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Asymptotics for optimal investment with high-water mark fee

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Asymptotics for optimal investment with high-water mark fee

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Dedicated to my family.

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Asymptotics for optimal investment with high-water mark fee

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This dissertation studies the problem of optimal investment in a fund charging high-water mark fees. We consider a market consisting of a riskless money-market account and a fund charging high-water mark fees at rate λ, with share price given exogenously as a geometric Brownian motion. A small investor invests in this market on an infinite time horizon and seeks to maximize expected utility from consumption rate. Utility is taken to be constant relative risk aversion (CRRA). In this setting, we study the asymptotic behavior of the value function for small values of the fee rate λ. In particular, we determine the first and second derivatives of the value function with respect to λ. We then exhibit for each λ explicit sub-optimal feedback investment and consumption strategies with payoffs that match the value function up to second order in λ.

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Chapter 1

Introduction

Investment funds charge investors fees for their services. Common fee structures include *proportional fees*, in which the investor pays a fixed percentage of the total investment on a yearly basis, and *high-water mark fees* in which the investor pays a fixed proportion of any profit made from the investment. These fee structures are often combined; a typical fee structure for hedge funds is the "2/20 rule", a combination of a 2% proportional fee and a 20% high-water mark fee.

There is a growing subset of the finance literature examining how the high-water mark fee structure affects the behavior of the fund manager, the recipient of the fees. Panageas and Westerfield [22] consider a risk-neutral manager who seeks to maximize the present value of her fee stream on an infinite horizon. Although their payoff is convex in this setting, managers do not put unbounded weights on risky assets in this setting; rather, risk-seeking behavior occurs only when the fund manager is constrained to a finite time horizon. Brown, Goetzmann, and Park [4] find empirically that excessively risky investment on the part of the fund manager increases the likelihood of termination for the hedge fund, and that the reputation costs associated with termination offset the incentive toward risky behavior posed by the high-water mark fees. Goetzmann, Ingersoll Jr., and Ross [12] give a closed form description of the value of a high-water mark contract as a claim on the investor's wealth.

In the mathematical finance literature, Guasoni and Obłój [14] study the problem of fund manager who seeks to maximize his expected utility from high-water mark fees on a large time horizon. In [16], on the other hand, Janeček and Sîrbu examine the high-water mark fee structure from the perspective of the *investor*. They introduce a continuous-time model for optimal investment and consumption in a market consisting of a fund charging high-water mark fee at rate λ and a riskless money-market account with zero interest rate. Their model allows continuous trading in and out of the fund, and is a modification of the classical optimal investment and consumption problem of Merton introduced in the seminal papers [20], [21]. This model yields an optimal control problem in which the state process is a two-dimensional reflected controlled diffusion. Assuming power utility, infinite time horizon, and a market consisting of a riskless bond and the fund (with share price given exogenously as a geometric Brownian motion), they show that the value function v^λ of this problem is a classical solution of the corresponding HJB equation. As a result, the optimal investment and consumption strategies are given in feedback form in terms of the value function and its derivatives. This HJB equation cannot be solved in closed form, however, so this result is of limited use for making explicit quantitative statements about how the model parameters, the high-water mark fee rate λ, and the current high-water mark (or equivalently, the "distance to paying fees") affect the investor's behavior. For this, we must rely on numerical approximations.

The problem of optimal investment in a fund charging high-water mark fee is closely related to the problem of optimal investment with drawdown constraint. In the *absence of consumption*, investment strategies which satisfy a drawdown constraint with proportion λ (*with no fees imposed*) are exactly those which satisfy a no-bankruptcy constraint when used to invest in a fund charging high-water mark fee at rate $\frac{\lambda}{1+\lambda}$. In the mathematical finance literature, optimal investment with drawdown constraint was first studied by Grossman and Zhou [13]; in their work, the payoff of an investment strategy was given by the long-term growth rate of expected utility of wealth. Cvitanić and Karatzas [7] then extended these results by approximating the long-term growth with auxiliary finite horizon utility maximization problems *without drawdown constraint*, which could then be solved using convex duality techniques. More recently, Roche [23] and Elie and Touzi [9] explicitly solved the infinite horizon investment/consumption problem with drawdown constraint.

The present work carries out an asymptotic analysis of the value function v^λ for small λ in the framework of [16]. In particular, we characterize the first and second derivatives v^1 and v^2 of v^λ with respect to λ as solutions of linear PDE which can be explicitly solved. We are therefore able to quantify how the "loss" due to high-water mark fees depends both on the the fee rate λ, the model parameters, and the high-water mark itself. In addition, we use this asymptotic expansion to produce explicit feedback investment/consumption strategies $(\hat{\theta}^\lambda, \hat{c}^\lambda)$ with the property that:

$$\hat{v}^\lambda = v^\lambda + o(\lambda^2) \ \textit{as} \ \lambda \to 0$$

where $\hat{v}^\lambda$ is the expected payoff corresponding to the feedback strategy $(\hat{\theta}^\lambda, \hat{c}^\lambda)$.

Conceptually, our work is analogous to a number of works in the literature of optimal investment in markets with *frictions* such as transaction costs. Often in this literature, one can characterize the value function as a (possibly smooth) solution of an HJB equation, but cannot solve the HJB equation explicitly. For instance, in the context of the investment/consumption problem with proportional transaction costs introduced by Davis and Norman [8], Shreve and Soner [24] show that the value function is a smooth solution of a certain free boundary problem, and that the optimal investment/consumption strategies are given in terms of the free boundaries. Neither the value function nor the free boundaries can be determined explicitly, however. Whalley and Wilmott [26] carry out a formal asymptotic analysis of the pricing and hedging of European options with proportional transaction costs. Janeček and Shreve [15] then exhibit a rigorous asymptotic expansion of the value function of the problem of [24] in powers of $\epsilon^{1/3}$, where ϵ is the rate of proportional transaction cost, and obtain asymptotic results on the location of the free boundary.

More recently, Altarovici, Muhle-Karbe, and Soner [1] obtained a similar expansion in powers of $\epsilon^{1/4}$ for the value function of the optimal investment problem with fixed transaction costs. Again, this expansion also leads to an explicit sub-optimal strategy, the payoff of which matches the value function upto order $\epsilon^{1/2}$.

At the technical level, our approach is perhaps most similar to that of [1], in that it is uses the techniques introduced by Barles and Perthame [3] and Evans [10]. First, we follow [3] in defining upper and lower relaxed semilimits for the derivatives v^1 and v^2 of v with respect to λ. We then show that the upper (resp. lower) relaxed semilimit is a viscosity subsolution (resp. supersolution) of an appropriate differential equation by adapting the perturbed test function method of [10]. Finally, we use a comparison principle to conclude that the upper and lower relaxed semilimits coincide, so that the first and second derivatives of v^λ with respect to λ are well-defined.

The relaxed semilimit approach of [3] was intended to deal with the case of discontinuous viscosity solutions and is in principle a more general approach than is needed for our case; the value functions v^λ at fixed λ are known to be smooth, thanks to [16]. However, most of the technical effort of our approach lies in checking the *boundedness* of the difference quotients

$$Q^{1,\lambda} = \frac{v^\lambda - v^0}{\lambda}, \quad Q^{2,\lambda} = \frac{v^\lambda - (v^0 + \lambda v^1)}{\lambda}$$

and their derivatives for small λ (and not in verifying the viscosity solution property of the resulting relaxed semilimits, which is rather straightforward). Some boundedness of this kind would be necessary for more classical approaches as well, so there is little additional cost to our more general method.

The works on proportional and fixed transaction costs cited above employ the PDE-based methods of stochastic control theory. On the other hand, there is a considerable literature treating the problem of optimal investment with proportional transactions costs via martingale methods and convex duality techniques; see the seminal papers of Jouini and Kallal [17] and Cvitancić and Karatzas [7]. In this spirit, Gerhold, Muhle-Karbe and W. Schachermayer [11] use techniques of convex duality to generalize the work of [15], obtaining fractional Taylor expansions of arbitrarily high order for the value function and the location of free boundary. In our setting, the value function v^λ is concave, so it is natural to ask whether the problem for fixed λ can be solved using convex duality techniques, and also whether duality can be applied to an asymptotic expansion. The state process in the case of high-water mark fees depends in a highly non-linear way on the strategies used, however, so it is not obvious how to adapt the usual arguments of convex duality. In particular, no analog of the classical martingale representation theorem or the optional decomposition theorem of [19] used in other settings is available.

Chapter 2

Asymptotics for optimal investment with high-watermark fee

2.1 The optimal investment problem for fixed fee rate λ

In this section, we introduce the model and principal results of [16].

2.1.1 A basic model for investment with high-water mark fees

To begin with, we present a model of investment *without consumption* in a fund charging high-water mark fee. We consider a market consisting of a riskless asset with zero interest rate and a risky fund with exogenously given share price F_t dollars per share at time t. A small investor chooses between these two assets and can freely and continuously rebalance his investment in them. In the absence of fees, the investor's accumulated profit from investment in the fund has the dynamics

$$\begin{aligned} dP_t &= \alpha_t \frac{dF_t}{F_t} \\ P_0 &= 0 \end{aligned}$$

where α_t is the dollar amount invested in the fund at time t. Since the interest rate of the riskless asset is zero and the investor does not consume, the investor's wealth is given by

$$X_t = x + P_t$$

where x is the initial wealth endowment.

In order to assess the high-water mark fee, the fund manager keeps track of the profits the investor has made from investment in the fund. That is, the fund manager keeps track of the high-water mark

$$P_t^* \triangleq \sup_{0 \le s \le t} P_s.$$

Whenever his profit P exceeds the historical high-water mark P^*, the investor pays the fund manager a proportion λ of the excess profit. The investor's profit under high-water mark fee thus evolves as

$$\begin{aligned} dP_t &= \alpha_t \frac{dF_t}{F_t} - \lambda d \left(\sup_{0 \le s \le t} P_s \right) \\ P_0 &= 0. \end{aligned} \tag{2.1}$$

To work in the dynamic programming framework, we must also introduce the notion of an initial high-water mark $i \geq 0$, a profit level which the investor must achieve before any fees are applied. This is a mathematical convenience; in practice, one has $i = 0$. With initial high-water mark i, the investor's profit evolves according to

$$\begin{aligned} dP_t &= \alpha_t \frac{dF_t}{F_t} - \lambda d \left(\sup_{0 \le s \le t} (P_s \vee i) \right) \\ P_0 &= 0. \end{aligned} \tag{2.2}$$

In this case, the high-water mark is

$$P_t^* = \sup_{0 \le s \le t} (P_s \vee i)$$

While equation (2.2) is implicit, it turns out that there is a unique solution that has a closed form expression in terms of the process $I_t = \int_0^t \alpha_t \frac{dF_t}{F_t} dt$. We reproduce Proposition 2.1 of [16] below.

Proposition 2.1.1. *Assume the share price process F_t is a continuous, strictly positive semimartingale, and that the predictable process α_t is chosen so that*

$$I_t = \int_0^t \alpha_t \frac{dF_t}{F_t}$$

is well defined. Then equation (2.2) *has the unique solution*

$$P_t = I_t - \frac{\lambda}{1+\lambda} \max_{0 \le s \le t} [I_s - i]^+ \tag{2.3}$$

$$P_t^* = i + \frac{1}{1+\lambda} \max_{0 \le s \le t} [I_s - i]^+. \tag{2.4}$$

Remark 2.1.1. Recall the famous Skorohod equation (see [18], chapter 3, page 210 and following); given $i \ge 0$ and a continuous function $f : [0, \infty) \to \mathbb{R}$ with $f(0) = 0$, there exists a unique continuous function k such that

1. $g(t) = i + f(t) + k(t) \ge 0$ for all t.

2. k is non-decreasing with $k(0) = 0$.

3. $\int_0^t \mathbf{1}_{[g(s)>0]} dk(s) = 0$ for all t.

Explicitly, the solution is given by

$$k(t) = \max_{0 \le s \le t} [-f(s) - i]^+.$$

Set $\Delta = P^* - P$. In other words, Δ is the "distance to paying fees". Note that $\Delta \ge 0$, and Δ satisfies the equation

$$\begin{aligned} d\Delta_t &= -\alpha_t \frac{dF_t}{F_t} + (1+\lambda) dP_t^* \\ \Delta_0 &= i. \end{aligned}$$

We also have

$$\int_0^t \mathbf{1}_{[\Delta_s > 0]} dP_s^* = 0 \quad \text{for all } t.$$

Therefore, $(1+\lambda)P^*$ is the solution k to the Skorohod equation above, with $f(t) = -\int_0^t \alpha_s \frac{dF_s}{F_s}$.

2.1.2 High-water mark fees with consumption

We are interested in modeling optimal investment and consumption with a no-bankruptcy constraint. As in the previous section, let x denote the investor's initial wealth, i the initial high-water mark. As before, the market consists of a riskless asset with interest rate zero and a fund with share price F_t. Let γ_t denote the investor's rate of consumption (as a dollar amount per unit of time), and let C_t denote the accumulated consumption $C_t = \int_0^t \gamma_s ds$. The investor's profit from investment now evolves according to

$$\begin{aligned} dP_t &= \alpha \frac{dF_t}{F_t} - \lambda d \left(\sup_{0 \le s \le t} (P_s \vee i) \right) \\ P_0 &= 0. \end{aligned}$$

Due to the presence of consumption, the investor's wealth is given by

$$X_t = x + P_t - C_t.$$

In this case, the high-water mark should track not the investor's wealth, but instead the profit P; that is, the investor's choice to consume some of his wealth should not affect the fund manager's record of the profit the investor has made through investing in the fund. Setting $n = x + i$, the high-water mark is therefore

$$\begin{aligned} P_t^* &= \sup_{0 \le s \le t} (P_s \vee i) \\ &= \sup_{0 \le s \le t} [(X_s + C_s - n) \vee i]^+ \end{aligned}$$

Following [16], we take (X, N) as a state, where X is the investor's wealth and $N = x + P^* - C$, so that the state process is given in differential notation by

$$\begin{aligned} dX_t &= \alpha_t \frac{dF_t}{F_t} - \gamma_t dt - \lambda dP_t^* \\ X_0 &= x, \\ dN_t &= dP_t^* - \gamma_t dt \\ N_0 &= n \ge x. \end{aligned} \tag{2.5}$$

The rationale for this choice of state is as follows; as usual, the wealth X should be a state. However, we cannot use (X, P^*) as a state because this choice does not encode information about past consumption. The choice of state (X, N) also leads to a natural choice of domain for the problem; we always have $X \le N$, and high-water mark fees are paid exactly when the state is on the boundary $X = N$ of the domain. The equation (2.5) represents a two-dimensional reflected diffusion on the domain

$$D = \{(x, n) : 0 < x \leq n)\}$$

with reflection of size P^* along the line $\{x = n\}$ in the oblique direction $(-\lambda, 1)$.

In our model, the fund price per share is given exogenously as a geometric Brownian motion; that is,

$$dF_t = F_t(\mu dt + \sigma dW_t)$$

where $(W_t)_{0 \leq t < \infty}$ is a Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t < \infty})$. The filtration $(\mathcal{F}_t)_{0 \leq t < \infty}$ is assumed to satisfy the usual conditions. We then have the following result on existence and uniqueness of solutions of (2.5) (Proposition 2.2 of [16]).

Proposition 2.1.2. *Suppose that the predictable processes α_t and γ_t satisfy the integrability condition*

$$\mathbb{P}\left(\int_0^t \left(\alpha_u^2 + \gamma_u\right) du < \infty \text{ for all } t \in [0, \infty)\right) = 1, \tag{2.6}$$

and make the notation

$$I_t = \int_0^t \alpha_u \frac{dF_u}{F_u}, \quad C_t = \int_0^t \gamma_u du,$$

Then equation (2.5) has a unique solution, which can be written as

$$X_t = x + I_t - C_t - \frac{\lambda}{1 + \lambda} \max_{0 \leq s \leq t} [I_s - i]^+, \tag{2.7}$$

$$N_t = n + \frac{1}{1 + \lambda} \max_{0 \leq s \leq t} [I_s - i]^+ - C_t. \tag{2.8}$$

The high-water mark is given by

$$\begin{aligned} P_t^* &= N_t + C_t - x \\ &= i + \frac{1}{1+\lambda} \max_{0 \le s \le y} [I_s - i]^+. \end{aligned}$$

2.1.3 The optimal investment/consumption problem

Definition 2.1.1. A pair (α, γ) of predictable processes is called *admissible* with respect to the initial conditions (x, n) if the integrability conditions of Proposition 2.1.2 are satisfied, the consumption stream γ_t is non-negative for all t, and X_t is strictly positive for all t. We denote by $\mathcal{A}^\lambda(x, n)$ the collection of strategies which are admissible for fee level λ and initial condition (x, n).

The preferences of the investor are modeled using expected (discounted) utility from consumption. Explicitly, for $(x, n) \in D = \{(x, n) | 0 \le x \le n\}$, the value function for the optimal investment problem with fee level λ is given by

$$v^\lambda(x, n) = \sup_{(\alpha,\gamma)\in\mathcal{A}^\lambda(x,n)} \mathbb{E}\left[\int_0^\infty e^{-\beta t} U(\gamma_t) dt\right] \tag{2.9}$$

Here $\beta > 0$ is a discount factor and utility is taken to be of constant relative risk aversion (CRRA) with

$$U(x) = \frac{x^{1-p}}{1-p}, \quad p > 0, p \neq 1.$$

2.1.4 Homotheticity and one-dimensional notational conventions

The value function v^λ has a homotheticity property which will be used extensively.

Proposition 2.1.3. *We have*

$$v^\lambda(x,n) = x^{1-p} v^\lambda(1, n/x).$$

As a result, we can express v_x^λ *and* v_{xx}^λ *in terms of* $v^\lambda, v_n^\lambda,$ *and* v_{nn}^λ

$$\begin{aligned} v_x^\lambda(x,n) =& x^{-p}\left((1-p)v^\lambda(1,n/x) - \left(\frac{n}{x}\right) v_n^\lambda(1,n/x)\right) \\ v_{xx}^\lambda(x,n) =& x^{-1-p}\Bigg(-p(1-p)v^\lambda(1,n/x) + 2p\left(\frac{n}{x}\right) v_n^\lambda(1,n/x) \\ & + \left(\frac{n}{x}\right)^2 v_{nn}^\lambda(1,n/x)\Bigg) \\ v_n^\lambda(x,n) =& x^{-p} v_n^\lambda(1,n/x) \end{aligned} \tag{2.10}$$

In light of the relations (2.10), we introduce the following one-dimensional notation for the value function v and its derivatives.

$$\begin{aligned} w^\lambda(y) &\triangleq v^\lambda(1,y) \\ w_x^\lambda(y) &\triangleq v_x^\lambda(1,y) \\ &= (1-p)w^\lambda(y) - y w_y^\lambda(y) \\ w_{xx}^\lambda(y) &\triangleq v_{xx}^\lambda(1,y) \\ &= -p(1-p)w^\lambda(y) + 2py w_y^\lambda(y) + y^2 w_{yy}^\lambda(y) \\ w_y^\lambda(y) &\triangleq w_n^\lambda(y) \\ \nabla w^\lambda(y) &\triangleq (w_x^\lambda, w_n^\lambda) \end{aligned} \tag{2.11}$$

With $y = n/x$, we therefore have

$$\begin{aligned} v^\lambda(x,n) &= x^{1-p} w^\lambda(y) \\ v_x^\lambda(x,n) &= x^{-p} w_x^\lambda(y) \\ v_{xx}^\lambda(x,n) &= x^{-1-p} w_{xx}^\lambda(y) \\ v_n^\lambda(x,n) &= x^{-p} w_y^\lambda(y) \end{aligned} \tag{2.12}$$

This slight abuse of notation will eventually allow us to write various PDEs for functions with homotheticity properties as one-dimensional ODEs, but in a way that resembles the more recognizable two-dimensional equation. We will always use y as the coordinate n/x on the line $\{(1, n/x) : n \geq x\}$. On the other hand, the letter z will be used as an abbreviated way of referring to a point (x, n) in the two dimensional domain $D = \{(x, n) : 0 < x \leq n\}$.

In general, the letter v (plus other additional decorations) will be used to denote a two-dimensional function with the homotheticity property, and the letter w (plus the same decorations) will be used to denote the one-dimensional version of this function, evaluated along $\{(1, n/x) : n \geq x\}$. The same conventions (2.11) for the partial derivatives of w will hold. This abuse of notation will apply to operators as well; in other words, if $v : D \to \mathbb{R}$ has the homotheticity property and $\mathcal{H}$ is a differential operator defined for functions on D, then we will make the notation

$$(\mathcal{H}w)(y) = (\mathcal{H}v)(1, y).$$

In the case that $\mathcal{H}$ has the structure

$$\mathcal{H}v = A(y)x^{1-p} + B(y)v + C(y)xv_x + D(y)x^2v_{xx} + E(y)xv_n \tag{2.13}$$

where $y = n/x$, then we have

$$\mathcal{H}v(x,n) = x^{1-p}\mathcal{H}w(n/x),$$

a fact that will later be used to reduce the HJB equation for v^λ to an ODE for w^λ.

2.1.5 The HJB equation and the main results of [16]

We can formally derive an HJB equation for v^λ by applying Itô's lemma to the process $v^\lambda(X_t, N_t)$, where (X_t, N_t) is a state process corresponding to some strategy (α, γ). For convenience, we will write the strategy in terms of proportions $\theta_t = \alpha_t/X_t$ and $c_t = \gamma_t/X_t$. Define the process

$$Z_t = e^{-\beta t}v^\lambda(X_t, N_t) + \int_0^t e^{-\beta u}U(c_uX_u)du,$$

which we expect to be a supermartingale if (θ, c) is a suboptimal strategy, and a martingale if (θ, c) is optimal. Then we have

$$\begin{aligned} dZ_t &= e^{-\beta t}\Big(-\beta v(X_t,N_t) + \mu\theta_tX_tv_x^\lambda(X_t,N_t) + \frac{1}{2}(\sigma\theta_tX_t)^2v_{xx}^\lambda(X_t,N_t) \\ &\quad +U(c_tX_t) - c_tX_t(v_x(X_t,N_t) + v_n(X_t,N_t))\Big)dt \qquad (2.14)\\ &\quad +e^{-\beta t}(\sigma\theta_tX_t)\,dW_t + e^{-\beta t}(v_n^\lambda(x,n) - \lambda v_x^\lambda(x,n))dP_t^* \end{aligned}$$

Now dP_t^* is supported on the set of times $\{t : X_t = N_t\}$ so we expect dP_t^* to be a singular measure. If Z_t is to be a martingale, we expect the singular and absolutely continuous drift terms of (2.14) to be separately zero. We can therefore formally derive the following HJB equation for v^λ:

$$\begin{aligned}
\sup_{\theta\in\mathbb{R},c\geq 0} \mathcal{L}^{\theta,c}v^\lambda &= 0 \text{ on } \{(x,n) : 0 < x < n\} \\
v_n^\lambda(x,x) - \lambda v_x^\lambda(x,x) &= 0 \text{ for } x > 0 \\
\lim_{n\to\infty} v^\lambda(x,n) &= v^0(x)
\end{aligned} \tag{2.15}$$

where we have defined

$$\mathcal{L}^{\theta,c}u \triangleq -\beta u + \mu\theta x u_x + \frac{1}{2}(\sigma\theta x)^2 u_{xx}^\lambda + U(cx) - cx(u_x + u_n) \tag{2.16}$$

Maximizing over c, we may write rewrite the interior condition in (2.15) as

$$-\beta u + \sup_{\theta\in\mathbb{R}}\left[\mu\theta x v_x^\lambda + \frac{1}{2}(\sigma\theta x)^2 v_{xx}^\lambda\right] + \tilde{U}((v_x^\lambda + v_n^\lambda)x) = 0. \tag{2.17}$$

Here $\tilde{U}$ is the Legendre transform of the utility function:

$$\begin{aligned}
\tilde{U}(\tilde{x}) &= \sup_{x>0}\left[U(x) - x\tilde{x}\right] \\
&= U(I(\tilde{x})) - \tilde{x}I(\tilde{x})
\end{aligned}$$

where $I = (U')^{-1}$ is the inverse of the marginal utility.

Remark 2.1.2. Note that the operators $\mathcal{L}^{\theta,c}$ have the structure of (2.13). Therefore, we have

$$\mathcal{L}^{\theta,c}v^\lambda(x,n) = x^{1-p}\mathcal{L}^{\theta,c}w^\lambda(n/x).$$

A similar remark applies to the boundary condition of (2.15), and we therefore have the one-dimensional version of the HJB equation:

$$\begin{aligned} \sup_{\theta\in\mathbb{R},c\geq 0} \mathcal{L}^{\theta,c} w^\lambda &= 0 \text{ on } (1,\infty) \\ w_y^\lambda(1) - \lambda w_x^\lambda(1) &= 0 \\ \lim_{y\to\infty} w^\lambda(y) &= w^0 \end{aligned} \tag{2.18}$$

where, following our notational conventions, $w^0 = v^0(1)$. If w^λ is a classical solution of (2.18), then it follows from the homotheticity properties that v^λ is a classical solution of (2.15).

We now give the principal result of [16].

Theorem 2.1.4. *1) The function $v^\lambda(x,n) = x^{1-p} w^\lambda(n/x)$ is C^2 on $\{(x,n) : 0 < x \leq n\}$ and a classical solution of* (2.15). *We also have*

$$v_x^\lambda(x,n) > 0, \quad v_n^\lambda(x,n) > 0, \quad v_{xx}^\lambda(x,n) > 0, \ \textit{for } 0 < x \leq n$$

and further,

$$\lim_{n\to\infty} n v_n^\lambda(x,n) = \lim_{n\to\infty} n^2 v_{nn}^\lambda(x,n) = 0 \tag{2.19}$$

2) Define the feedback proportions

$$\theta^\lambda(x,n) = -\frac{\mu}{\sigma^2}\frac{v_x^\lambda(x,n)}{x v_{xx}^\lambda(x,n)} \tag{2.20}$$

$$c^\lambda(x,n) = \frac{I(v_x^\lambda(x,n) + v_n^\lambda(x,n))}{x} \tag{2.21}$$

where $I = (U')^{-1}$. Then the closed loop equation

$$\begin{aligned}
dX_t^\lambda &= \theta^\lambda(X_t^\lambda, N_t^\lambda)X_t^\lambda \frac{dF_t}{F_t} - c^\lambda(X_t^\lambda, N_t^\lambda)X^\lambda dt - \lambda\left(dN_t^\lambda + c(X_t^\lambda, N_t^\lambda)X_t^\lambda dt\right), \\
X_0^\lambda &= x \\
N_t &= \sup_{0\le s\le t}\left[\left(X_s^\lambda + \int_0^s c^\lambda(X_u^\lambda, N_u^\lambda)X_u^\lambda du\right)\vee n\right] - \int_0^t c^\lambda(X_u^\lambda, N_u^\lambda)X_u^\lambda du \\
N_0 &= n
\end{aligned}$$

has a unique global strong solution (X^λ, N^λ) such that $0 < X^\lambda \le N^\lambda$, and the resulting payoff is optimal, i.e.

$$v^\lambda(x,n) = \mathbb{E}\left[\int_0^\infty e^{-\beta t}U(c^\lambda(X_t^\lambda, N_t^\lambda)X_t^\lambda)dt\right]$$

Remark 2.1.3. Note that from the relations (2.10) and the fact that $I(\tilde{x}) = \tilde{x}^{-p}$, we actually have

$$\begin{aligned}
\theta^\lambda(x,n) &= -\frac{\mu}{\sigma^2}\frac{w_x^\lambda(y)}{w_{xx}^\lambda(y)} &(2.22)\\
c^\lambda(x,n) &= I(w_x^\lambda(y) + w_n^\lambda(y)) &(2.23)
\end{aligned}$$

where $y = n/x$. In other words, the optimal strategy depends only on n/x, and can be expressed solely in terms of w. We will therefore commit the slight abuse of notation

$$\begin{aligned}
\theta^\lambda(y) &= \theta^\lambda(x,n) \\
c^\lambda(y) &= c^\lambda(x,n). &(2.24)
\end{aligned}$$

For reference, we recall the explicit solution to the optimal investment problem *without* fees.

Proposition 2.1.5. *Let v^0 denote the Merton value function. Then we have $v^0 < \infty$ if and only if*

$$\beta - \frac{1}{2}\left(\frac{1-p}{p}\right)\frac{\mu^2}{\sigma^2} > 0 \tag{2.25}$$

Suppose that (2.25) *hold. Then the optimal feedback investment and consumption proportions FOR $\lambda = 0$ are given by*

$$\theta^0 = \frac{\mu}{p\sigma^2}, \quad c^0 = \frac{\beta}{p} - \frac{1}{2}\left(\frac{1-p}{p^2}\right)\frac{\mu^2}{\sigma^2} \tag{2.26}$$

Moreover, v^0 is a C^2 solution of the HJB and is given in closed form by

$$v^0(x,n) = \frac{1}{1-p}(c^0)^{-p}x^{1-p}, \quad 0 < x \leq n \tag{2.27}$$

Finally, the concavity of the value function will be used throughout. To simplify the proof of concavity, we will make use of the existence of optimal strategies for each (x, n) proven in Theorem 2.1.4, though this result is not strictly needed for the argument.

Proposition 2.1.6. *The value function v^λ is concave on D.*

Proof. The argument relies on the convexity of the pathwise running maximum which appears in the state equation. Pick points $(x_1, n_1), (x_2, n_2)$ in D. Let (α^i, γ^i) be the optimal investment strategy for initial state (x_i, n_i), given in *dollar amounts*. Let $b \in [0, 1]$. Since the utility function is concave, it will suffice to show that

$$(\alpha, \gamma) \triangleq (b\alpha^1 + (1-b)\alpha^2, b\gamma^1 + (1-b)\gamma^2) \in \mathcal{A}^\lambda(x, n),$$

where $(x,n) = (bx_1 + (1-b)x_2, bn_1 + (1-b)n_2)$. In other words, we need to show the wealth process X corresponding to the initial condition $bx_1 + (1-b)x_2, bn_1 + (1-b)n_2$ and strategy (α, γ) is non-negative. Let $Y_t^i = \int_0^t \alpha^i \frac{dF_u}{F_u} du$ and $C_t^i = \int_0^t \gamma_u du$, and let X^i denote the wealth process corresponding to strategy (α^i, γ^i) and initial condition (x_i, n_i). Then

$$\begin{aligned} X_t &= b(x^1 + Y_t^1 - C_t^1) + (1-b)(x^2 + Y_t^2 - C_t^2) \\ &\quad - \frac{\lambda}{1+\lambda} \sup_{0\le s\le t} b([Y_s^1 - (n_1 - x_1)) + (1-b)(Y_s^2 - (n_2 - x_2))]^+ \end{aligned}$$

Then observe that since the map $x \mapsto x^+$ is convex,

$$\begin{aligned} \sup_{0\le s\le t} [bY_s^1 + (1-b)Y_s^2 - (n-x)]^+ &\le \sup_{0\le s\le t} \big(b[Y_s^1 - (n_1 - x_1)]^+ \\ &\quad +(1-b)[Y_s^2 - (n_2 - x_2)]^+\big) \\ &\le b \sup_{0\le s\le t} [Y_s^1 - (n_1 - x_1)]^+ \\ &\quad +(1-b) \sup_{0\le s\le t} [Y_s^2 - (n_2 - x_2)]^+ \end{aligned}$$

Therefore

$$\begin{aligned} X_t &\ge b\left(x^1 + Y_t^1 - C_t^1 - \frac{\lambda}{1+\lambda} \sup_{0\le s\le t} [Y^1 - (n_1 - x_1)]^+\right) \\ &\quad +(1-b)\left(x^1 + Y_t^1 - C_t^1 - \frac{\lambda}{1+\lambda} \sup_{0\le s\le t} [Y^1 - (n_1 - x_1)]^+\right) \\ &= bX_t^1 + (1-b)X_t^2 > 0 \end{aligned} \tag{2.28}$$

which completes the proof. □

2.2 Heuristic derivation of the corrector equations

In this subsection, we give formal derivations of the PDEs which we expect to characterize the derivatives of the value function with respect to λ:

$$\begin{aligned} v^1(x,n) &= \frac{d}{d\lambda} v^\lambda(x,n)\Big|_{\lambda=0} \\ v^2(x,n) &= \frac{d^2}{d\lambda^2} v^\lambda(x,n)\Big|_{\lambda=0} \end{aligned}$$

We then construct feedback proportions $\hat{\theta}^\lambda, \hat{c}^\lambda$ given explicitly in terms of v^1 for which we expect to have

$$\hat{v}^\lambda(x,n) = v^\lambda(x,n) + o(\lambda^2),$$

where $\hat{v}^\lambda(x,n)$ is the payoff of investing and consuming at the feedback proportions $\hat{\theta}^\lambda, \hat{c}^\lambda$ while facing fees at rate λ.

2.2.1 Derivation of the corrector equations and approximately optimal strategies

Typically in the literature on asymptotics, one identifies two different sources of loss due to frictions. On one hand, there is the loss of wealth due to a transaction cost itself. On the other, there is also a loss due to "displacement" from the Merton proportion. In the case of proportional transaction costs, for example, an investor cannot maintain the Merton proportion; the continuous rebalancing of assets required to maintain the Merton proportion would result in bankruptcy. Janecek and Shreve [15] give heuristics to quantify the trade-off between loss due to transaction costs and loss due to displacement; if the investor faces transaction costs of size ϵ and keeps his state within a "no-transaction region" of size ϵ^q around the Merton proportion, then loss due to displacement should be of order ϵ^{2q}, and loss due to transaction costs should be of order ϵ^{1-q}. The choice $q = 1/3$ *balances* these losses; the investor should use a no transaction region of size $\epsilon^{1/3}$, leading to the minimal total loss of order $\epsilon^{2/3}$ in the value function. From these heuristics, the authors assume a formal expansion of the value function in powers of $\epsilon^{1/3}$ (where the coefficient of $\epsilon^{1/3}$ is zero).

In our case, there is a loss of roughly order λ due to the fees themselves. On the other hand, there is no analogous loss due to displacement; the Merton proportion is an admissible strategy. For this reason, we suppose that the value function has an expansion in powers of λ:

$$v^\lambda = v^0 + \lambda v^1 + \frac{\lambda^2}{2} v^2 + o(\lambda^2).$$

To derive PDEs which characterize the derivatives v^k of the value function with respect to λ, we further assume formally that derivatives with respect to the state variables (x, n) and the fee level λ commute, in the sense that

$$\begin{aligned} v_x^\lambda &= v_x^0 + \lambda v_x^1 + \frac{\lambda^2}{2} v_x^2 + o(\lambda^2) \\ v_{xx}^\lambda &= v_{xx}^0 + \lambda v_{xx}^1 + \frac{\lambda^2}{2} v_{xx}^2 + o(\lambda^2) \\ v_n^\lambda &= v_n^0 + \lambda v_n^1 + \frac{\lambda^2}{2} v_n^2 + o(\lambda^2) \\ & \dots \end{aligned} \tag{2.29}$$

We derive PDEs for v^1 and v^2 by plugging this expansion into the HJB equation for v^λ and equating distinct powers of λ. To begin with, the boundary condition

$$v_n^\lambda(x,x) - \lambda v_x(x,x) = 0$$

gives rise to the separate boundary conditions

$$v_n^1(x,x) - v_x^0(x,x) \quad = \quad 0 \tag{2.30}$$

$$v_n^2(x,x) - 2v_x^1(x,x) \quad = \quad 0 \tag{2.31}$$

On the interior, v^λ satisfies

$$\begin{aligned} 0 &= -\beta v^\lambda + \sup_{\theta,c\geq 0}\left[\mu\theta x v_x^\lambda + \frac{1}{2}(\sigma\theta x)^2 v_{xx}^\lambda + \tilde{U}(v_x^\lambda + v_n^\lambda)\right] \\ &= -\beta v^\lambda + \mu\theta^\lambda(x,n)xv_x^\lambda + \frac{1}{2}(\sigma\theta^\lambda(x,n)x)^2 v_{xx}^\lambda + \tilde{U}(v_x^\lambda + v_n^\lambda). \end{aligned} \tag{2.32}$$

To determine which equations v^1 and v^2 should satisfy on the interior of the domain, we expand $\theta^\lambda, c^\lambda$ in terms of λ and separate the various powers of λ in the resulting expression. Computing the derivatives of $\theta^\lambda, c^\lambda$ with respect to λ, we have the formal expansions

$$\begin{aligned} \theta^\lambda(x,n) &= \theta^0 + \lambda\theta^1(x,n) + o(\lambda) \\ c^\lambda(x,n) &= c^0 + \lambda c^1(x,n) + o(\lambda) \end{aligned}$$

where θ^0 and c^0 are the optimal Merton investment/consumption proportions and

$$\theta^1 \quad = \quad -\frac{\mu}{\sigma^2}\frac{v_{xx}^0 v_x^1 - v_{xx}^1 v_x^0}{x v_{xx}^0} \tag{2.33}$$

$$c^1 \quad = \quad \frac{I'(v_x^0(x,n))(v_x^1(x,n) + v_n^1(x,n))}{x} \tag{2.34}$$

Plugging this into equation (2.32) and separating distinct powers of λ, we obtain

$$\begin{aligned}
0 \quad = \quad & -\beta v^0 + \mu\theta^0 x v^0 + \frac{1}{2}(\sigma\theta^0 x)^2 v^0_{xx} + \tilde{U}(v^0_x) \\
& +\lambda\left(-\beta v^1 + \mu\theta^0 x v^1_x + \frac{1}{2}(\sigma\theta^0 x)^2 v^1_{xx} + \tilde{U}'(v^0_x)(v^1_x + v^1_n)\right) \\
& +\frac{\lambda^2}{2}\left(-\beta v^2 + \mu\theta^0 v^2_x + \frac{1}{2}(\sigma\theta^0 x)^2 v^2_{xx} + \tilde{U}'(v^0_x)(v^2_x + v^2_n)\right) \qquad (2.35) \\
& +\lambda^2\left(\mu\theta^1 x v^1_x + \sigma\theta x^2 v^1_{xx} + \frac{1}{2}(\sigma\theta^1 x)^2 v^0_{xx} + \frac{1}{2}\tilde{U}''(v^0_x)(v^1_x + v^1_n)^2\right) \\
& +o(\lambda^2)
\end{aligned}$$

Now, the first line of (2.35) is equal to zero since v^0 satisfies the usual frictionless HJB equation. Separately equation of distinct powers of λ to zero, we can now write down the first- and second-order corrector equations for v^1 and v^2,

$$\begin{aligned}
\mathcal{A}v^1 &= 0 \\
v^1_n(x,x) - v^0_x(x,x) &= 0 \qquad (2.36) \\
\lim_{n\to\infty} v^1(x,n) &= 0
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{A}v^2 + g &= 0 \\
v^2_n(x,x) - v^1_x(x,x) &= 0 \qquad (2.37) \\
\lim_{n\to\infty} v^2(x,n) &= 0
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{A}u &\triangleq -\beta u + \mu\theta^0 x u_x + \frac{1}{2}(\sigma\theta^0 x)^2 u_{xx} + \tilde{U}(v_x^0)(u_x + u_n) \\
g &\triangleq 2\left[\mu\theta^1 x v_x^1 + \sigma\theta^0\theta^1 x^2 v_{xx}^1 + \frac{1}{2}(\sigma\theta^1 x)^2 v_{xx}^0 + \frac{1}{2}\tilde{U}''(v_x^0)(v_x^1 + v_n^1)^2\right]
\end{aligned}$$

2.2.2 Approximately optimal strategies

Let $\hat{v}^\lambda$ denote the payoff of the feedback investment/consumption proportions $\hat{\theta}^\lambda = \theta^0 + \lambda\theta^1$, $\hat{c}^\lambda = c^0 + \lambda c^1$ at fee level λ. In this section, we discuss the reasoning behind the claim that $\hat{v}^\lambda = v^\lambda + o(\lambda^2)$. For simplicity, we will restrict our attention to the investment strategy θ, ignoring consumption. Usually, one expects that if a parametric family of investment strategies $\bar{\theta}^\epsilon$ are within $o(\epsilon)$ of the optimal strategy, then the resulting payoffs are within $o(\epsilon^2)$ of the optimal payoff. Roughly speaking, this is because we expect the payoff to be locally quadratic around its maximum. By construction, we do have

$$|\hat{\theta}^\lambda(x,n) - \theta^\lambda(x,n)| = o(\lambda) \tag{2.38}$$

In this section, we check formally that the path-by-path strategies (i.e. open-loop controls) resulting from the feedback strategies θ^λ and $\hat{\theta}^\lambda$ match up to first order as well. Let $\Theta(\lambda, x, n) = \theta^\lambda(x,n)$, so that the optimal investment strategy at fee level λ has the path-by-path expression $\Theta(\lambda, X^\lambda, N^\lambda)$, where (X^λ, N^λ) is the trajectory of the optimal state process. Let $(\hat{X}^\lambda, \hat{N}^\lambda)$ denote the state process determined by the feedback strategy $\hat{\theta}^\lambda, \hat{c}^\lambda$ and fee level λ. Then the path-by-path investment strategy determined by the feedback control

$\hat{\theta}^\lambda, \hat{c}^\lambda$ is $\hat{\theta}^\lambda(\hat{X}^\lambda, \hat{N}^\lambda)$. We expect that

$$\Theta(\lambda, X^\lambda, N^\lambda) = \hat{\theta}^\lambda(\hat{X}^\lambda, \hat{N}^\lambda) + o(\lambda) \tag{2.39}$$

Indeed, formally, we have

$$\begin{aligned}
\frac{d}{d\lambda}\Theta(\lambda, X^\lambda, N^\lambda)\bigg|_{\lambda=0} &= \Theta_\lambda(0, X^0, N^0) \\
&\quad + \Theta_x(0, X^0, N^0)\frac{dX^\lambda}{d\lambda}\bigg|_{\lambda=0} + \Theta_n(0, X^0, N^0)\frac{dN^\lambda}{d\lambda}\bigg|_{\lambda=0} \\
&= \Theta_\lambda(0, X^0, N^0) + \Theta_x(0, X^0, N^0)\frac{dX^\lambda}{d\lambda}\bigg|_{\lambda=0} \\
&= \theta^1(X^0, N^0) + \theta^0_x(X^0)\frac{dX^\lambda}{d\lambda}\bigg|_{\lambda=0}
\end{aligned} \tag{2.40}$$

because $\Theta(0, x, n)$ is constant in n (in the case of power utility, $\Theta(0, x, n)$ is a constant). So formally writing a Taylor expansion, we should have

$$\Theta(\lambda, X^\lambda, N^\lambda) = \theta^0(X^0) + \lambda\left(\theta^1(X^0, N^0) + \theta^0_x(X^0)\frac{dX^\lambda}{d\lambda}\bigg|_{\lambda=0}\right) + o(\lambda)$$

On the other hand, we also formally compute that

$$\begin{aligned}
\frac{d}{d\lambda}\hat{\theta}^\lambda(\hat{X}^\lambda, \hat{N}^\lambda)\bigg|_{\lambda=0} &= \frac{d}{d\lambda}\left(\theta^0(\hat{X}^\lambda) + \lambda\theta^1(\hat{X}^\lambda, \hat{N}^\lambda)\right)\bigg|_{\lambda=0} \\
&= \theta^0_x(X^0)\frac{d\hat{X}^\lambda}{d\lambda}\bigg|_{\lambda=0} + \theta^1(X^0, N^0)
\end{aligned}$$

Therefore, we should have

$$\begin{aligned}
\hat{\theta}^\lambda(\hat{X}^\lambda, \hat{N}^\lambda) &= \theta^0(X^0) \\
&\quad + \lambda\left(\theta^0_x(X^0)\frac{d\hat{X}^\lambda}{d\lambda}\bigg|_{\lambda=0} + \theta^1(X^0, N^0)\right) + o(\lambda)
\end{aligned}$$

As a result, we obtain

$$\Theta(\lambda, X^\lambda, N^\lambda) - \hat{\theta}^\lambda(\hat{X}^\lambda, \hat{N}^\lambda) = \lambda\theta_x^0(X^0)\frac{d}{d\lambda}\left(X^\lambda - \hat{X}^\lambda\right)\Big|_{\lambda=0} + o(\lambda) \quad (2.41)$$

Heuristically, we expect to have

$$\frac{dX^\lambda}{d\lambda}\Big|_{\lambda=0} = \frac{d\hat{X}^\lambda}{d\lambda}\Big|_{\lambda=0}$$

because X^λ and $\hat{X}^\lambda$ are determined by feedback controls θ^λ and $\hat{\theta}^\lambda$ with

$$|\theta^\lambda - \hat{\theta}^\lambda| = o(\lambda).$$

In this case, the desired equality (2.39) should follow from (2.41). In the case of power utility, this is even more straightforward; in that case θ^0 is a constant, so $\theta_x^0 = 0$ and equation (2.41) does not actually involve derivatives of X^λ and $\hat{X}^\lambda$ with respect to λ.

2.2.3 Comparison with an iterative approach

Let $\tilde{v}^\lambda$ denote the payoff of using the Merton feedback proportions while facing fee rate λ. Suppose that, as argued above, the optimal strategy $\Theta(\lambda, X^\lambda, N^\lambda)$ is within $O(\lambda)$ of the Merton proportion θ^0. Then we expect that $v^\lambda - \tilde{v}^\lambda = O(\lambda^2)$, since differences at first order in the choice of strategy should result in differences at second order in the payoff. So, assuming that $\tilde{v}^\lambda$ is concave in (x, n) for small λ, we can consider the suboptimal control

$$\begin{aligned}\tilde{\theta}^\lambda &= \arg\max_{\theta\in\mathbb{R}}\left[\mu\theta x\tilde{v}_x^\lambda + \frac{1}{2}(\sigma\theta x)^2\tilde{v}_{xx}^\lambda + \tilde{U}(\tilde{v}_x^\lambda(x,n) + \tilde{v}_n^\lambda(x,n))\right] \\ &= -\frac{\mu}{\sigma^2}\frac{\tilde{v}_x^\lambda(x,n)}{x\tilde{v}_{xx}^\lambda(x,n)}, \qquad (2.42)\end{aligned}$$

and similarly for $\tilde{c}^\lambda$. If $\tilde{v}^\lambda = v^\lambda + O(\lambda^2)$ then, taking derivatives, it is easy to see $\tilde{\theta}^\lambda = \hat{\theta}^\lambda + o(\lambda)$ as feedback expressions. We may therefore expect the payoff of $\tilde{\theta}^\lambda$ to match v^λ up to second order in λ, and could take $\tilde{\theta}^\lambda$ as an approximately optimal strategy.

We choose not to pursue this approach, however, because it is more difficult to make rigorous. For example, it is not easy to even show that $\tilde{v}^\lambda$ is concave.

2.3 The first derivative

The goal of this section is find a closed form expression for v^1, the first derivative of the value function v^λ with respect to λ. We will move between one- and two-dimensional notation as needed, using the conventions of (2.11). Recall that z will refer to a point $(x, n) \in D$, and y will be reserved for one-dimensional notation, i.e. $y = n/x \geq 1$.

Definition 2.3.1. Following [3], we begin by defining the upper and lower relaxed semi-limits for v^1.

$$\underline{v}^1(z_0) = \liminf_{z \to z_0, \lambda \to 0} \frac{v^\lambda(z) - v^0(z)}{\lambda}, \quad \overline{v}^1(z_0) = \limsup_{z \to z_0, \lambda \to 0} \frac{v^\lambda(z) - v^0(z)}{\lambda} \tag{2.43}$$

Obviously $\underline{v}^1 \leq \overline{v}^1$. By construction, $\underline{v}^1$ is lower-semicontinuous and $\overline{v}^1$ is upper-semicontinuous.

Remark 2.3.1. The relaxed semilimits $\underline{v}^1$ and $\overline{v}^1$ are readily seen to inherit the homotheticity properties of v^λ. For example,

$$\begin{aligned}
\underline{v}^1(x_0,n_0) &= \liminf_{(x,n)\to(x_0,n_0),\ \lambda\to 0} \frac{v^\lambda(x,n)-v^0(x)}{\lambda} \\
&= \liminf_{(x,n)\to(x_0,n_0),\ \lambda\to 0} \frac{x^{1-p}(v^\lambda(1,n/x)-v^0(1))}{\lambda} \\
&= x_0^{1-p} \liminf_{y\to n_0/x_0,\ \lambda\to 0} \frac{w^\lambda(y)-w^0}{\lambda}
\end{aligned}$$

Recall here that w^0 is a constant. Therefore, making the definition

$$\underline{w}^1(y_0) = \liminf_{y\to y_0,\lambda\to 0} \frac{w^\lambda(y)-w^0}{\lambda},$$

we have

$$\underline{v}^1(x_0,n_0) = x_0^{1-p}\underline{v}^1(1,n_0/x_0)$$

We therefore apply the notational conventions of (2.11) to $\underline{w}^1$. Obviously, the same remarks apply to $\overline{v}^1$ and $\overline{w}^1$.

The argument then proceeds as follows:

1. We show that $\underline{v}^1$ and $\overline{v}^1$ (and in particular $\underline{w}^1$ and $\overline{w}^1$) are finite.

2. We then show that $\underline{w}^1$ (respectively $\overline{w}^1$) is a finite viscosity supersolution (respectively subsolution) of a linear ODE which we call the first-order corrector equation.

3. A comparison principle for the first-order corrector equation will then imply

$$\underline{w}^1 = w^1 = \overline{w}^1.$$

4. Finally, we will find an explicit smooth solution to the first-order corrector equation.

2.3.1 Bounds for v^1

We now show that the relaxed semilimits $\underline{v}^1, \overline{v}^1$ are finite on the domain $D = \{(x,n) : 0 < x \leq n\}$. Obviously $v^\lambda \leq v^0$, so we have

$$\underline{v}^1 \leq \overline{v}^1 \leq 0$$

It will therefore suffice to show that $\underline{v}^1 > -\infty$.

Throughout this subsection, $(X^{\lambda,z}, N^{\lambda,z})$ will denote the state process with initial condition $z = (x,n)$ corresponding to investing the Merton proportion θ^0 in the fund and consuming at the Merton rate (as a proportion of current wealth) c^0 (in particular, $X_t^{0,z}$ will denote the optimal wealth process for the $\lambda = 0$ investment problem). The accumulated consumption under this strategy will be denoted $C^{\lambda,z}$. We begin by obtaining some path-by-path lower bounds on the wealth process $X^{\lambda,z}$.

Lemma 2.3.1. *We have the bounds*

$$\begin{aligned} X_t^{\lambda,z} &\geq \frac{n^\lambda X_t^{0,z}}{[e^{c^0 t} N_t^{\lambda,z}]^\lambda} \\ &\geq \frac{X_t^{0,z}}{(e^{c^0 t} H_t^z)^\lambda} \end{aligned} \tag{2.44}$$

where $H^z = n^{-1} \sup_{0 \leq s \leq t} [(X_s^{0,z} + C_s^{0,z}) \vee n]$

Proof. To lighten the notation, we suppress the initial condition z. Applying Itô's lemma, we see that

$$\begin{aligned} d(\log(X_t^\lambda)) &= \left(\mu\theta^0 - \frac{1}{2}(\sigma\theta^0)^2 - c^0\right)dt + \sigma\theta^0 dW_t - \frac{\lambda}{X_t} d(N_t^\lambda + C_t^\lambda) \\ &= d(\log X_t^0) - \frac{\lambda}{N_t^\lambda} d(N_t^\lambda + C_t^\lambda) \\ &= d(\log X_t^0) - \lambda\left(d\log(N_t^\lambda) + \frac{c^0 X_t^\lambda}{N_t^\lambda} dt\right) \end{aligned} \tag{2.45}$$

From the above (and the initial condition $X_0 = x$), we can conclude that

$$\begin{aligned} X_t^\lambda &= \frac{n^\lambda X_t^0}{\left(\exp\left[\int_0^t c^0 \frac{X_u^\lambda}{N_u^\lambda} du\right] N_t^\lambda\right)^\lambda} \\ &\geq \frac{n^\lambda X_t^0}{\left(e^{c^0 t} N_t^\lambda\right)^\lambda} \end{aligned} \tag{2.46}$$

To conclude, we just need to show that $N_t^\lambda \leq n^\lambda H_t$ for all λ and all t. From equation (2.45) and the fact that $N^\lambda + C^\lambda$ is an increasing process, we see that $X^\lambda \leq X^0$. Since the rate of consumption rate is proportional to wealth, we then have $C^\lambda \leq C^0$ as well, so that $X^\lambda + C^\lambda \leq X^0 + C^0$. Combining this with (2.5), we conclude that

$$\begin{aligned} N_t^\lambda &\leq (X_t^\lambda + C_t^\lambda) \vee n \\ &\leq (X_t^0 + C_t^0) \vee n = H_t \end{aligned}$$

which completes the proof. □

Remark 2.3.2. There was nothing special about using the Merton feedback proportions (θ^0, c^0) in Lemma 2.3.1. Using the same arguments, a similar result will hold for any feedback proportions $(\bar{\theta}(x,n), \bar{c}(x,n))$ such that the closed loop equation for the corresponding state process has a solution. Explicitly, let $(\bar{X}^{\lambda,z}, \bar{N}^{\lambda,z})$ be the state process determined by $(\bar{\theta}(x,n), \bar{c}(x,n))$ at fee level λ , and let $\bar{X}^{0,z}$ be the wealth process determined by the control $(\bar{\theta}(\bar{X}^{\lambda,z}, \bar{N}^{\lambda,z}), \bar{c}(\bar{X}^{\lambda,z}, \bar{N}^{\lambda,z}))$ at fee level 0 (note this is *not* a feedback strategy in terms of $\bar{X}^{0,z}$). Then we have:

$$\bar{X}_t^{\lambda,z} \quad \geq \quad \frac{n^{\lambda} \bar{X}_t^{0,z}}{\left(\exp[\int_0^t \bar{c}(\bar{X}_u^{\lambda,z}, \bar{N}_u^{\lambda,z}) du] \bar{H}_t\right)^{\lambda}}$$

$$\bar{H}_t^z \triangleq n^{-1}[(\bar{X}^{0,z} + \bar{C}^{0,z}) \vee n]^*, \quad \bar{C}_t^{0,z} \triangleq \int_0^t \bar{c}(\bar{X}_u^{\lambda,z}, \bar{N}_u^{\lambda,z}) \bar{X}_u^{0,\lambda} du$$

The following lemma will be used in this section and elsewhere. Since it deals only with the frictionless optimal investment problem, we relegate its proof to the Appendix.

Lemma 2.3.2. *Let $(\bar{\theta}_t, \bar{c}_t)$ be an admissible strategy for fee level $\lambda = 0$ given in proportions (though not necessarily in feedback form). Suppose that*

$$|\bar{\theta}_t - \theta^0| + |\bar{c} - c^0| < \epsilon \quad dt \times d\mathbb{P}\text{-almost surely.}$$

Let $\bar{X}$ be the wealth process with controls given by $(\bar{\theta}_t, \bar{c}_t)$ with initial wealth x and with no high-watermark fees. *There is a constant M depending on the model parameters μ, σ, β, p such that for $\epsilon > 0$ sufficiently small, we have*

$$\mathbb{E} \int_0^{\infty} e^{-\beta t} U(\bar{c}_t \bar{X}) dt \geq (c^0 - M\epsilon)^{-1} \frac{(c^0 - \epsilon)^{1-p} x^{1-p}}{1-p}$$

We are now ready for the principal result of this subsection:

Proposition 2.3.3. *Let* $(x_0, n_0) \in D$. *In the notation of Lemma 2.3.1, we have*

$$\begin{aligned} \underline{v}^1(x_0, n_0) &\geq -\mathbb{E}\left[\int_0^\infty \mathbf{1}_{[\tau_z \leq t]} e^{-\beta t}(1-p)U(c^0 X_t^{0,z})(c^0 t + \log(H_t^z))dt\right] \\ &> -\infty \end{aligned} \tag{2.47}$$

where $z = (x, n)$ *is an arbitrary point in* $\overline{D} - \partial_y D$ *with* $x < x_0,\ n < n_0$ *and* $n/x < n_0/x_0$

$$\tau_z = \inf\{t \geq 0 : X_t^{0,z} + C_t^{0,z} \geq n\}$$

Proof. First we show the right-hand side of (2.47) is indeed finite. Pick $q > 1$ such that $q(1-p) < 1$, and let q' denote the Holder conjugate of q. Applying the Holder inequality on $[0, \infty) \times \Omega$, it will suffice to show that that for small $\delta > 0$,

$$\mathbb{E}\int_0^\infty e^{-\beta(1-\delta)qt}|U(c^0 X_t^0)|^q dt < \infty, \tag{2.48}$$

$$\mathbb{E}\int_0^\infty e^{-\beta\delta q' t}(c^0 t + \log(H_t^z))^{q'} dt < \infty. \tag{2.49}$$

First suppose $(1-p) > 0$. Then the left hand side of (2.48) is non-negative and bounded above by the value function of the Merton consumption/investment problem with the modified utility function and discount factor given by

$$\begin{aligned} V(x) &= (U(x))^q, \\ \rho &= \beta(1-\delta)q. \end{aligned}$$

We assume that U and the model parameters are chosen so that the well-posedness condition (2.25) holds:

$$\beta > \left(\frac{1}{2}\right)\frac{1-p}{p}\frac{\mu^2}{\sigma^2}.$$

Choosing $q > 1$ and $\delta > 0$ sufficiently small, the same condition does indeed hold for the modified utility function $V(x)$ and discount factor ρ, hence the first inequality of (2.48) holds.

$$\mathbb{E}\int_0^\infty \operatorname{sign}(1-p)e^{-\beta(1-\delta)qt}|U(c^0 X_t^0)|^q dt < \infty.$$

On the other hand, if $(1-p) < 0$, we once again consider the Merton consumption/investment problem with modified utility function and discount factor

$$\begin{aligned} V(x) &= -|U(x)|^q, \\ \rho &= \beta(1-\delta)q. \end{aligned}$$

Let $(\overline{\theta}^0, \overline{c}^0)$ denote the optimal feedback proportions for the investment/consumption problem with the modified utility function and discount factor V and ρ, as given in Proposition 2.1.5. Then for any ϵ, we can choose $q > 1$ sufficiently small that

$$|\overline{\theta}^0 - \theta^0| + |\overline{c}^0 - c^0| < \epsilon.$$

As a result, Lemma 2.3.2 implies that

$$\mathbb{E}\int_0^\infty e^{-\beta(1-\delta)qt}|U(c^0 X_t^0)|^q dt > -\infty.$$

To check (2.49), note that

$$C_t^{0,z} = \int_0^t c^0 X_u^{0,z} du \leq c^0 t \sup_{0\leq s\ \leq t} X_s,$$

and therefore

$$\begin{aligned} H_t^z &= n^{-1} \sup_{0\leq s\leq t} \left[(X_s^{0,z} + C_s^{0,z}) \vee n\right] \\ &\leq n^{-1}(n \vee (1 + c_0 t) \sup_{0\leq s\leq t} X^{0,z}). \end{aligned}$$

So there is a constant $K_0(q') > 0$ such that

$$|c^0 t + \log(H_t^z)|^{q'} \leq K_0(q') \left(1 + c_0 t^{q'} + \left| \sup_{0 \leq s \leq t} \log(X_s^{0,z}) \right|^{q'} \right) \tag{2.50}$$

Note that $\log(X^{0,z})$ is a Brownian motion with drift:

$$\log(X_t^{0,z}) = \left(\mu\theta^0 - c^0 - \frac{(\sigma\theta^0)^2}{2} \right) t + \sigma\theta^0 W_t$$

Now, Doob's maximal inequality tells us that

$$\begin{aligned} \mathbb{E}\left[\left| \sup_{0 \leq s \leq t} \sigma\theta^0 W_t \right|^{q'} \right] &\leq \frac{q'(\sigma\theta)^{q'}}{q'-1} \mathbb{E}[W_t^{q'}] \\ &= \frac{q'(\sigma\theta t)^{q'}/2}{q'-1} \mathbb{E}[\chi^{q'}] \end{aligned}$$

where χ is some $N(0,1)$ random variable. As a result, there are constants K_1, b with

$$\mathbb{E} \sup_{0 \leq s \leq t} |\log(X_t^{0,z})| \leq K(1 + t^b)$$

From (2.50), we therefore have

$$\mathbb{E}\left[|c^0 t + \log(H_t^z)|^{q'} \right] \leq K(1 + t^b)$$

after possibly enlarging K_1, b. Putting everything together, we conclude

$$\mathbb{E} \int_0^\infty e^{-\beta\delta q' t} (c^0 t + \log(H_t^z))^{q'} dt \;\; \leq \;\; \int_0^\infty e^{-\beta\delta q' t} K(1 + t^b) dt < \infty. \tag{2.51}$$

Now fix $z_0 = (x_0, n_0) \in D$ and pick $z_k \to z_0$ and $\lambda_k \to 0$ such that

$$\underline{v}^1(z_0) = \lim_{k \to \infty} \frac{v^{\lambda_k}(z_k) - v^0(z_0)}{\lambda_k}$$

Fix $z = (x, n) \in \overline{D} - \partial_y D$ with $n/x < n_0/x_0$. Note that τ_z is increasing in the ratio n/x, and that obviously $X^{\lambda,z} = X^{0,z}$ until τ_z. Applying Lemma 2.3.1, we see that for k sufficiently large,

$$\begin{aligned}\frac{U(c^0 X_t^{\lambda_k,z_k}) - U(c^0 X_t^{0,z_k})}{\lambda_k n} &\geq \mathbf{1}_{[\tau_{z_k}\leq t]} U(c^0 X_t^{0,z_k}) \frac{\left((e^{c^0 t} H_t^{z_k})^{-(1-p)\lambda_k} - 1\right)}{\lambda_k} \\ &\geq \mathbf{1}_{[\tau_z \leq t]} U(c^0 X_t^{0,z}) \frac{\left((e^{c^0 t} H_t^{z})^{-(1-p)\lambda_k} - 1\right)}{\lambda_k}\end{aligned}$$

By convexity of the function $\lambda \mapsto m^{-\lambda}$, we have the monotone convergence

$$\mathbf{1}_{[\tau_z \leq t]} U(c^0 X_t^{0,z}) \frac{\left((e^{c^0 t} H_t^{z})^{-(1-p)\lambda_k} - 1\right)}{\lambda_k} \nearrow -\mathbf{1}_{[\tau_z \leq t]}(1-p) U(c^0 X_t^{0,z})(c^0 t + \log(H_t^z)) \tag{2.52}$$

which holds $dt \otimes d\mathbb{P}-$almost surely on the set $[0, \infty) \times \Omega$. To conclude, we apply the monotone convergence theorem:

$$\begin{aligned}\underline{v}^1(x_0, n_0) &\geq \liminf_{k\to\infty} \mathbb{E} \int_0^\infty e^{-\beta t} \frac{U(c^0 X_t^{\lambda_k,z_k}) - U(c^0 X_t^{0,z_k})}{\lambda_k} dt \\ &\geq \lim_{k\to\infty} \mathbb{E} \int_0^\infty e^{-\beta t} \mathbf{1}_{[\tau_z \leq t]} U(c^0 X_t^{0,z}) \frac{\left((e^{c^0 t} H_t^{z})^{-(1-p)\lambda_k} - 1\right)}{\lambda_k} dt \\ &= -(1-p)\mathbb{E} \int_0^\infty e^{-\beta t} \left[\mathbf{1}_{[\tau_z \leq t]} U(c^0 X_t^{0,z})(c^0 t + \log(H_t^z))\right] dt \\ &> -\infty \end{aligned} \tag{2.53}$$

□

Proposition 2.3.3 also allows us to determine the limiting behavior of v^1 as $n \to \infty$.

Corollary 2.3.4. *We have*

$$\lim_{n\to\infty} \underline{v}^1(x,n) = 0$$

Proof. By Proposition 2.3.3, we have

$$\underline{v}^1(x_0,n_0) \geq -(1-p)\mathbb{E}\left[\int_0^\infty \mathbf{1}_{[\tau_z \leq t]} e^{-\beta t} U(c^0 X_t^{0,x})(c^0 t + \log(H_t^z))dt\right]$$

where $z = (x,n)$ is an arbitrary point in D with $x < x_0,\ n < n_0$ and $n/x < n_0/x_0$. Letting $n_0 \to \infty$ and choosing z_k with $n \to \infty$, while x, x_0 remain fixed, we see that

$$-\mathbf{1}_{[\tau_{z_k} \leq t]} e^{-\beta t}(1-p)U(c^0 X_t^{0,x})(c^0 t + \log(H_t^{z_k})) \nearrow 0$$

because $\mathbf{1}_{[\tau_z \leq t]}$ and H^z are both decreasing in n, and $\tau_z \to \infty$ as $n \to \infty$. From Proposition 2.3.3, we also have

$$\mathbb{E}\int_0^\infty \mathbf{1}_{[\tau_{z_k} \leq t]} e^{-\beta t} U(c^0 X_t^{0,x})(c^0 t + \log(H_t^{z_k}))dt > -\infty$$

We then apply the monotone convergence theorem on $[0,\infty) \times \Omega$ to conclude. $\square$

Remark 2.3.3. Let $\tilde{v}^\lambda(x,n)$ denote the payoff of using the Merton proportion (θ^0, c^0) at fee level λ, with initial condition (x,n). Obviously we have

$$\tilde{v}^\lambda(x,n) \leq v^\lambda(x,n)$$

Now define

$$\underline{\tilde{v}}^1(z_0) = \liminf_{z\to z_0,\ \lambda\to 0} \frac{\tilde{v}^\lambda(z) - v^0(z)}{\lambda}$$

Examining the proof of 2.3.3, we see that we have in fact proven the slightly stronger result

$$\underline{\tilde{v}}^1(x_0,n_0) \geq -(1-p)\mathbb{E}\left[\int_0^\infty \mathbf{1}_{[\tau_z \leq t]} e^{-\beta t} U(c^0 X_t^{0,z})(c^0 t + \log(H_t^z))dt\right] \quad (2.54)$$
$$> -\infty$$

where $z = (x,n)$ is an arbitrary point in $\overline{D} - \partial_y D$ with $x < x_0,\ n < n_0$ and $n/x < n_0/x_0$

$$\tau_z = \inf\{t \geq 0 : X_t^{0,z} + C_t^{0,z} \geq n\}$$

This fact will be used in the construction of approximately optimal strategies.

2.3.2 The first-order corrector equation

In this section we introduce the first-order corrector equation and show that the relaxed semilimit $\underline{w}^1$ (respectively $\overline{w}^1$) is a viscosity supersolution (resp. subsolution) of the first-order corrector equation. The following is just a one-dimensional version of (2.36).

The first-order corrector equation. The *one-dimensional first order corrector equation* for a C^2 function $w : [1,\infty) \to \mathbb{R}$ is given by

$$\begin{aligned}
\mathcal{A}w &\triangleq -\beta w + \mu\theta^0 w_x + \frac{1}{2}(\sigma\theta^0)^2 w_{xx} + \tilde{U}'(w_x^0(1))(w_x + w_n) = 0 \\
\mathcal{B}w &\triangleq w_y(1) - w_x^0(1) = 0 \qquad (2.55) \\
\lim_{y\to\infty} w(y) &= 0
\end{aligned}$$

where w_x and w_{xx} are as defined following the conventions of (2.11). If a function w satisfies the one-dimensional first corrector equation, then $u(x,n) =$

$x^{1-p}w(n)$ satisfies (2.36):

$$\begin{aligned}
-\beta u + \mu\theta^0 x u_x + \frac{1}{2}(\sigma\theta^0 x)^2 u_{xx} + \tilde{U}'(v_x^0(x))(u_x + u_n) &= 0 \text{ on int}(D) \\
u_n(x,x) - v_x^0(x) &= 0 \text{ for all } x \geq 0 \\
\lim_{n\to\infty} u(x,n) &= 0
\end{aligned}$$

For future reference, we record the coefficients of the first-order corrector equation in true one-dimensional notation.

Lemma 2.3.5. *We have*

$$(\mathcal{A}w)(y) = \frac{1}{2}(\sigma\theta^0 y)^2 w_{yy}(y) + c^0(y-1)w_y(y) - c^0 w(y) \tag{2.56}$$

Proof. First note that

$$\tilde{U}(\tilde{y}) = \frac{y^{-q}}{q}, \quad q = \frac{1-p}{p}$$

so that

$$\begin{aligned}
\tilde{U}'(w_x^0) &= \tilde{U}'((c^0)^{-p})) \\
&= -(c^0)^{(1-p)+p} = -c^0
\end{aligned} \tag{2.57}$$

Recalling the definitions of (2.11), we have

$$\begin{aligned}
\mathcal{A}w &= -\beta w + \mu\theta^0(1-p)w - \mu\theta^0 y w_y \\
&\quad + \frac{1}{2}(\sigma\theta^0)^2\left(-p(1-p)w + 2pyw_y + y^2 w_{yy}\right) \\
&\quad + c^0((1-p)w + (y-1)w_y) \\
&= \left(-\beta + \frac{1}{2}\frac{(1-p)\mu^2}{p\sigma^2} - (1-p)c^0\right) w \qquad (2.58) \\
&\quad + \left(-\frac{\mu^2 y}{p\sigma^2} + \frac{\mu^2 y}{p\sigma^2} + c^0(y-1)\right) w_y \\
&\quad + \frac{1}{2}(\sigma\theta^0 y)^2 w_{yy} \\
&= \frac{1}{2}(\sigma\theta^0 y)^2 w_{yy} + c^0(y-1)w_y - c^0 w
\end{aligned}$$

where the second equality (2.56) follows the expressions for θ^0 and c^0 recorded in Proposition 2.1.5. □

Following [5], we introduce the notion of a viscosity subsolutions and supersolutions of the first-order corrector equation.

Definition 2.3.2. Let $\underline{w} : [1, \infty) \to \mathbb{R}$ be a lower semicontinuous function. The inequality $\mathcal{A}\underline{w} \leq 0$ holds in the viscosity sense at $y_0 > 1$ if, for every C^2 function $\phi; [1, \infty) \to \mathbb{R}$ such that $\underline{w} - \phi$ has a local minimum of 0 at y_0 (we say ϕ *touches* $\underline{w}$ *below at* y_0), we have

$$\mathcal{A}\phi(y_0) \leq 0.$$

The inequality $\mathcal{A}\underline{w} \leq 0$ at $y_0 = 1$ holds in the viscosity sense if, for all C^2 functions $\phi : [1-\epsilon, \infty)$ such that such that $w - \phi$ achieves a local minimum of 0 *on* $[1,\infty)$ at 1 (we say ϕ *touches* $\underline{w}$ *below at* 1), we have

$$\mathcal{A}\phi(1) \le 0.$$

Finally, $\mathcal{B}\underline{w} \le 0$ holds in the viscosity sense if for all C^2 functions ϕ touching $\underline{w}$ below at 1, we have

$$\mathcal{B}\phi \le 0.$$

We say $\underline{w}$ is a viscosity supersolution of the first-order corrector equation if

$$\liminf_{y\to\infty} w(y) \ge 0$$

and the following set of inequalities hold in the viscosity sense:

$$\begin{aligned} \mathcal{A}\underline{w} &\le 0 \text{ on } (1,\infty) \\ \mathcal{A}\underline{w} \wedge \mathcal{B}\underline{w} &\le 0 \text{ at } y = 1 \end{aligned}$$

Definition 2.3.3. Let $\overline{w} : [1,\infty) \to \mathbb{R}$ be an upper semicontinuous function. The inequality $\mathcal{A}\overline{w} \ge 0$ holds in the viscosity sense at $y_0 > 1$ if, for every C^2 function $\phi; [1,\infty) \to \mathbb{R}$ such that $\overline{w} - \phi$ has a local maximum of 0 at y_0 (we say ϕ *touches* $\overline{w}$ *above at* y_0), we have

$$\mathcal{A}\phi(y_0) \ge 0$$

The inequality $\mathcal{A}\overline{w} \ge 0$ holds at $y_0 = 1$ in the viscosity sense if, for all C^2 functions $\phi : [1-\epsilon,\infty)$ such that such that $w - \phi$ achieves a local maximum of 0 *on* $[1,\infty)$ at 1 (we say ϕ *touches* $\overline{w}$ *above at* 1), we have

$$\mathcal{A}\phi(y_0) \ge 0$$

Finally, $\mathcal{B}\overline{w} \geq 0$ holds in the viscosity sense if for all C^2 functions ϕ touching $\overline{w}$ above at 1, we have

$$\mathcal{B}\phi \geq 0$$

We say $\overline{w}$ is a viscosity subsolution of the first-order corrector equation if

$$\limsup_{y\to\infty} w(y) \leq 0$$

and the following set of inequalities hold in the viscosity sense:

$$\begin{aligned} \mathcal{A}\overline{w} &\geq 0 \text{ on } (1,\infty) \\ \mathcal{A}\overline{w} \vee \mathcal{B}\overline{w} &\geq 0 \text{ at } y = 1 \end{aligned}$$

Remark 2.3.4. In Definition 2.3.2 all instances of "local minimum" may equivalently be replaced with "strict local minimum", "global minimum on $[1,\infty)$", or "strict global minimum on $[1,\infty)$" (and similarly with all instances of "local maximum" in Definition 2.3.3).

For reference, we include the following characterization of smooth viscosity sub- and supersolutions of the first order corrector equation.

Lemma 2.3.6. *Let $w : [1,\infty) \to \mathbb{R}$ be a C^2 function on on $(1,\infty)$ such that the right-hand derivatives*

$$\begin{aligned} w_{y+}(1) &\triangleq \lim_{y\to 1+} \frac{w(y) - w(1)}{y-1} \\ w_{yy+}(1) &\triangleq 2\lim_{y\to 1+} \frac{w(y) - (w(1) + (y-1)w_{y+}(1)}{(y-)^2} \end{aligned}$$

are well-defined and finite. The w is a viscosity subsolution of the first-order corrector equation if and only if we have

$$\begin{aligned} \mathcal{A}w \geq 0 \quad &\textit{on } (1,\infty) \\ w_{y+}(1) - w_x^0(1) \geq 0 & \end{aligned} \tag{2.59}$$

in the classical sense. Similarly, w is a viscosity supersolution if and only if

$$\begin{aligned} \mathcal{A}w \leq 0 \quad &\textit{on } (1,\infty) \\ w_{y+}(1) - w_x^0(1) \leq 0 & \end{aligned} \tag{2.60}$$

Proof. We deal only with the subsolution case. The only part of the conclusion which is not immediate is that if w is a smooth viscosity subsolution of the first-order corrector equation, so that

$$\mathcal{A}w \vee \mathcal{B}w \geq 0$$

in the viscosity sense at $y = 1$, then we must actually have $w_{y+} - w_x^0 \geq 0$. Suppose that $w_{y+} - w_x^0 < 0$. Pick a constant a with $w_{y+}(1) < a < w_x^0$ and define the function $\phi(y) = w(1) + a(y-1) - b(y-1)^2$ for some $b > 0$. By (2.59) and the definition of w_{y+}, we know $w - \phi$ achieves a local maximum of 0 at $y = 1$ for arbitrarily large choices of b. We then have $\phi_y(1) - w_x^0(1) < 0$ and for b sufficiently large

$$\mathcal{A}\phi(1) = -\frac{1}{2}(\sigma\theta^0)^2(2b) - c^0 w(1) < 0$$

contradicting the viscosity subsolution property of w at 1.

□

Definition 2.3.4. For $\theta \in \mathbb{R}$ and a function $w : [1, \infty) \to \mathbb{R}$, define the operator

$$L^\theta w \triangleq -\beta w + \mu\theta w_x + \frac{1}{2}(\sigma\theta)^2 w_{xx} + \tilde{U}(w_x + w_n)$$

using the notational conventions of (2.11).

Remark 2.3.5. With the notation of 2.3.4, we have

$$\begin{aligned} \sup_{\theta\in\mathbb{R}} L^\theta w^\lambda &= \sup_{\theta\in\mathbb{R},c>0} \mathcal{L}^{\theta,c} w^\lambda \\ &= 0 \end{aligned} \tag{2.61}$$

We also note that

$$L^{\theta^\lambda} w^\lambda = 0,$$

where θ^λ is the optimal investment proportion for fee level λ.

Proposition 2.3.7. *The lower relaxed semi-limit $\underline{w}^1$ is a viscosity supersolution of the one-dimensional first-order corrector equation.*

Proof. Let ϕ be a C^2 function such that $\underline{w}^1 - \phi$ has a *strict* local minimum of 0 at y_0. Make the notation

$$Q^{1,\lambda} \triangleq \frac{w^\lambda - w^0}{\lambda}.$$

Pick a subsequence $\lambda_k \to 0$ and $y_k \to y_0$ such that $Q^{1,\lambda_k}(y_k) \to \underline{w}^1(y_0)$. Let $\hat{y}_k$ be the minimizers of $Q^{1,\lambda_k} - \phi$ on a small, closed ball around y_0. We must have $\hat{y}_k \to y_0$. If not, then there is a subsequence $\hat{y}_{k_i} \to y \neq y_0$. Then

$$\begin{aligned} 0 < \underline{w}^1(y) - \phi(y) &\leq \liminf_{i\to\infty} Q^{1,\lambda_{k_i}}(\hat{y}_{k_i}) - \phi(\hat{y}_{k_i}) \\ &\leq \liminf_{i\to\infty} Q^{1,\lambda_{k_i}}(y_{k_i}) - \phi(y_{k_i}) \\ &= \underline{w}^1(y_0) - \phi(y_0) = 0 \end{aligned}$$

a contradiction. We conclude $\hat{y}_k \to y_0$. Since ϕ is continuous and $\hat{y}_k$ minimizes $\underline{w}^1 - \phi$ on a ball around y_0, we see that

$$\begin{aligned}
\underline{w}^1(y_0) - \phi(y_0) &= \lim_{k\to\infty} (Q^{1,y}(y_k) - \phi(y_k)) \\
&\geq \liminf_{k\to\infty} (Q^{1,y}(\hat{y}_k) - \phi(\hat{y}_k)) \\
&= \liminf_{k\to\infty} Q^{1,y}(\hat{y}_k) - \phi(y_0)
\end{aligned}$$

So after possibly extracting a subsequence of the $\hat{y}_k$, we have $Q^{1,y}(\hat{y}_k) \to \underline{w}^1(y_0)$. In other words, we may take $y_k = \hat{y}_k$. Now construct the smooth functions

$$\psi^k(y) = w^{\lambda_k}(y_k) + \lambda_k(\phi(y) - \phi(y_k)) \tag{2.62}$$

We claim that ψ^k touches w^{λ_k} below at y_k. To see this, note that since w^0 is constant in y_k, we have

$$\begin{aligned}
(w^{\lambda_k} - \psi^k)(y) &= (w^{\lambda_k}(y) - w^0(y) - \lambda_k\phi(y)) + (\lambda_k\phi(y_k) + w^{\lambda_k}(y_k) + w^0(y)) \\
&= \lambda_k(Q^{1,\lambda_k}(y) - \phi) + (\lambda_k\phi(y_k) - w^{\lambda_k}(y_k) + w^0(y))
\end{aligned}$$

Since $(\lambda_k\phi(y_k) - w^{\lambda_k}(y_k))$ is a constant for fixed k and $Q^{1,\lambda_k} - \phi$ has a local minimum at y_k, we conclude that $w^{\lambda_k} - \psi^k$ has a local minimum at y_k. By construction we have $(w^{\lambda_k} - \psi^k)(y_k) = 0$, so ψ^k touches w^{λ_k} below at y_k.

To begin with, we suppose that $y_0 > 1$. Note that

$$\begin{aligned}
\psi^k_x(y_k) &= (1-p)w^{\lambda_k}(y_k) - \lambda_k y_k \phi_y(y_k) \\
\psi^k_y(y_k) &= \lambda_k\phi_y(y_k) \\
\psi^k_{xx}(y_k) &= -p(1-p)w^{\lambda_k}(y_k) + \lambda_k(2py_k\phi_y(y_k) + y_k^2\phi_{yy}(y_k))
\end{aligned} \tag{2.63}$$

From Remark 2.3.5 and (2.63), we have at y_k

$$
\begin{aligned}
0 &= \sup_{\theta\in\mathbb{R}} L^{\theta^0} w^{\lambda_k}(y_k) \\
&\geq L^{\theta^0} w^{\lambda_k}(y_k) \\
&\geq L^{\theta^0} \psi^k(y_k) \qquad (2.64) \\
&= -\beta w^{\lambda_k} + \mu\theta^0((1-p)w^{\lambda_k} - \lambda_k y_k \phi_y) \\
&\quad +\frac{1}{2}(\sigma\theta^0)^2(-p(1-p)w^{\lambda_k} + \lambda_k(2py_k\phi_y + y_k^2\phi_{yy})) \\
&\quad +\tilde{U}((1-p)w_x^{\lambda_k} + \lambda_k(\phi_x + \phi_n)).
\end{aligned}
$$

Note that $L^{\theta^0} w^0 = 0$ since w^0 satisfies the $\lambda = 0$ HJB equation. Therefore, we have at y_k

$$
\begin{aligned}
L^{\theta^0}\psi^k &= L^{\theta^0}\psi^k - L^{\theta^0} w^0 \\
&= -\beta(w^{\lambda_k} - w^0) + \mu\theta^0((1-p)(w^{\lambda_k} - w^0) - \lambda_k y_k \phi_y) \qquad (2.65) \\
&\quad +\frac{1}{2}(\sigma\theta^0)^2(-p(1-p)(w^{\lambda_k} - w^0) + \lambda_k(2py_k\phi_y + y_k^2\phi_{yy})) \\
&\quad +\tilde{U}((1-p)w_x^{\lambda_k} + (1-y_k)\phi_y) - \tilde{U}(w_x^0).
\end{aligned}
$$

We now plan to divide by λ_k in (2.64) and send $k \to \infty$ to see that $\mathcal{A}\phi(y_0) \leq 0$. First, recall that

$$
Q^{1,\lambda_k}(y_k) = \frac{w^{\lambda_k}(y_k) - w^0(y_k)}{\lambda_k} \to \underline{w}^1(y_0) = \phi(y_0).
$$

Examining the terms of (2.65), we have

$$
\begin{aligned}
\frac{(1-p)(w^{\lambda_k} - w^0) - \lambda_k y_k \phi_y}{\lambda_k} &\to \phi_x(y_0), \\
\frac{-p(1-p)(w^{\lambda_k} - w^0) + \lambda_k(2py_k\phi_y + y_k^2\phi_{yy})}{\lambda_k} &\to \phi_{xx}(y_0).
\end{aligned} \qquad (2.66)
$$

From Taylor's theorem with remainder and the fact that $w_x^0 = (1-p)w^0$, we have

$$\begin{aligned}
\tilde{U}((1-p)w_x^{\lambda_k} + (1-y_k)\phi_y) &= \tilde{U}(w_x^0) \\
&\quad + \tilde{U}'(w^0)\left((1-p)(w^{\lambda_k} - w^0) + \lambda_k(1-y_l)\phi_y\right) \\
&\quad + \frac{\lambda_k^2}{2}\tilde{U}''(\xi)\left((1-p)(w^{\lambda_k} - w^0) + \lambda_k(1-y_k)\phi_y\right)^2
\end{aligned}$$

for ξ between $w_x^0(y_k)$ and $w_x^0(y_k) + \lambda_k(\phi_x + \phi_n)(y_k)$. Since the y_k are bounded and $w^0, w^{\lambda_k},$ and ϕ are C^2, we observe

$$\frac{\tilde{U}((1-p)w_x^{\lambda_k} + (1-y_k)\phi_y) - \tilde{U}((1-p)w^0)}{\lambda_k} \quad\to\quad \tilde{U}'(w_x^0(y_0))(\nabla\phi \cdot \mathbf{1})$$

Combining the above and (2.66) with (2.65), we conclude that

$$0 \le \frac{L^{\theta_0}\psi^k(y_k)}{\lambda_k} \to \mathcal{A}\phi(y_0) \tag{2.67}$$

so that the supersolution property for $\underline{w}^1$ holds at y_0.

Now suppose that $y_0 = 1$. If $y_k > 1$ for infinitely many y_k as above, then we may apply the same argument to show that $\mathcal{A}\underline{w}^1 \le 0$ in the viscosity sense at y_0. Otherwise, we may assume $y_k = 1$, so that $w_y^{\lambda_k}(y_k) - \lambda_k w_x^{\lambda_k}(y_k) \le 0$ in the viscosity sense. Touching w^{λ_k} below by the same ψ^k at $y_k = 1$, we see that

$$\begin{aligned}
0 = w_y^{\lambda_k}(y_k) - \lambda_k w_x^{\lambda_k}(y_k) &= (1+\lambda_k)w_y^{\lambda_k}(1) - \lambda_k(1-p)w^{\lambda_k}(1) \\
&\ge (1+\lambda_k)\psi_y^k(1) - \lambda_k(1-p)\psi^k(1) \\
&= (1+\lambda_k)\lambda_k\phi_n(1) - \lambda_k(1-p)(w^{\lambda_k}(1) + \lambda_k\phi(1))
\end{aligned}$$

Dividing by λ_k and letting $k \to \infty$, we conclude that

$$\begin{aligned}
0 &\le \lim_{k\to\infty} \frac{-\lambda_k(1-p)(w^{\lambda_k}(1) + \lambda_k\phi(1)) + (1+\lambda_k)\lambda_k\phi_n(1)}{\lambda_k} \\
&= -(1-p)w^0(1) + \phi_y(1)
\end{aligned}$$

Since $v^0(x,n) = (c^0)^{-p}\frac{x^{1-p}}{1-p}$, we have $w_x^0(1) = v_x^1(1,1) = (1-p)w^0(1)$. We conclude that $\phi_y(1) - w_x^0(1) \le 0$, so the boundary condition $(\mathcal{A}\underline{w}(1) \wedge \mathcal{B}w) \le 0$ holds in the viscosity sense. $\square$

Proposition 2.3.8. *The upper relaxed semi-limit $\overline{w}^1$ is a viscosity subsolution of the one-dimensional first-order corrector equation.*

Proof. First, suppose that $y_0 \in (1,\infty)$. Construct $\lambda_k \to 0$ and $y_k \to y_0$ as in the proof of Proposition 2.3.7. Let ϕ be a C^2 function such that $\overline{w}^1 - \phi$ has a strict local maximum at y_0. Construct C^2 functions ψ^k such that w^{λ_k} has a strict local maximum at y_k, as was done in the proof of Proposition 2.3.7. First suppose that $y_k > 1$ for infinitely many k. In this case, we can repeat the arguments of Proposition 2.3.7, (2.64) and following, to show that

$$\lim_{k\to\infty} \frac{L^{\theta^0}\psi^k(y_k)}{\lambda_k} = \mathcal{A}\phi(y_0).$$

On the other hand, since ψ^k touches w^{λ_k} above at y_k, we know that

$$\sup_{\theta\in\mathbb{R}} L^\theta\psi^k(y_k) \ge 0$$

for all k. To verify the subsolution property for $\overline{w}^1$ at y_0, it will therefore suffice to show that

$$\liminf_{k\to\infty} \frac{(L^{\theta^0}\psi^k)(y_k) - \left[\sup_{\theta\in\mathbb{R}}(L^\theta\psi^k)(y_k)\right]}{\lambda_k} \ge 0. \tag{2.68}$$

Now, recall that

$$\psi^k(y) \quad = \quad w^{\lambda_k}(y_k) + \lambda_k(\phi(y) - \phi(y_k))$$

As in Proposition 2.3.7, we have

$$
\begin{aligned}
\psi^k_y(y_k) &= \lambda_k \phi_y(y_k) \\
\psi^k_x(y) &= (1-p)w^{\lambda_k}(y_k) - y_k\phi_y(y_k) \\
\psi^k_{xx}(y) &= -p(1-p)w^{\lambda_k}(y_k) + 2py_k\phi_y(y_k) + y^2\phi_{yy}(y_k)
\end{aligned}
$$

Since w^λ is increasing in y and w^0 is constant, Proposition 2.3.3 implies there is a constant $M > 0$

$$
\begin{aligned}
0 &\geq w^{\lambda_k}(y_k) - w^0(y_k) \\
&\geq w^{\lambda_k}(1) - w^0(1) \geq -M\lambda_k,
\end{aligned}
\tag{2.69}
$$

for sufficiently large k. As a result, we have

$$
\sup_{y\in[1,\infty)} |\psi^k - w^0| + |\psi^k_y - w^0_y||\psi^k_x - w^0_x| + |\psi^k_{xx} - w^0_{xx}| \leq M\lambda_k
$$

for large k, after possibly enlarging M. For k sufficiently large that

$$
\psi^k_{xx}(y_k) < 0,
$$

we have

$$
\sup_{\theta\in\mathbb{R}} L^\theta \psi^k = -\beta\psi^k + \tilde{U}(\psi^k_x + \psi^k_n) - \frac{1}{2}\left(\frac{\mu}{\sigma}\right)^2 \frac{(\psi^k_x)^2}{\psi^k_{xx}} \geq 0
$$

Define the functions of $(a, b) \in \mathbb{R}^2$

$$
\begin{aligned}
\Phi^1(a,b) &= -\frac{1}{2}\left(\frac{\mu}{\sigma}\right)^2 \frac{(w^0_x + a)^2}{(w^0_{xx} + b)} \\
\Phi^2(a,b) &= \mu\theta^0(w^0_x + a) + \frac{1}{2}(\sigma\theta^0)^2(w^0_{xx} + b)
\end{aligned}
$$

We will be done if we show

$$\lim_{k\to\infty} \frac{\Phi^1(\phi_x - w_x^0, \phi_{xx} - w_{xx}^0) - \Phi^2(\phi_x - w_x^0, \phi_{xx} - w_{xx}^0)}{\lambda_k} = 0 \tag{2.70}$$

because no other terms contribute to (2.68). Once can then check that

1. The definition of θ^0 and the closed-form expression for w^0 imply the first-order derivatives of the Φ^i are equal at $(a, b) = (0, 0)$; that is,

$$\nabla\Phi^1(0, 0) = \nabla\Phi^2(0, 0).$$

2. The second-order derivatives of the Φ^i are bounded in a neighborhood of $(0, 0)$.

As a result, we obtain (2.70) from (2.69) by a Taylor expansion of $\Phi^1 - \Phi^2$ around $(0, 0)$.

If $y_k = 1$ for infinitely many k, we may repeat the arguments of Proposition 2.3.7. $\square$

2.3.3 A general comparison principle for the corrector equations

In this subsection, we apply a comparison principle for the first-order corrector equations to conclude that w^1 is well-defined, and is indeed the first derivative of w^λ with respect to λ. This comparison principle will also be the main tool for obtaining bounds on the relaxed semilimits corresponding to the second derivative of w^λ with respect to λ.

The following lemma provides explicit subsolutions and supersolutions of the first-order corrector equation which will be useful both in the proof of the comparison principle and for obtaining later bounds for the second derivative of w^λ with respect to λ.

Lemma 2.3.9. *Let $f(y) = y^{-q}$. Then for $q > 0$ sufficiently small we have*

$$\mathcal{A}f \leq 0 \text{ on } [1, \infty)$$

Proof. Just note that

$$\mathcal{A}f = y^{-q}\left(-c^0 - \frac{c^0 q(y-1)}{y} + \frac{q(q+1)}{2}(\sigma\theta^0)^2\right) \tag{2.71}$$

so that $\mathcal{A}f \leq 0$ for q small. □

We now introduce a slight generalization of the first-order corrector equation and state the comparison result; the proof will be given in the Appendix. We introduce this result to provide a unified proof of comparison for both the first- and second-order corrector equations.

Generalized corrector equation: We say a function $w : [1, \infty) \to \mathbb{R}$ satisfies the *generalized corrector equation* with non-homogeneity $h : [1, \infty) \to \mathbb{R}$ smooth, boundary condition η, and limit L if

$$\begin{aligned} \mathcal{A}w + h &= 0 \\ w_y(1) &= \eta \\ \lim_{y\to\infty} w(y) &= L \end{aligned} \tag{2.72}$$

We define the notions of viscosity sub- and supersolutions to the generalized corrector equation as we did for the first-order corrector equation in Definitions 2.3.3 and 2.3.2.

Theorem 2.3.10. *Suppose there exists a smooth solution W to the generalized corrector equation* (2.72) *for some choice of h, η and L with h continuous. Fix $L \in \mathbb{R}$ and let W^- be an upper semicontinuous viscosity subsolution of* (2.72), *with*

$$\limsup_{y \to \infty} W^-(y) \leq L.$$

Then we have $W^- \leq W$ on $[1, \infty)$. Similarly, if W^+ is a lower semicontinuous viscosity supersolution of (2.72), *with*

$$\liminf_{y \to \infty} W^+(y) \geq L.$$

then we have $W \geq W^+$.

We verify that the first-order corrector equation does indeed have an explicit smooth solution. Again the proof is deferred to the Appendix.

Proposition 2.3.11. *There exists a smooth solution W of the first-order corrector equation, satisfying*

$$\begin{aligned} (\mathcal{A}W)(y) &= 0 \text{ on } [1, \infty) \\ W_y(1) - w_x^0 &= 0 \\ \lim_{y \to \infty} W(y) &= 0 \end{aligned}$$

Explicitly, W has the form

$$W(y) = C\left((y-1)\left[\Gamma\left(\frac{1}{\alpha}, \frac{1}{\alpha y}\right) - \Gamma\left(\frac{1}{\alpha}\right)\right] + (\alpha y)^{1-\frac{1}{\alpha}} e^{-\frac{1}{\alpha y}}\right) \tag{2.73}$$

where $\alpha = \frac{(\sigma\theta^0)^2}{2c^0}$ *and* C *is the unique constant such that the boundary condition* $W_y - w_x^0 = 0$ *holds. Here* $\Gamma(s,y)$ *is the upper incomplete gamma function, and* $\Gamma(s)$ *is the usual gamma function; that is,*

$$\begin{aligned}\Gamma(s,y) &= \int_y^\infty t^{s-1}e^{-t}dt \\ \Gamma(s) &= \int_0^\infty t^{s-1}e^{-t}dt\end{aligned}$$

Remark 2.3.6. The constant C (and hence the function W) is given in terms of the model parameters, exponential functions, and specific values of the incomplete and usual gamma functions. The solution W is therefore explicit up to the integration needed to determine these values of the gamma functions.

The following is an immediate consequence of the comparison principle Theorem 2.3.10.

Theorem 2.3.12. *The limit*

$$w^1(y) = \lim_{\lambda\to 0} \frac{v^\lambda(1,y) - v^0(1)}{\lambda}$$

is well-defined and finite for all $y \in [1,\infty)$*. That is,* $w^1 = \underline{w}^1 = \overline{w}^1$ *is equal to the classical solution* W *of the first-order corrector equation given in Proposition 2.3.11.*

Corollary 2.3.13. *The function* w_1 *is concave, and we have*

$$yw_y^1(y),\ y^2 w_{yy}^1(y) \to 0 \text{ as } y \to \infty$$

Proof. Suppose $t \in (0,1)$ and $y, y' \in [1,\infty)$ Then since w^λ is concave and $w^0(y)$ is constant, we have

$$\begin{aligned} tw^1(y) + (1-t)w^1(y') &= \lim_{\lambda\to 0} \frac{t(w^\lambda(y) - w^0(y)) + (1-t)(w^\lambda(y') - w^0(y'))}{\lambda} \\ &\le \lim_{\lambda\to 0} \frac{w^\lambda(ty + (1-t)y') - w^0(ty + (1-t)y')}{\lambda} \\ &= w^1(ty + (1-t)y') \end{aligned} \tag{2.74}$$

Now we know that $w^1 \to 0$ as $y \to \infty$. This is a consequence of Lemma 2.3.3 and was used implicitly in Theorem 2.3.12. Since w^1 satisfies the first-order corrector equation, we must also have

$$\lim_{y\to\infty} c_0(y-1)w^1_y(y) + \frac{1}{2}y^2(\sigma\theta^0)^2 w^1_{yy}(y) = 0 \tag{2.75}$$

Note that $w^1_y(y) \geq 0$, since w^1 is concave, $w^1(1) \leq 0$, $w^1_y(1) > 0$, and $w^1(y) \to 0$ as $y \to \infty$. Now we show $yw^1_y(y) \to 0$ as $y \to \infty$. Suppose for contradiction that there exist $y_k \to \infty$ such that $w^1_y(y_k) \geq \epsilon/y_k$. Assume without loss of generality that $y_k/y_{k-1} > 1 + \delta$ for some $\delta > 0$. Since w^1_y is decreasing in y, we then have

$$
\begin{aligned}
w^1(y_k) &= -\int_{y_k}^{\infty} w^1_y(y)dy \\
&= -\lim_{K\to\infty}\sum_{i=k}^{K}\int_{y_i}^{y_{i+1}} w^1_y(y)dy \\
&\leq -\lim_{K\to\infty}\sum_{i=k}^{K}(y_{i+1}-y_i)w^1_y(y_{i+1}) \\
&\leq -\lim_{K\to\infty}\sum_{i=k}^{K}\left(1-\frac{1}{1+\delta}\right)y_{i+1}w^1_y(y_{i+1}) \\
&= -\lim_{K\to\infty}\sum_{i=k}^{K}\left(1-\frac{1}{1+\delta}\right)y_{i+1}\epsilon = -\infty
\end{aligned}
\tag{2.76}
$$

a contradiction. So we have $yw^1_y(y) \to 0$ as $y \to \infty$, and it follows from (2.75) that $y^2 w^1_{yy}(y) \to 0$ as well. □

2.4 The second derivative

In this section, we characterize the second derivative v^2 of the value function v^λ with respect to λ as the solution of a linear PDE, i.e. the second-order corrector equation 2.37.

Definition 2.4.1. The relaxed semi-limits for v^2 are given by

$$
\begin{aligned}
\underline{v}^2(z_0) &= 2\liminf_{z\to z_0,\lambda\to 0}\frac{v^\lambda(z)-(v^0(z)+\lambda v^1(z))}{\lambda^2}, \\
\overline{v}^2(z_0) &= 2\limsup_{z\to z_0,\lambda\to 0}\frac{v^\lambda(z)-(v^0(z)+\lambda v^1(z))}{\lambda^2}.
\end{aligned}
$$

By construction $\underline{v}^2$ and $\overline{v}^2$ are lower- and upper-semicontinuous, respectively.

We follow the same line of argument used to determine v^1:

1. We show that $\underline{v}^2$ and $\overline{v}^2$ are finite, and that

$$\lim_{y\to\infty} \underline{v}^2 = \lim_{y\to\infty} \overline{v}^2 = 0.$$

2. We show that $\underline{v}^2$ (respectively $\overline{v}^2$) is a viscosity supersolution (respectively subsolution) of the second-order corrector equation.

3. We find a smooth solution V of the second-order corrector equation.

4. A comparison principle will then imply that

$$\underline{v}^2 = v^2 = \overline{v}^2$$

and $v^2 = V$

Remark 2.4.1. The relaxed semilimits $\underline{v}^2, \overline{v}^2$ are readily seen to inherit the homotheticity properties of v^λ and v^1. That is, by the same reasoning as Remark 2.3.1, we see that if we define

$$\underline{w}^2(y) \quad = \quad 2 \liminf_{y\to y_0,\ \lambda\to 0} \frac{w^\lambda(y) - (w^0 + \lambda w^1(y))}{\lambda^2},$$

then we have

$$\underline{v}^2 = x^{1-p}\underline{v}^2(1, n/x). \tag{2.77}$$

As a result, we apply the notational conventions of (2.11) to $\underline{w}^2$. Obviously, the same remarks apply to $\overline{w}^2$. Throughout this section, we will also use the following notation:

$$
\begin{aligned}
Q^{1,\lambda}(y) &= \frac{w^\lambda(y) - w^0(y)}{\lambda} \\
Q^{2,\lambda}(y) &= \frac{w^\lambda(y) - (w^0 + \lambda w^1)(y)}{\lambda^2} && (2.78) \\
&= \frac{Q^{1,\lambda}(y) - w^1(y)}{\lambda} && (2.79)
\end{aligned}
$$

and will use the notation of (2.11) for the $Q^{i,\lambda}$.

2.4.1 Bounds for v^2

Proposition 2.4.1. *If $1-p>0$, then we have $\underline{w}^2 \geq 0$. If $1-p<0$, then $\underline{w}^2$ is uniformly bounded below with*

$$\liminf_{y\to\infty} \underline{w}^2(y) \geq 0.$$

Proof. We will use Theorem 2.3.10 to compare $Q^{1,\lambda}$ and w^1. First we compute that

$$
\begin{aligned}
\mathcal{A}Q^{1,\lambda} &= \frac{1}{\lambda}(\mathcal{A}w^\lambda(y) - \mathcal{A}w^0) \\
&= \frac{1}{\lambda}\left(L^{\theta^0}w^\lambda + \tilde{U}'(w_x^0(y))(w_x^\lambda + w_n^\lambda)(y) - \tilde{U}(w_x^\lambda + w_n^\lambda)(y))\right) \\
&\quad -\frac{1}{\lambda}\left(L^{\theta^0}w^0 + \tilde{U}'(w_x^0(y))w_x^0(y) - \tilde{U}(w_x^0(y))\right)
\end{aligned}
$$

Since $\tilde{U}$ is convex, we have

$$\tilde{U}((w_x^\lambda + w_n^\lambda)(y)) \geq \tilde{U}(w_x^0(y)) + \tilde{U}'(w_x^0(y))((w_x^\lambda + w_n^\lambda)(y) - w_x^0(y)).$$

Recalling that $L^{\theta^0}w^\lambda \leq 0$ and $L^{\theta^0}w^0 = 0$, we conclude that $\mathcal{A}Q^{1,\lambda} \leq 0$. To check the boundary supersolution condition of the first-order corrector equation, note that

$$\begin{aligned} Q_y^{1,\lambda}(1) = \frac{1}{\lambda} w_y^\lambda(1) &= (1-p)w^\lambda(1) - w_y^\lambda(1) \\ &\leq (1-p)w^\lambda(1). \end{aligned}$$

If $1-p > 0$, then $(1-p)w^\lambda(1) \leq (1-p)w^0(1) = w_x^0(1)$, and therefore the boundary supersolution condition of the first-order corrector equation:

$$Q_y^{1,\lambda}(1) - w_x^0(1) \leq 0$$

is satisfied. Suppose instead that $(1-p) < 0$. Obviously we have

$$(1-p)w^\lambda(1) = (1-p)w^0(1) + (1-p)\lambda w^1(1) + o(\lambda).$$

Using Lemma 2.3.9, we can construct $f : [1, \infty) \to \mathbb{R}$ such that $\mathcal{A}f \leq 0$ and $f_y(1) < -(1-p)w^1(1)$. For example, we may take $f(y) = Cy^{-q}$ for a large constant C and q sufficiently small. Then $Q^{1,\lambda} + \lambda f$ is a supersolution of the first-order corrector equation for small λ, since

$$\begin{aligned} \mathcal{A}(Q^{1,\lambda} + \lambda f) &\leq 0 \\ Q_y^{1,\lambda}(1) + \lambda f_y(1) - w_x^0(1) &\leq (1-p)w^\lambda(1) + \lambda f_y(1) - (1-p)w^0(1) \\ &= \lambda f_y(1) + \lambda(1-p)w^1(1) + o(\lambda) \leq 0 \text{ for small } \lambda, \end{aligned}$$

so $Q_y^{1,\lambda} + \lambda f \geq w^1$, for small λ. As a result, $Q^{2,\lambda} \geq -f$ for small λ. Our choice of f also implies that $\underline{w}^2$ is uniformly bounded below and

$$\liminf_{y \to \infty} \underline{w}^2(y) \geq 0$$

$\square$

The following series of lemmas will be used to show that $\overline{w}^2 < \infty$ and $\overline{w}^2(y) \to 0$ as $y \to \infty$.

Lemma 2.4.2. *We have*

$$0 \le v_n^\lambda(x,n) \le \lambda v_x^\lambda(x,n)$$

for all $(x,n) \in D$. *Equivalently,*

$$0 \le w_y^\lambda(y) \le \lambda w_x^\lambda(y)$$

for all $y \in [1,\infty)$.

Proof. Let (α,γ) be an investment/consumption policy *given in dollar amounts* which is admissible for initial condition $(x, n+h)$. We will show that (α,γ) is admissible for $(x+\lambda h, n)$, so that $v^\lambda(x,n+h) \le v^\lambda(x+\lambda h, n)$. We can then take derivatives with respect to h to conclude.

It will suffice to show

$$X_t \triangleq x + \lambda h + Y_t - \int_0^t \gamma_u du - \frac{\lambda}{1+\lambda} \sup_{0\le s\le t} [Y_s - (n-x-\lambda h)]^+ \ge 0$$

where $Y_t = \int_0^t \alpha \frac{dF}{F}$. By assumption, the wealth process corresponding to initial condition $(x, n+h)$ is positive:

$$X_t' = x + Y_t - \int_0^t \gamma_u du - \frac{\lambda}{1+\lambda} \sup_{0\le s\le t} [Y_s - (n+h-x)]^+ \ge 0.$$

Now observe that

$$\begin{aligned} X_t - X_t' &\ge \lambda h - \frac{\lambda}{1+\lambda} \sup_{0\le s\le t} [Y_t - (n-x-\lambda h)]^+ \\ &\quad + \frac{\lambda}{1+\lambda} \sup_{0\le s\le t} [Y_t - (n+h-x)]^+ \\ &\ge \lambda h - \frac{\lambda}{1+\lambda}(n+h-x-(n-x-\lambda h)) = 0 \end{aligned}$$

$\square$

Lemma 2.4.3. *Let*

$$G^\lambda(y) = |w^\lambda(y) - w^0(y)| + |w_y^\lambda(y)| + |w_x^\lambda(y) - w_x^0(y)| + |w_{xx}^\lambda(y) - w_{xx}^0(y)|$$

There is a constant M such that for sufficiently small λ, we have

$$\sup_{y \in [1,\infty)} G^\lambda(y) \le M\lambda \tag{2.80}$$

for all y.

Proof. To begin with, since $w^\lambda(y)$ is increasing in y and $w^0(y)$ is constant in y with $w^\lambda(y) \le w^0(y)$, we see that

$$|w^\lambda(y) - w^0(y)| \;\; \le \;\; |w^\lambda(1) - w^0(1)|.$$

Since

$$\left.\frac{dw^\lambda}{d\lambda}\right|_{\lambda=0} = w^1,$$

we have

$$|w^\lambda(1) - w^0(1)| \le M_0\lambda$$

for small λ as long as $M_0 > |w^1(1)|$. Therefore

$$|w^\lambda(y) - w^0(y)| \le M_0\lambda$$

for all y, for λ sufficiently small independent of y. Since w is concave, $w_y^\lambda(y) \ge$

0, and $w_y^0 = 0$, we have

$$\begin{aligned} w_y^\lambda(y) - w_y^0(y) &\leq w_y(1) \\ &= \lambda w_x^\lambda(1) \\ &= \lambda\left((1-p)w^\lambda(1) - w_y^\lambda(1)\right) \\ &\leq \lambda(1-p)w^\lambda(1) \end{aligned}$$

Since $w^\lambda(1) \to w^0(1)$ as $\lambda \to 0$, we have

$$\sup_{y\in[1,\infty)} |w_y^\lambda(y) - w_y^0(y)| \leq M_1\lambda$$

for λ sufficiently small , as long as we take $M_1 > |(1-p)w^0(1)|$. Next we argue that there is a constant $\tilde{M}$ such that

$$\sup_{y\in[1,\infty)} |yw_y^\lambda(y)| \leq \tilde{M}\lambda$$

for λ sufficiently small. Note that the existence of $\tilde{M}$ immediately implies the existence of a M_2 such that

$$\sup_{y\in[1,\infty)} |w_x^\lambda(y) - w_x^0(y)| \leq M_2\lambda \tag{2.81}$$

for λ sufficiently small, because

$$w_x^\lambda = (1-p)w^\lambda - yw_y^\lambda,$$

Now, since w is concave, we have

$$\begin{aligned} w_y^\lambda(y) &\leq \frac{w^\lambda(y) - w^\lambda(1)}{y-1} &(2.82)\\ &\leq \frac{w^0(1) - w^\lambda(1)}{y-1} &(2.83) \end{aligned}$$

Therefore,

$$0 \leq y w_y^\lambda(y) \leq \frac{y}{y-1}\left(w^0(1) - w^\lambda(1)\right) \tag{2.84}$$

As a result, we can easily find $\tilde{M}_0$ such that

$$|y w_y^\lambda(y)| \leq \tilde{M}_0 \lambda$$

on the interval $[1+\epsilon, \infty)$ for λ sufficiently small. On the other hand, since w_y^λ is decreasing in y, we have the bound

$$\begin{aligned} 0 \leq y w_y^\lambda(y) &\leq (1+\epsilon) w_y^\lambda(1) \\ &\leq (1+\epsilon) M_1 \lambda \end{aligned}$$

so we conclude that there exists $\tilde{M}$ such that

$$\sup_{y \in [1,\infty)} |y w_y^\lambda(y)| \leq \tilde{M} \lambda$$

for small λ.

It remains to show that there exists M_3 such that

$$|w_{xx}^\lambda(y) - w_{xx}^0(y)| \leq M_3 \lambda$$

for all y, for λ sufficiently small. Writing down the HJB equation for w at fixed λ, we have

$$w_{xx}^\lambda(y) = F(w^\lambda(y), w_x^\lambda(y), w_y^\lambda(y)), \tag{2.85}$$

where

$$F(w^\lambda(y), w_x^\lambda(y), w_y^\lambda(y)) \triangleq 2\left(\frac{\sigma}{\mu}\right)^2 \frac{(w_x^\lambda)^2}{-\beta w^\lambda + \tilde{U}(w_x^\lambda + w_y^\lambda)}. \tag{2.86}$$

Note that F is a twice-differentiable function of $(w^\lambda(y), w_x^\lambda(y), w_y^\lambda(y))$ in a neighborhood of the point $(w^0(y), w_x^0(y), 0)$. The first- and second-order derivatives of F are bounded in a neighborhood of $(w^0(1), w_x^0(1), 0)$, and we've shown there is a constant M with

$$\sup_{y\in[1,\infty)} |(w^\lambda(y), w_x^\lambda(y), w_y^\lambda(y)) - (w^0(y), w_x^0(y), 0)| \leq M\lambda$$

for sufficiently small λ, which is enough to conclude there exists a constant M_3 such that

$$\sup_{y\in[1,\infty)} |w_{xx}^\lambda(y) - w_{xx}^0(y)| \leq M_3\lambda \tag{2.87}$$

for λ sufficiently small. □

Lemma 2.4.4. *Derivatives in λ and the state variables commute. That is,*

$$\left.\frac{d}{d\lambda} w_y^\lambda(y)\right|_{\lambda=0} = w_y^1(y) \tag{2.88}$$

$$\left.\frac{d}{d\lambda} w_{yy}^\lambda(y)\right|_{\lambda=0} = w_{yy}^1(y) \tag{2.89}$$

In fact, the convergence of the difference quotients

$$\begin{aligned} \frac{w_y^\lambda(y)}{\lambda} &\to w_y^1(y) \\ \frac{w_{yy}^\lambda(y)}{\lambda^2} &\to w_{yy}^1 \end{aligned}$$

is locally uniform in y.

Proof. Define $f^\lambda(y) = w^1(y) - Q^{1,\lambda}(y)$. Then f is C^2 and $f^\lambda \to 0$ pointwise on $[1,\infty)$ as $\lambda \to 0$. From the proof of Lemma 2.4.3 and Corollary 2.3.4, we know that $f_y^\lambda = w_y^1(y) - \frac{1}{\lambda}w_y^\lambda(y)$ and $f_{yy}^\lambda = w_{yy}^1(y) - \frac{1}{\lambda}w_{yy}^\lambda(y)$ are uniformly bounded for small λ; say that we have

$$|f_y^\lambda| \leq M_1$$
$$|f_{yy}^\lambda| \leq M_2$$

for small λ. It is then easy to show $f^\lambda \to 0$ locally uniformly as $\lambda \to 0$; that is, for any y_0, there exists an open neighborhood of $B(y_0)$ containing y_0 such that $f \to 0$ uniformly on $B(y_0)$. We claim that $f_y^\lambda \to 0$ locally uniformly. Suppose for contradiction there exist $\lambda_k \to 0$ and $y_0 \in [1,\infty)$ such that $|f_y^{\lambda_k}(y_0+\epsilon_k)| > \rho$ for $\epsilon_k \to 0$ and some $\rho > 0$. Pick ϵ' sufficiently small that $f^\lambda \to 0$ uniformly on a ball of radius ϵ' around y_0. Pick $\epsilon < \epsilon'$. By Taylor's theorem, we have

$$f^{\lambda_k}(y_0+\epsilon) = f^{\lambda_k}(y_0+\epsilon_k) + (\epsilon-\epsilon_k)f_y^{\lambda_k}(y_0+\epsilon_k) + (\epsilon-\epsilon_k)^2 f_{yy}^{\lambda_k}(y_0+\tilde{\epsilon}_k)$$

where $\tilde{\epsilon}_k$ is between 0 and ϵ_k. Since $f^\lambda \to 0$ uniformly on $[y-\epsilon', y+\epsilon']$, we have

$$\begin{aligned}
|f^{\lambda_k}(y_0+\epsilon)| &= |f^{\lambda_k}(y_0+\epsilon_k) + (\epsilon-\epsilon_k)f_y^{\lambda_k}(y_0+\epsilon_k) + (\epsilon-\epsilon_k)^2 f_{yy}^{\lambda_k}(y_0+\tilde{\epsilon}_k)| \\
&\leq \delta(\lambda_k) \qquad (2.90) \\
|f^{\lambda_k}(y_0)| &\leq \delta(\lambda_k)
\end{aligned}$$

where $\delta(\lambda_k) \to 0$ independent of the choice of $\epsilon < \epsilon'$. To obtain a contradiction, we pick $\epsilon < \max(\rho/M_2, 1)$ and note that

$$\begin{aligned}
|f^{\lambda_k}(y_0+\epsilon_k) + (\epsilon-\epsilon_k)^2 f_{yy}^{\lambda_k}(y_0+\tilde{\epsilon}_k)| &\leq \delta(\lambda_k) + (\epsilon-\epsilon_k)^2 M_2 \\
&< \delta(\lambda_k) + (\epsilon-\epsilon_k)\rho \\
|(\epsilon-\epsilon_k)f_y^{\lambda_k}(y_0+\epsilon_k)| &> (\epsilon-\epsilon_k)\rho
\end{aligned}$$

which contradicts (2.90). As a result, we must have $f^\lambda(y_0) \to 0$ as $\lambda \to 0$, that is (2.88) holds. The result (2.89) is then obtained using the relations between $w,^\lambda$ w_y^λ and w_{yy}^λ given by the HJB equation, as well as the fact that w^1 satisfies the first-order corrector equation.

$\square$

Lemma 2.4.5. *As in Lemma 2.4.3, set*

$$G^\lambda(y) = |w^\lambda(y) - w^0(y)| + |w_y^\lambda(y)| + |w_x^\lambda(y) - w_x^0(y)| + |w_{xx}^\lambda(y) - w_{xx}^0(y)|$$

Then we have

$$\lim_{y_0\to\infty} \limsup_{\lambda\to 0} \sup_{y\in[y_0,\infty)} \frac{G^\lambda(y)}{\lambda} = 0 \tag{2.91}$$

Proof. Since w^λ is increasing in y with $w^\lambda(y) \to w^0(y)$ as $y \to \infty$, we have

$$\begin{aligned} 0 \leq \lim_{y_0\to\infty} \limsup_{\lambda\to 0} \sup_{y\in[y_0,\infty)} \frac{w^\lambda(y) - w^0(1)}{\lambda} &= \lim_{y_0\to\infty} \limsup_{\lambda\to 0} \frac{w^\lambda(y_0) - w^0(1)}{\lambda} \\ &= \lim_{y_0\to\infty} w^1(y_0) = 0 \end{aligned}$$

Let $y \geq y_0$. Then since w^λ is concave

$$\begin{aligned} w_y^\lambda(y) &\leq \frac{w^\lambda(y) - w^\lambda(y_0)}{y - y_0} \\ &\leq \frac{w^0(1) - w^\lambda(y_0)}{y - y_0} \end{aligned}$$

Therefore,

$$\begin{aligned} yw_y^\lambda(y) &\leq y_0 w_y^\lambda(y) + w^0(1) - w_y^\lambda(y_0) \\ &\leq y_0 w_y^\lambda(y_0) + w^0(1) - w^\lambda(y_0) \end{aligned}$$

Combining the above with Lemma 2.4.4 and Corollary 2.3.4, we observe that

$$\begin{aligned}\lim_{y_0\to\infty}\limsup_{\lambda\to 0}\sup_{y\in[y_0,\infty)}\frac{yw_y^\lambda(y)}{\lambda} &\le \lim_{y_0\to\infty}\limsup_{\lambda\to 0}\frac{y_0w_y^\lambda(y_0)+w^0(1)-w^\lambda(y_0)}{\lambda}\\ &\le \lim_{y_0\to\infty} y_0 w_n^1(y_0)+w^1(y_0)=0\end{aligned}$$

Since $w_x^\lambda(y) = (1-p)w^\lambda(y) - yw_x^\lambda(y)$ and $w_x^0(1) = (1-p)w^0(1)$ it follows immediately that

$$\lim_{y_0\to\infty}\limsup_{\lambda\to 0}\sup_{y\in[y_0,\infty)}\frac{|w_x^\lambda(y)-w_x^0(1)|}{\lambda} = 0 \tag{2.92}$$

We conclude by recalling the bound (2.87) on w_{xx}^λ. □

Proposition 2.4.6. *We have* $\overline{w}_2(y) < \infty$ *on* $[1,\infty)$.

Proof. We will exhibit $f:[1,\infty)\to\mathbb{R}$ such that $Q^{1,\lambda}-\lambda f$ is a subsolution of the first corrector equation for small λ, then apply Theorem 2.3.10 to conclude that $Q^{1,\lambda}-\lambda f\le w_1$, hence $\overline{w}_2\le 2f<\infty$. To begin with, we show that there exists $M>0$ such that

$$\mathcal{A}Q^{1,\lambda}\ge -M\lambda.$$

First observe that

$$\begin{aligned}\mathcal{A}Q^{1,\lambda}(y) &= \frac{1}{\lambda}\left(\mathcal{A}w^\lambda(y)-\mathcal{A}w^0(y)\right)\\ &= \frac{1}{\lambda}\left(\mathcal{A}w^\lambda(y)-L^{\theta^\lambda}w^\lambda(y)-\mathcal{A}w^0(y)+L^{\theta^0}w^0(y)\right)\\ &= \frac{1}{\lambda}\left(\mu\theta^0w_x^\lambda(y)+\frac{1}{2}(\sigma\theta^0)^2w_{xx}^\lambda(y)+\frac{1}{2}\left(\frac{\mu}{\sigma}\right)^2\frac{(w_x^\lambda(y))^2}{w_{xx}^\lambda(y)}\right)\\ &\quad +\frac{1}{\lambda}\left(\tilde{U}(w_x^0(y))+\tilde{U}'(w_x^0(y))(\nabla w^\lambda(y)\cdot\mathbf{1}-w_x^0(1))-\tilde{U}(\nabla w^\lambda\cdot\mathbf{1})\right)\\ &= \frac{1}{\lambda}\left(\mu\theta^0w_x^\lambda(y)+\frac{1}{2}(\sigma\theta^0)^2w_{xx}^\lambda(y)+\frac{1}{2}\left(\frac{\mu}{\sigma}\right)^2\frac{(w_x^\lambda(y))^2}{w_{xx}^\lambda(y)}\right)\\ &\quad -\frac{1}{2\lambda}\tilde{U}''(\xi_\lambda(y))(\nabla w^\lambda\cdot\mathbf{1}-w_x^0(y))^2\end{aligned} \tag{2.93}$$

where the third line follows from the usual HJB equation for w^λ, and $\xi_\lambda(y)$ is some number between $\nabla w^\lambda \cdot \mathbf{1}$ and $w^0_x(y)$. Note that there are constants C_1, M_1 such that, for small λ

$$\frac{1}{2}\tilde{U}'(\xi_\lambda(y))(\nabla w^\lambda(y) \cdot \mathbf{1} - w^0_x(y))^2 \leq C_1 G^\lambda(y)^2 \leq M_1 \lambda^2 \tag{2.94}$$

where $G^\lambda(y)$ is as defined in Lemma 2.4.2. A similar bound hold for the other terms of (2.93). To see this, define

$$\rho_{\lambda,y}(\theta) = \mu\theta w^\lambda_x(y) + \frac{1}{2}(\sigma\theta)^2 w^\lambda_{xx}(y) \tag{2.95}$$

Since $\rho_{\lambda,y}$ is quadratic in θ with maximum at $\theta^\lambda = -\frac{\mu}{\sigma^2}\frac{w^\lambda_x(y)}{w^\lambda_{xx}(y)}$, we have

$$\rho_{\lambda,y}(\theta^0) - \rho_{\lambda,y}(\theta^\lambda) = \sigma^2 w^\lambda_{xx}(y)(\theta^0 - \theta^\lambda)^2 \tag{2.96}$$

Note that Lemma 2.4.3 readily implies there are constants C_2, M_2 such that

$$(\theta^0 - \theta^\lambda)^2 \leq C_2 G^\lambda(y)^2 \leq M_2 \lambda^2$$

for small λ. So applying (2.94) to (2.96) in the equality (2.97), we see there are constants C, M such that

$$\begin{aligned} \mathcal{A}Q^{1,\lambda}(y) &= \frac{1}{\lambda}\left(\rho_{\lambda,y}(\theta^0) - \rho_{\lambda,y}(\theta^\lambda) - \tilde{U}'(\xi_\lambda(y))(\nabla w^\lambda(y) \cdot \mathbf{1} - w^0_x(y))^2\right) \\ &\geq -C\frac{G^\lambda(y)^2}{\lambda} \\ &\geq -M\lambda \end{aligned} \tag{2.97}$$

Now, we want to construct $f : [1, \infty) \to \mathbb{R}$ such that

$$\mathcal{A}(Q^{1,\lambda} - \lambda f) \geq 0 \tag{2.98}$$

$$Q^{1,\lambda}_y(1) - \lambda f_y - w^0_x(1) \geq 0$$

$$\limsup_{y\to\infty} f(y) \geq 0 \tag{2.99}$$

We begin by addressing the boundary condition. Note that

$$\begin{aligned}
Q_y^{1,\lambda}(1) &= \frac{w_y^\lambda(1)}{\lambda} \\
&= w_x^\lambda(1) \\
&= (1-p)w^\lambda(1) - w_y^\lambda(1) = (1-p)w^\lambda(1) - \lambda Q_y^{1,\lambda}(1).
\end{aligned}$$

Thus we have

$$\begin{aligned}
Q_y^{1,\lambda}(1) = \frac{1-p}{1+\lambda}w^\lambda(1) &= (1-p)w^0(1) + \lambda(1-p)(w^1(1) - w^0(1)) + o(\lambda) \\
&= w_x^0(1) + \lambda(1-p)w^1(1) + o(\lambda). \qquad (2.100)
\end{aligned}$$

Now define $f(y) = K_0 + K_1 y^{-q}$ for q, K_0, and $K_1 > 0$ positive constants. Recall from Lemma 2.3.9 that $\mathcal{A}(y^{-q}) \leq 0$ for q sufficiently small. Choosing K_0 large and q sufficiently small, we therefore have $\mathcal{A}f \leq -M$ for any $M > 0$. Further, we may take K_1 sufficiently large that $-f_y(1) > |(1-p)(w^1(1) - w^0(1))|$, so that (2.100) implies $(Q_y^{1,\lambda} - \lambda f_y)(1) \geq w_x^0$ for small λ. Finally, since $Q^{1,\lambda}(y) \to 0$ as $y \to \infty$, we have

$$\limsup_{y\to\infty} Q^{1,\lambda} - \lambda f \leq 0.$$

Altogether the conditions of Theorem 2.3.10 are satisfied, and we conclude that for small λ, we have $Q_y^{1,\lambda} - \lambda f' \leq w^1$. As a result, $Q^{2,\lambda} \leq f$ for small λ, and so $\overline{w}^2 < \infty$. □

Lastly, we check that $\overline{w}^2$ has the appropriate limiting behavior.

Proposition 2.4.7. *We have*

$$\limsup_{y\to\infty} \overline{w}^2(y) \leq 0$$

Proof. First, note that for any $y \in (1, \infty)$, we have

$$\liminf_{\lambda \to 0} Q_y^{2,\lambda}(y) > -\infty$$

Once we've proven this, the argument resembles the proof of Proposition 2.4.6. If $y = 1$, then this is clear from (2.100), since we have

$$\begin{aligned} Q_y^{2,\lambda} &= \frac{Q_y^{1,\lambda} - w_y^1}{\lambda} \\ &= \frac{w_x^0(1) + \lambda(1-p)(w^1(1) - w^0(1)) + o(\lambda)}{\lambda} - \frac{w_y^1}{\lambda} \\ &= (1-p)(w^1(1) - w^0(1)) + o(1) \end{aligned}$$

Now take $y > 1$. Suppose for contradiction that there exist $\lambda_k \to 0$ with $Q_y^{2,\lambda_k}(y) \to -\infty$. By a simple Taylor expansion, we have

$$Q^{2,\lambda}(y \pm \epsilon) = Q^{2,\lambda}(y) \pm \epsilon Q_y^{2,\lambda}(y) + \epsilon^2 Q_{yy}^{2,\lambda}(y + \epsilon_{\pm,\lambda}) \tag{2.101}$$

for some $\epsilon_{\pm,\lambda}$ between 0 and $\pm\epsilon$. At a point of inflection $\tilde{y}_k$ of Q^{2,λ_k}, we have

$$\mathcal{A}Q^{2,\lambda}(\tilde{y}_k) = -c_0 Q^{2,\lambda}(\tilde{y}_k) + (\tilde{y}_k - 1)c_0 Q_y^{2,\lambda}(\tilde{y}_k) \tag{2.102}$$

In particular, $Q_y^{2,\lambda_k}(\tilde{y}_k)$ is bounded below for small λ, since both $\mathcal{A}Q_y^{2,\lambda_k}$ and Q^{2,λ_k} are (this is a consequence of Proposition 2.4.1 and the proof of Proposition 2.4.6). So for infinitely many k we must have $Q_{yy}^{2,\lambda_k}(y) \neq 0$. We may assume without loss of generality that $Q_{yy}^{2,\lambda_k}(y) < 0$ or $Q_{yy}^{2,\lambda_k}(y) > 0$ for all k; to begin with, suppose $Q_{yy}^{2,\lambda_k}(y) < 0$ for all k. If for infinitely many k there is a point of inflection $\tilde{y}_{\epsilon,k}$ of Q^{2,λ_k} on the half-interval $[y, y+\epsilon]$, then we would have $Q_y^{2,\lambda_k}(\tilde{y}_{\epsilon,k}) < Q_y^{2,\lambda_k}(y) \to -\infty$ as $k \to \infty$, a contradiction. So we assume

$Q_{yy}^{2,\lambda_k} < 0$ on $[y, y+\epsilon]$. In this case, $Q^{2,\lambda_k}(y+\epsilon) \to -\infty$ as a result of (2.101), contradicting Proposition 2.4.1. Similar arguments apply if we suppose that $Q_{yy}^{2,\lambda_k}(y) > 0$ for all k and consider the interval $[y-\epsilon, y]$.

Now we may proceed to use comparison arguments to show the desired limiting behavior of $\overline{w}^2$. Recall from Lemma 2.3.9 that the function $f^{K,q}(y) = -Ky^{-q}$ satisfies

$$\mathcal{A}f^{K,q} \geq 0$$

for q sufficiently small. Fix $\hat{y}_k \to \infty$ and pick K_k with

$$qK_k > -\liminf_{\lambda \to 0} Q_y^{2,\lambda}(\hat{y}_k)$$

and $M_k > 0$ with

$$\begin{aligned} -M_k &\leq \inf_{y \in [\hat{y}_k, \infty),\ \lambda \leq \lambda_k} \mathcal{A}Q^{2,\lambda} \\ \lim_{k \to \infty} M_k &= 0 \end{aligned} \tag{2.103}$$

where $\{\lambda_k\}$ is some sequence with $\lambda_k \to 0$ as $k \to \infty$. To see that such a choice of M_k is possible, note that

$$\begin{aligned} \mathcal{A}Q^{2,\lambda} &= \frac{\mathcal{A}Q^{1,\lambda}}{\lambda} \\ &\geq C\frac{G^{\lambda}(y)^2}{\lambda^2} \end{aligned} \tag{2.104}$$

where the second line is just the bound (2.97) from the proof of Proposition 2.4.6. We then apply Lemma 2.4.5 to conclude that

$$\lim_{y_0 \to \infty} \liminf_{\lambda \to 0} \inf_{y \in [y_0, \infty)} \mathcal{A}Q^{2,\lambda} = 0$$

which allows a choice of M_k as in (2.103). We then have for $\lambda \leq \lambda_k$

$$\begin{aligned}
\mathcal{A}(Q^{2,\lambda} - M_k + f^{K_k,q}) &\geq 0 \text{ on } [\hat{y}_k, \infty) \\
(Q^{2,\lambda} - M_k + f^{K_k,q})_y(\hat{y}_k) &> 0 \\
\limsup_{y\to\infty}(Q^{2,\lambda} - M_k + f^{K_k,q})(y) &\leq 0
\end{aligned}$$

We therefore use 2.3.10 to see that $Q^{2,\lambda} - M_k + f_{K_k,q} \leq 0$ on $[\hat{y}_k, \infty)$ for all k for $\lambda \leq \lambda_k$. Since $f^{K_k,q} \to 0$ as $y \to \infty$ and $M_k \to 0$ as $k \to \infty$, we conclude that

$$\lim_{y_0\to\infty} \sup_{y\in[y_0,\infty),\lambda\geq\lambda_k} Q^{2,\lambda} = 0 \tag{2.105}$$

Now, if $\limsup_{y\to\infty} \overline{w}^2 > \epsilon$, then there exist $y_k \to \infty$ and λ_k arbitrarily small such that $Q^{2,\lambda_k}(y_k) > \epsilon > 0$ for all k, which is impossible by (2.105). $\square$

2.4.2 The second-order corrector equation

Definition 2.4.2. In order to define the second-order corrector equation, we first recall the definitions of the leading-order corrections (θ^1, c^1) to the optimal strategy as in (2.33):

$$\begin{aligned}
\theta^1(x,n) &= -\frac{\mu}{\sigma^2}\frac{v^0_{xx}v^1_x - v^1_{xx}v^0_x}{xv^0_{xx}} \\
c^1(x,n) &= \frac{I'(v^0_x(x,n))(v^1_x(x,n) + v^1_n(x,n))}{x}
\end{aligned} \tag{2.106}$$

One can easily check that, due to the homotheticity properties of v^0 and v^1, we have

$$\begin{aligned}
\theta^1(x,n) &= \theta^1(1, n/x) \\
c^1(x,n) &= c^1(1, n/x)
\end{aligned}$$

We therefore write θ^1, c^1 in one-dimensional notation, taking

$$\begin{aligned} \theta^1(y) &\triangleq \theta^1(1, y) \\ c^1(y) &\triangleq c^1(1, y) \end{aligned}$$

Finally, we define the approximately optimal strategies $(\hat{\theta}^\lambda, \hat{c}^\lambda)$ as follows:

$$\begin{aligned} \hat{\theta}^\lambda(y) &= \theta^0 + \lambda\theta^1(y) \\ \hat{c}^\lambda(y) &= c^0 + \lambda c^1(y) \end{aligned}$$

Remark 2.4.2. Due to Lemma 2.4.3 and Lemma 2.4.5, it is easy to see that θ^1 and c^1 are uniformly bounded with

$$\lim_{y\to\infty} \theta^1(y) = \lim_{y\to\infty} c^1(y) = 0$$

We now give the one-dimensional version of the second-order corrector equation (2.37).

The second-order corrector equation. The *one-dimensional second-order corrector equation* for a function $w : [1, \infty) \to \mathbb{R}$ is given by

$$\begin{aligned} \mathcal{A}w + g &= 0 \\ \mathcal{C}w &\triangleq w_y(1) - 2w^1_x(1) = 0 \\ \lim_{y\to\infty} w(y) &= 0 \end{aligned}$$

where

$$g \triangleq 2\left[\mu\theta^1 w^1_x + \sigma^2(\theta^0\theta^1)w^1_{xx} + \frac{1}{2}(\sigma\theta^1)^2 w^0_{xx} + \frac{1}{2}\tilde{U}''(w^0_x)(w^1_x + w^1_y)^2\right] \quad (2.107)$$

The notions of viscosity subsolutions and viscosity supersolutions of the second-order corrector equation are given in the same way as in Definitions 2.3.3 and 2.3.2.

Proposition 2.4.8. *The lower relaxed semilimit $\underline{w}^2$ is a viscosity supersolution of the second-order corrector equation.*

Proof. The proof closely follows that of Proposition 2.3.7. Let ϕ be a C^2 function such that $\underline{w}^2 - \phi$ has a strict local minimum of 0 at y_0. We pick a subsequence $\lambda_k \to 0$ such that $2Q^{2,\lambda_k}(y_k) \to \underline{w}^2(y_0)$ for some sequence $y_k \to y_0$. Let $\hat{y}_k$ be the minimizers of $2Q^{2,\lambda_k} - \phi$ on a closed ball $\overline{B}$ around y_0 taken sufficiently small that y_0 is the minimum of $\underline{w}^2 - \phi$ on $\overline{B}$. We must have $\hat{y}_k \to y_0$. If not, then there is a subsequence $\hat{y}_{k_i} \to y \neq y_0$. Then we have

$$\begin{aligned} 0 < \underline{w}^2(y) - \phi(y) &\leq \liminf_{i\to\infty} 2Q^{2,\lambda_{k_i}}(\hat{y}_{k_i}) - \phi(\hat{y}_{k_i}) \\ &\leq \liminf_{i\to\infty} 2Q^{2,\lambda_{k_i}}(y_{k_i}) - \phi(y_{k_i}) = \underline{w}^2(y_0) - \phi(y_0) \end{aligned}$$

This contradicts the assumption that the strict minimum of $\underline{w}^2 - \phi$ on $\overline{B}$ is achieved at y_0. So $\hat{y}_k \to y_0$, and we readily see (by minimality of the $\hat{y}_k$ and continuity of ϕ) that $2Q^{2,\lambda_k}(\hat{y}_k) \to \underline{w}^2(y_0)$. We may therefore take $y_k = \hat{y}_k$. Now construct the C^2 functions

$$\psi^k(y) = w^{\lambda_k}(y_k) + (\lambda_k w^1(y) - \lambda_k w^1(y_k)) + \frac{\lambda_k^2}{2}(\phi(y) - \phi(y_k)) \tag{2.108}$$

Observe that ψ^k touches $\underline{w}^{\lambda_k}$ below at y_k. To see this, note we clearly have

$\psi^k(y_k) = w^{\lambda_k}(y_k)$. Since w^0 is constant in y, we also have

$$
\begin{aligned}
w^{\lambda_k}(y) - \psi^k(y) &= w^{\lambda_k}(y) - (w^0(y) + \lambda_k w^1(y) + \frac{\lambda_k^2}{2}) \\
&\quad +(w^0(y_k) + \lambda_k w^1(y_k) + \frac{\lambda_k^2}{2}\phi(y_k)) \\
&= \frac{\lambda^2}{2}(2Q^{2,\lambda_k}(y_k) - \phi) + (w^0(y_k) + \lambda_k w^1(y_k) + \frac{\lambda_k^2}{2}\phi(y_k))
\end{aligned}
$$

By construction, $2Q^{2,\lambda_k} - \phi$ has a local minimum at y_k, so $w^{\lambda_k} - \psi^k$ does as well. For reference, we note that at y_k, we have

$$
\begin{aligned}
\psi_y^k &= \lambda_k w_y^1 + \frac{\lambda_k^2}{2}\phi_y \\
\psi_x^k &= (1-p)w^{\lambda_k} - y_k(\lambda_k w_y^1 + \frac{\lambda_k^2}{2}\phi_y) \\
\psi_{xx}^k &= -p(1-p)w^{\lambda_k} + 2py_k(\lambda_k w_1^y + \frac{\lambda_k^2}{2}\phi_y) + y_k^2(\lambda_k w_y^1 + \frac{\lambda_k^2}{2}\phi_y)
\end{aligned}
\tag{2.109}
$$

To begin with, suppose $y_0 > 1$. We plug ψ^k into the HJB equation for w^{λ_k} at y_k:

$$
\begin{aligned}
0 &\geq (\sup_\theta L^\theta \psi^k)(y_k) \\
&\geq L^{\hat{\theta}^\lambda}\psi^k(y_k) \\
&= L^{\theta^0}\psi^k(y_k) + \eta^k
\end{aligned}
\tag{2.110}
$$

where

$$\eta^k \triangleq \lambda_k\theta^1\left(\mu\psi_x^k + \lambda_k\theta^0\theta^1\sigma^2\psi_{xx}^k\right) + \frac{\lambda_k^2}{2}(\theta^1)^2\sigma^2\psi_{xx}^k \tag{2.111}$$

First, we show that

$$\frac{L^{\theta^0}\psi^k(y_k)}{\lambda_k^2} \to \frac{1}{2}\mathcal{A}\phi(y_0) + \frac{1}{2}\tilde{U}''(w_x^0)(w_x^1 + w_y^1)^2 \tag{2.112}$$

where $\mathcal{L}^\theta$ is the operator of Definition 2.3.4. Expanding each term of $L^{\theta^0}\psi^k(y_k)$ using (2.109), we have

$$\begin{aligned} L^{\theta^0}\psi^k &= -\beta w^{\lambda_k} + \mu\theta^0\left[(1-p)w^{\lambda_k} - y_k\left(\lambda_k w^1_y + \frac{\lambda_k^2}{2}\phi_y\right)\right] \\ &+\frac{1}{2}(\sigma\theta^0)^2\Bigg[-p(1-p)w^{\lambda_k} + 2py_k\left(\lambda_k w^1_y + \frac{\lambda_k^2}{2}\phi_y\right) \\ &+y_k^2\left(\lambda_k w^1_{yy} + \frac{\lambda_k^2}{2}\phi_{yy}\right)\Bigg] \\ &+\tilde{U}\left((1-p)w^{\lambda_k} - (1-y_k)\left(\lambda_k w^1_y + \frac{\lambda_k^2}{2}\phi_y\right)\right). \end{aligned} \tag{2.113}$$

Recalling that

$$L^{\theta^0}w^0 = \mathcal{A}w^1 = 0,$$

we subtract $L^{\theta^0}w^0(y_k) + \lambda_k\mathcal{A}w^1(y_k) = 0$ from (2.113). For brevity, we write the result in terms of Q^{2,λ_k}:

$$\begin{aligned} L^{\theta^0}\psi^k(y_k) &= -\beta\lambda_k^2 Q^{2,\lambda_k} + \mu\theta^0\lambda_k^2\left[(1-p)\ Q^{2,\lambda_k} - \frac{y_k}{2}\phi_y\right] \\ &+\frac{\lambda_k^2}{2}(\sigma\theta^0)^2\left[-p(1-p)Q^{2,\lambda_k} + \frac{1}{2}\left(2py_k\phi_y + y_k^2\phi_{yy}\right)\right] \\ &+\tilde{U}\left((1-p)w^{\lambda_k} - (1-y_k)(\lambda_k w^1_y + \frac{\lambda_k^2}{2}\phi_y)\right) \\ &-\left[\tilde{U}(w^0_x) + \lambda_k\tilde{U}'(w^0_x)((1-p)w^1 + (1-y_k)w^1_y)\right] \end{aligned} \tag{2.114}$$

Examining the first two lines of (2.114), we note that as $k\to\infty$, we have

$$\begin{aligned} -\beta Q^{2,\lambda_k} &\to -\frac{\beta}{2}\phi(y_0) \\ \mu\theta^0\left[(1-p)\ Q^{2,\lambda_k} - \frac{y_k}{2}\phi_y\right] &\to \frac{1}{2}\mu\theta^0\phi_x(y_0) \\ \frac{1}{2}(\sigma\theta^0)^2\left[-p(1-p)Q^{2,\lambda_k} + \frac{1}{2}\left(2py_k\phi_y + y_k^2\phi_{yy}\right)\right] &\to \frac{1}{4}(\sigma\theta^0)^2\phi_{xx}(y_0). \end{aligned} \tag{2.115}$$

We then treat the last two lines of (2.114) with the usual Taylor expansion argument: We have

$$\begin{aligned}\tilde{U}\left((1-p)w^{\lambda_k}-(1-y_k)(\lambda_k w_y^1+\frac{\lambda_k^2}{2}\phi_y)\right) &= \tilde{U}(w_x^0) \\ &\quad +\tilde{U}'(w_x^0)\delta^k \\ &\quad +\frac{1}{2}\tilde{U}''(\xi^k)(\delta^k)^2\end{aligned} \tag{2.116}$$

where

$$\delta^k=(1-p)w^{\lambda_k}-w_x^0+(1-y_k)\left(\lambda_k w_y^1+\frac{\lambda_k^2}{2}\phi_y\right)$$

and ξ^k is some number between w_x^0 and $w_x^0+\delta^k$. Adding the *first two terms* of the right-hand side of (2.116) and the last line of (2.114), then dividing by λ_k^2 and sending yields the expression

$$\begin{aligned}\tilde{U}'(w_x^0)((1-p)Q^{2,\lambda_k}+\frac{1-y_k}{2}\phi_y) &\to \frac{1}{2}\tilde{U}'(w_x^0)((1-p)\phi(y_0)-(1-y_0)\phi_y(y_0)) \\ &= \frac{1}{2}\tilde{U}'(w_x^0)(\phi_x(y_0)+\phi_y(y_0))\end{aligned} \tag{2.117}$$

as $k\to\infty$. To take care of the last term of (2.116), note that

$$\begin{aligned}\frac{(\delta^k)^2}{\lambda_k^2} &= \frac{\left((1-p)w^{\lambda_k}-(1-p)w^0-(1-y_k)\left(\lambda_k w_y^1+\frac{\lambda_k^2}{2}\phi_y\right)\right)^2}{\lambda_k^2} \\ &\to (1-p)w^1(y_0)+(1-y_0)w^1(y_0)=(w_x^1+w_y^1)(y_0)\end{aligned} \tag{2.118}$$

as $k\to\infty$. As a result,

$$\frac{\frac{1}{2}\tilde{U}''(\xi^k)(\delta^k)^2}{\lambda_k^2}\to\frac{1}{2}(w_x^1+w_y^1)(y_0) \tag{2.119}$$

as $k\to\infty$. Combining (2.114) with (2.115), (2.116), and (2.119), we conclude that (2.112) holds.

We now claim

$$\lim_{k\to\infty}\frac{\eta^k}{\lambda_k^2}=\mu\theta^1 w_x^1(y_0)+\sigma(\theta^0\theta^1)w_{xx}^1(y_0)+\frac{1}{2}(\sigma\theta^1)^2 w_{xx}^0(y_0) \tag{2.120}$$

On one hand, we have

$$\lim_{k\to\infty}\frac{1}{2}(\sigma\theta^1)^2\psi_{xx}^k=\frac{1}{2}(\sigma\theta^1)^2 w_{xx}^0(y_0) \tag{2.121}$$

so the last term of the right-hand side of (2.111) accounts for the last term of the right hand side of (2.121). To treat the first two terms of (2.111), recall that from Theorem 2.3.12 that

$$\lim_{k\to\infty}\frac{w^{\lambda_k}(y_k)-(w^0(y_k)+\lambda_k w^1(y_k))}{\lambda_k}=0$$

Therefore, we have

$$\begin{aligned}
\lambda_k\theta^1\left(\mu\psi_x^k+\sigma^2\theta^0\psi_{xx}^k\right) &= \lambda_k\mu\theta^1((1-p)(w^0+\lambda_k w^1)-y_k(\lambda_k w_y^1+\frac{\lambda_k^2}{2}\phi_y))\\
&\quad +\lambda_k\sigma^2\theta^0\theta^1\Big(-\sigma p(1-p)(w^0+\lambda_k w^1)\\
&\quad +2py_k(\lambda_k w_y^1+\frac{\lambda_k^2}{2}\phi_y)\\
&\quad +y_k^2(\lambda_k w_{yy}^1+\frac{\lambda_k^2}{2}\phi_{yy})\Big)+o(\lambda_k^2)\\
&= \lambda_k^2\left(\mu\theta^1 w_x^1+\sigma^2\theta^0\theta^1 w_{xx}^1\right)+o(\lambda_k^2) \qquad (2.122)
\end{aligned}$$

where the w^0 terms above cancel due to the explicit form of θ^0. Combining (2.122) and (2.121) yields the desired limiting behavior (2.120) of the η^k. Putting (2.110) together with (2.120) and (2.112), we conclude that

$$\mathcal{A}\phi(y_0)+g(y_0)\le 0.$$

That is, the interior supersolution property for the second-order corrector equation holds for $\underline{w}^2$ at y_0.

Suppose now that $y_0 = 1$. Construct the y_k and ψ^k as before. Then either $y_k = 1$ for infinitely many k or $y_k > 1$ for infinitely many k. In the second case, we may argue as above to show that $(\mathcal{A}\phi + g)(y_0) \leq 0$. If infinitely many y_k are equal to 1, for these k, we have

$$\begin{aligned}
0 \geq (\psi_y^k - \lambda_k \psi_x^k)(y_k) &= \left(\lambda_k w_y^1(1) + \frac{\lambda_k^2}{2}\phi_y(1)\right) \\
&\quad -\lambda_k(1-p)w^{\lambda_k}(1) + \left(\lambda_k^2 w_y^1(1) + \frac{\lambda_k^3}{2}\phi_y(1)\right) \\
&= \frac{\lambda_k^2}{2}\phi_y(1) - \lambda_k(1-p)\left(w^{\lambda_k}(1) - w^0(1)\right) - \lambda_k^2 w_y^1(1) \\
&\quad + \frac{\lambda_k^3}{2}\phi_y(1)
\end{aligned}$$

where the second line follows from the fact that $w_y^1(1) = w_x^0(1) = (1-p)w^0(1)$. Dividing by λ_k^2 and sending $k \to \infty$ yields

$$\begin{aligned}
0 &\geq \frac{1}{2}\phi_y(1) - \left((1-p)w^1(1) - w_y(1)\right) \\
&= \frac{1}{2}\phi_y(1) - w_x^1(1) \\
&= \frac{1}{2}\mathcal{C}\phi.
\end{aligned}$$

In other words, the boundary subsolution property of the second-order corrector equation holds.

$\square$

Lemma 2.4.9. *We have*

$$\frac{L^{\hat{\theta}^\lambda} w^\lambda}{\lambda^2} \to 0$$

locally uniformly in y as $\lambda \to 0$.

Proof. Let θ^λ be the optimal control for fee level λ. Since θ^λ is given as a continuous function of the derivatives of w^λ, Lemma 2.4.4 implies that

$$\frac{\theta^\lambda - (\theta^0 + \lambda\theta^1)}{\lambda} \to 0 \tag{2.123}$$

locally uniformly in y. Now define $\rho^{\lambda,y}(\theta)$ as in (2.95). Recalling that $\rho^{\lambda,y}(\theta)$ is quadratic in θ and is maximized at $\theta^\lambda(y)$, we have

$$\begin{aligned} L^{\hat{\theta}^\lambda} w^\lambda(y) &= \left(L^{\hat{\theta}^\lambda} - L^{\theta^\lambda}\right) w^\lambda(y) \\ &= \rho^{\lambda,y}(\theta^0 + \lambda\theta^1) - \rho^{\lambda,y}(\theta^\lambda) \\ &= \sigma^2 w^\lambda_{xx}(y)(\theta^0 + \lambda\theta^1 - \theta^\lambda)^2 \end{aligned}$$

where the last line follows from equation (2.96). Combining this with (2.123), we conclude. $\square$

Proposition 2.4.10. *The upper relaxed semi-limit $\overline{w}^2$ is a viscosity subsolution of the second-order corrector equation.*

Proof. Let ϕ be a C^2 function such that $\overline{w}^2 - \phi$ achieves a strict local maximum of 0 at y_0. To begin with, suppose that $y_0 > 1$. Following Proposition 2.4.8, we generate points $y_k \to y_0$ such that $2Q^{2,\lambda_k}(y_k) \to \overline{w}^{\lambda_k}$ and smooth functions ψ^k such that $w^{\lambda_k} - \psi^k$ has a local maximum of 0 at y_k. To begin with, assume $y_0 > 1$. Since $w^{\lambda_k} - \psi^k$ has a local maximum of 0 at y_k, we have

$$\begin{aligned} w^{\lambda_k}(y_0) &= \psi^k(y_0) \\ w^{\lambda_k}_y(y_0) &= \psi^k_y(y_0) \\ w^{\lambda_k}_{yy}(y_0) &\le \psi^k_{yy}(y_0) \end{aligned}$$

As a result, we have

$$L^{\hat{\theta}^{\lambda_k}}\psi^k(y_k) \geq L^{\hat{\theta}^{\lambda_k}}w^{\lambda_k}(y_k)$$

Combining this with Lemma 2.4.9, we see that

$$\liminf_{k\to\infty} \frac{L^{\hat{\theta}^{\lambda_k}}\psi^k(y_k)}{\lambda_k^2} \geq 0 \tag{2.124}$$

Note that the locally uniform convergence of Lemma 2.4.9 really was needed above, since we can't take the y_k to be fixed. We may now repeat the arguments of Proposition 2.4.8 to show that

$$\lim_{k\to 0} \frac{L^{\hat{\theta}^{\lambda_k}}\psi^k}{\lambda_k^2}(y_k) = \frac{1}{2}\left(\mathcal{A}\phi(y_0) + g(y_0)\right)$$

so that the interior subsolution property holds at y_0. In the case where $y_0 = 1$, the argument is identical to that of Proposition 2.4.8. □

The proof of the following result will be given in the appendix.

Proposition 2.4.11. *There exists a smooth solution W of the second-order corrector equation*

$$\begin{aligned} \mathcal{A}W + g &= 0 \text{ on } (1,\infty) \\ W_y(1) - 2w_x^1(1) &= 0 \\ \lim_{y\to\infty} W(y) &= 0 \end{aligned}$$

We are now ready to prove the principal result of the section.

Theorem 2.4.12. *We have $\underline{w}^2 = \overline{w}^2$, so that the limit*

$$w^2 = 2\lim_{\lambda\to 0} \frac{w^\lambda - (w^0 + \lambda w^1)(y)}{\lambda^2}$$

is well-defined, finite, and continuous in y, with

$$\lim_{y\to\infty} w^2(y) = 0$$

Further, w^2 is the unique viscosity solution of the second-order corrector equation, and so w^2 is equal to the W of Proposition 2.4.11.

Proof. In light of Proposition 2.4.11, we can simply apply Theorem 2.3.10 to the viscosity sub- and supersolutions $\overline{w}^2$ and $\underline{w}^2$ of the second-order corrector equation. □

2.5 Approximately optimal strategies

In this section, we show that the payoff $\hat{w}^\lambda$ of the strategy $(\hat{\theta}^\lambda, \hat{c}^\lambda)$ matches the value function w^λ up to second order in λ. We apply the same approach of previous sections, checking that the derivatives

$$\hat{w}^1 = \frac{d\hat{w}^\lambda}{d\lambda}\bigg|_{\lambda=0}, \quad \hat{w}^2 = \frac{d^2\hat{w}^\lambda}{d\lambda^2}\bigg|_{\lambda=0}$$

are finite and satisfy the first- and second-order corrector equations, respectively. We will then have $\hat{w}^1 = w^1$ and $\hat{w}^2 = w^2$, so that

$$\frac{w^\lambda - \hat{w}^\lambda}{\lambda^2} \to 0 \text{ as } \lambda \to 0.$$

In order exhibit the bounds needed for this type of argument, it will be convenient to first obtain the corresponding bounds for $\tilde{w}^\lambda$, the payoff of the Merton investment/consumption proportions (θ^0, c^0), and then to bound the difference $\tilde{w}^\lambda - \hat{w}^\lambda$. We begin by introducing some basic properties of $\tilde{w}^\lambda$ and $\hat{w}^\lambda$.

Proposition 2.5.1. *The feedback proportions* (θ^0, c^0) *result in an admissible strategy with high-water mark fee at rate* λ*. Let* $\tilde{v}^\lambda(x,n)$ *be the payoff of this strategy at fee rate* λ*; then* $\tilde{v}^\lambda$ *is a smooth solution of*

$$\begin{aligned} \mathcal{L}^{\theta^0,c^0}\tilde{v}^\lambda &= 0 \\ \tilde{v}_n(x,x) - \lambda\tilde{v}_x(x,x) &= 0 \text{ for all } x > 0 \\ \lim_{n\to\infty}\tilde{v}^\lambda(x,n) &= v^0(x) \end{aligned} \tag{2.125}$$

where the operator $\mathcal{L}^{\theta,c}$ *is defined in* (2.15) *and following. Moreover, we have*

$$\lim_{n\to\infty} n\tilde{v}^\lambda_n(x,n) = \lim_{n\to\infty} n^2\tilde{v}^\lambda_{nn}(x,n) = 0 \tag{2.126}$$

and

$$\lim_{\lambda\to 0}\tilde{v}^\lambda(x,n) = v^0(x) \tag{2.127}$$

Similarly, the feedback proportions $(\hat{\theta}^\lambda, \hat{c}^\lambda)$ *result in an admissible strategy with high-water mark fee at rate* λ*. Let* $\hat{v}^\lambda(x,n)$ *be the payoff of this strategy at fee rate* λ*; then* $\hat{v}^\lambda$ *is a smooth solution of*

$$\begin{aligned} \mathcal{L}^{\hat{\theta}^\lambda,\hat{c}^\lambda}\hat{v}^\lambda &= 0 \\ \hat{v}_n(x,x) - \lambda\hat{v}_x(x,x) &= 0 \text{ for all } x > 0 \\ \lim_{n\to\infty}\hat{v}^\lambda(x,n) &= v^0(x) \end{aligned} \tag{2.128}$$

and we have

$$\lim_{n\to\infty} n\hat{v}^\lambda_n(x,n) = \lim_{n\to\infty} n^2\hat{v}^\lambda_{nn}(x,n) = 0 \tag{2.129}$$

as well as

$$\lim_{\lambda\to 0}\hat{v}^\lambda(x,n) = v^0(x) \tag{2.130}$$

Lastly, we have

$$\begin{aligned}\tilde{v}^\lambda(x,n) &= x^{1-p}\tilde{v}^\lambda(1,n/x)\\ \hat{v}^\lambda(x,n) &= x^{1-p}\hat{v}^\lambda(1,n/x)\end{aligned}$$

Therefore we define

$$\begin{aligned}\tilde{w}^\lambda(y) &= \tilde{w}^\lambda(1,y)\\ \hat{w}^\lambda(y) &= \hat{w}^\lambda(1,y)\end{aligned}$$

and use the notational conventions of (2.11).

Proof. See Appendix. □

Definition 2.5.1. We define the relaxed lower semilimits

$$\begin{aligned}\underline{\tilde{w}}^1(y_0) &= \liminf_{y\to y_0}\frac{\tilde{w}^\lambda(y)-w^0(y)}{\lambda}\\ \underline{\hat{w}}^1(y_0) &= \liminf_{y\to y_0}\frac{\hat{w}^\lambda(y)-w^0(y)}{\lambda}\\ \underline{\tilde{w}}^2(y_0) &= 2\liminf_{y\to y_0}\frac{\tilde{w}^\lambda(y)-(w^0+\lambda w^1)(y)}{\lambda^2}\\ \underline{\hat{w}}^2(y_0) &= 2\liminf_{y\to y_0}\frac{\hat{w}^\lambda(y)-(w^0+\lambda w^1)(y)}{\lambda^2}\end{aligned}$$

As usual, these inherit the homotheticity properties of $\tilde{w}^\lambda, \hat{w}^\lambda$, and we will apply the notational conventions of (2.11) to these functions as well.

Remark 2.5.1. Since $\hat{w}^\lambda \leq w^\lambda$, it is obvious that

$$\begin{aligned}\underline{\hat{w}}^1 &\leq w^1\\ \underline{\hat{w}}^2 &\leq w^2\end{aligned}$$

Therefore, we only show that $\underline{\hat{w}}^1 \geq w^1$ and $\underline{\hat{w}}^2 \geq w^2$. It is then immediate that the derivatives

$$\begin{aligned} \hat{w}^1 &= \left.\frac{d\hat{w}^\lambda}{d\lambda}\right|_{\lambda=0} \\ \hat{w}^2 &= \left.\frac{d^2\hat{w}^\lambda}{d\lambda^2}\right|_{\lambda=0} \end{aligned}$$

are well-defined and equal to w^1 and w^2, respectively. In other words, *there is no need to work with the upper relaxed semilimits for* w^1 and w^2.

We now begin with the argument that $\underline{\tilde{w}}^1 > \infty$ and $\underline{\tilde{w}}^2 > \infty$, which will then be used to show that $\underline{\hat{w}}^1 > \infty$ and $\underline{\hat{w}}^2 > \infty$

Proposition 2.5.2. *Define*

$$\tilde{G}^\lambda(y) = |\tilde{w}^\lambda(y) - w^0(y)| + |\tilde{w}_x^\lambda(y) - w_x^0(y)| + |\tilde{w}_{xx}^\lambda(y) - w_{xx}^0(y)| + |\tilde{w}_y^\lambda(y)|$$

Then for λ sufficiently small there is a constant $\tilde{M}$ such that

$$\sup_{y\in[1,\infty)} \tilde{G}^\lambda(y) \leq \tilde{M}\lambda$$

Proof. First note that there is a constant M_0 such that

$$\sup_{y\in[1,\infty)} |\tilde{w}^\lambda(y) - w^0(y)| \leq M_0\lambda$$

for λ sufficiently small (we say that $|\tilde{w}^\lambda(y) - w^0(y)|$ is *uniformly* $O(\lambda)$). This is a consequence of Remark 2.3.3 and the fact that the right hand side of (2.47) is uniformly bounded below for all choices of $z = (1, y)$.

Suppose that $|y\tilde{w}_y|$ is *not* uniformly $O(\lambda)$. Then for any α, the function $f^{\alpha,\lambda}(y) = y\tilde{w}_y(y) + \alpha(\tilde{w}^\lambda(y) - w^0(1))$ also fails to be uniformly $O(\lambda)$. In this case, there must be $\lambda_k \to 0$ such that f^{α,λ_k} achieves its global extrema on $(1,\infty)$ for small λ_k, because

$$\begin{aligned}\tilde{w}_y^\lambda(1) &= \lambda \tilde{w}_x^\lambda(1) \\ &= \lambda\left((1-p)\tilde{w}^\lambda(1) - w_y^\lambda(1)\right)\end{aligned}$$

and therefore

$$\begin{aligned}\tilde{w}_y^\lambda(1) &= \frac{\lambda}{1+\lambda}(1-p)\tilde{w}^\lambda(1) \\ \lim_{y\to\infty} y\tilde{w}_y &= 0\end{aligned} \tag{2.131}$$

so that yw_y^λ is $O(\lambda)$ at $y=1$. Now suppose $|f^{\alpha,\lambda_k}|$ achieves its global maximum at y_k, so that

$$f_y^{\alpha,\lambda}(y_k) = (1+\alpha)\tilde{w}_y(y_k) + y_k\tilde{w}_{yy}(y_k) = 0 \tag{2.132}$$

Plugging $\tilde{w}$ into the equation $\mathcal{L}^{\theta^0,c^0}\tilde{w}^{\lambda_k} = 0$ at y_k and using the relation (2.132), we see that

$$\begin{aligned}0 &= U(c^0) + \left(-\beta - c^0(1-p) + \mu\theta^0(1-p) - \frac{1}{2}(\sigma\theta^0)^2 p(1-p)\right)\tilde{w}^{\lambda_k}(y_k) \\ &\quad + c_0(y_k-1)\tilde{w}_y^{\lambda_k} + \left(-\mu\theta^0 + p(\sigma\theta^0)^2\right)y_k\tilde{w}_y^{\lambda_k} + \frac{1}{2}(\sigma\theta^0 y_k)^2 w^{\lambda_k}(y_k) \\ &= U(c^0) + \left(-\beta - c^0(1-p) + \mu\theta^0(1-p) - \frac{1}{2}(\sigma\theta^0)^2 p(1-p)\right)\tilde{w}^{\lambda_k}(y_k) \\ &\quad + \left(c_0\left(1-\frac{1}{y_k}\right) - \frac{1+\alpha}{2}(\sigma\theta^0)^2\right)y_k\tilde{w}_y^\lambda\end{aligned} \tag{2.133}$$

Because w^0 satisfies the $\lambda=0$ HJB equation, we know that

$$\begin{aligned}U(c^0) + \left(-\beta - c^0(1-p) + \mu\theta^0(1-p) - \frac{1}{2}(\sigma\theta^0)^2 p(1-p)\right)w^0 &= 0 \\ \left(-\beta - c^0(1-p) + \mu\theta^0(1-p) - \frac{1}{2}(\sigma\theta^0)^2 p(1-p)\right) &< 0\end{aligned}$$

Since $|\tilde{w}^\lambda - w^0|$ is uniformly $O(\lambda)$ and $\tilde{w}^\lambda \leq w^0$, we conclude that there is a constant M_1 such that

$$\begin{aligned} 0 &\leq U(c^0) + \left(-\beta - c^0(1-p) + \mu\theta^0(1-p) - \frac{1}{2}(\sigma\theta^0)^2 p(1-p)\right)\tilde{w}^\lambda(y_k) \\ &= -\left(c_0\left(1-\frac{1}{y^\lambda}\right) - \frac{1+\alpha}{2}(\sigma\theta^0)^2\right) y^\lambda \tilde{w}^\lambda_y(y_k) \leq M_1\lambda \qquad (2.134) \end{aligned}$$

Choose α sufficiently negative that

$$c_0\left(1-\frac{1}{y_\lambda}\right) - \frac{1+\alpha}{2}(\sigma\theta^0)^2 > \epsilon > 0$$

for λ small. Then, since $|f^{\lambda,\alpha}(y_k)|/\lambda \to \infty$ as $\lambda \to 0$ and $\alpha\tilde{w}^\lambda$ is uniformly $O(\lambda)$, we must actually have $|y_k\tilde{w}^\lambda_y(y_k)|/\lambda \to \infty$ as $\lambda \to 0$. Since $c_0(1-\frac{1}{y_k}) - \frac{1+\alpha}{2}(\sigma\theta^0)^2$ is uniformly bounded away from 0, this contradicts (2.134).

We conclude that $y\tilde{w}^\lambda_y$ is uniformly $O(\lambda)$. Since $\tilde{w}^\lambda_x = (1-p)\tilde{w}^\lambda - y\tilde{w}^\lambda_y$, it follows immediately that $|\tilde{w}^\lambda_x - w^0_x|$ is uniformly $O(\lambda)$ as well. As a result, the PDE (2.125) implies that $y^2\tilde{w}^\lambda_{yy}$ (and hence $\tilde{w}^\lambda_{xx}$) are uniformly $O(\lambda)$ as well. $\square$

Proposition 2.5.3. *We have*

$$\begin{aligned} \underline{\tilde{w}}^2(y_0) &> -\infty \\ \liminf_{y\to\infty} \underline{\tilde{w}}^2 &\geq 0. \end{aligned}$$

for all $y_0 \in [1,\infty)$

Proof. Define

$$\tilde{Q}^{1,\lambda}(y) = \frac{\tilde{w}^\lambda(y) - w^0(1)}{\lambda}$$

As in Proposition 2.4.1, the goal is to exhibit a function $f : [1, \infty) \to \mathbb{R}$ such that $f(y) \to 0$ as $y \to \infty$ and $\tilde{Q}^{1,\lambda}(y) + \lambda f$ is a supersolution of the first-order corrector equation, allowing us to conclude

$$\underline{\tilde{w}}^2 \geq -2f > -\infty$$

First, note that:

$$\begin{aligned}
\mathcal{A}\tilde{Q}^{1,\lambda} &= \frac{1}{\lambda}\left(\mathcal{L}^{\theta^0,c^0}\tilde{w}^\lambda + \tilde{U}'(w_x^0)(\nabla\tilde{w}^\lambda \cdot \mathbf{1}) - (U(c^0) - c^0\nabla\tilde{w}^\lambda \cdot \mathbf{1})\right) \\
&\quad -\frac{1}{\lambda}\left(\mathcal{L}^{\theta^0,c^0}w^0 + \tilde{U}'(w_x^0)w_x^0 - (U(c^0) - c^0 w_x^0)\right) \\
&= 0 \qquad (2.135)
\end{aligned}$$

where the last line follows from the fact that $\tilde{U}'(w_x^0(1)) = -c^0$ and

$$\mathcal{L}^{\theta^0,c^0}w^0 = \mathcal{L}^{\theta^0,c^0}\tilde{w}^\lambda = 0$$

Using the usual homotheticity properties, the boundary condition for $\tilde{w}$ and the computations of (2.100), there is a constant $\tilde{M}$ such that for sufficiently small λ

$$\begin{aligned}
\tilde{Q}_y^{1,\lambda}(1) &= \frac{1}{\lambda}\tilde{w}_y^\lambda(1) \\
&= \frac{(1-p)}{1+\lambda}\tilde{w}^\lambda(1) \\
&\leq (1-p)w^0(1) + \tilde{M}\lambda \\
&\leq w_x^0(1) + \tilde{M}\lambda
\end{aligned}$$

where the third equality follows from the bound of Proposition 2.5.2. Therefore, we take f to be a supersolution to the first corrector equation such that $f'(1)$ is sufficiently large and negative. From Lemma 2.3.9, we can take $f(y) = Ky^{-q}$ for K large and q sufficiently small (both independent of λ). Then, noting that $\tilde{Q}^{1,\lambda} + \lambda f$ is a supersolution of the first corrector equation with

$$\lim_{y \to 0} \tilde{Q}^{1,\lambda} + \lambda f = 0$$

we conclude that $\tilde{Q}^{1,\lambda} + \lambda f \leq w^1$, hence

$$\frac{\tilde{Q}^{1,\lambda} - w^1}{\lambda} \geq -f$$

for λ sufficiently small. □

Corollary 2.5.4. *We have*

$$\underline{\tilde{w}}^1 = w^1$$

As a result, the limit

$$\tilde{w}^1 \triangleq \lim_{\lambda \to 0} \frac{\tilde{w}^\lambda - w^0}{\lambda}$$

is well defined, and $\tilde{w}^1 = w^1$.

Proof. Since $\tilde{w}^\lambda \leq w^\lambda$, we obviously have $\tilde{w}^1 \leq w^1$. On the other hand, if $\tilde{w}^1(y_0) < w^1(y_0)$, then

$$\begin{aligned} \underline{\tilde{w}}^2 &\leq 2 \liminf_{\lambda \to 0} \frac{\tilde{w}^\lambda(y_0) - (w^0 + \lambda w^1)(y_0)}{\lambda^2} \\ &= 2 \lim_{\lambda \to 0} \frac{\tilde{w}^1(y_0) - w^1(y_0)}{\lambda} = -\infty \end{aligned} \tag{2.136}$$

which contradicts Proposition 2.5.2. □

We now wish to use the boundedness results

$$\begin{aligned} \underline{\tilde{w}}^2 &> -\infty \\ \liminf_{y\to\infty} \underline{\tilde{w}}^2 &\geq 0 \end{aligned}$$

to show that

$$\begin{aligned} \underline{\hat{w}}^2 &> -\infty \\ \liminf_{y\to\infty} \underline{\hat{w}}^2 &\geq 0. \end{aligned}$$

We may then use arguments similar to Proposition 2.4.8 to show that $\underline{\hat{w}}^2$ is a supersolution of the second-order corrector equation, so that $\underline{\hat{w}}^2 = w^2$. The argument will proceed in the following steps:

1. We have $\underline{\hat{w}}^1 = \underline{\tilde{w}}^1 = w^1$.

2. There is a constant $M_2 > 0$ such that

$$\frac{\hat{w}^\lambda - \tilde{w}}{\lambda^2} \geq M_2.$$

   Since $\underline{\tilde{w}}^2 > -\infty$, this implies that $\underline{\hat{w}}^2 > -\infty$.

3. Modifying the arguments of Proposition 2.4.8, we see $\underline{\hat{w}}^2$ is a supersolution of the second-order corrector equation, so that $\underline{\hat{w}}^2 \geq w^2$. By the discussion of Remark 2.5.1, this completes the argument.

Steps 1 and 2 above will make use of the following easy lemma.

Lemma 2.5.5. *There is a constant $\tilde{M}$ such that*

$$\sup_{y\in[1,\infty)} |\mathcal{L}^{\hat{\theta}^\lambda,\hat{c}^\lambda}\tilde{w}^\lambda(y)| \le \tilde{M}\lambda^2$$

for λ sufficiently small. More strongly, we have

$$\lim_{y_0\to\infty} \limsup_{\lambda\to 0} \sup_{[y_0,\infty)} \frac{|\mathcal{L}^{\hat{\theta}^\lambda,\hat{c}^\lambda}\tilde{w}^\lambda(y)|}{\lambda^2} = 0 \tag{2.137}$$

Proof. Observe that

$$\begin{aligned}\mathcal{L}^{\hat{\theta}^\lambda,\hat{c}^\lambda}\tilde{w}^\lambda &= \mathcal{L}^{\theta^0,c^0}\tilde{w}^\lambda + U(c^0+\lambda c^1) - U(c^0) - \lambda c^1(\nabla\tilde{w}^\lambda\cdot\mathbf{1}) \\ &\quad +\lambda\theta^1\left(\mu\tilde{w}^\lambda_x + (\sigma\theta^0)\tilde{w}^\lambda_{xx}\right) + \frac{\lambda^2}{2}(\sigma\theta^1)^2\tilde{w}^\lambda_{xx}\end{aligned} \tag{2.138}$$

We know $\mathcal{L}^{\theta^0,c^0}\tilde{w}^\lambda = 0$. Recalling that $U'(c^0) = w^0_x$ and applying Taylor's Theorem, we see that

$$U(c^0+\lambda c^1) - U(c^0) - \lambda c^1(\nabla\tilde{w}^\lambda\cdot\mathbf{1}) = U''(\xi^\lambda)(\lambda c^1)^2 - \lambda c^1((\nabla\tilde{w}^\lambda\cdot\mathbf{1}) - w^0_x) \tag{2.139}$$

where ξ^λ is some number between c^0 and $c^0+\lambda c^1$. Recall from Remark 2.4.2 that c^1 is uniformly bounded. From Proposition 2.5.2, $((\nabla\tilde{w}^\lambda\cdot\mathbf{1}) - w^0_x)/\lambda$ is uniformly bounded as well. Therefore, there is a constant $\tilde{M}_0$ such that

$$|U(c^0+\lambda c^1) - U(c^0) - \lambda c^1(\nabla\tilde{w}^\lambda\cdot\mathbf{1})| \le \tilde{M}_0\lambda^2 \tag{2.140}$$

Recalling that $c^1 \to 0$ as $y\to\infty$, we see from (2.139) that for any $\tilde{\epsilon}_0 > 0$, we may in fact choose y_0 such that

$$\sup_{y\in[y_0,\infty)} |U(c^0+\lambda c^1) - U(c^0) - \lambda c^1(\nabla\tilde{w}^\lambda\cdot\mathbf{1})| \le \tilde{\epsilon}_0\lambda^2$$

for sufficiently small λ. For a similar bound on the second line of (2.138), note that for small λ we have

$$\begin{aligned}
\left|\lambda\theta^1\left(\mu\tilde{w}_x^\lambda + (\sigma^2\theta^0)\tilde{w}_{xx}^\lambda\right)\right| &= \left|\lambda\theta^1\mu\left(\tilde{w}_x^\lambda + \frac{1}{p}\tilde{w}_{xx}^\lambda\right)\right| \\
&= \left|\lambda\theta^1\mu\left(\tilde{w}_x^\lambda - w_x^0 - \frac{1}{p}\left(\tilde{w}_{xx}^\lambda - w_{xx}^0\right)\right)\right| \\
&\leq \tilde{M}_1\lambda^2 \qquad (2.141)
\end{aligned}$$

where the last line follows from Proposition 2.5.2. Since $\theta^1 \to 0$ as $y \to \infty$, we see that for $\tilde{\epsilon_1} > 0$ arbitrarily small we may choose y_0 such that

$$\sup_{y\in[y_0,\infty)}\left|\lambda\theta^1\left(\mu\tilde{w}_x^\lambda + (\sigma^2\theta^0)\tilde{w}_{xx}^\lambda\right)\right| \leq \tilde{\epsilon}^1\lambda^2$$

for small λ. Finally, since θ^1 is uniformly bounded, Proposition 2.5.2 also readily implies that there is a constant $\tilde{M}_2$ such that

$$\left|\frac{\lambda^2}{2}(\sigma\theta^1)^2\tilde{w}_{xx}^\lambda\right| \leq \tilde{M}_2\lambda^2$$

for small λ. Again, since $\theta^1 \to 0$, we for any $\tilde{\epsilon}_2 > 0$, we may choose y_0 such that

$$\sup_{y\in[y_0,\infty)}\left|\frac{\lambda^2}{2}(\sigma\theta^1)^2\tilde{w}_{xx}^\lambda\right| \leq \tilde{\epsilon}_2\lambda^2$$

for λ small. This completes the proof. □

Proposition 2.5.6. *We have* $\underline{\hat{w}}^1 = w^1$

Proof. Since $\hat{w}^\lambda \leq w^\lambda$, we know that $\underline{\hat{w}}^1 \leq w^1$. On the other hand, we will show that for arbitrarily small $\epsilon > 0$, we have

$$\sup_{y\in[1,\infty)}\frac{\tilde{w}^\lambda - \hat{w}^\lambda}{\lambda} < \epsilon \qquad (2.142)$$

for λ sufficiently small. Corollary 2.5.4 then immediately implies that we have

$$\underline{\hat{w}}^1 = \underline{\tilde{w}}^1 = w^1$$

As usual, we will obtain the bound (2.142) via a comparison argument. First, note that for sufficiently small λ, the zero-order coefficient of the operator $\mathcal{L}^{\hat{\theta}^\lambda,\hat{c}^\lambda}$ is negative; in other words, if $M_0 > 0$ is a constant, then $\mathcal{L}^{\hat{\theta}^\lambda,\hat{c}^\lambda} M_0 < 0$ for small λ. Explicitly, we can compute that

$$\mathcal{L}^{\hat{\theta}^\lambda,\hat{c}^\lambda} M_0 = A_0^\lambda M_0 + U(c^0 + \lambda c^1)$$

where

$$\begin{aligned}
A_0^\lambda &= -\beta + (1-p)\left[\mu(\theta^0 + \lambda\theta^1) - \frac{1}{2}p(\sigma(\theta^0 + \theta^1)^2) - (c^0 + \lambda c^1)\right] \\
&= -\beta + (1-p)\left[\mu\theta^0 - \frac{1}{2}p(\sigma\theta^0)^2 - c^0\right] \\
&\quad +\lambda(1-p)\left[\theta^1(\mu - p\sigma^2\theta^0 - c^1) - \left(\frac{\lambda^2}{2}\right)p(\sigma\theta^1)^2\right]
\end{aligned}$$

Recalling that

$$c^0 = \frac{\beta}{p} - \left(\frac{1}{2}\right)\frac{1-p}{p^2}\frac{\mu^2}{\sigma^2} = -\frac{1}{p}\left(-\beta + (1-p)\left[\mu\theta^0 - \frac{1}{2}p(\sigma\theta^0)^2\right]\right) > 0$$

we conclude that

$$A_0^\lambda = -c^0 + (1-p)\left[\lambda\theta^1(\mu - p\sigma^2\theta^0 - c^1) - \frac{\lambda^2}{2}p(\sigma\theta^1)^2\right] \quad (2.143)$$

From Lemma 2.5.5, there is a constant $\tilde{M}$ such that

$$|\mathcal{L}^{\hat{\theta},\hat{c}}\tilde{w}^\lambda| \leq \tilde{M}\lambda^2$$

As a result, for arbitrarily small $\epsilon_0 > 0$, there is a constant $C > 0$ such that

$$\mathcal{L}^{\hat{\theta}^\lambda,\hat{c}^\lambda}(\tilde{w}^\lambda - \epsilon_0\lambda) = \mathcal{L}^{\hat{\theta}^\lambda,\hat{c}^\lambda}\tilde{w}^\lambda + A_0^\lambda\epsilon_0\lambda \geq -\tilde{M}\lambda^2 + A_0^\lambda\epsilon_0\lambda \geq C\lambda \tag{2.144}$$

for λ small. for some $C' > 0$ and λ small. Now, the boundary conditions for $\hat{w}^\lambda$ and $\tilde{w}^\lambda$ may be written as

$$\begin{aligned} \tilde{w}_y^\lambda(1) &= \frac{\lambda(1-p)}{1+\lambda}\tilde{w}^\lambda(1) \\ \hat{w}_y^\lambda(1) &= \frac{\lambda(1-p)}{1+\lambda}\hat{w}^\lambda(1) \end{aligned} \tag{2.145}$$

Recalling (2.127) and (2.130), we see that for any $\epsilon > 0$, we have

$$|\tilde{w}_y^\lambda(1) - \hat{w}_y^\lambda(1)| = \frac{\lambda(1-p)}{1+\lambda}|\tilde{w}^\lambda(1) - \hat{w}^\lambda(1)| \leq \epsilon\lambda$$

for λ small. Now let $f(y) = \epsilon_1 y^{-q}$ for $q > 0$. It is easy to see that we can choose ϵ_1 sufficiently small that

$$|\mathcal{L}^{\hat{\theta}^\lambda,\hat{c}^\lambda}(-\lambda f) - U(c^0 + \lambda c^1)| \leq C\lambda$$

for λ sufficiently small. Therefore, we have

$$\begin{aligned} \mathcal{L}^{\hat{\theta}^\lambda,\hat{c}^\lambda}(\tilde{w}^\lambda - \epsilon_0\lambda - \lambda f) &> 0 \\ (\tilde{w}^\lambda - \epsilon_0\lambda - \lambda f)_y(1) &> \hat{w}_y^\lambda(1) \\ \lim_{y\to\infty} \tilde{w}^\lambda - \hat{w}^\lambda - \epsilon_0\lambda - \lambda f &\leq 0 \end{aligned} \tag{2.146}$$

for λ sufficiently small. We now make a standard comparison argument. Suppose for contradiction that there is a point $\hat{y}$ such that

$$(\tilde{w}^\lambda - \hat{w}^\lambda - \epsilon_0\lambda - \lambda f)(\hat{y}) > 0$$

By (2.146), we may assume that $\hat{y}$ is a global maximum of $\tilde{w}^\lambda - \hat{w}^\lambda - \epsilon_0\lambda - \lambda f$ with $\hat{y} > 1$. We must then have

$$\begin{aligned}(\tilde{w}^\lambda - \epsilon_0\lambda - \lambda f)_y(\hat{y}) &= \hat{w}^\lambda(\hat{y}) \\ (\tilde{w}^\lambda - \epsilon_0\lambda - \lambda f)_{yy}(\hat{y}) &< \hat{w}^\lambda_{yy}(\hat{y})\end{aligned} \tag{2.147}$$

On the other hand, we have

$$0 = \mathcal{L}^{\hat{\theta}^\lambda,\hat{c}^\lambda}\hat{w}^\lambda(\hat{y}) < \mathcal{L}^{\hat{\theta}^\lambda,\hat{c}^\lambda}(\tilde{w}^\lambda - \epsilon_0\lambda - \lambda f)(\hat{y}) \tag{2.148}$$

a contradiction, since at $\hat{y}$ we have

$$\begin{aligned}\mathcal{L}^{\hat{\theta}^\lambda,\hat{c}^\lambda}(\tilde{w}^\lambda - \epsilon_0\lambda - \lambda f) - \mathcal{L}^{\hat{\theta}^\lambda,\hat{c}^\lambda}\hat{w}^\lambda \quad &= \quad A_0(\tilde{w}^\lambda - \epsilon_0\lambda - \lambda f - \hat{w}^\lambda) \\ &\quad + \frac{1}{2}(\sigma^2\hat{\theta}^\lambda)^2(\tilde{w}^\lambda - \epsilon_0\lambda - \lambda f - \hat{w}^\lambda)_{yy} \\ &< 0\end{aligned} \tag{2.149}$$

where the last line of (2.149) follows from (2.147). We therefore conclude that

$$(\tilde{w}^\lambda - \hat{w}^\lambda - \epsilon_0\lambda - \lambda f)(\hat{y}) \leq 0$$

for λ sufficiently small. Since $|f| \leq \epsilon_1$, we have

$$\frac{\tilde{w}^\lambda - \hat{w}^\lambda}{\lambda} \leq \epsilon_0 + \epsilon_1$$

for small λ, so that (2.142) holds. □

Proposition 2.5.7. *We have*

$$\begin{aligned}\underline{\hat{w}}^2 \quad &> \quad -\infty \\ \liminf_{y\to\infty} \underline{\hat{w}}^2 \quad &\geq \quad 0\end{aligned}$$

Proof. The proof closely follows that of Proposition 2.5.6. We show that there is a constant M such that

$$\sup_{y\in[1,\infty)} \tilde{w}^\lambda - \hat{w}^\lambda \leq M\lambda^2 \tag{2.150}$$

It then follows from Proposition 2.5.3 that $\underline{\hat{w}}^2 > -\infty$. Once again, we obtain 2.150 by a comparison argument. From Lemma 2.5.5, there is a constant $\tilde{M}$ such that

$$|\mathcal{L}^{\hat{\theta}^\lambda,\hat{c}^\lambda}\tilde{w}^\lambda| \leq \tilde{M}\lambda^2$$

Arguing as in Proposition 2.5.3, we may choose M_0 such that

$$|\mathcal{L}^{\hat{\theta}^\lambda,\hat{c}^\lambda}\tilde{w}^\lambda - M_0\lambda^2| \leq C\lambda^2$$

for some $C > 0$ and λ sufficiently small. We may write the boundary conditions for $\tilde{w}^\lambda$, $\hat{w}^\lambda$ as

$$\begin{aligned}
\tilde{w}^\lambda_y(1) &= \frac{\lambda(1-p)}{1+\lambda}\tilde{w}^\lambda(1) \\
&= \frac{\lambda(1-p)}{1+\lambda}\left(w^0(1) + \lambda w^1(1) + \tilde{R}^\lambda\right) \\
\hat{w}^\lambda_y(1) &= \frac{\lambda(1-p)}{1+\lambda}\hat{w}^\lambda(1) \\
&= \frac{\lambda(1-p)}{1+\lambda}\left(w^0(1) + \lambda w^1(1) + \hat{R}^\lambda\right)
\end{aligned}$$

where $\tilde{R}^\lambda$, $\hat{R}^\lambda$ are numbers with

$$\lim_{\lambda\to 0}\frac{\tilde{R}^\lambda}{\lambda} = \lim_{\lambda\to 0}\frac{\hat{R}^\lambda}{\lambda} = 0$$

So for arbitrarily small $\epsilon > 0$, we have

$$|\tilde{w}^\lambda_y(1) - \hat{w}^\lambda_y(1)| \leq \epsilon\lambda^2 \tag{2.151}$$

for small λ.

Now set $f(y) = K_1 y^{-q}$ for K_1 chosen so that

$$|\mathcal{L}^{\hat{\theta}^\lambda,\hat{c}^\lambda}(-\lambda^2 f)| < K_0\lambda^2$$

As in the proof of Proposition 2.5.6, it is easy to show that such a choice of K_1 is possible. Then we have

$$\begin{aligned} \mathcal{L}^{\hat{\theta}^\lambda,\hat{c}^\lambda}(\tilde{w}^\lambda - K_0\lambda^2 - \lambda^2 f) &> 0 \\ (\tilde{w}^\lambda - K_0\lambda^2 - \lambda^2 f)_y(1) &> \hat{w}^\lambda_y(1) \\ \lim_{y\to\infty} \tilde{w}^\lambda - \hat{w}^\lambda - K_0\lambda^2 - \lambda^2 f &\leq 0 \end{aligned} \tag{2.152}$$

We may then make a comparison argument identical to that of Proposition 2.5.6 to conclude that

$$\frac{\tilde{w}^\lambda - \hat{w}^\lambda}{\lambda^2} \leq K_0 + f. \tag{2.153}$$

This completes the proof that $\underline{\hat{w}}^2 > -\infty$. To see that

$$\liminf_{y\to\infty} \underline{\hat{w}}^2(y) \geq 0$$

we adapt the arguments of Proposition 2.4.7 to our setting. The differences are as follows;

1. We check that

   $$\liminf_{\lambda\to 0} \frac{\tilde{w}^\lambda_y - \hat{w}^\lambda_y}{\lambda^2} > -\infty$$

   using similar arguments to those applied to show

   $$\liminf_{\lambda\to 0} Q^{2,\lambda}_y > -\infty$$

   in Proposition 2.4.7.

2. We apply the comparison argument given above on the interval $[y_k, \infty)$ for $y_k \to \infty$. We use the fact (proven in Lemma 2.5.5) that

$$\lim_{y_0 \to \infty} \limsup_{\lambda \to 0} \sup_{[y_0, \infty)} \frac{|\mathcal{L}^{\hat{\theta}^\lambda, \hat{c}^\lambda} \tilde{w}^\lambda|}{\lambda^2} = 0$$

to show that if we work on the interval $[y_k, \infty)$, the K_0 of equation (2.153) can be chosen arbitrarily small for large y_k. This is analogous to the use of Lemma 2.4.5 in Proposition 2.4.7. Since we have shown that

$$\liminf_{y \to \infty} \underline{\tilde{w}}^2 \geq 0,$$

we then obtain

$$\liminf_{y \to \infty} \underline{\hat{w}}^2 \geq 0$$

as a result.

$\square$

Theorem 2.5.8. *We have*

$$\underline{\hat{w}}^2 \quad = \quad w^2$$

As a result,

$$\begin{aligned} \left.\frac{d\hat{w}^\lambda}{d\lambda}\right|_{\lambda=0} &= w^1 \\ \left.\frac{d^2\hat{w}^\lambda}{d\lambda^2}\right|_{\lambda=0} &= w^2 \end{aligned}$$

In other words, the payoff of the approximately optimal strategies $(\hat{\theta}^\lambda, \hat{c}^\lambda)$ at fee level λ matches the value function up to second order in λ.

Proof. : From the reasoning of Remark 2.5.1, it will suffice to show that $\underline{\hat{w}}^2$ is a viscosity supersolution of the second-order corrector equation. Let ϕ be a C^2 function such that $\underline{\hat{w}}^2 - \phi$ has a strict local minimum at y_0. Choose $\lambda_k \to 0$, points $y_k \to y_0$ and functions

$$\psi^k \triangleq \hat{w}^{\lambda_k}(y_k) + \lambda_k \left(w^1(y) - w^1(y_k)\right) + \frac{\lambda_k^2}{2}\left(\phi(y) - \phi(y_k)\right)$$

touching w^{λ_k} below at y_k, just as we did in Proposition 2.4.8. First suppose we have

$$\mathcal{L}^{\hat{\theta}^\lambda, \hat{c}^\lambda} \psi^k(y_k) \leq 0$$

for infinitely many k. We can repeat the arguments of Proposition 2.4.8 to show that

$$\lim_{k\to\infty} \frac{L^{\hat{\theta}^{\lambda_k}} \psi^k(y_k)}{\lambda_k^2} = \frac{1}{2}\left(\mathcal{A} + g\right) \tag{2.154}$$

Therefore, to verify the interior supersolution property of the second-order corrector equation, it will suffice to show that

$$\limsup_{k\to\infty} \frac{L^{\hat{\theta}^{\lambda_k}} \psi^k(y_k) - \mathcal{L}^{\hat{\theta}^\lambda, \hat{c}^\lambda} \psi^k(y_0)}{\lambda_k^2} \leq 0 \tag{2.155}$$

Make the notation

$$\kappa^k \triangleq L^{\hat{\theta}^{\lambda_k}} \psi^k(y_k) - \mathcal{L}^{\hat{\theta}^\lambda, \hat{c}^\lambda} \psi^k(y_k) \tag{2.156}$$

and note that

$$\kappa^k \quad = \quad \tilde{U}(\nabla\psi^k \cdot \mathbf{1}) - \nu^k(\hat{c}^{\lambda_k})$$

where as usual all functions are implicitly evaluated at y_k, and

$$\nu^k(\tilde{y}) = U(\tilde{y}) - \tilde{y}(\nabla\psi^k \cdot \mathbf{1})$$

and note that ν^k has a maximum of $\tilde{U}(\nabla\psi^k \cdot \mathbf{1})$ at $I(\nabla\psi^k \cdot \mathbf{1})$ (recall that $I = (U')^{-1}$). Therefore, by a Taylor expansion, we have

$$\kappa^k = \frac{1}{2}(\nu^k)''(\xi^k)(\nabla\psi^k \cdot \mathbf{1} - \hat{c}^{\lambda_k})^2$$

where ξ^k is some number between $\nabla\psi^k \cdot \mathbf{1}$ and $\hat{c}^{\lambda_k}$. As a result, we will be done with the interior supersolution if we can show that

$$\lim_{k\to\infty} \frac{\nabla\psi^k \cdot \mathbf{1} - \hat{c}^{\lambda_k}}{\lambda_k} = 0 \tag{2.157}$$

so that

$$\lim_{k\to\infty} \frac{\kappa_k}{\lambda_k^2} = 0.$$

To verify (2.157), observe that

$$\begin{aligned} I(\nabla\psi^k \cdot \mathbf{1}) &= I(w_x^0) + I'(w_x^0)(\nabla\psi^k \cdot \mathbf{1} - w_x^0) + \frac{1}{2}I''(\xi^k)(\nabla\psi^k \cdot \mathbf{1} - w_x^0)^2 \\ &= c^0 + I'(w_x^0)\left((1-p)(\hat{w}^{\lambda_k} - w^0) - (1-y_k)(\lambda_k w_y^1 + \frac{\lambda_k^2}{2}\phi_y)\right) \\ &\quad \frac{1}{2}I''(\xi^k)(\nabla\psi^k \cdot \mathbf{1} - w_x^0)^2 \end{aligned} \tag{2.158}$$

for some ξ^k between w_x^0 and $\nabla\psi^k \cdot \mathbf{1}$. On the other hand, using the definition of c^1, we have

$$\hat{c}^{\lambda_k} = c^0 + \lambda_k I'(w_x^0)\left((1-p)w^1 - (1-y_k)w_y^1\right) \tag{2.159}$$

So we have

$$\begin{aligned} \lim_{k\to\infty} \frac{|I(\nabla\psi^k \cdot \mathbf{1}) - \hat{c}^{\lambda_k}|}{\lambda_k} &= \lim_{k\to\infty} \frac{(1-p)I'(w_x^0)(\hat{w}^{\lambda_k} - w^0)}{\lambda_k} - w^1(y_k) \\ &= 0, \end{aligned} \tag{2.160}$$

where the second line follows from Proposition 2.5.6 (it is easy to check that the second-order remainder terms of (2.158) make no contribution). We conclude that (2.157), and hence (2.155), holds.

If $y_k = 1$ for infinitely many k, we may argue as in Proposition 2.4.8, incorporating the result of Proposition 2.5.6 as needed.

$\square$

Chapter 3

Numerics

The paper [16] gives numerical results for the investment/consumption problem with high-water mark fee. Specifically, the authors numerically solve the HJB equation for the value function *for fixed* λ, then use the results to describe the optimal investment/consumption proportions, as well as the certainty equivalent wealth and the certainty equivalent excess return (which we will define below). In this section we will

1. compare these "exact" numerical approximations of the optimal strategy with our closed form, approximately optimal strategies.
2. numerically solve for $\hat{v}^\lambda$, the payoff of the approximately optimal strategy, and compare it to the "exact" numerical solution of [16] for v^λ (recall we showed in Chapter 2 that v^λ and $\hat{v}^\lambda$ are equal up to second order).

Notation. Throughout this chapter, we will use the notation developed in Chapter 2.

3.1 Certainty equivalent analysis

The values of the function v^λ are given in abstract units of utility, so it is difficult interpret the impact of fees by examining v^λ itself. Instead we consider the *zero fee certainty equivalent wealth* and *zero fee certainty equivalent excess return*. The zero fee certainty equivalent wealth is the quantity $\tilde{x}$ such that the investor would be indifferent between initial wealth $\tilde{x}$ when paying no fees and initial state (x, n) when paying high-water mark fee λ. Note that since

$$\begin{aligned} v^0(x) &= (c^0)^{-p} \frac{x^{1-p}}{1-p} \\ v^\lambda(x,n) &= x^{1-p} v^\lambda(1, n/x) \\ &= x^{1-p} w^\lambda(n/x) \end{aligned}$$

it will suffice to take $y = n/x$ and consider $\tilde{x}(y)$ such that the investor is indifferent between initial wealth $\tilde{x}(y)$ and paying no fees and initial wealth 1 when paying high watermark fee λ with initial high watermark y; the certainty equivalent for initial state (x, n) is then equal to $x \cdot \tilde{x}(n/x)$. We will therefore use the one-dimensional notation of Chapter 2 as convenient. Equating $v^0(\tilde{x}(y))$ and $w^\lambda(y)$, we have

$$\tilde{x}(y) = \left[(c^0)^p (1-p) w^\lambda(y)\right]^{\frac{1}{1-p}} \tag{3.1}$$

The zero fee certainty equivalent excess return $\tilde{\mu}(y)$ is defined similarly. Let $w^{0,\tilde{\mu}}$ be the Merton value function with initial wealth 1, for the modified excess return $\tilde{\mu}$. In other words,

$$\begin{aligned} w^{0,\tilde{\mu}} &= \frac{(c^{0,\tilde{\mu}})^{-p}}{1-p} \\ c^{0,\tilde{\mu}} &= \frac{\beta}{p} - \left(\frac{1-p}{2p^2}\right)\frac{\tilde{\mu}^2}{\sigma^2}. \end{aligned} \tag{3.2}$$

The certainty equivalent excess rate of return $\tilde{\mu}(y)$ is a solution to the equation

$$\tilde{w}^{0,\tilde{\mu}(y)} = w^\lambda(y)$$

From (3.2), we can write certainty equivalent rate of excess return (relative to the original rate μ) as

$$\frac{\tilde{\mu}(y)}{\mu} = \frac{\sqrt{2}\sigma p}{\mu}\left(\frac{\beta}{p} - ((1-p)w^\lambda(y))^{-\frac{1}{p}}\right)^{\frac{1}{2}} \tag{3.3}$$

Analogously, we can define the certainty equivalent wealth $\hat{x}(y)$ and rate of return $\hat{\mu}(y)$ for the payoff $\hat{w}^\lambda$ of the suboptimal strategy $(\hat{\theta}^\lambda, \hat{c}^\lambda)$. Explicitly, these are given by

$$\begin{aligned} \hat{x}(y) &= \left[(c^0)^p(1-p)\hat{w}^\lambda(y)\right]^{\frac{1}{1-p}} \\ \frac{\hat{\mu}(y)}{\mu} &= \frac{\sqrt{2}\sigma p}{\mu}\left(\frac{\beta}{p} - ((1-p)\hat{w}^\lambda(y))^{-\frac{1}{p}}\right)^{\frac{1}{2}} \end{aligned}$$

We present four graphs below:

- The optimal and approximately optimal investment proportions relative to the Merton investment proportion $\theta^\lambda(y)/\theta^0$ and $\hat{\theta}^\lambda(y)/\theta^0$ respectively.
- The optimal and approximately optimal consumption proportions relative to the Merton consumption proportion $c^\lambda(y)/c^0$ and $\hat{c}^\lambda(y)/c^0$ respectively.

- The relative zero-fee certainty equivalent wealth levels $\tilde{x}(y)$ and $\hat{x}(y)$.
- The relative zero-fee certainty equivalent excess return rates $\tilde{\mu}(y)/\mu$ and $\hat{\mu}(y)/\mu$

To obtain these graphs, we follow [16] in fixing $\sigma = 30\%$ and choosing the benchmark parameters

$$p = 3, \beta = 5\%, \mu = 10\%, \lambda = 20\%.$$

We then vary the parameters p, β, μ and λ around the benchmark parameters. Note that the choice to fix $\sigma = 30\%$ is not restrictive, since the value function of a model with some choice of μ and σ results in the same value function as a model with scaled rate of return and standard deviation $k\mu$ and $k\sigma$ (in addition, the optimal investment proportion is scaled by $1/k$).

Examining the resulting graphs, we have the following observations.

1. Most importantly, the certainty equivalent wealth and rate of return for the payoffs $\hat{w}^\lambda$ of the approximately optimal strategies $(\hat{\theta}^\lambda, \hat{c}^\lambda)$ closely track those for the true value function w^λ, but are slightly lower. The difference between the two does appear to decay faster than linearly as $\lambda \to 0$, reflecting the fact that $\hat{w}^\lambda$ matches w^λ up to second order in λ.

2. The approximately optimal strategies share some qualitative features with the optimal strategies. In particular, both investment proportions are greater than the Merton proportion when y is near 1. As noted

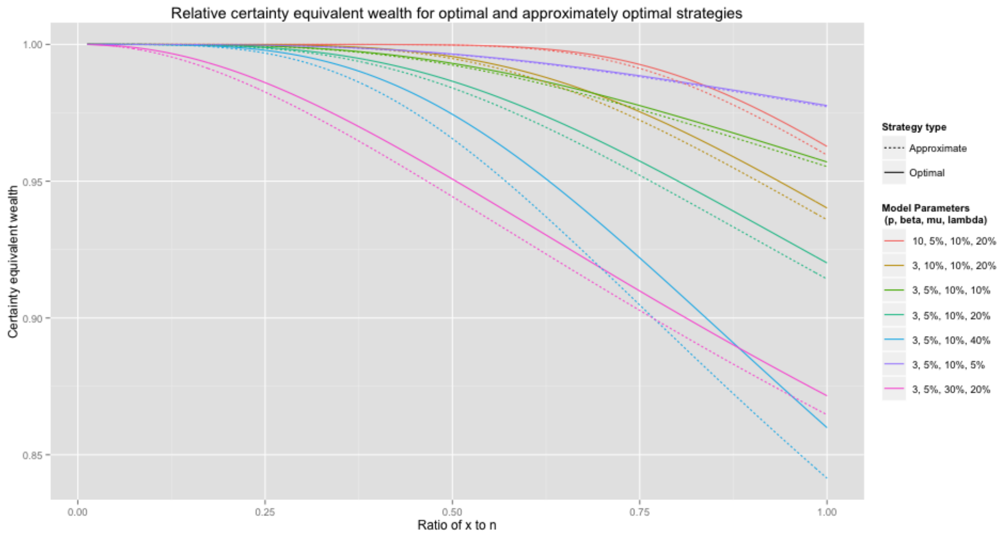

Figure 3.1: Relative certainty equivalent zero-fee initial wealth for the payoffs of the true optimal strategy $(\theta^\lambda, c^\lambda)$ and the approximately-optimal strategy $(\hat{\theta}^\lambda, \hat{c}^\lambda)$. For each choice of model parameters, we take $\sigma = 30\%$.

in [16], once the investor is near the high-water mark, she is willing to sacrifice a small amount of wealth to drive up the high-water mark by over-investing in the short term; she then benefits from the increased high-water mark in the future.

3. On the other hand, the approximately optimal investment and consumption proportion are both typically greater than the optimal proportions. In particular, the approximately optimal consumption strategy $\hat{c}^\lambda(y)$ is typically greater than the Merton proportion for large values of y, but the optimal consumption strategy $c^\lambda(y)$ is not.

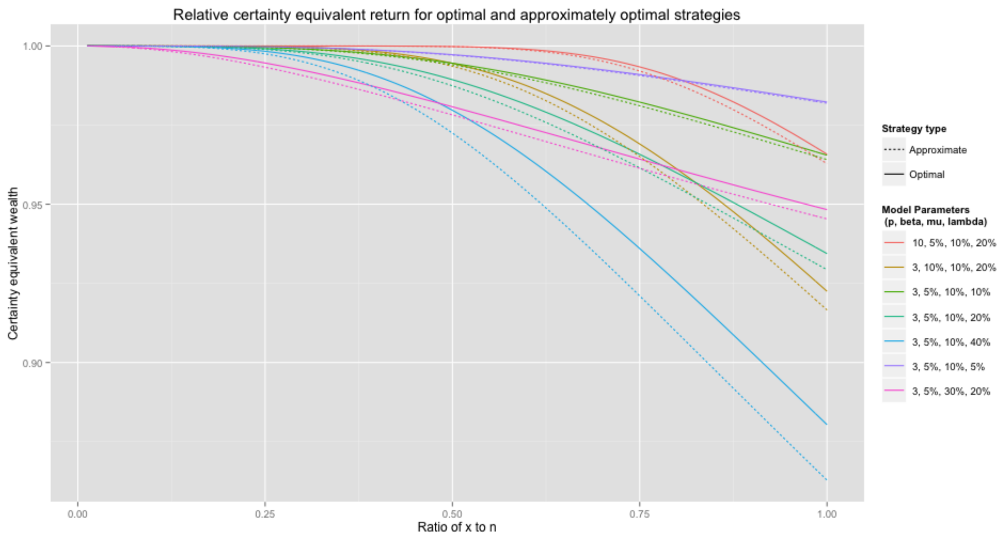

Figure 3.2: Relative certainty equivalent zero-fee rate of return for the payoffs of the true optimal strategy $(\theta^\lambda, c^\lambda)$ and the approximately-optimal strategy $(\hat{\theta}^\lambda, \hat{c}^\lambda)$.

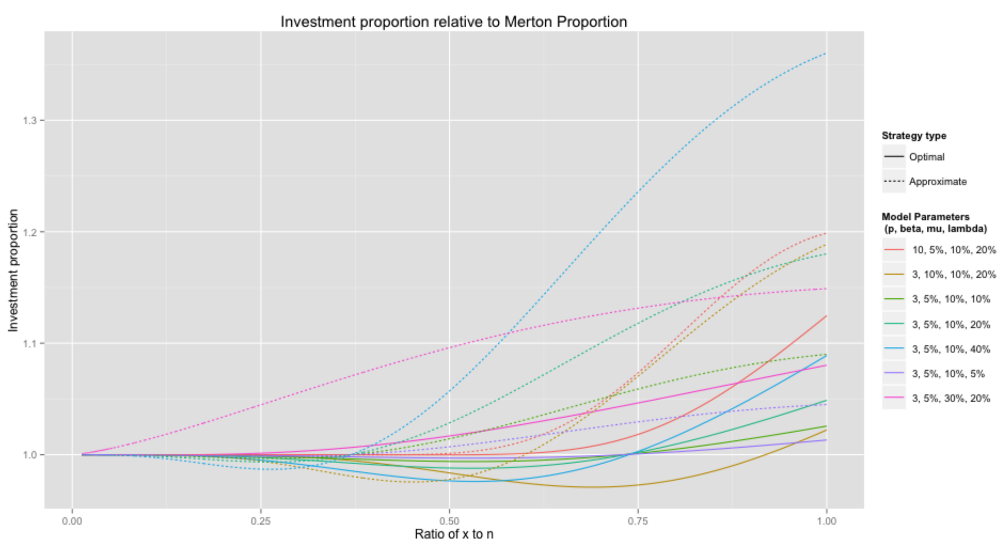

Figure 3.3: Investment proportion relative to the Merton proportion for true optimal strategy $(\theta^\lambda, c^\lambda)$ and the approximately-optimal strategy $(\hat{\theta}^\lambda, \hat{c}^\lambda)$.

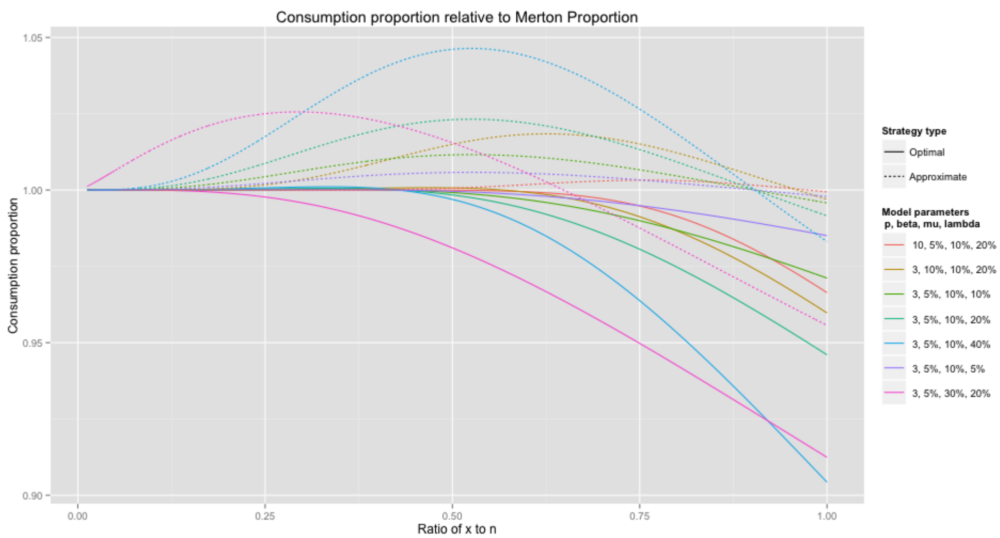

Figure 3.4: Consumption proportion relative to the Merton consumption proportion for the true optimal strategy $(\theta^\lambda, c^\lambda)$ and the approximately-optimal strategy $(\hat{\theta}^\lambda, \hat{c}^\lambda)$.

Appendix

Appendix 1

Appendix

1.1 Proof of Lemma 2.3.2

Lemma. *Let $(\bar{\theta}_t, \bar{c}_t)$ be an admissible strategy for fee level $\lambda = 0$ given in proportions (though not necessarily in feedback form). Suppose that*

$$|\bar{\theta}_t - \theta^0| + |\bar{c} - c^0| < \epsilon \quad dt \times d\mathbb{P}\text{-almost surely.}$$

Let $\bar{X}$ be the wealth process with controls given by $(\bar{\theta}_t, \bar{c}_t)$ and initial wealth x and with no high-watermark fees*. There is a constant M depending on the model parameters μ, σ, β, p such that for $\epsilon > 0$ sufficiently small, we have*

$$\mathbb{E} \int_0^\infty e^{-\beta t} U(\bar{c}_t \bar{X}_t) dt \geq (c^0 - M\epsilon)^{-1} \frac{(c^0 - \epsilon)^{1-p} x^{1-p}}{1-p}$$

Proof. Let $Y = U(\bar{X})$, and use $\mathcal{E}$ to denote the usual stochastic exponential. We readily compute that

$$\begin{aligned} Y_t &= \frac{x^{1-p}}{1-p} \mathcal{E}(L)_t A_t, \\ L_t &\triangleq (1-p) \int_0^t \sigma \bar{\theta}_t dW_u \end{aligned}$$

where

$$
\begin{aligned}
A_t &= \exp\left[(1-p)\int_0^t \mu\bar{\theta}_u - \bar{c}_u - \frac{p}{2}(\sigma\bar{\theta}_u)^2 du\right] \\
&= \exp\left[(1-p)\int_0^t \mu\theta^0 - c^0 - \frac{p}{2}(\sigma\theta^0)^2 du\right] \\
&\quad \cdot \exp\left[(1-p)\int_0^t \mu(\bar{\theta}_u - \theta^0) - \bar{c}_u + c^0 - \frac{p}{2}\sigma^2\left(\hat{\theta}_u^2 - (\theta^0)^2\right) du\right]
\end{aligned}
$$

We compute that

$$
(1-p)(\mu\theta^0 - c^0 - \frac{p}{2}(\sigma\theta)^2) = -c^0 + \beta
$$

so that there exists a constant $M > 0$ depending only on the model parameters such that

$$
e^{(-c^0+\beta-M\epsilon)t} \leq A_t \leq e^{(-c^0+\beta+M\epsilon)t}.
$$

Note that the boundedness conditions on $\bar{\theta}$ imply that $Z \triangleq \mathcal{E}(L)$ is martingale. Since Y is either non-negative or non-positive, we apply Fubini's theorem to see that:

$$
\begin{aligned}
\mathbb{E}\int_0^\infty e^{-\beta t}\bar{c}_t^{1-p}Y_t dt &\geq \int_0^\infty e^{-\beta t}x^{1-p}\frac{(c^0-\epsilon)^{1-p}}{1-p}\mathbb{E}[A_t Z_t]dt \\
&\geq \frac{(c^0-\epsilon)^{1-p}x^{1-p}}{1-p}\int_0^\infty e^{-\beta t}\mathbb{E}[e^{(-c^0+\beta\mp M\epsilon)t}Z_t]dt \\
&= \frac{(c^0-\epsilon)^{1-p}x^{1-p}}{1-p}\int_0^\infty e^{(-c^0\mp M\epsilon)t}dt \\
&= (c^0 \mp M\epsilon)^{-1}\frac{(c^0-\epsilon)^{1-p}x^{1-p}}{1-p}
\end{aligned}
$$

for ϵ sufficiently small. Here the $-$ holds in $\mp$ if $(1-p) > 0$, and the $+$ holds if $(1-p) < 0$. $\square$

1.2 Proof of Theorem 2.3.10, the comparison theorem

In this section we prove a general comparison principle that covers the first- and second-order corrector equations, as well as some other cases arising in the proofs of the boundedness of the relaxed semilimits.

Generalized corrector equation: We say a function $w : [1, \infty) \to \mathbb{R}$ satisfies the *generalized corrector equation* with inhomogeneity $h : [1, \infty) \to \mathbb{R}$, boundary condition η, and limit L if

$$\begin{aligned} \mathcal{A}w + h &= 0 \\ w_y(1) &= \eta \\ \liminf_{y \to \infty} w(y) &= L \end{aligned} \tag{1.1}$$

We define the notions of viscosity sub- and supersolutions to the generalized corrector equation as we did for the first-order corrector equation in Definitions 2.3.3 and 2.3.2. The statement of the comparison result, Theorem 2.3.10, is reproduced below.

Theorem. *Suppose there exists a smooth solution W to the generalized corrector equation for some choice of h, η and L with h continuous. Fix $L \in \mathbb{R}$ and let W^- be an upper semicontinuous viscosity subsolution of the generalized corrector equation with*

$$\limsup_{y \to \infty} W^-(y) \leq L$$

Then we have $W^- \leq W$ on $[1, \infty)$. Similarly, if W^+ is a lower semicontinuous *viscosity supersolution of the first-order corrector equation with*

$$\liminf_{y\to\infty} W^+(y) \geq L$$

then we have $W \leq W^+$.

Remark 1.2.1. Because we assume the existence of a smooth solution W of the generalized corrector equation, the proof of the comparison principle is elementary; in particular, it does not involve the "doubling argument" typically used in viscosity comparison results.

Proof. Suppose for contradiction that $W^- - W$ is positive at some point. Let $f(y) = y^{-q}$ for $q > 0$ small. Since

$$\limsup_{y\to\infty} W^-(y) \leq \lim_{y\to\infty} W(y),$$

the function $W^- - W$ must attain a finite, positive global maximum. Choose $\epsilon > 0$ with

$$\epsilon < \max_{y\in[1,\infty)} W^-(y) - W(y).$$

Then the function $W^- - W - \epsilon f$ must attain a positive global maximum at some $\tilde{y}$, and the test function

$$\phi^\epsilon(y) = W + \epsilon f + (W^-(\tilde{y}) - W(\tilde{y}) - \epsilon f(\tilde{y}))$$

then touches W^- strictly above at $\tilde{y}$. First suppose $\tilde{y} > 1$. Recall from Lemma 2.3.9 that $\mathcal{A}f \leq 0$, so we have

$$\begin{aligned} \mathcal{A}\phi^\epsilon(\tilde{y}) + h(\tilde{y}) &\leq \mathcal{A}W(\tilde{y}) + h(\tilde{y}) - c^0(W^-(\tilde{y}) - W(\tilde{y}) - \epsilon f(\tilde{y})) \\ &< 0 \end{aligned} \tag{1.2}$$

contradicting the subsolution property of W^-. Now suppose $\tilde{y} = 1$. The boundary subsolution property of W^- implies that we must have either $\phi^\epsilon_y(1) \geq \eta$ (impossible because $W_y(1) = \eta$ and $f_y(1) < 0$) or $\mathcal{A}\phi^\epsilon(1) + h(1) \geq 0$ (impossible for the same reasons as when $\tilde{y} > 1$). So we must have $W^- - W - \epsilon f \leq 0$ on $[1, \infty)$, and taking ϵ arbitrarily small, we conclude that we must have $W^- \leq W$. $\square$

1.3 Solution of the first-order corrector equation

The goal of this section is to find an explicit, smooth solution to the first-order corrector equation, proving Proposition 2.3.11.

Lemma 1.3.1. *For a function $f : [1, \infty) \to \mathbb{R}$, define the operator $\mathcal{A}^\alpha$ by*

$$(\mathcal{A}^\alpha f)(y) = \alpha y^2 f''(y) + (y-1)f'(y) - f(y) \tag{1.3}$$

Then $\mathcal{A}^\alpha f = 0$ has the general solution:

$$f(y) = c_1(y-1) + c_2\left((y-1)\Gamma\left(\frac{1}{\alpha}, \frac{1}{\alpha y}\right) + (\alpha y)^{1-\frac{1}{\alpha}} e^{\frac{-1}{\alpha y}}\right) \tag{1.4}$$

Here $\Gamma(s, y)$ denotes the upper incomplete gamma function and the usual gamma function:

$$\Gamma(s, y) \;\; = \;\; \int_y^\infty t^{s-1} e^{-t} dt$$

The solution of Lemma 1.3.1 was obtained using Mathematica, and can easily be checked by hand. To prove Proposition 2.3.11, we now need only choose the constants c_1, c_2 of (1.4) so that the boundary conditions and limiting behavior of the first-order corrector equation are satisfied.

Proof. (of **Proposition 2.3.11**) We begin with a solution W of the general form of (1.4):

$$W(y) = c_1(y-1) + c_2\left((y-1)\Gamma\left(\frac{1}{\alpha}, \frac{1}{\alpha y}\right) + (\alpha y)^{1-\frac{1}{\alpha}} e^{\frac{-1}{\alpha y}}\right)$$

In order for W to satisfy the limiting behavior

$$\lim_{y\to\infty} W(y) = 0$$

we expect that we should have $c_1 = -C\Gamma(1/\alpha)$ and $c_2 = C$, for some constant C. We therefore set

$$W(y) = C\left((y-1)\left[\Gamma\left(\frac{1}{\alpha}, \frac{1}{\alpha y}\right) - \Gamma\left(\frac{1}{\alpha}\right)\right] + (\alpha y)^{1-\frac{1}{\alpha}} e^{-\frac{1}{\alpha y}}\right). \qquad (1.5)$$

First we check that we indeed have $W(y) \to 0$ as $y \to \infty$. We consider two cases. If $\alpha < 1$, then clearly $(\alpha y)^{1-\frac{1}{\alpha}} e^{-\frac{1}{\alpha y}} \to 0$ as $y \to \infty$. Also,

$$\begin{aligned}
(y-1)\left[\Gamma\left(\frac{1}{\alpha}, \frac{1}{\alpha y}\right) - \Gamma\left(\frac{1}{\alpha}\right)\right] &= -(y-1)\int_0^{\frac{1}{\alpha y}} t^{\frac{1}{\alpha}-1} e^{-t}\,dt \\
&\geq -(y-1)\left[\alpha t^{\frac{1}{\alpha}}\right]_0^{\frac{1}{\alpha y}} \qquad (1.6)\\
&= -\alpha^{1-\frac{1}{\alpha}}(y-1)y^{-\frac{1}{\alpha}}
\end{aligned}$$

so we conclude that $W(y) \to 0$ as $y \to \infty$. In the case that $\alpha = 1$, we have

$$\begin{aligned}
(y-1)\left[\Gamma\left(\frac{1}{\alpha}, \frac{1}{\alpha y}\right) - \Gamma\left(\frac{1}{\alpha}\right)\right] &= -(y-1)\int_0^{\frac{1}{y}} e^{-t}\,dt \\
&= -(y-1)(1-e^{-\frac{1}{y}})
\end{aligned}$$

Note that

$$\lim_{y\to\infty} (y-1)(1-e^{-\frac{1}{y}}) = 1,$$

so that

$$W(y) \;\; = \;\; C\left[-(y-1)(1-e^{-\frac{1}{y}})+e^{-\frac{1}{y}}\right] \to 0 \text{ as } y \to \infty.$$

Now suppose $\alpha > 1$. As in (1.6), we have the lower bound

$$(y-1)\left[\Gamma\left(\frac{1}{\alpha},\frac{1}{\alpha y}\right)-\Gamma\left(\frac{1}{\alpha}\right)\right] \geq -\alpha^{1-\frac{1}{\alpha}}(y-1)y^{-\frac{1}{\alpha}}$$

On the other hand,

$$\begin{aligned}(y-1)\left[\Gamma\left(\frac{1}{\alpha},\frac{1}{\alpha y}\right)-\Gamma\left(\frac{1}{\alpha}\right)\right] &= -(y-1)\int_0^{\frac{1}{\alpha y}} t^{\frac{1}{\alpha}-1}e^{-t}dt \\ &< -(y-1)e^{-\frac{1}{\alpha y}}\left[\alpha t^{\frac{1}{\alpha}}\right]_0^{\frac{1}{\alpha y}} \qquad (1.7) \\ &= -\alpha^{1-\frac{1}{\alpha}}e^{-\frac{1}{\alpha y}}(y-1)y^{-\frac{1}{\alpha}} \\ &\quad \to 0 \text{ as } y \to \infty\end{aligned}$$

As a result,

$$|W(y)| \;\; \leq \;\; C\alpha^{1-\frac{1}{\alpha}}\left(|(y-1)y^{-\frac{1}{\alpha}}-y^{1-\frac{1}{\alpha}}e^{-\frac{1}{\alpha y}}|+|(y-1)y^{-\frac{1}{\alpha}}e^{-\frac{1}{\alpha y}}-y^{1-\frac{1}{\alpha}}e^{-\frac{1}{\alpha y}}|\right)$$

Another straightforward application of l'Hôpital's rule shows that both terms on the right hand side go to 0 as $y \to \infty$.

It remains to show that $W_y(1) \neq 0$ if $C \neq 0$, so that we may choose C such that W satisfies the boundary condition of the first-order corrector equation. If $\alpha = 1$, we can compute directly that

$$W_y(1)/C = 1 + 2e^{-1} < 0$$

Now suppose that $\alpha \neq 1$. We compute that

$$\begin{aligned} W_y(1)/C &= -\int_0^{\frac{1}{\alpha}} t^{\frac{1}{\alpha}-1} e^{-t} dt + \alpha^{1-\frac{1}{\alpha}} e^{-\frac{1}{\alpha}} \\ &< -e^{-\frac{1}{\alpha}} \int_0^{\frac{1}{\alpha}} t^{\frac{1}{\alpha}-1} dt + \alpha^{1-\frac{1}{\alpha}} e^{-\frac{1}{\alpha}} \\ &= -\alpha^{1-\frac{1}{\alpha}} e^{-\frac{1}{\alpha}} + \alpha^{1-\frac{1}{\alpha}} e^{-\frac{1}{\alpha}} = 0 \end{aligned}$$

where the bound is the same used in (1.7). □

Recall from Theorem 2.3.12 that we then have $w^1 = \underline{w}^1 = \overline{w}^1$, and w^1 is equal to the W of Proposition 2.3.11. The following lemma concerning the decay of w^1 will be used to exhibit a smooth solution of the second-order corrector equation.

Lemma 1.3.2. *Fix $\epsilon > 0$. Let g be as in (2.107). Then the limits*

$$\lim_{y\to\infty} y^{\epsilon} w^1(y), \ \lim_{y\to\infty} y^{2\epsilon} g(y) \tag{1.8}$$

are well-defined, and are finite exactly when $\epsilon \leq 1/\alpha$ and nonzero exactly when $\epsilon = 1/\alpha$.

Proof. Fix $\epsilon > 0$. We'll show that

$$\lim_{y\to\infty} (y-1)^{\epsilon} w^1(y)$$

is finite exactly when $\epsilon \leq 1/\alpha$. Now, as a consequence of Theorem 2.3.12 and Proposition 2.3.11, we have

$$w^1(y) = C\left((y-1)\left[\Gamma\left(\frac{1}{\alpha}, \frac{1}{\alpha y}\right) - \Gamma\left(\frac{1}{\alpha}\right)\right] + (\alpha y)^{1-\frac{1}{\alpha}} e^{-\frac{1}{\alpha y}}\right)$$

It will suffice to show the limit

$$\lim_{y\to\infty}(y-1)^{1+\epsilon}\left[\frac{w^1(y)}{(y-1)}\right]$$

well-defined and is finite exactly when $\epsilon \le 1/\alpha$. We have

$$\lim_{y\to\infty}(y-1)^{1+\epsilon}\left(\frac{y^{1-\frac{1}{\alpha}}e^{-\frac{1}{\alpha y}}}{y-1}-y^{-\frac{1}{\alpha}}e^{-\frac{1}{\alpha y}}\right)=0 \tag{1.9}$$

Therefore, it is enough to show the limit

$$\lim_{y\to\infty}(y-1)^{1+\epsilon}\left(\left[\Gamma\left(\frac{1}{\alpha},\frac{1}{\alpha y}\right)-\Gamma\left(\frac{1}{\alpha}\right)\right]+\alpha^{1-\frac{1}{\alpha}}y^{-\frac{1}{\alpha}}e^{-\frac{1}{\alpha y}}\right)$$

is well-defined and is finite exactly when $\epsilon < 1/\alpha$. Applying l'Hôpital's rule, we instead consider

$$\lim_{y\to\infty}(y-1)^{2+\epsilon}\frac{d}{dy}\left(\left[\Gamma\left(\frac{1}{\alpha},\frac{1}{\alpha y}\right)-\Gamma\left(\frac{1}{\alpha}\right)\right]+(\alpha y)^{-\frac{1}{\alpha}}e^{-\frac{1}{\alpha y}}\right) \tag{1.10}$$

Computing derivatives, we have

$$\begin{aligned}(1.10) \quad &= \lim_{y\to\infty}(y-1)^{2+\epsilon}\left(\alpha^{-1}y^{-1-\frac{1}{\alpha}}+\alpha^{-1-\frac{1}{\alpha}}y^{-2-\frac{1}{\alpha}}-\alpha^{-1}y^{-1-\frac{1}{\alpha}}\right)e^{-\frac{1}{\alpha y}}\\ &= \lim_{y\to\infty}(y-1)^{2+\epsilon}\left(\alpha^{-\frac{1}{\alpha}}y^{-2-\frac{1}{\alpha}}\right)e^{-\frac{1}{\alpha y}},\end{aligned}$$

which is well-defined, is finite exactly when $\epsilon \le 1/\alpha$, and is equal to 0 if $\epsilon < 1/\alpha$.

To see that the limit

$$\lim_{y\to\infty}y^{2\epsilon}g(y)$$

is well-defined and finite exactly when $\epsilon \le 1/\alpha$, note that g is linear combination of two-fold products of w^1, yw^1_y and $y^2w^1_{yy}$ – that is, of $(w^1)^2, w^1yw^1_y, w^1y^2w_{yy}, \ldots$. In light of the relation

$$\alpha y^2w^1_{yy}+(y-1)w^1_y-w^1=0$$

coming from the first-order corrector equation, it will suffice to show that the limit

$$L = \lim_{y\to\infty} y^{1+\epsilon} w_y^1(y)$$

is well-defined and is finite exactly when $\epsilon \leq 1/\alpha$. Equivalently, we prove

$$L' = \lim_{y\to\infty} y^{1+\epsilon} h(y)$$

is well-defined and is finite exactly when $\epsilon \leq 1/\alpha$, where $h(y) = \delta y^{\epsilon} w^1(y) + y^{1+\epsilon} w_y^1(y)$. First we claim that L is well-defined. If not, then there exist $y_k \to \infty$ such that h has a local maximum or minimum at y_k and the $h(y_k)$ are bounded away from 0. Then we have

$$\begin{aligned} y_k h_y(y_k) &= \delta\epsilon y^{\epsilon} w^1(y_k) + (1+\epsilon+\delta) y_k^{1+\epsilon} w_y^1(y_k) + y_k^{2+\epsilon} w_{yy}^1(y_k) \\ &= 0 \end{aligned}$$

Since $\mathcal{A}^{\alpha} w^1 = 0$, we have

$$\begin{aligned} 0 = \alpha y_k h_y(y_k) &= \alpha y_k h_y(y_k) - y_k^{\epsilon} \mathcal{A} w^1(y_k) \\ &= (\alpha\delta\epsilon + 1) y_k^{\epsilon} w^1(y_k) + (\alpha(1+\epsilon+\delta) - 1)\, y_k^{1+\epsilon} \epsilon w_y^1(y_k) \\ &\quad + y_k^{\epsilon} w_y^1(y_k). \end{aligned}$$

Choosing δ so that $\alpha(1+\epsilon+\delta)-1 \neq 0$ and recalling that $y^{\epsilon} w^1(y) \to 0$ as $y \to \infty$, we conclude that $y_k^{1+\epsilon} w_y^1(y_k) \to 0$ as $k \to \infty$, a contradiction. Therefore, the limits L and L' are well-defined. By l'Hôpital's rule, we therefore have

$$\lim_{y\to\infty} y^{\epsilon} w^1 = \lim_{y\to\infty} -\epsilon y^{1+\epsilon} w_y^1.$$

In particular, L and L' are finite (resp. equal to 0) exactly when $\lim_{y\to\infty} y^{\epsilon} w_y^1$ is finite (resp. equal to 0). This completes the proof. $\square$

1.4 Solution of the second-order corrector equation

We now give the proof of Proposition 2.4.11, which we reproduce below.

Proposition. *There is a smooth solution $W : [1,\infty) \to \mathbb{R}$ of the second-order corrector equation*

$$\begin{aligned} \mathcal{A}W + g &= 0 \text{ on } (1,\infty) \\ W_y(1) - 2w_x^1(1) &= 0 \\ \lim_{y\to\infty} W(y) &= 0 \end{aligned}$$

has a smooth solution.

Proof. Suppose without loss of generality that $c^0 = 1$ and set $\alpha = \frac{1}{2}(\sigma\theta^0)^2$. It will suffice to exhibit a smooth solution f to the inhomogeneous equation $\mathcal{A}^\alpha f + g = 0$ with the appropriate limiting behavior; we may then take $W = f + C_2 w^1$ where $C_2 = \frac{2w_x^1(1) - f_y(1)}{w_y^1(1)}$. Suppose that $f(y) = w^1(y)\nu(y)$. Applying $\mathcal{A}$, we must have:

$$\alpha y^2 w^1 \eta_y + (2\alpha y^2 w_y^1 + (y-1)w^1)\eta = -g \tag{1.11}$$

where $\eta = \nu_y$. Solving formally, we guess that

$$\eta(y) = -e^{-\int \psi dy} \int e^{\int \psi dy} \frac{g(y)}{\alpha y^2 w^1} dy.$$

where

$$\psi = \frac{(2\alpha y^2 w_y^1 + (y-1)w^1)}{\alpha y^2 w^1}$$

We therefore try the solution

$$\nu(y) = \int_y^\infty e^{-\int_1^t \psi(s)ds} \left(\int_t^\infty e^{\int_1^s \psi(u)du} \frac{g(s)}{\alpha s^2 w^1(s)} ds \right) dt. \tag{1.12}$$

First we check that the inner integral is finite. Observe that

$$\begin{aligned} \psi &= 2\frac{w^1_y}{w^1} + \left(\frac{1}{\alpha}\right)\frac{y-1}{y^2} \\ &= 2\frac{d}{dy}\log(-w^1) + \left(\frac{1}{\alpha}\right)\frac{y-1}{y^2} \\ &\leq 2\frac{d}{dy}\log(-w^1) + \frac{1}{\alpha y} \end{aligned} \tag{1.13}$$

We therefore have

$$\begin{aligned} e^{\int_1^s \psi(u)du} &\leq e^{2\log(-w^1(s)) - 2\log(-w^1(1)) + \frac{1}{\alpha}\log(s)} \\ &\leq \left(\frac{w^1(s)}{w^1(1)}\right)^2 s^{\frac{1}{\alpha}} \\ &\leq K_0 s^{-\frac{1}{\alpha}} \end{aligned} \tag{1.14}$$

for some large constant K_0, by Lemma 1.3.2. So to bound the inner integral of (1.12), we note that

$$\begin{aligned} \int_t^\infty \left| e^{\int_1^s \psi(u)du} \frac{g(s)}{\alpha s^2 w^1} \right| ds &\leq K_0 \int_t^\infty \left| \frac{g(s)}{\alpha s^{2+\frac{1}{\alpha}} w^1(s)} \right| ds \\ &\leq K_0 \int_t^\infty s^{-2-\frac{2}{\alpha}} ds \qquad (1.15) \\ &= K_0 t^{-1-\frac{2}{\alpha}}, \end{aligned}$$

where (1.15) follows from Lemma 1.3.2 after possibly enlarging the constant K_0. To see that ν is finite and

$$\lim_{y\to\infty} |w^1(y)\nu(y)| = 0,$$

first note that, by similar arguments to those of (1.14), we have also have the reverse bound

$$e^{-\int_1^t \psi(u)du} \leq K_0 t^{\frac{1}{\alpha}}$$

after again possibly enlarging K_0. Therefore, we have

$$\begin{aligned} |w^1(y)\nu(y)| &\leq K_0^2 \left| w^1(y) \int_y^\infty t^{\frac{1}{\alpha}} t^{-1-\frac{2}{\alpha}} dt \right| \\ &\leq K_0^2 |w^1(y) y^{-\frac{1}{\alpha}}| \\ &\leq K_0^2 y^{\frac{-2}{\alpha}} \end{aligned}$$

where the last line Lemma 1.3.2, again after possibly enlarging K_0. By construction, $f = w^1\nu$ is C^2 and satisfies $\mathcal{A}^\alpha f + g = 0$, which completes the proof. □

Proposition 2.4.11, combined with the general comparison result Theorem 1.2, allows us to conclude that there is a well-defined second-derivative $w^2 = \underline{w}^2 = \overline{w}^2$. Moreover, w^2 is equal to the W of the proof of Proposition 2.4.11.

1.5 Proof of Proposition 2.5.1

The goal of this section is to show that the payoffs $\tilde{w}^\lambda$ and $\hat{w}^\lambda$ of the suboptimal strategies (θ^0, c^0) and $(\hat{\theta}^\lambda, \hat{c}^\lambda)$ are smooth solutions of appropriate PDEs and satisfy certain growth conditions. We will only prove the results concerning $\hat{w}^\lambda$, since the analogous results for $\tilde{w}^\lambda$ can be obtained by simpler versions of the same arguments. To begin with, we recall the following result on admissibility of feedback strategies (see Propositions 5.2 and 5.3 of [16]).

Proposition 1.5.1. *Let θ and c be two functions on $[1,\infty)$ with the property that*

$$\begin{aligned}\Theta(x,n) &= x\theta(x,n)\\ \gamma(x,n) &= xc(x,n)\end{aligned}$$

are globally Lipschitz functions on D. Then the feedback strategy $(\Theta(x,n),\gamma(x,n))$ given in dollar amounts *gives rise to an admissible strategy for any initial condition $(x,n)\in D$. Equivalently, the feedback proportions $(\theta(x,n),c(x,n))$ give rise to an admissible strategy for any $(x,n)\in D$.*

Lemma 1.5.2. *The feedback proportions $(\hat{\theta}^{\lambda}(x,n),\hat{c}^{\lambda}(x,n))$ give rise to an admissible strategy for any initial condition $(x,n)\in D$.*

Proof. By Proposition 1.5.1, it will suffice to show that the functions

$$\begin{aligned}\Theta(x,n) &= x\hat{\theta}^{\lambda}(x,n)\\ \gamma(x,n) &= x\hat{c}^{\lambda}(x,n)\end{aligned}$$

are globally Lipschitz functions on D. Recalling the expressions for θ^1 and c^1 of (2.106), we have

$$\begin{aligned}\hat{\theta}^{\lambda}(x,n) &= \theta^0+\lambda\theta^1(y)\\ \hat{c}^{\lambda}(x,n) &= c^0+\lambda c^1(y)\end{aligned}$$

where $y=n/x$. Therefore,

$$\begin{aligned}\Theta_x(x,n) &= \theta^0+\lambda\left(\theta^1(y)-y\theta^1(y)\right)\\ \Theta_n(x,n) &= \lambda\theta^1_y \qquad (1.16)\end{aligned}$$

and similarly for γ. As a result, it is enough to show that $y\theta^1_y(y)$ and $yc^1_y(y)$ are bounded on $[1,\infty)$. Recall from Corollary 2.3.4 that $w^1(y), yw^1_y(y),$ and $y^2w^1_{yy}(y)$ are bounded. Taking derivatives with respect to y of the equation $\mathcal{A}w^1 = 0$, we see that $y^3w^1_{yyy}$ is bounded as well. We may then differentiate the explicit expressions

$$\begin{aligned}\theta^1 &= -\left(\frac{\mu}{\sigma^2}\right)\frac{w^0_{xx}w^1_x - w^0_x w^1_{xx}}{(w^0_{xx})^2}\\ c^1 &= I'(w^0_x)(w^1_x + w^1_y)\end{aligned}$$

to conclude that $y\theta^1_y$ and yc^1_y are bounded as well. □

Lemma 1.5.3. *For λ sufficiently small, we have:*

$$\begin{aligned}\hat{v}^\lambda(x,n) &> -\infty, \quad n \geq x > 0\\ \lim_{n\to\infty}\hat{v}^\lambda(x,n) &= v^0(x)\end{aligned}$$

Proof. Recall from Remark 2.4.2 that θ^1 and c^1 are bounded. We repeat the arguments of Proposition 2.3.3. Let $(\hat{X}^{\lambda,z}, \hat{N}^{\lambda,z})$ be the state process determined by the feedback proportions $(\hat{\theta}^\lambda, \hat{c}^\lambda)$ and the initial condition $z = (x,n)$. We apply the bounds of Remark 2.3.2 to see that

$$\hat{X}^{\lambda,z}_t \geq \frac{n^\lambda \hat{X}^{0,z}}{\left(\exp[\int_0^t \bar{c}(\hat{X}^{\lambda,z}_u, \hat{N}^{\lambda,z}_u)du]\hat{H}^z_t\right)^\lambda}$$

$$\hat{H}^z_t \triangleq n^{-1}[(\hat{X}^{0,z} + \hat{C}^{0,z}) \vee n]^*, \quad \hat{C}^{0,z}_t \triangleq \int_0^t \hat{c}^\lambda(\hat{X}^{\lambda,z}_u, \hat{N}^{\lambda,z}_u)\hat{X}^{0,z}_u du$$

where $(\hat{X}^{0,z}, \hat{N}^{0,z})$ denotes the state process determined by the controls

$$(\overline{\theta}_t^\lambda, \overline{c}_t^\lambda) \triangleq (\hat{\theta}^\lambda(\hat{X}^{\lambda,z}, \hat{N}^{\lambda,z}), \hat{c}^\lambda(\hat{X}^{\lambda,z}, \hat{N}^{\lambda,z}))$$

at fee level 0. Therefore,

$$\hat{v}(x,n) \geq \mathbb{E}\left[\int_0^\infty e^{-\beta t} U\left(\frac{n^\lambda \bar{c}_t^\lambda \hat{X}_t^{0,z}}{\left(\exp[\int_0^t \bar{c}_u^\lambda du]\hat{H}_t^z\right)^\lambda}\right) dt\right] \tag{1.17}$$

Let $\epsilon_0 > 0$. Since θ^1 and c^1 are bounded, we have

$$|\bar{\theta}_t^\lambda - \theta^0| + |\bar{c}_t^\lambda - c^0| < \epsilon_0$$

for all t, for λ sufficiently small. We now apply a Hölder argument to check that the right-hand side of (1.17) is finite. Pick $q > 1$, and let q' be its Hölder conjugate. For q sufficiently close to 1 and $\delta > 0$ sufficiently small, we can apply Lemma 2.3.2 and arguments similar to the proof of Proposition 2.3.3 to check that for all λ sufficiently small, we have

$$\mathbb{E}\int_0^\infty e^{-\beta(1-\delta)qt}|U(\bar{c}_t^\lambda \hat{X}_t^{0,z})|^q dt < u^0(x) - \epsilon(\beta,\delta,\mu,\sigma,\lambda,q') > -\infty$$

where u^0 is the value function of the Merton optimal investment/consumption problem with modified utility function $V(x) = sign(1-p)|U(x)|^q$ and modified discount factor $\beta(1-\delta)q'$, and we have $\epsilon \to 0$ as $\lambda \to 0$. Note that this bound (along with the choice of sufficiently small λ) is independent of n for fixed x.

It remains to show that for λ sufficiently small, the quantity

$$A^{\lambda,z} \triangleq \mathbb{E}\int_0^\infty e^{-\beta\delta q' t}\left|U\left(e^{\int_0^t \bar{c}_t^\lambda du}\hat{H}_t^z\right)\right|^{-\lambda q'} dt$$

is finite (in fact, we will show $A^{\lambda,z}$ is uniformly bounded for all $n \geq x$, for λ sufficiently small). Recalling that $|\bar{c}^\lambda - c^0| \leq \epsilon_0$ for λ small, we argue as in the proof of Proposition 2.3.3 to obtain the bound

$$\hat{H}_t^z \leq n^{-1} \sup_{0 \leq s \leq t} [(1 + (c^0 + \epsilon))s\hat{X}_s^{0,z} \vee n] \tag{1.18}$$

Now recalling the boundedness of $\bar{\theta}^\lambda$ and $\bar{c}^\lambda$, we have

$$\begin{aligned} (1 + (c^0 + \epsilon))t\hat{X}_t^{0,z} &= x\mathcal{E}(L)_t \\ L_t &= \int_0^t a_u^\lambda du + b_u^\lambda dW_u \end{aligned}$$

where a^λ and b^λ are predictable processes uniformly bounded in (t, ω) for small λ. Then for $\rho = -(1-p)q'\lambda/2$ we have

$$\begin{aligned} \left|(1-p)U\left(e^{\int_0^t \bar{c}_u^\lambda du}\hat{H}_t^z\right)\right|^{-\lambda q'} &= \left|e^{\int_0^t \bar{c}_u^\lambda du}\hat{H}_t^z\right|^{2\rho} \\ &\leq n^{-1}e^{2\rho\int_0^t \bar{c}_u^\lambda du} \sup_{0 \leq s \leq t} [n \vee x\mathcal{E}(L)_s]^{2\rho} \\ &\leq e^{(|\rho|+\rho^2)\bar{a}t} \sup_{0 \leq s \leq t} \left[1 \vee (x/n)^\rho \mathcal{E}(\bar{L})_s^2\right] \end{aligned}$$

where $\bar{a}$ is a large constant independent of ρ and n, and $\bar{L}_t = \rho\int_0^t \bar{b}_u dW_u$ for some uniformly bounded, predictable process $\bar{b}$. By Doob's maximal inequality,

$$\begin{aligned} \mathbb{E}\left[\sup_{0 \leq s \leq t} \mathcal{E}(\bar{L})_s^2\right] &\leq 4\mathbb{E}\left[\mathcal{E}(\bar{L})_t^2\right] \\ &\leq 4\mathbb{E}\left[e^{(|\rho|+\rho^2)\bar{a}t}\right] \end{aligned}$$

after possibly enlarging $\bar{a}$. We conclude that there exists a large constant K independent of n and λ such that, for λ (and hence ρ) sufficiently small, we have

$$\left|(1-p)U\left(e^{\int_0^t \bar{c}_u^\lambda du}\hat{H}_t^z\right)\right|^{-\lambda q'} \leq Ke^{\bar{a}(|\rho|+\rho^2)t} \tag{1.19}$$

Therefore, for λ sufficiently small that $\bar{a}(|\rho| + \rho^2) < -\beta\delta q'$, we have

$$0 \leq A^{\lambda,z} \leq \frac{K}{\beta\delta q' - \bar{a}(|\rho| + \rho^2)}.$$

We conclude that $\hat{v}^\lambda(x, n)$ is uniformly bounded below for all $n \geq x$.

We now show

$$\lim_{n\to\infty} \hat{v}^\lambda(x, n) = v^0(x) \tag{1.20}$$

We'll assume without loss of generality that $x = 1$ and fix λ small. Now the above arguments can be repeated to show that the family

$$\mathcal{X}^\lambda = \{U(\bar{c}_t^\lambda \hat{X}_t^{\lambda,z}) : z = (1, n), n \in [1, \infty)\}$$

is bounded in $\mathbb{L}^{1+\kappa}([0, \infty) \times \Omega, \beta e^{-\beta t} dt \times d\mathbb{P})$ for κ sufficiently small. To see this, note that we have already shown $\mathcal{X}^\lambda$ is bounded in $\mathbb{L}^1([0, \infty) \times \Omega, \beta e^{-\beta t} dt \times d\mathbb{P})$; this is just another way of saying that $\hat{w}^\lambda(1, n)$ is uniformly bounded above (by $v^0(1)$) and below (as a result of the arguments given above). For κ sufficiently small, we may simply repeat the same arguments after possibly modifying q'. Since $\mathcal{X}^\lambda$ is bounded in $\mathbb{L}^{1+\kappa}([0, \infty) \times \Omega, \beta e^{-\beta t} dt \times d\mathbb{P})$, it is uniformly integrable on $([0, \infty) \times \Omega, \beta e^{-\beta t} dt \times d\mathbb{P})$. We now claim that

$$U(\hat{c}^\lambda(\hat{X}_\cdot^{\lambda,z}, \hat{N}_\cdot^{\lambda,z})\hat{X}_\cdot^{\lambda,z}) \to U(c^0 X_\cdot^{0,z})$$

in measure (with respect to the product measure $\beta e^{-\beta t} dt \times d\mathbb{P}$) as $n \to \infty$. Since the family $\mathcal{X}^\lambda$ is uniformly integrable, this will imply (1.20). Because $\hat{c}^\lambda \to c^0$ uniformly, it will suffice to show that

$$|\hat{X}_\cdot^{\lambda,z} - X_\cdot^{0,z}| \to 0$$

in measure as $n \to \infty$ with respect to the product measure $\beta e^{-\beta t} dt \times d\mathbb{P}$. In fact, it will be enough to show that there is an increasing sequence of stopping times $\tau_k \to \infty$ such that

$$|\hat{X}^{\lambda,z}_{\cdot\wedge\tau_k} - X^{0,z}_{\cdot\wedge\tau_k}| \to 0$$

in measure on $[0, \tau_k) \times \Omega$ as $n \to \infty$. Pick any sequence $n_k \to \infty$. Let

$$\tau_k = \inf\{t : \hat{X}^{\lambda,(1,n_k)}_t = \sqrt{n_k}\}$$

Of course, before time τ_k, we have $\hat{X}^{\lambda,(1,n_k)}_t = \hat{X}^{0,(1,n_k)}_t$ as no fees have been incurred. Note also that if $l \geq k$, then $\hat{X}^{\lambda,(1,n_l)}_t = \hat{X}^{\lambda,(1,n_k)}_t$ before time τ_k. We have $\tau_k \to \infty$ as $k \to \infty$, and upto time τ_k, we have

$$\hat{N}^{\lambda,(1,n_k)}_t / \hat{X}^{\lambda,(1,n_k)}_t \geq \sqrt{n_k}$$

Recalling that

$$\lim_{n\to\infty} \theta^1(x,n) = \lim_{n\to\infty} c^1(x,n) = 0,$$

we have

$$|\hat{\theta}^\lambda(\hat{X}^{\lambda,(1,n_k)}_{\cdot\wedge\tau_k}, \hat{N}^{\lambda,(1,n_k)}_{\cdot\wedge\tau_k}) - \theta^0| + |\hat{c}^\lambda(\hat{X}^{\lambda,(1,n_k)}_{\cdot\wedge\tau_k}, \hat{N}^{\lambda,(1,n_k)}_{\cdot\wedge\tau_k}) - c^0| < \epsilon_k$$

where $\epsilon_k \to 0$. It is therefore easy to see that for any fixed k_0, we have

$$\left|\hat{X}^{\lambda,(1,n_k)}_{\cdot\wedge\tau_{k_0}} - X^{0,(1,n_k)}_{\cdot\wedge\tau_{k_0}}\right| \to 0 \text{ as } k \to \infty$$

in measure on $[0, \tau_{k_0}) \times \Omega$ (the strategies which determine $\hat{X}^{(\lambda,n_k)}_{\cdot\wedge\tau_{k_0}}$ converge to uniformly to the Merton proportions on $[0, \tau_{k_0}) \times \Omega$ as $k \to \infty$, and fees are not present in the dynamics of $\hat{X}^{(\lambda,n_k)}_{\cdot\wedge\tau_{k_0}}$), which completes the proof.

□

In order to show that $\hat{w}^\lambda$ is the solution of the appropriate PDE, we will need the following analogue of the dynamic programming principle; since the controls corresponding to $\hat{w}^\lambda$ are in feedback form, a rigorous proof is tractable, and amounts to the strong Markov property for reflected diffusions.

Proposition 1.5.4. *Let $(\hat{X}, \hat{N}) = (\hat{X}^{\lambda,z}, \hat{N}^{\lambda,z})$ be the state process corresponding to using the feedback control $(\hat{\theta}^\lambda, \hat{c}^\lambda)$ with initial condition $z = (x, n)$ and fee level λ. Set $\hat{Y} = \hat{N}/\hat{X}$, and define*

$$Z_t^z = \int_0^t e^{-\beta u} U(\hat{c}^\lambda(\hat{X}_u, \hat{N}_u)\hat{X}_u) du + e^{-\beta t} X_t^{1-p} \hat{w}^\lambda(\hat{Y}_t) \tag{1.21}$$

Then Z^z is a local martingale.

Proof. We've seen that for any initial condition z, the equation defining $(\hat{X}, \hat{N})$ has a pathwise unique strong solution (this follows from Lemma 1.5.2). Because the control $(\hat{\theta}^\lambda, \hat{c}^\lambda)$ is in feedback form, we may adapt the arguments of Stroock and Varadhan [25] to see that a corresponding martingale problem for the reflected diffusion $(\hat{X}, \hat{N})$ is well-posed, and that as a result, a strong Markov property holds for $(\hat{X}, \hat{N})$. It then follows by standard arguments that Z^z is a local martingale. Note that the arguments of [25] do not deal with reflection and therefore cannot be applied directly.

□

Lemma 1.5.5. *Fix $a \in [1, \infty)$. For $b > a$ arbitrarily large, there is a C^2*

solution $s^{a,b}$ to the scale equation

$$\begin{aligned} \mathcal{L}^{\hat{\theta}^\lambda,\hat{c}^\lambda} s^{a,b} &= 0 \\ s^{a,b}(a) &= \hat{w}^\lambda(a) \\ s^{a,b}(b) &= \hat{w}^\lambda(b). \end{aligned} \tag{1.22}$$

Proof. Let $\bar{\mathcal{L}}$ be the *homogeneous* version of the operator $\mathcal{L}^{\hat{\theta}^\lambda,\hat{c}^\lambda}$; obtained from $\mathcal{L}^{\hat{\theta}^\lambda,\hat{c}^\lambda}$ by subtracting an appropriate constant. The coefficients of both $\bar{\mathcal{L}}$ and $\mathcal{L}^{\hat{\theta}^\lambda,\hat{c}^\lambda}$ are continuous, From general results on second-order, linear ODEs, for any b there is a solution s^b to

$$\begin{gathered} \mathcal{L}^{\hat{\theta}^\lambda,\hat{c}^\lambda} s^b = 0 \text{ on } [a,b] \\ s^b(a) = \hat{w}^\lambda(a) \end{gathered} \tag{1.23}$$

Let h be a solution of the initial value problem

$$\begin{aligned} \bar{\mathcal{L}} h^b &= 0 \text{ on } [a,b] \\ h(a) &= 0 \\ h_y(a) &= \eta \end{aligned} \tag{1.24}$$

for some $\eta \neq 0$. Note that there are b arbitrarily large such that $h(b) \neq 0$; if not, then there exists a b such that $h(b) = h_y(b) = 0$, and uniqueness for problem

$$\begin{aligned} \bar{\mathcal{L}} h &= 0 \text{ on } [a,b] \\ h(b) &= 0 \\ h_y(b) &= 0 \end{aligned}$$

implies that $h = 0$ on $[a, b]$, contradicting $h_y(a) = 0$. For those b at which $h(b) \neq 0$, we may therefore take

$$s^{a,b} = s^b + K^b h$$

for an appropriate choice of constant K^b. $\square$

Lemma 1.5.6. *Let $\hat{X}, \hat{N}, \hat{Y}$ be as in Proposition 1.5.4. Pick $a, b \in [1, \infty)$ such that $n/x \in [a, b]$ and there exists $s^{a,b}$ as in Lemma 1.5.5. Define the stopping time*

$$\tau = \inf\{t > 0 | Y_t \notin [a, b]\}.$$

Then τ is almost-surely finite, and the process

$$S^z_t = \int_0^t e^{-\beta u} U(\hat{c}^\lambda(\hat{X}_u, \hat{N}_u)\hat{X}_u)du + e^{-\beta t}\hat{X}_t^{1-p} s^{a,b}(\hat{Y}_t) \tag{1.25}$$

is a bounded martingale up to time τ.

Proof. Recall that fees are paid exactly when $\hat{Y} = 1$. Therefore, no fees have been incurred up to time τ, and the state process has the form

$$\begin{aligned} \hat{X} &= x\mathcal{E}(I)_t \\ \hat{N} &= n - \int_0^t \hat{c}^\lambda(\hat{X}_u, \hat{N}_u)\hat{X}_u du \end{aligned}$$

where

$$I_t = \int_0^t \hat{\theta}^\lambda(\hat{X}_u, \hat{N}_u)\frac{dF_u}{F_u} - \hat{c}^\lambda(\hat{X}_u, \hat{N}_u)du$$

Applying Itô's Lemma, we then have

$$\begin{aligned} d\hat{Y}_t &= \frac{1}{\hat{X}_t}dN_t + N_t\left(-\frac{1}{\hat{X}_t^2}dX_t + \frac{1}{\hat{X}_t^3}d\langle X\rangle_t\right) \\ &= -\hat{c}^\lambda(\hat{X}_t,\hat{N}_t)dt + \hat{Y}_t(-dI_t + d\langle I\rangle_t) \\ &= (\hat{Y}_t - 1)\hat{c}^\lambda(\hat{X}_t,\hat{N}_t)dt + \hat{Y}_t\left(\frac{1}{2}\hat{\theta}^\lambda(\hat{X}_t,\hat{N}_t)^2 - \mu\hat{\theta}^\lambda(\hat{X}_t,\hat{N}_t)\right)dt \\ &\quad -\hat{Y}_t\sigma\hat{\theta}^\lambda(\hat{X}_t,\hat{N}_t)dW_t \end{aligned}$$

Recall that $\hat{c}^\lambda \to c^0$ and $\hat{\theta}^\lambda \to \theta^0$ uniformly as $\lambda \to 0$. Therefore, we may take λ sufficiently small that $\hat{Y}$ admits a decomposition

$$d\hat{Y}_t = a_t^\lambda dt + b_t^\lambda dW_t \tag{1.26}$$

up to time τ, where $a^\lambda,\ b^\lambda$ are uniformly bounded, predictable, pathwise continuous processes, and b^λ is uniformly bounded away from 0. Applying the Girsanov theorem, there is a measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ with respect to which $\hat{Y}$ is a local martingale up to time τ. Since the volatility b^λ of $\hat{Y}$ is bounded uniformly away from 0 before time τ, it follows that $\hat{Y}$ exits the interval $[a,b]$ in finite time $\mathbb{Q}$-almost surely (hence $\mathbb{P}$-almost surely). In other words, τ is $\mathbb{P}$-almost surely finite.

To see that S^z is a local martingale up to time τ, simply apply Itô's Lemma to obtain a semi-martingale decomposition of S^z; the resulting drift term is exactly $\hat{X}^{1-p}\left(\mathcal{L}^{\hat{\theta}^\lambda,\hat{c}^\lambda}s^{a,b}\right)(\hat{Y}_{t\wedge\tau})dt$, and by definition $\mathcal{L}^{\hat{\theta}^\lambda,\hat{c}^\lambda}s^{a,b} = 0$. To see that S_t^z is uniformly integrable up to time τ, note that before time τ, we have $0 \le \hat{X} \le \hat{N} \le n$. As a result,

$$|S_{t\wedge\tau}^z| \;\le\; \left|\int_0^{t\wedge\tau} e^{-\beta u}U(\hat{c}^\lambda(\hat{X}_u,\hat{N}_u)\hat{X}_u))du\right| + \left|ne^{-\beta(t\wedge\tau)}s^{a,b}(\hat{Y}_{t\wedge\tau})\right|$$

Since $s^{a,b}$ is bounded, the term $\left|ne^{-\beta(t\wedge\tau)}s^{a,b}(\hat{Y}_{t\wedge\tau})\right|$ is uniformly bounded over all t. On the other hand, the term $|\int_0^{t\wedge\tau} e^{-\beta u}U(\hat{c}^\lambda(\hat{X}_u,\hat{N}_u)\hat{X}_u))du|$ is increasing in t, and from the definition of $\hat{w}$, we have

$$0 \leq \mathbb{E}\left|\int_0^\tau e^{-\beta u}U(\hat{c}^\lambda(\hat{X}_u,\hat{N}_u)\hat{X}_u))du\right| \leq x^{1-p}\hat{w}^\lambda(1,y).$$

We conclude that $S^z_{t\wedge\tau}$ is bounded, and in particular a true martingale. $\square$

Proposition 1.5.7. *With $s^{a,b}$ as in Lemma 1.5.5, we have*

$$\hat{w}^\lambda = s^{a,b}$$

on $[a,b]$. In particular, $\hat{w}^\lambda$ is C^2 on $[1,\infty)$ and

$$\mathcal{L}^{\hat{\theta}^\lambda,\hat{c}^\lambda}\hat{w}^\lambda = 0 \text{ on } [1,\infty).$$

Proof. Fix $y \in (a,b)$ and set $z = (1,y)$. Let $\hat{X},\hat{N},\hat{Y}$ and Z be as in Proposition 1.5.4, and τ, S as in Lemma 1.5.6. Note that $S^z_0 = s^{a,b}(y)$ and $Z^z_0 = \hat{w}^\lambda(y)$. Since τ is almost surely finite, $\mathbb{P}[\hat{Y}_\tau = a] + \mathbb{P}[\hat{Y}_\tau = b] = 1$. From Lemma 1.5.6, we also know that S^z is a uniformly integrable martingale up to time τ, so we conclude that

$$\begin{aligned} s^{a,b}(y) = \mathbb{E}[S^z_0] &= \mathbb{E}[S^z_\tau] \\ &= \mathbb{E}\left[\int_0^\tau e^{-\beta u}U(\hat{c}^\lambda(\hat{X}_u,\hat{N}_u)\hat{X}_u)du\right] \\ &\quad + s^{a,b}(a)\mathbb{E}\left[\mathbf{1}_{[\hat{Y}_\tau = a]}\hat{X}_\tau^{1-p}\right] + s^{a,b}(b)\mathbb{E}\left[\mathbf{1}_{[\hat{Y}_\tau = b]}\hat{X}_\tau^{1-p}\right] \end{aligned} \tag{1.27}$$

On the other hand, the local martingale Z^z is uniformly integrable martingale up to time τ; to see this, one can use an argument identical to that of

Proposition 1.5.6. Therefore, we have

$$
\begin{aligned}
\hat{w}^{\lambda}(y) = \mathbb{E}[Z_0^z] &= \mathbb{E}[Z_\tau^z] \\
&= \mathbb{E}\left[\int_0^\tau e^{-\beta u} U(\hat{c}^{\lambda}(\hat{X}_u, \hat{N}_u)\hat{X}_u) du\right] \\
&\quad + \hat{w}^{\lambda}(a)\mathbb{E}\left[\mathbf{1}_{[\hat{Y}_\tau = a]} \hat{X}_\tau^{1-p}\right] + \hat{w}^{\lambda}(b)\mathbb{E}\left[\mathbf{1}_{[\hat{Y}_\tau = b]} \hat{X}_\tau^{1-p}\right]
\end{aligned} \tag{1.28}
$$

From (1.27) and (1.28), we conclude that $\hat{w}^{\lambda}(y) = s^{a,b}(y)$ □

Proposition 1.5.8. *The function $\hat{w}^{\lambda}$ is a C^2solution of*

$$
\begin{aligned}
\mathcal{L}^{\hat{\theta}^{\lambda}, \hat{c}^{\lambda}} \hat{w}^{\lambda} &= 0 \text{ on } [1, \infty) \\
\hat{w}_y^{\lambda}(1) - \lambda \hat{w}_x^{\lambda}(1) &= 0 \\
\lim_{y \to \infty} \hat{w}^{\lambda}(y) &= w^0
\end{aligned} \tag{1.29}
$$

Proof. By Proposition 1.5.7 and Lemma 1.5.3, it remains only to check the second line of (1.29). Note that since we've shown $\hat{w}^{\lambda}$ is C^2 on $[1, \infty)$ in Proposition 1.5.7, we already know $\hat{w}_y^{\lambda}$ is well-defined and continuous at $y = 1$. Set $z = (1, 1)$ and define $\hat{X}, \hat{N}, \hat{Y},$ and Z as in Proposition 1.5.4. Suppose for contradiction that

$$\hat{w}_y^{\lambda}(1) - \lambda \hat{w}^{\lambda}(1) < 0$$

For some $\epsilon > 0$, define the stopping time

$$\tau = \inf\{t : \hat{Y} \geq 1 + \epsilon\}.$$

From Proposition 1.5.4 and the arguments of Lemma 1.5.6, we see Z is a bounded martingale up to time τ. On the other hand, applying Itô's lemma to $\hat{X}^{1-p}\hat{w}^{\lambda}(\hat{Y})$ and combining the results of Propositions 2.1.2 and 1.5.7, we obtain a decomposition

$$Z_t = \tilde{Z}_t + A_t \tag{1.30}$$

where $\tilde{Z}_t$ is a local martingale (and a bounded martingale up to time τ) and

$$\begin{aligned} dA_t &= e^{-\beta t}\hat{X}_t^{1-p}(\hat{w}_y^{\lambda}(\hat{Y}_t) - \lambda \hat{w}_x^{\lambda}(\hat{Y}_t))d\hat{M}_t \\ A_0 &= 0 \\ \hat{M}_t &= \max_{0\le s\le t}\left[\int_0^s \hat{\theta}^{\lambda}(\hat{X}_u, \hat{N}_u)(\mu du + \sigma dW_u)\right] \end{aligned}$$

The measure $d\hat{M}_t$ is supported on the set of times $\{t : \hat{Y}_t = 1\}$, so that

$$dA_t = e^{-\beta t}\hat{X}_t^{1-p}(\hat{w}_y^{\lambda}(1) - \lambda \hat{w}_x^{\lambda}(1))d\hat{M}_t.$$

Since $\hat{\theta}^{\lambda}$ is bounded away from 0, it is easy to see that

$$\inf\{t > 0 : M_t > 0\} = 0 \ \ \mathbb{P} - \text{almost surely.}$$

As a result, we have $A_\tau < 0$ almost surely, contradicting the martingale property of $\hat{Z}$ up to time τ and the decomposition 1.30. $\square$

Lemma 1.5.9. *We have*

$$\lim_{y\to\infty} y\hat{w}_y^{\lambda} = \lim_{y\to\infty} y^2\hat{w}_{yy}^{\lambda}(y) = 0$$

Proof. Recall that θ^1 and c^1 are smooth and bounded with

$$\lim_{y\to\infty} c^1(y) = \lim_{y\to\infty} \theta^1(y) = 0.$$

As a result, we can rewrite the equation $\mathcal{L}^{\hat{\theta}^\lambda,\hat{c}^\lambda}\hat{w}^\lambda = 0$ as

$$a^2(y)\hat{w}^\lambda_{yy} + a^1(y)\hat{w}^\lambda_y + a^0(y)\hat{a}^\lambda + b(y) = 0 \tag{1.31}$$

where $a^i(y)/y^i$ has a finite, well-defined limit as $y \to \infty$ for $i = 0, 1, 2$, which is nonzero for $i = 2$, and $b(y) \to 0$.

First, we claim that the limit $L = \lim_{y\to\infty} y\hat{w}^\lambda(y)$ is well defined. If so, then l'Hôpital's rule gives

$$\begin{aligned}\lim_{y\to\infty} \hat{w}^\lambda &= \lim_{y\to\infty} \frac{y\hat{w}^\lambda(y)}{y} \\ &= \lim_{y\to\infty} (y\hat{w}^\lambda_y(y) + \hat{w}^\lambda)\end{aligned}$$

so that $L = 0$. From (1.31), we immediately see

$$\lim_{y\to\infty} y^2\hat{w}^\lambda_{yy} = 0$$

as well. To see that L is well-defined, set $h(y) = \delta\hat{w}^\lambda + y\hat{w}^\lambda_y - w^0$ for some constant δ. If L is not well-defined, then there must be infinitely many local maxima and minima y_k of h such that $h(y_k)$ is bounded away from 0. We then have

$$0 = h_y(y_k) = (1+\delta)\hat{w}^\lambda_y(y_k) + y_k\hat{w}^\lambda_{yy}y_k \tag{1.32}$$

Combining this with equation (1.31), we see that

$$-a^2(y)(1+\delta)\frac{\hat{w}^\lambda_y(y_k)}{y_k} + a^1(y_k)\hat{w}^\lambda_y(y_k) + a^0(y_k)w^\lambda_(y_k) + b(y_k) = 0 \tag{1.33}$$

We know that $a^0(y_k)w^\lambda(y_k) + b(y_k) \to 0$ as $k \to \infty$. On the other hand, since $a^2(y_k)/y_k^2$ has a finite, nonzero limit, we can choose δ so that the remaining piece $-a^2(y)(1+\delta)\frac{\hat{w}^\lambda_y(y_k)}{y_k} + a^1(y_k)\hat{w}^\lambda_y(y_k)$ is bounded away from 0, contradicting the fact that $\mathcal{L}^{\hat{\theta}^\lambda,\hat{c}^\lambda}\hat{w}^\lambda = 0$. $\square$

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