant dividend yields, $\{\delta_i\}_{i=1}^{n_a}$. The evolutions of the ex-dividend stock prices, $\{S_t^i\}_{i=1}^{n_a}$, follow geometric Brownian motions with constant drift and volatility. We approximate the continuous-time dynamics by a discrete-time Markov chain using the discretization described in He (1990).

### 4.1.2 Wealth Evolution

We define $W_t$ as the wealth in dollars at time $t$: before consumption; before rebalancing the portfolio and paying capital gain tax; after receiving interest and paying interest tax; after receiving stock dividends and paying tax on dividends. Before rebalancing at time $t$, $W_t$ consists of a stock portfolio of weights $\{\underline{x}_t^i\}_{i=1}^{n_a}$. The remaining wealth is in the money market account. The investor chooses her consumption $c_t$ (as a fraction of $W_t$) and a new stock portfolio, $\{\bar{x}_t^i\}_{i=1}^{n_a}$, which will be held until time $t+1$. Rebalancing the stock portfolio from $\underline{x}_t$ to $\bar{x}_t$ is subject to paying capital gain tax $\phi_t$ (as a fraction of $W_t$). Following these definitions, we can write the wealth growth rate as

$$\frac{W_{t+1}}{W_t} = \sum_{i=1}^{n_a} \bar{x}_t^i \frac{S_{t+1}^i}{S_t^i} \left[1 + \delta_i \left(1 - \tau_D\right)\right] + \left(1 - c_t - \phi_t - \sum_{i=1}^{n_a} \bar{x}_t^i\right) \left[e^r - \left(e^r - 1\right) \tau_I\right]$$

where $\tau_D$ is the tax rate on dividends and $\tau_I$ is the tax rate on interest income.

### 4.1.3 Capital Gain Taxation and Limited Use of Capital Losses

To calculate capital gain taxes one needs to keep track of the average purchase price of each stock. Following the notation of Gallmeyer et al. (2006), we define the tax basis, $B_t^i$, as the weighted average of the purchase price of stock $i$ after rebalancing at time $t$, and define the relative tax basis, $b_t^i$, as

$$b_t^i = \frac{B_{t-1}^i}{S_t^i}, i = 1, \ldots, n_a$$

When stock $i$ has an embedded capital gain; i.e., $b_t^i < 1$, the investor can either increase or decrease her position, where reducing the position ($0 \leq \bar{x}_t^i \leq \underline{x}_t^i$) results in a realization of the embedded capital gain. The total realized capital gain of all stocks, $g_t$ ($g_t \geq 0$) is calculated as

$$g_t = \sum_{i=1}^{n_a} \left(1 - b_t^i\right) \left(\underline{x}_t^i - \bar{x}_t^i\right) \mathbf{1}_{\left\{b_t^i < 1,\ 0 \leq \bar{x}_t^i \leq \underline{x}_t^i\right\}}$$

where $\mathbf{1}_{\{.\}}$ is an indicator function that takes the value of 0 or 1. In the FUL case, all realized capital gains are subject to taxes. In the LUL case, the realized past capital losses, accumulated in the carry–over loss account, $l_t$ ($l_t \leq 0$), are deductible from current capital gains. Thus, the incurred capital gain tax, $\phi_t$, is calculated as

$$\phi_t = \tau_C \times \max\left\{0, g_t + l_t\right\} \tag{4.1}$$

where $\tau_C$ is the capital gain tax rate. Any unused capital losses are carried forward to the next period as

$$l_{t+1} = \left(\frac{W_{t+1}}{W_t}\right)^{-1} \times \min\left\{0, g_t + l_t\right\} \tag{4.2}$$

When stock $i$ has an embedded capital loss; i.e., $b_t^i > 1$, the investor's optimal strategy is to realize the loss immediately by liquidating the position $\underline{x}_t^i$ to 0. In the FUL case, realized capital losses are subject to tax rebates, or negative taxes, and result in an increase in financial wealth when the loss is realized. In the LUL case, realized capital losses augment the carry-over loss account, $l_t$, which can be used to offset current capital gains as in equation (4.1) or be carried forward as in equation (4.2).

The relative tax basis of stock $i$, $b_t^i$, evolves according to the investor's rebalancing decisions. If $b_t^i > 1$, because of the immediate liquidation of the position in stock $i$, any new purchase of this stock is at the price of $S_t^i$ and the relative tax basis next period is $b_{t+1}^i = S_t^i / S_{t+1}^i$. If $b_t^i < 1$ and the investor reduces her position in stock $i$, there is no new purchase of this stock and the average purchase price of this stock, measured in dollars, remains constant. In terms of the relative tax basis, $b_t^i$ needs to be adjusted to reflect the price change of stock $i$; i.e., $b_{t+1}^i = \left(S_t^i / S_{t+1}^i\right) b_t^i$. If $b_t^i < 1$ and the investor increases her position in stock $i$ by purchasing additional shares at the price of $S_t^i$, the average purchase price, measured in dollars, is updated with this new purchase. Thus, the relative tax basis next period is given by

$$b_{t+1}^i = \frac{S_t^i}{S_{t+1}^i}\left[\frac{\underline{x}_t^i b_t^i + (\bar{x}_t^i - \underline{x}_t^i)}{\bar{x}_t^i}\right] = \frac{S_t^i}{S_{t+1}^i}\left[\left(b_t^i - 1\right)\frac{\underline{x}_t^i}{\bar{x}_t^i} + 1\right]$$

When an investor dies, capital gain taxes are forgiven and the tax basis of the stock resets to the current market price. This is consistent with the reset provision in the U.S. tax code. Dividend and interest taxes are still paid at the time of death.

### 4.1.4 Optimization Problem

To finance consumption, the investor trades in the money market and the stocks. Given an initial equity endowment, a consumption and security trading policy is an admissible trading strategy if it is self-financing, involves no short selling of the stocks, and leads to non-negative wealth over the investor's lifetime. The investor lives at most $T$ periods and faces a positive probability of death each period $t$, measured by the conditional survival probability of $e^{-\lambda_t}$. The single-period hazard rate, $\lambda_t$, $t = 0, \ldots, T-1$, are calibrated to the 1990 U.S. Life Table, compiled by the National Center for Health Statistics. We assume $t = 0$ corresponds to age 20 and $T = 80$ corresponds to age 100.

At time $T = 80$, the investor exits the economy with certainty. At time $t < T$, the investor has the conditional probability of $e^{-\lambda_t}$ to survive and $1-e^{-\lambda_t}$ to die. If the investor survives at time $t$, she chooses the optimal consumption and investment plan to maximize the sum of the current utility from consumption and the expected future utility. If death occurs at time $t$, the investor's assets totaling $W(t)$ are liquidated and used to purchase a perpetuity that pays to her heirs a constant real after-tax cash flow of $R^*W(t)$ each period starting from t+1. The quantity $R^*$ is the one-period after-tax real riskless interest rate computed using simple compounding. In terms of the nominal riskless money market rate $r$ and the inflation rate $I$, $R^*$ is given by $R^* = ((1-\tau_D)\,e^r + \tau_D)\,e^{-I} - 1$. The investor's problem at time $t$ can be described by the Bellman equation:

**Bellman Equation**

$$
\begin{aligned}
V_t\left(\underline{x}_t, b_t, l_t\right) = \max_{c_t, \bar{x}_t} \Big\{ & \left(1-e^{-\lambda_t}\right)\tfrac{1}{1-\beta}u\left(R^*\right)+e^{-\lambda_t}u\left(c_t\right) \\
& +e^{-\lambda_t}e^{-I(1-\gamma)}\beta E_t\left[\left(\tfrac{W_{t+1}}{W_t}\right)^{1-\gamma}V_{t+1}\left(\underline{x}_{t+1}, b_{t+1}, l_{t+1}\right)\right]\Big\} \\
\text{s.t.} \quad \tfrac{W_{t+1}}{W_t} &= \sum_{i=1}^{n_a}\bar{x}_t^i\tfrac{S_{t+1}^i}{S_t^i}\left[1+\delta_i\left(1-\tau_D\right)\right]+\left(1-c_t-\phi_t-\sum_{i=1}^{n_a}\bar{x}_t^i\right)\left[e^r-\left(e^r-1\right)\tau_I\right] \\
\underline{x}_{t+1}^i &= \bar{x}_t^i\tfrac{S_{t+1}^i}{S_t^i}\left(\tfrac{W_{t+1}}{W_t}\right)^{-1}, i=1,\ldots,n_a \\
b_{t+1}^i &= \tfrac{S_t^i}{S_{t+1}^i}\mathbf{1}_{\left\{b_t^i\geq 1\right\}}+\tfrac{S_t^i}{S_{t+1}^i}b_t^i\mathbf{1}_{\left\{b_t^i<1, 0\leq\bar{x}_t^i\leq\underline{x}_t^i\right\}} \\
&\quad +\tfrac{S_t^i}{S_{t+1}^i}\left[\left(b_t^i-1\right)\tfrac{\underline{x}_t^i}{\bar{x}_t^i}+1\right]\mathbf{1}_{\left\{b_t^i<1, \bar{x}_t^i>\underline{x}_t^i\right\}}, i=1,\ldots,n_a \\
g_t &= \sum_{i=1}^{n_a}\left(1-b_t^i\right)\left(\underline{x}_t^i-\bar{x}_t^i\right)\mathbf{1}_{\left\{b_t^i<1,\ 0\leq\bar{x}_t^i\leq\underline{x}_t^i\right\}} \\
\phi_t &= \tau_C\times\max\left\{0, g_t+l_t\right\} \\
l_{t+1} &= \left(\tfrac{W_{t+1}}{W_t}\right)^{-1}\times\min\left\{0, g_t+l_t\right\} \\
m\sum_{i=1}^{n_a}\bar{x}_t^i &\leq 1-c_t-\phi_t \\
\bar{x}_t^i &\geq 0, i=1,\ldots,n_a
\end{aligned}
\tag{4.3}
$$

where $V(\cdot)$ is the value function; $u(\cdot)=(\cdot)^{1-\gamma}/(1-\gamma)$ is the power utility with risk aversion coefficient $\gamma$; $\beta$ is the time discount factor; $I$ is the inflation rate; $m$ is the margin requirement on long positions.

## 4.2 Solution Methodology

Problem (4.3) belongs to the class of high-dimensional singular control problems, which are challenging to solve because its high-dimensionality and singularity.

To model the capital gain taxation, a state-dependent transaction cost, both the initial stock positions and the relative tax basis need to be tracked for each stock;

i.e., two state variables for each stock. In addition, modeling the limited use of capital losses requires an additional state variable, the carry-over loss. In other words, $n_a$ stocks results in $2n_a + 1$ state variables, which are all endogenous state variables that cannot be solved efficiently by simulation-based methods. I address the problem of high-dimensionality by the meshfree discretization and interpolation methods, discussed in Section 2.7. More specifically, I discretize the carry-over loss using a regular grid, where the grid points are placed more densely towards zero. For each level of the carry-over loss, I use a quasi-random grid to discretize the state space, spanned by the initial position, $\underline{x}_t$, and the relative tax basis, $b_t$.

The problem of singularity is caused by the non-differentiability of $b_{t+1}$ and $l_{t+1}$ with respect to the control variables $\bar{x}_t$ at the boundaries, where $g_t(\underline{x}_t, \bar{x}_t, b_t) + l_t = 0$ or $\bar{x}_t^i = \underline{x}_t^i, i = 1, \ldots, n_a$. To overcome this difficulty, I partition the control space at those non-differentiable boundaries by defining the following sets:

| | |
|---|---|
| All stocks: | $A = \{1, \ldots, n_a\}$ |
| Degenerate stocks: | $A_t^D = \{i \in A : \underline{x}_t^i = 0 \text{ or } b_t^i \geq 1\}$ |
| Non-degenerate, reduce-position stocks: | $A_t^{RP} = \left\{i \in A \backslash A_t^D : 0 \leq \bar{x}_t^i \leq \underline{x}_t^i\right\}$ |
| Non-degenerate, increase-position stocks: | $A_t^{IP} = \left\{i \in A \backslash A_t^D : \bar{x}_t^i > \underline{x}_t^i\right\}$ |

It is obvious that $\left(A_t^D, A_t^{RP}, A_t^{IP}\right)$ forms a partition of $A$. Given a grid point $(\underline{x}_t, b_t, l_t)$, all the possible partitions of the set $A$ are enumerated. Under each partition, a locally optimal solution, associated with this partition, is found by solving the first order optimality conditions (4.4) using the TRIC method. The globally optimal solution is selected among all locally optimal solutions by comparing their objective functions.

**Karush–Kuhn–Tucker Conditions**[1]

$$
\begin{aligned}
0 &= e^{-\lambda_t} e^{-I(1-\gamma)} \beta \frac{\partial}{\partial \bar{x}_t^i} E_t \left[ \left( \frac{W_{t+1}}{W_t} \right)^{1-\gamma} V_{t+1} \left( \underline{x}_{t+1}, b_{t+1}, l_{t+1} \right) \right] && \text{FOC of } \bar{x}_t^i \\
&\quad -\lambda_t^{RP,i} \mathbf{1}_{\left\{ i \in A_t^{RP} \right\}} + \lambda_t^{IP,i} \mathbf{1}_{\left\{ i \in A_t^{IP} \right\}} + \lambda_t^{i+} \\
&\quad -\lambda_t^g \frac{\partial g_t}{\partial \bar{x}_t^i} + \lambda_t^m \left( -\frac{\partial \phi_t}{\partial \bar{x}_t^i} - m \right), i \in A \\
0 &= e^{-\lambda_t} e^{-I(1-\gamma)} \beta \frac{\partial}{\partial c_t} E_t \left[ \left( \frac{W_{t+1}}{W_t} \right)^{1-\gamma} V_{t+1} \left( \underline{x}_{t+1}, b_{t+1}, l_{t+1} \right) \right] && \text{FOC of } c_t \\
&\quad + e^{-\lambda_t} c_t^{-\gamma} - \lambda_t^m \\
0 &= \lambda_t^g \left( g_t + l_t \right) && \text{Complementarity} \\
0 &= \lambda_t^m \left( 1 - c_t - \phi_t - m \sum_{i \in A} \bar{x}_t^i \right) && \text{Complementarity} \\
0 &= \lambda_t^{RP,i} \left( \underline{x}_t^i - \bar{x}_t^i \right), i \in A_t^{RP} && \text{Complementarity} \qquad (4.4) \\
0 &= \lambda_t^{IP,i} \left( \bar{x}_t^i - \underline{x}_t^i \right), i \in A_t^{IP} && \text{Complementarity} \\
0 &= \lambda_t^{i+} \bar{x}_t^i, i \in A && \text{Complementarity} \\
0 &\leq \lambda_t^g, -\left( g_t + l_t \right) && \text{Feasibility} \\
0 &\leq \lambda_t^m, 1 - c_t - \phi_t - m \sum_{i \in A} \bar{x}_t^i && \text{Feasibility} \\
0 &\leq \lambda_t^{RP,i}, \left( \underline{x}_t^i - \bar{x}_t^i \right), i \in A_t^{RP} && \text{Feasibility} \\
0 &\leq \lambda_t^{IP,i}, \left( \bar{x}_t^i - \underline{x}_t^i \right), i \in A_t^{IP} && \text{Feasibility} \\
0 &\leq \lambda_t^{i+}, \bar{x}_t^i, i \in A && \text{Feasibility}
\end{aligned}
$$

where $\lambda_t^g$ is the Lagrange multiplier of the LUL constraint, $g_t + l_t \leq 0$; $\lambda_t^m$ is the Lagrange multiplier of the margin constraint, $m \sum_{i \in A} \bar{x}_t^i \leq 1 - c_t - \phi_t$; $\lambda_t^{RP,i}$, $i \in A_t^{RP}$, are the Lagrange multipliers of the reduce-position constraints, $\bar{x}_t^i \leq \underline{x}_t^i$, $i \in A_t^{RP}$; $\lambda_t^{IP,i}$,

[1]For illustration purpose only the Karush-Kuhn-Tucker conditions assuming $g_t + l_t \leq 0$ are provided. Similar conditions can be derived by assuming $g_t + l_t \geq 0$.

$i \in A_t^{IP}$, are the Lagrange multipliers of the increase-position constraints, $\bar{x}_t^i \geq \underline{x}_t^i$, $i \in A_t^{IP}$; $\lambda_t^{i+}$, $i \in A$, are the Lagrange multipliers of the no-short-sale constraints, $\bar{x}_t^i \geq 0$, $i \in A$.

For each grid point, the first order optimality conditions (4.4) need to be solved repeatedly under different partitions of the state space within different iterations of the TRIC method. Although the conditional expectations in the first order optimality conditions have been approximated by linear functions, the resulting system of equations still contains one non-linear equation, which is the first order condition of $c_t$ with the term $e^{-\lambda_t} c_t^{-\gamma}$. The solution to the system of equations (4.4) is a pair of consumption and portfolio weights, $(c_t^*, \bar{x}_t^*)$. By observing the fact that the solution $c_t^*$ is very insensitive with respect to different choices of the portfolio weights $\bar{x}_t^*$, I suggest the following way to find the solution $(c_t^*, \bar{x}_t^*)$. Instead of treating (4.4) as a system of non-linear equations and solving for $c_t^*$ and $\bar{x}_t^*$ simultaneously, I separate the first order condition of $c_t$, a non-linear equation, from the other equations in (4.4), which are all linear equations. First, I assume an arbitrary portfolio weights $\bar{x}_t$ and update the value of $c_t$ by solving the first order condition of $c_t$. Then, I solve $\bar{x}_t$ from the remaining system of linear equations given the value of $c_t$. The process of updating is repeated until the pair of solutions, $(c_t, \bar{x}_t)$, converges.

It takes 200 CPUs (2.66GHz, 1.92GB memory) about 50 hours to solve a numerical example in parallel with: two stocks (five state variables), 80 periods (age 20 to age 100), 30 test solutions, 48,000 grid points each period, and 4,800 radial basis functions each period[2].

[2]The carry-over loss is discretized with 16 levels. For each level of the carry-over loss, the remaining state space is discretized with 3,000 quasi-random grid points. 300 of the grid points are selected as the centers of radial basis functions to form the approximation of the value function each period.

## 4.3 Numerical Results

### 4.3.1 Scenarios Considered

To understand how the LUL case influences optimal portfolio choice relative to the FUL case, we focus on several scenarios where an investor faces different risk-return tradeoffs between the stock and the money market as well as different tax trading costs. Relative to the LUL and the FUL assumptions, one benchmark we refer to is the case when the investor faces no capital gain taxation, abbreviated as NCGT. In this benchmark, the investor still pays dividend and interest taxes. Given the investment opportunity set is constant and the investor has constant relative risk aversion (CRRA) preferences in this benchmark, the optimal trading strategy is to hold a constant fraction of wealth in the stock at all times. Second, we also employ the FUL case as a benchmark to compare with the LUL case since the FUL assumption is commonly used in the academic literature.

In all our parameterizations, the investor begins investing at age 20 and can live to a maximum of 100 years. Hence, the maximum horizon for an investor is $T = 80$. The inflation rate is assumed to be $I = 3.5\%$. The investor's CRRA preferences are calibrated with a relative risk aversion coefficient $\gamma = 5$ and a time discount parameter $\beta = 0.96$. The bequest motive is calibrated such that the investor plans to provide a perpetual real income stream to her heirs.

Our base case choice of parameters, referred to throughout as the "Base Case", views the stock as an index fund with price dynamics consistent with a large-capitalization U.S. stock fund. We assume the expected return due to capital gains is $\mu = 8\%$, the dividend yield is $\delta = 2\%$, and the volatility is $\sigma = 16\%$. The money market's return $r$ equals 6%. The investor rebalances her portfolio once a year. The investor enters the economy at age 20 and exits no later than age 100. This choice of price system parameters is roughly comparable to those used in Dammon et al. (2001) and Gallmeyer et al. (2006). The tax rates used are set to roughly match those faced by a wealthy investor under the current U.S. tax code. We assume that interest is taxed at the investor's marginal income rate $\tau_I = 35\%$. Dividends are taxed at $\tau_D = 15\%$. The capital gain tax rate is set to the long-term rate $\tau_C = 20\%$. To be consistent with the U.S. tax code, capital gain taxes are forgiven at the investor's death.

We also consider several variations of the Base Case parameters. The first variation is meant to increase the value of the tax-loss selling option under the FUL case to capture an increased importance of tax-loss selling. An immediate way to increase the value of the FUL tax-loss selling option is just to increase the capital gain tax rate. In the "Capital Gain Tax 30% Case", the capital gain tax rate is increased to $\tau_C = 30\%$, roughly equal to the 28% rate imposed after the U.S. 1986 Tax Reform Act. This rate also provides a setting that is roughly consistent with the long-term capital gain tax rate paid in many foreign countries. For example, the capital gain tax rates in Finland, France, Sweden and Norway are currently 28%, 29%, 30%, and 28% respectively. In 2009, Germany's individual capital gain tax rate changed to approximately 28% from 0%. The second variation, the "Higher Risk Aversion Case", captures a case where stock holdings decrease for the NCGT investor and hence the dollar value of tax-loss selling decreases for the FUL investor by assuming that the relative risk aversion of the investor increases to $\gamma = 10$. Finally, we consider a variation with two stock indices instead of one, where both indices have the same drift and dividend yield as the stock index of the Base Case, and the correlation between the two indices is $\rho = 80\%$. We set the volatility of each individual stock index to $\sigma_i = \sigma/\sqrt{0.5\,(1+\rho)} = 16.87\%$, $i = 1, 2$, such that an equally weighted portfolio of the two stock indices has the same pre-tax Sharpe ratio as the stock index in the one asset cases.

### 4.3.2 The Conditional Structure of Optimal Portfolios

We begin our numerical analysis of the long-dated portfolio problem outlined in Equation (4.3) by characterizing the structure of optimal portfolios under the FUL and the LUL cases at a particular time and state. This analysis shows conditionally how the two tax cases differ as a function of the state variables. It however does not provide a full picture of the optimal strategies over the investor's lifetime as these strategies are just snapshots at a particular state and time. An analysis of lifetime portfolio holdings is undertaken in the Section 4.3.3.

**The Base Case**

Figure 4.1 presents optimal portfolio weights on stock conditional on the beginning period stock-to-wealth and basis-to-price ratios for the FUL and the LUL cases. The LUL case is assumed to have a zero carry-over loss. The parameters used are the Base Case parameters. Additionally, Table 4.1 provides the same information numerically as Figure 4.1 for a subset of the beginning period stock-to-wealth and basis-to-price ratios. The left (middle) panel is for the FUL (LUL) case. The right panel computes the percentage increase of the stock-to-wealth ratio for the FUL case relative to the LUL case. We do not report conditional trading strategies in the LUL case for nonzero carry-over losses entering the trading period. Based on our simulation analysis in the next section, nonzero carry-over losses mainly occur with stock positions with basis-to-price ratios close to one along the investor's optimal path. Hence, this situation is well-captured by just examining trading strategies with basis-to-price ratios bigger than one entering the period. For the NCGT benchmark, the optimal portfolio choice is a stock-to-wealth ratio of 0.498 at all times for these parameters.

The left panels of Figure 4.1 document optimal portfolio choice for the FUL case at ages 20 (top panel) and 80 (bottom panel). Such trading strategies are well-documented in Dammon et al. (2001). The investor's optimal equity position is a function of the beginning period allocation and the basis-to-price ratio. Tax trading costs play a large role in the investor's portfolio choice decision. When the marginal tax costs of trading are high due to a large embedded capital gain, the investor optimally chooses to hold more equity. For example at age 20 with a beginning period stock-to-wealth ratio of 0.70, the investor trades to an equity position of 0.64 when the basis-to-price ratio is 0.5 and an equity position of 0.57 when the basis-to-price ratio is 1.0. Given the forgiveness of capital gain taxes at death, the marginal tax cost of trading effectively increases with age. At age 80 and a beginning period stock-to-wealth ratio of 0.70, the optimal equity position increases to 0.70 when the basis-to-price ratio is 0.5 and 0.59 when the basis-to-price ratio is 1.0. Similar patterns are documented for the LUL case in the right panels of Figure 4.1. However, the LUL equity strategy is non-monotonic along the basis-to-price dimension when the basis-to-price ratio is around 1.0.

Overall, optimal equity allocations under the LUL case can fall significantly below the FUL allocations. This reduction in equity is driven by the restrictions on the use of capital losses placed on the LUL investor. Equity allocations when the basis-to-price ratio is around 1.0 are most strongly influenced. For example at age 20, a basis-to-price ratio of 1.0, and a beginning period stock-to-wealth ratio of 0.60, the LUL equity allocation is 0.46 while the FUL equity allocation is 0.57. Relative to the LUL case, this FUL allocation is 25% higher as can be seen in the right top panel of Table 4.1. At a basis-to-price ratio around one, a FUL investor has the potential to tax loss sell next period and receive a tax rebate. The LUL investor however will only potentially receive a capital loss to carry forward to the next trading period.

Equally interesting is how the two different assumptions on capital losses converge under extreme situations. The LUL equity strategy converges to the FUL equity strategy when the investor is over-invested in equity with a large capital gain. In this situation, the probability that the FUL investor engaging in any tax loss selling is greatly reduced given the large locked-in capital gain in the portfolio. Hence, the two strategies converge when the basis-to-price ratio is low and the beginning period equity allocation is high. For example, the FUL and the LUL equity positions both equal 0.70 at an initial stock allocation of 0.70 and a basis-to-price ratio of 0.5 at age 80.

The LUL strategy also converges to the NCGT strategy when the investor enters the period with a large capital loss. From Table 4.1 at age 80 and a basis-to-price ratio of 1.2, the LUL equity allocation converges to the NCGT equity allocation of 0.50 for all values of beginning period stock-to-wealth ratio between 0.30 and 0.70. Here the capital loss in the portfolio is large enough that the LUL investor can trade the same strategy as the NCGT investor and offset all future capital gain taxes. Note that this convergence of the trading strategy to the NCGT case would also occur when the LUL investor's carry-over loss is large.

Summarizing, under the Base Case parameters, the LUL optimal equity allocation is non-monotonic relative to the FUL optimal equity allocation along the basis-to-price dimension. When tax costs of trading are low, the LUL allocation falls significantly relative to both the FUL and the NCGT allocations. When the LUL investor has a position with a large embedded gain, he is locked into his position like the FUL investor. When large capital losses are accumulated by the LUL investor, her trading strategy resembles the NCGT investor's strategy.

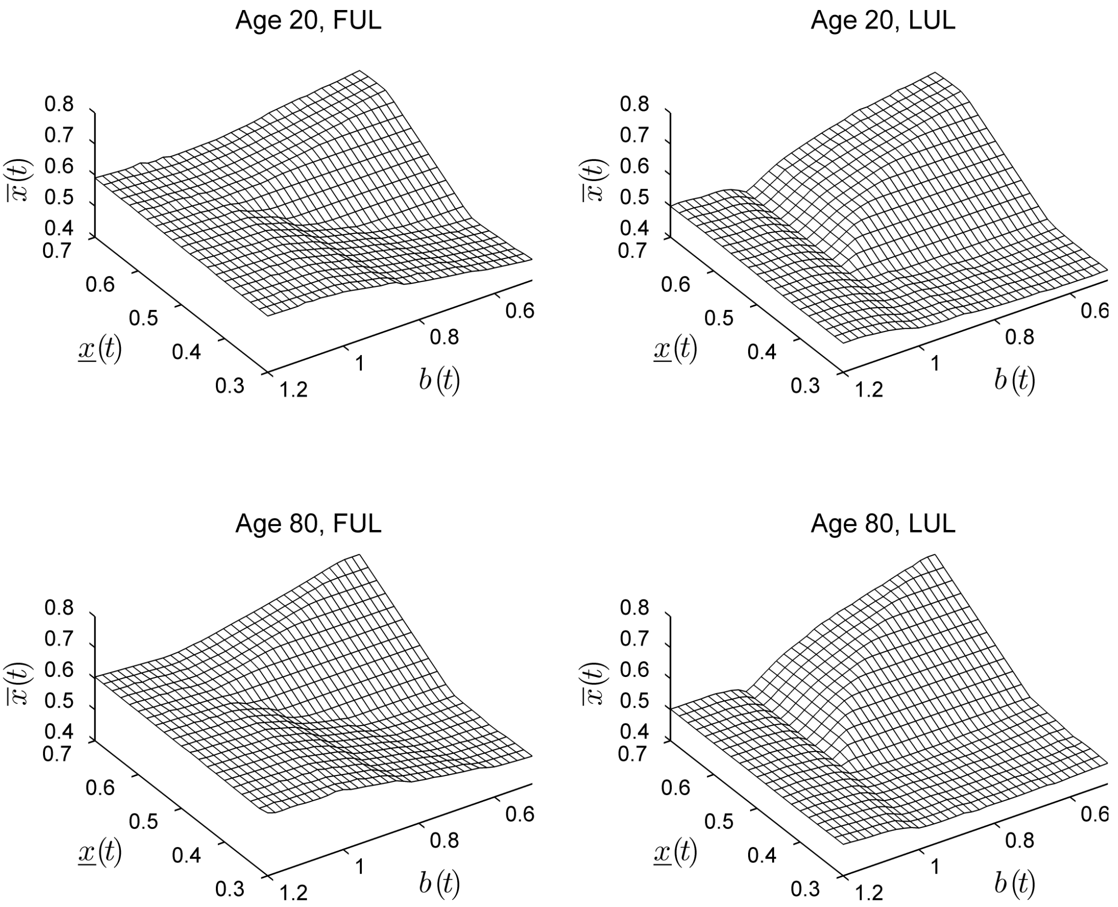


Figure 4.1: Application II - Optimal Strategies of the Base Case

The left (right) panels summarize the optimal portfolio weight, $\bar{x}$, as a function of the initial stock position, $\underline{x}$, and the initial tax basis, $b$, for the FUL (LUL) case. The top (bottom) panels present the optimal portfolio weight at age 20 (80). The LUL plots have a zero carry-over loss entering the trading period. The parameters used are the Base Case parameters summarized in Section 4.3.1.

Table 4.1: Application II - Optimal Strategies of the Base Case

The left (middle) panels summarize the optimal portfolio weight as a function of the initial stock position, $\underline{x}$, and the initial tax basis, $b$, for the FUL (LUL) case. The top (bottom) panels present the optimal portfolio weight at age 20 (80). The right panels compute the relative increase of the FUL portfolio weight with respect to the LUL portfolio weight. The LUL results have a zero carry-over loss entering the trading period. The parameters used are the Base Case parameters summarized in Section 4.3.1. All numbers in the table are in percentage.

Panel A - Age 20

| | FUL — $b_t$ | | | | | | | | LUL — $b_t$ | | | | | | | | (FUL-LUL)/LUL — $b_t$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\underline{x}_t$ | 50 | 60 | 70 | 80 | 90 | 100 | 110 | 120 | 50 | 60 | 70 | 80 | 90 | 100 | 110 | 120 | 50 | 60 | 70 | 80 | 90 | 100 | 110 | 120 |
| 30 | 47 | 48 | 50 | 52 | 55 | 57 | 58 | 58 | 43 | 43 | 44 | 44 | 45 | 46 | 49 | 49 | 8 | 11 | 15 | 18 | 22 | 25 | 20 | 18 |
| 35 | 46 | 47 | 49 | 52 | 55 | 57 | 58 | 58 | 43 | 43 | 43 | 44 | 45 | 46 | 49 | 49 | 5 | 9 | 14 | 17 | 21 | 25 | 18 | 18 |
| 40 | 45 | 46 | 48 | 51 | 54 | 57 | 58 | 58 | 43 | 43 | 43 | 44 | 45 | 46 | 49 | 49 | 4 | 7 | 11 | 17 | 21 | 25 | 18 | 18 |
| 45 | 45 | 46 | 47 | 50 | 54 | 57 | 58 | 59 | 45 | 45 | 45 | 45 | 45 | 46 | 49 | 49 | 0 | 1 | 5 | 11 | 19 | 25 | 19 | 19 |
| 50 | 50 | 50 | 50 | 50 | 53 | 57 | 58 | 59 | 50 | 50 | 50 | 50 | 51 | 46 | 49 | 50 | 0 | 0 | 0 | 0 | 5 | 25 | 18 | 18 |
| 55 | 56 | 56 | 56 | 56 | 56 | 57 | 58 | 59 | 56 | 56 | 56 | 56 | 53 | 46 | 49 | 50 | 0 | 0 | 0 | 0 | 5 | 25 | 18 | 18 |
| 60 | 61 | 61 | 60 | 59 | 58 | 57 | 59 | 59 | 61 | 61 | 60 | 57 | 53 | 46 | 49 | 50 | 0 | 0 | 1 | 4 | 9 | 25 | 19 | 18 |
| 65 | 64 | 63 | 61 | 59 | 58 | 58 | 58 | 59 | 64 | 62 | 59 | 57 | 53 | 46 | 49 | 50 | 1 | 1 | 2 | 4 | 10 | 25 | 18 | 18 |
| 70 | 64 | 63 | 61 | 59 | 58 | 57 | 58 | 59 | 63 | 62 | 60 | 57 | 53 | 46 | 49 | 50 | 1 | 1 | 2 | 4 | 10 | 25 | 18 | 18 |

Panel B - Age 80

| | FUL — $b_t$ | | | | | | | | LUL — $b_t$ | | | | | | | | (FUL-LUL)/LUL — $b_t$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\underline{x}_t$ | 50 | 60 | 70 | 80 | 90 | 100 | 110 | 120 | 50 | 60 | 70 | 80 | 90 | 100 | 110 | 120 | 50 | 60 | 70 | 80 | 90 | 100 | 110 | 120 |
| 30 | 49 | 50 | 52 | 54 | 57 | 59 | 59 | 60 | 47 | 46 | 46 | 46 | 47 | 47 | 49 | 50 | 5 | 7 | 12 | 16 | 22 | 26 | 20 | 20 |
| 35 | 49 | 49 | 51 | 53 | 57 | 59 | 59 | 60 | 47 | 46 | 46 | 46 | 46 | 47 | 49 | 50 | 4 | 6 | 10 | 15 | 22 | 26 | 20 | 19 |
| 40 | 48 | 49 | 50 | 52 | 56 | 59 | 59 | 60 | 47 | 47 | 46 | 46 | 47 | 47 | 50 | 50 | 3 | 4 | 8 | 14 | 21 | 25 | 20 | 20 |
| 45 | 49 | 49 | 49 | 52 | 56 | 59 | 60 | 60 | 48 | 47 | 46 | 46 | 46 | 47 | 50 | 50 | 2 | 3 | 6 | 12 | 21 | 25 | 20 | 20 |
| 50 | 50 | 50 | 50 | 51 | 55 | 59 | 59 | 60 | 50 | 50 | 50 | 50 | 50 | 47 | 50 | 50 | 0 | 0 | 0 | 2 | 9 | 25 | 19 | 20 |
| 55 | 55 | 55 | 55 | 55 | 55 | 59 | 60 | 60 | 55 | 55 | 55 | 55 | 55 | 47 | 50 | 50 | 0 | 0 | 0 | 0 | 0 | 25 | 19 | 20 |
| 60 | 60 | 60 | 60 | 60 | 60 | 59 | 60 | 60 | 60 | 60 | 60 | 60 | 55 | 47 | 50 | 50 | 0 | 0 | 0 | 0 | 9 | 25 | 19 | 20 |
| 65 | 65 | 65 | 65 | 63 | 60 | 59 | 60 | 60 | 65 | 65 | 64 | 61 | 55 | 47 | 50 | 50 | 0 | 0 | 1 | 4 | 9 | 25 | 19 | 21 |
| 70 | 70 | 69 | 66 | 63 | 61 | 59 | 60 | 61 | 70 | 68 | 64 | 61 | 55 | 47 | 50 | 50 | 0 | 1 | 2 | 4 | 10 | 25 | 19 | 21 |

**Comparative Statics**

Figure 4.2 and Table 4.2 explore the effect of increasing the capital gain tax to 30% on the FUL and the LUL optimal equity strategies. Given that changing the capital gain tax rate influences both the expected return and the volatility of the stock, it is not immediately apparent how portfolio allocations will be influenced. Under the LUL case, when the basis-to-price ratio is 1.0 and the capital gain tax rate is increased from 20% to 30%, the optimal equity allocation drops slightly at age 20 and remains the same at age 80. With the increase in the tax rate, the benefit of holding equity in the LUL case is reduced, which leads to lower allocations. However, from Constantinides (1983), we know that the benefit of tax loss selling is amplified by increasing the capital gain tax rate. This is consistent with our numerical results. Under the FUL case, increasing the capital gain tax rate from 20% to 30% significantly increases the optimal equity allocations at ages 20 and 80 when the basis-to-price ratio is 1.0.

The risk aversion of the investor is increased to $\gamma = 10$ in Figure 4.3 and Table 4.3 to capture a scenario where equity is a less important component of the investor's portfolio. In this case, the NCGT equity-to-wealth allocation is 0.248 as compared to 0.498 under the Base Case parameters. Increasing the risk aversion leads to largely the same patterns as in the Base Case and consistently lower equity allocations. The LUL optimal equity allocation is again non-monotonic along the basis-to-price dimension.

Figures 4.4 and 4.5 and Tables 4.4 and 4.5 report the optimal equity allocations when the investor can trade two stock indices with identical drift, volatility, and dividend yield, and a correlation of 80%. To summarize our numerical results of this high-dimensional case, we present the optimal equity allocations as a function of the basis-to-price ratios of both stocks when the beginning period stock-to-wealth ratios are 0.3 for stock 1 and 0.4 for stock 2. Given the beginning period total equity allocation fixed at a relatively high level of 0.7, the investor attempts to reduce her overall exposure to equity by selling the stock with a smaller marginal tax cost. For example, at age 20 when the basis-to-price ratios are 0.9 for Stock 1 and 0.5 for Stock 2, the

LUL investor reduces her Stock 1 holding to 0.19 and keeps her Stock 2 holding at 0.40. When the basis-to-price ratios are 0.5 for Stock 1 and 0.9 for Stock 2, the LUL investor keeps her Stock 1 holding at 0.30 and reduces her Stock 2 holding to 0.25. Another interesting finding is that the equity reduction behavior of the LUL investor in the one asset cases still persists in the two assets case in terms of the total equity allocation. Although crossovers are observed at the level of individual stocks, the total equity holdings of the LUL investor are consistently lower than the total equity holdings of the FUL investor at ages 20 and 80.

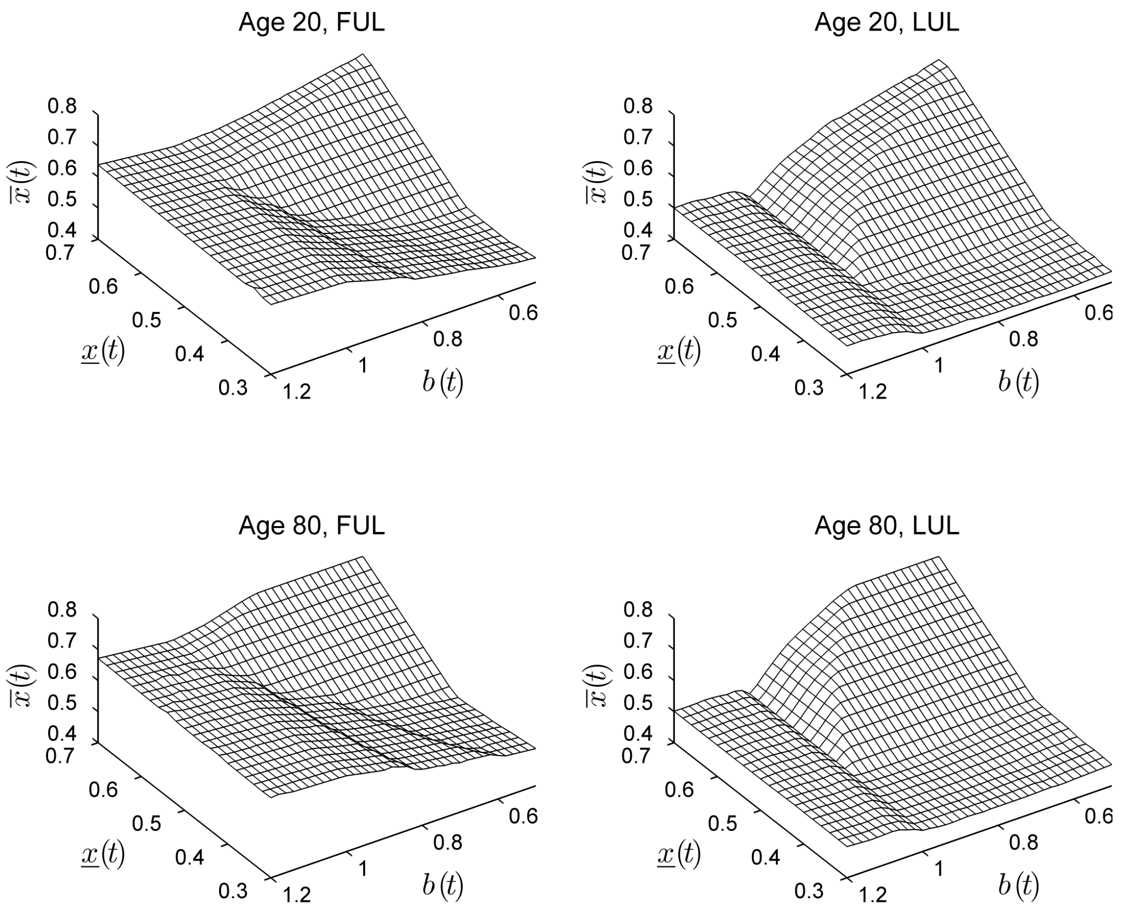


Figure 4.2: Application II - Optimal Strategies of the Capital Gain Tax 30% Case

The left (right) panels summarize the optimal portfolio weight, $\bar{x}$, as a function of the initial stock position, $\underline{x}$, and the initial tax basis, $b$, for the FUL (LUL) case. The top (bottom) panels present the optimal portfolio weight at age 20 (80). The LUL plots have a zero carry-over loss entering the trading period. The parameters used are the Base Case parameters summarized in Section 4.3.1 with a higher capital gain tax rate of 30%.

Table 4.2: Application II - Optimal Strategies of the Capital Gain Tax 30% Case

The left (middle) panels summarize the optimal portfolio weight as a function of the initial stock position, $\underline{x}$, and the initial tax basis, $b$, for the FUL (LUL) case. The top (bottom) panels present the optimal portfolio weight at age 20 (80). The right panels compute the relative increase of the FUL portfolio weight with respect to the LUL portfolio weight. The LUL results have a zero carry-over loss entering the trading period. The parameters used are the Base Case parameters summarized in Section 4.3.1 with a higher capital gain tax rate of 30%. All numbers in the table are in percentage.

Panel A - Age 20

| | FUL — $b_t$ | | | | | | | | LUL — $b_t$ | | | | | | | | (FUL-LUL)/LUL — $b_t$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\underline{x}_t$ | 50 | 60 | 70 | 80 | 90 | 100 | 110 | 120 | 50 | 60 | 70 | 80 | 90 | 100 | 110 | 120 | 50 | 60 | 70 | 80 | 90 | 100 | 110 | 120 |
| 30 | 48 | 49 | 52 | 54 | 58 | 61 | 62 | 62 | 44 | 43 | 43 | 43 | 44 | 45 | 48 | 49 | 10 | 15 | 20 | 25 | 31 | 37 | 28 | 27 |
| 35 | 47 | 48 | 51 | 53 | 57 | 61 | 62 | 63 | 44 | 43 | 43 | 43 | 44 | 45 | 49 | 49 | 7 | 12 | 18 | 23 | 30 | 37 | 27 | 28 |
| 40 | 46 | 47 | 49 | 53 | 57 | 61 | 62 | 63 | 44 | 44 | 43 | 43 | 44 | 45 | 49 | 49 | 5 | 9 | 16 | 22 | 30 | 37 | 27 | 27 |
| 45 | 46 | 47 | 49 | 52 | 56 | 61 | 62 | 63 | 45 | 45 | 45 | 45 | 45 | 45 | 49 | 49 | 2 | 3 | 7 | 14 | 24 | 36 | 27 | 27 |
| 50 | 50 | 50 | 50 | 51 | 55 | 61 | 62 | 63 | 50 | 50 | 50 | 50 | 50 | 45 | 49 | 50 | 0 | 0 | 0 | 2 | 9 | 36 | 27 | 27 |
| 55 | 56 | 56 | 56 | 56 | 56 | 61 | 62 | 63 | 56 | 56 | 56 | 56 | 54 | 45 | 49 | 50 | 0 | 0 | 0 | 0 | 3 | 36 | 27 | 26 |
| 60 | 61 | 61 | 61 | 61 | 61 | 61 | 62 | 64 | 61 | 61 | 61 | 59 | 54 | 45 | 49 | 50 | 0 | 0 | 0 | 2 | 12 | 36 | 27 | 27 |
| 65 | 66 | 66 | 65 | 64 | 62 | 61 | 63 | 64 | 66 | 65 | 63 | 59 | 54 | 45 | 49 | 50 | 0 | 0 | 4 | 7 | 14 | 36 | 27 | 27 |
| 70 | 70 | 68 | 66 | 63 | 62 | 61 | 63 | 64 | 68 | 65 | 62 | 60 | 54 | 45 | 49 | 50 | 3 | 4 | 5 | 6 | 15 | 36 | 27 | 28 |

Panel B - Age 80

| | FUL — $b_t$ | | | | | | | | LUL — $b_t$ | | | | | | | | (FUL-LUL)/LUL — $b_t$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\underline{x}_t$ | 50 | 60 | 70 | 80 | 90 | 100 | 110 | 120 | 50 | 60 | 70 | 80 | 90 | 100 | 110 | 120 | 50 | 60 | 70 | 80 | 90 | 100 | 110 | 120 |
| 30 | 52 | 54 | 56 | 58 | 62 | 64 | 65 | 66 | 47 | 46 | 46 | 46 | 46 | 47 | 50 | 50 | 11 | 17 | 21 | 26 | 34 | 38 | 31 | 32 |
| 35 | 51 | 52 | 55 | 57 | 62 | 64 | 65 | 66 | 47 | 46 | 46 | 46 | 46 | 47 | 50 | 50 | 8 | 12 | 20 | 25 | 33 | 38 | 31 | 31 |
| 40 | 50 | 51 | 54 | 57 | 61 | 65 | 65 | 66 | 48 | 47 | 46 | 46 | 46 | 47 | 50 | 50 | 5 | 9 | 17 | 23 | 33 | 38 | 31 | 31 |
| 45 | 49 | 51 | 52 | 56 | 61 | 64 | 65 | 66 | 48 | 47 | 47 | 46 | 46 | 47 | 50 | 50 | 3 | 7 | 12 | 21 | 32 | 37 | 32 | 32 |
| 50 | 50 | 50 | 52 | 56 | 60 | 64 | 65 | 66 | 50 | 50 | 50 | 50 | 50 | 47 | 50 | 50 | 0 | 0 | 3 | 11 | 20 | 36 | 31 | 33 |
| 55 | 55 | 55 | 55 | 56 | 59 | 64 | 65 | 67 | 55 | 55 | 55 | 55 | 55 | 47 | 50 | 50 | 0 | 0 | 0 | 0 | 6 | 37 | 31 | 33 |
| 60 | 60 | 60 | 60 | 60 | 60 | 64 | 66 | 67 | 60 | 60 | 60 | 60 | 59 | 47 | 50 | 50 | 0 | 0 | 0 | 0 | 2 | 36 | 31 | 33 |
| 65 | 65 | 65 | 65 | 65 | 65 | 64 | 66 | 67 | 65 | 65 | 65 | 65 | 59 | 47 | 50 | 50 | 0 | 0 | 0 | 0 | 11 | 36 | 31 | 34 |
| 70 | 70 | 70 | 70 | 70 | 67 | 65 | 66 | 67 | 70 | 70 | 70 | 66 | 59 | 47 | 50 | 50 | 0 | 0 | 0 | 5 | 13 | 37 | 31 | 34 |

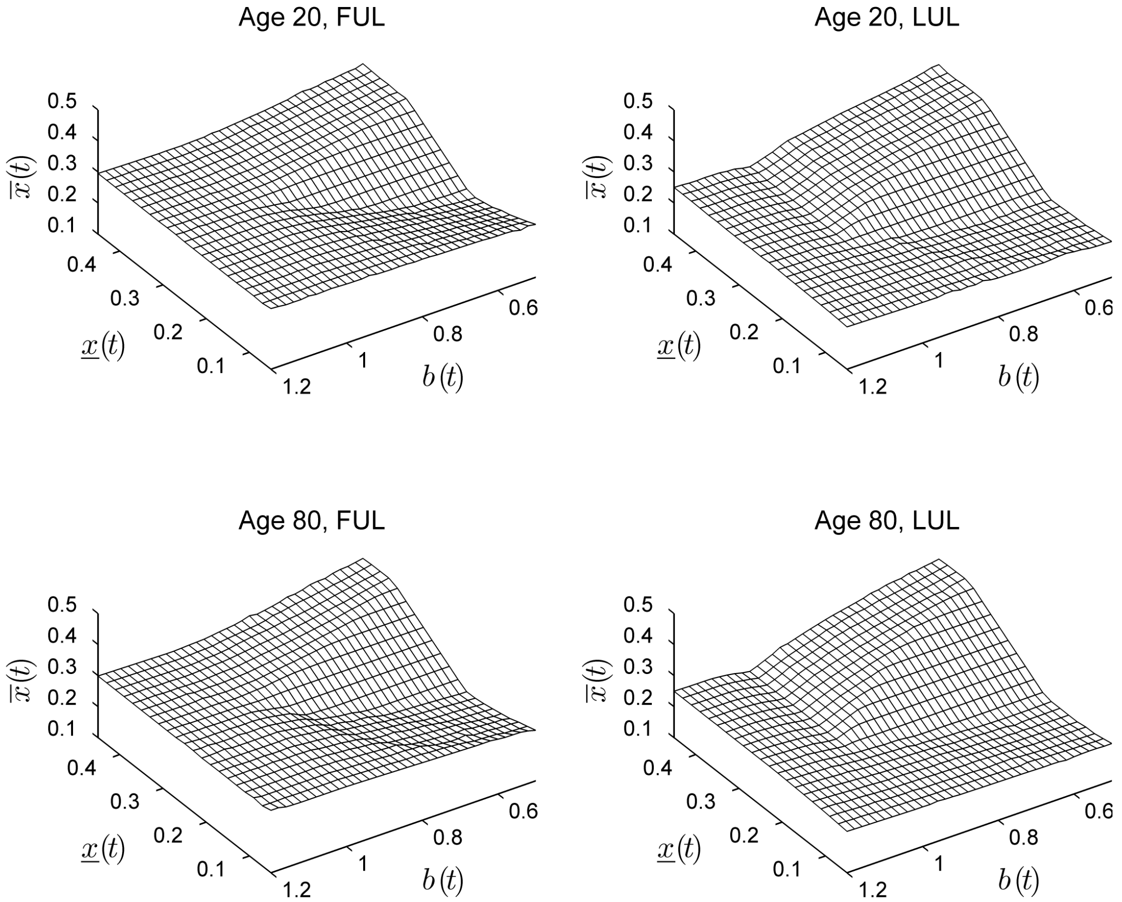


Figure 4.3: Application II - Optimal Strategies of the Higher Risk Aversion Case

The left (right) panels summarize the optimal portfolio weight, $\bar{x}$, as a function of the initial stock position, $\underline{x}$, and the initial tax basis, $b$, for the FUL (LUL) case. The top (bottom) panels present the optimal portfolio weight at age 20 (80). The LUL plots have a zero carry-over loss entering the trading period. The parameters used are the Base Case parameters summarized in Section 4.3.1 with a higher risk aversion coefficient of 10.

Table 4.3: Application II - Optimal Strategies of the Higher Risk Aversion Case

The left (middle) panels summarize the optimal portfolio weight as a function of the initial stock position, $\underline{x}$, and the initial tax basis, $b$, for the FUL (LUL) case. The top (bottom) panels present the optimal portfolio weight at age 20 (80). The right panels compute the relative increase of the FUL portfolio weight with respect to the LUL portfolio weight. The LUL results have a zero carry-over loss entering the trading period. The parameters used are the Base Case parameters summarized in Section 4.3.1 with a higher risk aversion coefficient of 10. All numbers in the table are in percentage.

Panel A - Age 20

| | FUL — $b_t$ | | | | | | | | LUL — $b_t$ | | | | | | | | (FUL-LUL)/LUL — $b_t$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 50 | 60 | 70 | 80 | 90 | 100 | 110 | 120 | 50 | 60 | 70 | 80 | 90 | 100 | 110 | 120 | 50 | 60 | 70 | 80 | 90 | 100 | 110 | 120 |
| 5 | 27 | 28 | 28 | 29 | 29 | 29 | 29 | 30 | 22 | 22 | 22 | 22 | 23 | 23 | 23 | 24 | 26 | 23 | 25 | 27 | 26 | 26 | 26 | 25 |
| 10 | 24 | 25 | 26 | 28 | 28 | 29 | 29 | 29 | 20 | 21 | 21 | 22 | 23 | 23 | 24 | 24 | 19 | 20 | 24 | 23 | 26 | 26 | 23 | 22 |
| 15 | 22 | 23 | 25 | 27 | 28 | 29 | 29 | 29 | 20 | 20 | 21 | 21 | 22 | 23 | 24 | 24 | 8 | 15 | 19 | 24 | 26 | 25 | 22 | 20 |
| 20 | 20 | 21 | 23 | 25 | 28 | 29 | 29 | 29 | 20 | 20 | 20 | 21 | 23 | 23 | 24 | 25 | 1 | 5 | 14 | 23 | 23 | 26 | 21 | 19 |
| $\underline{x}_t$ 25 | 25 | 25 | 25 | 25 | 27 | 29 | 29 | 29 | 25 | 25 | 25 | 25 | 25 | 23 | 24 | 25 | 0 | 0 | 0 | 0 | 7 | 25 | 20 | 19 |
| 30 | 30 | 30 | 30 | 30 | 29 | 29 | 29 | 29 | 30 | 30 | 30 | 29 | 27 | 23 | 24 | 25 | 0 | 0 | 0 | 2 | 8 | 25 | 20 | 19 |
| 35 | 35 | 34 | 32 | 31 | 30 | 29 | 29 | 29 | 35 | 33 | 31 | 30 | 27 | 23 | 24 | 25 | 1 | 2 | 3 | 4 | 8 | 25 | 19 | 18 |
| 40 | 35 | 34 | 32 | 31 | 30 | 29 | 29 | 29 | 35 | 33 | 31 | 30 | 27 | 23 | 25 | 25 | 1 | 2 | 3 | 4 | 9 | 26 | 19 | 18 |
| 45 | 35 | 34 | 32 | 31 | 30 | 29 | 29 | 30 | 35 | 33 | 31 | 29 | 27 | 23 | 25 | 25 | 1 | 3 | 3 | 5 | 9 | 26 | 19 | 18 |

Panel B - Age 80

| | FUL — $b_t$ | | | | | | | | LUL — $b_t$ | | | | | | | | (FUL-LUL)/LUL — $b_t$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 50 | 60 | 70 | 80 | 90 | 100 | 110 | 120 | 50 | 60 | 70 | 80 | 90 | 100 | 110 | 120 | 50 | 60 | 70 | 80 | 90 | 100 | 110 | 120 |
| 5 | 26 | 28 | 28 | 29 | 29 | 29 | 29 | 30 | 22 | 22 | 22 | 22 | 23 | 23 | 23 | 23 | 19 | 26 | 27 | 28 | 29 | 29 | 28 | 29 |
| 10 | 24 | 25 | 26 | 28 | 29 | 29 | 29 | 30 | 21 | 21 | 22 | 22 | 22 | 23 | 23 | 24 | 16 | 18 | 19 | 27 | 27 | 29 | 25 | 24 |
| 15 | 22 | 23 | 25 | 26 | 28 | 29 | 30 | 30 | 20 | 20 | 21 | 22 | 22 | 23 | 24 | 24 | 10 | 13 | 18 | 20 | 27 | 29 | 25 | 22 |
| 20 | 21 | 22 | 23 | 25 | 28 | 29 | 29 | 30 | 20 | 20 | 20 | 21 | 22 | 23 | 24 | 25 | 4 | 9 | 13 | 19 | 26 | 30 | 23 | 20 |
| $\underline{x}_t$ 25 | 25 | 25 | 25 | 25 | 26 | 29 | 29 | 30 | 25 | 25 | 25 | 25 | 25 | 23 | 24 | 25 | 0 | 0 | 0 | 0 | 5 | 28 | 22 | 21 |
| 30 | 30 | 30 | 30 | 30 | 30 | 29 | 30 | 30 | 30 | 30 | 30 | 30 | 28 | 23 | 24 | 25 | 0 | 0 | 0 | 0 | 8 | 27 | 21 | 20 |
| 35 | 35 | 35 | 34 | 32 | 30 | 29 | 30 | 30 | 35 | 35 | 34 | 31 | 28 | 23 | 24 | 25 | 0 | 0 | 0 | 2 | 9 | 29 | 21 | 19 |
| 40 | 38 | 36 | 34 | 32 | 30 | 29 | 30 | 30 | 38 | 36 | 34 | 31 | 28 | 23 | 25 | 25 | 1 | 0 | 0 | 2 | 10 | 29 | 20 | 20 |
| 45 | 38 | 36 | 34 | 32 | 30 | 29 | 30 | 30 | 38 | 36 | 34 | 31 | 28 | 23 | 25 | 25 | 1 | 0 | 1 | 4 | 9 | 28 | 20 | 20 |

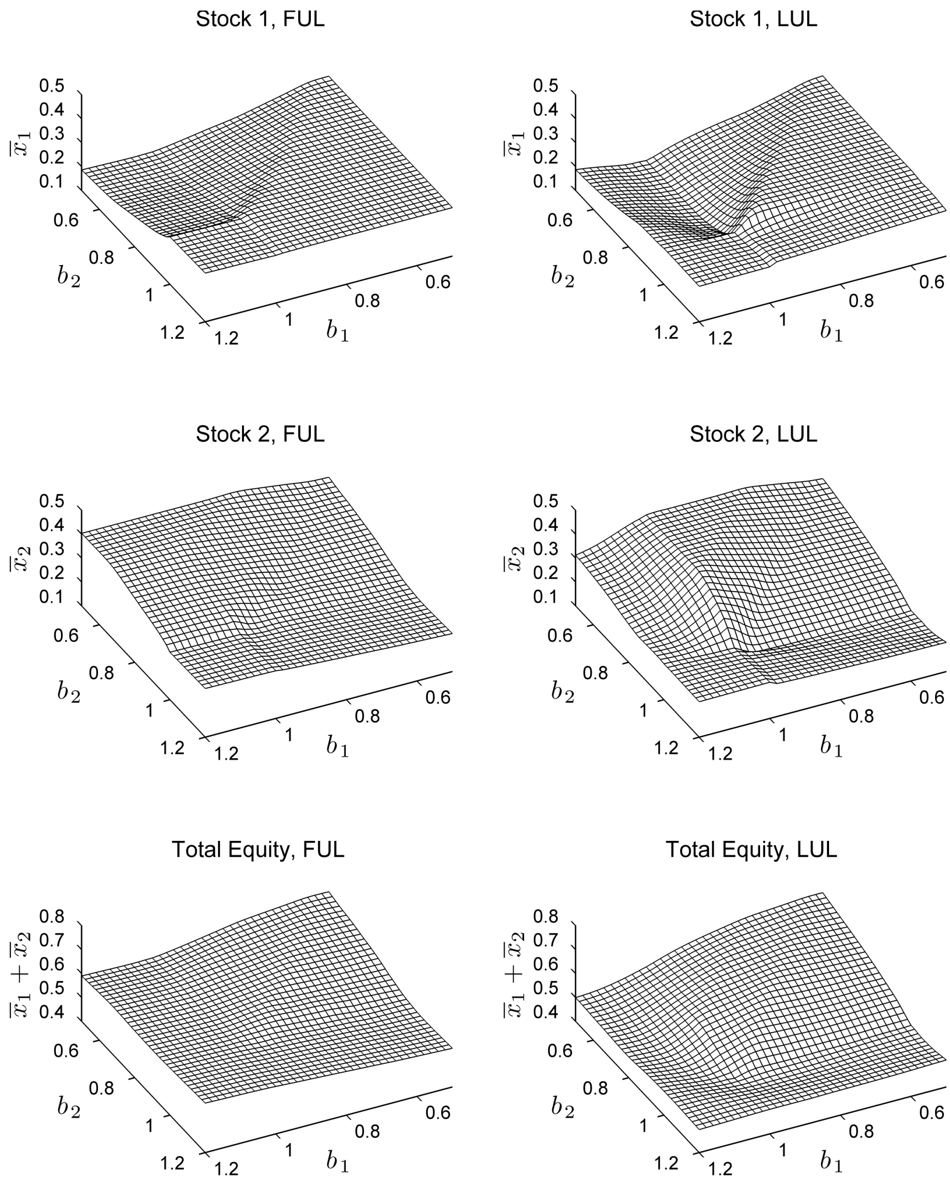


Figure 4.4: Application II - Optimal Strategies of the Two Assets Case at Age 20

The left (right) panels summarize the optimal portfolio weights as a function of the initial tax bases of the two stocks, $b_1$ and $b_2$, for the FUL (LUL) case. The initial positions of the two stocks are fixed at 30% and 40%. The top, middle, and bottom panels present the optimal portfolio weight for Stock 1, Stock 2, and the sum of the two stocks. The LUL plots have a zero carry-over loss entering the trading period. The parameters used are the Base Case parameters summarized in Section 4.3.1 except that the investment opportunity set is expanded to two stocks with correlation 80% and volatility 16.87%.

Table 4.4: Application II - Optimal Strategies of the Two Assets Case at Age 20

The top (bottom) panels summarize the optimal portfolio weight as a function of the initial tax bases of the two stocks, $b_1$ and $b_2$, for the FUL (LUL) case. The initial positions of the two stocks are fixed at 30% and 40%. The left, middle, and right panels present the optimal portfolio weight for Stock 1, Stock 2, and the sum of the two stocks. The LUL results have a zero carry-over loss entering the trading period. The parameters used are the Base Case parameters summarized in Section 4.3.1 except that the investment opportunity set is expanded to two stocks with correlation 80% and volatility 16.87%. All numbers in the table are in percentage.

Panel A - FUL

| | | Stock 1 — $b_2$ | | | | | | | | Stock 2 — $b_2$ | | | | | | | | Total Equity — $b_2$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | 50 | 60 | 70 | 80 | 90 | 100 | 110 | 120 | 50 | 60 | 70 | 80 | 90 | 100 | 110 | 120 | 50 | 60 | 70 | 80 | 90 | 100 | 110 | 120 |
| | 50 | 30 | 30 | 30 | 30 | 30 | 30 | 30 | 30 | 36 | 35 | 32 | 30 | 27 | 25 | 25 | 26 | 67 | 65 | 63 | 60 | 57 | 56 | 56 | 56 |
| | 60 | 29 | 30 | 30 | 30 | 30 | 30 | 30 | 30 | 38 | 35 | 33 | 30 | 28 | 26 | 26 | 26 | 66 | 65 | 63 | 60 | 58 | 56 | 56 | 57 |
| | 70 | 25 | 28 | 30 | 30 | 30 | 30 | 30 | 30 | 40 | 37 | 33 | 31 | 29 | 27 | 27 | 27 | 65 | 64 | 63 | 61 | 59 | 57 | 57 | 57 |
| | 80 | 23 | 24 | 27 | 29 | 30 | 30 | 30 | 30 | 40 | 38 | 36 | 32 | 29 | 27 | 27 | 28 | 63 | 63 | 62 | 61 | 60 | 58 | 58 | 58 |
| $b_1$ | 90 | 20 | 21 | 23 | 25 | 29 | 30 | 30 | 30 | 40 | 40 | 37 | 35 | 31 | 28 | 28 | 29 | 61 | 61 | 61 | 60 | 60 | 58 | 58 | 59 |
| | 100 | 18 | 19 | 20 | 23 | 25 | 29 | 30 | 30 | 40 | 40 | 39 | 36 | 34 | 29 | 30 | 30 | 59 | 59 | 59 | 59 | 59 | 59 | 59 | 60 |
| | 110 | 18 | 19 | 20 | 23 | 25 | 30 | 30 | 30 | 40 | 40 | 39 | 37 | 34 | 30 | 30 | 30 | 59 | 59 | 60 | 59 | 59 | 59 | 60 | 60 |
| | 120 | 19 | 19 | 20 | 23 | 25 | 30 | 30 | 30 | 40 | 40 | 40 | 37 | 34 | 30 | 30 | 30 | 59 | 60 | 60 | 60 | 59 | 60 | 60 | 61 |

Panel B - LUL

| | | Stock 1 — $b_2$ | | | | | | | | Stock 2 — $b_2$ | | | | | | | | Total Equity — $b_2$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | 50 | 60 | 70 | 80 | 90 | 100 | 110 | 120 | 50 | 60 | 70 | 80 | 90 | 100 | 110 | 120 | 50 | 60 | 70 | 80 | 90 | 100 | 110 | 120 |
| | 50 | 30 | 30 | 30 | 30 | 30 | 30 | 30 | 29 | 36 | 34 | 31 | 28 | 25 | 20 | 20 | 22 | 66 | 64 | 61 | 59 | 55 | 50 | 50 | 51 |
| | 60 | 28 | 30 | 30 | 30 | 30 | 30 | 29 | 29 | 38 | 34 | 32 | 29 | 25 | 19 | 20 | 22 | 66 | 65 | 62 | 59 | 55 | 50 | 50 | 51 |
| | 70 | 24 | 27 | 30 | 30 | 30 | 30 | 28 | 28 | 40 | 37 | 33 | 29 | 25 | 19 | 21 | 23 | 65 | 64 | 63 | 59 | 55 | 49 | 49 | 51 |
| | 80 | 22 | 23 | 25 | 29 | 30 | 30 | 27 | 27 | 40 | 40 | 36 | 31 | 25 | 19 | 21 | 23 | 62 | 63 | 61 | 60 | 55 | 49 | 49 | 50 |
| $b_1$ | 90 | 19 | 19 | 20 | 22 | 28 | 30 | 26 | 27 | 40 | 40 | 40 | 36 | 27 | 20 | 22 | 23 | 59 | 60 | 60 | 58 | 56 | 49 | 49 | 50 |
| | 100 | 15 | 15 | 15 | 15 | 17 | 24 | 25 | 25 | 40 | 40 | 40 | 40 | 35 | 24 | 25 | 25 | 55 | 55 | 55 | 55 | 52 | 48 | 49 | 50 |
| | 110 | 16 | 17 | 18 | 19 | 21 | 24 | 25 | 25 | 35 | 34 | 32 | 30 | 27 | 24 | 25 | 25 | 51 | 51 | 50 | 49 | 48 | 49 | 50 | 50 |
| | 120 | 19 | 19 | 21 | 23 | 24 | 25 | 25 | 25 | 31 | 30 | 28 | 26 | 25 | 25 | 25 | 25 | 50 | 50 | 49 | 49 | 50 | 50 | 50 | 50 |

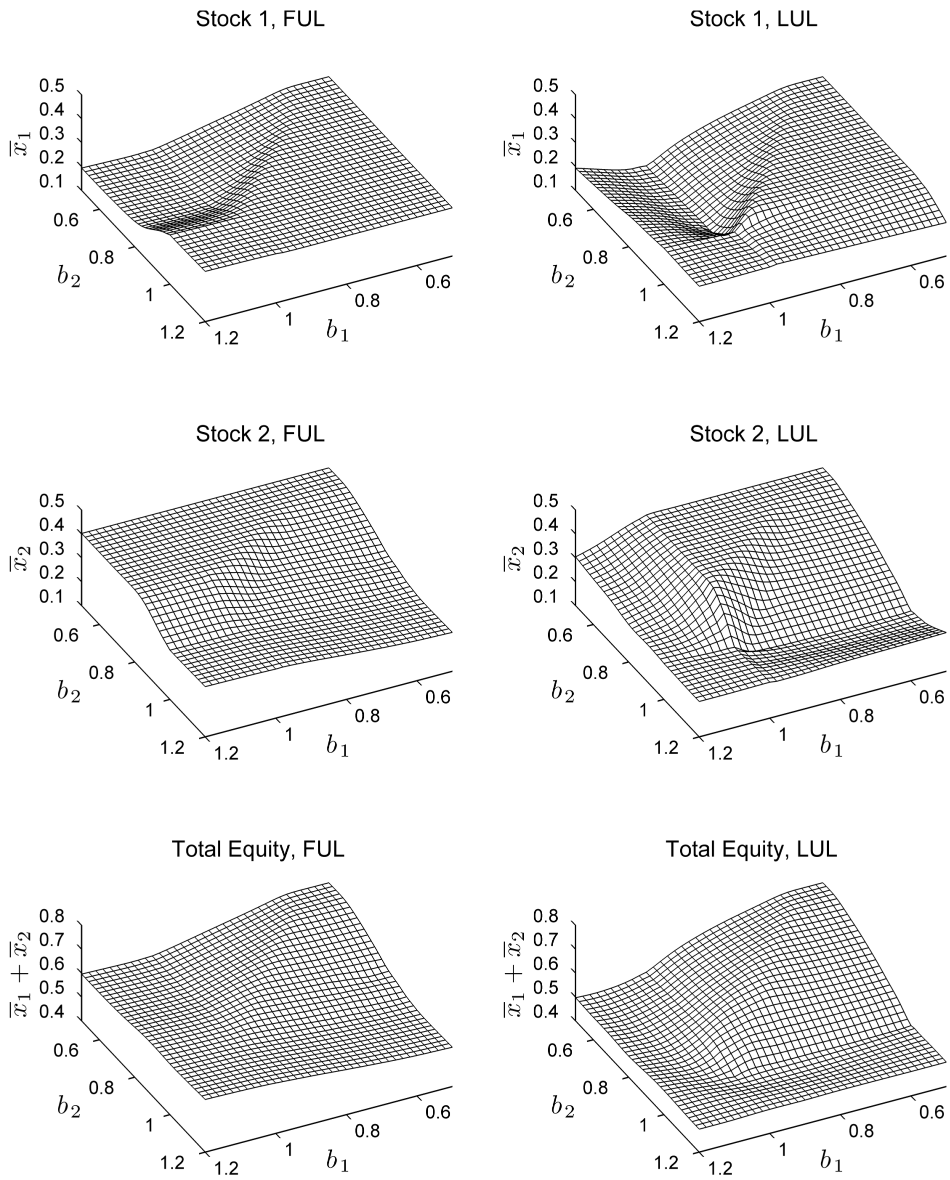


Figure 4.5: Application II - Optimal Strategies of the Two Assets Case at Age 80

The left (right) panels summarize the optimal portfolio weights as a function of the initial tax bases of the two stocks, $b_1$ and $b_2$, for the FUL (LUL) case. The initial positions of the two stocks are fixed at 30% and 40%. The top, middle, and bottom panels present the optimal portfolio weight for Stock 1, Stock 2, and the sum of the two stocks. The LUL plots have a zero carry-over loss entering the trading period. The parameters used are the Base Case parameters summarized in Section 4.3.1 except that the investment opportunity set is expanded to two stocks with correlation 80% and volatility 16.87%.

Table 4.5: Application II - Optimal Strategies of the Two Assets Case at Age 80

The top (bottom) panels summarize the optimal portfolio weight as a function of the initial tax bases of the two stocks, $b_1$ and $b_2$, for the FUL (LUL) case. The initial positions of the two stocks are fixed at 30% and 40%. The left, middle, and right panels present the optimal portfolio weight for Stock 1, Stock 2, and the sum of the two stocks. The LUL results have a zero carry-over loss entering the trading period. The parameters used are the Base Case parameters summarized in Section 4.3.1 except that the investment opportunity set is expanded to two stocks with correlation 80% and volatility 16.87%. All numbers in the table are in percentage.

Panel A - FUL

| | | Stock 1 — $b_2$ | | | | | | | | Stock 2 — $b_2$ | | | | | | | | Total Equity — $b_2$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | 50 | 60 | 70 | 80 | 90 | 100 | 110 | 120 | 50 | 60 | 70 | 80 | 90 | 100 | 110 | 120 | 50 | 60 | 70 | 80 | 90 | 100 | 110 | 120 |
| | 50 | 30 | 30 | 30 | 30 | 30 | 30 | 30 | 30 | 40 | 40 | 37 | 32 | 28 | 26 | 26 | 26 | 70 | 70 | 67 | 62 | 58 | 56 | 56 | 56 |
| | 60 | 30 | 30 | 30 | 30 | 30 | 30 | 30 | 30 | 40 | 40 | 37 | 33 | 29 | 26 | 27 | 27 | 70 | 70 | 67 | 63 | 59 | 57 | 57 | 57 |
| | 70 | 28 | 29 | 30 | 30 | 30 | 30 | 30 | 30 | 40 | 40 | 37 | 33 | 30 | 27 | 27 | 28 | 68 | 69 | 68 | 64 | 60 | 57 | 57 | 58 |
| | 80 | 25 | 25 | 26 | 30 | 30 | 30 | 30 | 30 | 40 | 40 | 40 | 34 | 31 | 28 | 28 | 29 | 65 | 65 | 66 | 65 | 61 | 58 | 58 | 59 |
| $b_1$ | 90 | 22 | 22 | 23 | 25 | 30 | 30 | 30 | 30 | 40 | 40 | 40 | 39 | 32 | 29 | 29 | 30 | 62 | 62 | 63 | 64 | 62 | 59 | 60 | 60 |
| | 100 | 19 | 19 | 20 | 20 | 25 | 30 | 30 | 31 | 40 | 40 | 40 | 40 | 36 | 30 | 30 | 31 | 59 | 59 | 60 | 61 | 61 | 60 | 61 | 61 |
| | 110 | 19 | 19 | 19 | 20 | 25 | 30 | 31 | 31 | 40 | 40 | 40 | 40 | 36 | 30 | 31 | 31 | 59 | 59 | 60 | 61 | 61 | 61 | 61 | 62 |
| | 120 | 19 | 19 | 20 | 21 | 25 | 31 | 31 | 31 | 40 | 40 | 40 | 40 | 36 | 31 | 31 | 31 | 60 | 60 | 60 | 61 | 62 | 61 | 61 | 62 |

Panel B - LUL

| | | Stock 1 — $b_2$ | | | | | | | | Stock 2 — $b_2$ | | | | | | | | Total Equity — $b_2$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | 50 | 60 | 70 | 80 | 90 | 100 | 110 | 120 | 50 | 60 | 70 | 80 | 90 | 100 | 110 | 120 | 50 | 60 | 70 | 80 | 90 | 100 | 110 | 120 |
| | 50 | 30 | 30 | 30 | 30 | 30 | 30 | 29 | 24 | 40 | 39 | 36 | 32 | 26 | 20 | 22 | 26 | 70 | 69 | 66 | 62 | 56 | 50 | 50 | 50 |
| | 60 | 30 | 30 | 30 | 30 | 30 | 30 | 28 | 24 | 40 | 40 | 36 | 32 | 26 | 19 | 22 | 26 | 70 | 70 | 66 | 62 | 56 | 49 | 50 | 50 |
| | 70 | 27 | 28 | 30 | 30 | 30 | 30 | 28 | 25 | 40 | 40 | 37 | 32 | 26 | 19 | 22 | 26 | 68 | 68 | 67 | 63 | 56 | 49 | 49 | 50 |
| | 80 | 24 | 24 | 25 | 30 | 30 | 30 | 27 | 25 | 40 | 40 | 40 | 34 | 27 | 19 | 22 | 25 | 64 | 65 | 66 | 64 | 57 | 49 | 49 | 50 |
| $b_1$ | 90 | 19 | 20 | 20 | 21 | 29 | 30 | 26 | 25 | 40 | 40 | 40 | 40 | 28 | 20 | 23 | 25 | 60 | 60 | 60 | 61 | 57 | 49 | 49 | 50 |
| | 100 | 12 | 13 | 13 | 14 | 15 | 24 | 25 | 25 | 40 | 40 | 40 | 40 | 38 | 24 | 25 | 25 | 53 | 53 | 53 | 54 | 53 | 48 | 49 | 50 |
| | 110 | 15 | 16 | 17 | 19 | 21 | 24 | 25 | 25 | 35 | 34 | 32 | 30 | 27 | 24 | 25 | 25 | 50 | 49 | 49 | 48 | 48 | 49 | 50 | 50 |
| | 120 | 19 | 20 | 21 | 23 | 24 | 25 | 25 | 25 | 30 | 29 | 28 | 26 | 26 | 25 | 25 | 25 | 50 | 49 | 49 | 49 | 50 | 50 | 50 | 50 |

### 4.3.3 The Lifetime Structure of Optimal Portfolios

While examining optimal portfolio choice at a particular time and state is useful in understanding the conditional differences in the FUL and the LUL trading strategies, it provides limited information about any differences in portfolio composition over an investor's lifetime. For example, it is very unlikely along an optimal trading path that an investor will simultaneously have a large carry-over loss and a large equity position with a low tax basis. Also, the conditional snapshots do not easily summarize the increase in wealth from tax rebates collected by the FUL investor. To gain insights about portfolio choice along the optimal strategy taken over an investor's lifetime, we perform Monte Carlo simulations starting at age 20 with no embedded stock gains, zero carry-over loss (the LUL cases), and an initial wealth of \$100 to track the evolution of the investor's optimal portfolio at ages 40, 60, and 80 conditional on the investor's survival. All simulations are over 50,000 paths and computed for the FUL and the LUL investors. Percentiles, means, and standard deviations are reported for the following variables at each age: wealth, $W$; stock-to-wealth ratio, $\bar{x}$; basis-to-price ratio, $b$; cumulative capital gain tax-to-wealth ratio, $G$; LUL carry-over loss-to-wealth ratio, $l$.

**The Base Case**

Table 4.6 reports the simulation results for the Base Case. The ability to tax loss sell in the FUL case leads to higher wealth relative to the LUL case at all ages and all percentiles of the wealth distribution. This increase in the FUL wealth distribution is driven by two effects — higher equity holdings and tax rebates collected through tax loss selling. From the stock-to-wealth ratio column, the FUL equity holdings dominate the LUL equity holdings at all ages until the 90th percentile leading to on average higher equity holdings in the FUL case. This difference in equity holdings is driven by the original divergence in equity holdings in the FUL and LUL cases when the basis-to-price ratio is around 1.0. Tax loss selling also plays a role in the higher FUL wealth distribution. Note that the cumulative capital gain tax-to-wealth ratios are negative for the lower percentiles at all ages for the FUL case. This is capturing the tax rebates being collected when the FUL investor tax loss sells.

The Base Case simulations also highlight that the LUL investor's carry-over loss is typically quite small. At age 40, the average carry over loss is 0.8% of wealth. Also, the distribution is heavily skewed toward zero as can be seen in the percentiles. Hence, the investor rarely has a significant carry-over loss in her portfolio over her lifetime.

Overall, the Base Case simulations highlight that under both the FUL and LUL cases, the investors get capital gains locked in quickly and actually pay little capital gain taxes over their lifetimes. Even though the capital gain tax rate is 20%, effectively it is significantly smaller.

**Comparative Statics**

Table 4.7 presents the simulation results for the Capital Gain Tax 30% Case. Given the capital gain tax rate is now higher, the effects seen in the Base Case now grow stronger. For example, the FUL investor's average equity holding at age 40 is 9.6% (relatively 15.6%) higher than that of the LUL investor compared to an increase of 4.4% (relatively 7.2%) in the Base Case. Also, the difference between the FUL and LUL wealth distributions increases due to the larger spread in equity holdings as well as the additional tax loss selling performed by the FUL investor. In particular, average cumulative capital gain taxes under the FUL case are negative at age 40 and approximately zero at ages 60 and 80.

Table 4.8 summarizes the simulation results for the Higher Risk Aversion Case. Given the decrease in equity holdings due to the higher risk aversion coefficient, the difference between the FUL and LUL wealth distributions is reduced relative to the Base Case parameters. The size of tax loss selling is also smaller in the FUL case now. At age 40 in the FUL case, the 5th percentile of the cumulative capital gain taxes is -0.8% compared to -2.0% for the Base Case.

Table 4.6: Application II - Base Case Simulations

This table presents simulation results for portfolio characteristics under the FUL and the LUL cases at ages 40, 60, and 80 over 50,000 paths. The investor starts at age 20 with no embedded capital gains and zero carry-over loss (the LUL cases). Percentiles, means, and standard deviations are reported for the following variables at each age: wealth $W_t$, stock-to-wealth ratio $\bar{x}_t$, basis-to-price ratio $b_t$, cumulative capital gain tax-to-wealth ratio $G_t$, and LUL carry-over loss-to-wealth ratio $l_t$.

| | | $W_t$ (\$) | | $\bar{x}_t$ (%) | | $b_t$ (%) | | $G_t$ (%) | | $l_t$ (%) |
|---|---|---|---|---|---|---|---|---|---|---|
| | | FUL | LUL | FUL | LUL | FUL | LUL | FUL | LUL | |
| Age 40 | Q5 | 161.3 | 160.5 | 52.3 | 46.0 | 7.8 | 7.8 | -2.0 | 0.0 | 0.0 |
| | Q10 | 183.4 | 178.8 | 55.3 | 48.5 | 10.0 | 10.0 | -1.1 | 0.0 | 0.0 |
| | Q15 | 202.8 | 194.7 | 57.3 | 49.9 | 11.7 | 11.7 | -0.6 | 0.0 | 0.0 |
| | Q20 | 218.7 | 207.9 | 58.8 | 50.9 | 13.8 | 13.8 | -0.3 | 0.0 | 0.0 |
| | Q30 | 250.9 | 235.1 | 61.6 | 53.9 | 16.1 | 16.1 | 0.1 | 0.0 | 0.0 |
| | Q40 | 281.8 | 261.3 | 63.9 | 57.7 | 19.7 | 20.1 | 0.2 | 0.0 | 0.0 |
| | Q50 | 316.5 | 290.9 | 66.0 | 61.1 | 23.8 | 23.8 | 0.4 | 0.0 | 0.0 |
| | Q60 | 356.4 | 325.1 | 67.8 | 64.3 | 27.9 | 28.4 | 0.6 | 0.0 | 0.0 |
| | Q70 | 409.3 | 367.4 | 69.6 | 67.1 | 34.1 | 35.0 | 0.9 | 0.1 | 0.0 |
| | Q80 | 485.4 | 449.8 | 71.1 | 69.9 | 42.1 | 42.1 | 1.2 | 0.5 | 0.0 |
| | Q85 | 542.2 | 503.4 | 71.8 | 71.2 | 47.0 | 48.7 | 1.4 | 0.8 | 0.0 |
| | Q90 | 610.1 | 568.0 | 72.6 | 72.4 | 54.5 | 56.6 | 1.6 | 1.1 | 1.5 |
| | Q95 | 707.4 | 647.9 | 73.2 | 73.3 | 67.7 | 69.7 | 2.0 | 1.6 | 4.9 |
| | Mean | 364.6 | 337.7 | 64.8 | 60.5 | 28.7 | 29.1 | 0.3 | 0.3 | 0.8 |
| | S.D. | 187.7 | 172.0 | 6.6 | 8.9 | 18.6 | 19.2 | 1.4 | 0.6 | 3.7 |
| Age 60 | Q5 | 375.9 | 356.3 | 51.5 | 48.2 | 1.2 | 1.2 | -0.6 | 0.0 | 0.0 |
| | Q10 | 465.3 | 429.7 | 55.3 | 51.6 | 1.8 | 1.8 | -0.2 | 0.0 | 0.0 |
| | Q15 | 540.5 | 494.1 | 58.1 | 54.7 | 2.2 | 2.2 | 0.0 | 0.0 | 0.0 |
| | Q20 | 613.3 | 555.1 | 60.3 | 57.1 | 2.6 | 2.6 | 0.1 | 0.0 | 0.0 |
| | Q30 | 759.2 | 681.8 | 64.1 | 61.5 | 3.6 | 3.6 | 0.2 | 0.0 | 0.0 |
| | Q40 | 925.7 | 828.3 | 67.1 | 65.1 | 4.9 | 4.9 | 0.4 | 0.0 | 0.0 |
| | Q50 | 1,109.8 | 988.5 | 69.5 | 68.1 | 6.1 | 6.2 | 0.6 | 0.2 | 0.0 |
| | Q60 | 1,342.5 | 1,216.5 | 71.9 | 70.7 | 8.0 | 8.0 | 0.8 | 0.4 | 0.0 |
| | Q70 | 1,649.7 | 1,464.4 | 74.0 | 72.8 | 10.5 | 11.0 | 1.1 | 0.7 | 0.0 |
| | Q80 | 2,070.4 | 1,847.0 | 75.6 | 75.3 | 14.2 | 15.1 | 1.3 | 1.1 | 0.0 |
| | Q85 | 2,424.8 | 2,234.1 | 77.1 | 76.5 | 17.6 | 18.3 | 1.5 | 1.3 | 0.0 |
| | Q90 | 2,847.1 | 2,614.2 | 77.9 | 77.9 | 22.6 | 24.6 | 1.8 | 1.6 | 0.0 |
| | Q95 | 3,841.1 | 3,420.4 | 78.3 | 78.7 | 32.6 | 36.3 | 2.1 | 1.9 | 0.0 |
| | Mean | 1,479.9 | 1,346.3 | 67.9 | 66.3 | 10.0 | 10.6 | 0.7 | 0.5 | 0.1 |
| | S.D. | 1,276.5 | 1,182.6 | 8.4 | 9.5 | 11.6 | 12.6 | 0.9 | 0.7 | 1.4 |
| Age 80 | Q5 | 983.7 | 904.6 | 52.2 | 51.0 | 0.2 | 0.2 | -0.2 | 0.0 | 0.0 |
| | Q10 | 1,311.7 | 1,182.1 | 56.4 | 55.2 | 0.3 | 0.3 | 0.0 | 0.0 | 0.0 |
| | Q15 | 1,601.3 | 1,427.2 | 59.5 | 58.3 | 0.5 | 0.5 | 0.0 | 0.0 | 0.0 |
| | Q20 | 1,885.6 | 1,674.1 | 62.1 | 60.9 | 0.6 | 0.6 | 0.0 | 0.0 | 0.0 |
| | Q30 | 2,496.4 | 2,212.8 | 66.3 | 65.2 | 0.8 | 0.9 | 0.1 | 0.0 | 0.0 |
| | Q40 | 3,209.5 | 2,848.6 | 69.5 | 68.6 | 1.2 | 1.2 | 0.2 | 0.0 | 0.0 |
| | Q50 | 4,083.6 | 3,621.8 | 72.4 | 71.6 | 1.6 | 1.7 | 0.3 | 0.1 | 0.0 |
| | Q60 | 5,208.0 | 4,647.2 | 74.9 | 74.2 | 2.2 | 2.3 | 0.4 | 0.2 | 0.0 |
| | Q70 | 6,803.4 | 6,072.7 | 77.3 | 76.8 | 3.2 | 3.4 | 0.5 | 0.4 | 0.0 |
| | Q80 | 9,235.1 | 8,387.0 | 79.8 | 79.4 | 5.0 | 5.3 | 0.6 | 0.5 | 0.0 |
| | Q85 | 11,150.3 | 10,142.6 | 81.1 | 80.8 | 6.6 | 7.2 | 0.7 | 0.6 | 0.0 |
| | Q90 | 14,155.2 | 12,928.9 | 82.6 | 82.3 | 9.7 | 11.1 | 0.9 | 0.8 | 0.0 |
| | Q95 | 20,245.4 | 18,514.4 | 84.5 | 84.3 | 16.7 | 19.5 | 1.1 | 1.0 | 0.0 |
| | Mean | 6,618.7 | 5,990.0 | 70.8 | 70.1 | 4.1 | 4.5 | 0.3 | 0.3 | 0.0 |
| | S.D. | 8,452.5 | 7,776.4 | 9.8 | 10.1 | 7.6 | 8.4 | 0.4 | 0.4 | 0.5 |

Table 4.7: Application II - Capital Gain Tax 30% Case Simulations

This table presents simulation results for portfolio characteristics under the FUL and the LUL cases at ages 40, 60, and 80 over 50,000 paths. The investor starts at age 20 with no embedded capital gains and zero carry-over loss (the LUL cases). Percentiles, means, and standard deviations are reported for the following variables at each age: wealth $W_t$, stock-to-wealth ratio $\bar{x}_t$, basis-to-price ratio $b_t$, cumulative capital gain tax-to-wealth ratio $G_t$, and LUL carry-over loss-to-wealth ratio $l_t$.

| | | $W_t$ (\$) | | $\bar{x}_t$ (%) | | $b_t$ (%) | | $G_t$ (%) | | $l_t$ (%) |
|---|---|---|---|---|---|---|---|---|---|---|
| | | FUL | LUL | FUL | LUL | FUL | LUL | FUL | LUL | |
| Age 40 | Q5 | 162.0 | 160.8 | 56.8 | 45.6 | 7.8 | 7.8 | -3.5 | 0.0 | 0.0 |
| | Q10 | 187.1 | 179.4 | 60.0 | 48.0 | 10.0 | 10.0 | -2.1 | 0.0 | 0.0 |
| | Q15 | 206.9 | 195.0 | 62.2 | 49.3 | 11.7 | 11.7 | -1.4 | 0.0 | 0.0 |
| | Q20 | 225.3 | 207.9 | 63.9 | 50.2 | 13.8 | 13.8 | -0.9 | 0.0 | 0.0 |
| | Q30 | 259.5 | 233.3 | 66.9 | 53.4 | 16.1 | 16.2 | -0.4 | 0.0 | 0.0 |
| | Q40 | 294.4 | 258.8 | 69.5 | 57.2 | 19.7 | 20.1 | 0.0 | 0.0 | 0.0 |
| | Q50 | 334.1 | 287.8 | 71.9 | 60.8 | 23.8 | 23.8 | 0.1 | 0.0 | 0.0 |
| | Q60 | 381.8 | 321.5 | 74.0 | 64.3 | 27.9 | 28.8 | 0.2 | 0.0 | 0.0 |
| | Q70 | 448.3 | 364.1 | 76.1 | 67.8 | 34.1 | 35.2 | 0.3 | 0.0 | 0.0 |
| | Q80 | 535.3 | 444.5 | 78.3 | 71.8 | 42.1 | 42.2 | 0.6 | 0.0 | 0.0 |
| | Q85 | 605.3 | 499.6 | 79.4 | 73.9 | 47.0 | 49.1 | 0.7 | 0.0 | 0.0 |
| | Q90 | 691.7 | 570.5 | 80.6 | 76.4 | 53.8 | 57.0 | 0.9 | 0.0 | 0.0 |
| | Q95 | 821.3 | 669.1 | 81.9 | 79.7 | 67.1 | 70.0 | 1.3 | 0.0 | 4.6 |
| | Mean | 397.9 | 339.6 | 70.9 | 61.3 | 28.6 | 29.2 | -0.4 | 0.0 | 0.7 |
| | S.D. | 231.0 | 185.1 | 7.8 | 10.7 | 18.5 | 19.3 | 1.8 | 0.2 | 3.6 |
| Age 60 | Q5 | 385.1 | 355.5 | 55.4 | 49.2 | 1.2 | 1.2 | -1.2 | 0.0 | 0.0 |
| | Q10 | 483.2 | 429.8 | 59.8 | 52.5 | 1.8 | 1.8 | -0.6 | 0.0 | 0.0 |
| | Q15 | 566.3 | 493.4 | 63.1 | 55.6 | 2.2 | 2.2 | -0.4 | 0.0 | 0.0 |
| | Q20 | 650.6 | 553.7 | 65.5 | 58.4 | 2.6 | 2.6 | -0.2 | 0.0 | 0.0 |
| | Q30 | 821.6 | 677.2 | 69.4 | 63.0 | 3.6 | 3.6 | -0.1 | 0.0 | 0.0 |
| | Q40 | 1,015.4 | 821.8 | 72.5 | 66.9 | 4.9 | 4.9 | 0.0 | 0.0 | 0.0 |
| | Q50 | 1,251.2 | 986.9 | 75.2 | 70.3 | 6.0 | 6.2 | 0.1 | 0.0 | 0.0 |
| | Q60 | 1,537.1 | 1,211.6 | 77.6 | 73.5 | 8.0 | 8.0 | 0.1 | 0.0 | 0.0 |
| | Q70 | 1,948.1 | 1,511.5 | 79.8 | 76.4 | 10.3 | 11.0 | 0.2 | 0.0 | 0.0 |
| | Q80 | 2,536.4 | 1,953.2 | 82.1 | 79.4 | 13.8 | 15.1 | 0.3 | 0.0 | 0.0 |
| | Q85 | 3,014.2 | 2,366.5 | 83.4 | 81.0 | 16.9 | 18.7 | 0.4 | 0.0 | 0.0 |
| | Q90 | 3,627.1 | 2,871.6 | 84.7 | 82.9 | 21.2 | 25.1 | 0.6 | 0.1 | 0.0 |
| | Q95 | 5,139.5 | 3,970.0 | 86.4 | 85.1 | 30.8 | 37.8 | 0.8 | 0.4 | 0.0 |
| | Mean | 1,807.9 | 1,447.6 | 73.6 | 69.0 | 9.8 | 10.8 | 0.0 | 0.1 | 0.1 |
| | S.D. | 1,860.9 | 1,504.9 | 9.4 | 11.0 | 11.2 | 12.9 | 0.9 | 0.2 | 1.4 |
| Age 80 | Q5 | 1,030.0 | 904.6 | 54.8 | 51.8 | 0.2 | 0.2 | -0.4 | 0.0 | 0.0 |
| | Q10 | 1,387.4 | 1,181.7 | 59.6 | 56.3 | 0.3 | 0.3 | -0.2 | 0.0 | 0.0 |
| | Q15 | 1,723.2 | 1,427.0 | 63.1 | 59.7 | 0.5 | 0.5 | -0.1 | 0.0 | 0.0 |
| | Q20 | 2,061.3 | 1,670.4 | 65.7 | 62.4 | 0.6 | 0.6 | -0.1 | 0.0 | 0.0 |
| | Q30 | 2,793.6 | 2,214.0 | 69.8 | 66.9 | 0.8 | 0.9 | 0.0 | 0.0 | 0.0 |
| | Q40 | 3,675.2 | 2,862.7 | 72.9 | 70.5 | 1.2 | 1.2 | 0.0 | 0.0 | 0.0 |
| | Q50 | 4,799.4 | 3,695.4 | 75.7 | 73.6 | 1.6 | 1.7 | 0.0 | 0.0 | 0.0 |
| | Q60 | 6,306.3 | 4,830.2 | 78.2 | 76.3 | 2.1 | 2.3 | 0.0 | 0.0 | 0.0 |
| | Q70 | 8,487.5 | 6,461.3 | 80.4 | 78.9 | 3.0 | 3.3 | 0.0 | 0.0 | 0.0 |
| | Q80 | 11,908.7 | 9,218.7 | 82.7 | 81.5 | 4.6 | 5.0 | 0.1 | 0.0 | 0.0 |
| | Q85 | 14,661.8 | 11,300.6 | 84.0 | 82.9 | 5.8 | 6.8 | 0.1 | 0.0 | 0.0 |
| | Q90 | 19,228.0 | 14,878.9 | 85.4 | 84.6 | 8.2 | 10.4 | 0.2 | 0.0 | 0.0 |
| | Q95 | 29,064.7 | 22,829.6 | 87.2 | 86.6 | 14.1 | 18.4 | 0.3 | 0.1 | 0.0 |
| | Mean | 8,828.3 | 6,907.8 | 74.0 | 71.9 | 3.7 | 4.3 | 0.0 | 0.0 | 0.0 |
| | S.D. | 13,881.4 | 11,131.1 | 9.8 | 10.5 | 6.7 | 8.2 | 0.4 | 0.1 | 0.5 |

Table 4.8: Application II - Higher Risk Aversion Case Simulations

This table presents simulation results for portfolio characteristics under the FUL and the LUL cases at ages 40, 60, and 80 over 50,000 paths. The investor starts at age 20 with no embedded capital gains and zero carry-over loss (the LUL cases). Percentiles, means, and standard deviations are reported for the following variables at each age: wealth $W_t$, stock-to-wealth ratio $\bar{x}_t$, basis-to-price ratio $b_t$, cumulative capital gain tax-to-wealth ratio $G_t$, and LUL carry-over loss-to-wealth ratio $l_t$.

| | | $W_t$ (\$) | | $\bar{x}_t$ (%) | | $b_t$ (%) | | $G_t$ (%) | | $l_t$ (%) |
|---|---|---|---|---|---|---|---|---|---|---|
| | | FUL | LUL | FUL | LUL | FUL | LUL | FUL | LUL | |
| Age 40 | Q5 | 175.9 | 174.7 | 25.0 | 22.3 | 7.8 | 7.8 | -0.8 | 0.0 | 0.0 |
| | Q10 | 188.2 | 184.7 | 26.9 | 24.0 | 10.0 | 10.0 | -0.3 | 0.0 | 0.0 |
| | Q15 | 197.6 | 192.6 | 28.5 | 25.1 | 11.7 | 11.7 | 0.0 | 0.0 | 0.0 |
| | Q20 | 206.0 | 199.6 | 29.5 | 26.5 | 13.8 | 13.8 | 0.2 | 0.0 | 0.0 |
| | Q30 | 220.5 | 212.6 | 31.6 | 29.0 | 16.2 | 16.2 | 0.4 | 0.0 | 0.0 |
| | Q40 | 234.7 | 225.8 | 33.4 | 31.4 | 19.7 | 19.8 | 0.7 | 0.1 | 0.0 |
| | Q50 | 249.7 | 240.2 | 35.1 | 33.6 | 23.8 | 23.8 | 1.0 | 0.4 | 0.0 |
| | Q60 | 266.4 | 256.0 | 36.6 | 35.8 | 27.9 | 28.3 | 1.3 | 0.8 | 0.0 |
| | Q70 | 287.7 | 274.9 | 37.9 | 37.5 | 34.1 | 34.7 | 1.8 | 1.3 | 0.0 |
| | Q80 | 313.2 | 304.6 | 39.1 | 39.5 | 42.1 | 42.1 | 2.3 | 1.8 | 0.0 |
| | Q85 | 332.3 | 323.5 | 39.6 | 40.1 | 47.3 | 48.2 | 2.6 | 2.1 | 0.0 |
| | Q90 | 354.2 | 345.1 | 40.0 | 40.7 | 55.1 | 55.6 | 3.0 | 2.6 | 0.0 |
| | Q95 | 385.2 | 371.9 | 40.4 | 41.3 | 68.5 | 69.1 | 3.6 | 3.2 | 2.1 |
| | Mean | 262.4 | 254.0 | 34.2 | 32.9 | 28.8 | 29.0 | 1.2 | 0.9 | 0.3 |
| | S.D. | 67.5 | 65.2 | 5.0 | 6.3 | 18.9 | 19.0 | 1.4 | 1.1 | 1.5 |
| Age 60 | Q5 | 371.7 | 357.7 | 24.1 | 22.9 | 1.2 | 1.2 | 0.1 | 0.0 | 0.0 |
| | Q10 | 414.9 | 396.4 | 26.9 | 25.6 | 1.8 | 1.8 | 0.4 | 0.0 | 0.0 |
| | Q15 | 449.1 | 428.1 | 29.0 | 27.8 | 2.2 | 2.2 | 0.6 | 0.1 | 0.0 |
| | Q20 | 481.0 | 458.0 | 30.7 | 29.9 | 2.6 | 2.6 | 0.8 | 0.4 | 0.0 |
| | Q30 | 537.8 | 514.1 | 33.7 | 33.1 | 3.6 | 3.6 | 1.3 | 0.9 | 0.0 |
| | Q40 | 596.5 | 574.0 | 36.1 | 35.7 | 4.9 | 4.9 | 1.7 | 1.4 | 0.0 |
| | Q50 | 658.6 | 630.3 | 38.0 | 37.8 | 6.2 | 6.2 | 2.2 | 1.8 | 0.0 |
| | Q60 | 727.0 | 703.0 | 40.0 | 40.3 | 8.0 | 8.0 | 2.6 | 2.3 | 0.0 |
| | Q70 | 814.0 | 781.4 | 41.4 | 41.5 | 10.7 | 10.9 | 3.1 | 2.9 | 0.0 |
| | Q80 | 919.7 | 881.5 | 43.0 | 43.2 | 14.8 | 14.9 | 3.7 | 3.5 | 0.0 |
| | Q85 | 998.8 | 971.6 | 43.5 | 43.7 | 17.8 | 17.8 | 4.1 | 3.9 | 0.0 |
| | Q90 | 1,088.8 | 1,056.2 | 43.8 | 43.9 | 23.2 | 23.3 | 4.5 | 4.3 | 0.0 |
| | Q95 | 1,284.5 | 1,233.6 | 44.1 | 44.2 | 34.1 | 34.4 | 5.1 | 5.0 | 0.0 |
| | Mean | 719.2 | 693.4 | 36.7 | 36.3 | 10.3 | 10.3 | 2.3 | 2.0 | 0.0 |
| | S.D. | 293.2 | 286.8 | 6.4 | 6.9 | 12.2 | 12.3 | 1.6 | 1.6 | 0.5 |
| Age 80 | Q5 | 833.9 | 790.6 | 25.6 | 25.0 | 0.2 | 0.2 | 0.3 | 0.0 | 0.0 |
| | Q10 | 974.0 | 923.9 | 28.8 | 28.3 | 0.3 | 0.3 | 0.5 | 0.3 | 0.0 |
| | Q15 | 1,085.7 | 1,033.8 | 31.3 | 30.9 | 0.5 | 0.5 | 0.8 | 0.6 | 0.0 |
| | Q20 | 1,188.3 | 1,131.1 | 33.3 | 33.1 | 0.6 | 0.6 | 1.0 | 0.8 | 0.0 |
| | Q30 | 1,385.0 | 1,325.8 | 36.7 | 36.6 | 0.9 | 0.9 | 1.5 | 1.3 | 0.0 |
| | Q40 | 1,587.2 | 1,523.0 | 39.5 | 39.4 | 1.2 | 1.3 | 1.9 | 1.8 | 0.0 |
| | Q50 | 1,807.7 | 1,735.4 | 42.1 | 42.3 | 1.7 | 1.7 | 2.3 | 2.2 | 0.0 |
| | Q60 | 2,058.9 | 1,983.1 | 44.5 | 44.6 | 2.4 | 2.4 | 2.8 | 2.7 | 0.0 |
| | Q70 | 2,378.9 | 2,292.7 | 46.1 | 46.4 | 3.4 | 3.5 | 3.3 | 3.2 | 0.0 |
| | Q80 | 2,796.1 | 2,710.4 | 49.2 | 49.1 | 5.5 | 5.6 | 3.8 | 3.7 | 0.0 |
| | Q85 | 3,098.0 | 3,005.5 | 49.7 | 49.5 | 7.7 | 7.7 | 4.1 | 4.0 | 0.0 |
| | Q90 | 3,519.8 | 3,413.2 | 49.9 | 49.7 | 11.6 | 11.8 | 4.5 | 4.4 | 0.0 |
| | Q95 | 4,265.4 | 4,150.1 | 50.1 | 49.8 | 19.4 | 19.7 | 5.0 | 5.0 | 0.0 |
| | Mean | 2,084.7 | 2,012.1 | 40.7 | 40.6 | 4.6 | 4.6 | 2.4 | 2.3 | 0.0 |
| | S.D. | 1,155.6 | 1,132.5 | 7.9 | 8.0 | 8.4 | 8.5 | 1.5 | 1.5 | 0.2 |

### 4.3.4 The Economic Costs of the FUL and the LUL Cases

Our analysis of the conditional structure of optimal portfolios in Section 4.3.2 demonstrates that optimal portfolios across the FUL and LUL cases can greatly differ when the basis-to-price ratio is around one. The lifetime portfolio analysis of Section 4.3.3 partially mitigates the differences though — over an investor's life, she typically holds stock positions with significant embedded gains regardless of the FUL or LUL assumption. However, optimal wealth distributions and taxes collected do differ.

We quantify the economic significance of the difference between the FUL and the LUL cases using a measure called wealth change. The wealth change is the required relative change in wealth for a 20 year old NCGT investor to be indifferent between two scenarios: a) switching to capital gain taxation with no embedded gains and zero carry-over loss (the LUL case); b) remaining untaxed on capital gains. If the wealth change is negative, it implies that the investor is worse off when forced to pay capital gain taxes. A positive wealth change implies that the investor is better off paying capital gain taxes as compared to being untaxed. Our measure of the cost of taxation is in contrast to most of the existing literature (Constantinides (1983); Dammon et al. (2001); Garlappi, Naik, and Slive (2001)) as we do not measure tax costs relative to an accrual-based capital gain taxation system where all gains and losses are marked-to-market annually. Instead, our wealth change measure is meant to capture the change in an investor's welfare by imposing a capital gain taxation scheme as compared to facing no capital gain taxation.

**The Base Case**

Under the FUL assumption, the wealth change of the Base Case shows a striking feature that an investor, not currently taxed on capital gains, would actually prefer to switch to paying capital gain taxes and the benefit of switching is equivalent to increasing her current wealth by 2.2% and staying untaxed. This counterfactual welfare result is driven by the stream of cash flows generated from the tax rebates. The same is not true for the LUL investor since the wealth change, -0.4%, is a negative value; i.e., switching to paying capital gain taxes is equivalent to decreasing her current wealth by 0.4%. Overall, imposing the FUL assumption exaggerates the investor's welfare by 2.6% compared to the more realistic assumption of LUL.

**Comparative Statics**

Increasing the capital gain tax rate to 30% amplifies the difference between the FUL and the LUL wealth changes from 2.6% to 4.0%. As the capital gain tax rate increases, the FUL wealth change increases to 3.6%. In contrast, the LUL wealth change remains unchanged at -0.4%. The increasing wealth change under the FUL case demonstrates the increased importance of the tax loss selling option in driving optimal portfolio decisions.

When the investor's risk aversion rises in the Higher Risk Aversion Case, the investor holds less equity and the gap between the FUL and the LUL wealth changes is reduced to 1.2%. The Two Assets Case has similar results as the Base Case. The counterfactual behavior of the FUL investor is justified again by a positive wealth change of 2.0%. The LUL wealth change, -0.6%, remains negative.

# Chapter 5

# Conclusions and Directions for Future Research

I develop a functional-approximation-based framework to solve high-dimensional stochastic control problems under general constraints by combining dynamic programming, the method of Lagrange multipliers, meshfree discretization and interpolation of a high-dimensional state space, and functional approximations of conditional expectations. To improve the accuracy of the conditional expectation approximation, I introduce the test region iterative contraction (TRIC) method. The basic idea is to approximate the conditional expectation using a second order polynomial basis within an approximation region, whose size is contracted iteratively.

The functional-approximation-based framework is general enough to solve a large class of high-dimensional discrete-time stochastic control problems arising from Finance, Economics, and other fields. I illustrate the capabilities of the method in two applications from Finance. The capability to handle constraints is demonstrated in the first application, where an investor chooses an optimal portfolio and consumption stream while facing margin constraints and receiving a non-tradable income stream. The second application focuses on the problem of high-dimensionality of the state space and the problem of singularity of the control space by considering a dynamic portfolio and consumption choice problem with capital gain taxation and multiple risky assets.

The functional-approximation-based framework can be extended in many directions. First, the development of the TRIC method is far from complete. The method depends on many inputs which can take arbitrary values. For example, the rule to contract and enlarge the test region, the shape of the test region, the number of test solutions, the iteration stopping criterion, etc. How do the complexity and the accuracy of the algorithm depend on these inputs? Is there any optimal way of choosing these inputs? The convergence and the speed of convergence of the TRIC method also need to be demonstrated. Second, the process of fitting a set of radial basis functions to the discretized state space can be implemented using different machine learning techniques. Instead of separating the discretization (sampling) step from the interpolation (learning) step as described in this work, an alternative is to apply the active learning technique and iteratively query for new samples in the regions where interpolation errors are relatively large. Third, as one of the main building blocks of the functional-approximation-based framework, the meshfree discretization and interpolation methods have been studied in many different fields. Other than the radial basis function method described in this work, there exist other meshfree methods, for example, the moving least squares method and the continuous blending method. Are these methods more appropriate for the purpose of approximating the value functions of stochastic control problems? I plan to address these issues in future research.

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# Vita

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