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Developing a Qualitative Geometry from the Conceptions of Young Children

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Developing a Qualitative Geometry from the Conceptions of Young Children

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Dedication

This dissertation is dedicated to the following people:

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- To Luna, whose countless wonderful ideas inspired my fascination with the ideas of all young children.
- To Sol, for his boundless energy and infinite spirit.

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Developing a Qualitative Geometry from the Conceptions of Young Children

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More than half a century ago, Piaget concluded from an investigation of children's representational thinking about the nature of space that the development of children's representational thought is topological before it is Euclidean. This conclusion, commonly referred to as the "topological primacy thesis," has essentially been rejected.

By giving emphasis to the ideas that develop rather than the order in which they develop, this work set out to develop a new form of non-metric geometry from young children's early and intuitive topological, or at least non-metric, ideas. I conducted an eighteen-week teaching experiment with two children, ages six and seven. I developed a new dynamic geometry environment called *Configure* that I used in tandem with clinical interviews in each of the episodes of the experiment to elicit these children's non-metric conceptions and subsequently support their development. I found that these children developed significant and authentic forms of geometric reasoning. It is these findings,

which I refer to as *qualitative geometry*, that have implications for the teaching of geometry and for research into students' mathematical reasoning.

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STATEMENT OF PROBLEM

Sit yourself down, take a seat. All you gotta do is repeat after me.

– The Jackson 5

The dominant focus of geometry instruction in elementary school is on knowledge of shapes (Clements, 2004, p. 8). Geometry is principally about identifying canonical shapes (e.g., squares, triangles, circles) and matching those shapes to their given names. Because the curriculum is so narrow and consequently disconnected from other domains of mathematics, the emphasis is typically on focusing and reinforcing what kids already know. Clements cites one study of kindergarten children (Thomas, 1982, as cited in Clements, 2004) that found that teachers added no new content or developed new knowledge beyond what kids already knew. Moreover, these picture-driven geometric experiences do nothing to move students beyond the level in which they identify shapes not by their properties – or even consider that shapes *have* properties – but by their appearance: “That’s a triangle, because it looks like a triangle.” As a result of this “low level” engagement, the targeted developmental level lies outside and “below” students’ zones of proximal development (Vygotsky, 1978). So it is not surprising that little change in children’s conceptions of shape occurs throughout the course of the elementary grades

(Lehrer & Chazan, 1998; Lehrer, Jenkins, & Osana, 1998). Still worse, there are other weaknesses to the traditional approach to the teaching of geometry in elementary school.

What makes matters detrimental to the development of children's learning geometry in this kind of environment is that concepts of shape are stabilizing as early as age six (Clements, 2004). This means that if young children's engagement is not expanded beyond a set of conventional shapes, these shapes develop into a set of visual prototypes that could rule children's thinking throughout their lives (Burger & Shaughnessy, 1986; Clements, 2004). Clements et al. (1999) suggest that the case of the square and the equivalent "diamond" is emblematic of this larger issue. Kids aged 4 to 6 classify them both as squares, but as they progress through elementary school, they're less likely to do so: "the limited number of exemplars [of shapes with multiple prototypes, such as triangles] common in school materials impedes, and possibly undermines, students' development of rich schemas for certain geometric shapes (p. 208). Along the same line, young children often require that squares and triangles have horizontal bases, that all triangles are acute, and that one dimension of a rectangle must be twice as long as the other (Clements, 2004). In this study, it appeared that what kids had learned in school was "getting in the way" of my investigation. They differentiated between rectangles and "long rectangles," and their primary mechanisms of naming shapes involved counting sides or "corners." They come to assume that these are fundamental properties and, accordingly, as shape concepts are stabilizing, the properties become distinguishing criteria used for classification (Lehrer, Jenkins, et al., 1998).

Similar unintended consequences for children's geometric thinking also arise because their interactions with shape are limited. Geometry does, of course, possess a significant visual component. However, as children see it, geometric thinking is restricted to passive observation of static images. The problem is, children don't *only* "see" shapes that way. They see shapes as malleable and often provide "morphing explanations" (Lehrer, Jenkins, et al., 1998, p. 142) for shapes they identify as similar. When geometry is about static images on paper, then engagement with, and understanding of, geometry is inevitably constrained to holistic representations of those shapes. Furthermore, attributes of shape emerge as fundamental properties when shapes are static. For instance, young children distinguish between a square and a regular diamond, because they see rotation as altering a figure's fundamental properties. "Half of the first- and second-grade children [in Lehrer et al.'s study] believed that a line oriented 50° from vertical was not straight. They characterized the line as "slanted" or 'bent'" (p.149). If interactions with shape do not allow for transformations of those shapes, then attributes of shape will arise as "false" fundamental properties, and actual fundamental properties cannot arise as significant.

Piaget's[1] (Piaget & Inhelder, 1956) groundbreaking investigation into the child's representational thinking about the nature of space revealed existence proofs of young children's intuitive and informal topological thinking. That means that not only do young children have metric, Euclidean ways of thinking about shapes, but at least as soon as they enter into school, they also have non-metric, topological ways of thinking. Unfortunately, these thoughts almost certainly go unacknowledged in their classrooms.

[1] I use "Piaget" to refer to both the man and his many collaborators.

There's really no place for them there. Elementary school instruction in geometry is inconsistent with, and inconsiderate of, children's non-metric ways of thinking. Its focus, albeit narrow, is metric. The consequence of disordering the content so as to only attend to children's Euclidean conceptions is that it may render a student-centered, cognitive-oriented, inquiry approach to instruction nonviable.

Mathematics, Interrupted

In order to support the reader's engagement, I provide a brief review of the mathematics. Without an even vague understanding of the domain, the reader will be at a loss as to how to make sense of qualitative geometry and the non-metric conceptions that structure it. Another sort of interruption appears in Chapter Two. Holding up young children's geometric conceptions as sufficiently significant to make a "sideways move" from an investigation of their conceptions of a "legitimate" mathematical domain provides an interruption not of the narrative but of the domain of mathematics itself.

Topology[2] is a branch of mathematics that deals with properties of a set of points that are invariant under bicontinuous transformations (i.e., both the transformation and its inverse are continuous). It is concerned with geometric properties that are dependent only upon the *relative* (as opposed to absolute) positions of the components of figures and not upon concepts such as length, curvature, size, or magnitude (which are Euclidean). Visualizing a set of points on an infinitely stretchable sheet of rubber may make the

[2] I will use "topology" throughout to denote the most basic subfield of topology called point-set topology.

geometry that is commonly referred to as "rubber sheet geometry"[3] more clear. The sheet may be stretched and twisted, but not torn or distorted so that two distinct points coincide. Straight lines may "become" curved and circles may become ellipses or even polygons. Bell (1951) elaborates:

> Imagine a tangle of curves drawn on a sheet of rubber. What properties of the curves remain unchanged as the sheet is stretched and twisted and crumpled in any way without tearing? Or what are the *qualitative* properties of the tangle as distinguished from its *metrical properties* – those depending upon measurements of distances and angles? (p. 156, italics in original)

Intuitively, but not entirely, this is the idea of topology. Euclidean geometry, in contrast, deals with properties that preserve distances between pairs of points, like length, curvature, size, and magnitude. Because topological transformations preserve neither shape nor size and Euclidean transformations preserve both size *and* shape, it follows that most Euclidean properties are lost under topological transformations. Using "rubber-sheet" language, topology is the study of "all those properties of extensible, flexible surfaces like sheets of rubber, which are invariant under stretching and bending without tearing" (Bell, 1951, p. 100). Euclidean transformations preserve congruence; topological transformations preserve nearness. Consequently, the set of Euclidean transformations is a subset of the set of topological transformations: topology is the more general geometry[4] and Euclidean geometry is, by definition, a special case of topology. Thus, the historical ordering of geometry instruction as Euclidean first and topological (for some) much,

[3] The term "rubber-sheet geometry" is attributed to Edward Kasner (Bell, 1951). The reader might be interested to know that Kasner is the mathematician whose five-year-old grandson named the googol, that "large" number that is a 1 followed by 1 hundred zeros.
[4] Some texts (e.g., (Smart, 1998)) have begun to establish topology as a division of mathematics distinct from geometry.

much later is logically reversed. Unfortunately, most students cannot see the superset for the subset. Much like the study of linear functions absent any consideration of the class of functions for which the linear function is a special case (Stroup, 1996; Stroup & Wilensky, 2000), they never get the chance to.

Back to Background

The common content of elementary school geometry is not only inconsistent with important elements of learners' developing thinking, it privileges a particular kind of formal geometry over other forms of logical and coherent geometry. When Piaget (Piaget & Inhelder, 1956) showed parallels between the "prelogical" (Piaget, as quoted in Bringuier, 1980, p. 33) structures young children construct spontaneously and topology, the responses were especially critical of his informal use of the mathematics. Kapadia (1974) was particularly harsh: "Topology is, contrary to accounts in popular recreational books, a very precise, systematic subject. To claim that a child's first vague and imprecise notions may be topological is a travesty" (p. 423). This critique demonstrates the privileging of a particular age-dependent account of what it means for mathematics to be formal on several counts. Besides the fact that what is useful among Piaget's findings is ignored at the expense of critique on formal mathematical grounds, if one assumes that knowledge is rooted in experience, there is no reason that children's conceptions should correspond entirely to those of formal mathematics (Lehrer & Jacobson, 1994). Inasmuch as their thinking more highly corresponds to their own real-world experiences than to formal mathematics, "children rarely attend to conventional properties because the conventions are more useful for more formal geometry but have few implications for children's everyday use of geometric concepts" (p. 8). This means that geometric ideas typically conceived as fundamental are not likely to coincide with those ideas that are fundamental to learners' natural conceptual development in geometry. "Children's topology," as conceptually significant as it might be, might not be expected to look exactly like some features of formal geometry.

Along the lines of Piaget's reference to young children's thinking as prelogical, Freudenthal (1971) sees their activity as "premathematics." Not that their thinking is amathematical, but that their activity, whether conceived as intuitive or informal, is fundamental to more formal mathematical activity. "Mathematics… used to be allowed to start as an activity," he argues, and "experiments… have shown that children can… develop an activity which on a higher level would be interpreted as mathematics" (the "real thing"). Finally, "this should be stressed against people [like Kapadia] who rightly object that it is no mathematics at all" (p. 417).

Another critique of the traditional approach to the teaching of geometry is that, perhaps not unlike any other domain, the content is given. It is principally about identifying shapes. The shapes are conventional as are their names. Teachers elicit this conventional knowledge of shape from students (Clements, 2004) and verify that it aligns with the curriculum. The opportunities for engaging students' thinking are restricted to only those occasions when elements of their thinking align with instruction, and the opportunities for pursuing connections to other domains of mathematics – even beyond geometry and topology – are severely limited. When geometric figures are seen as drawings on paper, then these connections are obscured. So geometry in elementary school becomes "elementary school geometry" – as if it were a domain in and of itself. It exists in isolation and is bound to be all about outlines of physical objects, nomenclature, and taxonomy (i.e., naming categories and putting things in those categories): "This is a circle, because it looks like the moon." The pictorial emphasis seems consistent with other naming tasks in elementary school, such as in biology: "This is a stamen because it looks like a stamen." The content is reduced to naming conventions as if children can only engage the iconic representation.

BENEFITS OF THIS RESEARCH

The situation presents itself quite differently in the context of arithmetic. Students solving problems may invent their own strategies, and if these strategies are logical, they are justified by well-understood, shared standards of sense-making (and also by fundamental theorems of arithmetic). This is what Cognitively Guided Instruction (CGI) (Carpenter, Fennema, Franke, Levi, & Empson, 1999) is about. Informed by a significant body of research that has helped us better understand the development of children's thinking about number concepts, CGI has now become a professional development program designed "to help teachers develop an understanding of their own students' mathematical thinking, its development, and how their students' thinking could form the basis for the development of more advanced mathematical ideas" (Fennema et al., 1996, p. 404). Simply stated, beginning with an assumption that "children enter school with a great deal of informal or intuitive knowledge of mathematics that can serve as the basis for developing much of the formal mathematics of the primary school curriculum" (Carpenter, Fennema, & Franke, 1996, p. 3), teachers use knowledge from research on children's mathematical thinking to generate a space for the consideration of children's informal and intuitive ideas about arithmetic, and instruction proceeds from there (Carpenter, Fennema, Franke, Chiang, & Loef, 1989). The formal mathematics is reconceived in such a way that considers and characterizes young children's ways of thinking (e.g., "join/separate" rather than "add/subtract"). As a result, learners have opportunities to meaningfully engage with the mathematics, because conventional features (e.g., symbols, notations) of formal mathematics do not get privileged over their ways of thinking about it. Nor does an assumed curriculum constrain the space of what gets talked about in their mathematics classroom. In contrast, as Lehrer and Jacobson (1994) argue, the research on learners' ways of thinking about geometry is lacking, particularly for children in the elementary grades. Clements and Battista (1992) seem to agree. Their call for research "to identify the specific, original intuitions and ideas that develop and the order in which they develop" (p. 426) is precisely what Piaget did over fifty years ago (Piaget & Inhelder, 1956), and with the exception of attention to the ordering of development, this is somewhat close to what I have attempted to accomplish herein.

Currently, students' mathematical potential is not being realized, because their teachers are not provided with the tools they need to build on the informal mathematical ideas their students bring into the classroom (National Research Council: Committee on Early Childhood Mathematics, 2009). If teachers do not understand how their students are thinking about geometry and if the curriculum is not connecting to the informal mathematical knowledge that kids bring into schools, then teachers have no capacity to engage their students' ideas and to subsequently support their development. By generating a space that lies beyond the dominant model of elementary school curriculum and instruction, we are prepared to better understand how children's geometric thinking develops.

METHODS AND RESEARCH QUESTIONS

To begin the process of developing models of young children's qualitative geometric conceptions, I used qualitative methods. In addition, because my focus is on qualitative geometric conceptions, I required a learning environment in which those conceptions were made salient and a tool with which to make them visible and subsequently support their development. In these and other ways, I conducted a teaching experiment (Steffe & Thompson, 2000) in which tasks were designed to model participants' current and developing knowledge in the domain. The *in situ* and *a posteriori* analyses contributed to the development of those models. My research questions were:

1. Given that early forms of topological, or at least non-metric, geometric reasoning have been identified and discussed in the research literature, can a software environment be developed in ways that support fundamental topological representations and transformations such that learners' reasoning about

topological ideas are made visible and are able to further develop in ways that could credibly be seen as both mathematical and significant?

2. What forms of topological or non-metric geometric ideas are made visible and can be seen to develop as a result of young learners' systemic engagement with a computer environment that makes topological representations and transformations accessible?

These questions structure the substance of this dissertation. In the following chapters, I further develop the rationale for the study, outline my conceptual framework, and discuss my methods, analysis, findings, and implications. In Chapter Two I interpret the relevant research on young children's topological conceptions and discuss the mathematics relevant to the investigation. In Chapter Three I discuss my methodological approaches, including the research design, sources of data and their analysis, as well as the perceived strengths and weaknesses of the study. In Chapter Four I provide a narrative of the development of the software I used in the study. In Chapter Five I present the study's findings in the forms of narratives of each of the participant's experiences. Finally, Chapter Six concludes the paper with a discussion of the conclusions and implications of the study, as well as potential future directions for this research.

I close this chapter with the presentation of the word cloud that was developed from the text of this dissertation. It is meant to foreshadow for the reader the stories that I tell here.

Figure 1. A word cloud of the text of this dissertation

Like the dissertation itself, this representation might attend to new ways of understanding what discourse about geometry could be like.

Chapter Two: Literature Review

In the previous chapter I provided an illustration of the dominant model of elementary geometry instruction and followed it with a variety of critiques of that model. I concluded by suggesting that, like CGI in arithmetic, a body of research into the development of children's geometric thinking can provide teachers with the information they need about their students' intuitive and informal geometric knowledge so as to be better prepared to engage and extend their students' geometric thinking.

In this chapter I provide a review of literature related to investigations of the development of children's representational thinking about space that yield existence proofs of their topological thinking. Then, I draw some conclusions about the topological ways that young children think about shape. These conceptions are used to structure the development of a new domain of mathematics, or at least a new analysis of learning, I call "qualitative geometry." Then, to frame the discussion of the impact, this literature review offers images of what a "qualitative move" beyond the dominant model might offer young students and their teachers.

Generating a space that lies beyond the dominant model of elementary school curriculum and instruction by identifying young children's non-metric geometric conceptions provides teachers with the capacity to better support the development of their students' geometric thinking. Now, the way I go about generating this space is of critical importance. Just as it is in the case of CGI, the primacy of mathematical content at the core of research and practice will be emphasized here. This sense of what research is to be about is one that Piaget took very seriously, as Papert (1980) explains:

> For Piaget, the separation between the learning process and what is being learned is a mistake. To understand how a child learns number, we have to study number. And we have to study number in a particular way: We have to study the structure of number, a mathematically serious undertaking (p. 158).

Accordingly, the initial justification for analyzing the development of qualitative geometry follows from the observation that Piaget (Piaget & Inhelder, 1956) carried out the first systematic investigation of children's representational thinking about the nature of space in a book titled, *The Child's Conception of Space*. Piaget finds from that investigation that:

> … representational thought or imagination at first appears to ignore metric and perspective relationships, proportions, etc. Consequently, it is forced to reconstruct space from the most primitive notions such as the topological relationships of proximity, separation, order, enclosure, etc., applying them to metric and projective figures yielded by perception. (p. 4)

This phenomenon that the development of children's representational thought is first topological (assuming neither constant size nor constant shape), then projective (assuming constant size, but not shape) and finally Euclidean (assuming both constant size *and* shape), has come to be referred to as the "topological primacy thesis" (cf. Martin, 1976b) Before considering two of Piaget's experiments that most significantly informed the work of this dissertation, it is necessary to explain the difference between "perceptual" and "representational" thought. As I elaborate below, that difference is significant for understanding both Piaget's methods and my own.

Perceptual and Representational Thought

It is important to acknowledge that Piaget was describing the development of children's representations of space and not their perceptions of that space. "Perception," as Piaget (Piaget & Inhelder, 1956) defines it, "is the knowledge of objects resulting from direct contact with them. Representation… involves the evocation of objects in their absence or, when it runs parallel to perception, in their presence" (p. 17). Representations are not mere copies of percepts – perceptual "reading[s] off" (Clements, et al., 1999; Flavell, 1963) – they are re/constructed from the coordination of percepts (or "centrations," in Piaget's words), each of which is initially acquired through the child's sensori-motor actions.

According to Piaget's framework of cognitive growth (1970b), in order for a child to build a representation of a particular shape, properties of that shape must be abstracted through a series of coordinated, reversible actions (or *operations* (Piaget, 1970b, p. 15)), and these properties must then be related and synthesized into a coherent whole. In contrast to ordinary abstraction – or generalization, which is a process of deriving properties *from* things and not from operations *on* things, this sort of abstraction – one that Piaget refers to as *reflective abstraction* (1970b, p. 28) – requires action on the objects – in this case shapes – that possess them. Relative to shapes, the properties or patterns that are abstracted provide the elements for the construction of mental representations, or images, that are not necessarily pictorial, but do exhibit some sort of mental or operational correspondence to the thing perceived. The construction of the representation of that shape from an active sense of perception of possible operations on objects is an instantiation of Piaget's model (1970b)[5] of the child's representational thought as developing from his or her sensori-motor intelligence. For Piaget's discussion of shape, he used a haptic environment, much less limited to visual perception than other investigations of geometric ideas.

To illustrate, Piaget's haptic perception experiments (Piaget & Inhelder, 1956) (involving tactile and kinesthetic perception, but not visual), presented in greater detail below, were conducted to analyze the child's construction of a mental representation from active perception. The child is presented with a number of objects and actively manipulates each one without being allowed to see it. Based on patterns in his or her haptic experience, the child constructs a mental representation of each object as he or she manipulates it. Each instance of contact forms a percept, and percepts are coordinated to form a mental representation of the object. Then, the child is asked to name, draw, or point out the object from a collection of visible objects or drawings of them. This sense connects with a broader literature related to constructions and perception. Some of this literature is consistent with a notion of seeing as active, as composed of possibilities. In this sense, a square is also seen as any of its images upon transformation, such as a non-square rectangle or even a triangle. When one actively sees the square, one also sees transformations of the square. As such, the square, rectangle, and triangle are not seen as distinct but as alike.

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[5] And its reflection in Gardner's presentation of Piaget's stage theory (Gardner, Kornhaber, & Wake, 1996a, pp. 105-112) as a theory of representations provides me with new appreciation for a theory I had otherwise had relatively little use for.

In a collection of lectures called *Ways of Worldmaking* (1978), Nelson Goodman describes the work of psychologist Paul A. Kolers. Goodman mentions an experiment Kolers had conducted in which two dots were flashed on a screen with variable times between flashes. He found that in a particular time interval, an observer would see a spot moving from the first position to the second. He wondered what this apparent motion would look like if figures such as squares, triangles, and circles, or even three-dimensional figures such as cubes, were flashed instead of dots. Noting that a notion of similarity only applies to two figures of the same shape, he wondered how *dissimilar* two figures of different shapes – and even different dimensions – would have to be before there could be no time interval in which there was apparent *change* from the first figure to the second. Given the diversity of shapes and dimensions, what he found was surprising: he showed that "almost any difference between two figures is smoothly resolved" (p. 75). One could use Kolers's findings as evidence to support a claim that at the perceptual level, under particular conditions, all figures are topologically equivalent. This equivalence is developed by the observer in relationship to active operations or possible transformations; it is not in the objects themselves.

As an example of how critiques could go wrong by not distinguishing between perception and representation, one could attempt to refute Piaget's thesis by operating at the perceptual rather than representational level as Dehaene and colleagues tried to do (2006). Participants in their study were given arrays of six images and asked in their language to point to the "weird" or "ugly" one. One of the six figures did not possess the property or concept of interest, and some of these were topological (e.g., connectedness, closed, holes). Among the authors' findings is a conclusion that "does not support Piaget's hypothesis of a developmental and cultural progression from topology to projective and Euclidean geometry, but rather suggests that geometrical intuition cuts across all of these domains" (pp. 382-383). Certainly, this exercise does not involve operations on representations, which is *the* essential feature of Piaget's notion of structure. Representations are constructed from internalized actions, and so it is particularly troubling that participants in this study were not active. They were pointing at shapes, not operating on them.

Like Piaget's investigations (1956), I am interested in the development of children's representational thinking. Having provided this clarification of the distinction between perception and representation, and I now return to review the literature related to those investigations.

Returning to Piaget's Investigations

Piaget (1956) identified five spatial relationships that he called topological: proximity, separation, order (or spatial succession), enclosure (or surrounding), and continuity. He defines them as follows:

- *proximity*: the nearby-ness of elements belonging to the same perceptual field
- *separation*: Two neighboring elements may be partly blended and confused. To introduce between them the relationship of separation has the effect of dissociating, or at least providing the means of dissociating them.
- *order* (or spatial succession): when two neighboring though separate elements are arranged one before another
- *enclosure* (or surrounding): In an organized series ABC, the element B is perceived as being 'between' A and C which form an enclosure along one dimension.

- *continuity*: In a series ABCDE etc., in which adjacent elements are confused or perceived without being distinguished (A = B, B = C, etc.), but where A and C or B and D are distinguished, the subject has an impression of continuity. (p. 8)

Piaget realized that these relationships are among the basic perceptual relationships analyzed by Gestalt theory[6]. These five relationships were first presented among a set of nine by Max Wertheimer in 1923 (as reported in Wertheimer, 1938):

- Law of Closure - Our mind adds missing elements to complete a figure.
- Law of Similarity – Our mind groups elements that appear similar (e.g., in terms of form, color, size and brightness).
- Law of Proximity - Our mind groups elements that are close together, even if they appear different.
- Law of Continuity - Our mind continues a pattern, even after it stops.

The five spatial relationships that are central to Piaget's investigation are compatible with formal mathematics to the extent that this geometry deals with properties that are dependent on the *relative* components of figures and not upon concepts such as length, size, or curvature. And these properties are at least as meaningful from the perspective of Gestalt laws of perception.

The consistency between Piaget's "topology" and Gestalt's basic perceptual relationships enables us to make sense of Piaget's findings of "topological primacy" by considering the relationship between perceptual and representational thought. His thesis of topological primacy derives from the claim that the actions that elicit topological relationships are more elementary than those needed to abstract Euclidean relationships, which require greater ability in ordering, organization, and coordination of action. So properties "like" proximity and separation are abstracted earlier than properties "like"

[6] Goodman, as discussed earlier, also draws upon the insights of Gestaltists.

side length and angle measure. Given that the topology referred to here is Piaget's, and that "Euclidean figures" refers to those figures for which Euclidean properties are most salient, I find this re-presentation of Piaget's findings more credible than "topological primacy."

Topology and "Piaget's Topology"

At some point it must be noted that, from a formal mathematical point of view, there is much about Piaget's (1956) mathematics that can be critiqued. His confusion of the mathematical terminology along with his use of terminology that is not mathematical is widespread. This is unfortunate for at least three reasons. First, Darke (1982) and Martin (1976b) advise that we need to be aware of the sense in which terms are used and not extrapolate beyond that sense of usage. For example, a concept of separation as Piaget uses it seems to precede a concept of separation as mathematicians use it. Or, a finding of topological primacy may have too little to do with mathematical conceptions of topological concepts. Second, if a term is used that is not mathematical (such as Piaget's uses of "irregular," "simple Euclidean shapes," and "topological figures"), it is not possible to provide the kind of mathematical definition that clarity requires. Third, it is with respect to terminology that most of the reactions to Piaget's investigations (Piaget & Inhelder, 1956) have been leveraged, and these reactions have clearly distracted us from what it is we might learn from those investigations about the development of the child's representational thinking. Consequently, as I elaborate later in this chapter, these critiques surrounding the issue of confusions around mathematical meanings provoked me, in part, to make a sideways move from an investigation of children's "capital T" topological conceptions to an investigation of their "qualitative" geometric conceptions, a cue I have taken from the journal *Nature* in which the word "topology" was first introduced to distinguish "qualitative geometry from the ordinary geometry in which quantitative relations chiefly are treated" (Wikipedia, 2007)[7].

That said, the "topological relationships" to which Piaget refers are actually "topological properties" (or "topological invariants"). Topological properties are those properties that remain invariant under continuous transformation. For example, a disc will never have a hole after a continuous transformation. On the other hand, a straight line could become curved upon transformation. Therefore, squares and circles are topologically equivalent. Finally, Euclidean transformations are those that preserve distances between pairs of points. A consequence of Euclidean transformation is that shape and size are preserved. Therefore, Euclidean properties *include* topological properties (because Euclidean properties preserve everything!), and this means that we cannot separate properties into distinct categories of topological and Euclidean.

Martin (1976b) and Kapadia (1974) devote entire papers to the issue of meanings, and each of those who has conducted replicate experiments (Esty, 1971; Laurendeau & Pinard, 1970; Lovell, 1959; Martin, 1976a, 1976b) also consider the issue. I demonstrate the significance of the issue by exploring Piaget's notions of separation and proximity.

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[7] Coincidentally, at one point in their text on the subject, Chinn and Steenrod (1966) provide a proof that "exhibits that peculiar blending of numerical precision and rough *qualitative geometry* [emphasis added] so characteristic of topology" (p. 1).

First, consider "separation." I propose that the deviation of separation in the Piagetian sense from separation in the mathematical sense could be dismissed as a technicality only if we simply accept that topology in the mathematical sense is not equivalent to topology in the Piagetian sense. If that is the case, then we could agree with Darke (1982) who argues that "the confusion of terminology alone seems to require a replacement of the topological primacy thesis by a 'weird shape primacy thesis'" (p. 121). But Martin (1976b) is more constructive, noting that loose everyday language is insufficient if these relationships are to be considered topological properties. Thus, we find it fitting to make sense of the Piagetian terminology.

Separation in the Piagetian sense seems to mean *disjoint*, which refers to a property of two or more sets whose intersection is empty. The formal mathematical sense requires a notion of limit points: a point P is a limit point of a set A if every open set around P contains at least one point of A that is distinct from P. Two sets are separated if they are disjoint *and* if each set contains no limit points of the other. For example, consider the infinite sets, $A = (0,1]$ and $B = (1,2]$. The intersection of A and B is empty,[8] so they are separate (disjoint) according to Piaget's definition. But they are not separate in the formal mathematical sense because 1 is an element of A and it is also a limit point of B.[9] As it turns out, in the case of Euclidean space and finite sets, it *is* true that Piaget's

[8] 1 is an element of A but not of B.
[9] Any open interval around 1 will "capture" points in B other than 1.

usage is equivalent to the mathematical sense.[10] Thus, Piaget's use of separation is acceptable in the cases where he uses it and the criticism on mathematical grounds is unwarranted.

Now consider proximity. This is the more useful case precisely because the confusion is harder to resolve. I could follow Kapadia (1974), who asserts that proximity is "certainly not a topological relationship… for it involves a vague idea of distance, a concept foreign to topology" (p. 420), but my reluctance to privilege some aspects of formal mathematics over other logical forms of reasoning requires that I be more considerate. Piaget defines proximity as "the nearby-ness of elements belonging to the same perceptual field" (1956, p. 8). This sensori-motor-based criterion for nearby-ness is as far as Piaget goes to define proximity but his intentions are clarified in the tasks he uses to assess it. Given that Piaget was interested in children's representational thinking and representations are constructions of mental representations through reflective abstraction, then nearby-ness must be arrived at in relation to perception. At this level, it makes little sense to talk about the proximity of two points since points are zero-dimensional and thus cannot be perceived. But even at the perceptual level, it is not clear how close two points must be to deem them proximal. This point, as Kapadia argues, does suggest that Piaget's proximity requires some notion of distance. But if proximity were a topological property, then, roughly, if two points are nearby one another in one set, their images under a topological transformation are also nearby one another. Indeed,

[10] Finite sets have no limit points. Consider $S = \{1,2,3\}$. It is trivial to find an open interval around any point *P* in *S* that captures only *P*. Simply pick ε such that $0 < \varepsilon < 1$. Then $(P-\varepsilon, P+\varepsilon)$ is just such an interval.

this could be the case, but the justification may not be so convincing. Recalling the rubber sheet and imagining that the images of nearby points can have images as far apart as we want gets us into trouble. Martin (1976b) provides just such an example (p. 20), shown in *Figure **2***. Note that the

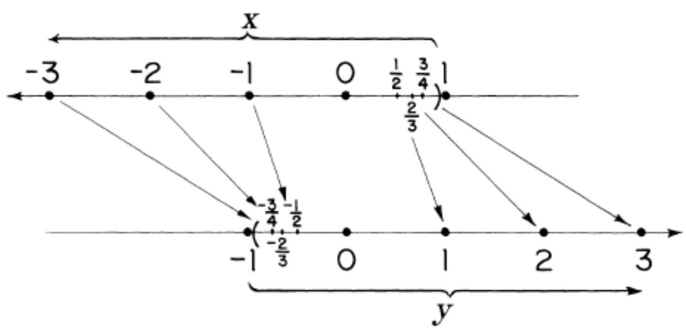


Figure 2. The homeomorphism defined by $f(x) = x/(1 - x)$

images of points in X between 0 and 1 cover the positive values in Y, and the images of points in X that are less than 0 are between –1 and 0 in Y. It would appear, then, that proximity as a proxy for nearness is *not* a topological property. But the demonstration is misleading, probably because in order to illustrate the nearness of two points, we must be able to visually perceive it. In other words, the figure does present proximal points "in the same perceptual field," but those points are not sufficiently nearby to satisfy a topological condition of nearness. I draw on one variety of the topological concept of neighborhoods to make the argument. In the case of two-dimensions (i.e., the plane), a neighborhood of a point P is the set of points inside a circle with center P and radius $\varepsilon > 0$. A reasonable criterion for nearness, then, is that two points are near each other if they lie in the same neighborhood, that is, the distance between them is less than the radius ε of the neighborhood. Because the concept of continuity makes use of the concept of neighborhood of a point, ε is typically thought of as infinitesimally small, although it need not be. Piaget's use of proximity, on the other hand, certainly suggests a more relaxed notion of nearness than this concept of neighborhood. Because we can choose a sufficiently "small" value of ε to define the neighborhood out of our perceptual field, the image of two points that are nearby each other in the Piagetian sense apparently need not be in the same neighborhood. To conclude, because Piaget's criterion for nearby-ness is perceptually based, it is not a topological property.

Consideration of the differences between the formal and informal concepts of separation and proximity may be demonstrative of the extent to which informal treatments of the subject matter become more accessible that more formal treatments. I elaborate this point later in the chapter when qualitative geometry is defined. For now, it suffices to say, this is one of the greatest advantages of a sideways move from an investigation of children's topological conceptions to their qualitative geometric ones. For if the investigation were of topological concepts, then findings lend themselves to binaries of topological or non-topological. But by relaxing the precisions, an investigation of children's qualitative geometric conceptions could produce findings that lend themselves to rich and nuanced descriptions without a felt need to classify.

Piaget's Experiments

My first analysis of the experiments of Piaget's investigations (1956) was for the sake of my own review of his conclusion of topological primacy. When I became uninterested in the order in which children's geometric ideas develop, my re-view was for the goal of identifying young children's topological conceptions of shape as revealed in the findings of Piaget's investigations and the replicate studies. Therefore, all the details that do not support this goal are omitted from the following presentation of those experiments and their replicate responses. Also, for the sake of maximizing clarity, I should note that where there is a potential for confusion, I have reinterpreted some of the mathematical language. All of the references are to his investigations as reported in *The Child's Conception Space* (1956).

Experiment 1: Piaget's Haptic Perception Tasks

Haptic perception involves tactile and kinesthetic perception, but not visual. The child is presented with a number of objects (such as a toothed semicircle, trapezoid, "irregular surface" with one or two holes, open and closed rings, intertwined and superimposed rings), familiar solids (ball, scissors, etc.) or "flat" shapes (squares, circles, etc.), and manipulates each one without being allowed to see it. The child is asked to name, draw, or point out the object from a collection of visible objects or drawings (*Figure **3***) of them. So that the experimenter can see the child's methods of tactile exploration, the child is placed before a screen behind which it feels the objects handed to him or her.

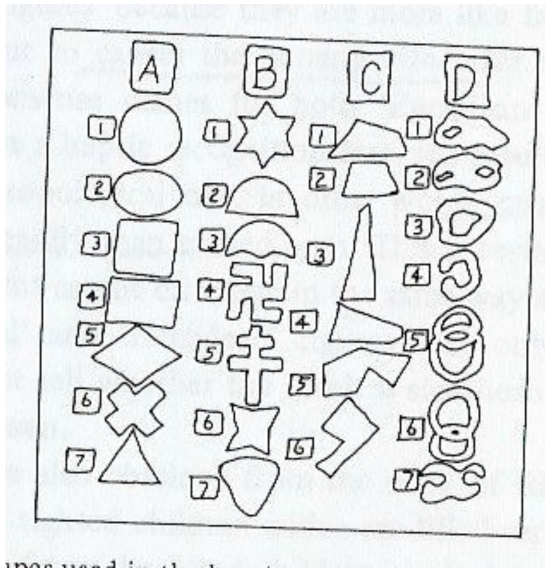


Figure 3. Shapes used in Piaget's haptic perception experiment

Piaget gives little indication, but his classification of "Euclidean forms" seems to include those shapes with straight sides in columns A, B, and C. His use of "purely topological form" seems to include the "irregular surface pierced by one or two holes" and the "open," "closed," or "intertwined rings" (p. 19) that appear in column D. He finds that by 3;6 – 4 (where *a*;*b* means *a* years and *b* months of age), children were able to recognize familiar objects, then "topological forms," but not "Euclidean forms." He argues that the first shapes recognized by the child are significant, not for the properties that are visually most salient, such as straight lines, curves, and angles, but rather for properties such as closed, open, and intertwined. The child easily distinguishes these features, but cannot differentiate "Euclidean shapes."

Piaget admits that errors arise from inadequate tactile exploration of the objects. For example, one child matched a held square to the model of a triangle, having touched only one of its corners. Another matched a held ellipse with the notched semicircle, only having felt its curved edge. This is unfortunate for the conduct of the study in that Piaget had apparently designed it with the expectation of an analysis of shapes by handling, by "passing it from one hand to the other, turning it over in all directions" (p. 10), that is, through full tactile exploration by the child. He suggests that the lack of exploration is a result of a deficiency in perceptual activity, referring to combinations of centrations (such as touching one part of the object) and decentrations (moving on to another part of the object). He concludes that the child's perceptions have yet to be integrated into a system that brings them together. This, he suggests, also explains the child's difficulty with copying the object by drawing, since the source of an image of an object is perceptual activity. Abstraction of shape is therefore more than the abstraction of properties; it also involves the actions taken by the child to coordinate those properties that allow him or her to grasp the shape as a single whole. This, he concludes, explains why the topological relationships of proximity and separation (which follow from openness and closure) arise earlier than Euclidean relationships. There is something fundamentally alike about topologically equivalent shapes.

At age 4 – 6, Piaget finds in children a progressive recognition of "Euclidean shapes." The child distinguishes between curvilinear and rectilinear shapes, but fails to distinguish the different sizes of the various shapes. The child explores the objects further, although not completely. Rectilinear shapes are identified by their angles, and "it is the analysis of the angle… that marks the child's transition from topological relationships to the perception of Euclidean ones" (p. 30). Reconstructing a rectilinear shape by drawing requires that the child consider that an angle is the intersection of two lines and not simply as "something that pricks" ("Leo," p. 29). Thus, abstraction of the angle must come from the action on the object rather than from the object itself. Specifically, it is the result of the coordination of hand and eye movements, which then gives the child an impression of straight sides.

By the end of 6;6 – 7, the perceptual activity is more organized than in the previous stage. The child demonstrates an operational (reversible[11]) method, which consists of grouping the perceived features in terms of a consistent search plan, and by starting at a particular point of reference and returning to it. Consequently, the distinctly perceived elements become connected with the others to form a single whole.

Responses to Piaget's Haptic Perception Tasks

Most of the responses to Piaget's investigations have been critiques around informal use of the terminology, an issue I've attended to briefly above, but one that deserves to be revisited here. Recall Darke's (1982) concern over false extrapolation and refer back to *Figure **3***. Note that all of the objects in columns A, B, and C, and object D7, are topologically equivalent. That is, each contains properties invariant under continuous transformation (i.e, each one can be stretched into any of the others). And so, if the topological primacy thesis were correct, a child could not distinguish between any of them. It's fair to argue then, as Martin (1976a) does, that the child's selection of topologically equivalent shapes could occur by chance. At a practical level, Kapadia

[11] I elaborate on Piaget's notion of reversibility on page 53 in the section where I introduce Qualitative Geometry.

(1974) suggests the topological equivalence of C, L, M, N, S, U, V, W, and Z (each sans serif), for example, would make it very difficult for children to learn the alphabet.

Replicate experiments were conducted by Lovell (1959) and Page (1959). Lovell gave 145 children aged 2;11 – 5;8 common objects and cardboard shapes like Piaget's. The subjects were asked to examine figures by touch and then select them by sight, as Piaget and Inhelder had done. He concluded that "topological shapes" were identified more easily than "Euclidean" ones, but if the topological shapes were compared only with Euclidean curvilinear shapes (like the circle and ellipse), there was no significant difference in the ease of identification. Page used 60 children aged 2;10 – 7;9. In general, his results agreed with Piaget's, except he noticed that children, even the youngest, *did* distinguish between curvilinear and rectilinear shapes.

I have already mentioned that, in the haptic perception experiment, errors arose because children did not adequately explore the objects. Piaget admits as much, and makes an effort to explain the nature of this behavior. Still, Page (1959) and Fisher (1965) are critical of Piaget's findings on these grounds. But Fisher also hypothesized that children should be more likely to identify objects whose names they know, so he conducted a replicate experiment wherein he canceled the naming effect by teaching children nonsense names for all the objects. His results showed a "linear primacy," that is, rectilinear shapes were more easily identified than curvilinear ones, but his study suffered from the same terminological issues as Piaget's. "Topological figures" and "Euclidean figures" were not well-defined.

Laurendeau and Pinard's (1970) investigation replicated Piaget's experiment. The authors used the same shapes as Piaget (*Figure **4***) and, like Piaget's, their design was informed by confused notions of the relevant mathematics. They offer that eight of the shapes are "of a topological character" and four are "metric." But they offer no indication as to which are which. I can see five in the first series that might map onto Piaget's "topological shapes," but in the second series I find only two.

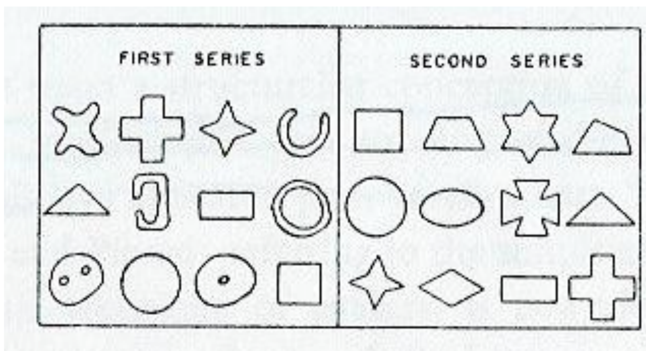


Figure 4. Shapes used by Laurendeau and Pinard (1970)

The authors find that participants most easily identified curvilinear shapes, then rectilinear and "topological," and finally "complex" rectilinear shapes. Unfortunately, I cannot make use of their findings, because I cannot make sense of them. Resorting to Piaget's terms, curvilinear shapes *are* "topological shapes," and I don't know what criterion differentiates a rectilinear shape from a complex rectilinear one.

In the next phase of analysis, Laurendeau and Pinard looked at children's errors. They categorized an error as either a "topological success" (one where the chosen shape was topologically equivalent to the correct shape) or a "topological error" (one where the chosen shape was not equivalent). This is problematic for two reasons. *Figure **5*** shows their groupings of equivalent shapes. First, note that the “open rectangle” appears in two groups. This means that the groups are not equivalence classes, because an equivalence class containing the “open rectangle” would contain all of the shapes in the sets in which it appears. Second, note that the groups are not formed according to topological equivalence as the authors suggest. If they had been, all of the shapes in groups 3, 4, 5, and 6 would be sorted together. The shapes seem to be grouped as such to simultaneously determine topological primacy and the primacy of straight-sidedness or curvilinearity.

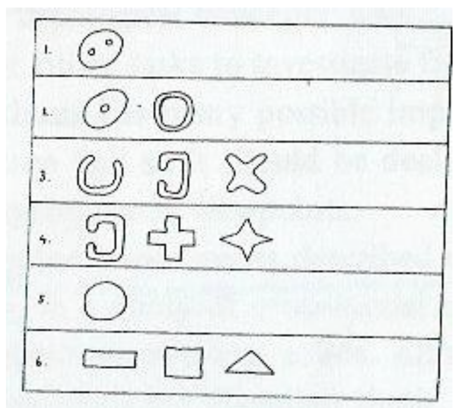


Figure 5. Laurendeau and Pinard’s groupings of equivalent shapes (1970)

Schipper (1983) also recognizes this issue. He finds it “interesting to note, as Laurendeau and Pinard already know, that their classification differs from the mathematically correct one. In a footnote (1970, p. 52) they try to defend their unusual classification with psychological arguments” (p. 288). Such a classification scheme is not permissible if the authors want to speak of children’s attention to topological properties. However, Schipper sees “an important hint” (p. 289) in that scheme and that is the relevance of a shape’s boundary: “Apparently children perceive indentations and protuberances of boundaries as especially striking characteristics” (p. 289).

Using their classification scheme, Laurendeau and Pinard found that "actual topological successes" nearly doubled "expected topological successes," and so they concluded that topological primacy existed. They argue, "it is quite obvious that, at least for some children, the errors are not randomly distributed but are guided by a search for homeomorphisms" (p. 61), or topological equivalents. But Darke's (1982) analysis proceeded differently. He analyzed their data with an assumption of topological primacy, which assumes that child would either choose the correct shape or one that is topologically equivalent. Then, an error would mean that the child chose a shape that is not equivalent. For example, of the objects in *Figure* ***5***, a closed ring and a disk with one hole are the only shapes in their topological equivalence class. And if topological primacy exists, a child is just as likely to choose one as the other. Now consider a second equivalence class that contains the nine topologically equivalent figures with no holes. We should expect a child to choose any shape in the class with equal likelihood. However, in both cases the actual responses are not as such. Subjects chose the same shape much more often than was expected under an assumption of topological primacy. Consequently, this analysis does *not* confirm topological primacy.

Martin's (1976a) haptic perception experiment is more considerate of the relevant mathematics. Six models are presented to 90 children – 30 each from ages four, six, and eight. The child is asked to select the one of three copies that is *most like* the model. The first copy is topologically equivalent. The other two copies are not topologically equivalent, but retain most of the Euclidean features of the model: the first copy loses the topological property of connectedness and the second loses closure. He found no age effect on children's choice of "most like" and no evidence of topological primacy.

Esty (1971) used a similar experimental design. Whereas Martin's (1976a) copies retained most of the Euclidean features of the model, Esty's copies keep at least one topological property of the model, but ignore almost all of the model's Euclidean features. Copy B allowed line segments to protrude, whereas in the model they did not. Otherwise, Copy B would be homeomorphic to the model. Copy C lost all of the model's topological properties. Esty also used "bandwidth" to control the amount that a copy would deviate from its model. That is, he drew a copy of the model with a fatter pen. Then, within the region formed by the fatter pen, he would form his copies of the model with a thinner pen. You might imagine that if the model were a circle and the bandwidth were large enough, the homeomorphic copy (Copy A) could be a square. Figure **6** shows the original model, two copies of different bandwidths, copy A (the homeomorphic copy), copy B, and copy C (the model minus the topological properties of connectedness and closure).

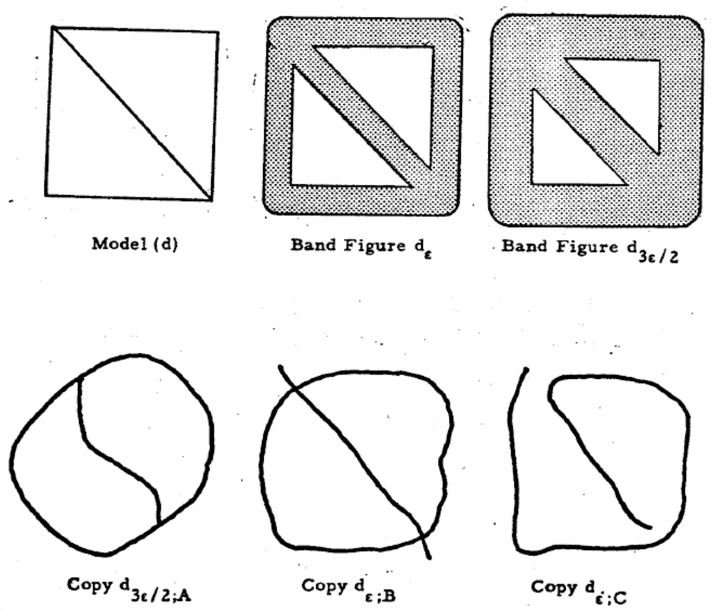


Figure 6. The model, two bandwidth figures, and copies A, B, and C.

Consistent with Piaget's findings, Esty found that four-year-olds tend to choose Copy A as "most like" the model and eight-year-olds choose Copy A as "least like." Then he makes some interesting conjectures as to why this is so. He wonders if the children are ignoring Euclidean relationships or whether it has to do with the way they look at pictures. The things that are "wrong" with copies B and C are confined to small parts of the area the picture covers; on the other hand, to notice the "non-straightness" of the sides in Type A copies requires the coordination of several centrations (in Piaget's terms). So maybe they chose Copy A not because of its topological inaccuracy, but simply because the kind of inaccuracy represented in homeomorphic copies is global rather than local, as it is in copies B and C. But this would require that they ignore the absence of angles, and Esty finds this unlikely, since there were other copies that represented the angles.

Experiment 2: Piaget's Drawing Tasks

The child is asked to produce drawings of a series of models, shown in *Figure **7***. Piaget suggests that some of the models emphasize topological relationships (1 – 3), while others are "simple Euclidean shapes," and still others combine both types of relationships. To eliminate the elements of skill and motor habits, children were also offered matchsticks with which to construct the straight-sided models.

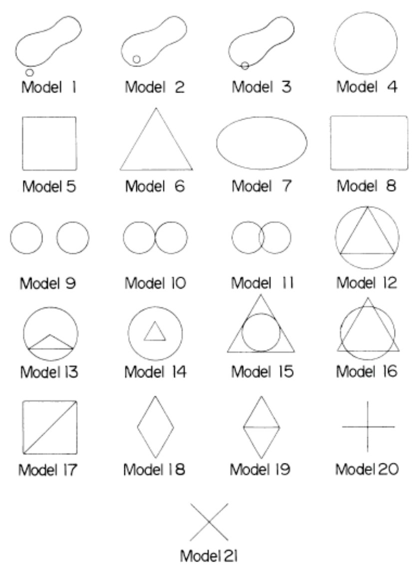


Figure 7. Models used in Piaget's drawing task.

Piaget finds that the drawings produced by children between 3;6 and 3;10 vary according to the models in that open shapes are distinguished from closed ones, and by age 3;10 – 4, they attend to some topological relationships, but "Euclidean relationships are completely ignored" (p. 55). In addition, he finds that squares and triangles are drawn as closed curves, but they are not distinguished from circles.

My interpretation of the presented drawings is different. *Figure **8*** shows drawings of models 4 (the circle) and 5 (the square) from a sample of drawings from children in

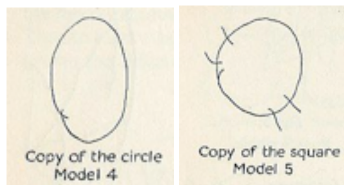


Figure 8. A sample from a child in the age range 3;10 – 4

this age range. These copies of the child's drawings of a square and a circle have the distinguishing feature of lines. These might be seen to suggest that there is a Euclidean relationship that the child attends to. The copy of the square suggests that the child seeks to represent a square as a closed (a topological relationship), four-sided (a Euclidean relationship) figure. The segmenting of the shape into four pieces suggests it is the sides of the square that distinguish it in the child's mind from a circle (which has no sides).

Also by 3;10 – 4, the child distinguishes the open-shaped cross from the circle and triangle, but is not able to distinguish rectangles, squares and triangles from circles and ellipses. Models 1 – 3 are drawn correctly, so attention is clearly paid to whether the smaller circle lies inside, outside, or upon the boundary of the larger figure. In topological terms, the child is representing properties of enclosure (by drawing the closed circle and larger figure), separation (by representing the distinction between the two figures), and proximity (by properly locating the circle in relation to the larger figure).

Darke (1982) suggests that at least some errors in the child's drawings should be attributed to a lack of motor ability. They can't always draw what they see. But Piaget disagrees, noting that children at this stage are able to copy the cross and draw vertical tree trunks linked to horizontal branches, for example, each of which requires the skills that are necessary for drawing a square. Furthermore, it is a matter of the composition of elements identified in the original figure that result in the construction of a copy of it, and since topological relationships "are inherent to the simplest possible ordering or organization of the actions from which the shape is abstracted" (p. 67), those appear prior to projective and metric relationships which require more complex types of organization of elements such as those involving angles and directions (e.g., parallel). Piaget supports this finding with a rather convincing argument:

> "…the 'abstraction of shapes' is not carried out solely on the basis of objects perceived as such, but is based to a far greater extent on the actions which enable objects to be built up in terms of their spatial structure" (p. 68) This abstraction "actually involves a complete reconstruction of physical space, made on the basis of the subject's own actions and to that extent, based originally upon a sensori-motor, and ultimately on a mental representational space determined by the coordination of these actions" (p. 76). "This is why the first shapes to be abstracted are topological rather than Euclidean in character, since topological relationships express the simplest possible coordination of the dissociated elements of the basic motor rhythms, as against the more complex regulatory process required for coordination of Euclidean figures" (p. 68).

By ignoring Piaget's use of "topological shapes" and considering "topological properties" instead, the argument seems sensible. Topological properties are more fundamental *qualities* of shape than Euclidean properties, which also have a metric component. Thus, they should be simpler to abstract – and these abstractions should be simpler to coordinate – than metric properties, which are characteristically more complex. Martin's (1976b) response to this argument is presented later.

The stage from 4 to 7 years is marked by progressive differentiation of shapes by "Euclidean" properties. Curved shapes begin to be distinguished from each other and from straight ones, although the latter still remain undifferentiated from each other, "notably the square and the triangle" (p. 56). It is the latter finding with which I cannot wholly agree. *Figure **9*** shows a sample of drawings of models 5 (the triangle) and 8 (the square) from a point just prior to this stage. According to Piaget, the child who drew

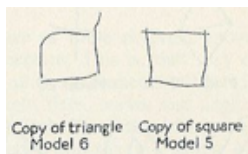


Figure 9. Drawings by a child just younger than 4 years of age

these copies fails to differentiate between the square and the triangle. My interpretation, however, is that the child who drew the copy of the triangle was attending to the three sides of the triangle.[12] She drew two straight sides and then the third curved side, which is curved so that the child can close the figure.

Early in this stage, lengths are considered so that squares are distinguished from non-square rectangles and circles are distinguished from non-circular ellipses. An effort is made to represent relationships of connectedness and separation (models 9 – 16, *Figure **7***) in drawings that are constructed fairly accurately, although the points of contact are

[12] I watched my child at age 3; 11 produce this same figure when asked to copy a triangle. Goodnow (1977, as cited in Darke, 1982) suggests, "a child's thought is often shown not so much in the end product of a drawing but in the process of producing it" (p. 134); I assert that I have an advantage if it is the case that Piaget missed the process while I did not.

not properly represented. Later in the stage, the child is able to represent inclinations and is finally able to draw the non-square rhombus (model 18, *Figure 7*).[13]

By 7 – 8 years all models are copied with consideration given to both topological and Euclidean relationships.

Responses to Piaget's Drawing Tasks

Lovell (1959) used Piaget's figures and found no significant difference in the accuracy of representing topological and Euclidean properties. Moreover, he found that children could construct the models six months earlier, on average, than they could draw them. Piaget's subjects were able to draw and construct the models at the same age. This is salient, because if Lovell's findings are correct, they suggest that the difficulties in drawing are not conceptual but sensori-motor.

Dodwell (1968, as cited in Darke, 1982) conducted a replicate experiment that was inconclusive. The three researchers involved in the experiment found it difficult to decide how well topological, projective, and Euclidean properties were reproduced in children's copies. Ninio (1979) argues that because the accuracy of a copy is not a mathematical problem, it must be the case that any scoring system will be based on the relative tolerance of judges for deviations on different spatial attributes. Martin (1976b) uses Piaget's samples of student work (*Figure*

[13] This phenomenon was surprising to me, also. After my child, aged 3; 11, had correctly copied my model of a square, I added a diagonal to my model and asked her to add it to her copy, as well. She drew a horizontal segment that joined the midpoint of opposite sides of the square. Later I asked her to copy a 'K', which she can easily name by sight. She drew the vertical segment correctly, but the legs were drawn horizontally. If I were to ask her to name the copy she had drawn, I suspect she would guess that it was an 'F'.

10) to make this point. Assuming as Piaget does that the child meant to close figures that should have been closed, "the evidence that topological representation precedes Euclidean representation is hazy" (p. 13). With respect to the copy of model 12 from "Late Stage 1" and the copies of models 12, 13, 15, and 16, from "early Stage II," Piaget finds that "the circumscribed figures are drawn correctly enough as regards their actual shapes but the points of contact are not properly represented" (1956, p. 56). Citing connectedness as a topological property, Martin (1976b) is correct that none of these drawings are topologically equivalent to the corresponding model (not even the copies of models 12 and 15 from "early Stage 2" which lack two points of tangency), because all of the models are connected [see *Figure* ***7***]. On the other hand, the copies of models 1 – 3 are topologically equivalent to the corresponding model, as are the copies of models 9 – 11.

Now, as for Piaget's claim, stated earlier, that the abstraction of shape from objects is not based on perception alone, but to a far greater extent on the actions one performs on the objects. Piaget has argued that the actions that elicit topological relationships are more elementary than those needed to abstract Euclidean relationships, which require greater ability in ordering, organization, and coordination of action. Moving beyond the then-fashionable critique of Piaget, Martin (1976b) suggests that the emphasis be placed on these abilities, rather than on trying to chronologically order spatial representations according to topology, Euclidean geometry, or projective geometry.

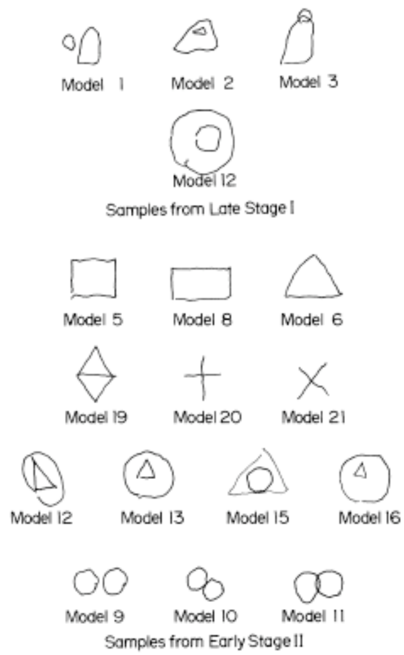


Figure 10. Sample drawings from Piaget's drawing task

> Perhaps coordination of curvilinear motions precedes coordination of rectilinear motions. Where it is necessary to coordinate only curvilinearity, as is the case for Models 1, 2, 3, 9, 10, and 11, the copies resemble the models sooner than when rectilinearity is involved. This would also explain the early resemblance of the children's rendering of Models 4, 5, 6, 7, and 8 to the circle rather than to their respective models. The later developed coordination of rectilinearity allows near Euclidean congruence to the models for drawings of Models 5, 6, 8, 17, 19, 20, and 21. Models 12, 13, 14, 15, and 16 involve both curvilinearity and rectilinearity; hence it is necessary to coordinate both types of action. (p. 15)

Notice how much is revealed when Martin attends to ideas rather than the order in which they develop. It's this move that informs my own investigation, which I elaborate upon later in Chapter Three.

Next, learning from the difficulties of others in interpreting children's drawings, Martin (1976a) used the same figures from his haptic perception experiment (where subjects were offered three copies per model) in a drawing test. This also allowed him to make comparisons in the results across the two experiments. Martin found that the drawings made by the four-year-olds after their tactile examinations of shapes were nonhomeomorphic 90% of the time, and most of those drawings showed no discernible properties of the corresponding models. Only 30% of eight-year-olds' drawings were nonhomeomorphic. Following visual examination, 60% of four-year-olds' drawings were nonhomeomorphic but they did reveal discernible properties. Thus, "drawings as analyzed in [the] study did not indicate that the representational space of four-year-olds is predominantly topological in nature" (p. 37). Then Martin (1976a) went a step further. By comparing the children's drawings to the figures they selected as "most like" and "worst" in his haptic perception experiment, he suggests that it is not attention to topological properties that enables children to draw homeomorphic copies, but the increasing coordination of Euclidean and projective properties that produces the homeomorphic drawings, because preservation of topological properties follows from the coordination of Euclidean and projective properties.

Take What You Need: Summarizing the Findings and Moving on

Investigations that are designed to confirm or deny topological primacy are not typically meant to inform our understanding about all the ways that young children reason about shapes. Fortunately, there is evidence among those findings that we can use in our investigations. What we gain from these studies is that young children *do* discriminate based on the informal topological properties of separation and connectedness, and the also-non-metric properties that include the number of corners, the orientation, and curvature (i.e., between (Euclidean) curvilinearity and rectilinearity).

In addition to critiques surrounding issues related to Piaget's confusion of the mathematical terminology, other reactions to Piaget's conclusion of topological primacy simply meant to confirm or deny it (Laurendeau & Pinard, 1970; Lovell, 1959; Martin, 1976a; Page, 1959). Darke's (1982) analysis does not confirm it; Vurpillot (1972, as cited in Darke, 1982) concludes that it is too extreme; and Martin (1976b) concludes it is "too strong" (p. 15). In fact, when the theory is presented in its totalizing form – "Representational space is first topological, then projective, and finally Euclidean," then my own review also reveals very little hope for topological primacy. But recall that Piaget's topology is not entirely consistent with formal topology. He had informal ways of describing shapes (as topological or Euclidean) that are indeed somewhat variably meaningful and, as such, completely inappropriate from a formal mathematical perspective.

I have mentioned Martin's (1976b) suggestion that emphasis be placed on abilities rather than on trying to chronologically order spatial representations according to topology, Euclidean geometry, or projective geometry. If I have been clear enough so far, it should be evident that his suggestion aligns with my goals for this study: I seek to better understand the nature of the child's representational space by developing a model of children's thinking in the domain of qualitative geometry.

Taken together, the results of Piaget's experiments and the replications of those experiments are critical to my own research in several ways. First, they provide existence proofs that young children represent the topological properties of separation and connectedness and the Euclidean property of curvature (i.e., between curvilinearity and rectilinearity). Moreover, while some of the distinctions young children make are topological and others are Euclidean, some appear to have qualities common to both topological and Euclidean properties. These distinctions have a comparative aspect that is not about length, for example, yet they are not entirely non-metric, either. In other words, the distinctions have features that are neither topological nor Euclidean; often they are based on properties fundamental to both topology and Euclidean geometry. For example, a square and a circle are topologically equivalent and "Euclidean-ly" distinct, but children (including those in my pilot study) often distinguish between them in a non-metric but non-topological way by identifying "corners" as the distinguishing characteristic. When angles are referred to as corners, this property is Euclidean in the same sense that angles are Euclidean, but the property is also topological, since children are not attending to the measure of those angles.

Papert's (1980) realization that, "for most people, nothing is more natural than that the most advanced ideas in mathematics should be inaccessible to children" (p. 161), and Bruner's (1963) claim that "any subject can be taught effectively in some intellectually honest form to any child at any stage of development" (p. 33), move me to my next point: these findings provide a basis for new opportunities to develop young children's geometric knowledge. In particular, a conceptual approach to the learning of non-metric geometry, one that is grounded in learners' rich, intuitive, conceptual thinking, not in prescribed, pre-determined standards, challenges the "Euclidean assumption" in the learning of geometry. And one of the strengths of a qualitative approach to geometry may be just that – a direct and explicit challenge to dominant views of what geometry and geometry instruction look like in schools.

By following the child's thinking, informal[14] understandings can provide the basis for new and developing understandings of this non-metric geometry. This approach is aligned with Piaget's view of the development of knowledge as described by Allardice and Ginsburg (1983): "The child's informal knowledge serves as a kind of cognitive underpinning or scaffolding for school learning" (p. 328). Furthermore, now that we realize that children "rarely attend to conventional properties perhaps because the conventions are more useful for more formal geometry but have few implications for children's everyday use of geometric concepts," (Lehrer & Jacobson, 1994) we can avoid the "great mistake" that Piaget warned us about: "the great mistake some people have made is going to formalization too quickly with students who aren't at all ready to assimilate it" (as quoted in Bringuier, 1980, p. 129). In other words, because the qualitative geometry that I am proposing is developed *from* the conceptions of young children, and because it is informal (albeit coherent and logical), there is no risk that a

[14] Piaget's view of informal knowledge (or *spontaneous concepts* (Piaget, 1970b)) are those concepts that children develop spontaneously in their natural surroundings, such as elementary notions of more and causality. Ginsburg (1977, as reported in Allardice & Ginsburg, 1983), on the other hand, describes the development of two kinds of informal knowledge. *Informal* and *natural* knowledge are those "intuitive" concepts that are developed before or outside the context of school. *Informal* and *cultural* knowledge, such as counting, are those concepts that are socially transmitted (in schools, on television, by adults, etc.). I have used "informal" and "intuitive" interchangeably to describe "children's topological concepts" with support from Clements and Battista (1992) who argue that the topological ideas I reference "are originally *intuitions* [italics added] grounded in action-building, drawing, and perceiving" (p. 426).

formal geometric framework gets privileged over the child's own way of knowing this new geometry.

Note how Kelly's (1971) vision of what an arguably informal Euclidean geometry could look like in schools problematizes this dichotomy between real-world experience and formal mathematics. He begins with a thought experiment:

> There is a simple and natural way of regarding Euclidean geometry that has been largely ignored in the teaching of geometry. This is the view that Euclidean geometry is the mathematical study of the size and shape attributes of physical objects. It is my belief that this functional approach to the subject… leads to surprisingly sophisticated and vital mathematical concepts. (p. 477)

Indeed, this creative approach to geometry *is* surprising, because it generates a space of pedagogical possibilities in which the mathematical concepts that are developed include those that are topological. The essay continues:

> The natural question, "What is an object?" translates to the mathematical question, "What properties of a geometric set make it the analog of a physical object?" An obvious requirement is that the set be bounded. But the key property of the "oneness" of the set as an object is connectedness. This can be expressed by the condition that each two points of the set be the ends of an arc in the set. However, this intuition simply shifts the difficulty to "What is an arc?" Though an answer to this presents difficulties, one partial answer is quite clear, namely, that a segment is an arc. Thus convexity provides a very simple form of connectedness. (p. 477)

By the time the experiment is concluded, Kelly also encounters topological concepts of boundary, interior, exterior, and simple closed curves. "As for the concepts themselves, leaving proof aside, I believe these are wholly accessible to grammar school children" (p. 478). Imagine handing out rubber sheets to a class of elementary school kids, having them draw a shape upon them, and then asking them in words they could understand, “What’s invariant?” Given her commitment to attending to children’s ideas and forms of reasoning, I’m pretty sure Eleanor Duckworth (1995) would be into it.

As if consideration of young children’s ways of thinking weren’t reason enough for the development of a qualitative geometry, its non-metric feature will give those who know it new access to proximal forms of geometry that an entirely Euclidean approach could not. For example, graph theory and topology are “close by.” Consider the case of electrical circuits: there are infinitely many topologically equivalent ways to build any circuit. In fact, there is a class of graphs called circuits. Similarly, nonhierarchical, nondirected concept maps are topologically equivalent. As part of some other research I have been a part of (Empson, Greenstein, Maldonado, & Chao, 2008), we developed a tool for discourse analysis called content maps, which are structurally similar to concept maps in that they have common topological properties. We have also developed a typology of connectedness (a topological construct) that has helped us characterize the maps we have created. On yet another project, a colleague and I have been trying to use topological concepts to better characterize the space of mathematical tasks than the commonly used spectrum of open and closed. Following one of our meetings, I walked down the hall to find that one of my peers is using social network theory in her research. Naturally (Scott, 2000), she used graph theory and topology in her analysis of several social networks. As a final example, consider the 3D online virtual world called Second Life. One feature of this world is that teleportation is possible. Consequently, the environment is a *discrete* metric space:[15] the distance between any two points X and Y is 1 when $X \neq Y$; it's 0 when $X = Y$.

Engagement with non-metric geometric conceptions also provides young children with access to the fundamental ideas of set theory. Here's how: Metz (1995) finds that children begin to use categories during preschool and that their categories are theory-laden. Also, attribute- and property-based reasoning about shape are two of the three most common ways that children reason about shape (Lehrer & Jacobson, 1994; Lehrer, Jenkins, et al., 1998)[16]. Finally, research with children as young as kindergarten age has confirmed that they are prepared to identify, define, and apply distinguishing criteria to given sets of figures (Burger & Shaughnessy, 1986; Clements, et al., 1999; Lehrer, Jenkins, et al., 1998). Therefore, it is reasonable to expect that if geometry instruction were to include categorization and classification tasks of the sort I have used in this study to elicit young children's non-metric conceptions, then they will be provided with new opportunities to engage with fundamental ideas from set theory. Additionally, if early geometry instruction is broadened to consider a greater variety of shapes than what learners are exposed to in a traditional prototype-only approach, and if this instruction also begins with an investigation of properties *and* the shapes that possess them, we might do a better job of scaffolding the development of children's hierarchical thinking.

[15] A metric space is a set of points (a topology) together with a notion of distance (called a metric).
[16] The third of the three most common forms of reasoning is resemblance-based.

"In all mathematical fields, the qualitative must precede the numerical (Piaget, 1950, p. 79-80)" (quoted in Munari, 1994, p. 314)[17]

INTRODUCING QUALITATIVE GEOMETRY

While Euclidean properties are those that are invariant under Euclidean transformations, and topological properties are those that are invariant under topological transformations, in the same sense, qualitative properties are those that are invariant under qualitative geometric transformations. These properties are non-metric; they are qualitative but not quantitative. They have a comparative aspect that is not about measure but about differences in measure. For example, triangles and squares are distinct in the *number* of corners each has, but actual angle measures are irrelevant from a qualitative perspective. Similarly, in a qualitative sense, we might say that ∠A is larger than ∠B, but the measure of an angle is not a qualitative property, so it would be irrelevant from a qualitative perspective to consider the measures of either of these angles. Likewise, we would be interested to know that $\overline{AB}$ is longer than $\overline{CD}$, but not that AB = 10 and CD = 5.

How might these comparisons be expressed in the context of my own study? Referencing another study, Piaget (de Zwaart, 1967, as cited in Piaget, 1970b) describes the language that children use to make comparisons of items in a collection of common

[17] This quote has been appropriated out of context. The full statement, still relevant, is here: "The mathematical operation derives from action, and it therefore follows that the intuitional presentation is not enough. The child itself must act, since the manual operation is necessarily a preparation for the mental one [...]. In all mathematical fields, the qualitative must precede the numerical" (Piaget, 1950, p. 79-80).

objects (e.g., blocks, pencils) as either "scalar" and unitary ("That one is big"; "This one is little") or "vector" and binary ("That one is bigger than the other"; "That one is longer and thinner, the other one is shorter and thicker"). The children who tended to use scalar language were identified as pre-operational (and therefore non-conserving); the children who tended to use vector language, on the other hand, were identified as operational and conserving. This latter group of children might be expected to use vector vocabulary to express a qualitative property in the comparative sense described above.

As I mentioned earlier, I took the term "qualitative" from a definition of topology that appeared in the journal, *Nature.* I also use it in a sense that is similar to the way Stroup (2002) uses it to describe a "qualitative" approach to calculus in which students engage with "qualitative or comparative" (p. 17) forms of rate-related reasoning. And his usage is drawn, at least in part, from Piaget, who used the phrase "qualitative speed" (1970b, p. 119) to refer to rate-based reasoning that is not ratio-based.

*Figure **11*** illustrates the way that qualitative geometry is situated among other geometries. While Euclidean properties are those that preserve distance, and topological properties are those that are dependent only upon the relative positions of the components of figures and not upon such concepts as length, size, or magnitude, qualitative properties are non-metric and preserve relative lengths, size, and magnitude. The set of Euclidean properties is the "largest" set, because Euclidean transformations preserve "everything."

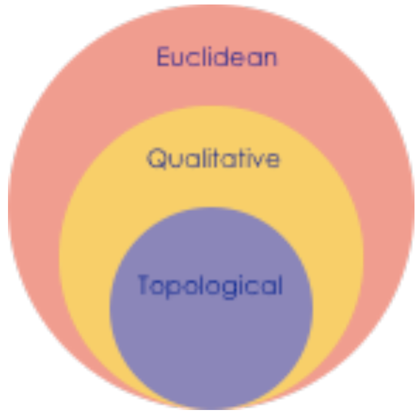


Figure 11. In terms of the number of properties each represents, qualitative properties lie "between" topological and Euclidean ones.

The set of topological properties is "smallest," because topological transformations preserve the fewest properties in comparison.

Qualitative geometry does have much in common with point-set topology. Again, one of the primary reasons for not referring to children's intuitive and informal non-metric conceptions as topological is to avoid the criticism incurred by Piaget for uses of terminology that are in some instances unclear and in others instances deviant from the definitions given by formal mathematics. But it is also true that topology and qualitative geometry are not equivalent, as the "corners and angles" example I provided earlier was meant to demonstrate. So although the notion of a "children's topology" to parallel a particular "children's mathematics" (Carpenter, et al., 1999) or a more general "students' mathematics"[18] (Steffe & Thompson, 2000)) is very appealing, it cannot adequately characterize the domain. Furthermore, by moving the investigation away from binaries

[18] Les Steffe (personal communication, March 16, 2010) doesn't see what "qualitative geometry" gets me. He suggests "geometry." When I explain that I'm only interested in children's non-metric conceptions, he suggests "spatial geometry."

that answer they yes/no question, “Is it a topological property?”, I am better positioned to explore young children’s thinking and subsequently provide the rich and textured characterizations that the exploration potentially yields.

My second research question, in part, seeks to determine the forms of young children’s qualitative conceptions of shape. To that end, I illustrate the significance and coherence of qualitative geometry by showing that it is a cognitive structure (Piaget, 1970b). To demonstrate this structure, I begin as Piaget does by viewing the algebraic group[19] as “a kind of prototype of structures” (p. 19). Showing that qualitative geometry is a cognitive structure means that it “behaves” as a group does, and this is accomplished by identifying a system of transformations – a patterning of actions – an organization of schemes or functions – an organized pattern of behavior – that gives it the group quality. Piaget generates the subject matter of topology in this way. Through the construction of new structures, Piaget establishes relationships between Euclidean and projective geometries, and topology. He starts with the group of displacements, which leaves invariant the dimensions, angles, parallels, straight lines, and so on of figures. This is the group that comprises Euclidean geometry. This group is a subgroup of higher groups that allow one or more of these properties to remain invariant while other properties are preserved. The group that constitutes projective geometry, for example, preserves straight lines while other properties vary. These are subgroups of the group I am interested in, which is topology. This group subjects all of these properties to transformation, leaving

[19] A group is a set of elements together with a binary operation that has the following properties: it is closed; it contains an inverse; it contains an identity; and the operation is associative.

none of them invariant. It is interesting in a structuring/"Big Ideas" kind of way that these geometries, which are traditionally seen as disconnected from one another,[20] "are thus reduced to one vast construction whose transformations under a graded series of conditions of invariance yield a 'nest' of subgroups within subgroups" (p. 22).

In addition to identifying the operations that are constituent of the cognitive structure that is qualitative geometry, I also seek to identify the patterns in how one engages with it that give it *wholeness*. Given what the Bourbaki have declared to be the three irreducible, "mother structures" of algebra, order, and topology, these are the sources of all other structures and substructures (Piaget, 1970b). New structures are formed by combining two parent structures, such as the combination of algebra and topology to produce algebraic topology. New substructures are formed by adding or subtracting conditions, that is, by fixing one property while varying another. This is the process by which we deduce the concept of similar figures, for example, by fixing shape (i.e., making shape invariant) and varying dimension.

As a cognitive structure, an understanding of qualitative geometry is typified by kinds of reversibility. The notion of reversible operations is central to Piaget's characterization of operationalized thought (Piaget, 1970b). For example, in the case of algebraic structures, reversibility takes the form of the ability to undo what has been done, like applying a function first and following it with the application of its inverse. Even in the case of searching figures in his haptic perception experiments, Piaget

[20] Topology, a branch of mathematics that initially grew out of geometry, is now often considered distinct from it. The rationale is that geometry is concerned with metric properties, but topology is not.

identified a kind of reversibility: beginning the search at a particular point of reference and later returning to it. In the case of qualitative geometry, one form of reversibility looks like applying a qualitative transformation to a figure (without altering its qualitative properties), and then transforming it back. In fact, Beth & Piaget (1966, as reported in Lesh, 1975) offer geometric transformations as the invertible cognitive operation that is roughly equivalent to the "mother structure" of topology.

While conceptually and cognitively significant in its own right, qualitative geometry's relationship to topology may "help situate and deepen students' understandings of the *basics* [emphasis added] as they are typically encountered in the traditional mathematics curriculum" (Stroup, 2005, p. 179) while also serving to lay the groundwork for *later* mathematical instruction (Carpenter, et al., 1989; diSessa, 2000). Given that a notion of transformation is so central to topology (not to mention the fact that transformations are *the* mechanism for the development of more complex cognitive structures from simple ones (Groen & Kieran, 1983; Piaget, 1970b)), and that transformational geometry with its consideration of the role of function is more relevant to just about every domain of mathematics than typical treatments of Euclidean geometry in elementary school, an understanding of qualitative geometry can be seen as transitional to these other domains. Chinn and Steenrod (1966) agree: Topology "is a kind of geometry… which underlies all geometries. A striking fact about topology is that its ideas have penetrated *nearly all areas of mathematics* [italics added]" (p. 1). Sowder and colleagues (2010) express a similar sentiment: "... from the mathematical point of view, transformation geometry is closer to the mainstream of modern mathematics than geometry á la Euclid. Transformations of the plane are special cases of mathematical *functions*, an idea that appears in some form in *every area of mathematics*" (p. 491, emphases in original).

As I have stated previously, a review of the literature related to the topological primacy thesis yields evidence that young children do possess early and intuitive ideas that are fundamental to both Euclidean geometry and topology. Recall that those who have conducted replicate experiments (Esty, 1971; Laurendeau & Pinard, 1970; Lovell, 1959; Martin, 1976a, 1976b) did not dispute Piaget's results (Piaget & Inhelder, 1956), only his conclusion of topological primacy. If we ignore the order in which these ideas develop, it may become easier to accept that, at least intuitively and eventually, children attend to notions of order, connectedness, enclosure, proximity, and separation. It is in the space common to Euclidean geometry and topology in which qualitative geometry is situated. Not coincidentally, each of these notions exists in that space. For this reason, it is a rather reasonable and straightforward conjecture to make that young children possess ideas fundamental to qualitative geometry.

Earlier in this chapter, I provided a review of the literature around Piaget's investigations into the development of representational thinking about the nature of space. Those studies and their replicates yielded findings that informed the mathematics I would investigate in this study. Now that an early definition of the domain of qualitative geometry has been provided, I conclude this section by providing two examples of empirical research findings that provide existence proofs of children's qualitative ways of thinking. The first example comes from the work of Burger and Shaughnessy (1986). The thinking they refer to as “imprecise” (p. 41) is exactly the sort of thinking I refer to as qualitative. They find that one student forms a class of figures that contains triangles that have a “small angle at the top, wider angles at the base.” This non-metric, comparative way of “measuring” angles gives it a qualitative property. A second example is provided from the findings of Lehrer, Jenkins, et al. (1998) and from Clements (1999). They find that young children describe lines that are neither vertical or horizontal as “slanty”; angles are described as “pointy”; shapes are described as “having corners,” skinny, fat, long, big, and small; and categories are formed by the number of sides or the number of angles. None of these descriptors are metric; rather, they all are qualitative.

Chapter Three: Methodology

Sure, go ahead and prepare your research design. Plan away. But my advice, don't hold on too tight. Constructivist/phenomenological methodologies require a certain spaciousness of thinking that allows for things to emerge on their own terms. Oh sure, we can review the literature and find those that have tromped this ground before us and conclude that things will go this way or that. Theories abound. Conclusions, too. Hitch your wagon. But in the end, let the kids or the teachers or the spawning salmon or whatever your 'subject' have their way with you; be present to their every word and wiggle. Pay attention. Resist that pressing urge to make sense of it all, to impose your questions, categories, and order too soon. (Thorp, 2005, p. 117)

The Pilot Study

Early in Spring 2008, I conducted a pilot study. It was meant to be a first pass at exploring young children's qualitative conceptions of shape and an evaluation of the effectiveness of the software for revealing those conceptions. A review of the literature related to the development of young children's representational thinking about the nature of space provided me with a vague but tangible sense of their intuitive and informal topological conceptions of shape. I had conjectured a qualitative geometric cognitive structure to frame these ways of thinking, and I had planned the pilot study, in part, to explore it.

The goals of the study were to answer the following questions:

1. In what ways do children distinguish between plane figures?
2. If these distinguishing features are qualitative, what are their characteristics?
3. What does it mean to understand these newly characterized distinctions?

I conducted the study at Zapata Elementary,[21] an urban school in which 93.2% of the students are identified as economically disadvantaged and 70.9% have been identified as at-risk. The school had been rated "academically acceptable" for 3 years prior. I chose this site through an existing relationship between the school's mathematics instructional specialist and myself. I chose to conduct my study in an elementary school so that I could select participants to closely align in age with those in Piaget's (Piaget & Inhelder, 1956) experiments, but also out of concern for the finding by Clements (2004) that conceptions of shape are stabilizing by age 6.

In order to select participants, I described the study and its purpose to the cooperating teachers of children in grades K through 3, and then asked the teachers to select two students whom they describe as average or above average (by ordinary school standards) and who are most articulate and likely to have interesting things to say. This

[21] A pseudonym.

selection procedure closely follows one used in a classic article by Burger and Shaughnessy (1986). Ten students were selected to participate, two at each grade level from grades K, 1, and 3, and four from grade 2. One of the goals of my pilot study was to reduce the size of my participant pool for the dissertation study from ten to no more than three. I planned to select the three most vocal and productive students while maximizing representation by age and gender. As these were initial explorations into qualitative geometry, I was interested to see what might be possible. So a smaller sample size with extended engagement is more useful to me than a larger one.

In preparation for the study, I developed an iteration of the software (*SoundTrack*, which I describe in Chapter Four) that was apparently suitable for eliciting participant's non-metric conceptions of shape. The inquiry was contextual in that investigations were situated within that software environment. I framed the interactions with participants with semi-structured clinical interviews. The "semi-structured" tasks and questions were informed by the review of the literature reported above. Guiding questions "seeded" the interview with tasks developed to elicit the features participants used to distinguish plane figures and apply those distinguishing features to classification and categorization tasks.

Pilot Informs the Study

The pilot study prepared me, both affectively and effectively, for the larger study. For all the ways it went right or wrong, it was useful in informing the design of the larger study. Rather than presenting those relevant outcomes here, they appear throughout this and the following chapters in places where "they do their work."

This larger study is an exploratory one that uses a teaching experiment to develop models of young children's thinking in the domain of qualitative geometry. Specifically, the research questions I address in my study are:

1. Given that early forms of topological, or at least non-metric, geometric reasoning have been identified and discussed in the research literature, can a software environment be developed in ways that support fundamental topological representations and transformations such that learners' reasoning about topological ideas are made visible and are able to further develop in ways that could credibly be seen as both mathematical and significant?

2. What forms of topological or non-metric geometric ideas are made visible and can be seen to develop as a result of young learners' systemic engagement with a computer environment that makes topological representations and transformations accessible?

To the extent that there are findings related to the second question, these will be seen as laying the foundation for the claim that there are forms of qualitative geometric reasoning available to young learners that can be identified as mathematical, are significant, and can develop in ways that would have implications both for research into students' mathematical reasoning and as a focus for further curriculum development and design. In its broadest sense, the phrase "qualitative geometry" will refer to these findings and the attendant possibilities for future investigation and curriculum development.

My goal in this study was to understand how children aged six and seven reason qualitatively about two-dimensional shapes. Thus, the investigation was exploratory in the sense that qualitative geometric conceptions of shape are both identified and characterized as such throughout the course of the study. In other words, as the title of this dissertation suggests – *Developing a Qualitative Geometry from the Conceptions of Young Children* – a subdomain of geometry called qualitative geometry is iteratively defined[22] as its elements are elicited from children and characterized in the conduct of the study.

Like other investigations into children's thinking that have traditionally involved qualitative methods (Mertens, 2005), the kinds of methodological approaches best suited to answer my research questions – and to allow for themes to emerge beyond them (Mertens, 2005, p. 420) – also involved qualitative methods. Quantitative methods yielding numerical summaries, in contrast, "throw out qualities of performance that are essential to getting adequate accounts of an individual's understanding" (Stroup & Wilensky, 2000, p. 878). Thus, they have absolutely no capacity for yielding the sort of textured, narrative characterization of a child's conceptual mathematical models that this study requires.

With its recurring emphasis on individuals' perceptions of experiences, my "exploratory" pilot study was at times phenomenological (Bogdan & Bilken, 2003, in Mertens, 2005), and the methods for data collection felt as if they had been borrowed from ethnography. Indeed, this was my thinking following the pilot study during the dissertation proposal phase, although I hadn't the language nor the tools to describe it. If

[22] The domain of Qualitative Geometry was preliminarily defined in Chapter 2 and some of its contents were presented there.

the microworld (Papert, 1980) of the software environment can be thought of as a mathematical place (R. Nemirovsky, 2005), then ethnographic techniques seem/ed well-suited to the task as defined by goals that also aligned with some form of the ethnographic question, '*What's going on here in terms of [kids' qualitative thinking in the space of a topological microworld]?*' or *"What do people in this setting have to know in order to do what they are doing?"* (Wolcott, 1999, p. 69). These questions are just broad enough to generate an observational space, and the centering ideas so focused as to orient the looking. Even Spindler and Spindler's (1987) standards for good phenomenological research resonate with my intentions:

> Observations are contextualized, both in the immediate setting in which behavior is observed and in further contexts beyond that context, as relevant. Hypotheses emerge in situ as the study goes on in the setting selected for observation. Judgment on what may be significant to study in depth is deferred until the orienting phase of the field study has been completed. Observation is prolonged and repetitive. Chains of events are observed more than once to establish the reliability of these observations. (pp. 18-19)

These phenomenological / ethnographic ways of thinking about my study kept me open to *what was going on* and to findings not necessarily linked to research questions, and led me to the methodology of the teaching experiment.

The Teaching Experiment: Developing Models of Children's Understanding

The aim of any clinical research in mathematics education should be the construction of models. (Thompson, 1982, p. 160)

The teaching experiment (Steffe & Thompson, 2000) is an "exploratory tool, derived from Piaget's clinical interview and aimed at exploring students' mathematics (…) Whereas the clinical interview is aimed at understanding students' current knowledge, the teaching experiment is directed toward understanding the progress students make over extended periods" (p. 273).

Steffe and Thompson (2000) "recommended strongly that the teacher-researcher engage in exploratory teaching before attempting to conduct a teaching experiment" (p. 283). That strong recommendation is only moderately appropriate for this study, since my study is essentially exploratory in terms of both children's thinking in the domain of qualitative geometry and the domain itself. While I did have the results of Piaget's (Piaget & Inhelder, 1956) experiments, other replicate experiments, and my own pilot study as starting points with which to launch the development of qualitative geometry, I felt that these points were too few to establish a sufficiently rich topography of schema associated with qualitative geometry. Thus, I intended for this sort of exploration right from the start and throughout all episodes of the teaching experiment.

My goal as a psychologist has been to "enter the child's mind" – to discover as best I could how the child thinks... (Ginsburg, 1997, p. ix)

As Ginsburg (1997) describes it, the interview is an exploration in which the interviewer aims to discover and identify thought processes by combining several methods, such as observation, experimentation, and "think aloud," to encourage the child to verbalize his or her thought processes as richly and explicitly as possible. Of the eight rationales provided by Ginsburg (1997) for using the clinical interview, I draw on these four to make the case for its use in this study:

- *Acceptance of constructivist theory requires that clinical interview methods be used.* He is quite adamant on this point: "If you accept the constructivist position, you have no choice. You *must* use the clinical interview or some other method that attempts to capture the distinctive nature of the child's thought" (p. 58, emphasis in original).

- *They offer the possibility of dealing with the problem of subjective equivalence.* Traditional standardized testing methods cannot guarantee that all children interpret a problem in the way that it was intended, but the flexible nature of clinical interviews allows the interviewer to vary the questions and tasks for each student and then to determine whether the child has interpreted them in the way that was intended.

- *They are useful for understanding thinking in its personal context.* Piaget developed the clinical interview, in part, to account for mental context by determining children's motivation to deal with the task, their understanding of the question, and the strength of their convictions; and to identify their underlying thought processes.

- *They embody a special kind of methodological fairness, especially appropriate in a multicultural society.* "If we want to understand what [children] are capable of and what lies beneath their performance, we must test them in flexible ways, acting like the artist attempting to capture their individuality" (p. 69).

Simply stated, I think of my teaching experiment as a sequence of clinical interviews. But conceptions of the clinical interview vary, so I will clarify. Given the emphasis on verbalization in Ginsburg's clinical interview, I draw on Lythcott and

Duschl's (1990) differentiation between the clinical interview method, which is interview/verbal data only, and the clinical exploratory method, which also includes a task. Because participants in my study also completed tasks on paper and in/using a software environment in ways that made their thinking not only verbal but visible, my episodes may be thought of as employing the *clinical exploratory method* as Lythcott and Duschl define it. This method feels more like Cobb and Steffe's (1983) teaching episodes, which they distinguish from clinical interviews.

> We use teaching episodes *as well as* [emphasis added] occasional clinical interviews as an observational technique. The interviews are used when we want to update our models of the children's current mathematical knowledge... However, the main emphasis is on the teaching episodes, as these give us a better opportunity to investigate children's mathematical constructions. Our primary objective is to give the children opportunities to abstract patterns or regularities from their own sensory-motor and conceptual activities. (p. 86)

Related to Cobb and Steffe's use of "teaching," one person offered this interpretation of my study: "So you're teaching kids topology?" Actually, my claim is that kids have topological conceptions and they bring these with them to my investigations. But as they spend time on the playground of the software I use in tandem with the interviews, current conceptions are developed and new ones are assimilated. I say more about this in Chapter Four when I present the software piece, but it should suffice to say that I had no plans to either implicitly or explicitly teach the kids topology. But the topologically-informed tasks and software design, and the assessment-slash-instructional nature of clinical interviews in general, provoked opportunities for participants to learn topology. This notion of teaching resonates with Cobb and Steffe's illustration of a teaching episode above and thus clarifies the "teaching" in "teaching experiment."

To summarize, I use "episode" to name each of the meetings I conducted with participants. These episodes may be thought of as hybrids of clinical interviews á la Ginsburg (1997) and teaching episodes á la Cobb and Steffe (1983), each featuring the clinical exploratory method of Lythcott and Duschl (1990).

Consistent with my research questions, a goal of this study was to develop *models* of children's thinking in the domain of qualitative geometry. Thompson (1982) explains: "By "model" I mean a conceptual system held by the modeler which provides an explanation of the phenomenon of interest, in this case a student's behavior within some portion of mathematics. The conceptual system held by the modeler, when applied to a particular student as an explanation of his behavior, is a model of that student's understanding" (p. 162).

In order to construct models of children's thinking, the researcher comes to the episodes with tools and ideas about activities that the child will work on. These pre-planned *analytic* interactions (Steffe & Thompson, 2000) loosely structure the interviews by serving as starting points for an unfolding conversation, and their outcomes serve as a resource to inform the subsequent episodes and the ongoing development of models. I want to be clear about my use of the phrase "loosely structure the interviews," which I invented for the purposes of this study. As I mentioned early in this chapter when I described the pilot study, my coursework in research methods led me to believe that a semi-structured interview could productively frame my interactions with the participants in my study. This was my approach in the pilot study. Indeed, a script is just what the Office of Research Support demanded in the IRB application. What happened as a result is that the space I meant to explore – that of children's qualitative conceptions – was over-structured. A *semi*-structure provided too much structure. Planning, and a desire to execute the plans, dominated the interactions. My discussion above of the ethnographic quality of the study makes a similar point about the often overly constraining nature of the overarching research questions themselves. Not to say that an unconstrained, anything-goes sort of situation would be more productive, but rather to argue that less is more. Again, I submit that the loose structure of an interview allows for a richer exploratory environment than a semi-structured one. My pilot study was meant to be exploratory of a "smooth space" wherein any sort of thought or action was possible, but instead it collapsed under the weight of planning and preparation and a motivation to "get it right." Any tendency on the part of the student toward "deterritorializing lines of flight" were "captured and reterritorialized as striated space," "where options for movement are limited to rigid strata and uniform lines of thought and action" (Deleuze and Guittari's notion of "lines of flight" as interpreted by Lerner, n.d.). In other words, a child's wonderings/wanderings were too often grounded by the intentions of a researcher with an agenda. I recall, for instance, wanting to grab the stylus of the tablet PC from a child who really just wanted to play around, and feeling anxiety that I might subsequently be left with "bad" data if I failed to do so.

In contrast to the semi-structuring of interviews, a researcher implementing loosely structured interviews realizes a priori that it is neither the analytic questions nor the hypotheses that informed them that provide much of the richness and texture that a qualitative study requires. The extent of the power of loosely structured interviews lies in their capacity to “get at the good stuff” by “setting research hypotheses aside and focusing on what actually happens” (Steffe & Thompson, 2000, p. 276), that is, by exploring the child’s responses and following up, on, and around them. In retrospect, I now realize that what the pilot study did not sufficiently allow for – but the larger study did – were *responsive and intuitive interactions* (2000, p. 278) informed by the researcher’s *current* model of the child’s thinking and moment-to-moment hypotheses about patterns in the child’s reasoning. With an emphasis on exploring the child’s reasoning, these unplanned actions are made by the researcher without explicit awareness or apparent expectation of how the child will respond. They’re often impromptu moves made “on the fly” (Ackerman, 1995, as cited in Steffe & Thompson, 2000, p. 276) as the researcher gets lost in the space between his or her own thinking and the thinking of the child. The interview’s loose structure generates those spaces and occasions opportunities to exploit and explore them.

Thompson (1982) continues with his description of how models are constructed:

> As he or she watches a student (…), the researcher asks himself, “What can this person be thinking so that his actions make sense from his perspective? What organization does the student have in mind so that his actions seem, to him, to form a coherent pattern?” This is the ground floor of modeling a student’s understanding. The researcher puts himself into the position of the student and attempts to examine the operations that he (the researcher) would need and the constraints he would have to operate under in order to (logically) behave as the student did. (pp. 160-161)

The effort to develop models of each participant’s thinking in the domain is an effort to answer my second research question, which is essentially about characterizing an understanding of qualitative geometry. As I described in Chapter Two in my discussion of what it means for qualitative geometry to be a cognitive structure, developing this model of understanding requires attention to participants' interactions on shapes, making sense of their explanations, identifying patterns in their actions, and considering the reversible operations underlying those actions. Thompson (1982) concludes by explaining how we know when the process of constructing models of participants' understandings is complete:

> One does this for each student in the investigation, and as soon as one begins to see a pattern in ones mode of explanation, the job must be expanded to reflectively abstracting the operations that one applies in constructing explanations. When the researcher comes to the point that he is reflectively aware of these operations, and he can relate one with another, he has his explanatory framework (of the moment). That is, he has isolated the components and relationships among components which allow for explanations of individual children's behaviors. (p. 161)

My second research question also seeks to model the development of understanding qualitative geometry. A note on the parallels between the design of my teaching experiment and the applications of Vygotsky's "instrumental method" to model conceptual development provides an illustration of how I aim to answer this question.

> An experimentally evoked process of concept formation never reflects in mirror form the real genetic process of development as it occurs in reality. But this is not, in our eyes, a drawback, but rather a definite advantage of experimental analysis. Experimental analysis allows a dissection in abstract form of the very essence of the genetic process of concept formation. It provides us with the key to a true understanding and comprehension of the real process of concept development as it actually occurs in children. (Vygotsky, as quoted in El'konin, 1967, p. 36, as it appears in Cobb & Steffe, 1983, p. 86).

Whereas my first research question is related to the role of the software in eliciting children's qualitative conceptions and subsequently supporting their development, my second question is related to the coherence of those conceptions and the domain they inform. I draw on Steffe and Thompson (2000) to link the mathematics to the methodology:

> We regard the mathematics of students as a legitimate mathematics to the extent we can find rational grounds for what students say and do. Looking behind what students say and do in an attempt to understand their mathematical realities is an essential part of a teaching experiment (…) For us, this awareness is essential because teaching experiment methodology is based on the necessity of providing an ontogenetic justification of mathematics; that is, a justification based on the history of its generation by individuals…. (p. 269)

This methodological choice is, in my mind, a necessary and essential way to proceed. An investigation into young children's qualitative geometric conceptions is consistent with Steffe and Thompson's view of "mathematics as a living subject rather than as a subject of being" (p. 269). Because qualitative geometry is developed from young children's conceptions, it, too, is living.

The Episodes of the Teaching Experiment

Informed by reports of teaching experiments in the mathematics education literature (cf. Kuhn et al., 1995; Lehrer & Chazan, 1998; Schauble, Glaser, Duschl, Schulze, & John, 1995; R. S. Siegler & Crowley, 1991; Steffe & Thompson, 2000), which ranged in duration from three weeks to ten weeks, my teaching experiment lasted approximately eighteen weeks. I met with participants between eight and ten times for episodes lasting between thirty and fifty minutes. The first episodes were conducted in the first week of October, 2009, and the last episodes were conducted by the third week of January, 2010, for a period of data collection lasting approximately 18 weeks.

I had developed a general plan (*Figure **12***) for each of those episodes. Some would be exploratory, some would be framed by a loosely structured interview, and others would feature 2 or more participants.

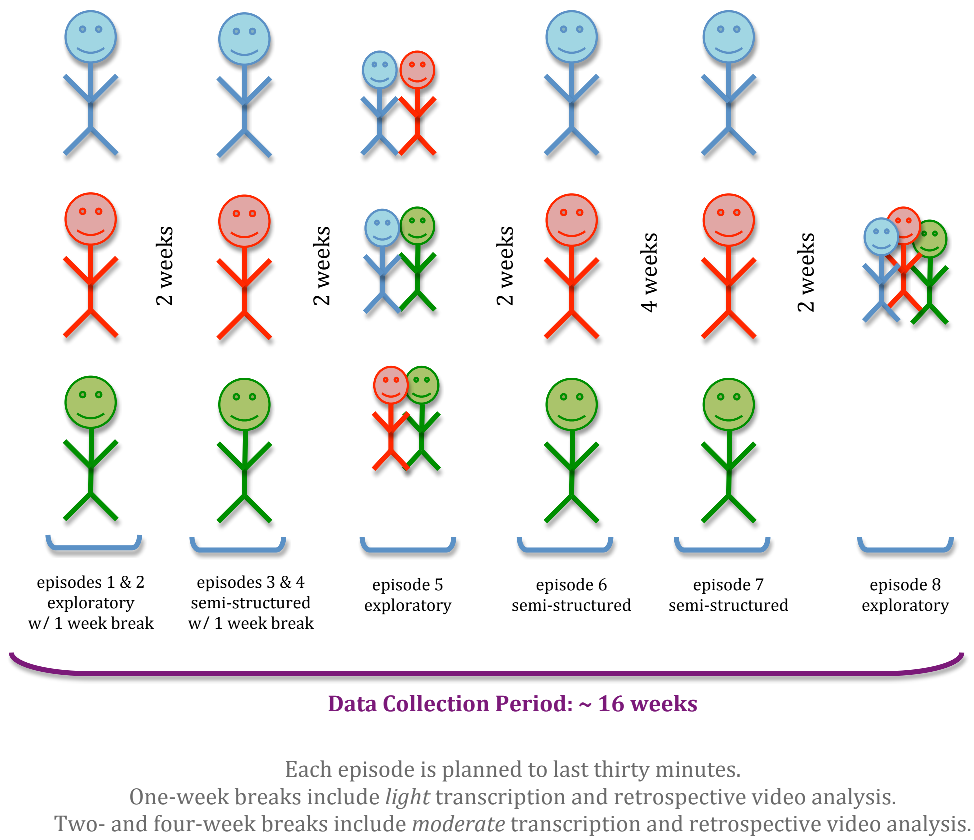


Figure 12. Plans for the episodes of the teaching experiment

But it didn't play out that way. Analytic tasks launched the explorations in each of the episodes and those tasks were followed up by responsive and intuitive ones. My current and developing model of each child's thinking and hypotheses about patterns in the child's reasoning informed each of the tasks. Following the conclusion of each episode, findings from a retrospective analysis informed new hypotheses, and new hypotheses informed the design of tasks for the subsequent episode. *Figure **13*** is the template of the record I used to summarize the analysis of each episode and plan for the subsequent episode.

NAME
ep X
12/XX/09

Overarching theme:

Leads followed:

Activities completed:

Notes/Transcript:

Leads to follow:

Activities to follow:

Figure 13. Template of the record used to summarize and plan for episodes

To illustrate how these records were used, here is a sample of a completed one. It is the template for Amanda's third episode.

Amanda
ep 3
11/03/09

Overarching theme:
There was a bit of free-play at the end of episode 2, but this is the first "real" use of the software. Explore a conception of "most like" as transformational from the pretest in Configure.

Leads followed:
In episode 2, Amanda identified pairs of shapes as most like in three ways: transformational, common attributes, and resemblance/looks like: on the topological congruence of segments and curves: Use Amanda's suggestions that S and / are alike because you can straighten S to get /.

Take one of Amanda's equivalence classes and have her perform the transformation using Configure (e.g., S into / and T into “epsilon”).

Activities completed:
What is it about two shapes that makes them homeomorphic?: T or Y into X, square and triangle, sail and triangle, clover and shape of your choice.

Notes:
Multiple conceptions of likeness: Amanda identified Y and T as more alike than X. She notes that they're alike in the sense that they're all letters, and they're alike in the sense that they all have lines. But in the software environment, Y and T are alike and neither is like X.

Successfully transforms Y and T, sail and triangle, square and triangle, clover and rectangle. Some progress toward identifying what makes a transformation possible. It requires that she pay attention to properties.

In the X, Y, T case, she says she needs one more line to transform the Y into the X.

In the sail, triangle case, she says she needs boxes [helper points] on the triangle in order to be able to bend them.

In the X, Y case, she attempts a transformation of Y into X and ultimately realizes it's not possible: So what would it take for her to be able to recognize that before proceeding, in the planning stage? Does she develop what she believes is an adequate plan? Or does she proceed without first developing a plan or with developing only the initial stages of the plan? DOES she consider whether two shapes are homeomorphic before proceeding? Or, is anything possible/worth trying? What would make it NOT worth trying?

Leads to follow:
What does it mean for two shapes to be homeomorphic in this environment?
Conjecture: ??? Amanda's plan for transforming two homeomorphic shapes...

Activities to follow:
Begin with one shape and have Amanda construct a homeomorphic one (rather than selecting from a set).
Begin with one shape and ask Amanda to draw/select a shape that is NOT homeomorphic.

The “overarching theme” is a brief description of what the episode was essentially about. “Leads followed” describes something Amanda did or said in the prior episode that was further explored in this episode. “Activities completed” describes the tasks that were conducted in this episode. “Notes” is used to record reflections about salient moments and often includes segments of the transcript. “Leads to follow” provides new questions or conjectures that might inform the design of the subsequent episode. Finally, “Activities to follow" gives the activities whose designs are informed by the "leads to follow." These components of the episode record should be seen as reflective of the teaching experiment's "recursive cycle" of "hypothesis formulation, experimental testing, and reconstruction of the hypothesis" (Steffe & Thompson, 2000, p. 296).

The Pretest

I conducted a pretest in the first episode of the teaching experiment to assess participants' conceptions of "most like." These conceptions are suggestive of the criteria participants use to distinguish between plane figures. The design of the pretest is informed by the "triads" of geometric shapes I used in my pilot study (*Figure **14***). These were the same triads that were developed by Lehrer, Jenkins et al. (1998) in collaboration with Clements and Battista whose research in children's conceptions of geometry has significantly influenced the field over the past 15 years (cf. Clements & Battista, 1992).

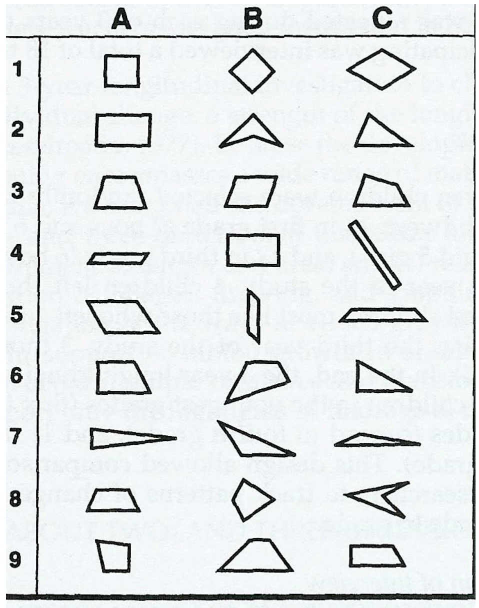


Figure 14. Pilot study “triads”

I redesigned these triads for the larger study, since they had been designed to elicit particular distinguishing characteristics that had more to do with Euclidean geometry than non-metric geometry. Some of the new triads (*Figure **15***) were designed to elicit metric conceptions, such as curvilinearity (row 1) and concavity (row 6); non-metric conceptions, such as closure (row 1) and connectedness (row 5); and both metric and non-metric conceptions, as in row 7, which considers the metric conception of curvilinearity and the non-metric conception of connectedness.

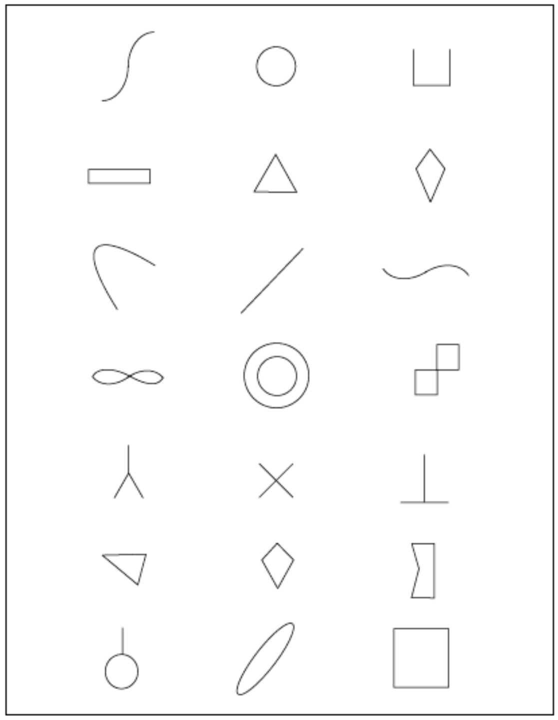


Figure 15. Pretest triads

Knowing full well that it isn't entirely possible to do so, I also wanted to get my conceptions out of the way of kids' ideas. So I cut out the shapes and printed them on square cards. I handed participants five randomly chosen cards and asked them to sort them into groups of "most like." Then, I realized that the square-ness of the cards was suggestive of only four ways to orient the cards, and I wanted to move participants beyond a conception of orientation as a fundamental property of shape. So I developed a second set of "follow-up" cards (*Figure* ***16***) and printed each one on a circular card so that they, too, could be used in five-card sorting tasks. In addition to an interest in moving beyond orientation, I found in the implementation of the first set of shapes that participants were highly attendant to curvature and resemblance to everyday objects and canonical shapes. As the shape of the cards was informed by findings relative to conceptions of orientation, the design of the follow-up shapes themselves was informed by these two additional findings. As a result, the follow-up shapes were drawn to possess both curvilinear and rectilinear attributes. Only the first shape in the last row of the first set of shapes (*Figure **15***) has both curved and straight attributes. Second, the attempt was made to design them so as not to readily connote in children's minds resemblance to everyday objects or canonical shapes. [Suffice it to say for now that the attempt was hardly successful.]

Figure 16. Pretest follow-up shapes

Finally, the pretest served the additional purpose of identifying the most vocal of the six students nominated by their teachers for participation in the study. Also, since the investigation is related to children's non-metric conceptions as elicited and developed using the software, I was interested, in a practical (as opposed to statistical) sense, in the effect of the software on the development of children's geometric conceptions. Thus, the pretest served as a pre-software launching point of the trajectory of participants' experiences in the study. The use of a post-test was considered but deemed inappropriate in gauging the effect of the software on participants' conceptions, since those conceptions are informed by – and most salient in – the software environment. A post-test of equivalent design and implementation to the pre-test would necessarily be implemented outside of the software environment.

The Participants

The first criterion [for case study selection] should be to maximize what we can learn.

(Stake, 1995, p. 4)

I conducted this study at Zapata Elementary where I had also conducted the pilot study. Similarly, as I had done in that study, in order to select participants, I returned to the teachers of participants in the pilot study who were in grades 1 and 2, and asked them to select two students whom they describe as average or above average (by ordinary school standards) and who are most articulate and likely to have interesting things to say.

The age-related criterion is informed by multiple sources. First, Clements (2004) finds that shape concepts begin forming in the preschool years and stabilize as early as age 6. Second, Piaget (Piaget & Inhelder, 1956) finds that the development of children's representational space occurs from 3;6 – 7. Third, some of the most cited studies of children's thinking in geometry (Clements & Battista, 1992; Clements, et al., 1999; Lehrer, Jenkins, et al., 1998) have been conducted with children who range in age from 3;6 – 8. Fourth, because I'm interested in characterizing the developing thinking of young children's qualitative geometric reasoning, and because it is not possible to pinpoint the occurrence and duration of development of these ways of thinking for the purposes of a longitudinal study with each of the participants, I think that a participant cross-section by age is likely to increase my opportunities to experientially abstract the elements required to develop a robust model of thinking in the domain (as long as I am explicit that this approach is not meant to substitute for a longitudinal study). Finally, it seems to me that a good starting point for the development of qualitative geometry from children's conceptions is an investigation with children in the late pre-operational stage and early concrete operational stage of development. Because the identification of qualitative properties is a function of children's actions on mental representations, children earlier in their development do not possess the abilities to engage in ways that are productive of the content of Qualitative Geometry.

A Loosely-Structured Interview Protocol

Each of the episode's analytic tasks was informed by my research questions, which I re-present here to remind the reader.

> 1. Given that early forms of topological, or at least non-metric, geometric reasoning have been identified and discussed in the research literature, can a software environment be developed in ways that support fundamental topological

representations and transformations such that learners' reasoning about topological ideas are made visible and are able to further develop in ways that could credibly be seen as both mathematical and significant?

2. What forms of topological or non-metric geometric ideas are made visible and can be seen to develop as a result of young learners' systemic engagement with a computer environment that makes topological representations and transformations accessible?

Consistent with these questions, these are some of the tasks that loosely structured the interviews. They were implemented by participants on paper and in/using the software.

a. Given a set of shapes, sort into groups those that are alike.
b. Given a class of shapes, identify a/the property that defines the class.
c. Given a class of shapes, construct a new shape that belongs to that class.
d. Given a class of shapes, determine whether this other shape does or does not belong to that class.
e. Given a shape, describe and/or transform it into one that is alike or equivalent.
f. Given two shapes, describe how one can be transformed into the other, if possible. Then complete the transformation.

Situating each of these tasks in relation to the research questions produces the following matrix:

		Loosely-structured Interview Tasks					
		a	b	c	d	e	f
RQ	1			•		•	•
	2	•	•	•	•	•	

Consistent with my research questions, each of these tasks is informed by conceptions of qualitative geometry. Consequently, the design of the software that participants used to complete the tasks simultaneously informs and is informed by the domain. Presentation of that software appears in Chapter Four.

DATA COLLECTION

I collected data from the following sources:

a) pretest
b) video recordings of the research space
c) participant loosely-structured interviews
d) screencasting software to produce a digital video and audio record of participants and their interactions using the software environment
e) jottings taken during and following observational episodes and later expanded into field notes (Van Maanen, 1988)
f) analytical memos (Strauss & Corbin, 1998) in the forms of local reflections on observations and more global reflections on salient and emerging themes

These methods of data collection were approved in my IRB application (IRB# 2007-10-0107; expires 02/18/2011). Situating each of these data sources in relation to the research questions produces the following matrix:

		Data Sources					
		a	b	c	d	e	f
RQ	1	•	•	•	•	•	
	2	•	•	•	•	•	•

As I mentioned above, the pretest was used to get some sense of the effect of participants' interactions in the software environment on their qualitative conceptions of shape. The video recordings and screencasts were used to conduct retrospective analyses of participants' activities on paper and in the software environment. The loosely structured interviews were designed to structure those activities so as to make participants' qualitative conceptions visible and further develop them. Finally, field notes and reflections supported my analyses of the data.

DATA ANALYSIS

As I mentioned in the section detailing my cases, a critical point in the development of children's thinking in the domain of qualitative geometry likely occurs at 4 – 6 years of age. An investigation of development over a time period approximating two years would obviously be too time-intensive for this study, but Siegler & Crowley (1991) offer alternatives: either 1) "provide a high concentration of tasks which are hypothesized to lead to changes in performance," or 2) "present a novel task and observe children's changing understanding over some number of observations" (p. 607). Analytic interactions framed some of the interactions while responsive and intuitive interactions framed others, so the approach I took in my teaching experiment is somewhere between the first and second options. As such, I adopted Siegler and Crowley's *microgenetic method* (1991) because, as they write, theirs is a "method that seems particularly well suited to studying change" (p. 606). In fact, Siegler and Jenkins (1989) is a report of what they refer to as a microgenetic experiment. Thus, and because Steffe and Thompson (1999) also suggest a "microanalysis" of the data, my methodology may be referred to as a *microgenetic teaching experiment.*

Stages of Analysis

The microanalytic methods I used to conduct the data analysis follow from this research design and are informed by Strauss and Corbin (2008; 1998). Consistent with their advice, analysis was ongoing during and following the data collection. One-week breaks between episodes allowed for *light* transcription and "retrospective conceptual analysis" (Steffe & Thompson, 2000, p. 273), and two- and four-week breaks allowed for *moderate* transcription and retrospective conceptual analysis. Light and moderate transcripts were "scanned" for salient themes (as opposed to applying a full "line-by-line" analysis) (Corbin & Strauss, 2008, p. 70).

Following the conclusion of data collection, reflective activities continued as light and moderate transcripts were elaborated to full ones and videos were re-viewed for a second (and an occasional third) round of retrospective analysis. Then, I generated codes (cf. Corbin & Strauss, 2008; Mertens, 2005; Miles & Huberman, 1994). Data analyses in studies of children's geometric conceptions by Lehrer et al. (1998) and Clements et al. (1999) informed the first iteration of coding (e.g., language of morphing, resembles everyday, suggests orientation), which was meant to be descriptive of participants' words and actions. Subsequent rounds of coding were implemented for the primary purpose of managing the data, of sitting down and spending time with it, of getting to know it and making sense of it so that I could craft narratives of the trajectory of each participant's experiences across all episodes from salient moments indicative or at least suggestive of conceptual development.

> *Writing forced me to see the whole theory and highlighted those parts that didn't fit so well…. So, I would go back to the data…. This kind of building and verifying of various aspects of the theory continued throughout the writing process…*
>
> (Strauss & Corbin, 1998)

This quote is the text of a memo written by Paul Alexander, a former student of Strauss and Corbin's. The goal of his analysis, that of developing grounded theory, differs from my own, which was to develop "some type of higher-order synthesis in the form of a descriptive picture" (Lincoln and Guba, 1985, as presented in Mertens, 2005) of participants' experiences. But the essence of the quote is remarkably similar to the way I thought about developing the narratives that I use to address my research questions. At the conclusion of multiple rounds of re-viewing and coding the data, no new information was emerging. So I decided to conclude the "formal" stages of the analysis, and I began to write. The process of crafting each of the narratives was aided by each of the following sources:

1. a clear analytic story as informed by the results of the analysis
2. a sense of what stories to write as informed by the research questions
3. a detailed outline as informed by the development of themes to characterize each of the episodes of the teaching experiment, and
4. the process of developing outlines for preliminary talks about the findings

In the same way that Paul thought about the writing as a tool for developing grounded theory, I saw the writing process as a tool for the development of narratives, each of which depends on the use I would make of the four sources above and the *Vygotksyan* sense (Bigge & Shermis, 1999) I would make as I translated emerging themes into the words that would eventually compose the narrative.

STRENGTHS AND LIMITATIONS OF THE RESEARCH

I believe that this study is worthy of attention. To make the argument, I address the strengths and limitations of the study by applying a set of criteria with which to judge the quality of *qualitative* research. These criteria have been put forth by Lincoln and Guba (1985) and endorsed by a host of others (cf. Anfara, Brown, & Mangione, 2002; Mertens, 2005). According to the authors (1982), they reflect another step in the move away from earlier positivist-sounding "rigor criteria" toward the more constructivist/interactionist-sounding "trustworthiness criteria," so as to provide a better fit with naturalistic research.

Table 1 Criteria for judging "trustworthiness" in qualitative research, as put forth by Lincoln & Guba (1985)

Criterion (parallel criterion in quantitative research)	**Strategies I employed to enhance the criterion**
Credibility (internal validity)	prolonged engagement and persistent observation, member checks
Transferability (external validity)	thick description, purposive sampling, multiple cases
Dependability (reliability)	audit trail, peer examination
Confirmability (objectivity)	reflexive journal, chain of evidence

Credibility is the criterion in qualitative research that parallels internal validity in quantitative research. Internal validity in an experimental situation has to do with the attribution of a change in the dependent variable to the independent variable. Broadly speaking, credibility as the parallel notion in qualitative research gets at the correspondence between the participants' actual perceptions of social constructs, phenomena, or experiences and the researcher's portrayal of participants' perspectives. In the context of this study, credibility may be thought of as the correspondence between "students' mathematics" – that is, the students' mathematical realities – and the "mathematics of students," which are my models and interpretations of students' mathematics (Steffe & Thompson, 2000, p. 268).

I aimed to establish credibility through prolonged and substantial engagement with – and persistent observation of – each of the three participants in the study. Senses of prolonged, substantial, and persistent that would be appropriate for my teaching experiment were informed by reports of teaching experiments in the mathematics education literature (cf. Kuhn, et al., 1995; Lehrer & Chazan, 1998; Schauble, et al., 1995; R. S. Siegler & Crowley, 1991; Steffe & Thompson, 2000), which ranged in duration from three weeks to ten weeks. My teaching experiment lasted approximately eighteen weeks. I met with participants between eight and ten times for episodes lasting between thirty and fifty minutes. The first episodes were conducted in the first week of October, 2009, and the last episodes were conducted by the third week of January, 2010, for a period of data collection lasting just longer than four months. By the time the last episodes were conducted, I felt I had identified salient and recurring themes, language, and patterns in operations suitable for the development of an adequate model of the three participants' thinking in the domain of qualitative geometry. Furthermore, at that point, I felt I had developed a rich – although certainly incomplete – conception of the developing domain of qualitative geometry itself.

A sort of member checking is naturally inherent to the design of my teaching experiment, since it consists essentially of a series of clinical interviews (Ginsburg, 1981), the purpose of which are to develop viable models of the interviewees' thinking. Furthermore, a sort of episodic feedback loop was established in that participants' responses informed and are further investigated in subsequent episodes.

The final strategy I consider with respect to credibility is triangulation. Mertens' (2005) notion of triangulation involves checking information that has been collected from different sources or methods for consistency across those sources of data. Again, the analog of credibility in quantitative research is internal validity, so this sense of transferability is about making the same measurement in different ways. Guba and Lincoln (1989, as cited in Mertens, 2005) do not support that view, since it "contradicts the notion of multiple realities" (p. 255). In the context of this study, Guba and Lincoln's objection is appropriate. The episodes I conducted with participants occurred predominantly within a "microworld" (Papert, 1980) called *Configure* (Steven Greenstein & Remmler, 2009) that operates according to a set of rules not found in other worlds. It's supposed to be that way. So I hold neither hope nor expectation of finding consistency in the information gathered across that and other sources.

Transferability is the criterion in qualitative research that parallels external validity in quantitative research. External validity concerns the degree to which the results may be generalized to other situations. In qualitative research, the transferability criterion is met by providing sufficient details about the research site so that a reader can judge the similarity between that site and another. In the context of a teaching experiment, then, transferability has nothing to do with the site. It is about the generalizability of my models of students' mathematics (and perhaps slightly about the students themselves).

Relative to the models, a model is generalizable if it is "useful in interpreting the mathematical activity of students other than those in the original teaching experiment" (Steffe & Thompson, 2000, p. 300). If in a new teaching experiment, the model is modified to produce a new model that supersedes my own, this further confirms the generalizability of the model, because the new model will have retained features of the old one.

It has been argued (Steffe & Thompson, 2000) that generalizability in the case of teaching experiments is not about the students but about the researchers' models of their mathematics. They don't discount the value of evaluating models by sampling for students in the same age group, but they don't see this as related to the issue of generalizability. Still, I include the students in matters of transferability, because I believe that doing so will support my intention that the findings of this study have implications for teaching. Therefore, a consideration of the students' ages is relevant. So I have established transferability relative to the participants through the purposive sampling method (Patton, 2002) I described above in Chapter Three, and by conducting data analysis throughout and following the data collection of this teaching experiment with an eye toward the observation of conceptual development across “multiple cases.” I did expect to find a group story. The literature predicts it (Darke, 1982; Esty, 1971; Laurendeau & Pinard, 1970; Lovell, 1959; Martin, 1976a, 1976b; Piaget, 1970b; Piaget & Inhelder, 1956), as do other investigations of children’s mathematical thinking (cf. Carpenter, et al., 1996). In addition, because clinical exploratory interviews were adequately “prolonged and engaging,” they yielded all the elements that “thick description” requires. Even Ginsburg (1997) goes so far as to argue that “the clinical interview can provide a kind of ‘thick description’ of the mind (to borrow from Geertz, 1973)” (p. ix). And fortunately, the literature provided me with extraordinary models (Ricardo Nemirovsky & Noble, 1997; Ricardo Nemirovsky, Tierney, & Wright, 1998) of thick description that guided the presentation of my own findings and which should meet or exceed the strongest requirements for transferability.

Dependability is the qualitative parallel to reliability or reproducibility or stability over time. In order to establish dependability in qualitative studies, Lincoln and Guba (1985) recommend overlapping methods and stepwise replication by taking multiple passes or using multiple researchers to do field work. Unfortunately, the scale of the study could not permit those strategies. But Guba and Lincoln (1989, as cited in Mertens, 2005) also suggest conducting a dependability audit, where each of the steps in the research process is documented and detailed. These were the expectations I assumed as I outlined my techniques for participant selection, data collection (including the design and functionality of the software), and data analysis in Chapter Three. Moreover, check-ins with Dr. Walter Stroup to discuss plans for interviews and their outcomes during data collection and emergent findings during data analysis, along with check-ins with colleagues Scott Eberle and Stephanie Peacock to review the mathematics of qualitative geometry, served as an admittedly weak but significant independent audit of the research process.

Still, data analysis has a personal component. The data and I do our analytic jazz dance, and when it's over – maybe the song stops or I unplug the jukebox – themes have emerged and I've made sense. Strauss and Corbin (1998) use "interplay" to name what goes on between the researcher and the data, noting that the interplay "is not entirely objective" and that operating objectively "is not entirely possible" (p. 58). In my mind, this way of talking positions "interplay" closer to objective than the "dance." Given the nature of the study as qualitative / clinical / exploratory and *my* efforts at presenting *their* mathematical reality, I suspect that things could have gone another way. Such is jazz, after all.

Confirmability is the qualitative parallel to objectivity in quantitative research. It concerns freedom from bias, that is, the minimization of the influence of the researcher's judgment on the outcomes. Confirmability means that the connection between the data and the interpretation is logical and that the "chain of evidence" that extends from the source of the data to its synthesis can be confirmed. Toward this end, I followed Lincoln and Guba's (1985) suggestion that the researcher keep a reflexive journal. I recorded in that journal reflections related to the design of the study, the development of the software, and happenings within and across episodes for each participant and across all participants. Entries were recorded almost daily during analysis as it was conducted both during and following data collection. If the purpose of analysis is to bring meaning, structure, and order to data (Anfara, et al., 2002, p. 31), then these reflections should be seen as playing a significant role in conducting the data analysis. The canvas of the blank page provided me with a space to reflect on the interactions between actions of participants, the supports that the software provided, and the development of the domain of qualitative geometry. These reflections, then, supported the crafting of the stories I tell in the findings in Chapter Five.

Chapter Four: Developing the Software

Considering Technology

I realized early in the development of this study that an investigation of young children's qualitative geometric thinking would require some sort of tool that could be used in tandem with a sequence of clinical interviews to make that thinking visible (Lesh, Hoover, Hole, Kelly, & Post, 2000), a tool that would simultaneously open a window into their mathematical thinking and allow them to explore geometric figures in ways not possible with physical objects (Clements, Sarama, Yelland, & Glass, 2008, p. 143), while also enabling me as the researcher to develop a better sense of their thinking and providing me with something tangible to discuss with the participants.

Andrew diSessa (2000) offers that "although you may think you are designing a computer system, you are really designing *mediated activities*. That is, you are designing the material context that supports particular activities" (p. 105). Similarly, Vygotsky's (1978) notion of mediated activity expresses the dialectic relationship between the tool and the sense-making. "The use of artificial means, the transition to mediated activity, fundamentally changes all psychological operations just as *the use of tools limitlessly broadens the range of activitie*s [emphasis added] within which the new psychological functions may operate" (p. 55). This new environment, then, could be a tool in the Vygotskyian sense in that it is an instrument that makes topological activities possible. Moreover, a tool that changes not only how mathematics is learned, but *what* mathematics can be learned could provide de facto challenges to 1) traditional forms of geometry instruction that are having little or no effect on the development of children's geometric concepts (Lehrer & Chazan, 1998), and 2) to van Hiele theory (1986), which I explicitly address in the findings section of Chapter Five.

Technology Takes Shape(s)

Domain-general, technology-centered approaches to teaching and learning tend to begin with a technology like a graphing calculator or spreadsheet software followed by considerations of "What might be taught / assessed / accomplished with this technology?" In contrast, a domain-centered approach to technology use in classrooms offers a move toward solutions to pedagogical problems by starting with the content and engaging with questions like, "How might we design a learning environment in which students participate in the kinds of experiences that support the coming to know of X?" The answer may or may not involve a form of computing technology. Accordingly, tools make sense in relation to particular kinds of activity. Thus, useful advice about which spreadsheet software is best or which graphing calculator is most appropriate must consider the purpose or the context in which these might be used.

In the following section, I document the iterative cycle of *"progressive refinement"* (Collins et al., 2004, p. 18) by which the software used in my pilot and dissertation studies came to be. This is just as much a tale about design-informed development of software as it is about a teacher-researcher's tumultuous, year-long relationship with educational technology and the variable and potential roles it might play in the conduct of the study.

The development of a tool to make children's qualitative ways of thinking visible was deemed a critical and essential way to proceed in this investigation. Without it, I would certainly have nothing to talk about. But it wasn't the simple matter I was prepared for. Development took far more than a year before I got to a version I could use in the pilot study. Then, it took several months beyond the conclusion of the pilot study to incorporate its findings into modifications of the software for use in the larger study. What follows is an account of the design-informing process.

The ongoing development of a web 2.0-based application called Distributed Biography (Olmanson, Greenstein, Smith, & Brewer, 2007) gave me the design-informing research experience I have drawn on to develop the environment. In collaboration with the other members of the Digital Spaces Working Group, we began with shared notions of *what it means to know* some one, some thing, some where, or some time, and proceeded by asking ourselves, "What might such a biography look like?", "What sort of technological tool might support users in its development?", and "How can relevant theories and epistemologies inform its design?" Informing the design of the software by exploring answers to questions like these resonated with the domain-centered view I take on knowing and learning. A description of the evolution of the project from initial design to alpha development is provided in Olmanson, Greenstein, Smith & Brewer, (2007, October), but the experiences I drew on most heavily to prepare me for the development of my own software occurred early on. In the fall of 2006, we created a white paper and began meeting biweekly to discuss overarching issues of the application. In the spring of 2007, we worked to convert the theory and beliefs into some guiding wireframes, and throughout the summer of 2007, we continued the iterative design and initial development process. The first iteration of my software was uninformed by these experiences, but by the second iteration I had met with a colleague who would be programming the software. That partnership put me in the role of primary designer and it allowed to me engage with her in discussions about my ideas for the design of the software and her knowledge of what's possible.

Aligned with the essentially Piagetian framework upon which this study is built – especially related to 1) his framework of genetic epistemology (Piaget, 1970a), 2) his model of conceptual development, particularly as it relates to the development of cognitive structures (Bringuier, 1980; Gardner, Kornhaber, & Wake, 1996b; Piaget, 1970b), and 3) his investigation of the development of children's representational thinking (Piaget & Inhelder, 1956) – I realized that I would require a software environment that could support the construction of (mental) representations of shape from perception. To summarize an earlier discussion (which begins on page 15), this distinction between representations and perceptions is significant, because conceptual thought proceeds from representational thought, and representational thought proceeds from perception. I'm interested in children's qualitative conceptions of shape, and I can only attend to those conceptions if I provide them with the means of constructing mental representations, the elements of which must be reflectively abstracted via actions on shape. In addition, conscious attention to those components of shape simultaneously supports my goal of getting kids to attend to fundamental properties.[23]

So the theoretical frame informs the design of the software to the extent that it should allow users to construct mental representations through their actions on shapes. But what shapes? And what actions? The review of the literature, particularly related to the models of children's thinking in the domain of informal topology as provided by Piaget (Piaget & Inhelder, 1956) and "the replicates" (Esty, 1971; Laurendeau & Pinard, 1970; Lovell, 1959; Martin, 1976a, 1976b) were useful in engaging these questions so as to further inform the design of the software.

Iteration One: SimTop

The first iteration in the development of a microworld (Papert, 1980) that I would use for the purposes of this study was a collection of "topological worlds" built in NetLogo (Wilensky, 1999) called *SimTop* (S. Greenstein, 2007) (to allude to a Simulated Topology). The idea was that there would be multiple worlds and that the shapes of these worlds would be topologically distinct. That is, they would vary in terms of their topological properties. For example, one of these worlds was equivalent to a circle, another was equivalent to a segment, and another was equivalent to a figure eight. This might be easier for users familiar with NetLogo to imagine if they consider an analog in that the built-in options for the shape of the "world" are a "box," a horizontal or vertical cylinder, and a torus. Each user would control a single spaceship, and the ultimate goal

[23] That's van Hiele Level 2: Abstraction (1986) for anyone who's keeping track.

would be to describe the shape of that space. Because these worlds are not globally visible, a snapshot of the environment cannot reveal them, but *Figure* ***17*** illustrates the operations that are available with which to control a spaceship.

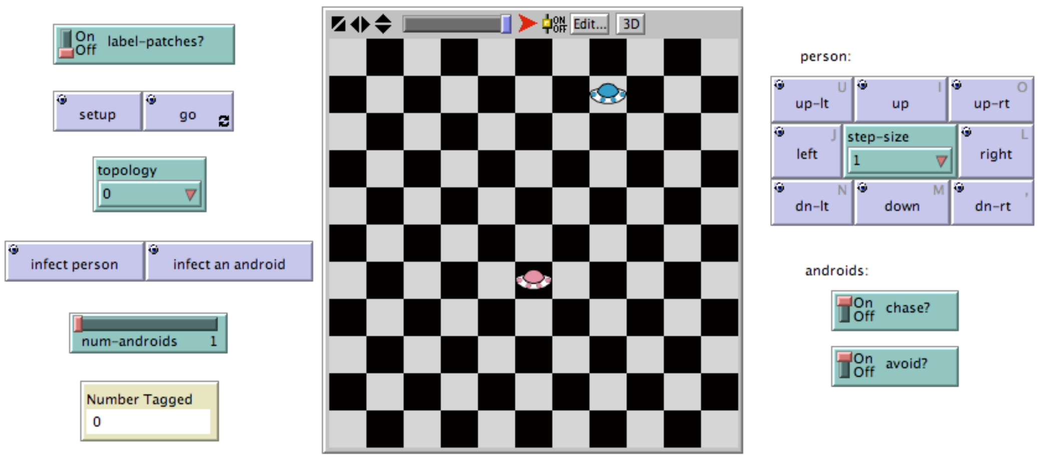


Figure 17. Screenshot of *SimTop*

The chessboard environment only appears to be a segment of the plane. The drop-down menu labeled "topology" allows the user to select from among the available topological worlds. In the world that is shaped like a ring, for example, only the spaces along that ring are accessible. At some points along its boundary, it is not possible to move to one of the adjacent spaces on the board. In worlds like the torus, at some points an attempt to move to an adjacent space results in landing at some other unexpected space across the board.

The blue spaceship is the turtle that the user controls. The pink spaceship is an android that, if turned on, will chase the blue spaceship around the (topological) space.

Similar androids (with similar agendas) appear in the NetLogo model *Disease* (Wilensky & Stroup, 2002). Hence, the leftover "infect person" and "infect an android" buttons in my model. My thinking was that the android would provide some incentive for the user to further explore the space.

I played around with this idea as I was developing my proposal and pilot study, and then I abandoned it. Translating increasingly "complex" shapes into code was too laborious, and might suggest that exploration of the world would draw more on cognitive resources associated with spatial visualization skills than those associated with the forms of reasoning I was interested in attending to (not that they should be entirely distinct). Furthermore, I found that the worlds weren't sufficiently interesting to explore, nor did I sense any "internal necessity" (Steffe & Thompson, 2000, p. 282) to explore them thoroughly. Children having a similar experience might only attend to the most salient features at the expense of a more thorough and adequate search of the worlds, just as in Piaget's haptic experiments (Piaget & Inhelder, 1956) where errors arose because children did not adequately explore the objects they were given.

Iteration Two: QualiGeo

The second iteration of the software, called *QualiGeo* (Steven Greenstein & Remmler, 2008a) (*Figure 18*), was also built in NetLogo Like *The Geometric superSupposer* (Schwartz & Yerushalmy, 1992b), the environment had two "facilities" (p. 178), one for construction (the "build space") and one for operation (the "play space"). New shapes could be transformed in the build space by dragging the blue "hot spots" of a built-in circle, triangle, quadrilateral, or pentagon. Then, the user would drag the shape into the play space and render them invisible before "scanning."

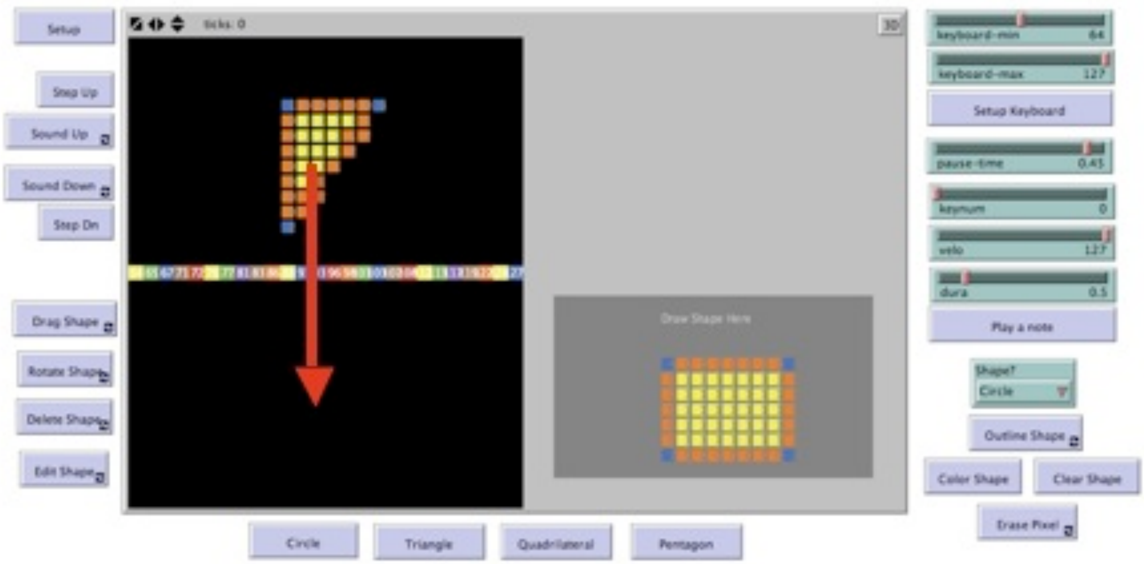


Figure 18. Screenshot of *QualiGeo* with an arrow reflecting the movement of the shape downward through the keyboard.

The utility of the build space makes possible an analog to the sort of "mental animation" of shape that Lehrer and colleagues (1998) find to be the "one exception" to the trend that geometry instruction throughout the elementary grades does little to change children's conceptions of shape. They identified among children "an increasing propensity to reason about similarity between and among figures by mentally animating the action of pulling or pushing on a vertex or side of a two-dimensional form" (p. 145). Not only does such a tool fit with this way that children think about shapes, Clements and colleagues go so far as to suggest that the development of geometric thinking may be achieved through engagement with images that are flexible, dynamic, and manipulable, and through the development of abilities to act on those images (Clements & Sarama, 2004; Clements, et al., 1999).

One goal of supporting kids' identification of qualitative properties of shape as one conceptual component in the development of a notion of figure via engagement with a dynamic geometry environment that focuses the user on underlying relationships rather than on the particulars of a specific drawing is to work toward a resolution of conventional claims about kids' analytical abilities: van Hiele (1986) predicts that the development of geometric conceptions is hierarchical and ordered, and that learners do not progress to the level in which they are able to coordinate and order properties, so conventional classifications of shapes cannot emerge until much later.[24] Another goal is to provide manipulable images so that children are more tolerant to the orientation effect (Clements, 2004). That is, to provide them with experiences that might "break" them of the inductive inference that orientation is a fundamental property (which would imply that squares and regular diamonds *are not* equivalent). One fortunate but unintended consequence that is common among participants in my own pilot study and in a study by Lehrer, Jacobson, et al. (1998) is that users come to see the environment as a playground for "morphing": "some children regarded figures as malleable objects that could be pushed or pulled to transform them into other figures" (p. 142). For example, one child stated that a chevron and a triangle are alike because "if you pull the bottom [of the

[24] Implications of this research for van Hiele theory are considered in Appendix E.

chevron] down, you make it into [the triangle]" (p. 142). Coincidentally, the transformation that turns a chevron into a triangle is topological: it demonstrates that a chevron and a triangle are topologically equivalent.

Once in the play space, when a shape is dragged across the keyboard (*Figure 18*), the keys that are "touched" by the shape are illuminated and tones are played that correspond to the keys' location on the keyboard. [A better illustration of this process is featured later in the presentation of the next iteration of the software, called *SoundTrack*.] That is, this kind of interaction provides simultaneous visual and auditory feedback. For instance, only the edges of an unfilled triangle will light keys on the keyboard and produce their corresponding tones, whereas the filled-in triangle will light the keyboard at all "points" between and including the edges and play the full set of corresponding tones. But the auditory and visual feedback induced by a variety of non-congruent triangles will have common features that could come to signify that topological equivalence. It is in this respect that *QualiGeo* provides a playground that focuses learners on metric and non-metric properties of shape.

The significance of this feedback mechanism is related to my earlier mention of Piaget's notion of reflective abstraction (Piaget & Inhelder, 1956) of properties of shape as something other than the ordinary abstraction – or generalization – of its properties. Reflected abstraction involves the (possible) actions taken to coordinate those properties into a single whole. *QualiGeo* was designed to support users' coordination of abstracted properties into a single whole. The role of invisible shapes corresponds to the role of hidden shapes in Piaget's haptic perception task (Piaget & Inhelder, 1956), but *QualiGeo* also allows invisible shapes to be used for engagement with one or more particular qualitative properties. This feature is meant to oppose the “What shape am I?” task used by Burger and Shaughnessy (1986) in which *the* properties are given and *the* shape is *the* answer.

The way that users employ the dragging capability together with the keyboard to “scan” figures is meant to resolve Piaget’s realization and Page’s later claim (1959) that young children only passively explored shapes in Piaget’s haptic perception tasks (Piaget & Inhelder, 1956); that is, shapes were inadequately searched for all of their parts. Searching via *QualiGeo* is more comprehensive so that more features of the figure are explored than those that are most salient (such as holes and corners), familiar, or are encountered by accident. The method also encourages greater “perceptual activity” (Piaget & Inhelder, 1956, p. 10), which refers to the coordination of centrations (such as touching one part of the object) and decentrations (moving on to another part of the object) in order to construct mental representations of objects from percepts. Furthermore, the search procedure yields immediate and coordinated visual and auditory feedback. In contrast, feedback provided to participants in Piaget’s haptic perception tasks was limited to the acknowledgment of direct contact with the object.

This iteration marked a significant improvement over the previous iteration in that it was designed to allow for active engagement with representations. As I mentioned earlier, Piaget’s stance is that in order for a child to build a representation of a particular shape, properties of that shape must be *reflectively* abstracted and coordinated into a coherent whole. From a constructivist perspective, not only does active engagement enable greater inference about learners' representational space, but because action (e.g., a reversible operation, or action which can return to its starting point) is the vehicle for developmental progress (Flavell, 1963; Piaget, 1970b; Piaget & Inhelder, 1956), it is an essential aspect of a study of the development of children's thinking.

Eventually, this "agent-based" model was retired. I decided it was inadequate and inappropriate, because the turtles moved in discrete – as opposed to continuous – steps. As a result, a shape composed of turtles could only be dragged across the keyboard in discrete steps, thereby producing discrete tones[25]. Thus, the feedback produced by the shapes was not true to the shapes themselves. Considering the fundamentality of continuity to topology, the microworld I would develop had to afford users opportunities for "reasoning by continuity" (Sinclair & Yurita, 2005, p. 5). This feature is critical to topological investigations, since continuity is so fundamental to topology. Reasoning by continuity is a concept identified by Sinclair and Yurita (2005) in a study that involved the use of a dynamic geometry environment called *The Geometer's Sketchpad* (Jackiw, 2001):

> Reasoning by continuity is a form of geometric reasoning that enacts Poncelet's Principle of Continuity: the properties and relations of a geometrical system or figure, be they metric or descriptive, remain valid in all of the successive stages of transformation during a motion that preserves the definition properties of that figure or system. The Poncelet Principle finds, perhaps for the first time, a physical embodiment through the use of *Sketchpad*, where dragging qualifies as a continuous motion that can preserve the system's initial properties. (p. 5)

If my software featured this *dynamic* quality, it could support the elicitation of users' non-metric conceptions by allowing the user to identify properties that remain

[25] Analogously, the interval $[0, 4]$ would be reported as the sequence $\{0, 1, 2, 3, 4\}$.

invariant under transformation and, as a result, to submit these *defining properties* as fundamental to qualitative geometry. “The principle of continuity guarantees that any member of the set of parallelograms may be obtained from another via dragging” (p. 5), and under the defining conditions of qualitative geometry, it could also guarantee, for example, that any closed figure may be obtained from another. It is worth noting that any transformation that changes one “breed” of parallelogram into another is not Euclidean, because magnitudes would not be preserved. For instance, a transformation that turns a rectangle into a non-square rhombus increases the lengths of a pair of the rectangle’s opposite sides and decreases the measures of a pair of its opposite angles. The transformation is topological, though, since both the preimage and the image of the transformation are simple closed curves. [Recall the rubber sheet if that helps.]

My next steps were informed by Judah Schwartz’s essay, “The Right Size Byte” (1995), in which he offers his own thinking about the design of educational software. These guidelines provided me with a check and refinement of my thinking about the design of the software and how it might help me accomplish my goal of making children’s geometric thinking visible. I had already decided that whatever form my software would take, it would use a “new approach to new content” (p. 172). Guided by a claim by Schwartz (1999) that important mathematical ideas can be introduced early on in the mathematical education of all students if the introduction is done in the context of interesting and powerful exploratory environments, the new approach would be via a *domain-specific exploratory environment* (Schwartz, 2007), like “*Geometric Supposer* [which] allows users to make Euclidean geometric constructions and explore the effects of modifying measures and constraints" (p. 163). Because the new content would be qualitative geometry, it would differ from *Geometric Supposer* in terms of the geometric constructions that would be possible. And maybe the approach is old to the far-too-few who know of it, but for everyone else (which is almost everyone), it is new. [For contrast, even the newest Computer-Assisted Instruction (CAI) environments can be seen to be using old approaches to old content.]

Next, I considered how much "instructional responsibility" the software should have. Schwartz delineates the opposite ends of the spectrum are:

> a) the software runs the conversation with the user, making inferences where necessary about the intentions of the user, and
>
> b) the software simply responds to the user, displaying the consequences of the user's actions (p. 174)

My sense is that the software would fit with (b), because I wanted an exploratory environment, not an instructional one. The design of my teaching experiment was informed by an interest in exploring children's developing thinking, so I expected learning to occur. But I had neither pre-specified objectives in mind nor any prepared trajectory to get them to objectives.

Then, I considered what would be the "intellectual scope" (p. 175) of the program. Young children's informal topological conceptions as I've identified in the literature parallel the "rubber sheet geometry" way of thinking about topology. Therefore, the ways that plane shapes could be manipulated so as to produce topological equivalence classes defined the boundary of the intellectual scope. These axioms could correspond to primitive commands like stretching and bending, but not breaking, since these operations on a particular shape maintain its fundamental topological properties.

The next step was to consider the pedagogical style that would be embedded in the software. This step was not of primary concern, since I wasn't yet interested in instructional implications. But because the environment would be exploratory, it would also support a teacher/student guided/inquiry style of pedagogy. So that decision was a consequence of earlier decisions. Finally, I considered "the problem of simplicity and complexity" (p. 177) by responding to the following questions:

> 1. What actions do you want users to be able to carry out in this software environment?
>
> 2. What elementary tools do you wish to put at the disposal of the users to make it possible to carry out these actions?

Again, the "rubber sheet" conception informed the response to question (1). "Simplicity" would come naturally, since the allowable actions would correspond to the axioms of rubber sheet topology. Furthermore, these axioms would simultaneously clarify qualitative geometry and open up a space for rule-based play (Ares, Stroup, & Schademan, 2009; Vygotsky, 1978) in a qualitatively geometric world, wherein "thought is separated from objects and action arises from ideas rather than things" (p. 97). It is Vygotsky's position that "the importance of play on a child's development is enormous" (p. 96):

> Play continually creates demands on the child to act against immediate impulse. At every step the child is faced with a conflict between the rules of the game and what he would do if he could suddenly act spontaneously…. Subordination to a rule and renunciation of action on immediate impulse are the means to maximum pleasure. Thus, the essential attribute of play is a rule that has become desire (p. 99). [Engaging

> that desire, then,] … creates a zone of proximal development of the child. In play a child always behaves beyond his average age, above his daily behavior; in play it is as though he were a head taller than himself. As in the focus of a magnifying glass, play contains all developmental tendencies in a condensed form and is itself a major source of development. (p. 102)

In essence, play in a qualitatively geometric microworld constrained by the axioms of “rubber sheet topology” could foster children’s qualitative ways of thinking about shape. In other words, as Sarama and Clements (2005) have it, “high-quality mathematics… can emerge from children’s play, their curiosity, and their natural ability to think” (p. 11) .

To respond to question (2), I considered the age range of the participants and then assumed that these kids would have no background experience using computers. From a usability perspective, the software had to be child-friendly – if not child-centric (Papert, 1980).

Iteration Three: SoundTrack

The next iteration of the software following *QualiGeo* was *SoundTrack* (Steven Greenstein & Remmler, 2008b) (*Figure 19*). This is the software that I used in my pilot study.

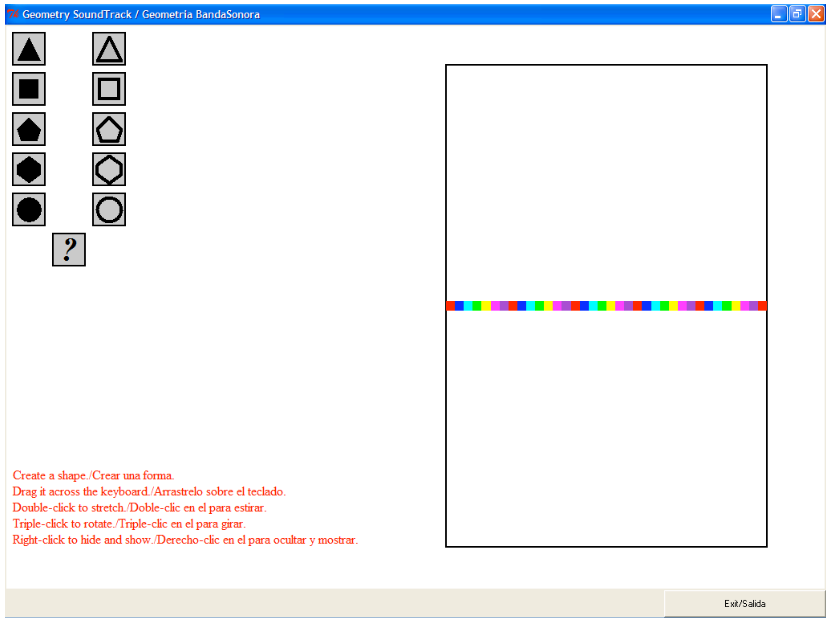


Figure 19. Screenshot of *SoundTrack*

SoundTrack essentially does what *QualiGeo* does, only better. It is a dynamic geometry environment featuring a build space that allows for active engagement with shapes, and a play space that features a keyboard (*Figure **20***) that provides the user with visual and auditory feedback to support the development of mental representations of shape. Written in the open source programming language called Python (Python Software Foundation, 1990-2009), *SoundTrack* solves *QualiGeo's* problem of discrete motion by allowing for the (apparently) continuous dragging of the components of shape.

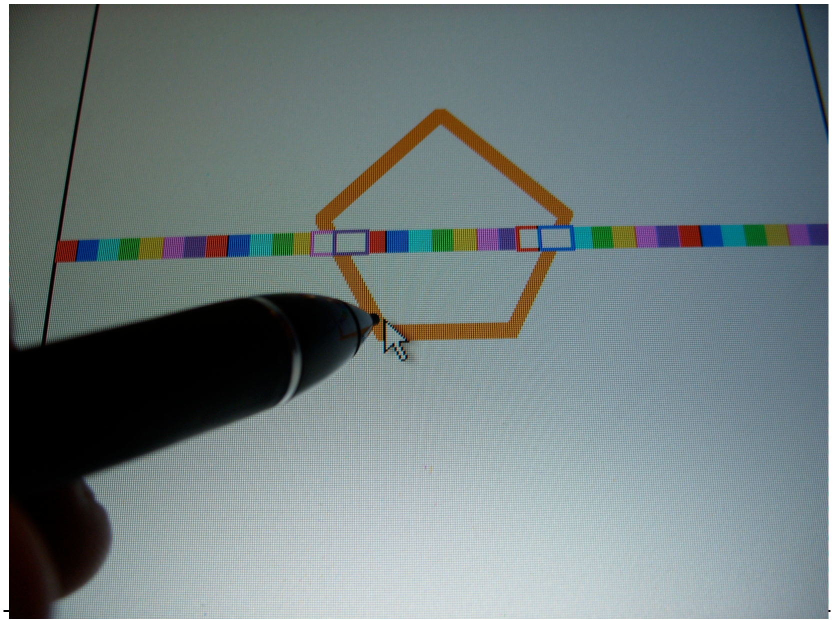


Figure 20. A shape dragged across the keyboard illuminates the keys that it touches and plays the tones that correspond to each key's location on the keyboard.

There are two sets of built-in shapes (*Figure **21***), one containing "solid" shapes and the

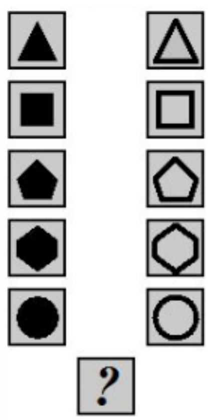


Figure 21. Built-in shapes available to users

other containing "empty" shapes. One reason that these sets were built in was to parallel the objects used in Piaget's haptic perception experiments that were designed to get at the topological distinction between shapes with differing numbers of holes. Each set contains a regular 3-, 4-, 5-, or 6-sided polygon, and a circle. When the user double-clicks on a circle, two hot spots appear at the end of a diameter. These hot spots may be dragged to alter its eccentricity and produce a variety of ellipses. When the user double-clicks on a polygon, hot spots appear at each vertex. These hot spots can be dragged to alter the shape by making it irregular or, by dragging one vertex on top of another, by reducing the number of sides. Although this capacity did render the equivalence of the triangle, square, and pentagon to each other and to the polygon (because a polygon with 6 sides could be transformed into one with 3, 4, or 5 sides), these four shapes were built in for a second reason of matching the likely geometric experiences of the children who would be using the software. An additional button reveals a hidden "mystery" shape. When a user presses the mystery shape button, a question mark appears over a hidden shape. This hidden shape is a random selection of one of the ten default shapes. Other features include rotation by triple-clicking – to counter the "orientation effect" (Clements, 2004) – and hiding a shape by right-clicking – so that I could design shapes and hide them for study participants to explore (not by visual perception of the shape itself, but by visual and auditory perception of feedback provided by the keyboard).

Pilot Study Findings Inform the Next Iteration

SoundTrack was implemented in the pilot study on a tablet PC. The participants jumped right in and were even willing to struggle through the variety of functions implemented through single clicks, double clicks, and right clicks. The hot spots were sufficiently large and the dragging was quick and continuous. In its capacity as a dynamic geometry environment, this was all good news. However, in terms of its utility at eliciting children's qualitative conceptions of shape, there were too many significant defects. In tandem with the given tasks, the build space proved to be a useful playground for participants to develop and elaborate on their responses in my replication of Lehrer and colleagues' "two most like among the triads" (Lehrer, Jenkins, et al., 1998) activity. But those triads (*Figure **22***) were developed to identify the "Euclidean" distinctions that

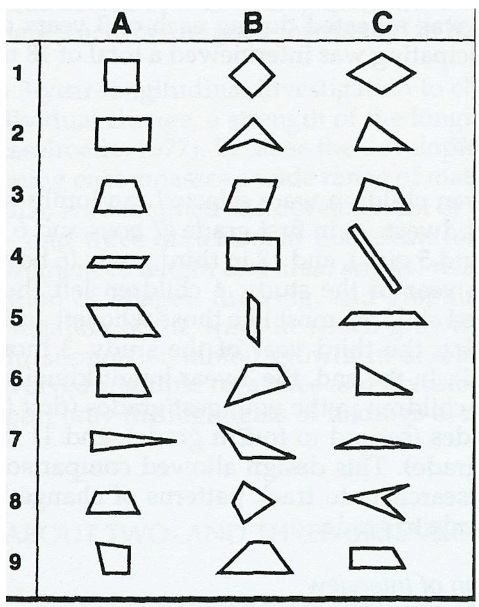


Figure 22. Triads developed by Lehrer et al. (1998)

children make. This is evidenced by the fact that all the shapes are polygons. If they had been developed to also get at children's topological distinctions, we would expect to also see "curvilinear" shapes in addition to ones formed solely from segments. Similarly, although *SoundTrack*'s polygons could be transformed into other polygons, the circle/ellipse could not be transformed into any one of the "rectilinear," topologically equivalent polygons, or vice versa. This lack of "topological" consideration in the design of the environment and replication of the triad activity was an unfortunate oversight, because it could result in children making a distinction across topologically equivalent shapes.

Furthermore, *SoundTrack*'s set of "solid" figures proved to be unuseful. Beyond the fact that the solid figures were different from empty ones because they illuminate additional lights on the keyboard, there really wasn't anything else to talk about. As a result, my thinking following the pilot study was that *the software really needs the capacity to construct new shapes, perhaps by adding segments and arcs and/or a capacity to transform segments to arcs and arcs to segments.*

Even more critical than the weakness in the designs of shapes, participants had trouble interpreting the auditory information provided by the keyboard. Thus, because shapes were built in, there were no opportunities to support the construction of a mental representation from perception. As a result, many of the interview questions could not be used. Moreover, it seemed that the kids did not have the language to describe what they were hearing. The juxtaposition of sound and shape was opaque, ambiguous, or, as with perceived volume differences, misleading. For example, many participants used "loud" to describe increases both in pitch (i.e., tones played further to the right on the keyboard) and in volume (i.e., more keys played simultaneously). Consequently, because the auditory feature didn't work, participants spent most of their time in the play space. On its own, that space proved so unconstrained that neither my participants nor I knew what to do when we got there. As a result, all I found is that it did not support a systematic engagement with shape.

Iteration Four: ShapeShifter

To prepare for the development of the next iteration of the software, I considered whether or not to impose a metaphor on the environment. Schwartz (1995) offers his thoughts on the matter:

> I believe it is important for the software designer to design software in such a way as to make the intellectual content of the software appealing and engaging. If it is necessary to have a "cover story" that drives users through the software by a series of external motivating "gimmicks," then, in my view, the design is not yet as good as it can be. (p. 177)

I considered a microworld featuring a "topologically sensitive troll." In one scenario, the user would carry shapes along a bridge guarded by a troll, and the troll would decide, based on a set of "topological criteria," whether the user could cross the bridge or not. The user would then decide, based on which figures were accepted or rejected, what criteria the troll was using. For instance, if the troll had accepted shapes like the topologically equivalent letters C, U, and I – each of which is a simple, open curve, and rejected the topologically equivalent T and Y – each of which is a *non*-simple open curve, the relevant properties would describe the shapes' connectedness. In another scenario, the user would play the role of a troll and decide the criteria the troll would use to determine which shapes would be allowed or rejected.

I went back and forth on this issue of metaphor. Going "forth," as Matthew Berland (Personal Communication, January, 29, 2009) suggested, a metaphor is useful "if it gets me something. Kids would rather hear the troll ask whether two shapes are alike than me." Going "back," I realized that the focus in that case would be on the troll's thinking instead of the user's, and I worried that the software might not support transfer when the trolls goes away. Moreover, it seemed unnecessary to import a troll given the pilot study participants' sustained interaction with the software. I didn't want to persuade them to participate via an attribute of the software that had nothing to do with the eliciting of their conceptions. Moreover, I have seen far too many pieces of educational software in the domain of mathematics where the goals of the game are the sugar that help the mathematical medicine go down. "Simplify the expression and save the Universe!" It's an approach that could be called "digital behaviorism" – a Flash-y dress on an old approach to creating a teaching machine (Skinner, 1968).

So I abandoned the idea of a troll before it was seriously considered, not only because the development of the software would be much too time- and cost-prohibitive for a dissertation study, but also more importantly, because the shapes would be given, the activities would essentially be categorization tasks that did not allow for opportunities for transformations of shape to identify fundamental topological properties.

Following the decision not to impose a metaphor on the environment, I considered a gaming environment. The software environment (*Figure **23***) would consist of a hybrid "build space" / "play space" that works a bit like Tetris. The shapes that appear are topologically equivalent to the shapes in one of several classes of shapes, each in a different bin surrounding the space. When a shape appears, the user would transform the shape and then drag the shape into the bin that contains the shapes to which it is equivalent. As more shapes are sorted, the duration with which shapes appear decreases. When the time expires, the shape disappears and a new one appears. Ideally, the time constraint would persuade users to act quickly and thereby allow me to witness qualitative differences between (preoperational / perceptual) goal-absent and (operational / representational) goal-oriented transformations of shape. Users could also design their own “levels” by designing the shapes that get dropped and/or the classes of shapes that appear in the bins.

The development of *ShapeShifter* proceeded as far as the construction of shapes and their transformation. The gaming aspect was abandoned because, again, its development would have been too time- and cost-prohibitive. But I was also concerned that the external motivations of a gaming environment might compromise the intrinsic motivation to explore.

The next iteration of the software is essentially *ShapeShifter* minus the gaming aspect.

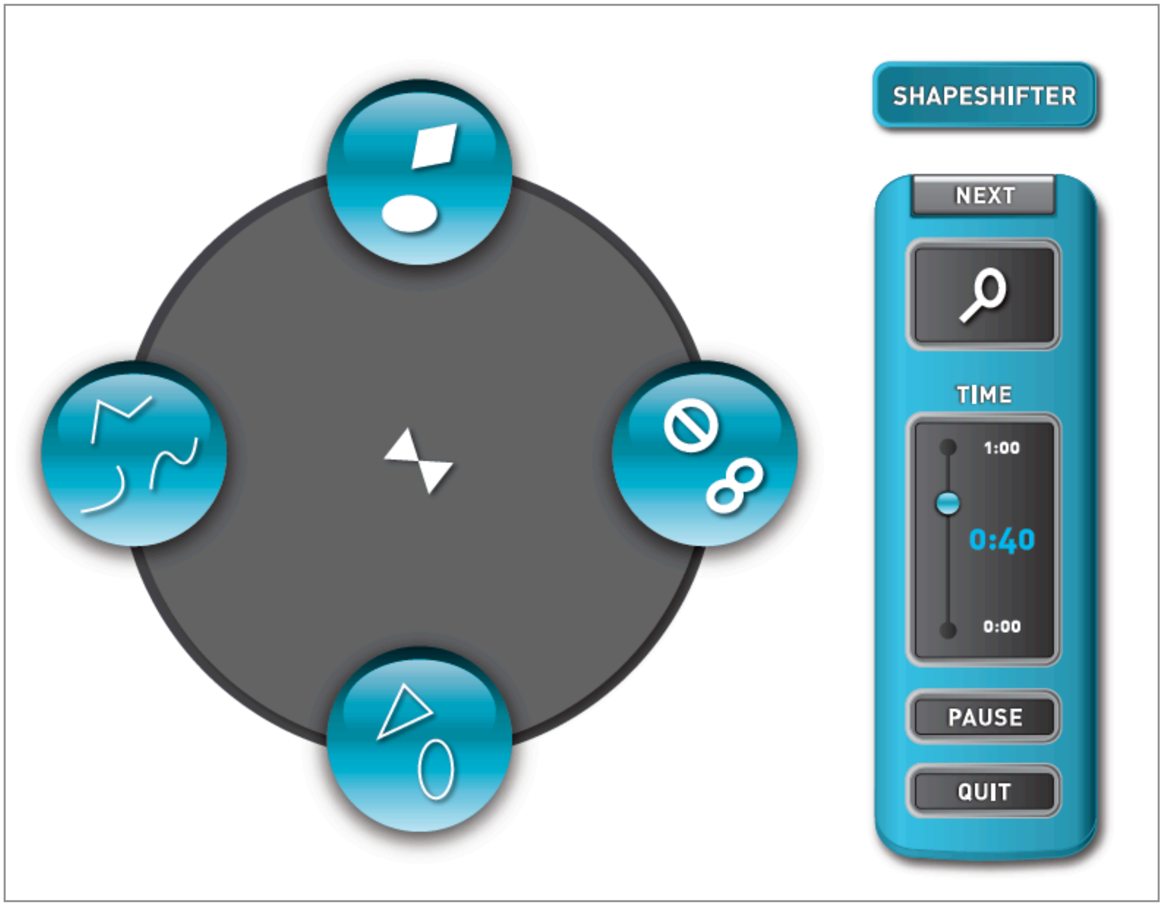


Figure 23. Screenshot of *ShapeShifter*

Iteration Five: Configure

Configure (*Figure* ***24***) (Steven Greenstein & Remmler, 2009) is the latest version of the software and the version that was implemented in the dissertation study. Like *SoundTrack,* it is a dynamic geometry environment featuring opportunities for reasoning by continuity and consisting of a hybrid "build space" / "play space."

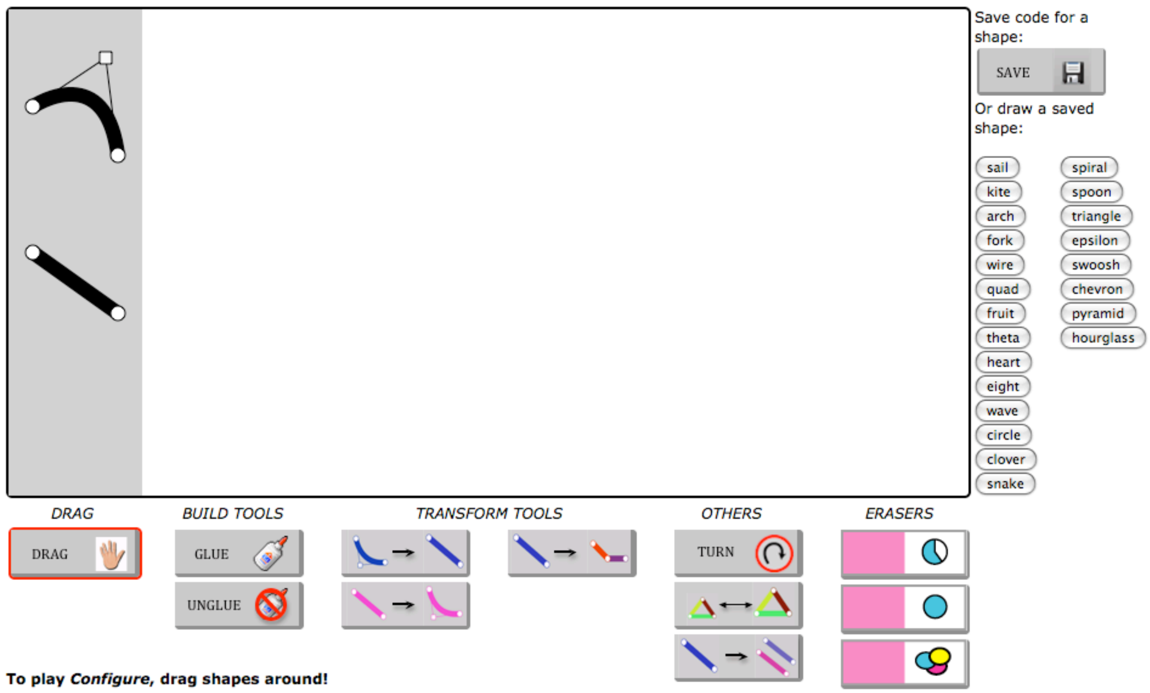
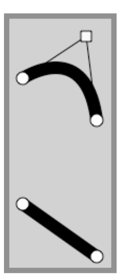


Figure 24. Screenshot of *Configure*

Fundamental Shapes



The *arc*[26] and the *segment* are Configure's only two fundamental, built-in shapes. While the environment is meant to be a "topological space," and the segment and the arc *are* topologically equivalent, elementary-age students have typically spent most of their "geometric time" with polygons and circles. Thus, the segment is provided to meet the students where they are, and the arc, potentially, gives them somewhere to go. That is, the environment is designed to support

[26] The term "arc" is used most often to refer to a portion of the circumference of a circle. I use it in a more general sense to refer to any curve joining two points.

the possible realization of their (topological) equivalence. This equivalence is instantiated in the environment with the operations that "make a segment into a curve" and "make a curve into a segment" (described below).

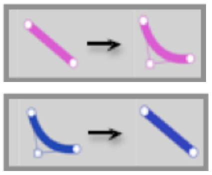
The arc consists of two endpoints and a "helper point." The endpoints may be dragged to change the length of the arc; the helper point may be dragged to alter the curvature of the arc. The segment consists of two endpoints, each of which may be dragged to either change the length of the segment or change its orientation.

The Tools

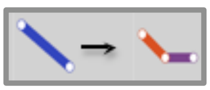
Some of *Configure's* tools are given intuitive icons and are otherwise unlabeled. In their descriptions below, quotes surround the ways I talked to kids about what the tools do.

Dragging is the default operation. When in "Drag" mode, a user can click on a hot spot and drag. For example, dragging an endpoint of a segment changes its length, dragging a corner varies the degree of the angle that contains it, and dragging a helper point alters the curvature of an arc.

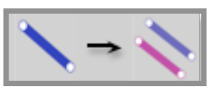
In my interactions with participants, the "glue" and "unglue" tools were used only to build shapes. Since topology is the study of the properties of geometric objects that are preserved under stretching and twisting but not under cutting and gluing, these tools were not used in the transformations of shapes.



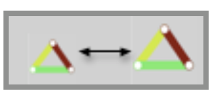
These tools were used to transform shapes. They will “make a segment a curve” or “make a curve a segment.” As I mentioned above, an arc could be transformed into a segment by altering its curvature, but for kids who don’t think about them as equivalent, these tools are provided to support the transformation in a conceptually prior way. That is, these are tools that children may eventually see no reason to use.



This “transform tool” will “add a corner” to a segment or a hot spot to an arc.

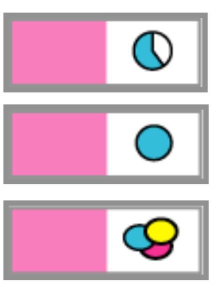


This tool will “make a copy” of a shape.



This tool will “change the size” of a figure by either enlarging or shrinking it.

This tool will “turn” or “rotate” a shape.



These are erasers. From top to bottom, these tools “erase part of a shape,” “erase a whole shape,” and “erase all shapes” on the screen.

The “save” tool was not made available to users. I used it to store a collection of shapes that I wanted to use during the episodes so that I wouldn’t have to take the time to construct them *in situ*. In order to save a shape, the shape is first constructed. Then, when the save button is pressed and the shape is clicked, code for that shape appears in a box further down the screen. That code is then inserted into the software's html code. It's an intricate process, because it's easier to program this way and, because I'm the only one who uses it, the save process need not be user-friendly. *Figure 25* is a screenshot of the shapes that were saved when the software was implemented.

Figure 25. Saved shapes that were used in the study

These shapes were chosen to provide opportunities for kids to identify the topological properties that were planned for the study. They were drawn first and named later. Some shapes were given obscure names in the hopes that the participants would pay more attention to their attributes than to their resemblance to everyday objects.

Improving Configure

Qualitative Geometry is topology as enacted in *Configure*. Whereas topological properties are those that are invariant under transformations, the properties of the non-metric geometry that is embodied in *Configure* are those that are invariant under whatever transformations are possible as determined by the rules, or “axioms,” of the environment. For example, Amanda’s transformation of a circle [G2, *Figure **15***] into the “lollipop” [G1, *Figure **15***] as illustrated in *Figure **26*** is possible in *Configure*, although it’s non-two-dimensional quality – dragging one object over or under another – means that it shouldn’t be. Ideally, the software should not allow these three-dimensional transformations.

Figure 26. Excerpts from Amanda’s transformation of a circle into a “lollipop.”

Similarly, when Eva wanted a “corner take away” tool to support her transformation of the diamond into a circle (in Episode 7), there was no tool that she could use to effectively “push” the corners of the diamond thereby transforming an angle into an arc. Given that the environment has a tool that will “add a corner” to a shape in a way that is equivalent to dragging a corner "into" it, ideally it should also have an inverse operation that will "remove a corner."

Lastly, since a "hot spot" is required for dragging a component of a shape, the software requires that a corner be added where a "hot spot" is desired. Unfortunately, adding a corner to an arc has the resultant effect of transforming a single arc into an angle, as illustrated in *Figure **27***. Ideally, the tool should allow the placement of a hot spot at any point on a shape.

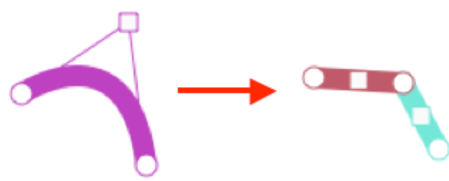


Figure 27. The addition of a corner to an arc (left) results in an angle (right).

Conclusion

The development of a tool that would be used for the "simple" task of making children's mathematical thinking visible was hardly a simple task in itself. Much of the complexity of the task arose from varying and variable considerations of "What might a topological experience look like?" and "What kinds of experiences make topological notions salient?", but also from the development of my thinking about the role the tool would play, from initial thought about its capacity to make topological thinking visible to the eventual realization that *the development of qualitative geometry* not only proceeds from *children's conceptions*, it is dialectically informed in the development of those conceptions as informed by the interactions between the tool as a form of topological representation and the topological operations afforded by it (Vygotsky, 1978).

Triangular considerations of the content, the design of the software, and the children who would and did use it (Figure **28**) were integral in the development of the final iteration of the environment. For example, the discrete movement of shapes in the

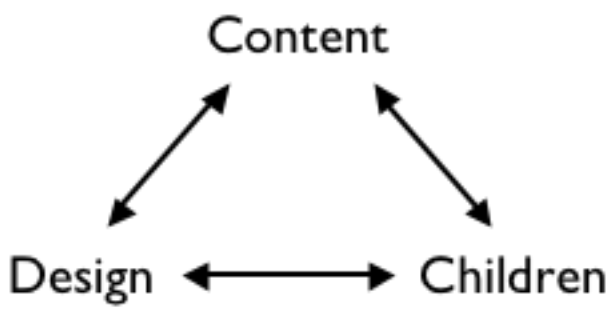


Figure 28. Software development is informed by the interactions between the content, the design, and the children.

second iteration, *QualiGeo* (*Figure 18*), signified insufficient attention to the fundamentality of continuity to topological investigations. Similarly, the failure of the auditory feature of the third iteration, *SoundTrack* (*Figure 19*), signified inadequate consideration of its usability by children. The clarity arrived at by the final iteration developed in response to shortcomings such as these in previous iterations of the environment. And the successful use of an iterative design process informed by these considerations resulted in an environment that now seems as if it is obviously as it should have been all along.

This entire chapter has been a response to my first research question, which I restate here:

> Given that early forms of topological, or at least non-metric, geometric reasoning have been identified and discussed in the research literature, can a software environment be developed in ways that support fundamental topological representations and transformations such that learners' reasoning about topological ideas are made visible and are able to further develop in ways that could credibly be seen as both mathematical and significant?

The answer to this question, then, is, "Yes." This chapter has documented the development of a software environment that "obviously" has these capacities. Illustrations provided in the following chapter of the dissertation lend further support to this response to my first research question. Now I move on to my second question.

Chapter Five: Findings

This chapter provides a response to my second research question, which I restate here:

> What forms of topological or non-metric geometric ideas are made visible and can be seen to develop as a result of young learners' systemic engagement with a computer environment that makes topological representations and transformations accessible?

Throughout the pretest, participants distinguished between shapes in ways that provide somewhat of a replication of a study by Lehrer, Jenkins, and Osana (1998). These researchers were also interested in the criteria children use to distinguish between plane figures, although I was only interested in a particular subset of those criteria. Thus, the triads I developed were different from theirs and my participants had the additional tasks of choosing the "most like" among five randomly selected shapes. Still, the tasks were essentially similar and the findings remarkably alike. In part because I wish to recognize the quality of their study and acknowledge its significance for informing aspects of my own, I quote directly from their findings section, and the only edits I provide are to replace their parenthetical quotes with ones from my own study.

> Children's reasoning varied from triad to triad and also varied within triads. [Their] reasoning most often involved the visual appearance of figures… This appearance-based reasoning, however, encompassed many distinctions. Sometimes children compared figures to prototypes of other figures ("kinda like a circle" and "just like a line") or to prototypes of real-world objects ("surfing board", "lollipop", and "fish head"). At other times, children focused on the size of the figure (they're different, because "that one is bigger than that one" and "those are more alike, because they're more small than that one") or to attributes that resulted from properties of angles (the triangle and the diamond are more alike than the rectangle, because the rectangle is not "pointy"). *Some children regarded figures as malleable objects that could be*

pushed or pulled to transform them into other figures [emphasis added]...” (pp. 141-142).

Their study clearly demonstrates that children use a wide variety of criteria to distinguish between shapes and they’re certainly not consistent in the application of those criteria. It is a particular subset of those criteria, however, that I make central in this investigation. Those are essentially related to the last statement in the paragraph of findings above, since transformations of shape are critical in this investigation for their capacity to support the development of participants’ qualitative reasoning (which depends on in the identification and application of qualitative properties of shape).

Consistent with my research questions, I provide accounts of moments in participants’ activities throughout the episodes of the teaching experiment, a trail of documentation of the development of their qualitative geometric reasoning.

To briefly review the fundamentals of qualitative geometry that were presented in Chapter Two, a qualitative geometric property is an attribute of shape that remains invariant under continuous qualitative transformations. The forms of those transformations are informed by topology and defined by the functionality of *Configure*, the software environment presented in Chapter Four. Using the “infinitely stretchable rubber sheet” conception of informal topology to describe these transformations, a shape printed on that rubber sheet may undergo stretching or shrinking but not tearing or gluing. Any attribute of shape that remains invariant as the shape stretches or shrinks is likely a qualitative property. Thus, the development of qualitative geometric reasoning may be characterized in part by the initial identification of qualitative properties.

Furthermore, since two shapes are deemed equivalent if each can be "qualitatively" transformed into the other (in the ways that the software environment allows), developing a conception of qualitative geometric equivalence is synonymous with the identification of these properties. This is because it is exactly those properties that remain invariant throughout the transformation that determine the equivalence. For example, a circle may be qualitatively transformed into a square. The qualitative property that remains invariant – that makes the transformation possible – may be called closed-ness. To reason qualitatively might look like using the property to identify other *equivalent* (or distinct[27]) shapes. Thus, attention to the equivalence of shapes is synonymous with determining that those shapes have common qualitative properties.

CASE 1: THE STORY OF AMANDA

"Hello, are you looking for shapes?"

At the time of the study, Amanda was seven years old and in second grade.

Episode 1: Everything's alike

In this episode the pretest was administered using the seven triads (*Figure* ***29***, also in Appendix A) and the five-card sorting task. When asked which of the two shapes in the first triad, row A, were most alike, Amanda had no response. Then, when asked to describe how they're different, she said that A1 was "like an S," A2 is "like a circle," and

[27] I use "distinct" as synonymous with "nonequivalent," a term that is used in the context of logic and logical operations to mean the opposite of equivalent.

A3 was “like a square.” This description of shapes as resembling everyday or canonical shapes was common across her responses in the pretest.

Phrasing the question that would be most productive at getting what I’m interested in – identifying the criteria participants use to distinguish between shapes – was complicated. The phrasing “Which two are more alike?” was most productive during the pretest. It seems to me that the form of the materials – printed shapes on paper – seemed more suited to that phrasing. When I gave participants cards with a shape printed on each one and asked them to “Put together the ones you think are more alike,” one child would pick up two and assemble them, and then talk about the shape that was produced. Inhelder and Piaget (1969) had a similar experience.

> “The instruction “Put together whatever is alike” gives a slight advantage to similarity relations of resemblance, but does not exclude spatial belonging, while the instruction “Put together whatever goes together” reinforces the latter without excluding the former” (p. 39).

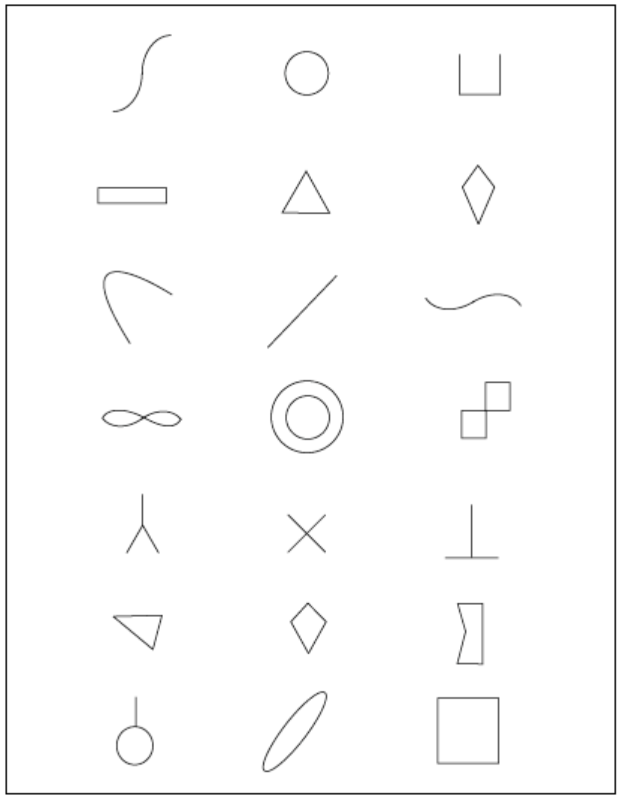


Figure 29. Pretest triads (rows A through G) [Also in Appendix A]

Amanda's distinction between an "awake" "S" [A1] and a "laying down" one [C3] are suggestive of attention to orientation. So to determine whether she sees orientation as a fundamental property of shape, I presented her with a picture of a triangle with a horizontal base – she called it a triangle. And when I asked her to name a triangle with a vertex pointed downward, she also called it a triangle. In four of the seven triads, she used the language of morphing to describe how one shape could be transformed into another by untwisting, bending, or pushing.

The following excerpt illustrates Amanda's working conception of alikeness by the end of episode 1. She has been presented with the following cards containing shapes A1/C3, B2, E2, E3, and G2 (*Figure **30***).

Figure 30. Amanda's sorting of all five cards into a single group (Episode 1) [left to right: B2, A1, E2, G3, G2]

Interviewer[28]: I want you to put together the ones you think are more alike. <pause> Make a group of the ones that you think are more alike.

Amanda sorts all of the shapes into a single group.

Interviewer: Put 'em all together?

Amanda nods to affirm and then sorts all shapes into a single group.

Interviewer: OK. How come?

Amanda: For this one [A1], I would say you just cut, like, this part off, and make it straight and then get the rest, put it straight, and then you got this one [E3].

– To make this one, you just – To make this one [E3], you just cut one off, and like, cut it off to <string?>, make it like that.

– And these two [E2 from B2], just cut two, like, one right here and, like, one right here. Put 'em together and then get a X [E2].

– For this one [A1 from E2], you just – to make this one [A1], just, like, get glue and attach 'em [the two "pieces" of E2], and then make a little curvy line.

– Done.

[28] I remind the reader that I was the sole interviewer in the study.

Amanda sees all five shapes as alike in the sense that she can transform any one into any of the others. The transformations that "put it straight" and "make a little curvy line" have a topological quality, while others – those that involve breaking, cutting, gluing, and attaching – do not. Later, in a way that parallels the conventional use of "equivalence" to describe homeomorphic copies, Amanda says that after the ellipse [G2] is transformed into the square [G3], "they'll be twins."

Episode 2: Multiple conceptions of alikeness

To follow up on questions provoked by aspects of Amanda's selections of most like in Episode 1, this episode commenced with a second round of the pretest that was administered using the "follow-up" set of shapes (*Figure **31***, also in Appendix B) described in "The Pretest" section of Chapter Three. These were printed on circular cards and placed in a bag with the pretest shapes that had been printed on square cards. Amanda chose five cards at random and was asked to "Put the ones together that you think are alike. You can make as many groups as you want." Then the task was repeated a second time.

Figure 31. Follow-up shapes (H1 through H6) [Also in Appendix B]

Amanda grouped two shapes [the inverted Y (E1) and the X (E2)] based on their resemblance to "letters." Although resemblance to everyday objects remained salient to Amanda throughout the teaching experiment, she was also interested in moving beyond it. When I followed up by asking, "When you look at the shapes of those letters, how are they alike?", Amanda replied, "They both have lines."

Amanda grouped a diamond [B3] and a concave pentagon [F3] together, because the pentagon could be transformed into the diamond by first "getting more string and gluing it." She then takes it upon herself to describe the transformation of the diamond into the pentagon, which is accomplished, in part, by cutting. From a topological perspective, two shapes are equivalent if they are homeomorphic under a *bi*-continuous transformation. That means that the transformation from one shape into the other must be, in the spirit of Piaget's (cf. 1970b) use of the word, reversible: a *doing* of the transformation followed by an *undoing* must produce the original image. Of course, Amanda is not operating from that perspective, so it is interesting to find that she feels the need to also express the inverse transformation.

As in Episode 1, Amanda used morphing language (e.g., "stretch", "straighten") to describe components of transformations. She could "straighten" the "S" [A1/C3] to produce the segment [C3], and the epsilon [H5] to produce the T [E3]. While these transformations are topological, other transformations involved "cuts" [B3 to B2] and "moves" [H3 to H2].

Amanda operated with at least three forms of alikeness in this episode: resemblance-based, attribute-based, and "transformational." Respectively, she said two shapes are alike if they are "both letters," if "they both have lines," and if they can be transformed into each other. The nature of the transformations varied, so the alikeness of a shape and its "transformational equivalent" was not based exclusively in a particular geometry. Again, some were topological and others were not. In Episode 3, my plan was for Amanda to spend more time with the third conception by further exploring the transformations she had described that could also be accomplished through transformations that are topological (e.g, the "S" and the segment; the Y and the T; and then the diamond and the triangle) and through those that are not (the T and the "S", the Y and the X).

Episode 3: Two shapes are alike if one can be transformed into the other.

In this episode, Amanda used the software for the first time. Here my goal was to have her explore a qualitative conception of alikeness (as defined by the transformations supported by the rules of the software). This exploration was meant to support the goal of having Amanda identify the properties of equivalent shapes. The shapes that are built into the software are presented in *Figure **32*** (also in Appendix C).

To begin the episode, I drew a Y [E1] and T [E3] in *Configure*. I reminded her that she chose these two shapes as more alike than the X [E2].

Interviewer: Do you remember why you said they were alike?
Amanda: 'Cuz they're letters.
Interviewer: Well, they're all letters. When you look at 'em now, which two would you say are more alike?
Amanda: None. [?]
Interviewer: Last time you said these [T and Y] are more alike. Do you remember why?
Amanda: Yes.
Interviewer: Why? What's alike about them?
Amanda: Look. Watch. You can make it [T] into a Y. <she drags two branches of the T to form the Y>
Interviewer: Exactly, that's why. You said you could bend it. Alright, can you make that one [Y] into a T?
Amanda drags the same 2 branches to form the T.
Interviewer: OK, we say that these are alike, because you can turn into the other.
Amanda: Mm hmm.

Figure 32. Shapes built into Configure [Also in Appendix C]

This is the first segment of the teaching experiment in which Amanda is presented with the conception of alikeness that is informed by the functionality of the software. She is told that the "transform tools" (i.e., the tools that either make a segment into a curve, make a curve into a segment, or add a "corner" to a segment or curve) will be used to transform shapes, and that the "build tools" (i.e., the tools that glue or unglue) will be used to build shapes. As Amanda has transformed shapes through gestures outside of the software environment by cutting or attaching, this distinction between tools that transform and tools that build is repeated throughout the experiment. It is a somewhat arbitrary distinction. But the space has rules and the exploration is informed by those rules. The arbitrary nature of the constraints on the mathematical activity that occurs in the space might be seen as a weakness, but these sorts of rule-based explorations should are entirely and authentically mathematical. For example, consider the differences between operating in universes that are spherical, hyperbolic, or flat. The activity that occurs within those universes is defined by the shape of the space. Consider the effectiveness of the Pythagorean Theorem in each of those universes. [Then, if you're prepared for the consequences, consider the fate of the universe itself.]

In the following segment, Amanda attempts a transformation between two topologically distinct shapes, the T or the Y, and the X.

Interviewer: Can you turn either one of these [T or Y] into the X without gluing or ungluing?

Amanda: I think, yes, but – <pause> How do you break it again?

Interviewer: We're not breaking. You would unglue, but we're not going to unglue it.

Amanda drags branches of the T around, apparently trying to find a way to drag out a third branch.

Interviewer: How are you thinking about doing it?

Amanda: Like this [has three branches of the X] but I need just one more line.

Interviewer: Can you do it without adding another line?

Amanda is persistent. She adds a corner to one branch.

Amanda: I caannn't.

Interviewer: Yeah, I can't do it, either, without adding another line. So we're going to be interested in which shapes can you turn into other shapes without breaking or adding a part. That's what we're looking at.

Following this segment, Amanda states that she wants to try to transform the Y into the "open square" [A3].

Amanda: I wanna do this one [A3].

Interviewer: Can you make one of those [T or Y] into this shape?

Amanda: Yeah. <drags 3 branches of Y> Uhhhhhh.

Amanda has attempted the transformation for two minutes when I interrupt. I'm interested in the extent to which she's planned the transformation or whether she is proceeding through unplanned trial and error.

Interviewer: What's going on?

Amanda: It's hard. I thought I could make this go to that, but I can't.

Interviewer: Do you see why you can't? <pause> What would you have to do in order to make it look like that?

Amanda: I would have to take this string off and replace it right here.

Only by attempting the transformation does Amanda realize that it's not possible. She does not articulate why the transformation is not possible, but she does identify what it would take to make it possible.

Following Amanda's successful transformation of a rectangle into a triangle, an "S" [A1/C3] into a segment [C2], and "sail" [the first shape in the first row *Figure **32***] into a triangle, I ask Amanda, "When you look at those two [sail, triangle], what do you see that's alike? What makes you think it's possible to turn that [triangle] into that [sail]?" She replies, "Mmmm. I need squares." In this response she is referring to the "helper points" that make it possible to change the curvature of an arc. Only by converting the segments of the triangle into arcs is the transformation of the triangle into the sail possible.

For the final transformation of the episode, I present Amanda with the clover [the second shape in the third row of *Figure **32***] and ask her, "Can you see one on this [pretest, *Figure **29***] page that you can turn it into?" Amanda's selection of the rectangle [B2] was surprising to me given that the two shapes are visually so distinct. But she proceeds with the transformation by adjusting the curvature of each of the clover's petals and then adding a corner to produce the rectangle. *Figure* ***33*** below provides the preimage and image of the transformation. Still, she provides no response to my query,

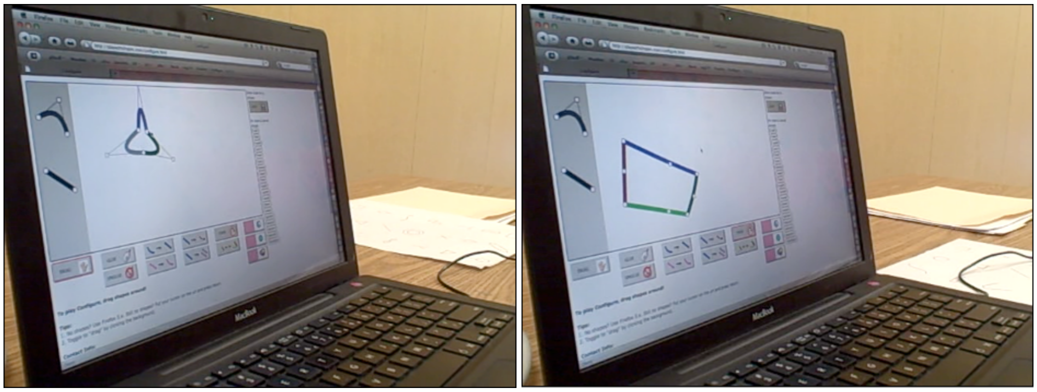


Figure 33. Amanda's transformation of the clover (left) into a rectangle (right)

"What makes you think you can turn it [the clover] into that [the rectangle]?" Thus, it appears that her operations on the clover are entirely concrete – as opposed to formal, which might be implicated by Amanda's explicit consideration of *possible* transformations of the clover – it seems that Amanda is still assimilating the space. She is fluent with the software and she has successfully transformed shapes into equivalents, but she has yet to articulate the properties of those equivalent shapes. Similarly, she has not yet expressed what it is about two shapes that makes them distinct. Each of these tasks requires attention to properties. The plan for Episode 4 is to make progress toward supporting Amanda's attention to properties, or, if she is indeed attending to properties, to support her verbalizations of those properties.

Episode 4: Developing equivalence

Episode 4 is essentially about transforming shapes in *Configure* and then reflecting on the outcomes. It is themed "developing equivalence," because it provides some evidence that Amanda is increasingly attentive to the properties of shapes that make transformations possible or impossible. Salient moments occurred during a subset of those transformations.

Amanda transformed the "open square" [A3] into the "arc" [C1] by converting the three segments into arcs and then adjusting the curvature. This appeared to be a simple task for her, but it was her response upon completion of the task that was salient. She appeared quite pleased with what she had accomplished. She leaned back in her chair and turned to smile at me. Because she had completed so many transformations prior to this one without concluding them with a smile, it seems to me that this smile points to cognitive structure (not to mention positive affect toward being a participant, the software, and perhaps even the mathematics).

In Chapter Two, I explained that a cognitive structure is a system of transformations – a patterning of actions (Stroup, 2002) – an organization of schemes or functions – an organized pattern of behavior. I also provided a demonstration of the way that Piaget generates the cognitive structure of topology. As evidenced by Amanda's behavior following this latest transformation, I argue that, likewise, children possess a cognitive structure of qualitative ways of thinking about shape.

To support my claim that this instance provides evidence of that structure, I draw from an episode of a teaching experiment conducted by Steffe and Thompson (2000) to identify the tools I need to do so. Two students, Jason and Patricia, are asked if each could break a (virtual) stick of candy into four "fair shares." After several attempts they recognize as failed, they eventually drew a stick that accomplished the task. Steffe and Thompson suggest that these attempts "confirm the internal necessity that the children must have felt that the four pieces together must be exactly the same length as the original stick. The two children had constructed an equipartioning scheme…" (p. 282). Dubinsky et al.'s (2001) eloquent framing of a scheme as a "collection of actions, processes, objects, and other schemas which are linked by some general principles to form a framework in the individual's mind that may be brought to bear upon a problem situation involving that concept" (p. 3), also contributes to my understanding of this interaction. That framework is the cognitive structure (Piaget, 1970b), which is implicated by Jason and Patricia's *felt need* for the four pieces of the stick to "sum" to the length of the original stick. Piaget elaborates in the following exchange (Bringuier, 1980):

> *Bringuier*: How do you know, experimentally, that you are dealing with a structure? In what form is it encountered?
>
> *Piaget*: How do we define it? It's the new feeling arising in the individual's consciousness, a feeling of *necessity* [emphasis added]. It's the links that, considered simply as given or observed, are experienced as necessary. The individual can't think otherwise. Take transitivity as an example. If $A = B$, if the child observes that $B = C$, does $A = C$? On can, for instance, ask a child to compare a stick of a certain weight with a rod of the same weight, then with a ball. For the child at the preoperatory level, before the construction of structures, there is no relationship among the three objects. He says he doesn't know. He saw A and B; he saw B and C, but he didn't see A and C together. Or he draws a conclusion about what he believes to be possible or probable. A child at the level of structures finds it obvious, necessary. He smiles and shrugs his shoulders at being asked such a simple question: If $A = B$ and $B = C$, then, obviously,

> $A = C$. Necessity is the criterion of the structure's closure, the achievement of a structure. (p. 40-41)

Jason and Patricia's assimilation of a scheme for dividing the stick of candy illustrates the process by which the problem situation provokes attention to the relevant conceptual structure. So if "assimilation is just the proof that structures exist" (Piaget, in Bringuier, 1980, p. 42), and structures are patterns of operations, how can we identify their existence in the ways that kids operate on shapes? An answer to this question informs this analysis.

To organize the evidence that qualitative geometry is a cognitive structure, I refer to the fundamental elements that form structures: reversible operations. Just as Jason and Patricia's *felt need* was expressed in their attempts to divide the stick of candy into fair shares, and Jason's concluding smile and shrug of his shoulders, Amanda's expression is suggestive of a similar sort of "obvious necessity" and evidence of a structure of non-metric geometry. She follows the completion of the transformation by sitting back and smiling at having been asked to perform such a simple task.

When asked if the "pyramid" [third shape in the last row of *Figure **32***] is "alike or different" from the theta [second shape in the second row of *Figure **32***], she says the pyramid is bigger. When I reduce its size, Amanda says, "You need to make it a little circle." I interpret this statement as a declaration that the straight sides of the pyramid would need to be curved. So I convert the segments to arcs. Then Amanda proceeds to complete the transformation. Obviously, Amanda is again distinguishing between shapes by size and curvature, neither of which is a topological property. Since the point is not to rid her of these ways of thinking about shapes, but to instead focus the activity on a particular subset of those distinguishing characteristics, I make accommodations and then continue the exploration. When asked about H6 and the diamond [B3] (Figure **34**), she does not provide a verbal response. Instead, she performs the transformation.

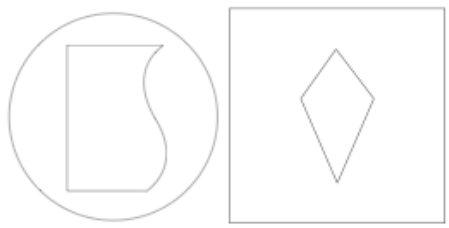


Figure 34. Shape H6 and the diamond [B3]

My next step is an attempt to provoke Amanda to perform the transformations internally through operations of her mental representations of shapes. I ask if the transformation of the epsilon [H5] to the "fork" [fourth shape in the first row of *Figure **32***] is "easy or hard." She says it's "easy." Then I ask her about the Y [E1] and the fork, and she also says it's "easy." So I ask her if either the Y or the "fork" can be transformed into the "S" [A1/C3]. Again she replies, "easy." Knowing that the shapes are actually not equivalent, I ask her to perform the transformations in *Configure.* She attempts to bend the segments that form the Y and then realizes that "That one [extra piece] is messing up my S." It turns out that even in her final episode, Amanda sorts shapes together that are "made out of stems," but she does not verbalize a distinction between ones that are actually distinct, such as the Y and the X.

Later in the episode, Amanda is asked whether a circle could be transformed into a "curve" [C1]. She attempts the transformation and then decides, "I can't make it." When I ask her why not, she replies, "It's glued." This moment is salient because it reflects Amanda's assimilation – as tentative as it may be – of the rules of the environment for informing what transformations are possible.

At this point, Amanda has transformed numerous shapes into equivalent ones, but is inconsistent in the identification of properties that make those transformations possible. Exploring shapes in pairs may not be so supportive of the development of Amanda's qualitative conceptions, because she is not required to attend to the properties that define their equivalence. She only has to determine whether a transformation from one to the other is possible. So the plan for Episode 5 is to further develop a conception of alikeness as equivalence in the software environment and then attempt to provoke Amanda's attention to prototypes of equivalence classes. Identifying a circle as a prototype for closed shapes or a segment as a prototype for open ones, for example, can support Amanda's identification of the property that defines each class. To elaborate, the elements of equivalence classes reveal transformations of shapes structured by the wholeness of the property that defines the class. They are representative instances of the patterning of actions (Stroup, 2002) that determine the cognitive structure of qualitative geometry.

Episode 5: Given a shape, identify an equivalent one.

At the beginning of this episode I presented Amanda with her "sorting worksheet" (*Figure* ***35***). The worksheet identifies the equivalence classes of shapes Amanda has transformed through the prior episode. Amanda hadn't transformed each shape into every other shape in the class, but every shape in the class is either the preimage or image of Amanda's transformation of a shape in the class.

The following excerpts portray my attempts to provoke Amanda's attention to properties that define equivalence classes of shapes.

Interviewer: So when you look at these shapes [in the first row and column of the sorting worksheet, *Figure **35***], do you see anything about them that's alike? Something that makes you think I can take any one of these and turn it into any of the other ones?

Amanda: This one and this one? <refers to the square and the triangle >

Interviewer: Yeah, you've actually transformed them all. So what is it about these [square, rectangle] that's alike?

Amanda: It's alike 'cause – if <?> was a triangle, but all you do is just make it a little longer, and then shorten it down. And to make this one [the square from the rectangle], you just push that over quite a bit.

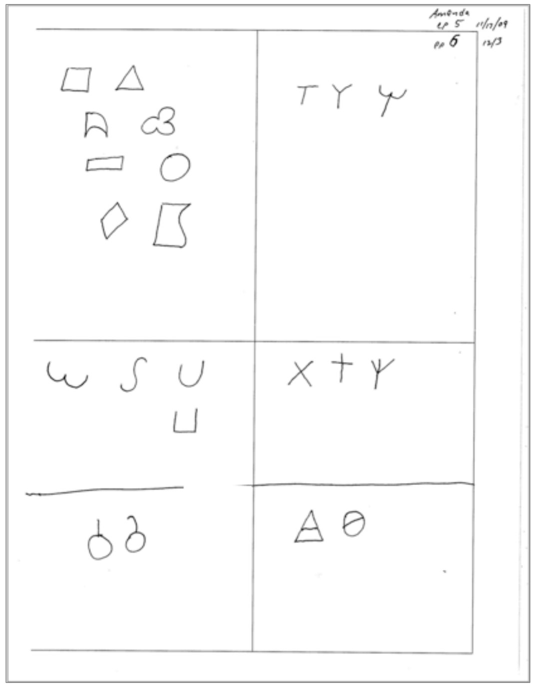


Figure 35. Amanda's “sorting worksheet” (Episode 5)

Although I asked Amanda to consider the entire class of eight shapes, she instead considers only two – the square and the triangle. Perhaps Amanda has simply accommodated our repeated pairwise consideration of shapes.

It turns out that attention to properties that define a class is a newly developing ability for children at this developmental level (Inhelder & Piaget, 1969, p. 17). The question I’m asking Amanda is meant to focus her attention on these properties, yet it is very difficult to phrase in a way that makes sense to her. Her responses to my queries indicate that she has indeed assimilated the conception of (pairwise) alikeness that means that one shape can be transformed into the other, since she is now operating with it. According to Piaget, Amanda is operating “at the level of graphic collections [where] relations of similarity and difference are applied pairwise and unconnected with part-whole relations" (p. 19). The level of graphic collections is developmentally prior to the level of class inclusion. "A class involves two kinds of properties or relations: 1) (a) properties which are common to the members of the given class and those of other classes to which it belongs, and (b) properties which are specific to the members of the given class and which differentiate them from members of other classes" (p. 17). Getting Amanda to attend to the property that defines the equivalence classes of shapes requires that she operate at the level of class inclusion. Toward that end, I ask her to consider three shapes simultaneously.

Interviewer: What's alike about these three ["arc" (C1), "S" A1/C3, "open square" A3] that makes it possible to take any one and turn it into any of the others?

Amanda: You can turn this one ["arc" (C1)] into this one ["open square" (A3)]. Just make that straight, and that one, and make that straight.

The outcome is the same. Alikeness to Amanda means that one shape can be transformed into another. This is just as the term was defined in earlier episodes. For instance, in episode 3, I tell Amanda, "We say that these are alike, because you can turn one into the other." Later in that same episode, I remind her, "What we're interested in is things that are alike, because you can turn one into the other." In retrospect, I realize the unintended consequence of defining alikeness in this way is its explicit consideration of shapes in pairs. What also should be stressed is that alikeness is also a property of any number of shapes.

Next, Amanda completes a handful of transformations of shapes in the same classes, and then I return to the effort to focus her attention on pairwise equivalence.

Interviewer: When you look at these shapes [the “pyramid” and the theta in row 3, column 2, of the sorting worksheet, *Figure **35***], what do you see that’s alike about them?

Amanda: They have a line in the middle.

Although she is only considering two shapes, that response is indicative of conceptual progress. The first time these two shapes were considered in episode 4, Amanda explained that she required that the size of the pyramid be reduced and its sides converted to arcs in order to complete the transformation. In this second investigation of those two shapes, her response does not involve the language of transformations. Instead, Amanda describes the property that defines the equivalence of the two shapes, although she may not see it that way. It could be argued that Amanda is simply attending to common attributes of those shapes, since she may not realize that the attribute is a fundamental property of each shape. My next step is to investigate how Amanda operates using that property.

Interviewer: Is there another one on this [pretest] page that you can turn either of these [“pyramid”, theta] into? <pause> You can make up one, you can take one of these, anything you want to.

Amanda: <No response>

Perhaps she could not find another shape with a “line in the middle.” Perhaps she was overwhelmed by the task.

Interviewer: So how about another one here [in the class of “open” shapes, row 2, column 1 of *Figure **35***]?

Amanda places her fingers at the endpoints of the “S” and gestures to close the shape. Then she sits back as if to reconsider, perhaps deciding that a closed shape like the circle that would result from “closing” the “S” would not belong in the class. A full minute later, she draws a segment.

Just as she did in Episode 4 following a successful transformation of the “open square” [A3] into the “arc” [C1], Amanda sits back and smiles as she reflects on the “obvious” accomplishment of drawing a shape that is equivalent to the others in the class. Now that Amanda has identified a property that defines an equivalence class, the primary focus of Episode 6 is to continue that effort by developing appropriate names for the classes of shapes on the sorting worksheet.

Episode 6: Identify the property that defines the class.

I initiate the episode with another attempt at directing Amanda’s attention to the property that defines each class of shapes on her sorting worksheet (*Figure **35***). Prior attempts have *apparently* been unsuccessful – although not entirely – so this time I appeal to Amanda for help.

Interviewer: I need you to help me out here. I’m trying to find a word -- I need a word to describe the shapes here [in the first row, first column of her sorting worksheet, *Figure **35***].

Amanda: What kind of word?

Interviewer: A word that describes how they’re all alike.

Amanda: Uhhhhh -

Interviewer: See, these [in another class] are different, right? So I need a different word for these two [classes], because they're not alike. I can't turn that into that. Or – Why don't we start with a word that you would use to talk about –

Amanda: I can probably make this ["fork", row 1, shape 4, *Figure* ***32***] into this [Ψ, *Figure* ***35***, row 2, column 2, third shape] if I just move that one up and turn it around.

Given a request to consider the property that defines the equivalence of a class of eight shapes, Amanda responds by describing how *one* shape in a class can be transformed into *one* other. Four minutes later, we repeat the attempt, but this time I ask Amanda to consider a second class of shapes that contains only two items, those that are in the third row and first column of her sorting worksheet (*Figure* ***35***).

Interviewer: Is there a word that you could use to describe these? <pause> Actually, let's just make a list of words that we could use to describe these shapes.

Amanda: Uhhhh. So you want me to name 'em?

Interviewer: No, just words that describe how they're alike. We can start with this group or any one you want. <pause> Say you're on the phone with somebody and you wanted to talk about the shapes in that set. How would you describe it over the phone?

Amanda: <pretends to pick up a phone> Uhhh, hello. I got one cherry that has a straight stem and I have one cherry that has a curved stem. Which one would you like to buy? OK bye, bye.

Earlier, Amanda described the "pyramid" and the theta as alike, because "they have a line in the middle." This characterization of the two shapes as "cherries" is similarly indicative of her attention to a common attribute of shape (since she may not realize that the attribute is a fundamental property of each shape). But this more holistic identification of the alikeness of two shapes could reasonably be seen as conceptually proximal to the identification of fundamental properties. Exploiting the momentum, I ask Amanda to continue.

Amanda <picks up the phone>: Hello. Are you looking for shapes?

Interviewer <on the phone>: Yes, I'm looking for candy-flavored shapes.

Amanda: OK. We have just the ones. We got a lot, lot, lot. We got ones with lines in the middle that are easy to cut open. One's a triangle <the "pyramid" and one's a circle <the theta>. That's all we have left. <There are only two shapes in this class.>

Interviewer: <labeling the set> Lines in the middle – that are easy to cut open. What makes them easy? Because they have the line in the middle?

Amanda: Yeah. They're already cut.

Interviewer: Alright, this is working.

Amanda <picks up the phone again>: Are you here for letters? OK. Lowercase? <referring to the class of shapes that are equivalent to an X, in row 2, column 2 of *Figure **35***> We got T in lowercase and a lowercase X. Yes, they are candy.

Interviewer: How are they alike? I need one word for both <x and t>, like you gave me here <the class of shapes with "lines in the middle">.

Amanda: They're both stems.

Interviewer: Stems?

Amanda: Yeah. Made out of stems.

Interviewer: <labeling the set> Made out of stems. Alright, keep going. How about these <that are equivalent to the segment, row 2, column 2 of *Figure **35***>?

Amanda <picks up the phone>: Are you here for candy? We got letters.

Interviewer: Well, that's a letter <in the class of shapes that are equivalent to a "T", row 1, column 2 of *Figure **35***>, but it doesn't go here. What's alike about those <in this set>?

Amanda <on the phone>: Are you here for candy letters? Yeah? A big T and a stem Y and a curvy fork?

Interviewer: How are these "stems" [equivalent to a T] different from these [equivalent to an X]?

Amanda: 'Cuz those [equivalent to an X] aren't straight.

Interviewer: What do you mean?

Amanda: They're like – they're not like this [gestures vertically].

Interviewer: That one's <"lowercase t"> straight. How is that one different from that one?

Amanda: This one's lowercase and this one's uppercase.

Interviewer: Let's talk about the shape, not the letter.

Amanda: OK? You called the wrong place then.

Amanda describes the class of shapes that are equivalent to an X as "made of stems." She initially distinguishes between the class of equivalents to the X and the class of equivalents to the T by appealing to the orientation of segments that compose the shapes, but ultimately abandons the argument (and hangs up the phone). Even though she realized in episode 3 that the T could be transformed into the Y, but in order to transform the T into the X she would "need just one more line," she is never able to determine the distinction.

We conclude the episode by considering one more class of shapes, those that are equivalent to a segment (row 2, column 1 of *Figure **35***, plus the segment she drew in episode 5).

Interviewer: How are these alike? How would you describe the ones in there if you were on the phone selling candy?

Amanda <on the phone>: Hello? Yes? How would you like to buy some candy? OK, we got W, S, U, and we got two more shapes that come with it.

Amanda: I can turn all these to this <Amanda describes how each of the other four shapes can be transformed into the segment.>.

Interviewer: Yes! Great. Exactly. So how are they alike that you can do that?

Amanda: I would call them – candy.

Interviewer: But I thought they were all candy. So if you're on the phone and you say, "I got candy," how are they going to picture that [set]?

Amanda: OK. Ummmm. Letter candy.

Interviewer: But that's letter candy [in another set].

Amanda: Ohhh, god.

Interviewer: Can I suggest a word and you tell me if it fits?

Amanda: Yeah.

Interviewer: How about – have you ever had gummy worms?

Amanda: Yeah.

Interviewer: How about worms? Does "worms" work for this? Can you –

Amanda: Oh yeah! They're worms – called worms.

Interviewer: OK. Alright, good.

It was important to me throughout the teaching experiment to use Amanda's own language so that my reports could do their best to represent *her* ideas and not my reinterpretation of them. Using words that made sense to her – because she invented them – also served to facilitate communication between us. In this excerpt, however, I decided to offer her a word since she had not come up with one of her own. My thinking was that her acceptance or rejection of the word could give me some indication as to the validity of my model of her thinking. It turns out that Amanda not only accepted the word, but her exclamation of "Oh yeah!" strikes me as suggestive of the goodness of fit of my model.

Episode 7: What's equivalent is determined by what doesn't change – by what remains invariant.

Topological properties are those that remain invariant under bicontinuous transformations. Similarly, qualitative properties are those that remain invariant under the transformational operations allowed in *Configure*. Therefore, we can sidestep the previous method for getting at qualitative properties of shape: those that we can identify as common across shapes identified through transformation as equivalent. Instead, since a qualitative property is one that remains invariant through *Configure's* dragging, bending, and stretching operations, participants can operate on shapes in these ways in order to identify those properties. That method defines the theme of this episode.

It also defines one of the "big ideas" of mathematics. Judah Schwartz (1993, Winter) identifies invariance as one of the "big ideas" included in the reform curriculum.[29] "Mathematical invariance is what allows us to see commonality in many situations that, on the surface, may appear to be very different" (Stroup, 2005, p. 193). Schwartz (1999) claims that although these ideas traditionally only appear in advanced courses, exploratory software environments can support their introduction early on in the education of *all* students. Similarly, Stroup (2005) draws a "significant "meta-lesson"" from seeing "basic concepts… as deeply interwoven with what are often considered advanced topics..." (p. 193). Schwartz's claim and Stroup's meta-lesson speak to this

[29] Coincidentally, transformation is another one. The others are representation, symmetry, scale, continuity, order and betweenness, boundedness, uniqueness, relaxation and constraint, successive approximation, and proof and plausibility.

study relative to its involvement of the youngest children exploring some of the most advanced mathematical ideas.

To begin the episode, I followed up on Amanda's consideration of shapes she describes as "made out of stems" by presenting her with other shapes made of stems (*Figure **36***) and eventually asking her sort them into equivalence classes. The idea is that

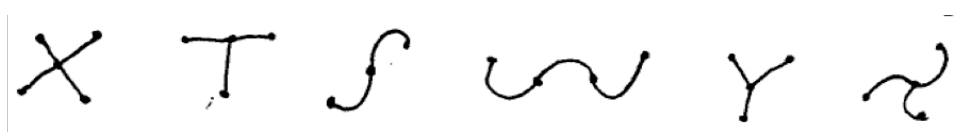


Figure 36. Shapes made of "stems"

experimental transformations of shapes will support Amanda's identification of qualitative properties of those shapes. Then, once properties have been identified, Amanda will be able to sort shapes into equivalence classes on the sorting worksheet.

A "T" is drawn to "seed" the first class of shapes; no other sets on the worksheet contain shapes.

Interviewer: This time I want to talk about what *doesn't* change. Let's start with this one [the built-in arc].

Amanda: That's simple.

Interviewer: Tell me what doesn't change about it.

Amanda: It doesn't break.

Interviewer: Tell me what doesn't change about *this* shape [the spiral, last row, fourth shape of *Figure **32***]. What stays the same?

Amanda: The little dots and all that. They don't come off.

Interviewer: What else?

Amanda: That doesn't come off, all these ["branches"]. Ooh, ooh. Put the spiral in the T section [on the sorting worksheet].

Amanda adds the spiral to the set on the worksheet that contains the T.

Interviewer: What about this one [the "wire", first row, fifth shape of *Figure **32***]

Amanda: Oooh, I can make a Y out of that thing!

Amanda's realization that the "branches" of the spiral "don't come off" is a realization of the spiral's topological property of connectedness. In the case of the spiral and the other shapes "made out of stems," it is this property that determines equivalence or distinction.

As in previous episodes when the task is about identifying equivalent shapes, Amanda proceeds to develop equivalence classes not by identifying (invariant) properties through transformations, but through attempts to transform one shape into another. She determines through these transformations that the Y and the Ψ [*Figure **35***, row 2, column 2, third shape] are distinct and that the X and the Ψ are equivalent. Then, she considers the "S" [A1/C3]. Without the use of the software, Amanda realizes that the "S" is distinct from the X and the Ψ, places it in a new set by itself, and congratulates herself by singing a short tune. Given that the development of this new equivalence class is accomplished outside of the software environment in which she developed the conception of alikeness with which she operates, this distinction between the "S" and the X and Ψ is cognitively and mathematically significant in that it is determined through mental transformations.

Next, Amanda uses the software to confirm her conjecture of the equivalence of the "wave" [last row, last shape of *Figure **32***], the segment, and the S. Then, I ask her to consider a circle. She uses the software to "close" an S by gluing its endpoints to form a circle, but when I remind her that gluing is not allowed in the transformation of shapes, she places the circle into a new equivalence class by itself, thus acknowledging its distinction from all of the shapes "made out of stems." Finally, in the task that concludes the episode, I ask Amanda to consider a triangle.

Interviewer: When I drag the triangle around, what about it stays the same?
Amanda: It stays attached.
Interviewer: What still stays the same, even if I bend things?
Amanda: It'll still be attached.

In this brief exchange, Amanda identifies a property of triangles that remains invariant under transformation. This "attached" quality of triangles corresponds to the same topological property that earlier determined the equivalence or distinction of shapes "made out of stems."

In the prior episode, Amanda identified the properties that define equivalence classes of shapes. The names she gave to equivalence classes of "cherries" and shapes with "lines in the middle" are suggestive of the property that defines the class, and my suggestion that shapes equivalent to a segment could be called "worms" seemed to resonate with the way she thought of the class. She also identified a group of shapes "made out of stems," but was either unable or unprepared to describe a fundamental distinction between the two equivalence classes that formed the group. In this episode, attention to fundamental properties was supported by dragging shapes so as to make those properties salient and visible. Amanda identified a correlate to the topology property of connectedness as instantiated in the distinction between shapes that feature four (e.g., X and Ψ) or three (e.g., T and Y) segments with a common endpoint, shapes that are equivalent to a segment, and shapes that are equivalent to a circle. In the next episode, the effort to identify qualitative properties that define equivalence classes is continued. Whereas in this episode, Amanda was given shapes to sort into equivalence classes, in the next episode the classes of shapes were given and Amanda is asked to name them as she did earlier when she named the classes of "cherries" and shapes with "lines in the middle."

Episode 8: Given a class of shapes, identify the property that defines the class.

This is the final exploratory episode with Amanda. In the activities of the prior episode, she was not given the opportunity to use what she learned, if anything, about the properties that determine the equivalence of shapes through dragging. Thus, the format of the sorting task that begins this episode should give her the opportunity to demonstrate what, if anything, she learned as a result of that exploration of shapes. The sorting worksheet as it appeared at the conclusion of the episode is shown in *Figure **37***.

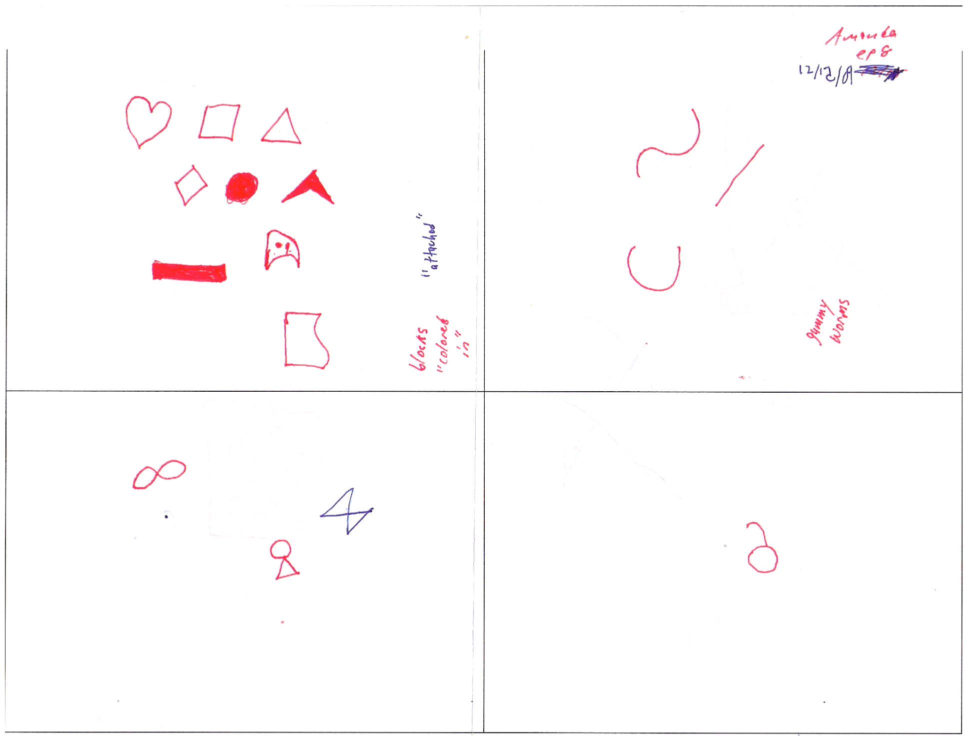


Figure 37. Amanda's sorting worksheet for Episode 8 (as it appeared at the conclusion of the episode)

I started by reviewing the names we've given to the sets of "cherries," shapes with "lines in the middle," and "worms." Then we considered the class of simple closed curves[30] that appears in the upper left quadrant of the worksheet.

> *Interviewer*: We haven't done shapes like these [simple closed curves] yet.
>
> *Amanda*: I would call 'em blocks.
>
> *Interviewer*: How are they all blocks? Tell me what that word means.
>
> *Amanda*: See the block - this, this, this <tapping the pen to simulate block-ing>
>
> *Interviewer*: How is this <"figure 8", the leftmost figure in the bottom left quadrant of the sorting worksheet> not a block?

---

[30] A simple curve is one that does not cross itself. A closed curve is one that has no endpoints and encloses and area. Thus, a simple closed curve, or "Jordan" curve, is one that is topologically equivalent to a circle.

Amand: 'Cuz how could it be a block when it's like this? <gestures to draw a figure 8>

Interviewer: I don't know. Is this <"figure 8"> a block?

Amanda: I would call this an 8.

Interviewer: Could you draw another shape that would go there, that's like the 8? <pause> Does this one go here <I draw the "hourglass," the rightmost figure in the bottom left quadrant of the worksheet>?

Amanda: Yeah, that one's there, that one's there.

Interviewer: OK, tell me why.

Amanda: 'Cuz all you gotta do is... make those straight... and for that one, you curve it. <describes and gestures the transformation from the "figure 8" to the "hourglass" and back>

Interviewer: Alright, so tell me about blocks. I think I know what you mean.

Amanda: Blocks. Blocks. They never lose their shape.

Interviewer: They never lose their shape?

Amanda: Yeah.

Interviewer: I can stretch them and bend them, though, right?

Amanda: Oooh! Blocks are made out of wood.

Interviewer: Yeah? OK. And? Couldn't I make that <"hourglass"> out of wood?

Amanda: No.

Interviewer: I like it. I wanna be clear about what you mean. <pause> What about a cherry? Is that a block?

Amanda: No.

Interviewer: OK, how is that different from a block? I'm going to call these <simple closed curves> blocks. If you just tell me what are and what aren't blocks, I can get a better sense of what you mean by it.

Amanda: Well, blocks, like, ... <turns the worksheet over> This is a block <draws a square >. This is not a block <draws the non-simple curve recreated in *Figure*

38>. They need to be, like, colored in. These <points to the non-simple curve> can't last, so they're just like that.

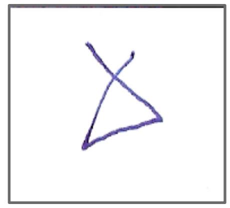


Figure 38. Amanda's drawing of a shape that is "not a block" (recreated)

Interviewer: <repeating> They need to be colored in. <I label the set of simple closed curves "colored in.">

Amanda: I'm coloring them in. <Amanda colors in the three now-shaded shapes in the set of simple closed curves (*Figure* ***37***)>

As I mentioned in the review of the literature in Chapter Two, Piaget's (1956) thesis of topological primacy derives from his claim that the actions that elicit topological relationships are more elementary than those needed to abstract Euclidean relationships. Action is originally sensori-motor, and "this is why the first shapes to be abstracted are topological rather than Euclidean in character, since topological relationships express the simplest possible coordination of the dissociated elements of the basic motor rhythms, as against the more complex regulatory process required for coordination of Euclidean figures" (p. 68). Similarly, I also expected that the first class of shapes that participants in the study would name – by identifying the property that defines the class – would be the class of simple closed curves. But it wasn't until the eighth episode that Amanda provided a name for them. Given my expectation, and its consistency with Piaget's claim, I would conclude that Amanda was attentive to the property of "closedness" early on.

Thus, I believe that this finding is worthy of further attention. Perhaps Amanda was attentive to the property, but she lacked the language to express it. Or perhaps she was not attentive and it was only through engagement in the activities throughout the episodes that made the property sufficiently salient to her.

I conclude the episode by asking Amanda to describe two classes of shapes that similarly "overlap."

Interviewer: <Amanda has scribbled over her drawing of a shape that is "not a block," so I redraw it.> Do you have a name for shapes that do that? Like this one <I draw a lowercase gamma>?
Amanda: That's – this <gamma> would be "swoosh."
Interviewer: And that [*Figure **38***]?
Amanda: This would be "guard," 'cuz you know how some people have, like, their sword? They go like this? <Amanda draws a person standing next to the "gamma.">
Interviewer: Do you see something similar about those two [*Figure **38*** and the "gamma"]?
Amanda: Uh, yeah.
Interviewer: What is that?
Amanda: They overlap.

I review the names she's given to other classes of shapes, and then we consider the equivalent "hourglass" and "figure 8."

Interviewer: Alright, so we have cherries, worms. These [the class that contains the "figure 8" and the "hourglass"]?
Amanda: <No response>

This marked the conclusion of the teaching experiment with Amanda.

Summarizing Amanda

These were the enacted themes that characterize each of Amanda's episodes. They describe the elements of the experiment's teaching trajectory.

1. *Everything's alike.*
2. *Multiple conceptions of alikeness*
3. *Two shapes are alike if one can be transformed into the other.*
4. *Developing equivalence.*
5. *Given a shape, identify an equivalent one.*
6. *Identify the property that defines the class.*
7. *What's equivalent is determined by what doesn't change – by what remains invariant.*
8. *Given a class of shapes, identify the property that defines the class.*

At the beginning of the teaching experiment, Amanda described how all of the shapes she was given were alike using resemblance-, attribute- and transformation-based conceptions of alikeness. She used morphing language to describe how any one shape could be transformed into any of the others. Some of these transformations were topological, while others were not.

As she progressed through the episodes of the teaching experiment, she moved further away from attribute- and resemblance-based conceptions of alikeness toward the assimilation of alikeness as determined by the rules of the environment. As those rules were correspondent to the axioms of qualitative geometry, this progression indicated the development of qualitative reasoning. For example, by Episode 4, Amanda was much more likely to say that a circle and a square were alike than she was when the experiment began.

By the conclusion of the experiment, Amanda had developed a conception of alikeness as "transformational" that was so robust that when she was asked if two shapes were alike, she would respond by describing the transformation of one into the other. The development of this conception prepared her to describe fundamental properties of equivalence classes of shapes and to use a class-defining property to either identify or draw new shapes that rightfully belonged to a given class.

The development of Amanda's conception of alikeness from the first to the final episode of the teaching experiment, and the forms of alikeness with which she operated in the later episodes of the experiment, are findings that will be collected with the findings from the second participant's teaching experiment to help me answer my second research question, which I restate here to remind the reader:

> Question 2. What forms of topological or non-metric geometric ideas are made visible and can be seen to develop as a result of young learners' systemic engagement with a computer environment that makes topological representations and transformations accessible?

CASE 2: THE STORY OF EVA

"They're all in the family."

At the time of the study, Eva was six years old and in first grade.

Episode 1: Curvilinearity is salient

In this episode the pretest was administered using the seven triads (*Figure **29***) and the five-card sorting task. In three out of four triads composed of "curvilinear" and "rectilinear" shapes (rows A, C, D, and G), Eva identified the curvilinear shapes as "most like." Only in the case of row D was the form of the compositions of those shapes more salient than curvilinearity. In addition, in both of the five-card sorting tasks, the distinction between the two groups she formed was according to the curvilinearity or rectilinearity of shapes. In the first of these tasks, she sorted the X, Y, and T [row 5] into one group and the circle [B2] and "figure 8" [D1] into the second group, Similarly, in the second task, she sorted the "open square", diamond, and "double square" [A3, B3/F2, D3] into one group and the "S", circle, concentric circles, and "figure 8" [D2, A1/C3, D1, B2] into the other. Then she added the previously chosen X, Y, and T [row 5] to the first group and the "arc" [C1] into the second group. Eva's "strong" distinction between shapes based on curvilinearity was so salient that it suggested and informed the design of the follow-up shapes (*Figure **31***) which are composed of both curvilinear and rectilinear attributes.

In her justifications of "most like," Eva described transformations involving cutting, squooshing, bending, unfolding, unwrapping, pushing, curving, and straightening. For example, she said, "you can just straighten" the "arc" [C1] and the "S" [C3] into the segment [C2], "you can just curve" the segment [C2] into the "arc" [C1] or the "S" [C3], "you could just bend" the branches of the X [E3] into the Y [E1], and "you could push" the ellipse [G2] to form the rectangle [B1]. These transformations served to justify the alikeness of shapes with curvilinear attributes, the alikeness of shapes with rectilinear attributes, and the distinction between the two groups. It is also worth noting that Eva, like Amanda, in her descriptions of the alikeness of the "arc", the "S", and the segment, offered descriptions of both the transformations and inverse transformations of each into the other that formal topological equivalence requires.

The language of transformation figures quite prominently in Eva's justifications of the alikeness of shapes, but it is not the nature of the transformations that determines alikeness, it is the forms of the shapes themselves. This is an attribute-based conception of alikeness that should be seen as conceptually and developmentally prior to a property-based one, which would depend on the transformations defined by the geometry (e.g., Euclidean, projective, topological). In other words, for Eva, "curved shapes" are distinguished from "straight shapes," and the general form of a transformation that produces one from another is essentially, "whatever it takes to turn one shape into another one like it." This, too, was the conception of alikeness with which Amanda operated as she sorted all five cards into one group in Episode 1.

Episode 2: Resemblance to everyday objects and canonical shapes is primary

The second round of the pretest was administered using the "follow-up" set of shapes (*Figure **31***). Eva chose two cards at random from the follow-up set [H2, H6] and three at random from the pretest triads (B3, C1, G2, of *Figure **29***). She sorted the shapes into the three groups shown in *Figure **39***. This arrangement of the groups is just as Eva had arranged them on the table.

Figure 39. Eva's first sorting of her first five-card sorting task (Episode 2) [left to right: B3, H2, H6, C1, G2]

These groupings are striking in that they appear to be well informed by a conception of alikeness as curvilinearity-based. That is, the leftmost group consists of a single shape (the diamond), which is a polygon (consisting of only straight sides). The rightmost group (the arc and the ellipse) consists of "curvilinear" shapes. The grouping in the center has rectilinear and curvilinear attributes. When I asked Eva, "How are those [arc and ellipse] alike?", she reflected for a moment and then re-sorted the shapes multiple times. Eventually, she settled on the two groups shown in *Figure **40***.

Figure 40. Eva's final sorting of her first five-card sorting task (Episode 2) [left to right: B3, H6, G2, C1, H2]

Interviewer: How come you put these together?

Eva: 'Cuz they both have corners and they both aren't as curvy as these [in the other group] are. Well, this one [H6] is kinda curvy, but that's o.k.

Interviewer: OK, and what about those three <in the other group>?

Eva: These, they're all really the same. See? If you put a magnifying glass on this [ellipse, G2], the bottom would look like this [arc, C1]. And if you cut off that [the arc] <Eva places her straightened hand across the arc, from one endpoint to the other>, it kinda looks like that [H2], except it's more stubby... and round. They're all in the family of this [ellipse, G2]. They're all a different half of [it].

In the midst of this sorting task, Eva seems to have shifted schemes for sorting from the attribute-based distinction of curvilinearity that she employed in Episode 1 to an attributed-based distinction that also considers the role of corners. The shapes in the pretest triads could not have provoked this latter criterion, since no shapes in that set are composed of both curves and corners.

Later in the episode, Eva similarly sorted a diamond [B3] together with a triangle [B2], because each "has corners." The epsilon [H5] was distinguished from the diamond [B3] and the triangle, because it has "no corners." In contrast, Eva grouped it with the segment [C2] and H4, because they are "made of lines." In another sorting task, Eva's reasoning shifted. Whereas earlier she identified the diamond and triangle as alike in that they have corners, a distinguishing criterion also identified by Piaget (1956), she also identified them as “ordinary,” and, thus, as distinct from “weird” shapes (Darke, 1982, p. 121) (in the follow-up set of shapes (*Figure **31***)), such as H3, which she described as “like nothing I’ve ever seen before.” In Episode 9, she expresses a similar sentiment when she describes “shapes” as “things you see all over the place.” Otherwise, figures are “not shapes.”

Eva also employed resemblance-based criteria for distinctions when grouping the “double square” [D3], “figure 8” [D1], and H3, which are “all masks.” In the task that followed, she grouped the “rounded square” [H1] with the “double square” [D3], because they’re “both squares.” Similarly, in the final sorting task of the episode, when Eva was asked to consider another participant’s groupings of five shapes (*Figure **41***), she said she agreed that the T [E3] and the epsilon [H5] belong together, because “they’re both

Figure 41. The final five-card sorting task of Episode 2 [from left to right: H5, E3, A1/C3, C2, H3]

kinda letterish.” Then she added, “they’re both lines, but I would have put this one [the T (E3)] with this one [the segment (C2)],” which had been grouped with the “S” (A1/C3)]. When asked, “Why do you think she put these [segment (C3), “S” (A1/C3)] together?”, Eva replied, “You could just straighten this [“S”] out and it would be this [segment]; you could just squiggle that [segment] a little bit and it would be this [“S”].” This moment is salient in that it marked the only instance in the episode in which Eva used a “transformational” conception of alikeness.

In Episode 1, Eva used morphing language to describe the alikeness of the topologically equivalent “arc” and segment, and the T and Y. In this episode, she grouped the topologically equivalent “masks” as well as the also-equivalent diamond and the triangle, but did not use morphing language to explain either of the groupings. This was her only use of morphing language in Episode 2, but it is not clear what provoked such a distinction. Like many children completing tasks like these (Lehrer, Jenkins, et al., 1998), Eva’s reasoning varied across tasks. It seems that the other participant’s groupings (*Figure **41***) of the “S” and the segment – a grouping that Eva says she *would not* have made – provoked the consideration of a “transformational” conception of alikeness. Thus, Eva operates from conceptions of alikeness that are attribute-based (e.g., “has curves,” “has corners,” “made of lines”), resemblance-based (e.g., to everyday objects or canonical shapes), and “transformational,” although the latter is less salient than the other two.

In this second episode, Eva identified topologically equivalent shapes as alike (e.g, the arc and the segment, and the T and the Y) although it was resemblance that determined the alikeness and not topological properties. In the following episode, the plan was for Eva to use *Configure* to transform the shapes in each of these equivalence classes.

Episode 3: Two shapes are alike if one can be transformed into the other.

In this episode, Eva used *Configure* for the first time, and through her activities, she became more familiar with the ways it worked. The theme of the episode was Eva's assimilation of what it means to be alike according to the rules of the software. The shapes she transformed were identified as alike in the prior episode using attribute- and resemblance-based conceptions of alikeness.

Whereas Amanda and Eva used attributes to determine the alikeness of – or distinction between – shapes, the *properties* of shapes that we're interested are qualitative. They are the non-metric transformations that are defined by the tools of *Configure*. To get a sense of the relationship between attributes and properties, consider that the T and the Y are similarly "connected," but the X is not. The form of connectedness of those shapes is a topological property, and it is one that is readily apparent in the transformational capacities of *Configure*: the T can be transformed into the Y (and vice versa), but neither can be transformed into (or from) the X.[31]

Following Eva's transformation of the T into the epsilon, I move toward that conception of alikeness by explaining, "These are alike, because you could take that one [the T] and you could turn it into that one [the epsilon]." In subsequent episodes, the focus will also move to the properties of shapes identified as equivalent through transformations. Next, I present Eva with the segment and the "S."

---

[31] It can also be determined by considering the effect of removing the "center" point of each of the letters. In the case of the T and the Y, removing that point leaves three pieces. In the case of the X, four pieces are left.

Interviewer: So one of the other questions I asked you was about these two shapes. How did you say that they're alike?

Eva: Because, if you curve that one [the segment], it would become that one. If you straighten that one [the "S"], it would become that one.

Then I present her with the "arc" [C1], the ellipse [G2], and the arch [H2] (*Figure **42***), which she identified as "all really the same" in the previous episode, because "they're all in the family of" the ellipse. When she explains that in order to transform the arch into

Figure 42. The ellipse, the arc, and the arch

the "arc," she would "push that [the eraser button] and then erase that [the segment-ed component of the arch] and turn it upside down and make it a little narrower," I clarify how we'll determine that two shapes are alike by reminding her, "I want to know which ones we can make into the other ones *without* gluing or erasing." Then I repeat the task.

Interviewer: Can you turn one into another without erasing or gluing?

Eva: If I could just – Hmmm. <moves the cursor to the ellipse> You can't get all these things off? That's impossible!

Eva identifies the ellipse and the "arc" as distinct, because the transformation of the ellipse into the "arc" requires that part of the ellipse be removed. In contrast, after completing the transformation of the ellipse into the arch by altering the curvature of one of the two arcs that forms the ellipse, she declares, "I'm finished."

In the final task of the episode, I was interested in the extent to which Eva identifies a property of two equivalent shapes. I presented her with the “sail” [first shape of row 1, *Figure **32***] and the triangle [fourth shape of row 3, *Figure **32***].

> *Interviewer*: What do you think about those two shapes? Do you see anything that’s alike about them?
>
> *Eva*: It [the sail] kinda looks like a triangle, a curved one.

Whereas in Episodes 1 and 2 Eva rarely used a conception of alikeness as “transformational,” in this episode she developed such a conception through multiple tasks in which she identified two shapes as alike because one could be transformed into the other. The next efforts are seen as promoting the development of qualitative reasoning by provoking Eva’s attention to properties of equivalent shapes. Her remark that the “sail” looks like a “curved” triangle is suggestive of such a property.

Episode 4: Given a shape, identify an equivalent one.

In the prior episode, Eva used the software to determine whether one shape could be transformed into another. The theme of this episode was developing the notion of topological equivalence by considering that equivalence is a function of invariant properties. That consideration requires not only the determination of the “transformational” alikeness of two shapes, but also attention to the properties of shapes that are “transformationally” alike.

At the conclusion of the previous episode, Eva remarked that the "sail" [first shape of row 1, *Figure **32***] "kinda looks like a triangle, a curved one." To follow up, I asked her to consider the arch [H2] and that same triangle.

Interviewer: What do you see that's alike about these two shapes?
Eva: This [arch] is kinda like a triangle. If you made that pointy, it would be a triangle.

Eva used curvilinearity as a criterion for distinction all throughout the pretest in Episode 1. In contrast, these observations that the arch looks like a "curved" triangle, and the triangle is like a "pointy" arch, draw on curvilinearity as a criterion for alikeness. Because *Configure* was new to Eva in Episode 2, and because curvilinearity is "lost" in that non-metric microworld, this "reversal" seems to have been supported by Eva's explorations in the software environment. This is salient, because it suggests that Eva's qualitative conceptions are developing as a result of those explorations. It also suggests a response to my first research question in terms of the capacity of the environment for supporting the development of children's qualitative reasoning.

Next, I ask Eva to consider the X, the Y, and the T. In the pretest of Episode 1, she chose the Y and the T as more alike than the X, because you can "bend" the T to get the Y and "straighten" the Y to get the T.

Interviewer: What makes these two shapes [T, Y] more alike than this one [X]?
Eva: Yeah, because -- swing that out [branches of Y] -- those two -- and it'll be like that [T]. Bend those [branches of T] like that and it'll be that [Y].
Interviewer: You can't do anything with that [X]?

Eva: Yeah. If there was one [branch] on top of the other [the Y or the T], you could open those up.

Interviewer: Can you show me with that Y?

Eva transforms the Y into the T.

Interviewer: And what about the X? Can you make any of those [T, Y] into the X?

Eva: <rotates three branches of the Y to make room for a fourth branch> But I need another one [branch] to do that [turn the Y into an X].

Interviewer: That's what we're interested in, whether you need to add pieces or take pieces away. So do you see that the only way to do it would be to add a piece?

Eva: Yeah. <grabs a new segment and attaches to the Y to form the X> There! Perfect.

Interviewer: Alright, so what did you have to do to make it into an X?

Eva: All I needed was one of these [another built-in segment] and that [a T], and then I could turn it into that by using glue and turn and drag.

This exploration in which Eva required glue in one instance but not in another was seen a preparatory for the subsequent conversation in which I inform her that we will no longer be using glue to attach pieces in our transformations.

Eva: [If you can't use glue,] then how *do* you do it?

Interviewer: That's the question, can you do it or not? Sometimes you can and sometimes you can't. Can you see a way to do it?

Eva: Can *you*?

Interviewer: I'll ask you first and then I'll tell you what *I* think.

Eva: Uhhh, no.

Interviewer: No, it's not possible.

Eva recognizes the Y and the T as distinct from the X in that the now-allowable transformations do not include gluing. Taking this realization together with her many experiences transforming one shape into another, my next attempt is to investigate the extent to which Eva considers aspects of shapes that make the transformation of one into the other possible.

Interviewer: Earlier, you did this [arch] and you turned it into a triangle. Right?

Eva: Yes.

Interviewer: Alright, so I want to know if you can draw another shape that you can turn into a triangle, here [on paper] or if you want to use that [computer] you can draw it on there. Draw a shape that's not a triangle but that you can turn into a triangle.

Eva: I could make a square, couldn't I?

Subsequently, Eva wants my help to unglue the square, but I remind her that gluing is not allowed. The rule still seems arbitrary, but Eva doesn't resist. Instead, she grabs a vertex and moves it around as if she believes it might be useful but she had no plan for it. Then, a drag of a vertex of the square results into two sides converging into one.

Eva: Is it like that? *That's* a kind of a triangle. There!

I continue explorations on the theme by presenting Eva with the epsilon [H5].

Interviewer: Pick one of these on the [pretest] page and tell me which one you can turn it into.

Eva: Maybe if I work really, really hard for a while, I could make it probably into – Maybe I could make into that [inverted "lollipop" (G1)], maybe, if I could just move that [outer arcs of the epsilon] into a circle.

Interviewer: How would you turn it into a circle?

Eva struggles for about a minute until she realizes that forming the circle would require gluing.

Apparently, Eva chose the "lollipop," because she had imagined how the components of the epsilon could correspond to the components of the "lollipop." Then, she realized that the transformation that would produce the "lollipop" could not be achieved without gluing. Next, I present her with the "clover" [second shape, row 3 of *Figure **32***].

Interviewer: Can you make it into one of these shapes [on the pretest]?

Eva: I *could* make it into a circle.

Interviewer: Alright, let's see it.

Eva completes the transformation.

Interviewer: Alright, let's do it again. [I give her a second clover.] Can you find another one?

Eva: I could turn it into a triangle?

Interviewer: Really!?

Eva: Uh huh. <She completes the transformation.>

Eva's confident identification of the circle and the triangle as transformable from the "clover" is significant. To say that Eva identified the clover as equivalent according to the rules of the environment is probably not going too far. She has had multiple experiences transforming one shape into one that is "alike." Furthermore, she has had experiences where a failed attempt to transform one shape into another is assimilated as, or at least compatible with, a "non-alikeness" of those shapes and, at times, a consideration of the "forbidden" transformations of gluing and breaking, which, when required, also indicate that the two shapes are not alike.

It may even be the case that in addition to operating with a conception of alikeness as "one can be transformed into the other," Eva's identification of the circle and the triangle as transformable from the "clover" points to a consideration of a common, fundamental property of the three shapes. But there is insufficient evidence to support the claim, especially since Eva was not able to transform the epsilon into her choice of the seemingly transformable "lollipop," which occurred just prior to these transformations, could arguably be seen as counterevidence. Thus, the next episode is designed, in part, to gather more evidence of property-based of reasoning (or at least scaffold its development).

Episode 5: Developing equivalence

The plan was to provoke Eva to attend to properties of shape by considering what it is that makes two figures homeomorphic. The episode began with Eva using the software to transform shape H6 into a diamond [B3], which was significant, but only relative to the complexity of the task. On paper, however, she sorted the segment [C2] together with the "S" [A1/C3] and subsequently justified the grouping by describing the transformations that would take one into the other, as she had done several times before. Still, none of the conversations got to what it was that made those transformations possible. In the following episode, the effort to provoke Eva's attention to properties is continued.

Episode 6: Developing equivalence

The effort to attend to properties is made in order to support the development of Eva's qualitative reasoning. Whereas she has demonstrated successful transformations of equivalent shapes, and has even confirmed her predictions that the transformations were possible, she has yet to articulate the properties of those equivalent shapes. This is likely because she was operating from a conception of alikeness which means that one shape can be transformed into the other. So when she is asked whether two shapes are alike, or even if one can be transformed into the other, she simply considers what actions would be necessary in order to turn one into the other. Worth noting, this consideration is obviously guided at least in part by mental operations on shape. Thus, it is likely that Eva's plans for transforming one shape into the other are substantive but not fully developed *a priori*. For if they had been, she wouldn't eventually be struck by a realization that, in the case of two shapes that are nonequivalent, gluing or ungluing is required to complete the transformation. I witnessed this phenomenon late in Episode 4 in Eva's inability to transform the epsilon into her choice of the seemingly transformable "lollipop." Two other examples occur in this episode. In the first, which occurs at the beginning of the episode, Eva asks if she can turn the "sail" [first shape of row 1, *Figure **32***] into a theta [second shape, row 2 of *Figure **32***]. In the second, which occurs at the end of the episode, Eva asks, "Can I turn a kite [B3] into a fork [fourth shape of row 1, *Figure **32***]?" When I ask her, "What makes you think you can do it?", she responds by initiating the transformation. Ultimately, she realizes she would need to glue components of the fork to form the kite.

The following excerpt illustrates an attempt to investigate Eva's developing conception of equivalence. The shapes are equivalent and the invariant property is "closedness."

1 *Eva*: I can turn *that* [the heart, third shape of row 2, *Figure **32***] into a square?

2 *Interviewer*: What makes you think you can turn it into a square? How did you

know that?

Eva: Because.

Eva completes the transformation.

Interviewer: I want to know if you can describe how you knew you could take that [heart] and turn it into that [square]?

Eva: I knew if I could turn them straight and be able to drag a lot, I'd be able to make a square.

Interviewer: Alright, so straightening and dragging lets you make it into a square.

Eva: Uh huh.

Interviewer: What about from here [square] to here [triangle]?

Eva: I knew that if I just put this like that <gestures to indicate the dragging of a vertex to reduce two sides into one> and stretched it a little bit, I would be able to make that [triangle].

At this point, I was interested in the extent to which Eva attends to properties – or at least common attributes – of equivalent shapes, in particular, "closed" shapes. Unfortunately, my phrasing of the question in line 2 is regrettable, since it clearly reinforces Eva's consideration of nothing other than the form of the transformation that determines the alikeness of two shapes.

In the tasks that followed, Eva explored other "closed" shapes. She transformed a triangle into a square and a square into a pentagon, each of which involved "adding a corner." I collected these shapes along with some of the other simple closed curves she explored in this episode on a sorting worksheet (*Figure **43***). The theta was crossed out

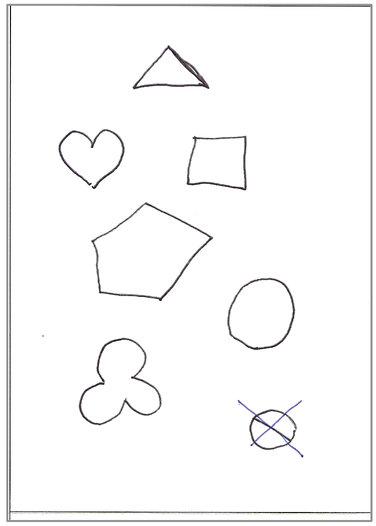


Figure 43. The set of simple closed curves explored by Eva at the conclusion of Episode 6

when Eva realized the "sail" could not be transformed into it. When I asked her, "What makes this shape [the theta] different from all these?", she did not respond. She played instead. So my plan for the subsequent episode was to follow up on Eva's considerations of "closed" shapes, like the "kite", square, and triangle, using activities that make properties of equivalent shapes more salient.

Episode 7: Identify the property that defines the class.

I begin this episode by presenting Eva with a diamond [B3] and a circle [A2]. I chose these two shapes to follow up on her considerations of "closed" shapes in the prior episode, and also because her distinctions between shapes on paper have so often been based on curvilinearity. My thinking is that if she's asked to comment on what *is* alike about two shapes that she earlier identified as not alike – one of which is "rectilinear" and the other which is “curvilinear” – she will be encouraged to “move past” that distinction and consider other criteria.

Interviewer: What’s alike about those?

Eva: Well, I can say that– I can’t really explain. What if you disappeared that --- maybe these corners [of the diamond] disappear. Let me see. <Eva goes to the computer. Apparently, she wants to remove the corners of the diamond, but *Configure* does not have such a tool.>

Interviewer: What are you trying to do?

Eva: I’m trying to take the corners away.

Interviewer: So how do you take the corners away?

Eva: Is there a “corner take away”?

I show her how to bend the sides in a way that essentially removes the corners.

Eva: It would turn into a circle! See? There.

Eva used “corners” as a distinguishing characteristic in each of the first two episodes. During the pretest, she identified the “S” [A1] and circle [A2] as more alike, because they are “curved,” and the “open square” [A3] as different, because it “has corners.” Then, in the second episode, she grouped H6 with the diamond [B3], and the diamond with the triangle [B2], because each group is composed of shapes that “have corners,” whereas the other shapes in each of the tasks did not. So, not only did this task require that Eva ignore the curvilinearity-based distinction between the two shapes, it also gave her the experience of identifying two shapes as alike, which she earlier identified as different. This is significant, because it represents a conceptual move toward the assimilation of a qualitative conception of alikeness.

Eva has employed attribute-, resemblance-, and “transformation”-based conceptions of alikeness. But another version of alikeness is instantiated in the tools of the software. Thus, the following tasks are designed to elicit from Eva a word that is similar to “alike” and that describes shapes that can be transformed into each other. Toward that end, I appeal to her to help me find words that name the shapes she has identified as alike using the software. Those are presented in the sorting worksheet in *Figure **44***.

I explain to Eva that the use of the word “alike” to describe shapes that can be transformed into each other hasn’t worked so well, because, for example, she has described “curvy” shapes as alike. But she also knows that the “curvy” “S” [A1] and the “straight” segment [C2] are alike, because she has transformed one into the other “about 189 times.”

> *Interviewer*: I need another word I can use to describe shapes that I can turn into each other.
>
> *Eva*: <referring to the set that contains the “S” and the “segment” [upper right quadrant of *Figure **44***]> They’re both lines.
>
> *Interviewer*: They’re both lines? We can use the words “both lines.”

This response marks the first time Eva has given a name to an equivalence class of shapes.

> *Interviewer*: You [transformed] this one earlier, the kite into the circle [the two shapes in the second row of the class of shapes in the upper left quadrant of *Figure **44***].
>
> *Eva*: They’re both simple.
>
> *Interviewer*: They’re both what?
>
> *Eva*: Simple.

Interviewer: Should I use the word “simple” to talk about these? What about the heart [the first shape in the first row of the class of shapes in the upper left quadrant of *Figure* ***44***]? Is that simple?

Eva: No.

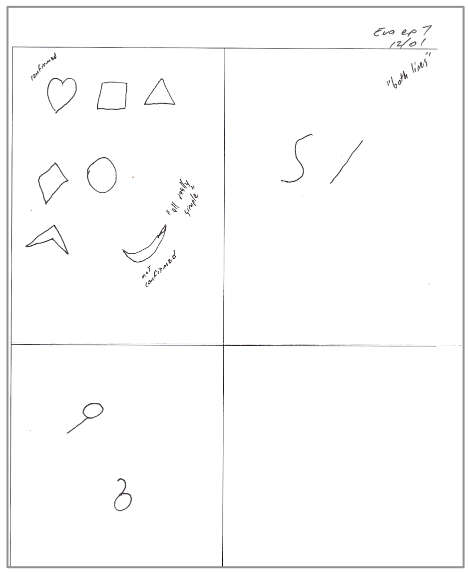


Figure 44. Eva’s sorting worksheet (Episode 7)

Eva goes on to explain that the circle cannot be transformed into the heart, because the circle “doesn’t have these two lines [the two arcs that form the heart] that could turn curvy. It just is a plain circle.” When I suggest that she attempt the transformation, she says she “[doesn’t] think it’s gonna work out so good,” which certainly seems to indicate that she has developed no plans for a suitable transformation. Still, she attempts a transformation and eventually completes it by performing a relatively complicated move, using the “add a corner” tool to turn one of the arcs that composes the circle into the two that form the heart. So, perhaps it is the complexity of a suitable transformation of the circle into the heart that informs Eva's conception of topological alikeness. To explore this conjecture, in a later episode I will guide Eva in an exploration of properties of equivalent shapes in ways that do not depend on the consideration of a transformation that turns one into the other.

I missed the opportunity to explore Eva's conceptions of "simple" when I asked her whether the heart is simple, so I follow up on it in Episode 9. My conjecture at this point is that "simple" refers to convex canonical or "near canonical" "closed" shapes given Eva's rejection of the heart, which she essentially identified as distinct from the "simple" (and "plain") circle. This conjecture is informed by Piaget's investigations (1956) and the replicates which, in their own ways, made some reference to a distinction between simple and more complex shapes. For example, as I wrote earlier, given Piaget's (1956) claim that "topological relationships express the *simplest* [emphasis added] possible coordination of the dissociated elements of the basic motor rhythms, as against the more complex regulatory process required for coordination of Euclidean figures" (p. 68), he might have argued that the actions needed to reflectively abstract the form of the circle are more elementary that those required of the heart, whose actions require relatively greater ability to abstract and coordinate. Next, we consider "open" shapes before returning to "closed" ones.

Interviewer: What about these ["S", segment]? Can you think of a word that describes how these are alike?

Eva: They're both lines!

Interviewer: They're both lines. Alright. So what word do you want to use over here [in the class of closed shapes]?

Eva: Mmmmm. They're all really simple shapes, and they're not like, uh, they're all not like a "swoosh" [last figure in the bottom row of the class of shapes in the upper left quadrant of *Figure **44***].

Similar to the way she thought about the heart, which she identified as distinct from the "simple" circle, Eva sees the "swoosh" as also distinct. Perhaps the identification of the heart and swoosh as distinct from the circle is a matter of concavity. Even so, it is reasonable that such a distinction be left to an issue of the "complexity" of shapes and thus ignored since it's non-topological nature deems it less worthy of our attention.

On several occasions in this episode, Eva's attention to properties of shapes was distracted by a requisite consideration of the forms of transformations of one shape into a (supposed) equivalent. So as to redirect her attention, in the following episode, Eva will engage in activities that support the identification of topological properties of shape.

Episode 8: What's equivalent is determined by what doesn't change – by what remains invariant.

Until now, Eva has been tasked with identifying fundamental properties by class membership (whose members were determined by successful transformations). She has only slightly been able to do so, probably because, as I mentioned above in Amanda's case (because she experienced a similar difficulty), the identification of a property that defines a class is a newly developing ability for children at this developmental level (Inhelder & Piaget, 1969, p. 17). In the following episodes, however, Eva's attention to properties will be supported not by attending to the transformations that confirm or deny the equivalence of two shapes, but by activities informed by a conception of a topological property as one that remains invariant under bicontinuous transformations.

Where I present the *SoundTrack* and *Configure* software environments in Chapter Four I mentioned that dynamic geometry environments provide opportunities for reasoning by continuity that enact Poncelet's Principle of Continuity. This principle guarantees that properties of shapes remain invariant through dragging. Accordingly, Eva will be asked to identify those attributes of shape that do not change upon dragging. In more formal terms, she will be asked to identify properties that remain invariant under bicontinuous transformations.

In this segment of Episode 8, I present Eva with several of *Configure's* built-in shapes and we have a conversation about "what doesn't change" as I drag components of the shapes around and apply tools to them (which have the equivalent effect of dragging, like "adding a corner" and "changing a segment to an arc").

Interviewer: I'm going to drag this [arch] around. I can stretch and bend things. Can you tell me what *doesn't* change as I move this [the segment that composes the arch] around?

Eva: The line doesn't change.

Interviewer: I can change its length, right?

Eva: You just can't change that it's a line.

Interviewer: I can change it with this ["segment to arc"] button. I can bend it and change its length. So can you tell me something about it that *doesn't* change?

Eva: Oh, the dots. You can't move the way they're shaped.

Interviewer: That's true. Good. I'm looking for things like that.

My inference that Eva's use of "line" in line 6 refers to the segment that composes the arch was incorrect given that she also uses "line" in the following two excerpts to refer to the bending branches of the spiral and the curved-then-straight arcs that compose the "S".[32]

Interviewer: Let's look at this one [the spiral]. OK, I can change these [branches'] lengths, I can change how much they bend.
Eva: But they're always lines and you can't break the lines.
Interviewer: Alright, they're always lines and I can't break 'em.
Eva: Yeah, you can't break 'em off.
Interviewer: They'll always be attached right there <at the center>?
Eva: Yeah.
Interviewer: Great. That's the thing I'm looking for. <then> I can make 'em straight, I can bend, I can turn this into the T like we talked about earlier, but they'll always be attached at the middle.
Eva: Mm hmm.
Interviewer: How many are attached at the middle?
Eva: 3. They'll always be 3.
Interviewer: What stays the same about this one [the "S"]?
Eva: There's always 2 lines. <The "S" is composed of two arcs.>
Interviewer: I can even add another one.
Eva: They're always curved!
Interviewer: I can make 'em straight.
Eva: OK.
Interviewer: You said there's two, but I can always add another one, right? <I "add a corner" so that the "S" is now composed of 3 arcs. Then I drag it around.> So what's always the same about this?

[32] Coincidentally, the common conception of a "curve" is any path, whether straight or curved.

33 *Eva*: They're all lines!
34 *Interviewer*: What's the same about this ["figure 8"]?
35 *Eva*: There's always two circles?
36 *Interviewer*: There's always two circles, alright.
37 *Eva*: Unless you erase it, which is not allowed.

Every shape has some attributes that remain invariant and some that do not. Thus, given the right/wrong quality of the questions, I felt that the activity would be most productive if I moved closer to participant than observer. For example, it was Eva who observed that the arcs that compose the "S" are "always curved." It was I who, in response, adjusted their curvature to make them straight so as not to leave Eva with the possible impression that curvature is invariant. At one point, in fact, I moved closer to teacher than participant by pointing out where the branches of the spiral are attached (lines 16, 20, 22) and asking Eva, "How many are attached…?" (line 22).

Not every line of geometrical exploration could be explored. Each episode saw several opportunities go uninvestigated. For instance, it is not true that the simple closed curves that compose the "figure 8" must always be circles. Curvilinearity is not a topological property. Still, the excerpt does reveal Eva's attention to the invariance of the arch's dots (an unfortunate consequence of the software's design), the "unbreakable" quality of the arcs of the spiral, the "line"-d quality of the "S", and the "two"-ness of the "figure 8." Significantly, each of these has a conceptual parallel in the topological property of connectedness.

As a sort of follow up assessment, I gave Eva a collection of shapes on a sorting worksheet. These were shapes I had also given to Amanda, who described them as shapes “made of stems.” To Eva, they’re “made of lines.” The worksheet as it appeared at the conclusion of the activity is shown in *Figure* ***45***.

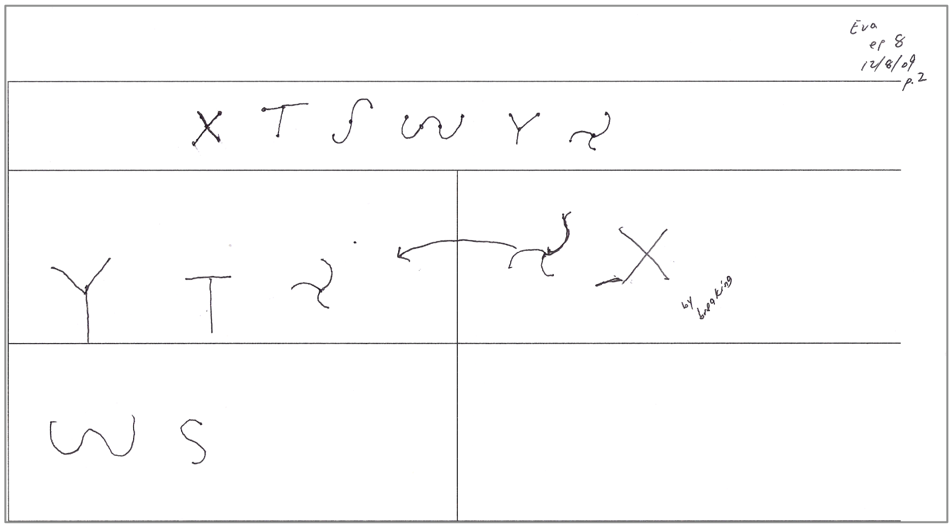


Figure 45. Eva’s final sortings of shapes “made of lines” (Episode 8)

Interviewer: Put the ones in the same box that can be turned into each other.

Eva: <thinking out loud> I could turn –

Interviewer: How would you do those two [the topologically distinct spiral and X]?

Eva begins to describe how she would use the software to accomplish the transformation.

Interviewer: Alright, let’s do it [on the computer].

Eva attempts to transform the X into the spiral but quickly realizes that she would need to “subtract” one of its branches to complete the transformation. Then she moves the spiral from its initial group on the sorting worksheet to the group that contains the Y and the T.

Eva: I could turn it into the Y and then turn the Y into the T.

Eva’s use of the software to realize the distinction between the X and the spiral may suggest that she was operating, at least at one point, from a more visual-holistic conception of alikeness. Her sorting of the spiral with the X, however, *does* suggest a significant conceptual move from earlier attributed-based sorts that depended entirely upon curvilinearity. Also, it's certainly worth noting[33] that the distinction between shapes made of 3 and 4 "branches" is not readily apparent.

Quick! Sort the letters E *and* F, T, *and* X. *How many equivalence classes did you get?*

Although both Amanda and Eva used the software to identify the distinction between nonequivalent shapes "made of stems" (Amanda) or "made of lines" (Eva), neither ever verbalized it. With the exception of the spiral, the groups Eva formed are topological equivalence classes. When I asked her describe a transformation between X and the spiral, she moved to the software for help, but then she no longer needed it to realize she "could turn [the spiral] into the Y and then turn the Y into the T."

Episode 9: Given a class of shapes, identify the property that defines the class.

In Episode 2, Eva described a group of shapes "made of lines," language she used throughout the episodes of the experiment. In Episode 8, which was designed primarily to support Eva's attention to the properties that determine the equivalence of shapes, she recognized the "unbreakable" quality of the arcs of the spiral, the "line"-d quality of the "S", and the "two"-ness of the "figure 8." In this final exploratory episode, a sorting

[33] It's also worth noting that Eva is upset that the school makes her eat "that fake American cheese" on field trips" and that they make her sing "Five Little Snowmen."

activity is designed to assess Eva's qualitative property-based reasoning. The sorting worksheet as it appeared at the conclusion of the episode is shown in *Figure **46***.

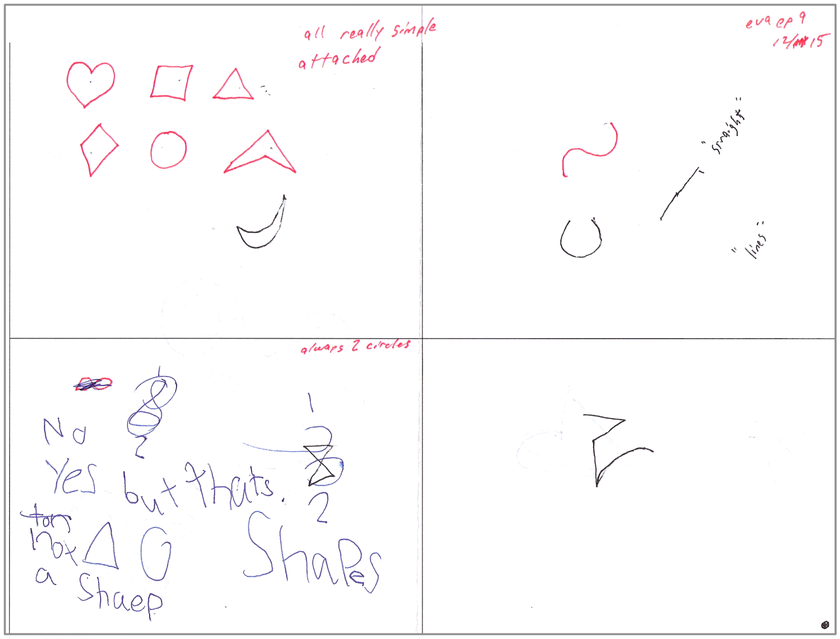


Figure 46. Eva's sorting worksheet as it appeared at the conclusion of Episode 9

We begin by considering the group of simple closed curves in the upper left quadrant of the sorting worksheet. In Episode 2, Eva said the diamond and the triangle were "ordinary," in contrast to shapes that she described as "like nothing I've ever seen before." In Episode 7, she said that the kite and the circle, also in this class, were "simple," but that the heart was not. My conjecture then was that "simple" referred to convex canonical or "near canonical" shapes.

Interviewer: At one point you said these are "all really simple." I'm wondering if you're still ok–

Eva: <reading my notes on the sorting sheet> Simple. A–

Interviewer: Attached.

Eva: Attached. What do you mean by -- oh, it has no broken lines. They all have a space in the middle.

Interviewer: They all have a space in the middle?

Eva: Yeah. They all have a space.

Interviewer: What about that [“figure 8”]? Does that have a space in the middle?

Eva: It has *two* spaces.

Interviewer: And what about this [“S”]?

Eva: It has *none* space.

Episode 7 marked the first instance where Eva gave a name to a set of shapes. Those were shapes “made of lines,” and in Episode 8 she sorted them into equivalence classes, which indicated that she was using a qualitative conception of alikeness to complete the task. But this characterization of shapes having “a space in the middle” marks the first instance where Eva names a class of simple closed shapes. This accomplishment came late in the teaching experiment for Amanda, as well.

One might argue that Eva’s interpretation of “attached” was a simple translation given out of context, that is, *without* consideration of the class to which it was given. But this cannot be the case. “Attached” is Eva’s word. She used it in Episode 7 to describe “what stays the same” about the triangle, and in Episode 8 to describe “what doesn’t change” about the branches of the spiral. While a triangle *does* “have a space in the middle,” a spiral does not. So the interpretation Eva offered must have come from a simultaneous consideration of the class of shapes to which it was attached.

Furthermore, it’s worth considering that Eva’s earlier uses of “ordinary” and “simple” were given in the same sense. Then, if it turns out that the word “attached” that labeled the collection on the sorting worksheet prompted the connection and provoked Eva's interpretation – *which she states as if it is obvious* – this phenomenon would be significant in that it points to a qualitative geometric cognitive structure of the sort I've considered in this and in earlier chapters.

Coincidentally, the conception of "attached" that Eva gives to the class of simple closed shapes on the sorting worksheet (*Figure* ***46***) as having "no broken lines" and "a space in the middle" is remarkably similar to the formal definitions of conceptually similar terms given on one of the most popular websites about mathematics in the field. MathWorld (www.mathworld.com) defines a curve as a "one-dimensional continuum" (Wolfram Research, 1999-2010b) and a closed curve as one "which completely encloses an area" (Wolfram Research, 1999-2010a).

Next, we reconsider "open" shapes, that is, those that are equivalent to the segment Eva has just described as having no space in the middle.

Interviewer: Did we ever come up with a word for these kinds of ["U", "S", segment; *Figure* ***46***] shapes?

Eva: <NR>

Interviewer: I don't remember if we ever came up with a word for those.

Eva: Lines.

Interviewer <I label the set "lines">: Is this ["S"] like a line?

Eva: <NR>

Eva nods to affirm.

Interviewer: These are all lines?

Eva: Uh huh. But this one's [the segment] straight.

Eva gives the name “lines” to the set that contains the equivalent “U”, “S”, and segment. Recall that earlier she also used the term to describe the curved branches of the spiral. Given the curvature-tolerant property of these shapes, Eva’s “lines” are equivalent to the common conception of “curves” that I mentioned above – any path, whether straight or curved – and as provided by Wolfram MathWorld: “a one-dimensional continuum.”

In the following brief excerpt, we reconsider the “swoosh” [first shape of the last row of *Figure **32***]. In Episode 7, Eva distinguished between it and the shapes she described as “simple” [those simple closed shapes in the upper left quadrant of *Figure **44***]. At the time, I hypothesized that her distinction between it and the heart was a matter of concavity.

Interviewer: The swoosh. Does that have a space inside?

Eva: Yes. But it’s just not the same. I can’t see why I can’t think it’s the same! It’s just not the same!

Interviewer: Can you make that [“open” curve, *Figure **47***] to a swoosh without gluing or breaking?

Eva: I can’t glue, but I *can* connect. <She connects the endpoints of the open shape, but does not glue them.> There!

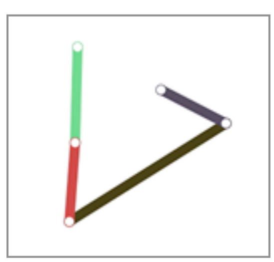


Figure 47. Eva is asked whether this shape can be transformed into the “swoosh.”

It was Eva's decision to connect the endpoints of the open shape to begin to form the swoosh. So apparently she realizes that the swoosh is "connected" whereas the open shape is not. So it needs to be.

Summarizing Eva

These were the enacted themes that characterize each of Eva's episodes. They describe the elements of the experiment's teaching trajectory.

1. *Curvilinearity is salient.*
2. *Resemblance to everyday objects and canonical shapes is primary.*
3. *Two shapes are alike if one can be transformed into the other.*
4. *Given a shape, identify an equivalent one.*
5. *Developing equivalence*
6. *Developing equivalence (cont.)*
7. *Identify the property that defines the class.*
8. *What's equivalent is determined by what doesn't change – by what remains invariant.*
9. *Given a class of shapes, identify the property that defines the class.*

At the beginning of the teaching experiment, Eva's primary criterion for determining the alikeness of shapes was curvilinearity, and the forms of the transformations she would use to justify the attributed-based conception of the alikeness of "curvilinear" or "rectilinear" shapes followed from, as opposed to determined, her identifications of "most like." In Episode 2, when new shapes were introduced to try and resolve or – *dis*solve – Eva's strong curvilinear-based distinctions, she was provoked to more strongly consider resemblance in her selections of "most like."

Then, in Episode 3, Eva's assimilation of the rules of the software environment supported a developing qualitative conception of alikeness, that is, as determined by the rules of the software. For example, her remark that the "sail" looks like a "curved" triangle is suggestive of that development, since neither shape is canonical nor everyday, and the "sail" has both curved and straight attributes, whereas the triangle is a polygon. These attributes had figured more prominently in Eva's earlier selections of "most like." Following the first round of activities conducted in the software environment, she realized the equivalence of the apparently distinctive circle and triangle, and identified the clover as also equivalent.

Eva spent two episodes operating with a conception of alikeness as "transformational," before eventually naming a class of shapes according to the property that determines the equivalence. Her use of "lines" to describe the class of "open" shapes was clear enough, but she did not (and perhaps, could not) elaborate on what she meant by "simple," the name she gave to the class of (simple closed) shapes equivalent to a circle.

Because Eva had used transformations so often and in such flexible ways in earlier episodes to describe the alikeness of two shapes, the effort was then made to move beyond it, or at least beside it, to identify the invariant properties of shapes. These activities supported Eva's identification of the "unbreakable" quality of the arcs of the spiral, the "line"-d quality of the "S", and the "two"-ness of the "figure 8." Whereas she had earlier sorted all shapes "made of lines" into the same group, Eva subsequently sorted them into equivalence classes for the first time.

Ultimately, Eva elaborated on her earlier use of "simple shapes" by describing them as those shapes that "have a space in the middle." Furthermore, she refined that property to distinguish between shapes having zero, one, or two holes.

Cross-case Analysis

My second research question seeks to determine what forms of qualitative geometric ideas are made visible and can be seen to develop as a result of participants' engagement with *Configure* as structured by the tasks of the teaching experiment. Those ideas are presented in each of the presentations of the participant cases above and in the summaries that follow them. In this section, I look across both cases for commonalities and divergence. The enacted themes that characterize each of Amanda's and Eva's episodes are listed here. An illustrative side-by-side comparison of these trajectories is also presented in *Figure **48***. The '~' indicates episodes with similar themes; the '=' indicates episodes with equivalent themes. A generic trajectory derived from these two trajectories, along with a brief description of the goals of each episode, appears in Appendix D.

Amanda's Trajectory

1. *Everything's alike.*
2. *Multiple conceptions of alikeness*
3. *Two shapes are alike if one can be transformed into the other*
4. *Developing equivalence*
5. *Given a shape, identify an equivalent one.*
6. *Identify the property that defines the class.*
7. *What's equivalent is determined by what doesn't change – by what remains invariant.*
8. *Given a class of shapes, identify the property that defines the class.*

Eva's Trajectory

1. *Curvilinearity is salient.*
2. *Resemblance to everyday objects and canonical shapes is primary.*
3. *Two shapes are alike if one can be transformed into the other.*
4. *Given a shape, identify an equivalent one.*

5. Developing equivalence
6. Developing equivalence (cont.)
7. Identify the property that defines the class.
8. What's equivalent is determined by what doesn't change – by what remains invariant.
9. Given a class of shapes, identify the property that defines the class.

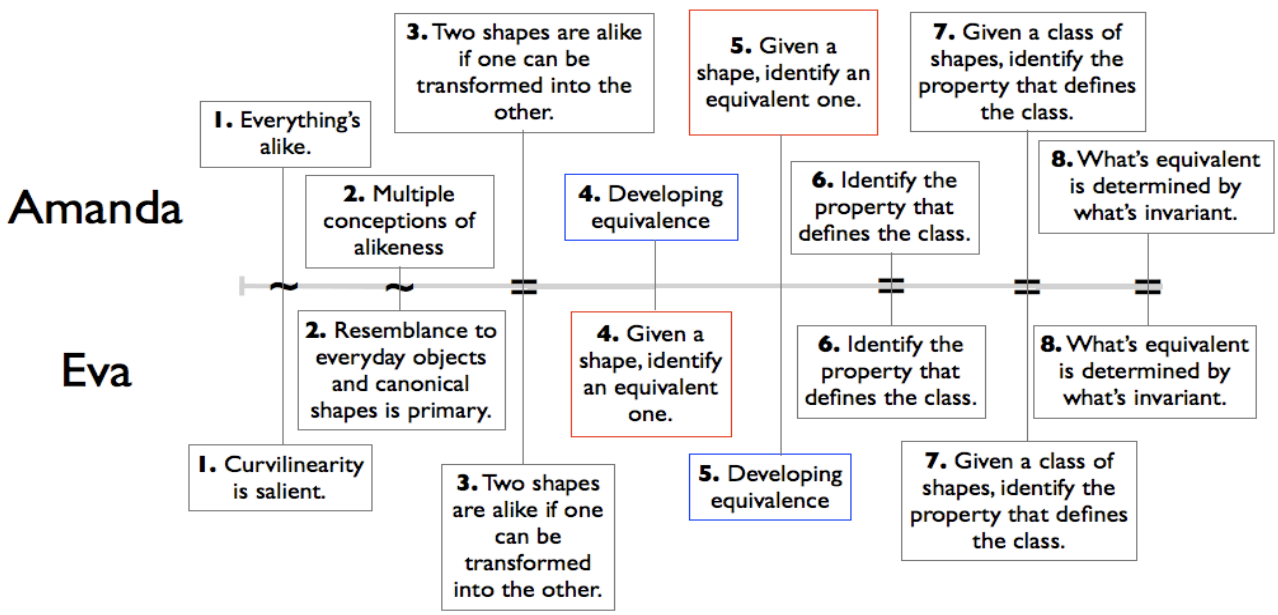


Figure 48. A side-by-side comparison of Amanda's and Eva's trajectories

I should restate that, in keeping with my participant-centered agenda for the exploration of non-metric ideas, with the exception of the pre-test which was the only task designed *a priori*, the design of the tasks that structured each of the episodes was informed by what happened in the previous episode or episodes. This consideration of participant's ideas accounts for both the similarities and distinctions between the two trajectories. The finding that they are so similar is, I believe, significant, and is not the result of my telling or leading them to a set of mathematically important insights or conclusions. In this way, it is similar to the "group stories" that organize Piaget's entire body of work (see especially Gardner, Kornhaber, & Wake, 1996b; Piaget, 1970a).

Worth noting, structural similarities between the two trajectories may be attributed primarily to two objectives. Given the goal of identifying and developing participants' non-metric conceptions, and the fact that this development was seen as being developed through participants' engagement with the software, there had to be occasions in which 1) participants would need to become familiar with the software, and 2) participants would need to be supported in their efforts to "get to reasoning." To do so required that they identify what, for them, are meaningful or task-relevant qualitative properties of shapes. When the efforts to move beyond conceptions of equivalence as "transformational" were not sufficiently productive, participants were tasked with identifying properties through the equivalent exercise of dragging. Still, these "moves" could not be accomplished unless 1) participants developed fluency with the software, and 2) their qualitative conceptions were identified and, in some sense, seen as useful to the participant. That they could also be developed is evidence of the common endpoint across the two trajectories. The very fact that this endpoint can be seen as mathematically significant lends further support to the sense that the activities these students were involved in was, and is, mathematical. It also resonates with NCTM's (1989) vision of a strands-based curriculum for school mathematics. That is, seeing geometry, for example, as a strand across the entire mathematics curriculum carries with it an attendant claim that young children have the capacity to engage early on in some mathematically authentic form of geometric reasoning. Furthermore, the features, changes, and outcomes of participants' sense-making are of-a-kind with that of mathematicians. Thus, the burden of proof would seem to shift to those disinclined to call this activity mathematics or mathematical. In the next and final chapter, I discuss the implications of these findings and possible directions for future research.

Chapter Six: Conclusions and Implications

Research Questions and Conclusions

By way of review, the first research questions guiding this investigation was:

> Question 1. Given that early forms of topological, or at least non-metric, geometric reasoning have been identified and discussed in the research literature, can a software environment be developed in ways that support fundamental topological representations and transformations such that learners' reasoning about topological ideas are made visible and are able to further develop in ways that could credibly be seen as both mathematical and significant?

In response to this question and based on the findings from the analyses reported in Chapter Five, I draw these conclusions. Given the software's role in supporting the development of participants' qualitative reasoning from intuitive and informal non-metric conceptions of alikeness to qualitative property-based reasoning, the answer to this question is, "Yes." First, the conceptual move from attribute- and resemblance-based conceptions of alikeness to qualitative conceptions of equivalence was supported by participants' transformations of shapes into equivalent ones and the development of equivalence classes of those shapes. Second, participants' identifications of properties that define equivalence classes could only be achieved once participants had developed those classes. Third, further development of qualitative reasoning was supported by participants' identifications of qualitative properties through "reasoning by continuity," or dragging, in a way that is consistent with the relevant and more formal mathematics.

As I mentioned in the conclusion to Chapter Four, the development of the final iteration of the software required an iterative process to ensure the integration of the mathematics content with a designed that supports the users' mathematical engagement with that content. This final iteration might seem "obvious" or simple, but such an assessment misrepresents the time, effort, and commitment to making improvements based on ongoing analyses. This finding leaves me suspicious of *a priori* or domain-general design principles such as those that inform the design of computer-assisted instruction environments.

Also by way of review, I restate my second research question here:

> Question 2. What forms of topological or non-metric geometric ideas are made visible and can be seen to develop as a result of young learners' systemic engagement with a computer environment that makes topological representations and transformations accessible?

A response to this question is provided in a summary of the participants' experiences throughout the teaching experiment. As early as the very first episode of the teaching experiment, both Amanda and Eva used the language of "morphing" in their very first episodes. For example, Amanda described how one shape could be transformed into another by untwisting, bending, or pushing. Similarly, Eva described transformations involving cutting, squooshing, bending, unfolding, unwrapping, pushing, curving, and straightening. And not only did they make the same sorts of attributed-based distinctions between shapes that Lehrer and colleagues (1998) identified in the performances of participants in their own study, Amanda and Eva also assimilated qualitative, or non-metric, conceptions of equivalence as they progressed through the episodes of the experiment. By their final episode, their qualitative conceptions of equivalence were developed to the extent that they were able to identify properties of equivalence classes of shapes and using a class-defining property to either identify or draw new shapes that rightfully belonged to a given class. Amanda's property-based distinctions were evident in the names she assigned to the classes of "cherries," shapes with "lines in the middle," "worms" (although initially mine), and "blocks." Eva made property-based distinctions, as well, specifically in her identification of the "unbreakable" quality of the arcs of the spiral, the "line"-d quality of the "S", the "two"-ness of the "figure 8," and the "space in the middle"-ness of shapes equivalent to a circle. These are not generalizations. Rather they are structural in character, since they are organized in terms of properties that are meaningful in relation to possible transformations. These properties, then, are mathematically significant, which suggests that participants are, in fact, doing mathematics. It is not quite the mathematics with which university professors might engage, but the "family resemblance" (Wittgenstein, 1958, p. 32) is sufficient to warrant the label of "real" mathematics. Thus, it is indeed the case that software environments can and should be built that support this kind of authentic mathematical activity.

IMPLICATIONS FOR CURRICULUM DESIGN

Any subject can be taught effectively in some intellectually honest form to any child at any stage of development. (Bruner, 1963, p. 33)

Young children's experiences in geometry throughout elementary school are entirely Euclidean with a particular emphasis on naming shapes. As this study has revealed, children also have non-metric ideas. Through their engagement in authentic mathematical activities, these conceptions have been seen to develop in significant ways. This finding should be beneficial to students whose non-metric ideas have typically gone ignored, to teachers who wish to engage and extend their students' geometric ideas, to the community of mathematics educators whose research has only nominally investigated them, and to curriculum developers who wish to develop materials to support their development.

For most people, nothing is more natural than that the most advanced ideas in mathematics should be inaccessible to children. (Papert, 1980, p. 161)

By considering a domain of mathematics typically thought of as advanced and then investigating how the youngest children think about it, the domain can rightly be thought of as a vehicle for telling a possibly greater and more significant story, one that is not exclusively about geometry. This lends credence to the claim that not only children but *everyone* possesses powerful understandings of mathematical ideas and they can develop those understandings well in advance of their traditional placement in the standard university curriculum sequence. They might come to discover academic strengths and interests not known to them otherwise. They might even identify with the mathematics and see it as a deeply engaging and important form of human activity.

IMPLICATIONS FOR RESEARCH

Lehrer and colleagues (Lehrer, Jenkins, et al., 1998) reported in the results of their "most like" tasks using the triads (*Figure **22***) they had developed in collaboration with Clements and Battista that children's "reasoning varied from triad to triad and within triads" (p.140). In essence, these varying criteria for determining "which two are most like" were developed and applied haphazardly. As such, they are not reversible operations (Piaget, 1970b, p. 15).

According to Piaget's characterization of operationalized thought (Piaget, 1970b), the fundamental elements of cognitive structures are reversible operations. These operations typify an understanding of the structure. They give the structure its characteristic wholeness. In the case of qualitative geometry, as I mentioned earlier in my discussion of cognitive structures in Chapter Two, one form of reversibility looks like applying a qualitative transformation to a figure (without altering its qualitative properties), and then transforming it back. To illustrate this point with the general case, an equivalence class of shapes may be developed from the constructions of mental representations of shapes that are generated via the active sense of perception of possible transformations of that shape. Particularly, the equivalence class containing C and all the possible qualitative transformations of C – including V, S, and L – is defined by any element of the class and the kinds of transformations that are allowed. Taken together, the shape and the allowable transformations give the shape the property that defines the class. That's because a property is exactly an attribute of shape that remains invariant under transformation. Here, it is the non-metric quality of the geometry that determines the properties that define equivalence classes, and the equivalence class containing C, in particular, contains all simple open curves.

Just as participants in the study by Lehrer and colleagues (Lehrer, Jenkins, et al., 1998) had done, in the early episodes of the teaching experiment in which the pretest was administered, the reasoning of participants in this study varied within and across triads. As such, their ir-reversible nature *does not* implicate a cognitive structure. That is, there seems to be no geometric conceptual framework that structures their activity. Nothing about the nature of the task – i.e., the problems, the tools, and the environment – problematizes their assessments of most like. *Any thing goes*. There's nothing more to say about the differences between Amanda and Eva's selections of "most like" in a given triad other than to simply say, "They're different." Thus, even though it wasn't their intent to do so, the only conclusions Lehrer and colleagues could draw about the criteria their participants used to distinguish between shapes were about their form and their frequency. There could be nothing to make sense of, since sense making would require some sort of cognitive structure with which to guide or direct the sense making.

In contrast, the non-metric environment of *Configure* structured participants' selections of "most like." Like the arithmetic problem types that are central to CGI (Carpenter, et al., 1999), there are right and wrong answers. The "Join – Result

Unknown" problem, "Torin had 5 snails. Odin gave Torin 6 more snails. Now how many snails does Torin have?", has a single correct answer. These problems have a specific goal, an "agreed upon endpoint" (Stroup, Ares, Hurford, & Lesh, 2007). And these endpoints are justified by well-understood, shared standards of sense-making (and also by fundamental theorems of arithmetic). Otherwise it wouldn't be possible to provide teachers with the strategies children are likely to use to solve them. Similarly, the non-metric quality that structures participants' activity in *Configure* generates a well-defined conception of alikeness that is inherent to the space and is likely taken as shared by anyone who plays in the space. And this conception of alikeness, or qualitative equivalence, determines the properties that define the alikeness. So when Eva and Amanda use this conception of alikeness to determine which shapes are alike, and then develop equivalence class of these shapes, there's something to talk about. For example, it makes sense to talk about Amanda's developing conception of qualitative equivalence as evidenced by her selections of "most like." And it makes sense to analyze Eva's identification of a shape that she says is equivalent to a given shape. These sorts of analyses were not possible in the results of Lehrer and colleagues' (Lehrer, Jenkins, et al., 1998) investigation. In fact, these sorts of analyses might not be so rich in a Euclidean environment, since Euclidean equivalence classes contain only congruent figures. Shapes that are most like a given shape in such an environment are those that are exactly alike.

CGI (Carpenter, et al., 1999) is designed "to help teachers develop an understanding of their own students' mathematical thinking, its development, and how their students' thinking could form the basis for the development of more advanced mathematical ideas" (Fennema, et al., 1996, p. 404). The findings of this study can be similarly helpful to teachers. But instead of providing them with particular problem types and the strategies their students are likely to use to solve them, or even particular non-metric properties and the equivalence classes children are likely to generate, this study provides teachers with a sense of the course of development of their students' geometric reasoning (where geometric reasoning is conceived as the application of geometric properties and relationships in problem solving). Whereas axioms of non-metric geometry framed the activities in this study, the development of geometric reasoning can equivalently occur in the context of other geometries, even the existing Euclidean space of elementary school geometry. The non-metric properties that Amanda and Eva identified are significant not so much in their form, but in the function they serve. That is, they are evidence of the development of their geometric reasoning.

POSSIBILITIES FOR FUTURE RESEARCH

This study is an initial foray into the development of qualitative geometry and the identification and characterization of young children's conceptions of it. Future directions that I might like to take this research include:

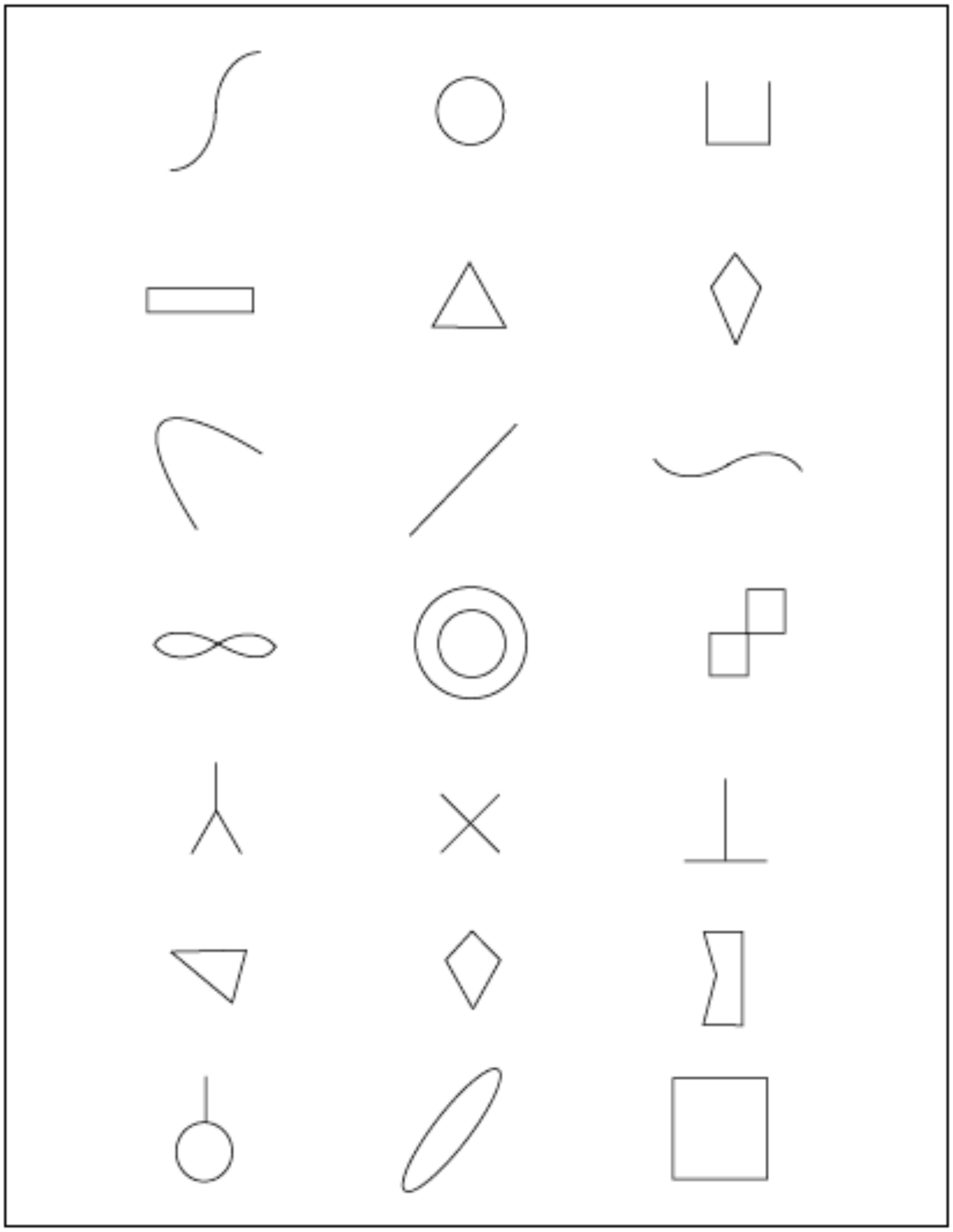
- Replicates of this teaching experiment to develop "superseding models" (Steffe & Thompson, 2000) of children's thinking in the domain of qualitative geometry. Simply stated, a superseding model performs better than a preceding one.
- Further characterization of the structure of qualitative geometry (Piaget, 1970b), (where structure refers to patterns of reasoning that learners come to enact).
- Further development of the software to better align with the fundamental properties of qualitative geometry and to better support young children's engagement with it (Barab & Squire, 2004; Brown, 1992; Collins, Joseph, & Bielaczyc, 2004; Kafai, 1999).
- The development of a hypothesized trajectory for the learning of qualitative geometry (Clements & Sarama, 2004; Cobb & Steffe, 1983).
- The development of new tools and activities to support learners' engagement with – and teachers' understanding of – the content of qualitative geometry (Scardamalia & Bereiter, 1991; Schwartz, 1995; Stroup, Ares, & Hurford, 2005; Vygotsky, 1978).
- A reconceptualization of traditional approaches to Euclidean geometry in elementary schools in light of this study's findings (Lehrer, Jenkins, et al., 1998).

- A progressive alignment of this research with other results of a similar kind, such as Stroup's qualitative calculus (2002), Papert's recursive structures (1980), and Schwartz's function-based algebra (Schwartz & Yerushalmy, 1992a).

I see this research agenda as significant for several reasons. First and perhaps foremost in terms of learners and their school experiences, this research and the trajectories it makes possible lie beyond the dominant model of geometry curriculum, a model that has been principally about identifying and matching names to canonical shapes and has historically ignored children's intuitive conceptions and offered them few or no opportunities to develop their geometric knowledge throughout elementary school. Second, qualitative geometry is a re/conception of topology that characterizes young children's conceptions of shape and privileges those ideas over the content of the canon of formal mathematics. Third, learners are provided with new opportunities to engage with the mathematics in ways that connect with the rich informal mathematical knowledge they bring with them to school. And fourth, the developers of standards and curricula are given the information they need to better align the curriculum to children's developing geometric thinking, and teachers, given the knowledge about how children's geometric ideas develop, are afforded new opportunities to engage and extend their students' thinking.

Appendix A: Pretest "triads"

(appears in the text as *Figure* ***29***)



Triads are named according to the rows (A to G) in which they appear.

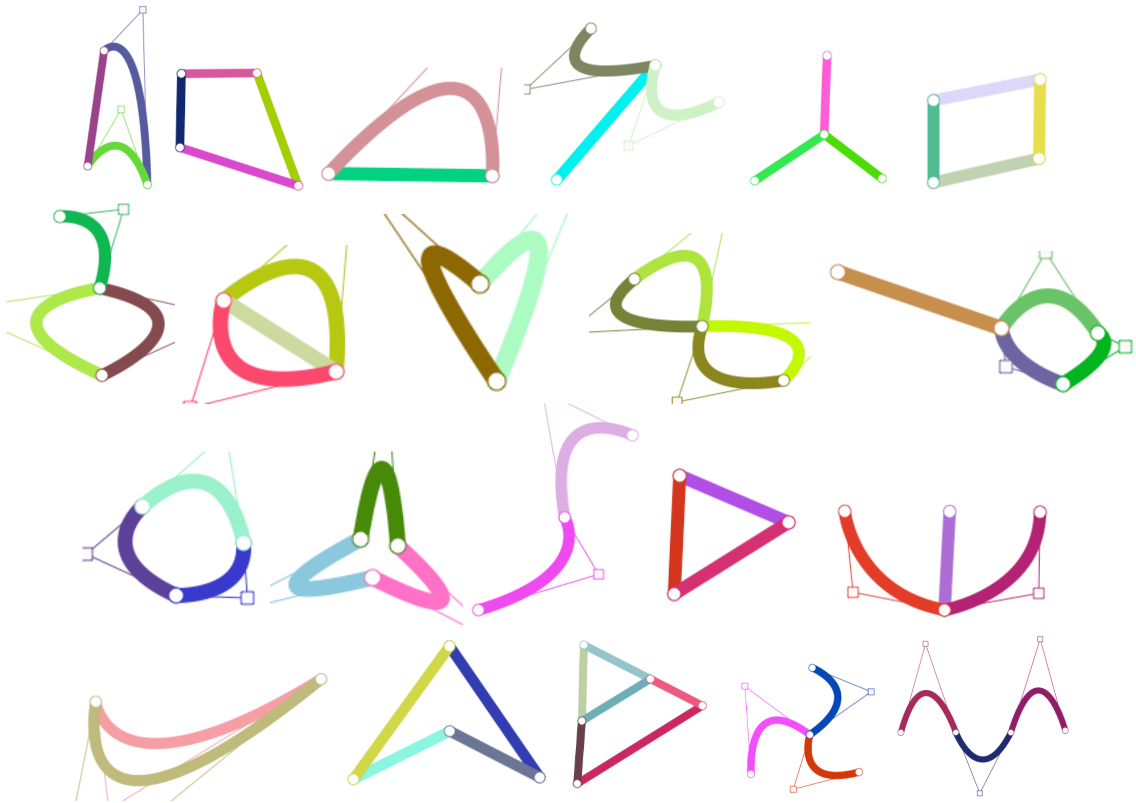
Appendix B: Pretest Follow-up Shapes

(appears in the text as *Figure* ***31***)

From top to bottom: H1 to H6.

Appendix C: *Configure's* built-in shapes

(appears in the text as *Figure 32*)



Appendix D: A Generic Trajectory of Teaching Episodes

Figure **49** shows a generic trajectory of teaching episodes. This "average" trajectory is derived from the trajectories of each of the two participants that participated in the teaching experiments that were reported in this paper. Thus, the trajectory describes the likely experiences of new participants in a similar teaching experiment.

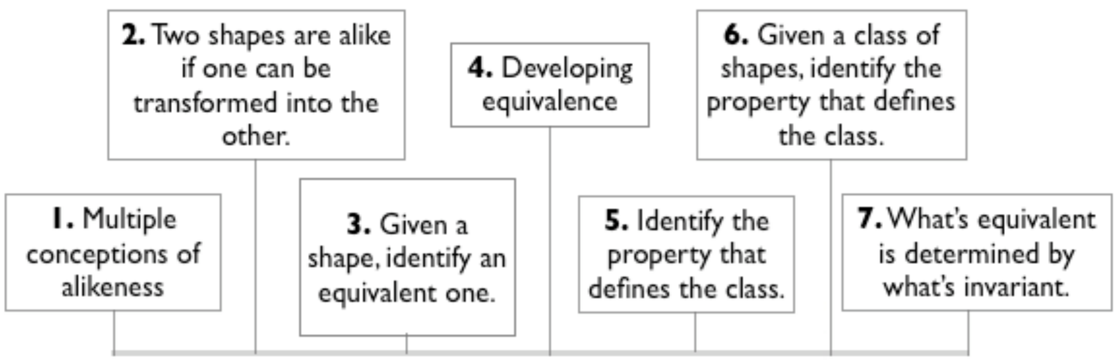


Figure 49. A generic trajectory of teaching episodes

The following list gives descriptions of the goals of each of the episodes shown in the generic trajectory. Note that episodes may last for more than one meeting. For instance, Episode 1 of the generic trajectory describes the theme of the first two episodes that occurred in the teaching experiments reported in this paper.

1. *Multiple conceptions of alikeness:* The pretest is administered.
2. *Two shapes are alike if one can be transformed into the other:* In this episode, the software is introduced and used for the first time. The plan is to explore a

conception of alikeness as defined by the rules of the software. This exploration is meant to support the goal of having the learner identify the properties of equivalent shapes.

3. *Given a shape, identify an equivalent one:* In this episode, the plan is for the learner to spend more time with the conception of alikeness as "transformational," that is, two shapes are alike if each can be transformed into the other. This is accomplished by further exploring the transformations the learner has described that can and cannot be performed using the software.
4. *Developing equivalence:* The plan for this episode is to make progress toward supporting the learner's attention to properties. The episode is essentially about transforming shapes in *Configure* and then reflecting on the outcomes. It is themed "developing equivalence," because it is this episode in which the learner becomes increasingly attentive to the properties of shapes that make transformations possible or impossible.
5. *Identify the property that defines the class:* The plan for this episode is to further develop a conception of alikeness as equivalence in the software environment and then attempt to provoke the learner's attention to prototypes of equivalence classes. Identifying a circle as a prototype for closed shapes or a segment as a prototype for open ones, for example, can support the learner's identification of the property that defines each class.
6. *Given a class of shapes, identify the property that defines the class:* Assuming that in the prior episode the learner identified a property that defines an

equivalence class, the primary focus of this episode is to continue that effort by developing appropriate names for the equivalence classes of shapes that have been produced throughout the previous episodes.

7. *What's equivalent is determined by what's invariant:* Thus far, the dominant strategy for provoking the learner's attention to qualitative properties of shape is by looking across shapes identified through transformations as equivalent. Instead, since a qualitative property is one that remains invariant through *Configure's* dragging, bending, and stretching operations, learners can operate on shapes in these ways in order to identify those properties. That method defines the theme of this episode.

Appendix E: Activity for Teachers

Each of the cooperating teachers whose students participated in this study happily agreed to give me time to work with those students at least once a week over the course of an entire semester. And they asked for nothing in exchange. To demonstrate my appreciation for their support, I promised to provide them with free copies of the software and a set of activities they could implement in their classes. The software was delivered when the data collection period had ended and the activity and an offer to implement it was delivered on May 1, 2010. This is that activity. It can also be downloaded from www.playwithshapes.com/activity.pdf.

Instructions:

- The following activity is to be implemented with your students. They will be using dynamic geometry software called *Configure* to complete the tasks. That software is free and web-based, and can be found at www.playwithshapes.com. I would suggest that students complete the activity in pairs, but a single student working alone could also have a wonderful and productive experience.
- To begin, a "sorting worksheet" should be drawn on a blank sheet of paper by dividing the paper into six or eight regions of equal size. Students will be drawing "alike" shapes in each of those regions.

- The instructions for the activity are present in the flow chart. Basically what the students will be doing is determining which shapes are alike according to the ways that the software will allow them to transform shapes. This way of thinking about "alikeness" is most likely going to be a new way of thinking about alikeness for your students. It is similar to the way it's thought about in an advanced mathematics course called topology. Essentially, two shapes are alike if one can be stretched and bent – without tearing or breaking – to form the other. This way of thinking about alikeness is why topology is informally referred to as "rubber sheet geometry." Basically, *Configure* is a tool that allows users to stretch and bend, but not break or tear, two-dimensional shapes. As students progress through the activity, they will be developing groups of shapes that are alike in each region of the sorting worksheet.

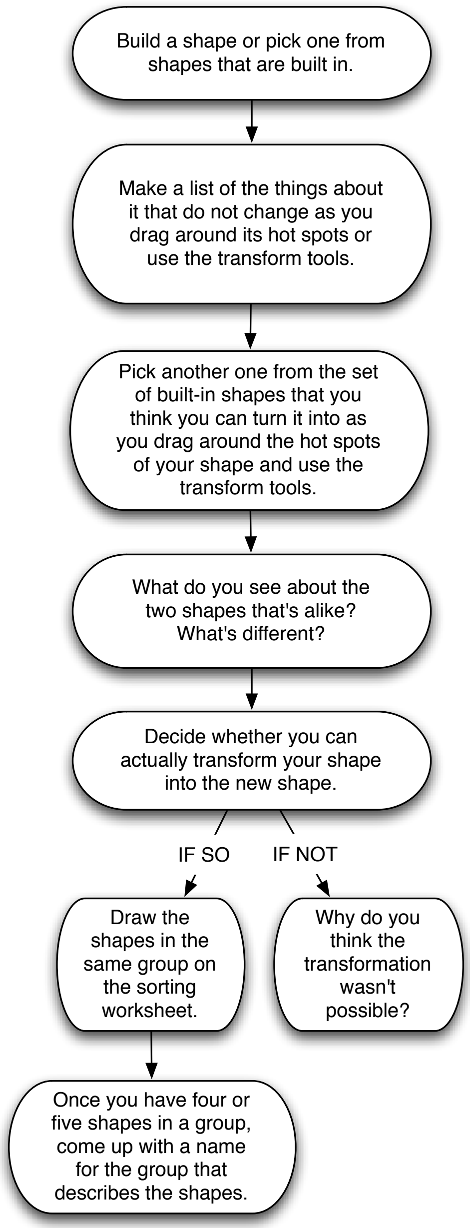


Figure 50. Activity for Teachers

Appendix F: Implications for van Hiele Theory

The focus of this study was not on the evaluation of the van Hiele model (van Hiele, 1959, 1986). But given the dominance of the model for describing the progression of children's geometric reasoning, I felt compelled to consider its fit, because my study is essentially an investigation of children's thinking in geometry. Briefly, the levels most relevant to this investigation are, on a scale from 0 to 4[34]:

- Level 0 (*Visualization*) refers to the stage in which learners recognize figures holistically, by their appearance, without any consideration of their properties. "That is a circle, because it's round like a cookie."
- Level 1 (*Analysis*) refers to the stage in which learners identify figures by their properties, but these properties are independent. That is, they realize that shapes have properties, but they do not see relationships between those properties. "That is a square, because it has four equal sides. Because it has four equal sides, it also has four equal angles."
- Level 2 (*Abstraction*) refers to the stage in which learners reason using properties of shapes and understand connections among properties. "Isosceles triangles have lines of symmetry, *so* their base angles must be congruent."

"Unlike Piaget's model of development, the van Hiele model ties development not to the growth of general mental structures, but to [experiences resulting from] *particular forms of instruction* [emphasis added]" (Lehrer, Jenkins, et al., 1998, p. 138). I present the clause in that form to stress an assumption of the theory that is too often forgotten or ignored. To dismiss the theory as viable in predicting the behaviors of participants in my study, I illustrate the differences between the design of the learning environment in which my study was carried out and those "particular forms." Furthermore, due to the dominance of van Hiele and its misuse to constrain the geometric

[34] The levels have also been presented on a scale from 1 to 5.

experiences of students in school, a more robust way to address my research in light of it may be required. In a canonical piece by Clements et al. (1999), the authors suggest that their finding of contradictions to the predictions of van Hiele "raises the question of whether the strictly visual prototype approach to teaching geometric shapes is a necessary prerequisite to more flexible categorical thinking or a detriment to the early development of such thinking" (p. 204). I have provided their response to the question in Chapter One when I make a case for this study, but it deserves to be repeated here: "The strictly visual prototype-only approach with limited exemplars should be rejected" (p. 208). Of greater concern in light of these findings is that van Hiele theory has, in fact, influenced the development of the geometry strand of the *Standards* published by the National Council of Teachers of Mathematics (NCTM), a group of "90,000 people who care about mathematics education" (NCTM, 2010).

Despite the high level of acceptance and veneration within the field, compelling arguments and empirical data call it into question (cf. Clements, Battista, Sarama, Swaminathan, & McMillen, 1997). Clements et al. (1999) simply state, "van Hiele theory does not adequately describe young children's conceptions" (p. 194). For example, a study by Lehrer and colleagues (1998) – situated in a context where the model should make its most viable predictions – offers evidence particularly relevant to this study. Their finding that children provide "morphing explanations" (p. 142) of similarity contrasts predictions by van Hiele, since children's distinctions "appear to defy description by a single, "visual" level of development." In addition, they found that "children referred to properties of figures or the number of vertices or "corners," as well as to classes of figures (squares and rectangles)." This varying attention to either visual components or properties of shape means "children's justifications often "jumped" across nonadjacent levels of the van Hiele hierarchy" (p. 142). This study produced similar findings that counter the model's predictions.

During the teaching experiment, Amanda was in first grade and Eva was in second. Prior to the study, both children had the kinds of geometric experiences characterized as "traditional" in Chapter One. In contrast, their experiences in this study could be described as "progressive." As such, the van Hiele model predicts performances that counter the theory. In particular, both Amanda and Eva provided "morphing explanations" throughout the experiment. The criteria they used to determine alikeness developed from attributed-based and resemblance-based to the qualitative sense instantiated in the design of the software. Then these qualitative conceptions were organized into schema that they then used to develop qualitative equivalence classes of shapes. In terms of van Hiele levels, their activity jumped across Levels 0, 1 and 2 (e.g., identifying properties and using them to develop classes) – a level that many students do not reach until grades four or five – throughout the 18 weeks of the teaching experiment.

Van Hiele theory offers a *model* with which to predict aspects of children's performances not a set of guidelines or expectations about what they're able to do. Taking the latter perspective naturally results in a self-perpetuating cycle of children doing what we expect them to do because that's all we allow them to do. This might account for the reason that throughout elementary school, children spend all their "geometric" time naming conventional shapes. The model does a mediocre job in contexts most similar to those "traditional" contexts in which it was developed (indeed, its Gagné-esque "skills before thinking" quality as evident in the "visual before reasoning" hierarchy seems deserving of the label), and it does not generalize well given how poor are its predictions beyond them.

Recall that part of the significance of this study is that it advocates for the generation of a space beyond those contexts. In tandem with activities such as the tasks implemented in this teaching experiment, the software alone provided such a context. And in that context participants engaged in rich and authentic mathematical experiences. If the van Hiele model is used to constrain children's geometric experiences to fit with its predictions – by informing standards, which inform the design of curricula, which influence teachers, we run the risk of cheating students out of these kinds of experiences.

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