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Application of a
Two-Dimensional Potential Analog
to the
Solution of Root-Locus Problems

BY

D. R. ZIEMER

Henry Beckman Conservation Bulletin No. 1

BUREAU OF ENGINEERING RESEARCH

THE UNIVERSITY OF TEXAS : AUSTIN

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The benefits of education and of useful knowledge, generally diffused through a community, are essential to the preservation of a free government.

SAM HOUSTON

Cultivated mind is the guardian genius of Democracy, and while guided and controlled by virtue, the noblest attribute of man. It is the only dictator that freemen acknowledge, and the only security which freemen desire.

MIRABEAU B. LAMAR

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Contents

THE UNIVERSITY OF TEXAS

BUREAU OF ENGINEERING RESEARCH

A recent "List of Publications" issued by the Bureau of Engineering Research is available upon request.

Preface

In 1956, Mr. Henry Beckman of Austin, Texas, donated to The University of Texas a sum of money for the "Henry Beckman Engineering Conservation Creative Graduate Award" for the most acceptable creative research in the College of Engineering. The award is based upon the most creative and ingenious contribution offered by a graduate student of the College of Engineering in the form of a thesis study and is to be selected from theses submitted during the calendar year. The selection is made by the Graduate Faculty of the College of Engineering from Master's theses on some facet of engineering related to the conservation of resources in Texas. This competitive award is to be made annually from the proceeds of the invested funds. A summary of the winning award is to be published in the annual Henry Beckman Conservation Bulletin of the Bureau of Engineering Research.

This is the first bulletin of the Henry Beckman Conservation series and contains the summary of the winning thesis for the 1956 calendar year. In addition, abstracts of other theses considered in this competition are included as a supplement to this publication.

It is with genuine pleasure that the College of Engineering inaugurates this award for outstanding work among the graduate students in the College of Engineering.

W. R. Woolrich
Dean of Engineering

Application of a Two-Dimensional Potential Analog to the Solution of Root-Locus Problems

BY

D. R. ZIEMER*

I. Introduction

The field of automatic control has witnessed a tremendous growth of interest and activity in the past fifteen years. The many military applications of servomechanisms were no doubt the largest contributing factor. However, because of the military necessity for higher performance control systems, the required servomechanisms became more complex, and the resulting design problems grew more complicated.

During and immediately after World War II, the main emphasis was placed on the frequency response method of analysis. This was due in no small way to the extensive work of men like Nyquist and Bode in the fields of feedback amplifier design and related circuit theory.

As the complexity of the systems increased, the transient response of the system became a very important criterion for design. Since the transient response is not easily obtained from the Nyquist and Bode techniques, a new design tool was urgently needed. About 1950 Evans[6] introduced his root-locus method utilizing the complex frequency plane concept as a means of obtaining transient and frequency responses with equal ease. The root-locus method is straightforward and has obtained rather wide acceptance.

The purpose of this paper is to discuss a two-dimensional potential analog utilizing the concepts of the root-locus method as a further tool in the design and synthesis of automatic control systems.

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* Master of Science in Electrical Engineering, January, 1956; now with Temco Aircraft Corporation, Dallas, Texas.

II. Review of the Root-Locus Method

Design in terms of the complex singularities of a system, which is essentially what the root-locus method does, is greatly facilitated if the design is accomplished in terms of two parameters, the undamped natural frequency ω_n and the damping ratio ζ.

Consider the transfer function relating Laplace Transforms of the output and input functions.

$$G(s) = \frac{c}{as^2 + bs + c} \tag{1}$$

$$s = \sigma + j\omega$$

If the damping term is set equal to zero (*i.e.*, b $=0$), the characteristic equation becomes

$$as^2 + c = 0 \tag{2}$$

The quantity ω_n is defined as the magnitude of the roots of Eq. (2).

$$\omega_n = s = \pm\sqrt{\frac{c}{a}} \tag{3}$$

The parameter ζ or damping ratio is defined as the ratio of actual damping to critical damping. Critical damping is defined from Eq. (2) to be that value of the damping coefficient b which makes $b^2 = 4\,ac$, or

$$b = 2\sqrt{ac}$$

The damping ratio is then

$$\zeta = \frac{\text{actual damping}}{\text{critical damping}} = \frac{b}{2\sqrt{ac}} \tag{4}$$

If ω_n and ζ are used as parameters to replace appropriate combinations of a, b, and c, Eq. (1) becomes

$$G_s = \frac{\omega_n^2}{s^2 + 2\omega_n \zeta s + \omega_n^2} \tag{5}$$

$$G_s = \frac{\omega_n^2}{[(s + \omega_n \zeta)^2 + (\omega_n \sqrt{1 - \zeta^2})^2]} \tag{6}$$

The locus of the variation of the roots of the characteristic equation of Eq. (6) as a function of fixed ω_n and variable ζ $(0 \le \zeta < \infty)$ are shown in Fig. 1.

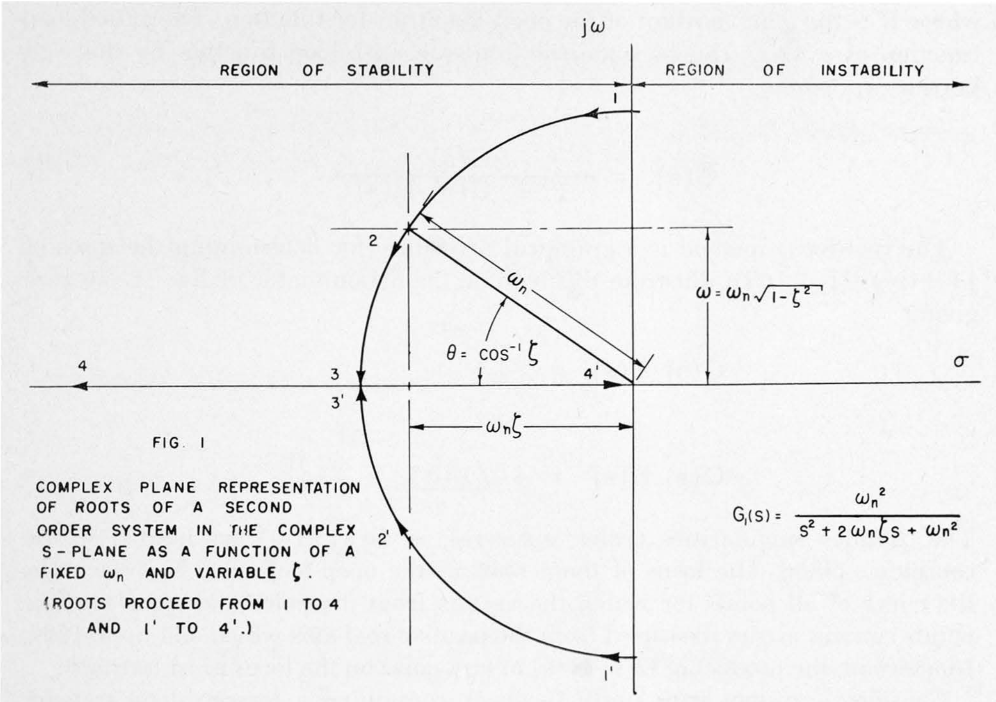


FIG. I

COMPLEX PLANE REPRESENTATION OF ROOTS OF A SECOND ORDER SYSTEM IN THE COMPLEX S-PLANE AS A FUNCTION OF A FIXED ω_n AND VARIABLE ζ.
(ROOTS PROCEED FROM 1 TO 4 AND 1' TO 4'.)

As motivation for the development of the root-locus method, consider Fig. 2.

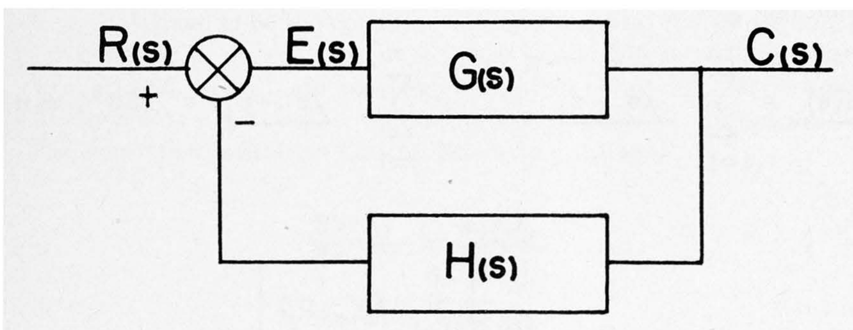


FIG. 2
SINGLE-LOOP FEEDBACK SYSTEM

In the usual feedback control system, G(s) is the ratio of two polynomials in s which is a rational algebraic function,

$$G(s) = \frac{K \prod_{i=1}^{m} (s - z_i)}{\prod_{j=1}^{n} (s - p_j)} \tag{7}$$

where K is the gain constant of the open-loop transfer function. The closed-loop function of s, $\overline{G}(s)$ can be obtained from the open-loop function by the well known expression

$$\overline{G}(s) = \frac{G(s)}{1 + G(s)\,H(s)} \tag{8}$$

The root-locus method is a graphical procedure for determining the zeros of $[1 + G(s)\,H(s)]$. To illustrate this, equate the denominator of Eq. (8) to zero giving

$$G(s)\,H(s) = -1$$

$$G(s)\,H(s) = 1 \angle 180° = 1\,\epsilon^{j\pi} \tag{9}$$

The complex singularities (poles and zeros) of $G(s)\,H(s)$ are located on the complex s-plane. The locus of these roots as the open-loop gain K is varied is the curve of all points for which the vectors from the poles and zeros to these points contain angles measured from the positive real axis which add up to 180°. In addition, the product of $G(s)\,H(s)$ at any point on the locus must be unity.

Consider a system with unity feedback containing a forward-loop transfer function given by Eq. (7). The conditions of the root-locus implied by Eq. (9) demand that

$$\angle G(s) = \sum_{i=1}^{m} \angle s - z_i \;-\; \sum_{j=1}^{n} \angle s - p_j = 180° \pm n360°$$

and

$$K = \left| \frac{\prod_{j=1}^{n} (s - p_j)}{\prod_{i=1}^{m} (s - z_i)} \right| \tag{10}$$

III. A Two-Dimensional Potential Analog

To motivate the development of a two-dimensional potential analog, we may start by taking the natural logarithm of both sides of Eq. (7)

$$V + j\phi = \ln G(s) = \ln K + \sum_{i=1}^{m} \ln (s - z_i) - \sum_{j=1}^{n} \ln (s - p_j) \qquad (11)$$

Equating the real and the imaginary parts gives

$$V(s) = \ln K + \sum_{i=1}^{m} \ln \left| s - z_i \right| - \sum_{j=1}^{n} \ln \left| s - p_j \right| \qquad (12)$$

and

$$\phi(s) = \sum_{i=1}^{m} \angle s - z_i - \sum_{j=1}^{n} \angle s - p_j \qquad (13)$$

where V(s) is defined to be the potential function and $\Phi(s)$ the stream function. If V(s) is the gain in nepers, then $\Phi(s)$ is the phase shift in radians. Furthermore, $V(\sigma, \omega)$ and $\Phi(\sigma, \omega)$ are conjugate functions, since Eq. (7) is analytic except at the points $s = z_i$ and $s = p_j$, and therefore generate orthogonal trajectories. These functions atisfy the Cauchy-Riemann conditions

$$\frac{\partial V(s)}{\partial \sigma} = \frac{\partial \phi(s)}{\partial \omega}$$

$$\frac{\partial \phi(s)}{\partial \sigma} = \frac{-\partial V(s)}{\partial \omega} \qquad (14)$$

The nomenclature of Eqs. (12) and (13) is illustrated in Fig. 3.

In view of the simplicity of Eqs. (12) and (13), it would be highly desirable if a practical physical analog could be constructed to mechanize their solution. Consider the electric field surrounding a small wire filament carrying a current of I amperes into a thin homogeneous sheet effectively infinite in extent. Since the current is restricted to two dimensions, the relationship between the current I entering or leaving the sheet and current density $\vec{J}$ in amperes per unit length is for any closed curve C enclosing the wire filament

$$I = \oint_C (\vec{J} \cdot \vec{n})\, dl \qquad (15)$$

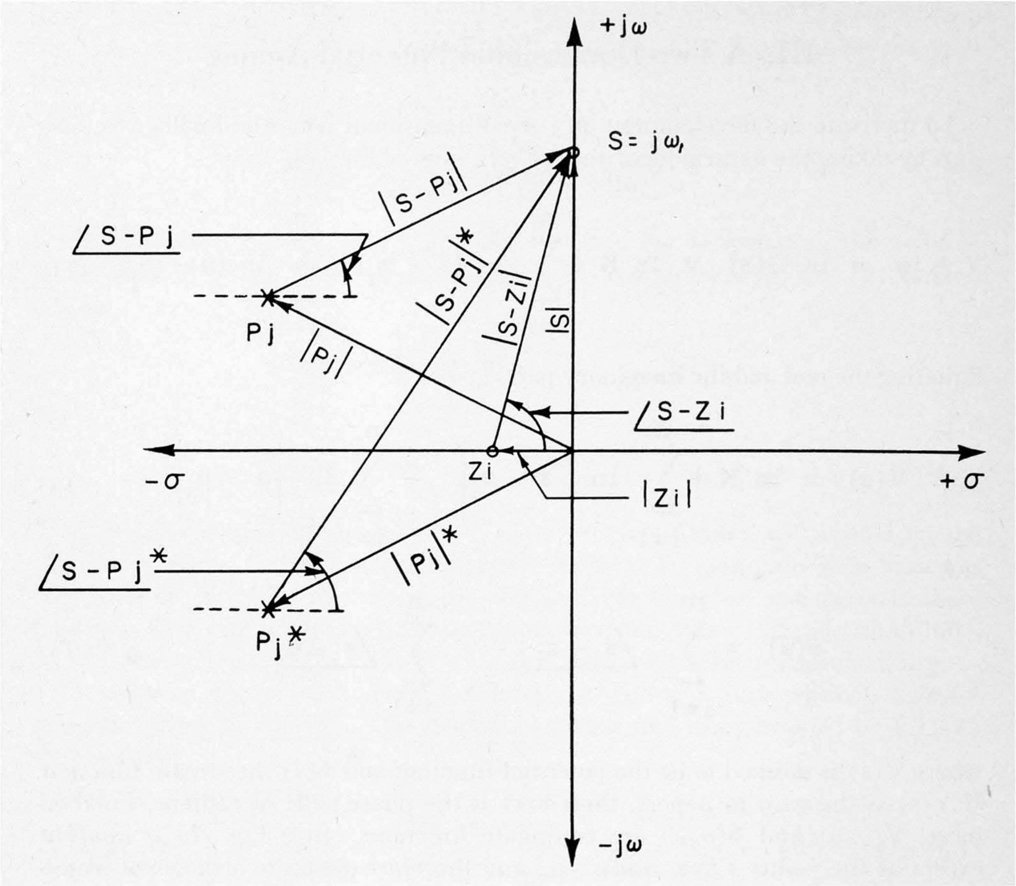


FIG. 3

VECTOR DIAGRAM ILLUSTRATING EQUATIONS NO. 12 AND 13

where $\vec{n}$ is a unit vector whose direction is normal to the curve at any point. If the path of integration is conveniently chosen to be a circle, then Eq. (15) becomes

$$I = 2\pi r J.$$

Then,

$$\vec{J} = \frac{I}{2\pi r}\vec{A}_r \qquad (16)$$

where $\vec{A}_r$ is a unit vector in the radial direction. The relationship between the electric field intensity $\vec{E}$ and the current density is by Ohm's law

$$\vec{E} = \rho\vec{J} \tag{17}$$

where ρ is the resistivity of the homogeneous sheet. Furthermore, the potential $\overline{V}$ as a function of r, the distance from the filament, is

$$\overline{V} = -\int^{r} \vec{E} \cdot \vec{dr} \tag{18}$$

Finally, substituting Eq. (16) in Eq. (17) and the result in Eq. (18) and integrating gives

$$\overline{V} = \ln r + c \tag{19}$$

where c is the potential at some reference point.

To obtain a general potential function consider several wire filaments passing into the homogeneous sheet. Let the currents into the sheet be positive and those out be negative. Let the location of the positive electrodes be designated by the complex numbers $\overline{p_j}$ and the location of the negative electrodes by $\overline{z_i}$. The general expression for the potential at some point on the sheet designated by the complex number $\overline{s}$ due to the positive and negative electrodes is from Eq. (19)

$$\overline{V(\overline{s})} = k \sum_{i=1}^{m} \ln \left| \overline{s} - \overline{z_i} \right| - k \sum_{j=1}^{n} \ln \left| \overline{s} - \overline{p_j} \right| \tag{20}$$

where k is a constant of the homogeneous sheet and contains the constant c in Eq. (19).

Eq. (20) is identical in form to Eq. (12).

An expression analogous to Eq. (13) can also be obtained quite easily. Examine Fig. 4.

The current flow across any line connecting A and B is given by the integral

$$I_{AB} = \int_{AB} \vec{J} \cdot \vec{n}\, dl \tag{21}$$

Since

$$\vec{J} \cdot \vec{n}\, dl = Jr\, d\alpha,$$

$$I_{AB} = \int_{AB} \frac{I}{2\pi r}\, r\, d\alpha = \frac{I}{2\pi}\,\alpha = \frac{I}{2\pi} \underline{/\overline{s} - \overline{p}} \tag{22}$$

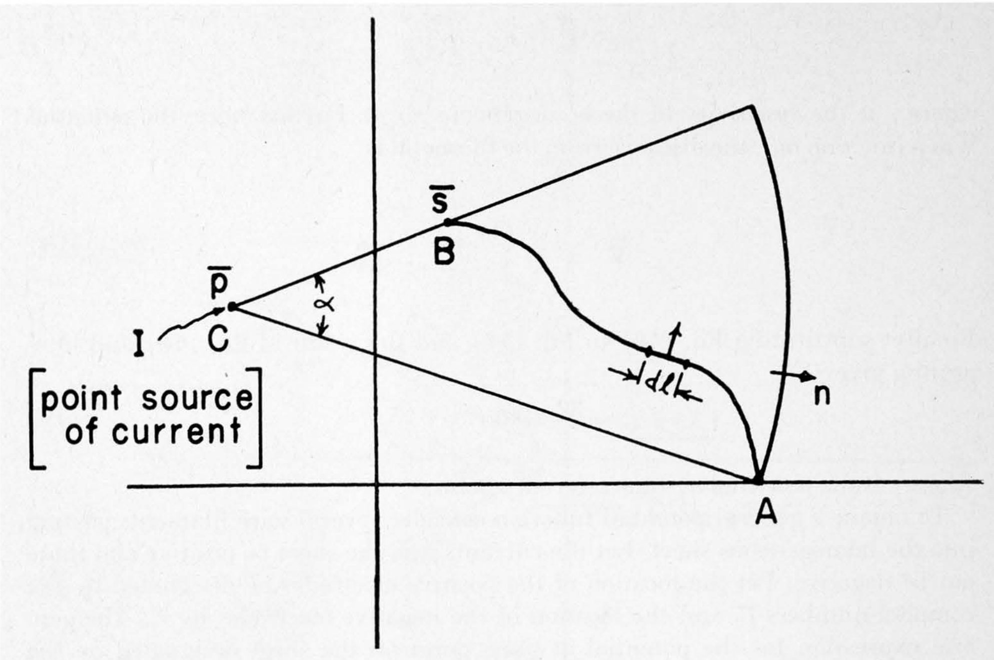


FIG. 4

Applying this to the current source configuration shown in Fig. 5, it can be seen that the total current flowing across the line AB depends upon the sum of angles a and a'. Furthermore, it is obvious from the geometry of the configuration that

$$a = \angle \bar{s} - \bar{p}_j + \beta,$$

$$a' = \angle \bar{s} - \bar{p}_j{}^* - \beta',$$

and $$a + a' = \angle \bar{s} - \bar{p}_j + \angle \bar{s} - \bar{p}_j{}^* \tag{23}$$

since $\beta = \beta'$.

It then follows that the current flowing across a curve such as AB due to a complex of current sources and sinks located at points designated by complex numbers p_j and z_i, respectively, is

$$I_{AB} = \bar{\phi} = \frac{I}{2\pi}\left\{\sum_j \angle(\bar{s} - \bar{p}_j) - \sum_i \angle(\bar{s} - \bar{z}_i)\right\} \tag{24}$$

Eq. (24) is identical in form to Eq. (13). The potential analog is based on the analogy which exists between Eq. (24) and Eq. (13) and between Eq. (20) and Eq. (12).

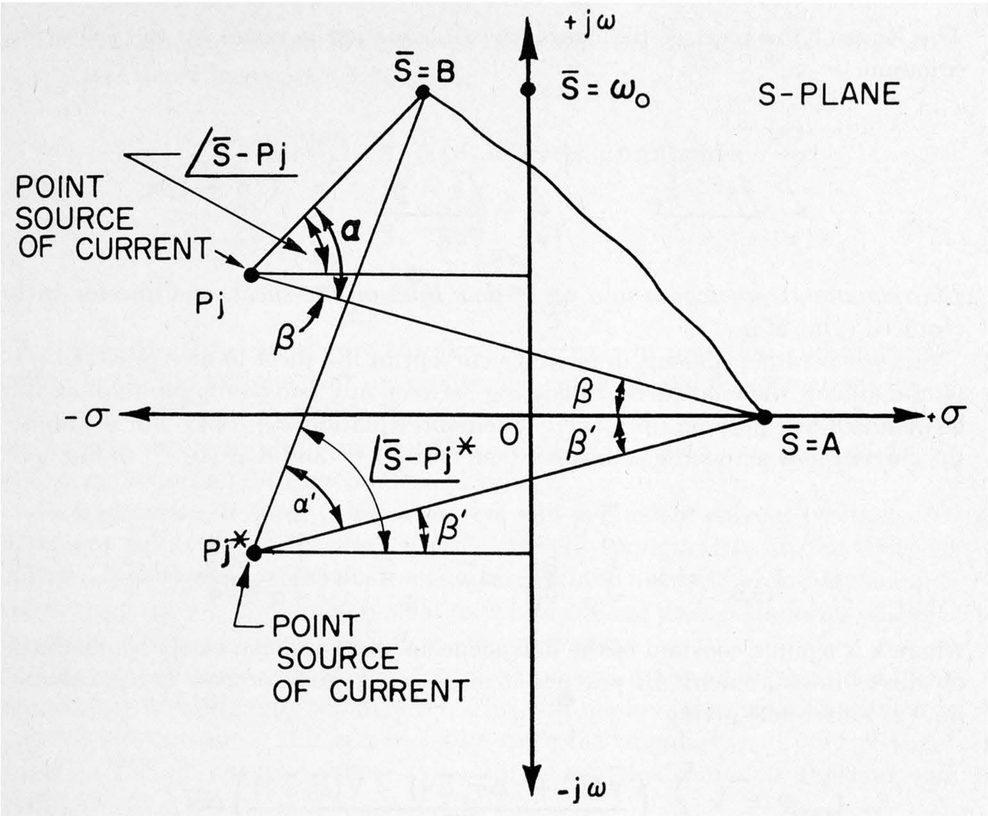


FIG. 5

ILLUSTRATION FOR DEVELOPMENT OF PHASE ANALOG

If s lies at some point on the $j\omega$ axis; *i.e.*, $s = j\omega_o$, then $\Phi(s) = \Phi(j\omega_o)$ which is the phase shift for sinusoidal excitation at radian frequency ω_o. $V(s) = V(j\omega_o)$ is the gain in nepers for sinusoidal excitation. Correspondingly, if $\bar{s}$ lies at some point on the sheet designated by $\overline{j\omega_o}$, then $\bar{\Phi}(\bar{s}) = \bar{\Phi}(\overline{j\omega_o})$ is analogous to $\Phi(j\omega_o)$ and $\bar{V}(\bar{s}) = \bar{V}(\overline{j\omega_o})$ is analogous to $V(j\omega_o)$.

Consider the logarithm of a characteristic equation, $G(s) = -1$, which may be written as

$$\ln k + \sum_{i=1}^{m} \ln \left| s - z_i \right| - \sum_{j=1}^{n} \ln \left| s - p_j \right| +$$

$$j \sum_{i=1}^{m} \underline{/s - z_i} - j \sum_{j=1}^{n} \underline{/s - p_j} = 0 + j\pi\,(2n + 1) \qquad (25)$$

The locus of the roots of the characteristic equation is given by the following equation

$$\sum_{i=1}^{m} \angle s - z_i \; - \sum_{j=1}^{n} \angle s - p_j \;=\; (2n + 1)\pi \tag{26}$$

This equation is analogous to a set of flow lines on the sheet, one line for each required value of n.

Since it is rather difficult to measure current in the sheet in any practical potential analog, the total current I flowing between any two points can most easily be obtained by applying the Cauchy-Riemann equation, Eq. (14). For example, the current flow across the $j\omega$ axis between O at (0, 0) and B at $(0, \bar{\omega}_o)$ in Fig. 5 is

$$I_{OB} = k \int_{o}^{\bar{\omega}_o} \frac{\partial \bar{\phi}}{\partial \bar{\omega}}\, d\bar{\omega} = k \int_{o}^{\omega_o} \frac{\partial \bar{V}}{\partial \bar{\sigma}}\, d\bar{\omega} \tag{27}$$

where k is again a constant of the homogeneous sheet and can easily be obtained by direct measurement. In practice a more satisfactory process is to evaluate Eq. (27) discretely giving

$$I_{OB} = k \sum_{i=1} \left(\frac{\bar{V}(\bar{\sigma} + \Delta\bar{\sigma}, \bar{\omega}_i) - \bar{V}(\bar{\sigma}, \bar{\omega}_i)}{\Delta\bar{\sigma}} \right) \Delta\bar{\omega}_i \tag{28}$$

Since a potential analog to Eq. (12) and Eq. (13) has been shown to exist, there remains the problem of inherent errors in a practical application of the mechanization of the analog. Other than the usual errors in measurement and lack of homogeneity, the major sources of error are the lack of point source electrodes and the necessity of a finite size of the conductive sheet.

Considering the problem of finite size electrodes first, the conclusion is reached that as long as the equipotential lines are concentric circles about the electrodes no error results. If a pole and zero are located in close vicinity of one another, an error will result since the correct equipotential lines are no longer concentric circles. The amount of this error is easily calculated since it is the same problem as finding the field around a pair of transmission lines. As an indication of the amount of correction required, consider an electrode about 0.02 inch in diameter (*e.g.*, a pointed welding rod) then for 0.5-inch spacing the correction required is 0.002 inch which is negligible.

The problem of simulating an infinite medium by using finite dimensions is not so easily disposed of as was the problem of finite-sized electrodes. Of course, one could always calculate the error due to the use of a finite medium by the methods of potential theory.[8] Fortunately, a medium that is effectively infinite in extent can be realized by two uniform sheets, finite in size, insulated from one another, and joined at their boundaries. Consider Figs. 6 and 7. The regions D and $\bar{D}$ in these figures are related by the conformal transformation $z\bar{z} = a^2$.

It can be shown that on the circle of inversion; *i.e.*, the circle of radius a in each plane, the following equations hold.

$$V(a, \theta_o) = \overline{V}(a, \theta_o) + \text{constant} \tag{29}$$

$$\left.\frac{\partial V}{\partial r}\right|_{\substack{r=a \\ \theta=\theta_o}} = -\left.\frac{\partial \overline{V}}{\partial \overline{r}}\right|_{\substack{\overline{r}=a \\ \theta=\theta_o}} \tag{30}$$

Consequently, if the sheets are cut along the circle of inversion and joined together along the periphery of the circles, the potential and flow functions in the regions C and $\overline{C}$ will not be perturbed from the values existing when C and $\overline{C}$ are parts of sheets of infinite extent. This operation is analogous to the application of the compensation theorem in network theory.

To summarize, if sheet C has n sources and m sinks of current positioned at arbitrary points to satisfy the desired pole-zero configuration in the complex s-plane, then sheet C can be made effectively infinite in extent by joining another conducting sheet $\overline{C}$ of diameter equal to that of c along their boundaries where $\overline{C}$ has a single current sink of strength, n − m, at its origin.

There is one other very important point concerning the analog and this is the desirability of using only a half sheet. This will readily be recognized when it is pointed out that the σ-axis is a flow line since the singularities of any physical function that do not lie on the σ-axis must be complex conjugate and hence be symmetrical about the σ-axis.

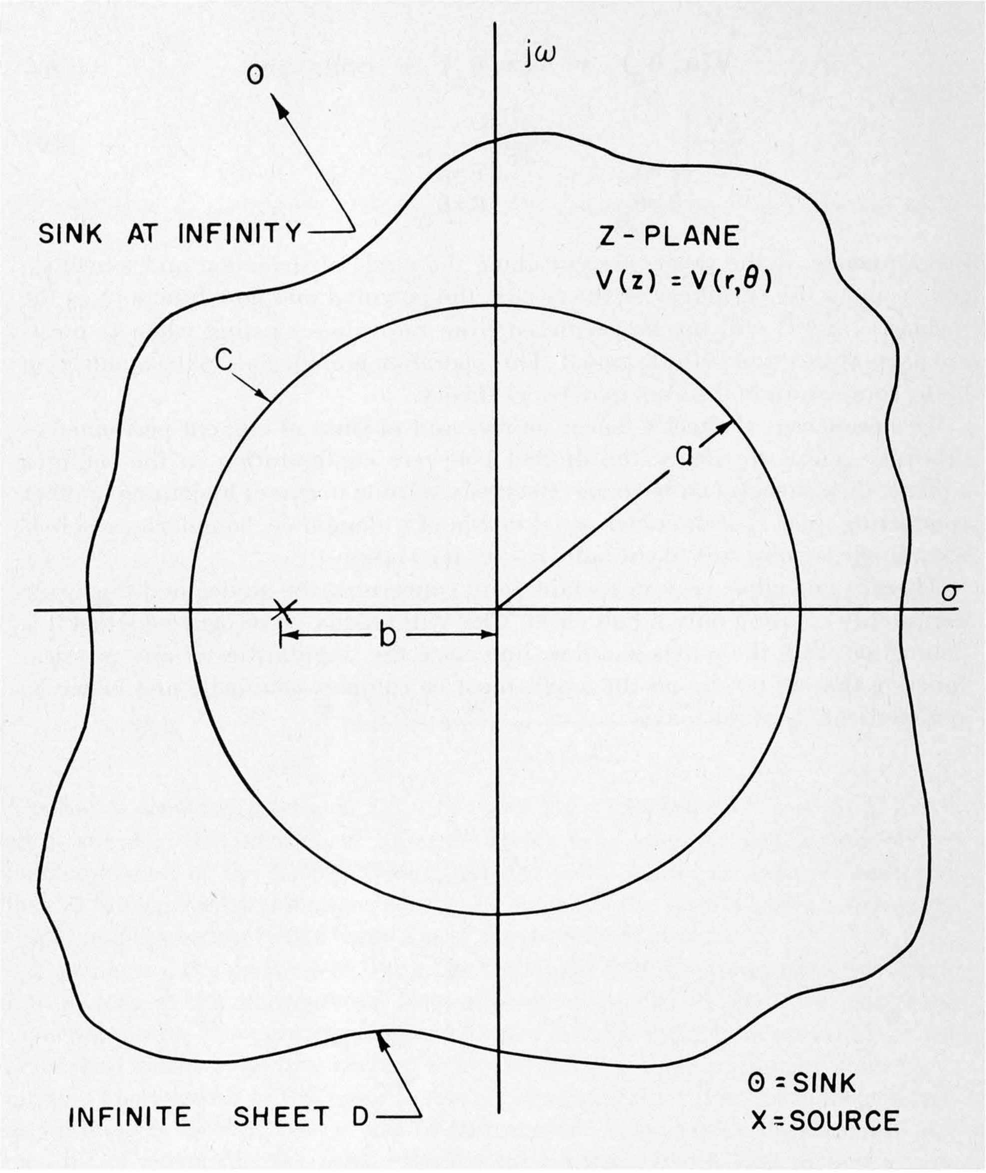


FIG. 6
SOURCE AND SINK IN THE Z-PLANE
(NOT TO SCALE)

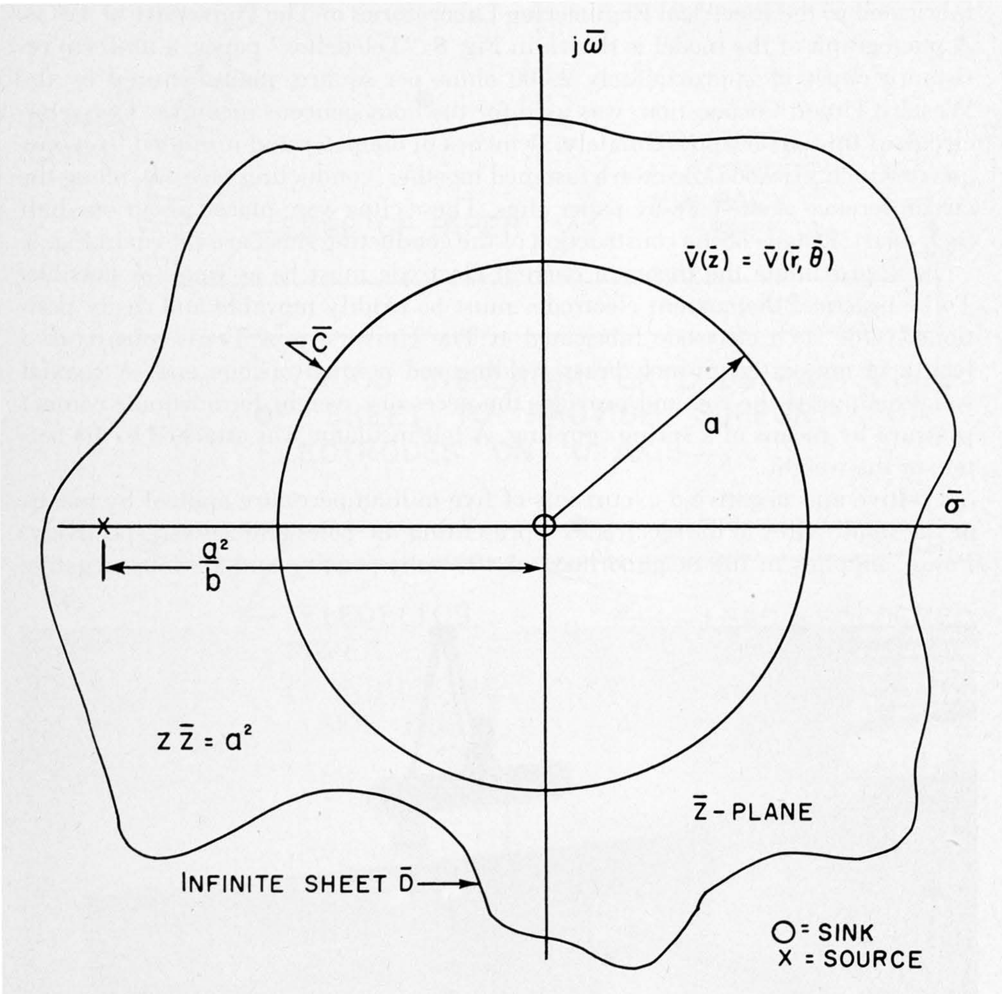


FIG. 7
SOURCE AND SINK IN THE $\bar{Z}$ PLANE
(NOT TO SCALE)

IV. A Practical Laboratory Potential Analog

A laboratory model based on the potential analog theory described above was fabricated in the Electrical Engineering Laboratories of The University of Texas. A photograph of the model is shown in Fig. 8. "Teledeltos" paper, a uniform resistance paper of approximately 2,000 ohms per square, manufactured by the Western Union Corporation, was used for the homogeneous medium. Two semicircles of this paper approximately 34 inches in diameter and insulated by a one-quarter-inch plywood core were fastened together, conducting face out, along the circumference of the core by paper clips. These clips were placed about one-half inch apart. Details of the construction of the conducting sheet are shown in Fig. 9.

To approximate the theory a current electrode must be as small as possible. To be practical the current electrodes must be readily movable and easily positioned. One such electrode fabricated at The University of Texas consists of a length of one-sixteenth-inch brass welding rod pointed on one end. A coaxial weight supports the rod and provides the necessary weight for adequate contact pressure by means of a spring coupling. A felt insulator was attached to the bottom of the weight.

Positive and negative d-c. currents of five milliamperes are applied by means of the small wires to the electrodes representing the poles and zeros respectively. Power supplies in the neighborhood of 100 volts positive and 25 volts negative

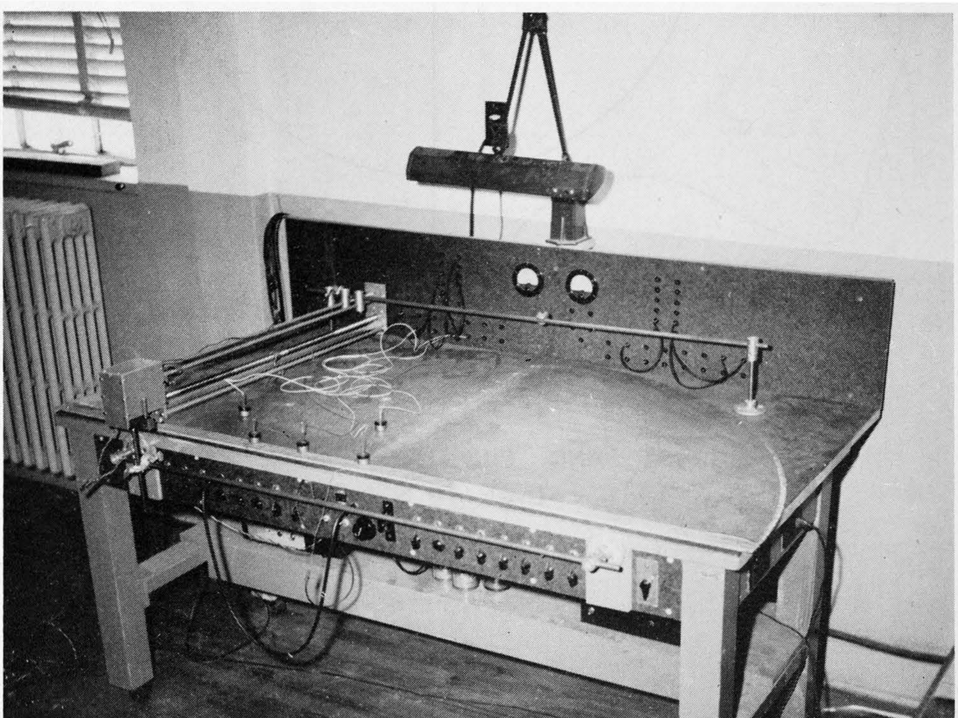


Fig. 8. Laboratory Model of the Potential Analog

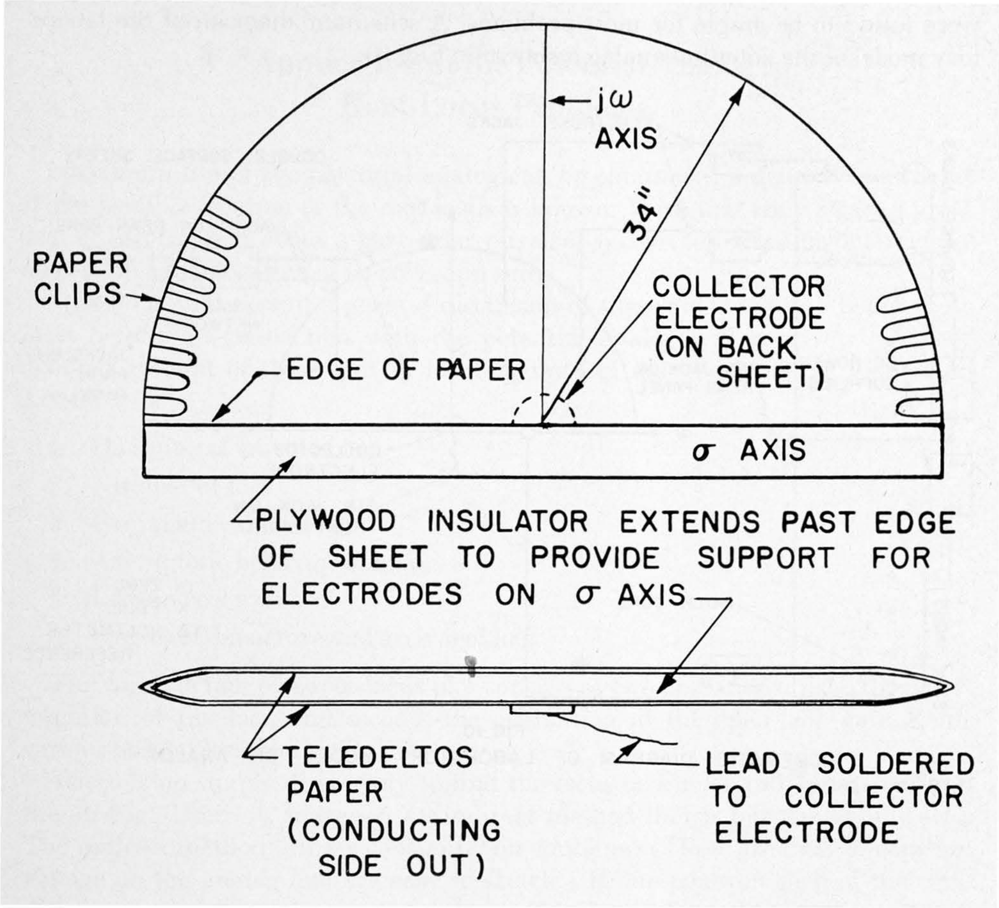


FIG 9

DETAILS OF CONDUCTING SHEET

were found to be ample for most problems. A schematic diagram of the laboratory model of the potential analog is shown in Fig. 10.

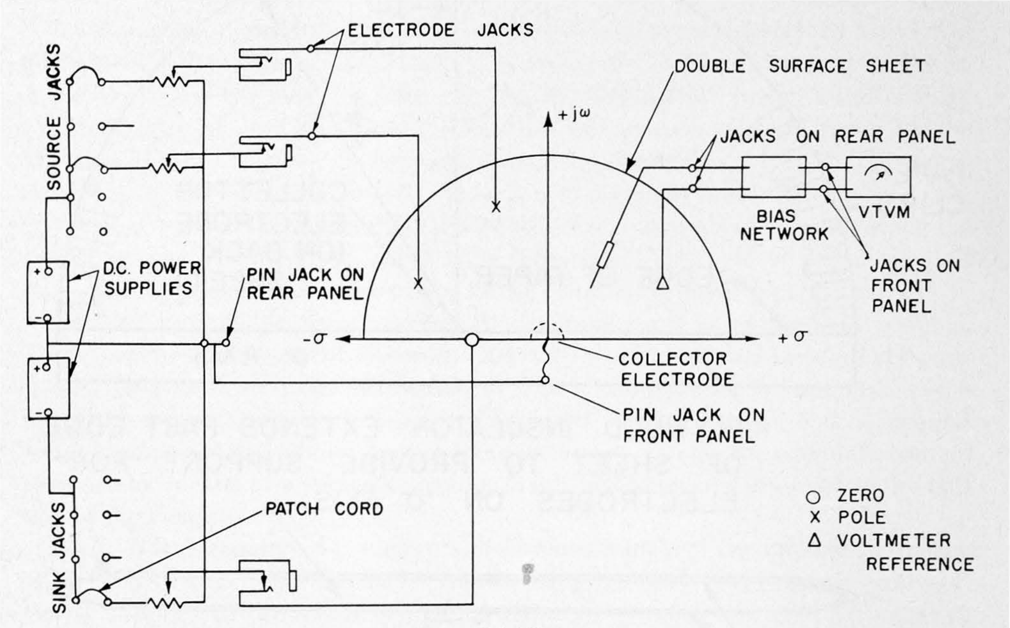


FIG. 10
SCHEMATIC DIAGRAM OF LABORATORY MODEL OF ANALOG

V. Application of the Potential Analog to Root-Locus Problems

Maximum use of the potential analog can be obtained if a general knowledge of the possible location of the root-locus is known. With this knowledge, a great deal of information about a particular physical system can often be obtained by inspection of the location of its poles and zeros.

Space does not permit a general discussion of these aids so only a list of those most helpful in connection with the potential analog will be given. (A very detailed account of these aids in locating the loci may be found in Evans and Truxal.[6, 11])

1. Direction of travel of loci
2. Number of loci
3. Conjugate value of loci
4. Asymptotic behavior of loci
5. Loci on real axis
6. Angles of departure and arrival of loci

The construction of a root-locus plot consists of two operations; first, the determination of the loci, and second, the evaluation of the open-loop gain K for various points along the loci.

There is no simple direct way to find the locus or line of 180° phase shift on the analog. There is, however, an indirect method that is not too complicated. The indirect method utilizes constant gain contours. (These are lines of constant voltage on the analog and are easy to sketch.) If one point on each of the complex loci can be located accurately, then a line drawn from these points orthogonal to the constant gain contours will give the required loci. The loci along the real axis can, of course, be drawn from inspection of the singularities lying along the real axis. Probably the easiest way to locate one point on each of the complex loci is to make a guess at the location of the point and then check the location by algebraically adding the phase angles from the poles and zeros. If the angle sums to $180° \pm n\ 360°$, the desired point is determined. Finding the location of the probable region in which the complex loci lie can be enhanced by utilizing the asymptotic behavior of the loci.

Evaluation of the open-loop gain constant K for various points along the loci is particularly simple on the analog and constitutes one of the chief advantages of the analog method in obtaining root-locus plots. To evaluate the open-loop constant K, Eqs. (20) and (10) will be utilized. If Eq. (20) is evaluated at two points on the sheet, *e.g.*, at $s = 1$ and $s = 2$, then the constant k of the homogeneous sheet can be evaluated.

$$V_1 = k \ln G(1) \tag{31}$$

$$V_2 = k \ln G(2) \tag{32}$$

Subtracting Eq. (31) from Eq. (32) gives

$$k = \frac{V_2 - V_1}{\ln\left(\frac{G(2)}{G(1)}\right)} \tag{33}$$

If the voltmeter reference is placed at s = 1, Eq. (33) reduces to

$$k = \frac{V_2}{\ln\left(\frac{G(2)}{G(1)}\right)} \tag{34}$$

and the sheet constant k can be evaluated. From Eq. (10) we can obtain the result that

$$K_1 = \left|\frac{1}{G(1)}\right| \qquad K_2 = \left|\frac{1}{G(2)}\right| \tag{35}$$

Substituting Eq. (35) in (34) gives

$$\ln\left(\frac{K_1}{K_2}\right) = \frac{V_2}{k}$$

$$\frac{K_1}{K_2} = \varepsilon^{\frac{V_2}{k}} \tag{36}$$

The general expression for the open-loop gain K for any point along the loci can be obtained by writing the general expression for K_n with the help of Eq. (36)

$$K_n = K_1 \varepsilon^{-\frac{V_n}{k}} \tag{37}$$

where $K_1 = \left|\frac{1}{G(1)}\right|$ and the voltmeter reference is at s = 1.

As soon as the root-locus has been sketched, we are in a position to obtain immediately the closed-loop frequency and transient responses for any value of open-loop gain K of the original open-loop configuration of poles and zeros.

Suppose the closed-loop frequency and transient responses are desired for an open-loop gain K sufficient to give the complex conjugate poles in a closed-loop system a damping ratio of $\zeta = 0.4$. See Fig. 11.

To obtain this information it is first necessary to lay off a radial line from the origin making an angle of $\theta = \cos^{-1}\zeta = \cos^{-1}0.4$ or 66.4 degrees. The potential is noted at the point of intersection of the locus and the radial line of constant damping. The same potential is located on the other two loci in order to determine the position of the remaining two closed-loop poles. The closed-loop poles are shown corresponding to a gain of K = 1.46 in Fig. 11. If the three poles are now placed at these points on the conducting sheet, the closed-loop frequency response can be obtained by measuring the potential along the $j\omega$-axis proceeding from the origin toward the edge of the sheet. These voltages must be multiplied by the constant k of the sheet given by Eq. (20) and Eq. (28). The resulting products will be the gain response in nepers and the phase response in radians.

The transient response can most easily be determined for a unit-impulse or a unit-step function input. The process of convolution may be applied to obtain the output response for any desired input. The response for a unit-impulse input is given by Eq. (38).

$$g_2(t) = \sum_{i=1}^{n} A_{-i}\,\varepsilon^{(p_i t)} \tag{38}$$

For complex conjugate roots Eq. (38) becomes

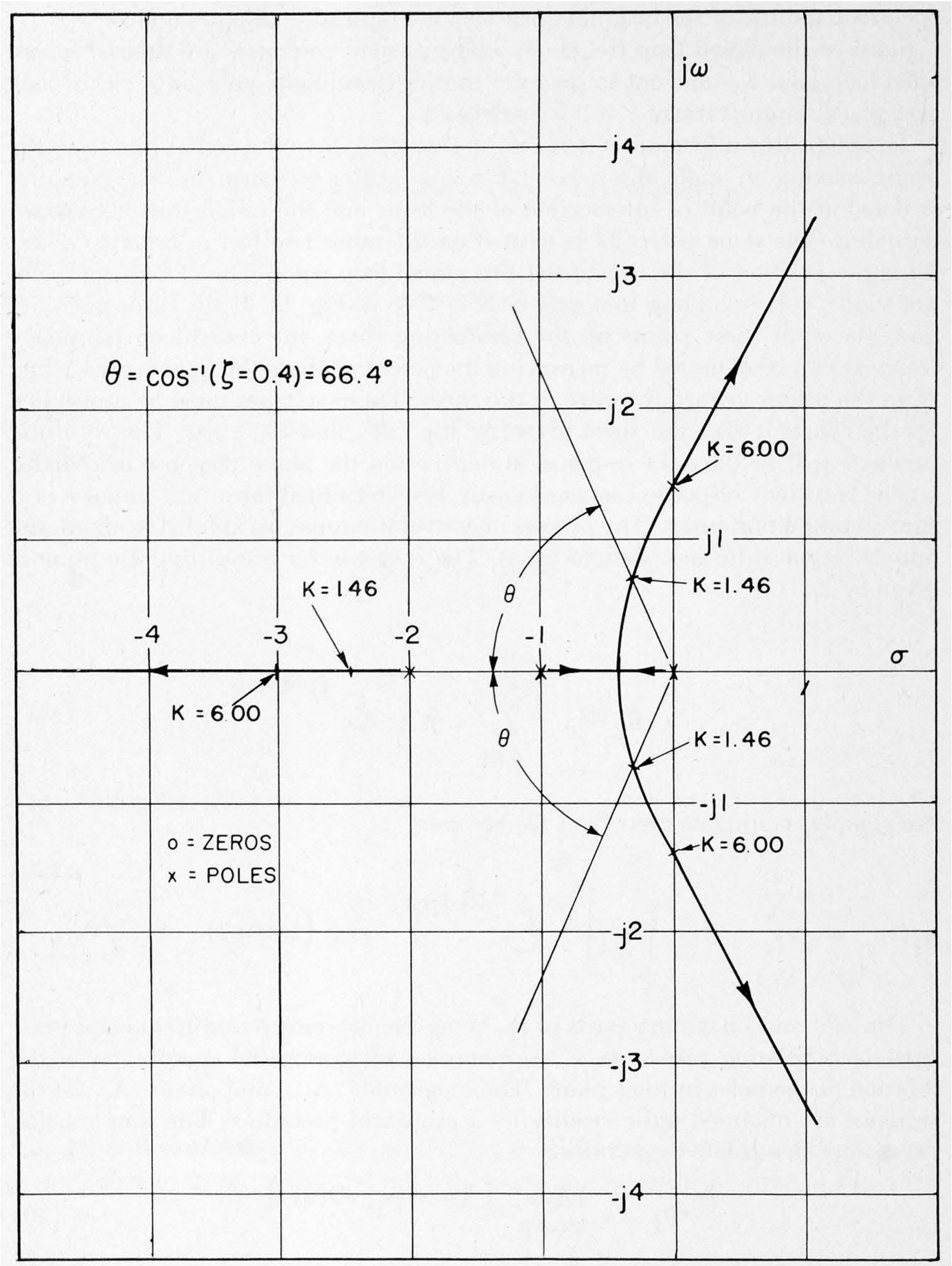
$$g_i(t) = \sum_{i=1}^{n} 2\,|A_{-i}|\,\varepsilon^{Re(p_i t)} \cos\left(Im(p_i t) + \underline{/A_{-i}}\right) \tag{39}$$

The real and imaginary parts of p_i, being the decrement and frequency functions of oscillation respectively, are obtained by noting the coordinates of the location of the poles in the s-plane. The magnitude $|A_{-i}|$ and phase $\underline{/A_{-i}}$ of the residues are obtained quite readily by a graphical procedure. For simple poles, the residue in a pole is by definition

$$A_{-p_i} = \lim_{s \to p_i} \left[(s - p_i)\,G(s) \right] \tag{40}$$

From Eq. (7) this becomes

$$A_{-p_i} = \lim_{s \to p_i} K \left[\frac{(s - z_i)(s - z_2)\cdot\ \cdot\ \cdot(s - z_m)}{(s - p_2)(s - p_3)\cdot\ \cdot\ \cdot(s - p_n)} \right] \tag{41}$$



ROOT-LOCUS PLOT FOR $G_1(s) = \dfrac{K}{s(s+1)(s+2)}$

FIG. 11

The factor $(s - z_i)_{s=p_i}$ can be represented by a vector from the zero at $s = z_i$ to the pole at $s = p_i$. Therefore, A_{-p_i} is simply the open-loop constant K multiplied by the vectors from the various zeros to $s = p_i$ divided by vectors from the various poles to $s = p_i$. Thus, for a location of closed-loop poles given by an open-loop gain of K = 1.46 in Fig. 11, the three residues are

$$A_{-1} = \frac{1.46}{(2.2)(2.2)} \angle 0° = 0.302 \angle 0°$$

$$A_{-1} = \frac{1.46}{(1.37)(2.2)} \angle -108° = 0.485 \angle -108° \tag{42}$$

$$A_{-1} = \frac{1.46}{(1.37)(2.2)} \angle 108° = 0.485 \angle 108°$$

The closed-loop transient response by virtue of Eq. (38) and (39) is

$$g(t) = 0.302\, \epsilon^{-2.4t} + 2(0.485)\, \epsilon^{0.3t} \cos(0.69t - 108) \tag{43}$$

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Abstracts

RELATION OF THE HEAD LOSS AT BRIDGE PIERS AND THE DRAG RESISTANCE OF THE PIERS

BY JERRY GARRETT*

The investigation was undertaken to find a reliable method of predicting the head loss in an open channel due to the presence of piers.

An equation giving the head loss as a function of the drag coefficient, the boundary geometry, and the velocity of flow was derived. It was checked in the Hydraulic Laboratory at The University of Texas by the use of models for symmetrically placed piers in a single row with conditions in which the drag coefficient was independent of the Froude and Reynolds numbers.

Values of the head loss calculated from the equation using measured values of the drag coefficient were in reasonable agreement with measured values of the head loss.

PHASE EQUILIBRIA IN THE SODIUM CHLORIDE—SODIUM SULFIDE—WATER SYSTEM

BY SALIHS ALPARGUN†

To maintain the balance in production of and demand for the different products of a process is a major concern of the chemical industry. An increased demand for one product might cause the accumulation of another and create a surplus product problem. In recent years, because of an increased demand for chlorine, electrolytic caustic producers had such a problem, surplus caustic soda. At times this material has been taken by barge some distance out into the Gulf of Mexico and dumped into the ocean. This represents an economic waste.

A process that would make it possible to use the effluent liquor from the diaphram of the electrolytic caustic-chlorine cell without requiring the evaporation of this effluent would make possible the utilization of this solution.

By converting the caustic soda into sodium sulfide a number of advantages would be realized: (1) the surplus caustic problem would be solved, (2) evaporation would not be required, (3) the sodium sulfide could be used instead of salt cake in the kraft pulp industry, and (4) steam production in the recovery unit would be increased due to the decrease in reduction load. Industrial feasibility of this process depends upon the possible economical ways of separating sodium sulfide and sodium chloride. For this reason the phase relationships in the system, sodium chloride—sodium sulfide—water, were investigated at 0°, 25°, 40°, and 60° C.

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Results. The solubility of pure sodium chloride and pure sodium sulfide were determind over the range 0° to 60° C. At 48.3° C., $Na_2S \cdot 9H_2O$ undergoes a transition to $Na_2S \cdot 5H_2O$. Solubilities in the three component system are reported in tables.

From the solubility data it is seen that concentration of the solution by evaporation will cause crystallization of sodium chloride until the invarient point is reached. Cooling will then cause crystallization of $Na_2S \cdot 9H_2O$ after which the solution may be concentrated again by evaporation to remove more sodium chloride. Using this cyclic process all of the sodium sulfide may be separated from the sodium chloride.

Conclusion. A process has been devised based on physical-chemical data for the separation of sodium sulfide from caustic-chlorine cell effluent to give a product which may be used in the kraft pulp industry.

FUNDAMENTAL STUDIES OF PARAFFIN DEPOSITION IN THE PRODUCTION OF CRUDE OIL

BY JAMES N. HOWELL*

No paraffin deposits occur in tubing or flow lines when the existing temperature is above the cloud point of the oil. When the cloud point is reached, paraffin particles begin to appear and deposition takes place. The rate of deposition varies directly with the flow rate up to Reynolds' numbers of approximately 2,000 (linear flow) but decreases rapidly with increased velocity of flow, *i.e.*, in the region of turbulent flow.

Plastic pipe shows less tendency to accumulate paraffin, yet all the varieties of plastic pipe tested showed some deposition. Greater amounts of paraffin were deposited when the initial temperature of the oil passed through the pipes was above the cloud point thus effecting cooling by heat transfer through the pipe wall. This latter phenomenon indicates adherence is greater when initial deposition occurs as a thin film rather than when a solid phase is present prior to contact with the surface of the pipe.

Some correlation was found between the rate of deposition and the wettability characteristic of the crude oil as indicated by the contact angle.

A SOLID PARTICLE HEAT EXCHANGE SYSTEM FOR NUCLEAR POWERED AIRCRAFT

BY RUSSELL ARCHIBALD†

Air-borne silicon-carbide pellets about 5/16-inch diameter are proposed as a means to increase the heat transfer rate from the surfaces of the fuel element in a nuclear reactor. The pellets would be circulated through the reactor and to a

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fixed-bed, or helical-path, heat exchanger where the gases for the turbine would be heated. The pellet movement system would require a compressor, an energizing device such as an ejector, and a means of separating the pellets from the transporting air.

The arguments for such a system are the high heat transfer rate per unit volume in the reactor and air-to-gas heat exchanger, especially in the radiation region, and a flexible location of the reactor with respect to the propulsion system. This would permit shorter "cooling-off" time since the pellets would have low, induced radioactivity. It would also mean leaks in the gas system could occur without shut-down requirements. The pellets would also serve as a circulating moderator. The principal objections to such a system would be possible added weight and erosion or spalling of the pellets.