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Some Constructions Involving Surgery on Surfaces in 4–manifolds

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Some Constructions Involving Surgery on Surfaces in 4–manifolds

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Dedicated to my family.

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Some Constructions Involving Surgery on Surfaces in 4–manifolds

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This dissertation concerns embedded surfaces in smooth 4–manifolds and especially surgery on those surfaces. These cut and paste operations are a powerful tool in the study of smooth 4–manifolds, and we study these operations in several new contexts. We give applications to several different areas of low-dimensional topology, including embedding 3–manifolds into 4–manifolds, broken Lefschetz fibrations, slice knots, and the relationship between knots and 2–knots.

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Chapter 1

Introduction

Surfaces embedded in 4–manifolds form a central object of study in low-dimensional topology. For example, properly embedded surfaces (with boundary) in B^4 relate to knot theory by the study of knot concordance and slice genus. In another direction, perhaps the best source of potential counterexamples to the smooth Generalized Poincaré Conjecture in dimension 4 are the manifolds obtained by the Gluck twist surgery operation on 2–knots (embedded S^2's in S^4). Indeed, the general phenomenon of *exotic* 4–manifolds (manifolds that are homeomorphic but not diffeomorphic) is amenable to the tools provided by surgery on surfaces. In fact, as a consequence of Wall's theorem [74] every smooth structure on a simply-connected 4–manifold X can be obtained by spherical surgery on $X \#_k S^2 \times S^2$ for some k, and by [10], every pair of simply-connected exotic 4–manifolds can be related by a sequence of surgeries on embedded tori. Furthermore, the smooth structure is often sensitive to the minimum genus of a surface representing a particular homology class (see, for example, [48] [62]).

Surfaces in 4–manifolds and surgery on those surfaces provide a unifying theme for this dissertation. Chapters 2 and 3 focus on the torus surgery operation on 4–manifolds, while Chapter 4 relies on information about the Gluck twist and a type of a surgery on a disk that doubles to a Gluck twist.

In Chapter 2 we study the result of torus surgery on tori embedded in S^4. Key questions include which 4–manifolds can be obtained in this way and the uniqueness of such descriptions. We show that the possible fundamental groups these manifolds realize is quite large. We also apply torus surgery and Gluck twists to construct embeddings of 3–manifolds. While many constructions of embeddings of 3–manifolds into 4–manifolds depend on handlebody techniques and branched covers of (doubly) slice knots, here we take an alternative approach using surgery on surfaces.

In Chapter 3 we study torus surgery in the context of singular fibrations on 4–manifolds. Lefschetz and elliptic fibrations have a long history of study, first in algebraic geometry and then in symplectic geometry, topology, and gauge theory. Indeed, performing torus surgeries on regular fibers of the Lefschetz fibration $E(1)$ on the manifold $\mathbb{CP}^2\#_9\overline{\mathbb{CP}^2}$ gave the first counterexamples to the 4-dimensional h-cobordism conjecture [19]. The resulting exotic copies of $\mathbb{CP}^2\#_9\overline{\mathbb{CP}^2}$ no longer admit Lefschetz fibrations; performing torus surgeries introduces *multiple fiber* singularities to the fibration map.

Inspired by the powerful correspondence between Lefschetz fibrations and symplectic structures due to Donaldson [20] and Gompf [29], a host of authors (initiated by the paper [8]) sought to extend these techniques to the near-symplectic case with what are most commonly known as broken Lefschetz fibrations (BLFs). In fact, it is now known that every closed 4–manifold admits the structure of a BLF (see [6], [9], [24], [52]), which consists of a map to a surface with only Lefschetz singularities and indefinite fold singularities. It has been a long standing hope to detect exoticness from the point of view of BLFs (see relevant work by Perutz [67] [68], and also Williams [75]). A natural set of manifolds to try to pursue this approach would be the exotic families $E(n)_{p,q}$ of elliptic surfaces (with multiple fibers of multiplicity p and q).

In Chapter 3 we construct the first explicit BLFs on the manifolds $E(n)_{p,q}$ by showing how to replace a multiple fiber singularity with a series of indefinite fold singularities (these singularities are stable and generic). The construction relies on decomposing a torus surgery into a series of round handle attachments. As an intermediary step we construct generic fibrations in neighborhoods of exceptional fibers in Seifert fibered spaces. It remains to be seen if these fibrations will be useful in detecting exoticness of the underlying manifolds.

Finally, Chapter 4 consists of joint work with Jeffrey Meier [49], where we study various types of slice disks (with potential relevance to the Slice-Ribbon Conjecture), and the interplay between knots and 2–knots. In particular we focus on slice disks that are fibered, that is, whose complements fiber over S^1 (the boundary will be a fibered knot). By interpreting monodromy changes as surgeries on surfaces in the fiber we give several interesting decomposition theorems for fibered disks and 2–knots.

Chapter 2

Surgery on tori in the 4–sphere

2.1 Introduction

[1] Given an embedded torus $\mathcal{T}$ with trivial normal bundle in a 4–manifold X, torus surgery on $\mathcal{T}$ (also called a *log transform*) is the process of removing a neighborhood $\nu\mathcal{T}$ and re-gluing $T^2 \times D^2$ by some diffeomorphism ϕ of the boundary to form $X_{\mathcal{T}} = X \setminus \nu\mathcal{T} \cup_\phi T^2 \times D^2$. Torus surgery is the operation underlying almost all examples of exotic 4–manifolds (see [22] for a nice overview). While torus surgery is a well-studied operation, most of the work has focused on tori embedded in elliptic surfaces (or at least in neighborhoods that admit a special elliptic fibration). Here we restrict to the case where the tori are embedded in S^4.

There are two natural 4-dimensional analogues to Dehn surgery on knots in S^3. The first is the Gluck twist operation [25] on 2–knots, and the other is torus surgery (it is known that surgery on higher genus surfaces is a trivial operation since the gluing map will always extend over the tubular neighborhood of the surface). Now the possible 4–manifolds obtained by a

[1]The majority of this chapter has previously appeared in the following preprint, which has been submitted for publication: Kyle Larson. Surgery on tori in the 4-sphere (2015). Preprint available at http://arxiv.org/abs/1502.06834 .

Gluck twist in S^4 are quite limited; by an application of Freedman's theorem [23] the result will always be homeomorphic to S^4. Furthermore, by a theorem of Iwase [42], the result of a Gluck twist can also be obtained by a certain related torus surgery, and so for these reasons torus surgery seems to be the appropriate 4-dimensional generalization of Dehn surgery.

However, there is an immediate impediment to proving an analogue of the powerful Lickorish-Wallace theorem for Dehn surgery: torus surgery always preserves the Euler characteristic and signature of the 4–manifold. Therefore, the relevant question is:

Question 2.1.1. Which 4–manifolds with Euler characteristic 2 and signature 0 can be obtained by surgery on a link of tori in S^4?

As a preliminary result in this direction we can show that a large class of groups can be obtained as fundamental groups of such 4–manifolds: we prove that any finitely presented group with non-negative deficiency appears as the fundamental group of a 4–manifold obtained by surgery on a link of tori in S^4. While we will see that it is also possible to obtain groups with arbitrarily large negative deficiency, it is known that not all groups can be obtained in this way. Now at present a full answer to Question 2.1.1 remains out of reach. However, a theorem by Baykur and Sunukjian [10] is relevant here. A consequence of their theorem is that any 4–manifold with Euler characteristic 2 and signature 0 can be obtained by a *sequence* of torus surgeries starting in S^4 (in particular it may be necessary to have intermediate 4–manifolds). Question 2.1.1 asks when is it possible to replace such a sequence with a single *simultaneous* set of torus surgeries.

We will consider various spinning constructions to provide nice examples. We will produce an infinite family of distinct tori that admit non-trivial surgeries to S^4, suggesting the possibility that manifolds obtained by a single torus surgery in S^4 never have a unique such description. We will also see that cyclic branched covers of spun knots can always be obtained by torus surgery in S^4, and that two spun knots are always related by torus surgery in their exteriors.

Finally, we examine surgery on the unknotted torus, and use this to show that the 3–manifolds obtained by p/q Dehn surgery on a knot in S^3 always embed in either $S^1 \times S^3 \# S^2 \times S^2$ or $S^1 \times S^3 \# S^2 \widetilde{\times} S^2$. If we puncture the 3–manifold then we can eliminate the $S^1 \times S^3$ connect summand. We also construct embeddings of 3–manifolds into S^4 by considering cross sections of Gluck twists. The statement of our result is simplest if we start with a *ribbon link* L in S^3. If M_L is the 3–manifold obtained by surgery on L with all the surgery coefficients belonging to the set $\{1/n\}_{n \in \mathbb{Z}}$, then we have that M_L smoothly embeds in S^4. M_L will be a homology 3–sphere, and so we see that this theorem allows us to construct embeddings for a large family of homology spheres into S^4. We finish by giving an alternative proof of a theorem of Gompf about embedding punctured homology spheres in S^4.

2.1.1 Organization

In Section 2.2 we give definitions and consider the basic algebraic invariants related to torus surgery. A discussion of several spinning constructions and their connection to torus surgery takes place in Section 2.3. Our results regarding fundamental groups appear in Section 2.4 in the context of interpreting torus surgeries as round cobordisms. Lastly, Section 2.5 contains our results about surgery on the unknotted torus and embeddings of 3–manifolds.

2.2 The basics

We will assume that all manifolds and maps are smooth, and that homology is calculated with integer coefficients unless otherwise noted.

2.2.1 Torus exteriors

A *surface knot* K is an embedded submanifold in S^4 that is diffeomorphic to some closed surface. When K is diffeomorphic to S^2 it is called a 2–knot. This chapter is concerned with the case that K is diffeomorphic to the torus T^2, and we will simply say that K is a torus in S^4 (henceforth we switch to the notation $\mathcal{T}$ for a torus in S^4). Let $E_{\mathcal{T}} = \overline{S^4 \setminus \nu\mathcal{T}}$ denote the *exterior* of $\mathcal{T}$. We can compute the homology of $E_{\mathcal{T}}$ by the long exact sequence of the pair $(S^4, E_{\mathcal{T}})$, using the isomorphism $H_n(S^4, X_{\mathcal{T}}) \cong H_n(\nu\mathcal{T}, \partial\nu\mathcal{T})$ from excision. The result is that $H_n(E_{\mathcal{T}})$ is isomorphic to $\mathbb{Z}$ for $n = 0$ or 1, $\mathbb{Z} \oplus \mathbb{Z}$ for $n = 2$, and 0 otherwise. The calculation shows that $H_1(E_{\mathcal{T}})$ is generated by the homology class of a meridian of $\mathcal{T}$ and generators of $H_2(E_{\mathcal{T}})$ are given by the *rim tori* $S^1 \times \{pt\} \times \partial D^2$ and $\{pt\} \times S^1 \times \partial D^2$ in $\partial E_{\mathcal{T}} = \partial\nu\mathcal{T}$ under the identification $\nu\mathcal{T} = S^1 \times S^1 \times D^2$ (and these tori have algebraic intersection number 0 in $E_{\mathcal{T}}$).

The fundamental groups of torus exteriors in S^4 (and for surface knots in general) have been widely studied. The collection of such groups includes all 2–knot groups, and hence all classical knot groups (for an overview see [16]). Among other things, it is known that this collection contains groups of arbitrarily large negative deficiency [53]. (The *deficiency* of a finite group presentation is the number of generators minus the number of relations. The deficiency of a group is the maximum deficiency of all presentations for the group.)

2.2.2 Torus surgery

Let $\mathcal{T}$ be an embedded torus in S^4. We want to think of $\mathcal{T}$ as a particular embedding of $S^1 \times S^1$ into S^4, so that we have fixed curves $\alpha = S^1 \times \{pt\}$ and $\beta = \{pt\} \times S^1$ in $\mathcal{T} \subset S^4$ (whose homology classes provide a preferred basis for $H_1(\mathcal{T})$). Note that it is possible for there to be infinitely many distinct isotopy classes of embeddings $S^1 \times S^1 \hookrightarrow S^4$ with the *same* submanifold as their image [38]. A framing for $\mathcal{T}$ is a particular identification of a tubular neighborhood $\nu\mathcal{T}$ with $T^2 \times D^2$. Given our fixed embedding, there is a canonical framing for $\mathcal{T}$, specified by requiring that the pushoffs $\alpha \times \{pt\}$ and $\beta \times \{pt\}$ in $T^2 \times \partial D^2$ are nullhomologous in the exterior $E_{\mathcal{T}}$ of $\mathcal{T}$. (Framings are identified with $H^1(T^2) \cong \mathbb{Z} \oplus \mathbb{Z}$, and since the first homology of the exterior is generated by a meridian we can twist the D^2 factor along α and β so that the pushoffs are nullhomologous.)

We now define the operation of interest in this chapter. *Torus surgery on* $\mathcal{T}$ is the process of removing $\nu\mathcal{T}$ from S^4 and re-gluing $T^2 \times D^2$ by a diffeomorphism $\phi\colon T^2 \times \partial D^2 \to T^2 \times \partial D^2$, using our canonical framing to identify $\partial\nu\mathcal{T}$ with $T^2 \times \partial D^2$. We will momentarily denote the resulting closed 4–manifold by $S^4_{\mathcal{T}}(\phi)$. Since $T^2 \times D^2$ admits a handle decomposition relative its boundary with one 2-handle, two 3-handles, and a 4-handle, we can construct $S^4_{\mathcal{T}}(\phi)$ from $E_{\mathcal{T}}$ by adding one 2-handle, two 3-handles, and a 4-handle. There is a unique way to attach 3- and 4-handles for a closed 4-manifold ([51], [61]), and so $S^4_{\mathcal{T}}(\phi)$ is determined up to diffeomorphism by the attaching circle of the 2-handle (the framing must be the product framing). The attaching circle will be the image of the meridian $\{pt\} \times \partial D^2$ under ϕ, and this is determined up to isotopy by its homology class $[\phi(\{pt\} \times \partial D^2)] = p[m] + a[\alpha] + b[\beta]$, where m is the meridian of $\mathcal{T}$. Therefore, given our fixed embedding of $\mathcal{T}$ and the resulting canonical framing, $S^4_{\mathcal{T}}(\phi)$ is determined up to diffeomorphism by the integers p, a, and b (and such a triple of integers can be realized if and only if they share no common factors). Hence we will denote a torus surgery on $\mathcal{T}$ by $S^4_{\mathcal{T}}(p, a, b)$. It turns out that the integer p is particularly important, and it is called the *multiplicity* of the surgery. If we think of $T^2 \times \partial D^2$ as $\mathbb{R}^3/\mathbb{Z}^3$, then we can represent our gluing map ϕ by a matrix in $GL(3, \mathbb{Z})$. Since the resulting diffeomorphism type only depends on the image of $\{pt\} \times \partial D^2$, we can choose the gluing map to be any integral matrix (with determinant ± 1)

of the form:

$$\phi = \begin{pmatrix} * & * & a \\ * & * & b \\ * & * & p \end{pmatrix}$$

Now there is another common notation to specify a particular torus surgery. The homology class $a[\alpha]+b[\beta] \in H_1(T^2)$ equals $q\gamma$ for some primitive element $\gamma \in H_1(T^2)$. We call q the *auxiliary multiplicity* and γ the *direction* of the surgery. Then specifying the multiplicity, auxiliary multiplicity, and direction determines the resulting diffeomorphism type of the surgery. If $q = 1$ we will say the surgery is *integral*.

The trivial surgery is $S^4_{\mathcal{T}}(1, 0, 0)$, which returns $(S^4, \mathcal{T})$. Note that if we choose different embeddings of the same $\mathcal{T}$ (thought of as a submanifold), we will get the same set of possible torus surgeries but the surgery data could be different. We will also consider surgery on links of tori (a collection of multiple disjoint embeddings of tori into S^4), and the resulting manifold will be determined by the surgery data for each individual torus surgery.

2.2.3 Algebraic topology

Next we examine the basic algebraic topology of $S^4_{\mathcal{T}}(p, a, b)$. Since we have already computed the homology of $E_{\mathcal{T}}$, we can compute the homology of $S^4_{\mathcal{T}}(p, a, b)$ using the long exact sequence of the pair $(S^4_{\mathcal{T}}(p, a, b), E_{\mathcal{T}})$. For this calculation it is useful to change our identification $\nu\mathcal{T} = T^2 \times D^2$ by a self-diffeomorphism of $T^2 \times D^2$ that is the identity on the second factor but on the T^2 factor sends the direction γ to $[\{pt\} \times S^1]$. Then we can choose our gluing map $\phi\colon\ T^2 \times \partial D^2 \to T^2 \times \partial D^2$ to be:

$$\phi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & q \\ 0 & d & p \end{pmatrix}$$

for some c and d satisfying $cp - dq = 1$. Following the calculation we see that for $p \neq 0$, $H_n(S^4_{\mathfrak{I}}(p, a, b))$ is isomorphic to $\mathbb{Z}_p$ for $n = 1, 2$, and vanishes for $n = 3$. Furthermore, $H_1(S^4_{\mathfrak{I}}(p, a, b))$ is generated by the original meridian m in $E_{\mathfrak{I}}$ and $H_2(S^4_{\mathfrak{I}}(p, a, b))$ is generated by the glued-in torus $T^2 \times \{0\}$. In particular, we observe that multiplicity 1 surgery produces a homology 4–sphere.

Similar computations show multiplicity 0 surgery results in a 4–manifold with the homology of $S^1 \times S^3 \# S^2 \times S^2$.

There is a simple relationship between the fundamental group of $S^4_{\mathfrak{I}}(p, a, b)$ and the fundamental group of the torus exterior $E_{\mathfrak{I}}$. We start with a presentation of $\pi_1(E_{\mathfrak{I}})$ and add a single relation corresponding to the attaching circle of the 2-handle.

2.2.4 Spin structures

Recall that a 4–manifold X is spin if and only if its second Stiefel-Whitney class $w_2(X) \in H^2(X; \mathbb{Z}_2)$ vanishes. If X is spin, the set of distinct spin structures can be identified with $H^1(X; \mathbb{Z}_2)$. For *odd* multiplicity p, we can calculate from the integral homology of $S^4_{\mathfrak{I}}(p, a, b)$ that $H^2(S^4_{\mathfrak{I}}(p, a, b); \mathbb{Z}_2) \cong H^1(S^4_{\mathfrak{I}}(p, a, b); \mathbb{Z}_2) \cong 0$. Hence $S^4_{\mathfrak{I}}(p, a, b)$ has a unique spin structure for odd p, regardless of the particular choice for a or b.

For even p the situation is more subtle. Here we follow Iwase [42]. Suppose we take a curve on $\mathcal{T}$ and push off using our canonical framing to obtain a curve c in $\partial E_{\mathcal{T}}$ such that $[c] = 0$ in $H_1(E_{\mathcal{T}}; \mathbb{Z}_2)$. Let c' be a pushoff of c in $\partial E_{\mathcal{T}}$ using the product framing of the boundary. Now let D and D' be 2–chains in $E_{\mathcal{T}}$ such that $[\partial D] = [c]$ and $[\partial D'] = [c']$ (mod 2), and D and D' intersect transversely. Then $q([c]) = D \cdot D'$ (mod 2) is a well-defined function. In fact, q is the Rokhlin quadratic form [69] (see also [60]) for $\mathcal{T}$ and so it satisfies $q([c_1]+[c_2]) = q([c_1]) + q([c_2]) + [c_1] \cdot [c_2]$ (mod 2). Furthermore, if the kernel of the inclusion map $H_1(\partial E_{\mathcal{T}}; \mathbb{Z}_2) \to H_1(E_{\mathcal{T}}; \mathbb{Z}_2)$ is $\{0, e_1, e_2, e_3\}$, then Iwase shows that $q(e_i) = 1$ for exactly one e_i (in other words the Arf invariant is 0 for tori in S^4; see also [69]). This motivates the following definition.

Definition 2.2.1. A particular embedding $S^1 \times S^1 \hookrightarrow S^4$ will be called a *spin embedding* if $q([\{pt\} \times S^1]) = q([S^1 \times \{pt\}]) = 0$. We will say the resulting torus $\mathcal{T}$ in S^4 is *spin embedded.*

Note that we can always change our embedding of a torus so that it is spin embedded. Now we can determine when the result of an even multiplicity surgery is spin. Iwase [42] worked this out for a special class of tori obtained by spinning torus knots in S^3, and in fact his proof works in this more general context, which we give here.

Proposition 2.2.2. $S^4_{\mathcal{T}}(p, a, b)$ is spin if p is odd. If p is even, assume that $\mathcal{T}$ is spin embedded. Then $S^4_{\mathcal{T}}(p, a, b)$ is spin if and only if $ab = 0$ (mod 2).

Proof. We saw above that $S^4_{\mathcal{T}}(p,a,b)$ is spin if p is odd, so assume p is even. The Mayer-Vietoris sequence with $\mathbb{Z}_2$ coefficients (we use these coefficients for the rest of the argument) gives us:

$$H_2(T^2\times D^2)\oplus H_2(E_{\mathcal{T}}) \xrightarrow{\Psi} H_2(S^4_{\mathcal{T}}(p,a,b)) \xrightarrow{\partial} H_1(\partial(T^2\times D^2)) \xrightarrow{\Phi} H_1(T^2\times D^2)\oplus H_1(E_{\mathcal{T}})$$

Now the kernel of $\Phi = \{0,[m]\}$ for a meridian m of $\mathcal{T}$. Hence we have an induced split exact sequence and isomorphism $H_2(S^4_{\mathcal{T}}(p,a,b)) = im\Psi \oplus \langle[D_m + D_\sigma]\rangle$ for D_m the class of the meridinal disk in $T^2\times D^2$ and D_σ a 2–chain in $E_{\mathcal{T}}$ bounded by the surgery curve $\sigma = \phi(\{pt\}\times\partial D^2)$ (whose homology class $p[\mu]+a[\alpha]+b[\beta]$ is 0 since p is even). Now the $\mathbb{Z}_2$ intersection form is trivial on $im\Psi$ and on $\langle[D_m + D_\sigma]\rangle$ we have $[D_m+D_\sigma]^2 = q([\sigma]) = a^2q(\alpha)+b^2q(\beta)+ab$ (mod 2). If $\mathcal{T}$ is spin embedded then this equals ab (mod 2) and so by the Wu formula we get that $w_2(S^4_{\mathcal{T}}(p,a,b)) = 0$ (and hence $S^4_{\mathcal{T}}(p,a,b)$ is spin) if and only if $ab = 0$ (mod 2).

□

2.2.5 Special neighborhoods

The fishtail neighborhood F and cusp neighborhood C are compact 4-manifolds that admit elliptic fibrations over the disk with a single fishtail or cusp singular fiber, respectively. We can describe handle decompositions for these manifolds as follows (see Figure 2.1). We start with a handle decomposition for $T^2\times D^2$, and to form F we add another 2-handle attached along a pushoff of an S^1 factor of $T^2\times\{0\}$, where the framing of the 2-handle

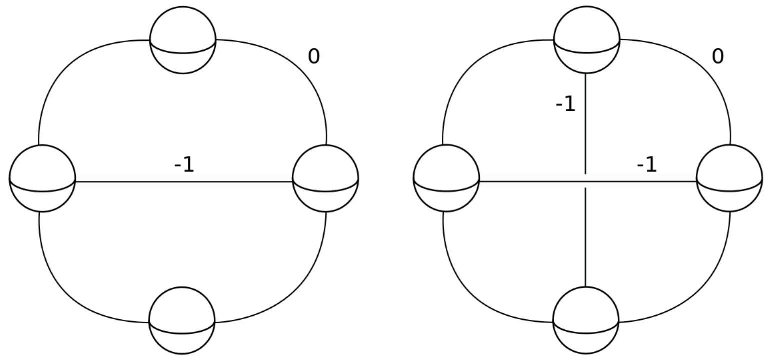

Figure 2.1: Here are handle diagrams for the fishtail neighborhood F (left) and the cusp neighborhood C (right).

will be obtained from the product framing of the boundary by adding a single left-handed twist. To form C we add one more 2-handle along a pushoff of the other S^1 factor of $T^2 \times \{0\}$, where again the framing will be given by taking the product framing and adding a single left-handed twist. The attaching circles for these extra 2–handles are called *vanishing cycles*.

Performing torus surgery on regular fibers of fishtail and cusp neighborhoods has been an important operation in the theory of 4–manifolds. In this context we have a fixed framing for the torus fiber coming from the fibration map, and so we have well-defined notions of the multiplicity, auxiliary multiplicity, and direction of the surgery as before. Here we state two theorems that demonstrate nice properties satisfied by torus surgeries in these neighborhoods. For proofs we refer the reader to [29] for the first theorem and to [26] for the second.

Theorem 2.2.3. The result of performing torus surgery on a regular fiber of a cusp neighborhood C depends only on the multiplicity p of the surgery, up to diffeomorphism relative to the boundary.

This theorem says that the result of torus surgery in C is independent of the auxiliary multiplicity or direction of the surgery. We observe that C does not admit an embedding into S^4. For example, by Proposition 2.2.2 we see that whether the result of performing an even multiplicity surgery on a torus in S^4 is spin depends on the auxiliary multiplicity and the direction. If such a torus was a regular fiber of a cusp neighborhood there could be no such dependence. However, there do exist embeddings of fishtail neighborhoods into S^4, and we will apply the following result in Section 2.3.

Theorem 2.2.4. The result of performing a multiplicity 1 surgery on a regular fiber of a fishtail neighborhood F, with direction given by the vanishing cycle, is diffeomorphic to F relative to the boundary.

2.3 Spinning constructions

First we introduce a nice family of tori in S^4. Let K be a knot in S^3. Remove from S^3 a 3–ball disjoint from K, and consider the resulting pair (B^3, K). Then we get a torus $\mathcal{T}_K$ in S^4 by taking $K \times S^1 \subset B^3 \times S^1$ in the decomposition $B^3 \times S^1 \cup_{\mathrm{id}} S^2 \times D^2$ of the 4–sphere (see Figure 2.2). Note that we get a different torus, denoted $\mathcal{T}'_K$, if we glue $S^2 \times D^2$ to $B^3 \times S^1$ by the Gluck twist map $\rho\colon S^2 \times S^1 \to S^2 \times S^1$. (Recall ρ is defined by sending (x, θ) to $(rot_\theta(x), \theta)$, where rot_θ is rotation of S^2 about a fixed axis through angle θ.) We will call $\mathcal{T}_K$ the *spun torus* of K and $\mathcal{T}'_K$ the *twisted spun torus* of K (Boyle calls $\mathcal{T}'_K$ a *turned torus* [13]). We will choose our embedding $S^1 \times S^1 \hookrightarrow \mathcal{T}_K$ (resp. $\mathcal{T}'_K$) so that α is identified with $K \times \{pt\}$ and β is identified with $\{pt\} \times S^1$ in $K \times S^1$.

For a nontrivial knot $K \subset S^3$, the exteriors of $\mathcal{T}_K$ and $\mathcal{T}'_K$ will have the same fundamental group (the knot group for K), but they are neither isotopic [55] nor have diffeomorphic exteriors [13]. We remark that the special case of spinning torus knots in S^3 (and surgery on the resulting tori) was extensively studied by Iwase in [42] and [40].

Proposition 2.3.1. Given a knot $K \subset S^3$, the twisted spun torus $\mathcal{T}'_K$ is a regular fiber of a fishtail neighborhood in S^4 with the vanishing cycle given by a push off of β.

Proof. Isotope K in B^3 so that a point $x \in K$ lies near the boundary ∂B^3. In particular, arrange so that $x \times \partial D^2 \subset \partial \nu K$ is tangent to ∂B^3 at a point $x_0 \in \partial B^3$. Now we obtain $\mathcal{T}'_K$ in S^4 by crossing (B^3, K) with S^1 and gluing in $S^2 \times D^2$ by the map ρ. In terms of handles, we glue in $S^2 \times D^2$ by first attaching a 2-handle h_2 along $\{pt\} \times S^1$ on the boundary $S^2 \times S^1$ with framing given by adding a left-handed twist to the product framing (if we glue by the identity map instead of ρ we would take the product framing), and then capping off with a 4-handle. We can choose the attaching circle of h_2 to be $x_0 \times S^1$, and then it follows directly from the definition that $(\nu K \times S^1) \cup h_2 = \nu \mathcal{T}'_K \cup h_2$ is a fishtail neighborhood. We have attached h_2 along a push off of β, and this attaching circle is the vanishing cycle. □

In contrast to the case of knots admitting non-trivial S^3 surgeries (only the unknot admits such a surgery [31]), we can construct infinitely many tori in S^4 that admit non-trivial S^4 surgeries. In particular, as a corollary of the preceding proposition we see that each twisted spun torus $\mathcal{T}'_K$ admits infinitely many non-trivial surgeries to S^4.

Corollary 2.3.2. $S^4_{\mathcal{T}'_K}(1,0,b)$ is diffeomorphic to S^4.

Proof. This follows directly from Theorem 2.2.4. □

The author first observed the existence of these surgeries follows from a more general result appearing in unpublished work by Gompf [27]. Since S^4 does not admit a unique surgery description, it is natural to ask how widespread is this phenomenon.

Question 2.3.3. If X is a 4–manifold obtained by surgery on a torus $\mathcal{T}$ in S^4, can X be obtained by surgery on an infinite family of distinct tori $\{\mathcal{T}_i\}$ in S^4?

The above examples also suggest another question. In contrast with the classical dimension [31], it is known that there exist inequivalent 2–knots with the same complement [30]. However, by [25] there can be at most two 2–knots with the same complement. The case for tori in S^4 is unknown.

Question 2.3.4. Do there exist (perhaps infinitely many) distinct tori in S^4 with the same complement?

Torus surgery appears to be the right perspective to answer this question positively. The goal would be to find non-trivial tori that admit surgeries to S^4 such that the surgery gluing map does not extend over the exterior or the neighborhood of the torus (this rules out the examples from Corollary 2.3.2).

Next we define the spin of a manifold.

Definition 2.3.5. Let M be a closed n–manifold, and let M° denote M with an open ball removed. Then the *spin* of M is the closed $(n+1)$–manifold defined by $spin(M) = \partial(M^\circ \times D^2)$. This is equivalent to taking $M^\circ \times S^1 \cup_{\mathrm{id}} S^2 \times D^2$.

The following is an easy observation.

Proposition 2.3.6. Let M be a closed, orientable 3–manifold. Then $spin(M)$ can be obtained by surgery on a link of tori in S^4.

Proof. The Lickorish-Wallace theorem states that M can be obtained by Dehn surgery on a link L in S^3. Remove a 3–ball away from L in S^3, so that we now think of L as sitting in B^3. We obtain a link of tori T_L in S^4 by taking $L \times S^1 \subset B^3 \times S^1$ inside $spin(S^3) = B^3 \times S^1 \cup S^2 \times D^2 = S^4$. We can now perform surgery on T_L where we take the gluing maps to be S^1 times the Dehn surgery gluing maps on L that give M. Hence we transform $B^3 \times S^1 \cup S^2 \times D^2$ to $M^\circ \times S^1 \cup S^2 \times D^2 = spin(M)$. □

Here we recall the process of spinning knots, a construction due to Artin (see Figure 2.2).

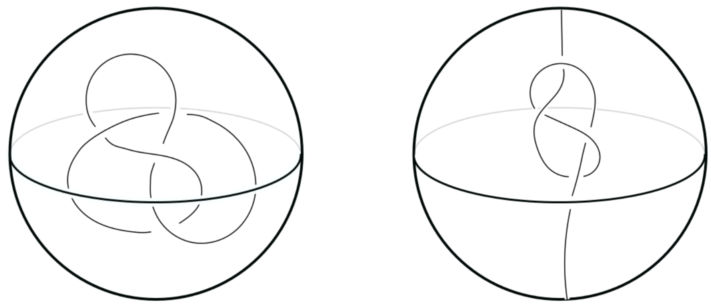

Figure 2.2: Spinning the knot $K \subset B^3$ on the left will result in a torus $\mathcal{T}_K$ in S^4. Spinning the knotted arc $\hat{K} \subset B^3$ on the right will result in the 2–knot *spin*(K) in S^4.

Definition 2.3.7. Let K be a knot in S^3. If we remove an open ball around a point of K, we get a knotted arc $\hat{K} \subset B^3$. The *spin* of K (denoted *spin*(K)) is the 2–knot obtained by taking the annulus $\hat{K} \times S^1 \subset B^3 \times S^1$ and capping off with two disks in $S^2 \times D^2$ inside the decomposition $S^4 = B^3 \times S^1 \cup_{\mathrm{id}} S^2 \times D^2$.

This is a well-studied operation that generalizes to higher-dimensional knots. The following is a corollary to Proposition 2.3.6.

Corollary 2.3.8. Let K be a knot in S^3. The d-fold cyclic branched cover of *spin*(K) can be obtained by surgery on a link of tori in S^4.

Proof. Let $M_d(K)$ denote the d-fold cyclic branched cover of K. If $\widetilde{K}$ is the lift of K in $M_d(K)$, let B denote a fibered neighborhood of a point $x \in \widetilde{K}$. Now if $M_d(K)^\circ = \overline{M_d(K) \setminus B}$, then $spin(M_d(K)) = M_d(K)^\circ \times S^1 \cup S^2 \times D^2$. Observe that the d-fold branched covering $S^2 \to S^2$ (with branch points the two poles) times the identity on the D^2 factor fits together with the induced branched covering of $M_d(K)^\circ \times S^1$ to form a branched covering of $spin(M_d(K))$ with branch locus $spin(K)$. Hence $spin(M_d(K))$ is the d-fold cyclic branched cover of $spin(K)$, and the result then follows from Proposition 2.3.6. □

Next we prove a generalization of a theorem appearing in [49], where the authors considered the case of spinning *fibered* knots and gave a proof relying on interpreting monodromy changes as surgeries.

Theorem 2.3.9. Let K_1 and K_2 be two knots in S^3. Then $(S^4, spin(K_2))$ can be obtained from $(S^4, spin(K_1))$ by surgery on a link of tori in the complement of $spin(K_1)$.

Proof. Consider knots K_1 and K_2 in S^3. We can obtain K_2 from K_1 by surgery on a link L in the exterior of K_1, where each component is unknotted and with framing ± 1, since such surgeries allow one to change overcrossings to undercrossings and vice versa. If we remove a small ball around a point of K_1 we see L and $\hat{K}_1$ inside B^3. Then upon spinning we get $spin(K_1)$ and a link $\mathcal{T}_L$ of tori in S^4. By construction, performing multiplicity ± 1 surgeries on $\mathcal{T}_L$ (using the Dehn surgery maps times the identity map in the S^1 direction) in the exterior of $spin(K_1)$ will return $spin(K_2)$. □

2.4 Round cobordisms

A useful way to study torus surgeries is by round handles (for a full development of this perspective see [10]). Recall that an n-dimensional round

k-handle is a copy of $S^1 \times D^k \times D^{n-1-k}$ attached along $S^1 \times \partial D^k \times D^{n-1-k}$. It is a basic fact that an n-dimensional round k-handle can be decomposed into an n dimensional k-handle and an n dimensional $(k+1)$-handle. Consider in particular the case of a 5-dimensional round 2-handle $S^1 \times D^2 \times D^2$ attached along $S^1 \times \partial D^2 \times D^2$ to $X \times I$ for some 4–manifold X. This defines a *round cobordism* between X and X', where X' can be obtained from X by removing the attaching region $S^1 \times \partial D^2 \times D^2$ and gluing $S^1 \times D^2 \times \partial D^2$ by the identification of their boundary. Observe that this is simply an integral torus surgery on the torus $\mathcal{T} = S^1 \times \partial D^2 \times \{0\}$. Furthermore, the converse is also true: any integral torus surgery corresponds to a cobordism given by a 5-dimensional round 2-handle (see [10] Lemma 2). The torus, multiplicity, and direction of the surgery determine how the attaching region $S^1 \times \partial D^2 \times D^2$ of the round 2-handle is embedded.

For our purposes the most important tool will be the Fundamental Lemma of Round Handles, due to Asimov [7]. The Lemma states that if we attach a k-handle h_k and a $k+1$-handle h_{k+1} to a manifold *independently*, then we can combine h_k and h_{k+1} to form a single round k-handle. This means that if we form a 5-dimensional cobordism W from X to X' by independently adding a 5-dimensional 2-handle and a 5-dimensional 3-handle to $X \times I$ (so that the attaching sphere of the 3-handle is disjoint from the belt sphere of the 2-handle), then W can be decomposed as $X \times I$ plus a 5-dimensional round 2-handle. By our remarks above, we see that X' can be obtained from X by an integral torus surgery.

We can use this method to construct 4–manifolds that can be obtained by surgery on a link of tori in S^4. The following theorem is proved using a technique similar to that found in [45], where Kervaire gives a characterization of the fundamental groups of homology spheres of dimension greater than 4.

Theorem 2.4.1. Any finitely presented group with non-negative deficiency appears as the fundamental group of a 4–manifold obtained by integral surgery on a link of tori in S^4.

Proof. The important observation is that if we form a 5-dimensional cobordism by first attaching a 3-handle h_3 and then a 2-handle h_2, the 2- and 3-handles will be attached independently (by transversality we can isotope the attaching sphere of the 2-handle off the belt sphere of the 3-handle and then off the 3-handle completely). Then by the Fundamental Lemma of Round Handles h_3 and h_2 can be isotoped to form a single 5-dimensional round 2-handle.

Now let $G = \langle g_1, g_2, \ldots, g_m | r_1, r_2, \ldots, r_n \rangle$ be a finitely presented group with deficiency $m - n \geq 0$. We construct a 4–manifold with fundamental group G as follows. First attach m 5-dimensional 3-handles to $S^4 \times I$ along attaching 2–spheres that form the m-component unlink in $S^4 \times \{1\}$. This gives a cobordism from S^4 to $\#_m S^1 \times S^3$. Note that $\pi_1(\#_m S^1 \times S^3) \cong F_m$, the free group on m letters. We can represent each relation r_i of G by an embedded curve ρ_i in $\#_m S^1 \times S^3$, and we can assume these curves are disjoint. Then we attach n 5-dimensional 2-handles along the ρ_i. This has the affect of surgering out a neighborhood of each ρ_i, which is a copy of $S^1 \times D^3$, and gluing in a copy of $D^2 \times S^2$. We see that in the resulting 4–manifold each ρ_i will be nullhomotopic, and so the new fundamental group will be exactly G. Lastly we attach $m - n$ more 2-handles along disjoint nullhomotopic curves in the boundary. This will leave the fundamental group unchanged, but now we have a cobordism W between S^4 and a 4–manifold X formed by first attaching m 3-handles, and then attaching m 2-handles. By our comments above, we see that these handles are attached independently, and so W is formed by attaching m round 2-handles to $S^4 \times I$ (whose attaching regions are disjoint). Therefore X is obtained by integral surgery on a link of tori in S^4, and $\pi_1(X) \cong G$. □

For groups of negative deficiency we have the following result.

Proposition 2.4.2. One can produce 4–manifolds with fundamental groups of arbitrarily large negative deficiency by surgery on tori in S^4.

Proof. Start with the family $\mathcal{K}_m$ of 2–knots constructed in [53]. Levine showed that the knot group of $\mathcal{K}_m$ has deficiency $1-m$. Then we can produce a family of tori $\mathcal{T}_m$ by adding a trivial tube to each $\mathcal{K}_m$ in a small 4–ball neighborhood of a point in $\mathcal{K}_m$ (see Figure 2.3). This doesn't change the fundamental group of the exterior, and the T^3 boundary of $E_{\mathcal{T}_m}$ will have two S^1 factors that are nullhomotopic in the exterior (the third S^1 factor is the meridinal direction). We can then perform a multiplicity 0 torus surgery on $\mathcal{T}_m$ with surgery direction either of the nullhomotopic S^1 factors of the boundary. The result of this surgery will have the same fundamental group as the exterior since the

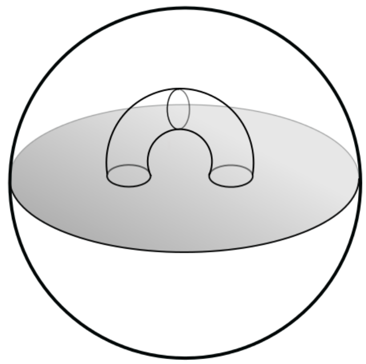

Figure 2.3: Adding a trivial tube to a 2–knot. Here we see a a slice $B^3 \times \{1/2\}$ of a 4–ball neighborhood $B^3 \times I$ of a point on a 2–knot. The disk bounded by the equator is a small patch of the 2–knot, and we add a small tube to increase the genus of the surface. Notice that each of the S^1 factors bounds a disk in the exterior.

attaching curve of the 2-handle is already nullhomotopic in the exterior, and so we get groups with arbitrarily large negative deficiency. □

However, it is not possible to achieve all finitely presented groups. Hausmann and Weinberger [35] constructed a finitely presented group that cannot be realized as the fundamental group of a 4–manifold with Euler characteristic 2. Therefore it seems difficult to give a complete characterization of the possible fundamental groups of 4–manifolds obtained by torus surgery in S^4.

We can use round handles to give another description of 4–manifolds obtained by torus surgery in S^4. Here *spherical surgery* means replacing an embedded copy of $S^2 \times D^2$ with $S^1 \times D^3$.

Proposition 2.4.3. Let X be a 4–manifold obtained by integral surgery on a single torus in S^4. Then X can also be obtained by a spherical surgery on an embedded S^2 in $S^2 \times S^2$ or $S^2 \mathbin{\widetilde{\times}} S^2$.

Recall that $S^2 \widetilde{\times} S^2$ is the twisted S^2 bundle over S^2, and is diffeomorphic to $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$.

Proof. We saw above that X can be obtained by attaching a 5-dimensional round 2-handle R to $S^4 \times I$. Furthermore, R can be decomposed into a 2-handle h_2 and 3-handle h_3. A 5-dimensional 2-handle is a copy of $D^2 \times D^3$ attached along $\partial D^2 \times D^3$. Up to isotopy there is a unique circle in S^4, and there are two surgeries on this circle corresponding to a $\mathbb{Z}_2$ choice of framing. The resulting 4–manifolds are $S^2 \times S^2$ and $S^2 \widetilde{\times} S^2$. Hence h_2 gives a cobordism from S^4 to M, where M is one of the S^2 bundles over S^2. We complete our cobordism to X by attaching the 3-handle h_3 to M. A 5-dimensional 3-handle is a copy of $D^3 \times D^2$ attached along $\partial D^3 \times D^2$. Observing how the boundary changes, we see that X is obtained by spherical surgery on $\partial D^3 \times \{0\}$ in M, and so the result follows.

□

Remark 2.4.4. Using similar techniques it is not hard to show that if X is the result of surgery on a *link* of tori, then X can be obtained by a set of spherical surgeries in $\#_k S^2 \times S^2$ or $\#_k S^2 \mathbin{\widetilde{\times}} S^2$ for some k. For non-integral surgeries we use the fact pointed out in [10] that the result of a non-integral surgery can be obtained as a set of integral surgeries.

2.5 The unknotted torus and embedding 3–manifolds

The unknotted torus is the unique torus in S^4 that bounds a solid torus $S^1 \times D^2$. In [60], Montesinos analyzed which gluing diffeomorphisms extend over the exterior of the unknotted torus (this exterior is the so-called *standard twin*). He used this to prove the following theorem; here instead we give a short proof using Cerf's theorem.

Theorem 2.5.1. [60] Let $\mathcal{T}$ be the unknotted torus in S^4. Then the result of any multiplicity 1 surgery on $\mathcal{T}$ is diffeomorphic to S^4.

Proof. Fix a particular multiplicity 1 surgery on $\mathcal{T}$, and let q and γ be the corresponding auxiliary multiplicity and direction of the surgery. Since $\mathcal{T}$ is unknotted, we can isotope it so it lies embedded in the standard S^3 equator of S^4. Then a neighborhood of $\mathcal{T}$ in S^4 is $\mathcal{T} \times I_0 \times I_1$, where $\mathcal{T} \times I_0$ is a neighborhood of $\mathcal{T}$ in S^3 and $I_1 = [0,1]$ is the interval induced from a collar neighborhood of S^3 in S^4. We perform the surgery as follows. We can remove $\mathcal{T} \times I_0 \times I_1$ and re-glue by any diffeomorphism of the boundary with multiplicity 1, auxiliary multiplicity q, and direction γ. Therefore we can choose the gluing map to be the identity map on $\mathcal{T} \times \partial I_0 \times I_1 \cup \mathcal{T} \times I_0 \times \{1\}$ and on $\mathcal{T} \times I_0 \times \{0\}$ to be the map that twists $\mathcal{T}$ in the γ direction q times. Finally, we observe that this surgery is equivalent to cutting S^4 along $S^3 \times \{0\}$ and re-gluing by the diffeomorphism of S^3 given by twisting $\mathcal{T}$ in the γ direction q times. By Cerf's theorem this diffeomorphism extends over B^4 and so we get back S^4. □

Now we will use Montesinos' work to show that the result of multiplicity 0 surgery on the unknotted torus is also quite restrictive, although as we saw in Proposition 2.2.2 we should obtain both spin and non-spin manifolds. This proposition was proved by Pao [65] (at least in the topological category) in the context of torus actions on 4–manifolds. Iwase also gives a proof in [41]; in fact, Iwase gives a similar classification for surgery on the unknotted torus for any multiplicity.

Theorem 2.5.2. The result of multiplicity 0 surgery on the unknotted torus $\mathcal{T}$ is either $S^1 \times S^3 \# S^2 \times S^2$ or $S^1 \times S^3 \# S^2 \widetilde{\times} S^2$. Indeed, $S^4_{\mathcal{T}}(0, a, b)$ is diffeomorphic to $S^1 \times S^3 \# S^2 \times S^2$ if ab is even, and $S^1 \times S^3 \# S^2 \widetilde{\times} S^2$ if ab is odd.

Here we choose our embedding of $\mathcal{T}$ by realizing the unknotted torus as the spin of the unknot $U \subset S^3$, so that the first S^1 factor α is identified with a longitude of U and the other S^1 factor β is identified with the S^1 direction of the spin.

Proof. Montesinos [60] showed that gluing maps of the form

$$\psi = \begin{pmatrix} c & d & * \\ e & f & * \\ 0 & 0 & 1 \end{pmatrix}$$

(where $c+d+e+f$ is an even number) extend over the exterior of the unknotted torus and hence don't affect the resulting diffeomorphism type. We will show that any choice of a and b (necessarily relatively prime) can be obtained by starting with a gluing map that has direction $\gamma = [\alpha]$ or $\gamma = [\alpha] + [\beta]$ and then post-composing with a gluing map of the form ψ above. Then by Montesinos' result we see that ψ extends over the exterior and so the two possible resulting diffeomorphism types are $S^4_{\mathcal{T}}(0,1,0)$ and $S^4_{\mathcal{T}}(0,1,1)$. In the following lemma we will show that these manifolds are diffeomorphic to $S^1 \times S^3 \# S^2 \times S^2$ and $S^1 \times S^3 \# S^2 \widetilde{\times} S^2$, respectively.

Suppose the direction of the surgery is $\gamma = a[\alpha] + b[\beta] \in H_2(\mathcal{T})$. Then we can choose the gluing map to be

$$\phi = \begin{pmatrix} 0 & m & a \\ 0 & n & b \\ 1 & 0 & 0 \end{pmatrix}$$

for some integers m and n satisfying $mb - na = 1$. Post-composing with a map ψ as above has the effect of changing the direction $\begin{pmatrix} a \\ b \end{pmatrix}$ by multiplying by the even matrix $\begin{pmatrix} c & d \\ e & f \end{pmatrix}$ (we will say a matrix is *even* if the sum of its entries is even). In light of this it is sufficient to show that for any relatively prime pair $\begin{pmatrix} a \\ b \end{pmatrix}$ there is an even matrix A such that if ab is an odd number then we have $\begin{pmatrix} a \\ b \end{pmatrix} = A \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and if ab is an even number then $\begin{pmatrix} a \\ b \end{pmatrix} = A \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. First assume that ab an odd number. Then $a + b$ is an even number. There exists some matrix in $GL(2, \mathbb{Z})$ such that $\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} c & d \\ e & f \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} c+d \\ e+f \end{pmatrix}$. Since $a + b = (c + d) + (e + f)$ is even, we see that this matrix is even.

Now assume that ab is an even number. Then $a + b$ is an odd number. Now we can write $\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a & -d \\ b & c \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ for some integers c and d that are solutions to the equation $ax + by = 1$. Given one such pair of solutions (c, d), it is known that all other solutions are of the form $(c + kb, d - ka)$ for some

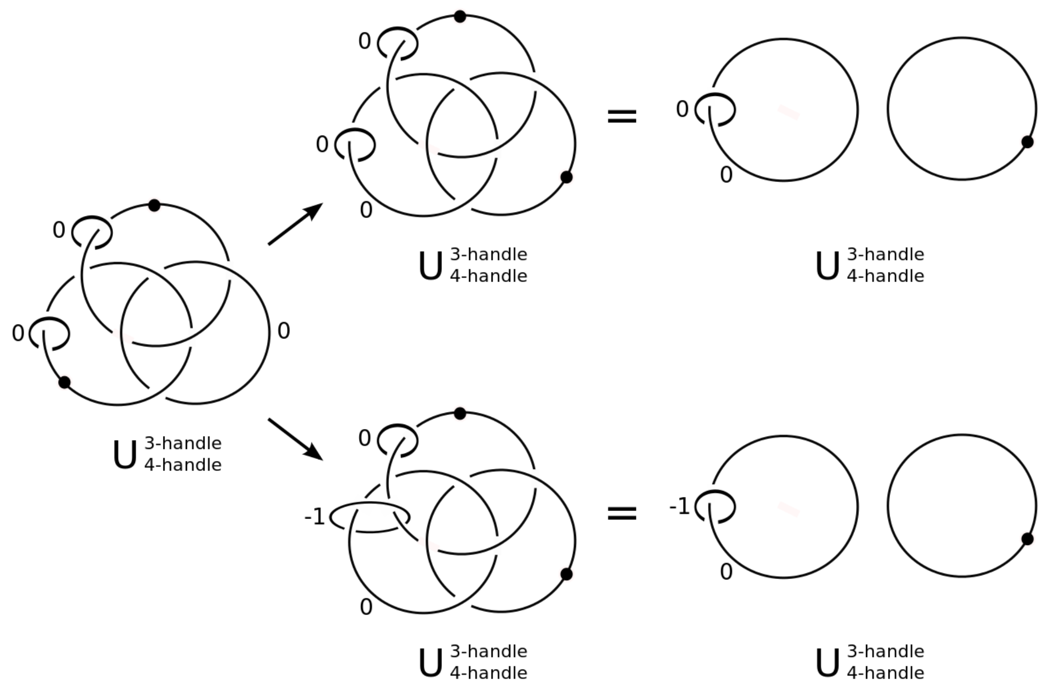

Figure 2.4: Multiplicity 0 surgery on the unknotted torus.

integer k. In particular, this implies that it is always possible to choose a pair of solutions with opposite parity (since a and b necessarily have opposite parity), and so we can choose the above matrix to be even.

To complete the proof we need to show that $S^4_{\mathcal{T}}(0,1,0)$ is diffeomorphic to $S^1 \times S^3 \# S^2 \times S^2$ and $S^4_{\mathcal{T}}(0,1,1)$ is diffeomorphic to $S^1 \times S^3 \# S^2 \widetilde{\times} S^2$. We will do in the following lemma. $\square$

Lemma 2.5.3. For the unknotted torus $\mathcal{T} \subset S^4$, we have $S^4_{\mathcal{T}}(0,1,0)$ is diffeomorphic to $S^1 \times S^3 \# S^2 \times S^2$ and $S^4_{\mathcal{T}}(0,1,1)$ is diffeomorphic to $S^1 \times S^3 \# S^2 \widetilde{\times} S^2$.

Proof. We prove this using a handlebody description of torus surgery (see [5] or [29] Chapter 8 for a thorough explanation of this perspective). On the left hand of Figure 2.4 we see the unknotted torus embedded in S^4. The Borromean rings consisting of two 1-handles in dotted circle notation and one 0 framed 2-handle give a handle decomposition of $\nu\mathfrak{T}$ (here we clearly see the boundary is T^3), and the rest of the handles give the embedding into S^4. The top row of Figure 2.4 corresponds to doing multiplicity 0 surgery with direction $[\alpha]$. We cut out $\nu\mathfrak{T}$, and re-glue by the diffeomorphism of T^3 that cyclically permutes the three S^1 factors, giving the middle top picture. A handle cancellation results in $S^1 \times S^3 \# S^2 \times S^2$, proving the first part of the lemma.

The bottom row of Figure 2.4 corresponds to multiplicity 0 surgery with direction $[\alpha] + [\beta]$. Here we cut out $\nu\mathfrak{T}$ and re-glue by the diffeomorphism of T^3 that first cyclically permutes the three S^1 factors and then applies ϕ, where ϕ is the map that cuts along $T^2 \times \{pt\} \subset T^2 \times \partial D^2$ and re-glues by a map that puts a full rotation in the direction of the first S^1 factor and is the identity on the second factor (this is analogous to a Dehn twist). Observe that ϕ sends any curve intersecting $T^2 \times \{pt\}$ to a curve that also wraps around the first S^1 factor, and adds a twist to the framing. The result is the bottom middle picture, and handle slides and a cancellation show that this is diffeomorphic to $S^1 \times S^3 \# S^2 \widetilde{\times} S^2$.

$\square$

2.5.1 Embeddings

We can now use Theorem 2.5.2 to prove a theorem about embedding 3–manifolds obtained by surgery on a knot in S^3.

Theorem 2.5.4. Let K be a knot in S^3 and let $S^3_{p/q}(K)$ denote p/q Dehn surgery on K. Then $S^3_{p/q}(K)$ smoothly embeds in $S^1 \times S^3 \# S^2 \times S^2$ if pq is even, and embeds in $S^1 \times S^3 \# S^2 \widetilde{\times} S^2$ if pq is odd.

Proof. Consider K as sitting in the standard S^3 equator of S^4. Then $\partial\nu K \subset S^3$ gives an unknotted torus $\mathcal{T}$ when we include $S^3 \hookrightarrow S^4$. We do multiplicity 0 surgery on $\mathcal{T}$ as follows. We will choose our embedding of $\mathcal{T}$ so that the first S^1 factor is the meridian of K and the second S^1 factor is the longitude of K. The collar neighborhood of $\partial\nu K$ in S^3 and the collar neighborhood of S^3 in S^4 provide a framing for $\mathcal{T}$. We will choose our surgery direction γ to be the homology class of p times the meridian and q times the longitude. We claim that $S^3_{p/q}(K)$ embeds in $S^4_{\mathcal{T}}(0,p,q)$, which by Theorem 2.5.2 is diffeomorphic to $S^1 \times S^3 \# S^2 \times S^2$ if pq is even, and $S^1 \times S^3 \# S^2 \widetilde{\times} S^2$ if pq is odd.

Our goal is to see the Dehn surgery concurrently with the torus surgery. To obtain $S^4_{\mathcal{T}}(0,p,q)$ we remove our chosen neighborhood of $\mathcal{T}$ and glue back $S^1 \times S^1 \times D^2$ by a gluing map of the form

$$\phi = \begin{pmatrix} 0 & m & p \\ 0 & n & q \\ 1 & 0 & 0 \end{pmatrix}$$

(for suitable m and n). Hence we are gluing in a solid torus $\{pt\} \times S^1 \times D^2$ to each $S^1 \times S^1 \times \{pt\} \subset S^1 \times S^1 \times \partial D^2$, where the boundary of the meridinal disk gets mapped to p times the first factor and q times the second factor. Since the boundary of the knot exterior $\partial E_K \subset \partial E_T$ is exactly one of these tori, we see that we are gluing a solid torus to the knot exterior in a manner that gives p/q Dehn surgery on K .

$\square$

Observe from the proof that the meridian of $\mathcal{T}$ intersects $E_K \subset S^4$ in a single point. Hence this curve intersects $S^3_{p/q}(K)$ in a single point after the torus surgery, and this curve generates the fundamental group of the resulting 4–manifold. It follows that surgery on this curve (replacing $S^1 \times D^3$ with $D^2 \times S^2$) will kill the $S^1 \times S^3$ connect summand of the resulting 4–manifold. This surgery will puncture $S^3_{p/q}(K)$, and so we get the following corollary.

Corollary 2.5.5. If $S^3_{p/q}(K)^\circ$ denotes the 3–manifold obtained by puncturing $S^3_{p/q}(K)$, then $S^3_{p/q}(K)^\circ$ smoothly embeds in $S^2 \times S^2$ if pq is even, and embeds in $S^2 \widetilde{\times} S^2$ if pq is odd.

See [21] for a similar statement when K is the unknot (and so the 3–manifolds are lens spaces). Note that any 3–manifold obtained by *integral* surgery on a knot always embeds in $S^2 \times S^2$ or $S^2 \widetilde{\times} S^2$ (just double the 4–dimensional 2-handlebody). An interesting thing about the above construction is that it does not distinguish between integral and non-integral Dehn surgery.

Next we look at embeddings of 3–manifolds into S^4. Recall that a link is slice if the components bound disjoint slice disks in B^4, and a link is ribbon if the components bound disjoint ribbon disks in B^4 (see Chapter 4 for definitions of slice disks and ribbon disks).

Theorem 2.5.6. Let L be a ribbon link in S^3. If M_L is the 3–manifold obtained by surgery on L with all the surgery coefficients belonging to the set $\{1/n\}_{n\in\mathbb{Z}}$, then M_L smoothly embeds in S^4. If L is only *slice*, then we get an embedding into a *homotopy* 4–sphere. However, if we restrict the surgery coefficients to the set $\{1/(2n)\}_{n\in\mathbb{Z}}$, then again we get an embedding into the standard S^4.

Budney and Burton [14] observed that if L is slice and the coefficients are ± 1, then M_L embeds in a homotopy 4–sphere by blowing down the resulting 2–spheres in the 2-handlebody formed by attaching 2-handles to L with the corresponding framings. We obtain this generalization by proceeding in a different direction; we consider cross sections of Gluck twists on the 2–knots obtained by doubling the ribbon or slice disks.

Proof. First we consider the case where K is a slice knot in S^3. Let $\mathcal{D}_K$ be the slice disk in B^4, and let $\mathcal{S}_K$ be the 2–knot in S^4 obtained by doubling the pair $(B^4, \mathcal{D}_K)$. We will do surgery on $\mathcal{S}_K$ and see Dehn surgery on K as a cross section. Identify a neighborhood of $\mathcal{S}_K$ with $S^2 \times D^2$ such that *equator* $\times D^2$ is identified with a tubular neighborhood of K in $S^3 \subset S^4$ (and the induced framing is the zero framing). We will cut out $S^2 \times D^2$ and re-glue by the map $\rho\colon\ S^2 \times \partial D^2 \to S^2 \times \partial D^2$ defined by sending (x, θ) to $(rot_\theta(x), \theta)$, where rot_θ is the map that rotates S^2 through an angle θ about a fixed axis (we choose this to send the equator to itself). Now the result of this surgery is by definition the Gluck twist on $\mathcal{S}_K$ in S^4, and hence returns a homotopy 4–sphere. In fact this is true for all odd powers of ρ. However, if we instead re-glue by an even power ρ^{2n} of ρ, then since ρ^2 is isotopic to the identity map [25] the result of the surgery will be the *standard* S^4. Furthermore, if D_K is a *ribbon* disk, then $\mathcal{S}_K$ is a ribbon 2–knot and re-gluing by any power ρ^n will return the standard S^4 (see, for example, [29]).

Now we examine what happens to a neighborhood of K. For a point x in K (thinking of K as the equator of $\mathcal{S}_K$), consider the effect of ρ^n on the boundary of the meridinal disk $x \times D^2$. As θ varies the curve (x, θ) on the boundary will map to a curve that wraps once around the meridinal direction and n times around the longitudinal direction of $K \times \partial D^2$. This is exactly Dehn surgery on K with surgery coefficient $1/n$, and so we get an embedding into the 4–manifold obtained by the corresponding surgery on $\mathcal{S}_K$. We can extend this to surgery on a slice link by performing surgery on each of the 2–knots obtained by doubling the multiple slice disks. Finally, we finish by applying the comments in the previous paragraph about the result of the various 2–knot surgeries.

$\square$

As mentioned in the introduction, we note that the above theorem provides many embeddings of homology 3–spheres into S^4.

Proposition 2.5.7. If M_L is a 3–manifold obtained by surgery on a slice link

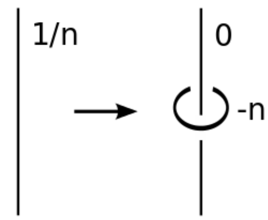

Figure 2.5: Reverse slam dunk move.

L with all the surgery coefficients belonging to the set $\{1/n\}_{n\in\mathbb{Z}}$, then M_L is a homology 3–sphere.

Proof. First we realize M_L as *integral* surgery on a link by performing the reverse slam dunk move as in Figure 2.5 (see [29] Section 5.3) on each component of L. Since any two components in a slice link have 0 linking number, the corresponding linking matrix for this new surgery diagram will be block diagonal, where each component K_i of L (with surgery coefficient $1/n_i$) corresponds to a block on the diagonal of the form $\begin{pmatrix} -n_i & 1 \\ 1 & 0 \end{pmatrix}$. It then follows that the determinant of the linking matrix is ± 1, and so the resulting 3–manifold M_L will be an integral homology 3–sphere (again see [29] Section 5.3). $\square$

Finally, by looking at multiplicity 1 surgeries on the unknotted torus it is possible to give an alternate proof of an unpublished theorem of Gompf ([27]) about embedding *punctured* homology spheres in S^4. First we give a modification of the spinning constructions in Section 2.3 to show that the unknotted torus can be constructed in a particularly interesting way.

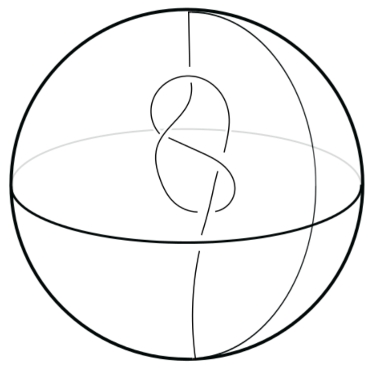

Figure 2.6: $\hat{K} \cup a$ in B^3. $\hat{K}$ is the knotted arc with endpoints at the poles and a is the arc on the boundary S^2.

Given a knot K in S^3, let $\hat{K}$ be the corresponding knotted arc in B^3 with endpoints at the poles, as in Definition 2.3.7. Let a be an arc in $S^2 = \partial B^3$ connecting the poles, so that $\hat{K} \cup a$ is a knot in B^3 (see Figure 2.6). Now if we glue $(B^3, \hat{K}) \times S^1$ to $(S^2, a) \times D^2$ by the identity map then we get the spun torus $\mathcal{T}_K$, but instead we want to glue these two pieces by the Gluck twist map ρ (we appropriately smooth the torus in either case). Denote the resulting torus by $\mathcal{U}_K$, so that we have $(S^4, \mathcal{U}_K) = (B^3, \hat{K}) \times S^1 \cup_\rho (S^2, a) \times D^2$. One can think of this as taking $\hat{K} \cup a$ in B^3 and as we go around the S^1 direction wrapping a once around the boundary S^2.

Lemma 2.5.8. $\mathcal{U}_K$ is the unknotted torus, for any knot K.

Proof. This follows from Zeeman's twist spinning theorem [77]. Zeeman showed that if we form a 2–knot by taking the annulus $\hat{K} \times S^1 \subset B^3 \times S^1$ and capping off with two disks (at the poles) in $S^2 \times D^2$ inside the decomposition

$S^4 = B^3 \times S^1 \cup_\rho S^2 \times D^2$, then the resulting 2–knot (called the 1-twist spin of K) is in fact the unknotted 2–knot. We then obtain $\mathcal{U}_K$ by attaching the handle $a \times D^2 \subset S^2 \times D^2$ to the 1-twist spin of K in $B^3 \times S^1 \cup_\rho S^2 \times D^2$. Finally, a theorem of Boyle ([12], Corollary 5) states that a torus formed by attaching a handle to the unknotted 2–knot is itself unknotted, giving the result. $\square$

Theorem 2.5.9 (Gompf, [27])**.** If $M = S^3_{1/n}(K)$ is a homology sphere that is surgery on a knot, then if we puncture M it smoothly embeds in S^4. In fact, M° appears as the fiber of a fibered 2–knot in S^4.

Proof. Consider the torus $\mathcal{U}_K \subset B^3 \times S^1 \cup_\rho S^2 \times D^2$ constructed above. Isotope $\mathcal{U}_K$ so that it lies in the interior of $B^3 \times S^1$. Then in each $B^3 \times \{pt\}$ we see a copy of $K = \hat{K} \cup a$, where the arc a wraps once around $\hat{K}$ as we go around the S^1 direction. We then perform a torus surgery on $\mathcal{U}_K$ by performing $1/n$ Dehn surgery on K in each $B^3 \times \{pt\}$. However, we must check that this is well-defined; that is, we must verify that the isotopy of K (wrapping a around $\hat{K}$) behaves well with respect to the Dehn surgery gluing map. Dehn surgery is determined by the image of a meridian under the gluing map. Denote this curve by γ. Since the isotopy of K wrapping a around $\hat{K}$ preserves γ (up to isotopy), we get a well-defined torus surgery. Therefore we are performing a multiplicity 1 surgery on $\mathcal{U}_K$, which is unknotted by Lemma 2.5.8, and so by Theorem 2.5.1 the manifold obtained by this surgery is diffeomorphic to S^4. In conclusion, we performed a torus surgery in $B^3 \times S^1 \cup_\rho S^2 \times D^2$ that returned S^4, such that in each $B^3 \times \{pt\}$ we see $1/n$ surgery on K. The result then follows, where $S^2 \times \{0\} \subset S^2 \times D^2$ is our fibered 2–knot (see Chapter 4 for the definition of fibered knots). $\square$

Chapter 3

Generic fibrations around multiple fibers

3.1 Introduction

[1] In this chapter we change our perspective and consider the relationship between torus surgery and various types of singular fibrations on 4–manifolds. These fibrations have proved to be powerful tools in the study of smooth 4-manifolds. The classical theories of Lefschetz and elliptic fibrations are a rich source of interesting examples and provide connections to algebraic geometry, symplectic geometry, and gauge theory. More recently it has been shown that every smooth closed 4-manifold admits a broken Lefschetz fibration (see, for example, Akbulut and Karakurt [6], Baykur [9] , Gay and Kirby [24], and Lekili [52]), or alternatively a purely wrinkled fibration (which are also called indefinite Morse 2-functions). On the other hand, we have seen that torus surgery (often called a *log transform* in this context) is perhaps the most important surgical tool for 4-manifolds. In this chapter we integrate these two perspectives by studying the result of torus surgery on a regular fiber of a map to a surface. In particular, we construct nice fibrations in a neighborhood of the glued in torus that agree with the original fibration on

[1]The material in this chapter has previously been published in: Kyle Larson. Generic fibrations around multiple fibers. *New York J. Math.*, 20:1161–1173, 2014.

the boundary. The existence of such fibrations follows from a more general result of Gay and Kirby [24], but here we produce the first explicit examples. Our work also fits nicely into the context of Baykur and Sunukjian [10], where the authors discuss when broken Lefschetz fibrations on different manifolds can be related by torus surgery and homotopy modifications of the fibration. Our construction illustrates this for some specific examples.

3.2 Torus surgery on a fiber

Here we give a definition of torus surgery convenient to this context. Let X be a smooth 4-manifold and Σ a smooth surface, with $f\colon X \to \Sigma$ some type of fibration map (e.g. an elliptic fibration or broken Lefschetz fibration, but in general we just require f to be proper and smooth). If $T \subset X$ is a regular fiber diffeomorphic to a torus, then we can identify a tubular neighborhood νT with $T^2 \times D^2$ and a neighborhood of $f(T)$ with D^2 such that $f|_{T^2 \times D^2}$ is projection onto the second factor. Let $\phi\colon \nu T \to T^2 \times D^2$ be such an identification. Torus surgery on T is the operation of cutting out νT and gluing in $T^2 \times D^2$ by $\phi^{-1} \circ \psi$, where ψ is a self-diffeomorphism of $\partial(T^2 \times D^2)$. Let X_T be the resulting manifold $X \setminus \nu T \cup_{\phi^{-1}\circ\psi} T^2 \times D^2$. Since gluing in $T^2 \times D^2$ amounts to attaching a 2-handle, two 3-handles, and a 4-handle, the diffeomorphism type of X_T is determined by the attaching sphere of the 2-handle: $\phi^{-1} \circ \psi(\{pt\} \times \partial D^2)$ (the framing is canonical). The isotopy class of this curve is then determined by the homology class $\gamma = \psi_*[\{pt\} \times \partial D^2] \in H_1(T^2) \oplus \mathbb{Z}$, where the $\mathbb{Z}$ factor is generated by $m = [\{pt\} \times \partial D^2]$. Now γ must be a primitive element, so $\gamma = q\alpha + pm$ for relatively prime integers p and q and α a primitive element of $H_1(T^2)$. Hence, given our identification ϕ, X_T is determined up to diffeomorphism by the data p, q, and α, which are called the *multiplicity*, the *auxiliary multiplicity*, and the *direction*. We say the surgery is *integral* if $q = \pm 1$.

Now fixing a specific torus surgery determined by p, q, and α, we may change our identification ϕ so that the direction α corresponds to the second S^1 factor of $T^2 \times D^2 = S^1 \times S^1 \times D^2$. To be precise, we compose ϕ with a map $g \times id \colon T^2 \times D^2 \to T^2 \times D^2$, where g is some self-diffeomorphism of T^2 that sends a curve representing α to $\{pt\} \times S^1$. We abuse notation by renaming this new identification ϕ. In doing this we have not changed the surgery, but we have changed how we look at a neighborhood of T in order to make things more convenient for what follows.

We are interested in which surgeries on T allow the fibration $f|_{X \setminus \nu T}$ to be extended over X_T. By our above remarks, up to diffeomorphism we can choose our gluing map ψ to be (thinking of $\partial(T^2 \times D^2)$ as $\mathbb{R}^3/\mathbb{Z}^3$):

$$\psi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & (qk+1)/p & q \\ 0 & k & p \end{pmatrix}$$

where k is an integer satisfying $qk + 1 \equiv 0 \bmod p$ (if $p = 0$, set the center entry to 0). If we instead think of $T^2 \times D^2$ as $\{(\xi_1, \xi_2, z) \subset \mathbb{C}^3 \mid \xi_i \in S^1 \subset \mathbb{C}, z \in D^2 \subset \mathbb{C}\}$, then we can write ψ multiplicatively as $\psi(\xi_1, \xi_2, z) = (\xi_1, \xi_2^{(qk+1)/p} \cdot z^q, \xi_2^k \cdot z^p)$ (see Harer, Kas, Kirby [34] for more information). Now we can see that if $p \neq 0$

the fibration extends over the glued in $T^2 \times D^2$ by defining $\hat{f}\colon\ X_T \to \Sigma$ by

$$\hat{f}(x) = \begin{cases} f(x) & \text{if } x \in X \setminus \nu T \\ \xi_2^k \cdot z^p & \text{if } x = (\xi_1, \xi_2, z) \in T^2 \times D^2 \end{cases}$$

One can check that the fibration on $T^2 \times D^2$ is exactly S^1 times the fibration around a $(p, -k)$ exceptional fiber in a Seifert fibered space. Hence the central fiber $T = T^2 \times \{0\}$ is p-times covered by nearby fibers and the homology class of a nearby fiber $[F] = p \cdot [T]$. Furthermore, if $p > 1$ then one can compute in local coordinates that $d\hat{f}$ vanishes on T and is a submersion everywhere else in $T^2 \times D^2$ (if $p = 1$ the fibration extends over $T^2 \times D^2$ with no singularity). For $p > 1$ we say that T is a *multiple fiber* singularity of $\hat{f}$. Since $d\hat{f}$ vanishes on a 2-dimensional subspace, $\hat{f}$ cannot be a generic map to a surface (near T).

The purpose of this chapter is to construct indefinite generic fibrations on $T^2 \times D^2$ that agree with $\hat{f}$ on $\partial(T^2 \times D^2)$.

3.3 Constructing generic fibrations

Our strategy will be to construct generic fibrations using round handles. An $(n+1)$-dimensional round k-handle is $S^1 \times h_k^n$, where h_k^n is an n-dimensional k-handle, and it is attached along S^1 times the attaching region of h_k^n (see Baykur [11], Baykur and Sunukjian [10] for more information about round handles). If we are attaching a round handle to a manifold whose boundary fibers over S^1, so that a single h_k^n is attached to each fiber, then we can extend the boundary fibration over the round k-handle by taking the Morse level sets of each h_k^n (and adjusting the fibration in a collar neighborhood of the boundary). However, this fibration will have a *fold singularity*, which by definition is a singular set that locally looks like $\mathbb{R}$ times a Morse singularity. More precisely, there exist local coordinates $(t, x_1, \cdots, x_n)$ around each critical point such that the fibration map is given by $(t, x_1, \cdots, x_n) \mapsto (t, x_1^2 \pm \cdots \pm x_n^2)$ in these coordinates. Importantly for our purposes, fold singularities of maps to surfaces are a generic type of singularity. The fold singularity is called *indefinite* if the Morse singularity in the above coordinates is indefinite (i.e. the Morse critical point does not have index equal to 0 or n).

Remark 3.3.1. In what follows we will abuse terminology and call a fibration *generic* if its singularities consist of only indefinite fold singularities. It is a fact from singularity theory that such fibrations do belong to the set of generic (and stable) maps, but these fibrations are actually a very special subset of all generic maps. Indeed, maps from a 4-manifold to a surface with only these singularities are a subset of both broken Lefschetz fibrations (which can also contain Lefschetz singularities) and purely wrinkled fibrations (which can also contain indefinite cusp singularities).

First we do our construction for a neighborhood of an exceptional fiber in a Seifert fibered space. Recall that the neighborhood of a (p, q) exceptional fiber can be formed by taking a solid cylinder and identifying the two ends with a $2\pi q/p$ twist. In fact, we start with the simplest possible case: the neighborhood of a (2,1) exceptional fiber. Let N be a tubular neighborhood of a (2,1) exceptional fiber. Then N is diffeomorphic to a solid cylinder with

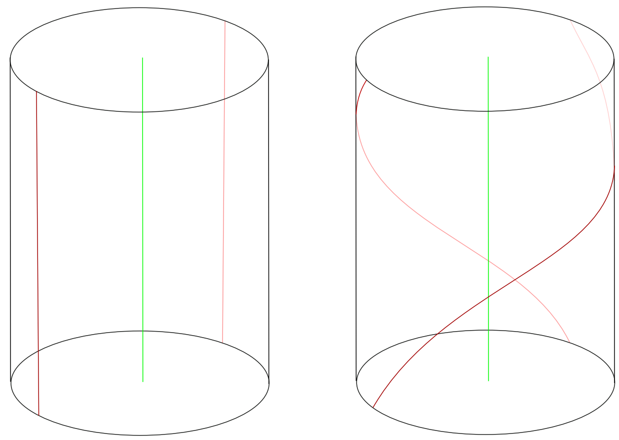

Figure 3.1: A neighborhood of a (2,1) exceptional fiber. On the left the ends of the solid cylinder are identified by a 180 degree twist, whereas on the right the ends are identified by the identity map. The green arc becomes the exceptional fiber under the identification, and the red arcs become a single fiber on the boundary.

ends identified with a 180 degree twist (see Figure 3.1). The exceptional fiber is the circle formed by identifying the two ends of the central arc. A regular fiber of N consists of two arcs opposite each other and equidistant from the central arc, which form a single circle after the identification of the ends of the cylinder. We can also view N as a solid cylinder with ends identified by the identity map, but now regular fibers twist around the central fiber (see the second picture of Figure 3.1). Let $f\colon N \to D^2$ be the fibration map (note that f is not simply the projection of the solid cylinder; we have to compose with the 2 to 1 branched covering map of the disk).

Lemma 3.3.2. N admits a generic fibration $\hat{f}\colon N \to D^2$ such that $\hat{f}\Big|_{\partial N} = f|_{\partial N}$, with one indefinite fold singular locus. The image of the critical set is an embedded circle in D^2, and the preimage of a point in the interior of this

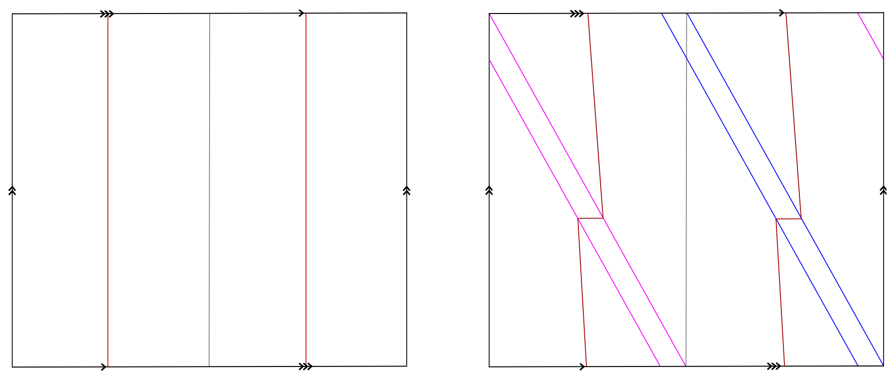

Figure 3.2: The fibering on ∂N. The top edges are identified with the bottom edges with a 1/2 unit shift to the right. The red arcs form a single fiber (note that the gray middle arc is *not* part of the fibration), and we see the result of the isotopy of the fibration on ∂N in the second picture. The diagonal strips form the attaching region for the 3-dimensional round 1-handle, and the two horizontal sections of the red fiber are the attaching regions for the 2-dimensional 1-handle.

circle is two disjoint circles.

Proof. Our strategy will be to delete $\mathrm{int}N$, and then fill it back in (relative to the boundary) with one round 1-handle and two trivially fibered solid tori in such a way as to extend the fibration on ∂N. Recall a 3-dimensional round 1-handle is a copy of $S^1 \times D^1 \times D^1$ attached along an embedding of $S^1 \times \partial D^1 \times D^1$. We can think of this as adding a circle's worth of 2-dimensional 1-handles. In our case we attach a single 2-dimensional 1-handle to each S^1 fiber of ∂N. Now there are two ways to attach a 2-dimensional 1-handle to S^1, resulting in either one or two components (depending on whether the 1-handle preserves or reverses orientation). We will attach the round 1-handle so that the resulting fibers have two components. Before we attach the round 1-handle we modify

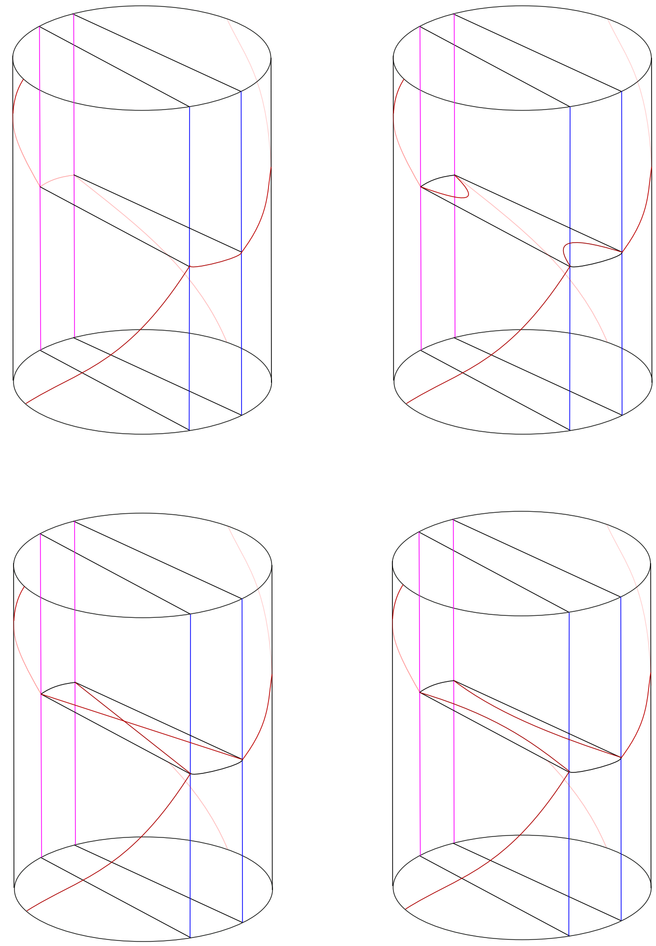

Figure 3.3: Extending the fibration across R. The top and bottom of the cylinders are identified by the identity map, and we see R as the rectangular prism with top and bottom identified. In each picture the red arcs form a single fiber. The first picture shows the fibration on ∂N (after the isotopy), and in the following pictures the fiber get pushed across the 2-dimensional 1-handle (while the arcs on the boundary of the cylinder actually live in a collar $\partial N \times I$). In the first two pictures the fiber is a single circle wrapping twice around the cylinder. The third picture is the singular level, where the fiber consists of the wedge of two circles. The last picture shows a fiber past the singular level, and the fiber consists of two disjoint circles. Here we see that after extending the fibration across R we get two "chambers" with torus boundaries, each fibered by (1,1) curves.

the fibration on ∂N by an isotopy.

If we restrict our attention to the fibration on ∂N, thought of as the boundary of the cylinder with top and bottom identified with a 180 degree twist, then we can cut the cylinder open and think of ∂N as a square with left and right edges identified by the identity map and the top edge identified to the bottom edge by a 1/2 unit shift to the right (see Figure 3.2). We see the fibers of ∂N as a pair of vertical arcs separated by 1/2 units in the horizontal direction. Our modification of the fibration on ∂N involves isotoping the fibers (in a collar $\partial N \times I$) so that each fiber is horizontal along the two diagonal strips in the second picture of Figure 3.2. The diagonal strips will form the attaching region of the round 1-handle, and the two horizontal sections of each fiber will be the attaching region of the 2-dimensional 1-handle to each fiber.

Now we can extend the fibration on ∂N across the round 1-handle (which we will denote by R) as follows (see Figure 3.3): ∂N consists of a circle's worth of S^1 fibers, and attaching R has the effect of attaching a 2-dimensional 1-handle to each fiber. We extend the fibration over each of these 1-handles by taking the level sets corresponding to the natural Morse function on 1-handle $\cup$ (fiber $\times I$), where the I factor comes from a collar neighborhood $\partial N \times I$. So before the critical level the fibers will be circles, the critical level will be the wedge of two circles, and after the critical level the fibers will be a disjoint union of two circles. Therefore, adding the round 1-handle R introduces a fold singularity C ($S^1\times$ the Morse critical point of the 2-dimensional 1-handle), and $\hat{f}$ maps C to an embedded circle. The boundary $\partial(\partial N \cup R)$ is two disjoint tori (here we are only considering the "interior" part of the boundary, the exterior of course consists of another torus). Furthermore, as we can see in Figure 3.3, each of these tori are fibered with multiplicity 1 (i.e. fibered by (1,1) curves). So we see that adding the round 1-handle reduces the multiplicity from 2 to 1 at the expense of increasing the number of components of a fiber from 1 to 2. Now we can fill in these two tori with two trivially fibered solid tori (the (1,1) fibration on the boundary extends over the solid torus without singularities). Topologically we are just gluing back in the two solid tori of $N \setminus (\partial N \cup R)$, but in such a way as to extend the fibration. This completes our construction of a generic fibration on N.

□

It is quite easy to extend our construction to the case of a $(p, 1)$ exceptional fiber:

Proposition 3.3.3. If N is a tubular neighborhood of a (p,1) exceptional fiber and $f\colon N \to D^2$ is the fibration map, then N admits a generic fibration $\hat{f}\colon N \to D^2$ constructed with $p-1$ round handles such that $\hat{f}\big|_{\partial N} = f|_{\partial N}$. The image of the critical set is $p-1$ concentrically embedded circles in D^2, and the preimage of a point in the interior of these circles is p disjoint circles.

Proof. We proceed as before, by starting with the fibration on ∂N, and attaching a 3-dimensional round 1-handle along the two strips as in Figure 3.4 (after isotoping the fibration in a collar $\partial N \times I$ so that fibers are horizontal across the diagonal strips). We extend the fibration across the round handle

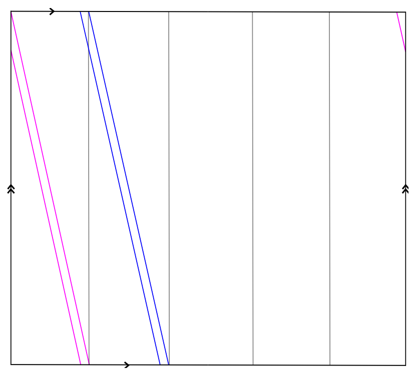

Figure 3.4: Attaching a 3-dimensional round 1-handle to ∂N. The diagonal strips form the attaching region for the first round 1-handle. Here we draw the case for a (5,1) exceptional fiber, but the picture obviously generalizes to a $(p, 1)$ exceptional fiber.

as before, and the resulting interior boundary is again two tori, but this time one has multiplicity 1 and the other has multiplicity $p-1$. The torus fibered with multiplicity 1 can be filled with a trivially fibered solid torus, and we repeat this procedure inductively with the torus fibered with multiplicity $p-1$. The result is that we consecutively attach $p-1$ round 1-handles (each one increasing the number of components of a fiber by 1) and glue in $p-1$ trivially fibered solid tori. This gives the required generic fibration on N.

□

One can construct generic fibrations in a neighborhood of a (p, q) exceptional fiber in a similar manner, but the author has not worked out a general algorithm.

We now proceed to the construction of generic fibrations around a torus multiple fiber. Here we describe the process for singular fibrations resulting from *integral* surgeries (which will suffice for our applications), but again, one could apply these techniques to non-integral surgeries as well.

Theorem 3.3.4. The fibration around a multiple fiber singularity resulting from an integral torus surgery of multiplicity p can be replaced with a generic fibration (extending the fibration on the boundary) composed of $(p-1)$ round 1-handles and $(p-1)$ round 2-handles. The image of the indefinite fold critical set is $2\cdot(p-1)$ concentrically embedded circles and the preimage of an interior point consists of p disjoint tori.

Proof. As before, we will start with the multiplicity 2 case and then generalize to multiplicity p. Let M be the neighborhood of the multiple fiber singularity. By our remarks in Section 3.2, M is fiber-preserving diffeomorphic to $S^1 \times N$, where N is the fibered neighborhood of a (2,1) exceptional fiber. Let us assume that the fibration on $\partial M = S^1 \times \partial N$ has been modified by isotopy so that the fibration is S^1 times the modified fibration on ∂N. We will use our generic fibration on N to construct a generic fibration on M, however, the fibration is *not* just S^1 times the generic fibration on N. In that case the singular set would be $S^1 \times C$, where C is the singular circle of the generic fibration on N, and 2-dimensional singular sets do not occur generically. In what follows it will be helpful to refer to Figure 3.3 and think of $M = S^1 \times N$ as a "movie" where time is the S^1 direction.

The generic fibration on N was constructed using a 3-dimensional round 1-handle R, which we thought of as a circle's worth of 2-dimensional 1-handles.

We use this family of 2-dimensional 1-handles to construct a 4-dimensional round 1-handle R_1^4 and a 4-dimensional round 2-handle R_2^4 as follows: Let θ parametrize the S^1 factor of $R = S^1 \times D^1 \times D^1$ (hence θ parametrizes the family of 2-dimensional 1-handles) and let x and t be coordinates on the two D^1 factors. Define two subsets $I_1, I_2 \subset S^1$ by $I_1 = \{(\cos\theta, \sin\theta) \in \mathbb{R}^2 \mid \theta \in [-\pi/4, \pi/4]\}$ and $I_2 = \overline{S^1 \setminus I_1}$. Let $g\colon R \hookrightarrow N$ denote the embedding map from our previous construction (note this is not simply the attaching map, but in fact embeds the entire round handle into N), and let $R_1^4 = S^1 \times I_1 \times D^1 \times D^1$ where φ parametrizes the S^1 factor.

Embed R_1^4 into M by the map $G_1\colon S^1 \times I_1 \times D^1 \times D^1 \hookrightarrow S^1 \times N$, $G_1(\varphi, \theta, x, t) = (\varphi + \theta, g(\varphi, x, t))$ (It is important to note that g now takes φ as input instead of θ). We can think of attaching R_1^4 to $\partial M = S^1 \times \partial N$ by $G_1|_{S^1 \times I_1 \times \partial D^1 \times D^1}$. For a fixed value of φ, say φ_0, $g|_{\{\varphi_0\} \times \partial D^1 \times D^1}$ maps to a single circle fiber c of ∂N, and so $G_1|_{\{\varphi_0\} \times I_1 \times \partial D^1 \times D^1}$ is an attaching map for a 3-dimensional 1-handle to the torus fiber $S^1 \times c \subset S^1 \times \partial N = \partial M$. Indeed we see that $G_1|_{\{\varphi_0\} \times I_1 \times D^1 \times D^1}$ embeds a 3-dimensional 1-handle into M by embedding the 2-dimensional 1-handle "slices" $\{\theta\} \times D^1 \times D^1$ into $\{\varphi_0 + \theta\} \times N \subset M$ for $\theta \in I_1$ (see Figure 3.5). Letting φ range over S^1 shows that attaching R_1^4 amounts to adding a 3-dimensional 1-handle to each torus fiber of $S^1 \times \partial N$, so that the genus of the fibers increases by one. However, this is done in a way such that if we look at the result of attaching R_1^4 in a single frame of our "movie," $(\partial M \cup R_1^4) \cap (\{pt\} \times N)$, we see a 3-dimensional 1-handle attached to ∂N, but which is composed of 2-dimensional 1-handle "slices," each slice belonging to a different $G_1|_{\{\varphi\}\times I_1\times D^1\times D^1}$. The result is that the fold singular set of R_1^4 intersects a frame of our movie, $\{pt\}\times N$, in a single point, corresponding to the Morse singularity of the 3-dimensional 1-handle $G_1|_{\{pt\}\times I_1\times D^1\times D^1}$ (which occurs at $\theta = 0 \in I_1$).

We embed the 4-dimensional round 2-handle $R_2^4 = S^1 \times I_2 \times D^1 \times D^1$ into M similarly, by the map $G_2(\varphi, \theta, x, t) = (\varphi + \theta, g(\varphi, x, t))$ (indeed this is the same map, except θ now takes values in I_2). We can think of attaching R_2^4 to $\partial(\partial M \cup R_1^4)$ by $G_2|_{S^1\times\partial(I_2\times D^1)\times D^1}$, and one can check that this amounts to attaching a 3-dimensional 2-handle $G_2|_{\{\varphi\}\times I_2\times D^1\times D^1}$ to each genus 2 fiber of $\partial(\partial M \cup R_1^4)$. Now $G_2|_{\{\varphi\}\times I_2\times D^1\times D^1}$ is actually a separating 3-dimensional 2-handle, and we can see this by looking at Figure 3.5 (here we again consider a fixed $\varphi = \varphi_0$). As θ varies over I_2 we add more 2-dimensional 1-handle slices to the picture whose attaching regions fill out the remainder of the two annuli. We see that this amounts to attaching a 3-dimensional 2-handle to the genus 2 fiber whose attaching circle runs twice over the 1-handle. From the picture we see that this is a separating 2-handle that results in two disjoint torus components.

Another way to see this is by considering the resulting fibration on the boundary: if we look at a frame of our movie after attaching R_1^4 and R_2^4, $(\partial M \cup R_1^4 \cup R_2^4) \cap (\{pt\}\times N)$, we see $(\{pt\}\times\partial N)\cup R$, but the 2-dimensional slices of R in this frame belong to different slices of R_1^4 and R_2^4. The point is that topologically $\partial M \cup R_1^4 \cup R_2^4$ gives a decomposition of $S^1 \times (\partial N \cup R)$, but in such a way that the natural fibrations on the boundaries agree. That is, the fibers of

$\partial(\partial M \cup R_1^4 \cup R_2^4)$ are exactly S^1 times the fibers of $\partial(\partial N \cup R)$. This means that after attaching R_1^4 and R_2^4 the fibers consist of two disjoint tori. Furthermore, we observe that since the fibration on N was completed by adding two trivially fibered solid tori to $\partial(\partial N \cup R)$, our fibration on M is completed by adding two trivially fibered $T^2 \times D^2$'s to $\partial M \cup R_1^4 \cup R_2^4 = S^1 \times (\partial N \cup R)$ (again we have reduced the multiplicity from 2 to 1).

To go from the multiplicity 2 case to the general case of multiplicity p, we repeat the above construction inductively using the generic fibration around a $(p, 1)$ exceptional fiber. The result will be a generic fibration with $p-1$ pairs of 4-dimensional round 1- and 2-handles added in succession. Each round 1-handle raises the genus by one on a single component of a fiber, and then the following round 2-handle splits the genus 2 component into two tori. Therefore the preimage of a point in the interior of the round singular images will be the disjoint union of p tori.

$\square$

Now that we have constructed one such generic fibration around a multiple fiber singularity, it is easy to produce others using the homotopy moves of Baykur [9], Lekili [52], and Williams [75].

3.4 An application

We conclude this chapter by applying our construction to give explicit broken Lefschetz fibrations (BLFs) on an important family of 4-manifolds: the

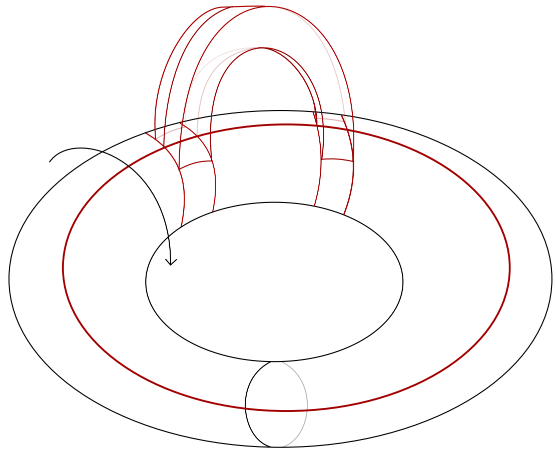

Figure 3.5: Here we change our perspective and consider the contribution of R_1^4 to a single torus fiber $S^1 \times c \subset S^1 \times \partial N$ (the arrow in the picture shows the S^1 direction). The red circle on the torus is a single $\{pt\} \times c$, and adding R_1^4 corresponds to attaching a 2-dimensional 1-handle to the circle in a single frame $\{pt\} \times \partial N$. As $\theta \in I_1$ varies we add an interval's worth of 2-dimensional 1-handles that fill out a 3-dimensional 1-handle attached to our torus fiber.

elliptic surfaces. Our construction may be helpful for studying exotic behavior from the point of view of BLFs.

An elliptic surface is a 4-manifold that admits a (possibly singular) fibration over a surface such that a regular fiber is diffeomorphic to a torus, and with the extra condition that the fibration is locally holomorphic. Up to diffeomorphism we can assume that an elliptic surface is equipped with a fibration map with only Lefschetz singularities and multiple fiber singularities coming from torus surgery (see, for example, Gompf and Stipsicz [29]). Hence we can use our construction to replace the fibration around a multiple fiber with a fibration with only indefinite fold singularities. The resulting fibration is by definition a BLF (since the only other singularities are Lefschetz singularities). Of particular interest are the families of simply-connected exotic elliptic surfaces $E(n)_{p,q}$. The notation means that torus surgery is performed on two separate regular fibers of $E(n)$, one with multiplicity p and the other with multiplicity q for relatively prime p and q. These regular fibers will lie in a cusp neighborhood, and so up to diffeomorphism the surgeries are determined by their multiplicity. Hence we can apply our construction using *integral* multiplicity p and multiplicity q surgery.

Lastly, we consider the special case of the Dolgachev surface $E(1)_{2,3}$.

Example 3.4.1. We construct a BLF on $E(1)_{2,3}$ by replacing the torus multiple fiber of multiplicity 2 with a generic fibration with two fold singularities coming from a round 1-handle and a round 2-handle. The torus multiple fiber of multiplicity 3 is replaced with a generic fibration with 4 fold singularities coming from two successive pairs of round 1- and 2-handles. In Figure 3.6 we draw the critical image on the base S^2 for the BLF on $E(1)_{2,3}$ resulting from our construction.

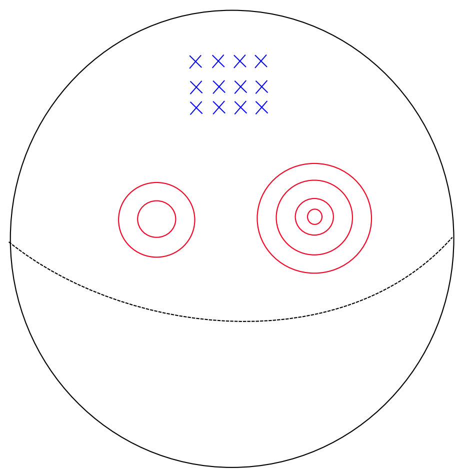

Figure 3.6: The critical image on the base S^2 for a BLF on $E(1)_{2,3}$. The blue x's are the images of the 12 Lefschetz critical points, and the red circles are the images of the fold singularities.

Chapter 4

Fibered ribbon disks

4.1 Introduction

[1] In this chapter we leave behind torus surgery and consider questions about fibered knots and disks. A knot K is fibered if its complement fibers over the circle (with the fibration well-behaved near K). Fibered knots have a long and rich history of study (for both classical knots and higher-dimensional knots). In the classical case, a theorem of Stallings ([71], see also [64]) states that a knot is fibered if and only if its group has a finitely generated commutator subgroup. Stallings [72] also gave a method to produce new fibered knots from old ones by twisting along a fiber, and Harer [33] showed that this twisting operation and a type of plumbing is sufficient to generate all fibered knots in S^3.

Another special class of knots are slice knots. A knot K in S^3 is slice if it bounds a smoothly embedded disk in B^4 (and more generally an n–knot in S^{n+2} is slice if it bounds a disk in B^{n+3}). If K bounds an immersed disk in S^3

[1] This chapter consists of joint work with Jeffrey Meier, the majority of which previously appeared in: Kyle Larson and Jeffrey Meier. Fibered ribbon disks. 2014. Accepted to *Journal of Knot Theory and Its Ramifications*. Preprint available at http://arxiv.org/abs/1410.4854 .

with only ribbon singularities we say K is a ribbon knot. Every ribbon knot is slice, and the famous Slice-Ribbon Conjecture states that every slice knot in S^3 is ribbon. Historically there have been few potential counterexamples due to the difficulty of producing knots that are slice but not obviously ribbon. For recent progress in this direction see [28] and [1]. In this chapter we study slice disks in B^4 whose complements fiber over the circle. The fiber will be a 3–manifold with surface boundary, and the boundary of the slice disk will be a fibered knot in S^3.

A potentially intermediate class of knots between slice and ribbon are homotopy-ribbon knots, which by definition bound slice disks whose complements admit a handle decomposition without handles of index three or higher. Classical work by Casson-Gordon [17] and Cochran [18] shows that for fibered 1–knots and 2–knots this condition can be characterized in terms of certain properties of the fiber. Furthermore, Casson and Gordon show that a fibered homotopy-ribbon 1–knot K bounds a fibered homotopy-ribbon disk in a homotopy 4–ball. Doubling the disk gives a fibered homotopy-ribbon 2–knot in a homotopy 4–sphere. To understand this relationship better we were motivated to study the intermediary case of fibered homotopy-ribbon disks. Our first result is a characterization of such disks in terms of their fiber.

Theorem 4.1.1. Let D be a fibered slice disk in B^4 with fiber H. Then D is homotopy-ribbon if and only if $H \cong H_g$, the solid genus g handlebody.

Next we consider how to produce new fibered disks from old ones, and so prove an analogue of the Stallings twist theorem [72]. The proof continues the broader idea of interpreting certain changes to the monodromy of a mapping torus as surgeries on the total space. The particular surgeries we use for fibered disks double to Gluck twists, so we see the phenomenon of an infinite order operation (twisting along a disk) collapse upon doubling to an order two operation (twisting along a sphere). As a result, we can show that many different fibered disks double to the same fibered 2–knot.

Theorem 4.1.2. Let $D_0 \subset B^4$ be a fibered disk with fiber H, and let $E \subset H$ be an essential, properly embedded disk that is unknotted in B^4. Then, changing the monodromy by twisting m times along E gives a new fibered disk $D_m \subset B^4$. Furthermore,

1. the collection $\{D_m\}_{m\in\mathbb{Z}}$ of disks obtained from twisting contains infinitely many pairwise inequivalent elements, and
2. the collection $\{\mathcal{D}D_m\}_{m\in\mathbb{Z}}$ of 2–knots in S^4 obtained by doubling contains at most *two* pairwise inequivalent elements.

Additionally, if the fiber H is a handlebody, then the disk exteriors $\{\overline{B^4 \setminus \nu D_m}\}_{m\in\mathbb{Z}}$ are all homotopy equivalent.

If we start with a fibered ribbon disk D_0 for a ribbon knot ∂D_0, we can twist m times along an unknotted disk E in the fiber to obtain D_m, which will be a *homotopy*-ribbon disk for ∂D_m, by Theorem 4.1.1. However, it's not obvious that D_m must be ribbon. Therefore, we see that the above procedure could, in principle, be used to produce potential counterexamples to the Slice-Ribbon Conjecture.

In the previous theorem we saw that it was possible for a fibered 2–knot to be obtained as the double of infinitely many different fibered disks (in a fiber-preserving way). In the next theorem we show that this is always the case for fibered homotopy-ribbon 2–knots, with the caveat that we cannot guarantee that the disks lie in B^4.

Theorem 4.1.3. Let $\mathcal{S}$ be a non-trivial fibered homotopy-ribbon 2–knot in S^4. Then $(S^4, \mathcal{S})$ can be expressed as the double of infinitely many pairs (W_m, D_m), where D_m is a fibered homotopy-ribbon disk in a contractible manifold W_m. Furthermore, infinitely many of the W_m are pairwise non-diffeomorphic.

Given a 2–knot $\mathcal{S} \subset S^4$, we call a 1–knot K a *symmetric equator* of $\mathcal{S}$ if $\mathcal{S}$ is the double of a disk along K (in some contractible 4–manifold). We have the following immediate corollary to Theorem 4.1.3.

Corollary 4.1.4. Any non-trivial, fibered homotopy-ribbon 2–knot has infinitely many distinct fibered symmetric equators.

The techniques in this chapter can be illustrated by considering the classical construction of spinning a fibered 1–knot. Spins of fibered knots provide examples to which Theorems 4.1.2 and 4.1.3 can be applied in a very nice way. For example, the collection of 2–knots produced by Theorem 4.1.2 contains only one isotopy class if D_0 is a half-spun disk (see Section 4.6).

4.1.1 Organization

In Section 4.2, we set up basic notation and conventions and introduce the main objects of study. In Section 4.3, we describe how the work of [17] allows us to pass from fibered homotopy-ribbon 1–knots to 2–disk-knots, and then 2–knots upon doubling. We prove Theorem 4.1.1 using the characterization of fibered homotopy-ribbon 2–knots in [18].

A main theme of the chapter is to interpret changes to the monodromy of fibrations as surgeries on the total space, as has classically been done with the Stallings twist. In Section 4.4, we explore this theme in depth, and prove some lemmas required for our main results. In Section 4.5, we prove and discuss Theorems 4.1.2 and 4.1.3, and raise a number of interesting questions. To illustrate the techniques and results found throughout the chapter, we conclude with a discussion of spinning fibered 1–knots in Section 4.6.

4.2 Preliminaries and notation

All manifolds will be assumed to be oriented and smooth, and all maps will be smooth. The boundary of a manifold X will be denoted ∂X. If X is closed, we will denote by X° the manifold obtained by puncturing X (that is, removing the interior of a closed ball from X). The *double* of X is the manifold $\mathcal{D}X = X \cup_{\partial X} (-X)$, where the gluing is done by the identity map. Note that we also have $\mathcal{D}X \cong \partial(X \times I)$. Similarly, the *spin* of a closed manifold X is $\mathcal{S}(X) = \partial(X^\circ \times D^2)$. We will denote the closed tubular neighborhood of an embedded submanifold N of X by νN.

An n–*knot* K is an embedded copy of S^n in S^{n+2}. We say that K is *unknotted* if it bounds an embedded D^{n+1} in S^{n+2}. The *exterior* of K is $E_K = \overline{S^{n+2} \setminus \nu K}$. An n–*disk-knot* D is a proper embedding of the pair $(D^n, \partial D^n)$ into $(D^{n+2}, \partial D^{n+2})$ (we will sometimes refer to these as just *disks*). Observe that ∂D is an $(n-1)$–knot in $\partial D^{n+2} = S^{n+1}$. We say D is *unknotted* if there is an isotopy fixing ∂D that takes D to an embedded disk in ∂D^{n+2} (in particular, this implies that ∂D is unknotted as well). Knots occurring as boundaries of disk-knots are called *slice* knots, and the disk the knot bounds is a called a *slice disk*. Embedded knots and disk-knots are considered up to the equivalence of pairwise diffeomorphism.

Throughout, we will let Σ_g denote the genus g surface, $H_g = \natural_g S^1 \times D^2$ denote the genus g handlebody, and M_g denote $\#_g S^1 \times S^2$.

Let Y be a compact and connected n–manifold with (possibly empty) connected boundary ∂Y, and let $\phi : Y \to Y$ be a diffeomorphism. The *mapping torus* $Y \times_\phi S^1$ of Y is the $(n+1)$–manifold formed from $Y \times I$ by identifying $y \times \{1\}$ with $\phi(y) \times \{0\}$ for all $y \in Y$. We see that a mapping torus is a fiber bundle over S^1 with fiber Y. The map ϕ is called the *monodromy* of the mapping torus. If ∂Y is non-empty, then $\partial(Y \times_\phi S^1)$ is a mapping torus with fiber ∂Y and monodromy the restriction $\phi|_{\partial Y}$. We are especially interested in the case where the exterior of a knot or disk admits such a fibration, so we highlight the following definition.

Definition 4.2.1. We say an n–knot K is *fibered* if E_K has the structure of a mapping torus with the additional condition that the boundary of the mapping torus is identified with $\partial(\nu K) = K \times \partial D^2$ such that the fibration map is projection onto the second factor. An n–disk-knot D is *fibered* if $\overline{D^n \setminus \nu D}$ has the structure of a mapping torus (again, trivial on $\partial(\nu D)$). In this case, we see that ∂D is a fibered knot.

Remark 4.2.2. While a fibered n–knot will have a *punctured* $(n+1)$–manifold as a fiber, we can fill in the punctures with a copy of $S^1 \times D^{n+1}$ to get a mapping torus without boundary. This closed mapping torus can be obtained by surgering the n–knot rather than removing it. Therefore, in what follows it may be convenient to switch between these two set-ups.

More generally we will say that a disk or sphere embedded in an *arbitrary* manifold is fibered if its complement admits the above structure. For example, in this chapter we will consider fibered knots in homology 3–spheres and fibered disk-knots in contractible 4–manifolds. The above set-up generalizes easily to these settings.

Example 4.2.3. Let K be a fibered knot in S^3, so E_K admits the structure of a mapping torus $\Sigma_g^\circ \times_\varphi S^1$. The boundary of Σ_g° is a longitude of K, so performing 0–surgery on K glues in a disk to each longitude, resulting in a closed surface bundle $S_0^3(K) = \Sigma_g \times_{\widehat{\varphi}} S^1$. The monodromy $\widehat{\varphi}$ is obtained by extending φ across the capped off surface Σ_g. The simplest example is when K is the unknot, in which case the fibers are disks, and 0–surgery results in $S^2 \times S^1$.

For an arbitrary fibered knot $K \subset S^3$, performing $(1/n)$–surgery on K results in a homology 3–sphere $S^3_{1/n}(K)$. Let K' be the core of the glued-in solid torus, also called the *dual knot* of the surgery. Since K' is the core of the surgery torus, we see that $E_{K'} = E_K$, and so K' is a fibered knot in $S^3_{1/n}(K)$. Note that the boundary compatibility condition is still satisfied because K' and K share the same longitudes in their shared exterior.

We conclude this section by examining a concept that will be central throughout the chapter: the relationship between fibered disk-knots and fibered 2–knots. Let H be a compact 3–manifold with $\partial H \cong \Sigma_g$. Let $\phi : H \to H$ be a diffeomorphism, and consider the mapping torus $X_0 = H \times_\phi S^1$. We can isotope ϕ so that it fixes a small disk $D^2 \subset \partial H$. This gives us an embedded solid torus $D^2 \times S^1$ in ∂X_0, and a fixed fibering of $\partial D^2 \times S^1$ by a preferred longitude $\lambda = \{pt\} \times S^1$. See Figure 4.1(a).

Consider the 4–manifold $X = X_0 \cup_f h$ obtained by attaching 4–dimensional 2–handle h along $D^2 \times S^1$. We say that h has framing k if the framing is related to the one induced by the product structure on the $D^2 \times S^1$ by taking k full right-handed twists (for negative k take left-handed twists). Observe that the cocore D of h is a fibered disk in X, since removing a neighborhood of D is equivalent to removing h, and results in the fibered manifold X_0.

If we double D we get a 2–knot $\mathcal{S} = \mathcal{D}D \subset \mathcal{D}X$, and notice that $\mathcal{S}$ is a fibered 2–knot in $\mathcal{D}X$ with fiber $M = \mathcal{D}H$. The monodromy $\Phi : M \to M$ is the *double* of the monodromy $\phi : H \to H$, so $E_{\mathcal{S}} = \overline{\mathcal{D}X \setminus \nu\mathcal{S}} \cong M^\circ \times_\Phi S^1$.

Lemma 4.2.4. The pair $(\mathcal{D}X, \mathcal{S})$ depends only on the parity of k, and the two possible pairs are related by a Gluck twist on $\mathcal{S}$.

Proof. The pair $(\mathcal{D}X, \mathcal{S})$ is obtained from $E_{\mathcal{S}}$ by gluing in $S^2 \times D^2$. Gluing in a copy of $S^2 \times D^2$ to a 4–manifold with $S^2 \times S^1$ boundary amounts to attaching a 2–handle and a 4–handle, since $S^2 \times D^2$ admits a handle decomposition relative to its boundary with one 2–handle and one 4–handle. We can choose the 2–handle to be h from the previous paragraph; in other words, we can isotope the attaching region of the 2–handle to lie in one hemisphere of $S^2 \times S^1$. There is a unique way to attach the 4–handle. Gluck [25] showed that up to isotopy there is a unique, non-trivial way to glue in $S^2 \times D^2$. This corresponds to the unique element ρ in the mapping class group of $S^2 \times S^1$ that doesn't extend over $S^2 \times D^2$. Choosing framing k corresponds to gluing by the map ρ^k, and since ρ^2 is isotopic to the identity, the result follows. Lastly, we point out that the manifolds coming from even or odd framings are related by a Gluck twist. (See Section 4.4 for more details.) $\square$

In this chapter we are interested in disk-knots with fiber $H_g = \natural_g S^1 \times D^2$ and 2–knots with fiber $M_g = (\#_g S^1 \times S^2)^\circ$, which are clearly related by doubling in the way we have just discussed.

Let $D \subset B^4$ be a fibered disk-knot with fiber H_g and monodromy ϕ, so $\overline{B^4 \setminus \nu(D)} \cong H_g \times_\phi S^1$. Note that $\partial(B^4, D) = (S^3, K)$ is a fibered slice knot with fiber Σ_g° and monodromy φ, where $\varphi = \phi|_{(\partial H_g)^\circ}$, as in Example 4.2.3 and

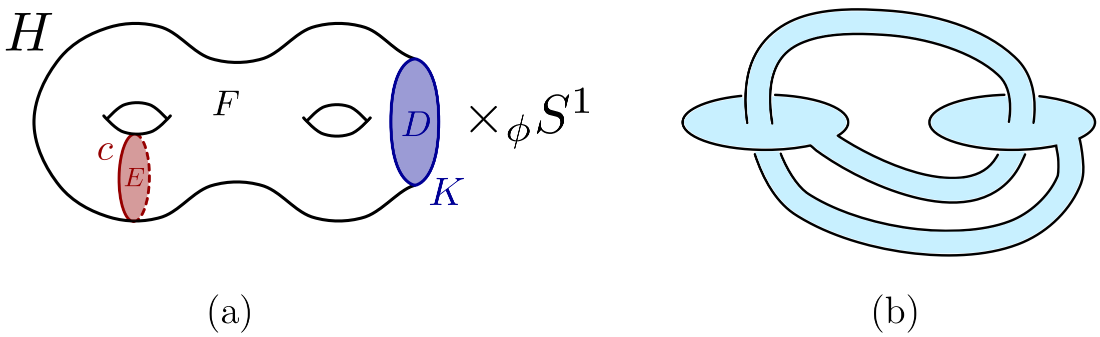

Figure 4.1: A schematic (a) for a handlebody bundle, and (b) an immersed ribbon disk in S^3. In (a), ϕ has been isotoped to preserve D, which we think of as a slice disk for the fibered knot K, whose fiber is F. The boundary of the handlebody bundle is the surface bundle $S^3_0(K)$, which has $\widehat{F} = F \cup_K D$ as fibers. A Stallings curve c is shown as the boundary of a disk E in H.

Figure 4.1(a). Doubling (B^4, D) results in a fibered 2–knot $\mathcal{S} \subset S^4$ with fiber M_g°.

The rest of the chapter is devoted to the analysis of these objects.

4.3 Homotopy-ribbon knots and disks

Following [18], we will say that an n–knot $K \subset S^{n+2}$ is *homotopy-ribbon* if it bounds a slice disk $D \subset D^{n+3}$ with the property that $\overline{D^{n+3} \setminus \nu D}$ admits a handle decomposition with only 0–, 1–, and 2–handles, in which case D is called a *homotopy-ribbon* disk for K[2]. The knot K is called *ribbon* if it bounds an immersed disk in S^{n+2} with only ribbon singularities. (See [43] for details.) These singularities can be removed by pushing the disk into D^{n+3}, giving a

[2]Although a more common definition for homotopy-ribbon is what we call *weakly* homotopy-ribbon, we use the definition given by Cochran in [18] since it makes our statements simpler.

ribbon disk for K. See Figure 4.1(b) for an example.

We call a slice knot $K \subset S^{n+2}$ *weakly homotopy-ribbon* if there is a surjection

$$\pi_1(S^{n+2} \setminus K) \twoheadrightarrow \pi_1(D^{n+3} \setminus D).$$

We have the following inclusions among n-knots:

$$\textit{ribbon} \subseteq \textit{homotopy-ribbon} \subseteq \textit{weakly homotopy-ribbon} \subseteq \textit{slice}$$

The Slice-Ribbon Conjecture postulates that every slice knot in S^3 is ribbon. In fact, in the classical case, it is not known whether any of the converse inclusions hold or not. On the other hand, it is known that every 2–knot is slice [44], while some 2–knots are not ribbon [76]. We now state the theorems of Cochran and Casson-Gordon that motivate the present work.

Theorem 4.3.1 (Cochran, [18])**.** Let $K \subset S^4$ be a fibered 2–knot with fiber M°. Then K is homotopy-ribbon if and only if $M \cong \#_g S^1 \times S^2$.

Theorem 4.3.2 (Casson-Gordon, [17])**.** Let $K \subset S^3$ be a fibered 1–knot with monodromy $\varphi : \Sigma_g^\circ \to \Sigma_g^\circ$.

1. If K bounds a homotopy-ribbon disk $D \subset B^4$, then the closed monodromy $\widehat{\varphi} : \Sigma_g \to \Sigma_g$ extends over a handlebody to $\phi : H_g \to H_g$.

2. If the closed monodromy $\widehat{\varphi}$ extends over a handlebody to $\phi : H_g \to H_g$, then there is a homotopy-ribbon-disk D' for K in a homotopy 4–ball B such that $\overline{B \setminus \nu(D')} \cong H_g \times_\phi S^1$.

We remark that the original theorem stated by Casson-Gordon is slightly more general, and uses the property we call "weakly homotopy-ribbon". However, the theorem can be strengthened to conclude homotopy-ribbon, since every mapping torus of H_g can be built with only 0–, 1–, and 2–handles (see Lemma 4.3.3).

We see there is a strong correspondence between fibered homotopy-ribbon knots and conditions on the fiber. Here, we expand the picture to include fibered 2–disk-knots.

Theorem 4.1.1. Let D be a fibered disk-knot in B^4 with fiber H. Then D is homotopy-ribbon if and only if $H \cong H_g$ for some g.

Proof. Suppose that D is a fibered homotopy-ribbon disk-knot. Let $\mathcal{S} \subset S^4$ be the 2–knot obtained by doubling D. Then $\mathcal{S}$ is a fibered homotopy-ribbon 2–knot with fiber $M = H \cup_F H$, where $F = \partial H$. To see this, consider the product $(B^4 \times I, D \times I)$ of (B^4, D). The 3–ball $D \times I$ is a slice disk in B^5 for $\mathcal{S} = \mathcal{D}D = \partial(D \times I)$. In fact, it is a homotopy-ribbon disk; the exterior of $D \times I$ is obtained by crossing the exterior of D with I, and this preserves the indices of the handle decompositions. By Theorem 4.3.1, this means that $M \cong M_h = \#_h S^1 \times S^2$ for some h.

Now, suppose that the genus of $F = \partial H$ is g. Since F is a closed, separating surface in M, it can be compressed g times (by the Loop Theorem [66]). Since $M = H \cup_F H$ is a double, these compressions can be done one by one and symmetrically to both sides. It follows that $H = H_g\#(\#_{\frac{h-g}{2}} S^1 \times S^2)$. However, we claim that there is a surjection $\pi_1(F) \twoheadrightarrow \pi_1(H)$. This implies that $h = g$, as desired.

To see where this surjection comes from, consider the infinite cyclic cover $\widetilde{W}$ of $W = \overline{B^4 \setminus \nu(D)}$. We have that $\widetilde{W} \cong H \times \mathbb{R}$ and $\partial\widetilde{W} \cong F \times \mathbb{R}$. Furthermore, we have the following identifications of the commutator subgroups:

$$\pi_1(H) \cong \pi_1(\widetilde{W}) \cong [\pi_1(W), \pi_1(W)]$$

and

$$\pi_1(F) \cong \pi_1(\partial\widetilde{W}) \cong [\pi_1(Y), \pi_1(Y)],$$

where $Y = \partial W \cong S^3_0(\partial D)$ is given by 0–Dehn surgery on the knot $\partial D \subset S^3$.

By assumption, $\pi_1(S^3 \setminus \partial D) \twoheadrightarrow \pi_1(W)$ (here we use the weaker definition of homotopy-ribbon, which follows from the stronger), which implies that $\pi_1(S^3_0(\partial D)) \twoheadrightarrow \pi_1(W)$ (since the class of the longitude includes to the trivial element). This gives a surjection between the corresponding commutator subgroups. It follows that $\pi_1(F) \twoheadrightarrow \pi_1(H)$, as desired.

Conversely, suppose that D is a fibered disk-knot with handlebody fiber. Since H_g admits a handle decomposition with only 0– and 1–handles, by Lemma 4.3.3 below, the exterior of D in B^4 can be built using only 0–, 1–, and 2–handles. This, by definition, implies that D is homotopy-ribbon.

$\square$

The following lemma is well known (see, for example, [61], or for more exposition and examples [5]).

Lemma 4.3.3. Let $X = Y \times_\phi S^1$ be a mapping torus with fiber an n–dimensional manifold Y and monodromy $\phi\colon Y \to Y$. Then a handle decompostion of Y induces a handle decomposition of X, with each k-handle of Y inducing a k-handle and a $(k+1)$-handle for X.

Proof. Recall that X is formed by taking the product $Y \times I$ and identifying $y \times \{1\}$ with $\phi(y) \times \{0\}$ for all $y \in Y$ (where $I = [0,1]$). Fix a handle decomposition of Y, and let h_k be a k-handle for Y. Then h_k is a copy of $D^k \times D^{n-k}$ attached along $\partial D^k \times D^{n-k}$. Let S denote the attaching sphere and C the core of h_k. Now h_k induces a k-handle H_k and a $(k+1)$-handle H_{k+1} for X as follows. We let $H_k = h_k \times [0,1/2] = D^k \times (D^{n-k} \times [0,1/2])$, attached along $\partial D^k \times (D^{n-k} \times [0,1/2])$. We form H_{k+1} from $h_k \times [1/2,1]$ by associating the extra interval with the D^k factor of h_k. That is, we define $H_{k+1} = (D^k \times [1/2,1]) \times D^{n-k}$, attached along $\partial(D^k \times [1/2,1]) \times D^{n-k}$. The attaching sphere for H_{k+1} will be the union of $S \times [1/2,1]$ and $C \times \partial[1/2,1]$. Note that $C \times \{1\}$ is identified with $\phi(C) \times \{0\}$ in X.

We think of this process as first constructing a handle decomposition for $Y \times [0,1/2]$ by taking the handle decomposition for Y and crossing each handle with $[0,1/2]$ (increasing the dimension of the handles but preserving the indices). Then we complete the handle decomposition for X by adding the second set of handles (where both the dimension and index of the handles of Y are increased by one). For example, if the monodromy ϕ is the identity map (so that $X = Y \times S^1$), then for a given handle h_k of Y we think of H_{k+1} as gluing $h_k \times \{1/2\}$ to $h_k \times \{0\}$ and so completing the mapping torus. $\square$

In Theorem 4.1.1 it is worth noting that the restriction to homotopy-ribbon disk-knots is important, since any fibered slice knot can bound infinitely many different fibered slice disks in the 4–ball, none of which are homotopy-ribbon. To see this, take any fibered slice disk D in the 4–ball with boundary some fibered slice knot and form the fiber-preserving connected sum $D\#\mathcal{S}$ for any fibered 2–knot $\mathcal{S}$. For example, if we take any fibered 2–knot $\mathcal{S} \subset S^4$ and remove a small 4–ball centered on a point in $\mathcal{S}$, then the result is a slice disk D for the unknot whose fibers are the same as those of $\mathcal{S}$.

Example 4.3.4. Many infinite families of handlebody bundles whose total space is the complement of a ribbon disk in B^4 were produced by Aitchison-Silver using *construction by isotopy*. Their main result states that the boundaries of these fibered ribbon disks are doubly slice, fibered ribbon knots that, collectively, realize all possible Alexander polynomials for such knots. See [3] for complete details.

The result of Casson-Gordon (Theorem 4.3.2) allows us to start with a fibered homotopy-ribbon 1–knot and construct a fibered homotopy-ribbon disk in a homotopy 4–ball B. This is accomplished by extending the (closed) monodromy to H_g and adding a 2–handle to the resulting mapping torus.

Next, we can double the resulting fibered disk to get a fibered homotopy-ribbon 2–knot in the homotopy 4–sphere $\mathcal{D}B$. However, it is not known in general whether B and $\mathcal{D}B$ must be standard. This suggests the following question.

Question 4.3.5. Does a fibered homotopy-ribbon knot always extend to a fibered homotopy-ribbon disk in B^4? Specifically, must B be diffeomorphic to B^4?

It is also not clear if the resulting fibered disk and 2–knot depend only on the original 1–knot, or if they also depend on the choice of monodromy extension. We remark that Long has given an example of a surface monodromy that extends over distinct handlebodies [56].

4.4 Changing monodromies by surgery

Let $K \subset S^3$ be a fibered knot with fiber Σ_g° and monodromy φ. An essential curve c on a fiber F_* is called a *Stallings curve* if c bounds a disk D_c in S^3 (called a *Stallings disk*), and the framing on c induced by F_* is zero. Given such a curve, we can cut open E_K along F_* and re-glue using the surface diffeomorphism $\tau_c : F_* \to F_*$, which is given by a Dehn twist along $c \subset F_*$. This operation, called a *Stallings twist* along c, produces a new fibered knot $K' \subset S^3$ with fiber Σ_g° and monodromy $\phi' = \phi \circ \tau_c^{\pm 1}$ [72].

The Stallings twist is a classical operation and provides the first instance of a more general theme: interpreting a change to the monodromy of

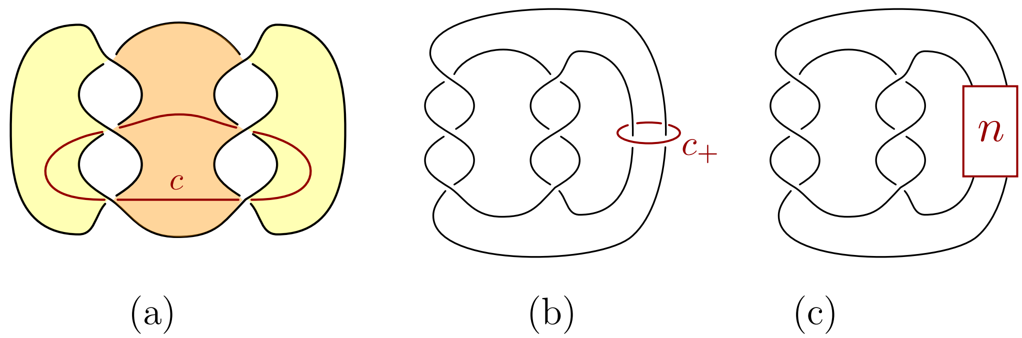

Figure 4.2: An example of Stallings twisting on a fibered ribbon knot along a curve c (shown in (a)) that extends to a disk in the handlebody. In (b), we see a push-off c_+ of c (after isotoping the knot), and (c) shows the fibered ribbon knots obtained by twisting n times along c. (The box represents n full-twists.)

a fiber bundle as a surgery on the total space. If K' is obtained from K by a Stallings twist, then the resulting mapping torus $E_{K'}$ is related to the original by ± 1 Dehn surgery on c in E_K. The following lemma allows us to conclude that Stallings twists can be used to create infinite families of distinct knots.

Lemma 4.4.1. Suppose that K' is obtained from K by twisting m times about a Stallings curve c for K, and assume that $g(K) \geq 2$. Then K' is distinct from K provided $|m| = 1$ or $|m| > 9g - 3$.

Proof. If $K' = K$, then the corresponding monodromies are conjugate [15]:

$$\varphi \circ \tau_c^m = \sigma^{-1} \varphi \sigma.$$

This implies that τ_c^m is a commutator. Corollary 2.6 of [47] states that a single Dehn twist is never a commutator, which yields the result when $|m| = 1$, and Corollary 2.4 of [46] states that τ_c^m cannot be a commutator when $|m| > 9g - 3$. □

There are other instances where interpreting a change to the monodromy as a surgery has proved useful. Gompf [26] used this approach to study Cappell-Shaneson 4–spheres, which are constructed from mapping tori of T^3. There, the corresponding operations are changing the monodromy by twisting along a torus and performing torus surgery. Nash [63] pursued this idea further in his thesis.

In what follows, we will interpret changing the monodromy of a handlebody bundle by twisting along a disk as a 2–handle surgery, and changing the monodromy of a $\#_g S^1 \times S^2$ bundle by twisting along a sphere as a Gluck twist. We first give some precise definitions of the diffeomorphisms by which we will change the monodromies.

Definition 4.4.2. Let $\omega_\theta : D^2 \to D^2$ and $\Omega_\theta : S^2 \to S^2$ be the maps given by clockwise rotation of D^2 and S^2 (about some fixed axis) through the angle θ. Let H be a compact 3–manifold with boundary and let M be a compact 3–manifold.

Given a properly embedded disk $E \subset H$, we can identify a neighborhood of E with $D^2 \times I$ and define a map $\tau_E : H \to H$ to be the identity map away from $D^2 \times I$, and $\tau_D(z,t) = (\omega_{2\pi t}(z), t)$ on $D^2 \times I$.

Similarly, given an embedded 2–sphere S in M, we can identify a neighborhood of S with $S^2 \times I$ and define a map $\tau_S : M \to M$ to be the identity map away from $S^2 \times I$, and $\tau_S(x,t) = (\Omega_{2\pi t}(x), t)$ on $S^2 \times I$.

We will call these maps *twisting along the disk D* and *twisting along* *the sphere* S, respectively. Observe that we can iterate these operations to twist m times along a disk or sphere, for any $m \in \mathbb{Z}$, where $m < 0$ should be interpreted as counterclockwise rotation.

Remark 4.4.3. The restriction $\tau_D|_{\partial H} : \partial H \to \partial H$ is a right-handed Dehn twist τ_c along the curve $c = \partial D^2$. If we double the operation of twisting along a disk we get twisting along a sphere. That is, the double of the map $\tau_D : H \to H$ is the map $\tau_S : M \to M$ where S is the double of D in $M = \mathcal{D}H$.

Later, we will mostly be interested in the case when $H = H_g$ and $M = M_g$, but we will continue to work in generality for the time being. Next, we introduce two types of surgery and examine how they relate to the diffeomorphisms defined above.

4.4.1 The Gluck twist

Definition 4.4.4. Given an embedded 2–sphere S in a 4–manifold X with trivial normal bundle, we can identify a neighborhood of S with $S^2 \times D^2$. We produce a new 4–manifold X_S by removing $S^2 \times D^2$ and re-gluing it using the map $\rho : S^2 \times S^1 \to S^2 \times S^1$ defined by $\rho(x, \theta) = (\Omega_\theta(x), \theta)$. The manifold X_S is said to be the result of performing a *Gluck twist* on S.

Gluck [25] defined the preceding operation and showed that, up to isotopy, ρ is the only non-trivial way to glue in an $S^2 \times D^2$ to a manifold with $S^2 \times S^1$ boundary. Furthermore, he showed that ρ^2 is isotopic to the identity map, so ρ has order two in the mapping class group. Therefore, performing two consecutive Gluck twists on a sphere gives you back the original 4–manifold.

Recall that gluing in a copy of $S^2 \times D^2$ to a 4–manifold with $S^2 \times S^1$ boundary amounts to attaching a 2–handle h and a 4–handle. The attaching circle for h will be $\{pt\} \times S^1$, and the framing will be zero (corresponding to the product framing) if we glue by the identity map, or ± 1 if we glue by the Gluck twist map ρ. There is a unique way to attach the 4–handle in either case. Furthermore, since ρ^2 is trivial, we see that the framing only matters mod 2.

Remark 4.4.5. Note that ρ is isotopic to a map taking $\Omega_{2\pi t}$ on half of the circle (identifying half the circle with I) and the identity map on the other half. Therefore we can substitute such a map in the gluing process without changing the result.

Performing a Gluck twist on a 2–knot $\mathcal{S} \subset S^4$ produces a homotopy 4–sphere $X_{\mathcal{S}}$, and it is not known, in general, when $X_{\mathcal{S}}$ is diffeomorphic to S^4. For certain types of 2–knots, however, it is known that a Gluck twist not only returns S^4, but also preserves the equivalence class of the 2–knot. The easiest case is for the unknotted 2–sphere, which we record here for later use.

Lemma 4.4.6. Let $\mathcal{U}$ be the unknotted 2–sphere in S^4. Performing a Gluck twist on $\mathcal{U}$ gives back $(S^4, \mathcal{U})$.

Proof. It is a basic fact that $\overline{S^4 \setminus \nu\mathcal{U}} = B^3 \times S^1$. If $X_{\mathcal{U}}$ is the result of performing a Gluck twist on $\mathcal{U}$, then $X_{\mathcal{U}} = B^3 \times S^1 \cup_\rho S^2 \times D^2$. Since ρ clearly extends over $B^3 \times S^1$, we see that $X_{\mathcal{U}} = B^3 \times S^1 \cup_\rho S^2 \times D^2$ is diffeomorphic to $B^3 \times S^1 \cup_{id} S^2 \times D^2 \cong S^4$, and the diffeomorphism is the identity map on $S^2 \times D^2$ (and so preserves $\mathcal{U}$). We think of this diffeomorphism as "untwisting" along $B^3 \times S^1$. $\square$

Remark 4.4.7. In fact, this can be generalized to any 2–knot $\mathcal{S}$ that bounds $(\#_g S^1 \times S^2)^\circ$. See Chapter 13 of [43] for details. It follows, for example, that if $\mathcal{S} \subset S^4$ is a homotopy-ribbon 2–knot, then $X_{\mathcal{S}} \cong S^4$, and the Gluck twist preserves the homotopy-ribbon 2–knot.

Our main observation here is that changing the monodromy of a 4–dimensional mapping torus by twisting along a sphere contained in a fiber corresponds to performing a Gluck twist on the total space.

Proposition 4.4.8. Let M be a compact 3–manifold, and consider the mapping torus $X = M \times_\Phi S^1$. Let S be an embedded 2–sphere in a fiber M_*. Let $X' = M \times_{\phi\circ\tau_S} S^1$ be the mapping torus formed by cutting X along M_* and re-gluing with the diffeomorphism $\tau_S : M_* \to M_*$. Then X' is obtained from X by performing a Gluck twist on S. Furthermore, applying τ_S twice gives $X'' = M \times_{\phi\circ\tau_S^2} S^1$, which is diffeomorphic to X.

Proof. Identify a neighborhood of S in M_* with $S \times I$. Since the mapping torus is locally a product $M_* \times I_0$, we can identify a neighborhood of S in X with $S \times I \times I_0$. Performing a Gluck twist on S corresponds to deleting $S \times I \times I_0$ and re-gluing it by ρ along $S \times \partial(I \times I_0)$. By Remark 4.4.5, we can isotope ρ so that it is the identity map except on $S \times I \times \{1\}$, where we take $\Omega_{2\pi t}$. The result is clearly diffeomorphic to cutting along the fiber M and re-gluing by τ_S, so we get the first claim. The second claim follows from the first claim and the fact that performing a Gluck twist twice on a sphere preserves the original diffeomorphism type. $\square$

4.4.2 2–handle surgery

Next, we introduce an operation on 4–manifolds with boundary that will be instrumental in proving our main results.

Definition 4.4.9. Suppose a 4–manifold X can be decomposed as $X = X_0 \cup_f h$, with a 2–handle h attached to some 4–manifold with boundary X_0 with framing f. We say X' is obtained from X by *2–handle surgery* on h with *slope* m if $X' = X_0 \cup_{f'} h$ is formed by removing h and re-attaching it with the same attaching circle but with framing f', where f' is obtained from f by adding m right-hand twists.

In other words, 2–handle surgery is the process of cutting out and re-gluing a B^4 along $S^1 \times D^2$. We make the following simple observations.

Lemma 4.4.10. Suppose X' is obtained from X by 2–handle surgery. Then $\pi_1(X') \cong \pi_1(X)$ and $H_*(X) \cong H_*(X')$.

Note that it is possible to change the intersection form of X via 2–handle surgery.

Proof. We get a handle decomposition for X by taking a handle decomposition for X_0 and adding h. A handle decomposition for a 4–manifold allows one to read off a presentation of the fundamental group: each 1–handle gives a generator, and the attaching circle for each 2–handle provides a relation. Now the framings for the 2–handles do not affect the relations, so removing h removes a relation and re–attaching h (with any framing) adds the same relation back. Thus, $\pi_1(X') \cong \pi_1(X)$. Similarly, the homology groups can be computed in a simple way from a handle decomposition, and one can check that they don't depend on the framings of the 2–handles. □

The next lemma gives a condition for when a 2–handle surgery preserves the diffeomorphism type of the 4–manifold. Recall that a properly embedded disk D in a 4–manifold X is said to be *unknotted* if it is isotopic rel ∂ into the boundary of X.

Lemma 4.4.11. Suppose X contains a 2–handle h whose cocore is unknotted in X. Then any 2–handle surgery on h will preserve the diffeomorphism type of X.

Proof. Removing h is equivalent to removing a neighborhood of the cocore D. If D is unknotted, then this can be thought of as adding a 1–handle to X. We form X' by attaching a 2–handle along the former attaching circle, which will intersect the belt sphere of the 1–handle geometrically once. Therefore, these two handles will cancel (regardless of framing). It follows that we have not changed X, so X' is diffeomorphic to X. In other words, D being unknotted is equivalent to the existence of a 1–handle that h cancels, and the cancellation does not depend on the framing of h. $\square$

Next, we consider how the boundary of a 4–manifold changes when doing 2–handle surgery. Recall that for $X = X_0 \cup h$, ∂X is obtained from ∂X_0 by performing integral surgery on the attaching circle K of h, with the surgery coefficient given by the framing. The belt sphere of h in ∂X is then the dual knot K' of K. If X' is the result of the 2–handle surgery, then $\partial X'$ is obtained from ∂X by doing surgery on K', which can be seen as the composition of two surgeries: first do the 'dual surgery' from ∂X back to ∂X_0, then do another surgery (corresponding to the new framing of h) from ∂X_0 to $\partial X'$. Note that the surgery from ∂X to $\partial X'$ won't be integral, in general.

Example 4.4.12. Suppose that $X = B^4$, and X_0 is the exterior of a slice disk for a knot $K \subset S^3 = \partial B^4$. Then, ∂X_0 is the 3–manifold obtained by 0–surgery on K. We see that X is obtained from X_0 by attaching a 0–framed 2–handle h along $K' \subset \partial X_0$, where K' is the dual knot to K. Now, X' is formed by 2–handle surgery on h with slope m if h is attached with framing m instead of framing 0, so $\partial X'$ is m–Dehn surgery on K' in ∂X_0. From this viewpoint, we see that $\partial X'$ is obtained by doing $(-1/m)$–Dehn surgery on K in S^3.

Now we show that changing the monodromy of a 4–dimensional mapping torus with boundary by twisting along a disk that is properly embedded in a fiber corresponds to performing a 2–handle surgery on the total space.

Proposition 4.4.13. Let H be a compact 3–manifold with boundary, and consider the mapping torus $X = H \times_\phi S^1$. Let D be a disk that is properly embedded in a fiber H_*. Let $X' = H \times_{\Phi \circ \tau_D} S^1$ be the mapping torus obtained by cutting X along H_* and re-gluing with the diffeomorphism $\tau_D : H_* \to H_*$. Then, X' can be obtained from X by performing a 2–handle surgery on a handle h in X where the cocore of h is D.

Proof. Similarly to Proposition 4.4.8, we identify a neighborhood of D in H_* with $D \times I$ and a neighborhood of D in X as $N = D \times I \times I_0$. Then N is diffeomorphic to B^4, and $N \cap (\overline{X \setminus N}) = D \times \partial(I \times I_0)$ is a solid torus. Thus, we can view N as a 2–handle attached to $\overline{X \setminus N}$ with cocore $D \times \{0\} \times \{0\}$. Cutting out N and re-gluing it is then a 2–handle surgery. We choose the gluing map to be the identity map except on $D \times I \times \{1\}$, where we take $\omega_{2\pi t}$. This is clearly diffeomorphic to cutting along the fiber and re-gluing by τ_D to obtain X'. □

4.5 Main results

Now we can apply the techniques from the previous section to prove our main results. Here we give slightly more detailed statements of the theorems described in the introduction.

Theorem 4.1.2. Let $D_0 \subset B^4$ be a fibered disk-knot with fiber H and monodromy ϕ, and let $E \subset H$ be an essential, properly embedded disk that is unknotted in B^4. Then, the result of twisting m times along E is a new fibered disk-knot $D_m \subset B^4$ with monodromy $\phi_m = \phi \circ \tau_E^m$. Furthermore,

1. $K_m = \partial D_m$ is a fibered knot in S^3 with monodromy $\varphi_m = \varphi_0 \circ \tau_c^m$, where $c = \partial E$ is a Stallings curve;

2. the collection $\{D_m\}_{m\in\mathbb{Z}}$ contains infinitely many pairwise inequivalent fibered disk-knots, where $D_0 = D$; and

3. the collection $\{\mathcal{D}D_m\}_{m\in\mathbb{Z}}$ of fibered 2–knots obtained by doubling contains at most two pairwise inequivalent elements.

If in addition the fiber H is the handlebody H_g, then the family of disk exteriors $\{\overline{B^4 \setminus \nu D_m}\}_{m\in\mathbb{Z}}$ will all be homotopy equivalent.

Note that we say a properly embedded disk $E \subset H$ is *essential* if ∂D doesn't bound a disk in ∂H.

Proof. Changing the monodromy by twisting along E is equivalent to performing a 2–handle surgery, by Proposition 4.4.13, where the cocore of the 2–handle is E. Since E is unknotted, Lemma 4.4.11 states that the diffeomorphism type of the total space doesn't change, and so the result is a fibered disk-knot in B^4. In fact there is a small subtlety: we are using the fact that performing the 2–handle surgery in the disk exterior $H_g \times_\phi S^1$ and then filling back in the the disk D is equivalent to performing the 2–handle surgery in $H_g \times_\phi S^1 \cup \nu D = B^4$. However, the order of these operations is insignificant, because $E \cap \nu D = \emptyset$.

Because E is unknotted, it can be isotoped to a disk E' lying in $S^3 = \partial B^4$ such that $c = \partial E'$ lies on a fiber surface F_0 for $K_0 = \partial D_0$. Furthermore, the framing on c induced by F is zero, since c bounds a disk in H_g. Therefore, E' is a Stallings disk for $K_0 = \partial D_0$. Since twisting along a disk restricts to a Dehn twist on the boundary, we see that we are changing the monodromy of the boundary surface bundle by $\tau_E^m|_{\partial H} = \tau_c^m$. This settles part (1).

Part (2) follows from the fact that infinitely many of the boundary knots ∂D_m are distinct. For example, the collection $\{K_{k(9g-2)}\}_{k\in\mathbb{Z}}$ contains pairwise distinct knots, by Lemma 4.4.1 (here we use the fact that there are no fibered genus 1 knots that are slice, and so the genus is at least 2).

Part (3) follows from Remark 4.4.3 and Proposition 4.4.8.

Finally, if $H = H_g$, then we will show that $Z_m = H_g \times_{\phi\circ\tau_E^m} S^1$ is a $K(G,1)$, where $G \cong \pi_1(Z_m)$ is independent of m, by Lemma 4.4.10. By the long exact sequence of a fibration, and since the base space S^1 satisfies $\pi_n(S^1) = 0$ for $n > 1$, we see that $\pi_n(Z_m) \cong \pi_n(H_g) = 0$ for all $n > 1$. Thus, the Z_m are homotopy equivalent for all m. $\square$

Remark 4.5.1. In many cases, it may be that $\mathcal{D}D_m = \mathcal{D}D_n$ for all $m, n \in \mathbb{Z}$. For example, when we consider spinning fibered 1–knots below, we will see examples where the resulting collection of 2–knots contains a *unique* isotopy class.

In regards to the last statement of the theorem we note that Gordon and Sumners [32] gave examples of disks whose exteriors have the homotopy type of a circle and double to give the unknotted 2–sphere.

Now given a fibered 2–knot $\mathcal{S} \subset S^4$ with fiber $\#_g S^1 \times S^2$ (i.e. a fibered homotopy-ribbon 2–knot by Cochran), we consider the ways in which it can be decomposed as the *double* of a fibered disk in a contractible 4–manifold. We have already seen in Theorem 4.1.2 that it is possible for a 2–knot to be the double of infinitely many distinct disk-knots in B^4, but this is a somewhat special situation. We next prove a sort of converse result, which is more general but comes with the trade-off that we can no longer guarantee that the fibered disks lie in B^4. Again, here we give a more detailed statement than in the introduction.

Theorem 4.1.3. Let $\mathcal{S}$ be a non-trivial fibered 2–knot in S^4 with fiber $(\#_g S^1 \times S^2)^\circ$. Then $(S^4, \mathcal{S})$ can be expressed as the double of infinitely many pairs (W_m, D_m), where

1. W_m is a contractible 4–manifold;
2. D_m is a fibered homotopy-ribbon disk-knot in W_m;
3. the boundaries $Y_m = \partial W_m$ are all related by Dehn filling on a common 3–manifold; and
4. infinitely many of the Y_m (and, therefore, the corresponding W_m) are non-diffeomorphic.

We remark that the theorem holds in a slightly more general setting.

One could consider $\mathcal{S}$ to be in a homotopy 4–sphere, for example, yet draw the same conclusions.

To prove Theorem 4.1.3, we must first record a fact about self-diffeomorphisms of $M_g = \#_g S^1 \times S^2$.

Proposition 4.5.2. Let $\Phi : M_g \to M_g$ be an orientation-preserving diffeomorphism. Then Φ can be isotoped so that it preserves the standard Heegaard splitting of M_g. Furthermore, Φ is isotopic to the double of a diffeomorphism $\phi : H_g \to H_g$.

Proof. Montesinos [61] gives representatives for generators of the mapping class group $\mathcal{M}(M_g)$, and, upon inspection, we see that each of the orientation-preserving representatives satisfy the above properties. The result follows, since any orientation-preserving diffeomorphism of M_g is isotopic to a composition of Montesinos' representatives.

□

Remark 4.5.3. Given such a Φ, the description of Φ as the double of a handlebody diffeomorphism ϕ is not unique. Indeed we can alter ϕ by twisting twice along any disk and the doubled map Φ will be unaffected (up to isotopy), by Proposition 4.4.8.

Given a fibered 2–knot $\mathcal{S} \subset S^4$ with fiber M_g° and monodromy Φ, we can apply Proposition 4.5.2 to obtain a (non-unique) handlebody bundle $H_g \times_\phi S^1$

which doubles to the exterior of $\mathcal{S}$, $E_{\mathcal{S}} = M_g^\circ \times_\Phi S^1$. More precisely, we double the handlebody bundle, except along a disk D_0 that is fixed by ϕ.

Recall from our comments on the Gluck twist that S^4 is recovered from $E_{\mathcal{S}}$ by attaching a 0–framed 2–handle h and a 4–handle. The attaching circle of h is $\{pt\} \times S^1$, and it can be isotoped to lie on the boundary of the sub-bundle $H_g \times_\phi S^1$ (in fact, we can choose the attaching region to be $D_0 \times S^1$). Let $W_0 = (H_g \times_\phi S^1) \cup h$ and observe that $S^4 = \mathcal{D}W_0$. This is most easily seen by noticing that the complement $E_{\mathcal{S}}$ is completed to S^4 by gluing in a $S^2 \times D^2$, where we think of the cocore D of h as the southern hemisphere and the doubled copy of D as the northern hemisphere. Then the proof of Theorem 4.1.3 will be completed by proving the following claim.

Claim 4.5.4. Let W_m be the manifold obtained by performing 2–handle surgery on h with slope m, and let D_m be the cocore of the re-glued h. Then, $\mathcal{D}W_m = S^4$, and $\mathcal{D}D_m = \mathcal{S}$. Furthermore, the W_m are contractible 4–manifolds, and infinitely many of the W_m are distinct.

Proof. The fact that $\mathcal{D}W_m = S^4$ and $\mathcal{D}D_m = \mathcal{S}$ follows from Lemma 4.2.4 and Remark 4.4.7: the pairs $(\mathcal{D}W_m, \mathcal{D}D_m)$ are related by Gluck twists on $\mathcal{D}D_m$, and Gluck twists on 2–knots bounding M_g° preserve both the 4–manifold and the 2–knot. Next, notice that $\pi_1(H_g \times_\phi S^1) \cong \pi_1(M_g^\circ \times_\Phi S^1)$, since Φ_* and ϕ_* are identical as automorphisms of the free group on g generators. It follows that the HNN extensions presenting these groups will be identical. Since $\pi_1(M_g \times_\Phi S^1 \cup h) \cong 1$, it follows that $\pi_1(H_g \times_\phi S^1 \cup h) \cong 1$.

One can calculate that W_m has trivial homology, and it then follows from Whitehead's theorem that W_m is contractible. (The inclusion of a point into W_m is a homology equivalence; hence, a homotopy equivalence.)

The boundaries $Y_m = \partial W_m$ are all related by Dehn filling on a common 3–manifold Y, as discussed in Example 4.4.12. More precisely, we have that $Y_m = Y(-1/m)$.

First suppose that Y is Seifert fibered and that the induced slope of the fibering of the boundary is a/b. If $a \neq 0$, then the $(-1/m)$–filling introduces a new exceptional fiber of multiplicity $\Delta(a/b, -1/m) = am - b$, and the spaces are thusly distinguished. If $a = 0$, then 0–filling is reducible [70]; however, this manifold is also a nontrivial closed surface bundle, a contradiction.

If Y is hyperbolic, then we have $vol(Y(-1/m)) < vol(Y)$ for all m and, by [73],

$$\lim_{m\to\infty} vol(Y(-1/m)) = vol(Y).$$

It follows that there is an infinite sequence $\{m_i\}_{i\in\mathbb{N}}$ such that $vol(Y_{m_1}) < vol(Y_{m_2}) < vol(Y_{m_3}) < \cdots$. Therefore, infinitely many of the Y_m are distinct. If Y is toroidal, then we can cut Y along all essential tori, and repeat the above argument on the atoroidal piece containing ∂Y. $\square$

Question 4.5.5. Is it possible that $Y_{m'} \cong Y_m$ for some $m' \neq m$? This would be an example of a cosmetic surgery on a fibered homotopy-ribbon knot in a $\mathbb{Z}$–homology 3–sphere.

Quite a few choices were made in the preceding construction, and this raises a number of questions. An obvious one is whether we can always arrange for one of the W_m to be the standard 4–ball. We can phrase the question as follows.

Question 4.5.6. Is every fibered homotopy-ribbon 2–knot in S^4 the double of a fibered homotopy-ribbon disk-knot in B^4?

A result of Levine [54] states that the double of any disk-knot is doubly slice, so to answer the above question negatively, it would suffice to find a fibered homotopy-ribbon 2–knot in S^4 that is not doubly slice. If we generalize to arbitrary fibers then we can observe that not all fibered 2–knots are doubles of fibered disks. This follows simply because there exist 2–knots that are fibered by 3–manifolds that are themselves not doubles (see [77]).

We introduce the term *halving* to describe the process of going from a fibered homotopy-ribbon 2–knot $\mathcal{S} \subset S^4$ to one of the cross-sectional fibered homotopy-ribbon 1–knots produced by Theorem 4.1.3. This raises the question of which 1–knots can result from this process for a given 2–knot.

Definition 4.5.7. Given a 2–knot $\mathcal{S} \subset S^4$, we call 1–knot K a *symmetric equator* of $\mathcal{S}$ if K can be obtained from $\mathcal{S}$ by halving. In other words, $\mathcal{S}$ is the double along K of a disk in a contractible 4–manifold (hence K lives in some homology 3–sphere).

From this point of view, we have the following corollary to Theorem 4.1.3.

Corollary 4.1.4. Any nontrivial, fibered homotopy-ribbon 2–knot has infinitely many distinct fibered symmetric equators.

4.6 Spinning fibered 1-knots

Here we give a method to produce many examples of fibered ribbon disks and fibered ribbon 2–knots. Recall that the *spin* of a manifold X is given by $\mathcal{S}(X) = \partial(X^\circ \times D^2)$. Equivalently, we can view the spin as a double, $\mathcal{S}(K) \cong \mathcal{D}(X^\circ \times I)$, as follows.

$$\mathcal{D}(X^\circ \times I) = \partial(X^\circ \times I \times I) \cong \partial(X^\circ \times D^2)$$

Notice that $\mathcal{S}(S^n) \cong S^{n+1}$, so the the spin of a knot (S^3, K) is a 2–knot $(S^4, \mathcal{S}(K))$, called the *spin* of K. Also, if we let $(B^3, K^\circ) = (S^3, K)^\circ$ be the punctured pair, then $(B^3, K^\circ) \times I = (B^4, D_K)$, where D_K is a ribbon disk for $K\#(-K)$, which we sometimes call the *half-spin* of K. The justification of this terminology is that $\mathcal{D}(B^4, D_K) \cong (S^4, \mathcal{S}(K))$.

Now, if (S^3, K) is a fibered knot, then $K\#(-K)$ is a fibered ribbon knot, (B^4, D_K) is a fibered ribbon disk for $K\#(-K)$, and $(S^4, \mathcal{S}(K))$ is a fibered ribbon 2–knot. Suppose that K has monodromy $\varphi : \Sigma_g^\circ \to \Sigma_g^\circ$. Then, $(B^4, D_K) \cong (S^3, K)^\circ \times I$ is clearly fibered with fibers $H_{2g} \cong \Sigma_g^\circ \times I$ and monodromy $\phi = \varphi \times \text{id}$. It is clear that the fibration restricts to the boundary to give a fibration of $(S^3, K\#(-K))$, where the fibers are $\Sigma_g^\circ \natural (-\Sigma_g^\circ)$ and the monodromy is $\varphi \natural (-\varphi)$. Finally, if we view $(S^4, \mathcal{S}(K))$ as the double of (B^4, D_K), then we see that the former is fibered with fiber $M_{2g} = \#_{2g} S^1 \times S^2$ (since

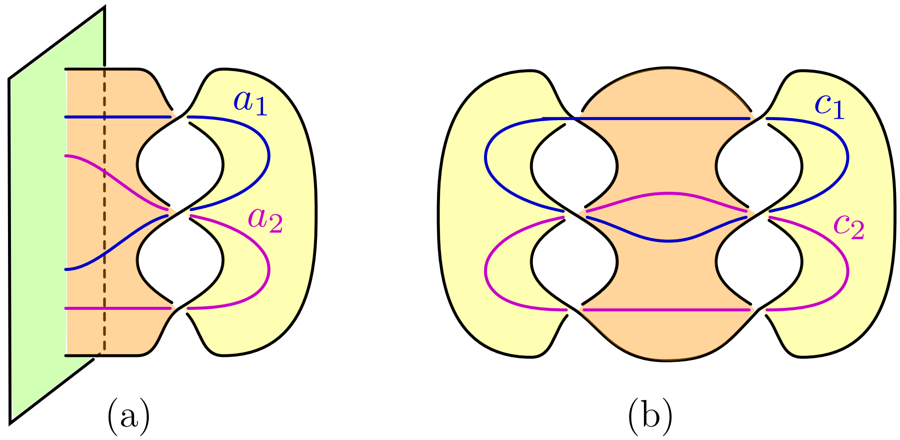

Figure 4.3: The right handed trefoil, shown (a) as an embedded arc in B^3. (A portion of the boundary 2–sphere is shown in green.) Lying on a Seifert surface for K°, we see two arcs, a_1 and a_2, which are boundary parallel in B^3. If we form the spin of the the picture in (a), we get the spun trefoil knot, together with two spheres S_1 and S_2 corresponding to the spins of the arcs a_1 and a_2. In (b), we see an equatorial cross section of the spun trefoil. Alternatively, (b) shows the boundary of the half-spun trefoil. The curves c_1 and c_2 are Stallings curves for the square knot, and bound disks in the genus two handlebody, thought of as a fiber of the half-spun trefoil.

$M_{2g} = \mathcal{D}H_{2g}$) and monodromy $\Phi = \mathcal{D}\phi$. This spinning construction provides a nice set of examples to apply the techniques and results from the previous sections.

The trivial pair (S^4, S^2) admits a natural fibration by 3–balls. One way to visualize the spin $\mathcal{S}(K)$ is to view K as a knotted arc $K^\circ \subset B^3$, and to identify this 3–ball with a fiber of (S^4, S^2). Then, $\mathcal{S}(K)$ is the trace of K° as the 3–ball sweeps out all of S^4 via this fibration. See Figure 4.3(a), or Chapter 2 for more information.

Let K be a fibered knot, and let F be a fiber of $K^\circ \subset B^3$. Let a be an unknotted arc on F, with endpoints in ∂B^3, as in Figure 4.3(a). Then $D_a = a \times I \subset (\overline{B^3 \setminus \nu K}) \times I$ is an unknotted disk in a fiber H_{2g} of D_K. We can apply Theorem 4.1.2 to obtain new fibered disks D_{J_m} in B^4 by twisting m times along D_a. The disks D_{J_m} will be homotopy-ribbon disks for knots J_m obtained from $K\#(-K)$ by doing m Stallings twists along $c = \partial D_a$ (see also Figure 4.2).

Choosing different arcs for a provides a wealth of examples. Some of these families of fibered ribbon knots have been studied elsewhere [4, 3, 39, 72], often with the goal of finding infinitely many distinct fibered ribbon knots with, say, the same Alexander module.

When we double this picture to get $(S^4, \mathcal{S}(K))$, we get an unknotted sphere $S_a = \mathcal{D}D_a$ inside a fiber $M \cong M_{2g}$ of $\mathcal{S}(K)$. We see that S_a is the spin of a, and twisting along S_a gives a new fibered ribbon 2–knot $\mathcal{S}(K)'$ in S^4, which is the double of the disk-knot D_{J_m}, for all odd m (Theorem 4.1.2). However, in this special setting, we see that twisting on S_a preserves the 2–knot $\mathcal{S}(K)$.

Proposition 4.6.1. $\mathcal{S}(K)' = \mathcal{S}(K)$

Proof. Since a is unknotted it bounds a semi-disk in B^3 (that is, a together with an arc on the boundary bound a disk). Perturb the disk so that it intersects K° transversely in k points. Spinning this semi-disk gives a ball B_a bounded by S_a, showing that S_a is, in fact, unknotted in S^4. Furthermore, we see that B_a intersects $\mathcal{S}(K)$ in k circles, which come from spinning the k intersection points. The circles form a k component unlink since they are the spins of isolated points. By an isotopy of $\mathcal{S}(K)$ taking place in a collar $B_a \times I$, we can assume the circles all lie concentrically in a plane in B_a.

Now, $\mathcal{S}(K)'$ is formed by changing the monodromy by twisting along S_a, and, by Proposition 4.4.8, this is equivalent to performing a Gluck twist on S_a. By the proof of Lemma 4.4.6, we see that the diffeomorphism taking $S^4_{S_a}$ (the result of performing a Gluck twist on S_a) back to S^4 is "untwisting" along the complement of S_a. This can be thought of as cutting $S^1 \times B^3 = \overline{S^4 \setminus \nu S_a}$ at B_a and re-gluing by a 2π twist about an axis, which we can choose to be perpendicular to the plane in B_a containing the concentric circles. Therefore the twist in fact preserves the circles, and hence sends $\mathcal{S}(K)'$ onto $\mathcal{S}(K)$, proving the claim. $\square$

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