University of Texas Bulletin

No. 2109: February 10, 1921

The Texas Mathematics Teachers' Bulletin

Volume VI, No. 2

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SECOND-CLASS MATTER AT THE POSTOFFICE AT AUSTIN, TEXAS,
UNDER THE ACT OF AUGUST 24, 1912

The benefits of education and of useful knowledge, generally diffused through a community, are essential to the preservation of a free government.

Sam Houston

Cultivated mind is the guardian genius of democracy. . . . It is the only dictator that freemen acknowledge and the only security that freemen desire.

Mirabeau B. Lamar

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The Texas Mathematics Teachers' Bulletin

Volume VI, No. 2

Edited by

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Instructor in Pure Mathematics,

and

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This Bulletin is open to the teachers of mathematics in Texas for the expression of their views. The editors assume no responsibility for statements of facts or opinions in articles not written by them.

PUBLISHED BY THE UNIVERSITY SIX TIMES A MONTH, AND ENTERED AS SECOND-CLASS MATTER AT THE POSTOFFICE AT AUSTIN, TEXAS, UNDER THE ACT OF AUGUST 24, 1912

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DR. E. R. HEDRICK'S ARTICLE

Last July Dr. E. R. Hedrick of the University of Missouri gave an address on the work of the National Committee on Mathematical Requirements to the students of the University of Texas Summer School.

Dr. Hedrick is a nationally recognized mathematical authority and one of the most devoted workers on this important Committee. The editors of the *Bulletin* consider themselves fortunate in being able to reprint Dr. Hedrick's address.

CONTRIBUTIONS BY HIGH SCHOOL TEACHERS

The editors very much desire to have High School teachers of mathematics contribute to this *Bulletin*. Could there be any better way of helping one teacher (and that is the sole object of this publication) than by displaying to him the troubles, problems and solutions of others in his profession? We believe not.

Consequently, it is with pleasure that we publish in this issue "Quantity or Quality?" by Miss Mary Campbell of the Beaumont High School, and also with the hope that it will be only one of many articles which our readers will send to us.

Let *your* article be long or short,—on any subject—just so it will be of interest to those who elsewhere are daily going over the same ground with you. We will publish it.

THE NATIONAL COMMITTEE ON MATHEMATICAL REQUIREMENTS

ADDRESS BY E. R. HEDRICK DELIVERED BEFORE THE UNIVERSITY OF TEXAS

I am not proposing to advocate the teaching of mathematics, nor shall I attempt to persuade you of its desirability this evening. Why should I not? You even expect it. Perhaps that is just the reason.

We have had all too much of urging in education; all too much oratory; all too much of impassioned appeal; all too much prejudice and unfairness. It is my belief, for example, that mathematics has been more harmed by its advocates than by its opponents. The ill-considered attacks of a few oratorical educators have done more to strengthen the cause of mathematics than have the equally ill-considered efforts of well-meaning friends whose point of view was unduly prejudiced in favor of mathematics.

If we are ever to place education upon a sound basis, in any subject, in all subjects, we must get rid of prejudice and of selfish appeals. We must discuss curricula and subjects on a scientific basis of fact without pre-judgments. If we do not, education and pedagogy will sink to a low level where they will indeed deserve the condemnation which some already express and feel.

My effort this evening will be to discuss mathematics in this scientific and unprejudiced mood, and to tell you about the work of a Committee whose labors are being conducted in just that spirit. I shall endeavor to convince you, not of the value of mathematics, but of the fact that this Committee is actually working in an unprejudiced and entirely scientific spirit upon the whole question of requirements in mathematics. It is perhaps unfortunate that this is striking and unusual. All such committees should work in this manner. But since those of which I have known in the past have not succeeded in doing so to any degree as has this Committee, I feel that I shall need my best powers to convince you that any such thing is possible, is actual.

The National Committee on Mathematical Requirements was first organized some four years ago under the auspices of the Mathematical Association of America. Its problem was to analyse and to weigh carefully all of the problems underlying requirements in mathematics by schools and colleges. The very existence of such a committee has a significance to which I call your particular attention. It means that the leading mathematicians of America are awake—that they are alive to the deficiences of our traditional courses in mathematics. It means that these men have set themselves the task of reforming the teaching of one of the great school subjects, which is evidently bound to be of increasing importance. It is only in this way that any lasting reform may be secured.

The original Committee, whose Chairman is Professor J. W. Young, of Dartmouth College, himself a distinguished mathematician, includes such eminent mathematicians as E. H. Moore of the University of Chicago and other men of national and international reputation whose names you will find upon the back of the Report of which copies have been given to you. It has extended its original membership to include seven well known secondary school teachers, whose names you will also find upon the back of the Report. These include such men as the Vice-Chairman of the Committee, Mr. J. A. Foberg, of the Crane Technical High School and Junior College of Chicago, and your own Mr. P. H. Underwood, of Galveston. The Committee has issued two reports which I shall discuss briefly tonight. One of these, which has already been printed, is in your hands.* The other will be issued through the United States Bureau of Education, as is this one, as soon as at all possible. Copies of these Reports may be secured through any of the members of the Committee, or through the Bureau of Education.

These reports demonstrate so thoroughly the scientific

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*Reprinted in Vol. V, No. 3, of *The Texas Mathematics Teachers' Bulletin.*

and unprejudiced attitude of the Committee that numerous bodies interested in education have given it their whole-hearted support. Thus the United States Bureau of Education has undertaken the publication and distribution of the reports as they appear, and the General Education Board has appropriated liberally of its funds to carry on the work of the Committee.

With such a committee earnestly working in a scientific rather than in a controversial manner, and with adequate financial support, there is strong reason to hope that the mathematicians of the country can join hands with the real educational leaders in a much needed investigation. I, for one, earnestly hope that the undignified and unscientific debating of these topics between mathematicians and educators will come to a speedy termination. Such questions cannot be settled properly in public debate by prejudiced parties to a controversy. They must be settled by amicable discussion at great length in private conference of scholars and teachers.

In order to lay before you briefly the point of view of this National Committee, and to convince you of their unprejudiced attitude, I will quote one passage from the first preliminary report, which is in your hands, and which is entitled "The Reorganization of the First Courses in Secondary School Mathematics." They say:

"The following two principles are made the basis of the discussion:

"1) The primary purpose of the teaching of mathematics should be to develop those powers of understanding and analyzing of quantity and of space which are necessary to a better appreciation of the progress of civilization, and a better understanding of life and of the universe about us, and to develop those habits of thinking which will make those powers effective in the life of the individual.

"2) The courses in each year should be so planned as to give the pupil the most valuable mathematical information and training which he is capable of receiving in that year, with little reference to the courses which he may or may not take in succeeding years."

This is new doctrine in mathematics. Here is no outworn phrase on disciplinary values. The demand is for better understanding of life, for a better appreciation of civilization. To those who do not know the myriad uses of mathematics in life, to those who do not know that our material civilization depends on mathematical knowledge, this platform may seem dangerous; it may seem to wreck the whole basis of mathematical training in secondary schools. It is an interesting commentary which will carry conviction to many that our leading mathematicians are quite willing to rest their whole case on such a platform.

It does wreck some of our traditional courses. Some of our traditional mathematics should have beeen wrecked. Thus the Report of the Committee proceeds, in detail that I will not impose upon you, to specify just what topics should be retained, and what should be omitted. Among omissions in Algebra are, of course, Highest Common Factor and Lowest Common Multiple, Square and Cube Root of Algebraic Expressions, and Simultaneous Quadratics. But such omissions will not surprise those of you who have kept in touch with the recent recommendations of the leaders in mathematical thought in America. We now find other additions to these worst offenders: the theorems on proportion, the theory of exponents, the theory of quadratic equations. In fact, if you will read this Report, you will be surprised to find that the Committee has honestly endeavored to carry out in full its fundamental principles quoted above.

Is then Algebra and possibly a part of Geometry to be adandoned? Far from it. Just as in the teaching of the modern languages, when the old drill on conjugations and declensions and rules was swept away, its place was taken by a real study of the living language as such, as a mode of expression, as a living reality. I venture to say that the disciplinary value of language study has not suffered in the process. And I know that the students taught under the present methods know fully as much about the declensions and the rules a year after they leave the course as did the older students who studied only those rules and declensions.

Similarly in Algebra and Geometry, if we sweep aside some of the outworn topics accumulated when discipline was considered to be the entire purpose and aim, we have to substitute something of the real life of mathematics, some of the ways in which actual people use it.

You will find recommended the study of the formulas that are used in science, you will find logarithms and the slide rule, you will find informal constructive geometry, you will find graphs and the actual uses of graphs, emphasized. The first elements of trigonometry, as used in surveying and in simple mechanics, should, under a modern theory of education, precede formal geometry and should be taught to all children.

To sum up all this, the Committee has insisted that the only single idea which will serve to unify the course is what we call the "function" idea. It is indeed significant that this idea, here emphasized as the one great fundamental idea underlying the entire course in both algebra and geometry, is practically not taught at all in the old traditional courses, nor is it in anywise hinted at in the older traditional text-books.

In passing, I may remark that the function idea is probably unknown to many who took the old traditional courses in mathematics. It seems certain that many of the so-called educators who have criticized mathematics most violently (and unguardedly) are themselves ignorant of the fundamental importance of this idea, not only to mathematics, but to every person who has to deal with or to think about quantities of any sort in any walk of life.

For the function notion is simply the idea that one of two related quantities determines the value of another. Thus the cost of labor affects the price of cotton. The height of water behind a dam affects the water pressure at the bottom of the dam, as you all know, at least vaguely, since the great dam in Austin broke. The diameter of a water pipe determines its area, and thence the amount of water it will carry.

Such ideas occur in every science, in every trade, in every home. It is a mistake to imagine that people will think accurately about affairs that concern them deeply without such training as that which is proposed. Mistakes concerning interest on money, mistakes on life insurance, mistakes on estimated costs and values are as common as mistakes in science, due always to lack of training in functional thinking.

A very interesting book that I can heartily recommend to all of you is SIR OLIVER LODGE, *"Easy Mathematics, Principally Arithmetic,"* written by that great scientist in his prime, before he undertook the more occult studies in which he is now engaged. This is a book for adults. It deals with simple topics in a novel, interesting, and valuable way. In it, he refers to the common mistakes that people make in thinking of relations between quantities. Some of his illustrations, while crude, are striking. Thus he asks:

"If a camel can go for six days after drinking twelve gallons of water, how much water would he have to drink to go for three months?"

He points out that many persons would thoughtlessly solve this problem by ordinary proportion. Or, he asks:

"If a boy can slide ten feet on the ice with a running start of fifteen feet, how far can he slide with a running start of half a mile?"

Again:

"If a horse can carry three hundred pounds when standing on four legs, how much can he carry when standing on one leg?"

Such problems, while ridiculous, illustrate perfectly the commonplace nature of problems not solvable by proportion. The common errors made in such problems exist in very practical affairs. That they may become extremely serious is illustrated by the Quebec Bridge disaster of a decade ago, when hundreds of human lives and seven millions of dollars were lost, all because one man supposed that a girder twice as large as another would be twice as strong. A homely case of the same error has become almost proverbial, in the statement that twice as much medicine does not give twice as much benefit to a sick man.

Between this homely case of doubling the dose of medicine and the awful case of the Quebec Bridge disaster lie hosts of other relations between quantities in everyday life, in business, in science, in which it behooves us to think with care and accuracy. To gain an appreciation of such vital matters, the student must be taught with patience and skill. It is our business to see that our mathematical courses in secondary schools do give training in such thinking. If they do, as they have not in the past, they will be more worthy of recognition by real educators, and of retention in school curricula. The National Committee desires to reform our teaching to this end, and it desires to carry the message to you, educators and students who will be teachers, in order that you may more rationally judge the fitness of mathematics as a school subject, and in order that your own teaching of mathematics should be more valuable to your students.

I have been able to give only a hasty and incomplete explanation of the ideas involved, and of the methods of presenting them to students. You will be able to find all these statements elaborated and explained in the Report mentioned above, and in other Reports now in preparation. I may say that I am myself preparing for the National Committee a pamphlet on the function idea, and on the methods of presenting this idea in elementary courses in Algebra and Geometry. This report may be published during the coming winter, if it receives the approval of the National Committee.*

Meanwhile another Report has just been issued, as yet only in mimeograph form, on the subject of Mathematics for Junior High Schools.† A few copies are available, and the printed form is to be expected soon. The general information contained in this Report, on Junior High Schools in America, is interesting entirely aside from mathematics.

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*Since this paper was read, this report has been presented to the National Committee and has been accepted. It will be published soon.

†This Report is now available in printed form. It was reprinted in Vol. VI, No. 1, of *The Texas Mathematics Teachers' Bulletin.*

I may not weary you again with the details of this second Report. In broad lines, it will be interesting to all of you. It gives first a résumé of the organization of such schools at present, and the detailed reason for their establishment.

In mathematics, its broad principles are the same as those of the first Report which I quoted at length. Thus, it emphasizes the understanding of life and of the world about us as the fundamental purpose of mathematical courses. If the Junior High School occupies the seventh, eighth, and ninth grades, as it does in the great majority of bona-fide Junior High Schools, the Committee recommends that the course be recognized to contain those essential elements of arithmetic, algebra, geometry, and trigonometry which do unquestionably assist toward the understanding and the mastery of life and of the processes of the actual world.

These subjects are analyzed in detail. In arithmetic, in algebra, in geometry, enormous cuts are made which will shock the conservative who is living in the past. They are the logical consequences of the principles accepted and announced above. Thus fractions are to be confined, in general, to those simple fractions most used in the world: fractions whose denominators are 2, 3, 4, 5 and simple combinations of these. Can you remember the fractions with which you yourself dealt? Can you conceive the immense simplification that will result in this one topic alone?

And so throughout Algebra and Geometry, very sweeping changes in our traditions are forced upon us by the same logical and scientific examination. But Trigonometry, in its simplest phases,—the so-called numerical trigonometry—is inserted, even before formal geometry. Why? Because these simplest phases are not difficult, and because they do lead to real appreciation and understanding of world processes, because of their connections with surveying, engineering, artillery, Physics, and many sciences.

I trust that I have now inspired you at least to read, if not accept, the conclusions of this remarkable Committee. Remarkable because unprejudiced. Remarkable because of scientific endeavor as opposed to controversial and arbitrary pronouncements. Remarkable because of its honestly carrying out its own first principles regardless of the consequences.

I believe that the type of work that this Committee is doing is the salvation of mathematics because it is honest and because I think that honesty does pay. I believe such a spirit of scientific enquiry is the salvation also of education in general and of the science of pedagogy. Let me propose to you that the work of this Committee will do as much to establish a good reputation for pedagogy as to establish one for mathematics.

For the principles announced are broader than mathematics. The problems are fundamentally the same in History, in English, in the Sciences, in Pedagogy itself. The main problem, as I see it, is to get away from matters of form, and to give emphasis to matters of spirit. In language, to have less of formal grammar and more of living speech; in History, to have less of dates and names, and more of historical causes; in mathematics, less drill on formal operations, and more real insight into relations between quantities; in pedagogy in general, less pedantry, and more LIFE.

E. R. HEDRICK.

QUANTITY OR QUALITY?

The other day, one of my pupils in a class which had been studying algebra for some six months attempted to solve a problem in addition of fractions. He correctly factored the denominators, found the lowest common denominator, and expressed the fractions with that denominator. Then, being ready to do the addition, he gave forth the astonishing information that $x^2+x^2=x^4$ and $x^4-x=x^3$.

I give this incident, not because it is of startling interest, but because I am coming to believe that it is typical of a great deal of the misconception that is "abroad in the land" as far as algebraic work is concerned. For several years I have been a teacher of mathematics in a city where conditions are such that there is considerable coming and going. Each year we have a number of pupils who come from other schools; and I have observed that it is no particular teacher and no particular climate that produces the $x^2+x^2=x^4$ idea. Pupils drift into second year algebra with no adequate understanding of the fundamentals of the subject. They come to Trigonometry with no idea of simplifying a complex fraction. And I dare say that they get to the University without knowing what a quadratic equation is.

Now what is the trouble, and what are we going to do about it? I have thought about the question a great deal; and it is my profound conviction that we have been attempting too much and therefore accomplishing far too little. Of what use can it be to teach a pupil to extract cube root when he doesn't know the difference between adding fractions and solving a fractional equation? What is the sense in worrying a pupil with the puzzle of factoring $x^4-2abx^2-a^4-a^2b^2-b^4$ while he is still uncertain about handling the factoring of a^3+b^3? Why bother about multiplying $a^{4n+1}-4a^{3n}+2a^{2n-1}-a^{n-2}$ by $2a^3-a^2+a$ when the pupil can't be sure about how to act when told to simplify: $(x+3)^2-(x+2)(x+1)$? Why make a long-suffering and innocent child use reams of paper in multiplying together polynomials a yard long when exactly the same principles are illustrated in shorter problems? Understand, I am not advocating doing away with everything difficult. I am just saying that the complicated work has had more than its fair share of attention and that as a result we have left a great many of our pupils with that foolish notion of "I haven't a mathematical mind and I simply can't learn Algebra."

I think, then, that the time has come for us to do some intelligent omitting. I am heartily in favor of the omissions suggested in the Preliminary Report by the National Committee on Mathematical Requirements about the reorganization of secondary school courses. And even in the part that they have left us, I think there is room for choice. I believe that time and efficiency will be gained if we disregard those strange and unusual factoring cases and other problems whose only claim to distinction lies in the fact that they get a lot of hard work out of the pupil. For my part, never again will I attempt to make any class solve all of the problems in Wentworth's *New School Algebra.* [I hereby apologize to the one on whom I tried it in former years.]

It is, then, my opinion that if we spend our time thoroughly teaching the fundamentals, we will have less occasion to wail about "too little time"; and we will also free ourselves from the humiliation of having our celebrities declare that $x^2+x^2=x^4$.

MARY CAMPBELL.

THE SLIDE RULE

In common with its many other recommendations looking toward the practical application and simplification of the graded school work in mathematics, the National Committee on Mathematical Requirements is strongly advising the introduction to the high school pupil of the Slide Rule. A facile use of the Rule will greatly reduce the labor of a large amount of mathematical calculation in school work, but, of course, the main object of the recommendation is that of giving the student a knowledge of an instrument which in many cases will be of inestimable value to him in his business life.

This short paper will touch on the invention and development of the Slide Rule as well as on the basic theories of the Rule; and is for the High Schoool teacher who will be somewhat interested in the theoretical side of the Rule in order more clearly to impress its practical value on his students.

Since 1800 there have been designed some 260 different types of Rule. A glance at a few of the names of these Rules will be the best criterion by which to judge how extensively this instrument has been put to practical use: "Slide Rule for Use in Chemistry," "Rule for Gauging Casks," "Circular Interest Rule," "Timber Contenting Rule," "Slide Rule for Calculation Blast Furnace Charges," "Rule for Electrical and Mechanical Engineers," "Rule for Nautical Calculations," "Rule for Unit Strains in Columns," "Hydraulic Slide Rule," "Rule for Traction of Locomotives," "Power Computer," "Piece-Work Balance Calculator," "Horse-Power Scale," "Shaft, Beam and Girder Scale," "Pump Scale," "Photo-Exposure Scale," "Rule for Lathe Settings for Maximum Output," "Rule for Strength of Spur Gears," "The U. S. Geological Survey Topographic Slide Rule," "Rule for Wiring Calculations," etc., etc.

However, the only Rule to be discussed here is the one most extensively used—the Polyphase Slide Rule. It is urged that the reader acquire one of these Rules before giving more than cursory attention to descriptions of their mechanism.

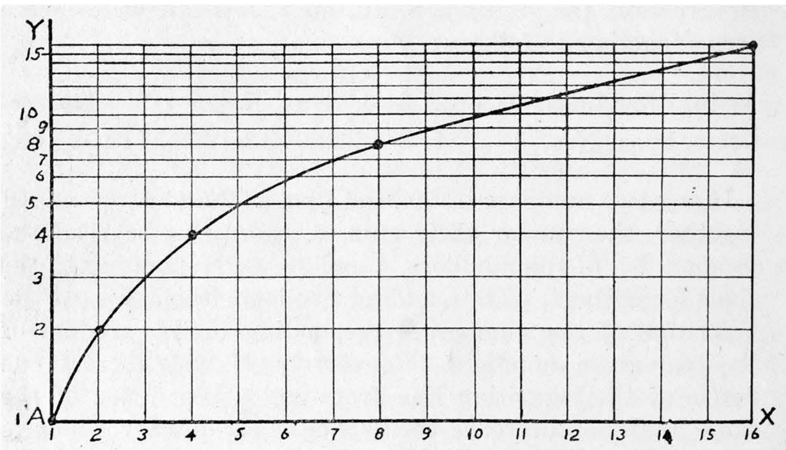
All Slide Rules are based on logarithms, and all calculations which can be performed by logarithms may be solved by the Rule. These fundamental operations include work in multiplication, division, powers, roots, proportion, reciprocals, logarithms of numbers, and trigonometry computations dealing with sines, cosines, and tangents.

The invention of the Slide Rule is attributed to Edmund Wingate, though Edmund Gunter invented in 1620 a logarithm scale, without sliding parts, which was the basis of Wingate's invention of 1630. Let us make a simple logarithm scale and see how the sliding parts perfected its use for computation. Since logarithms are a series of numbers in Arithmetical Progression (as, 0, 1, 2, 3, 4, etc.) corresponding to another series of numbers in Geometrical Progression (as, 1, 2, 4, 8, 16, etc.), we can write these series together as follows:

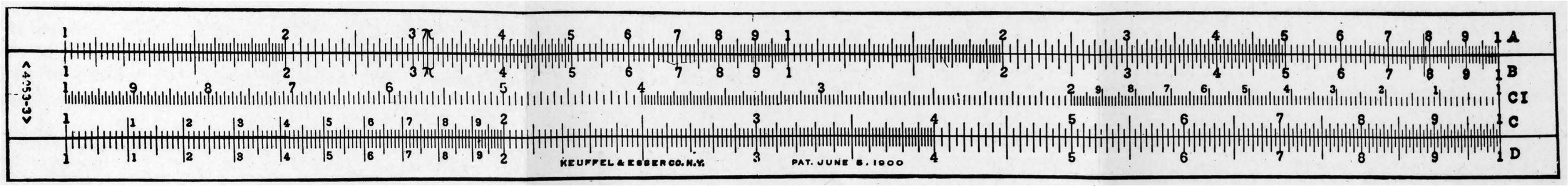
Arith. Prog.	Logs.	0	1	2	3	4	5	6	7	8	9	10	etc.
Geom. Prog.	Nos.	1	2	4	8	16	32	64	128	256	512	1024	etc.

If any two numbers in the first line, as 2 and 4, are added together, then under their sum, 6, there can be read the *product*, 64, of the numbers, 4 and 16, corresponding to the given logarithms. Thus, adding two logarithms, we get the logarithm of the number corresponding to the product of the two given numbers. Conversely, if we subtract one figure of the logarithm line from any other figure of the same line, the difference corresponds to a number which is the *quotient* of the two numbers corresponding to the given logarithms. For example, 8 minus 5 is 3, corresponding to a number 8, or the quotient found by dividing 256 by 32.

Now Gunter made use of this fact by simply laying off a line on which the digits 1, 2, 3, 4, to 10 were arranged in such a way that this proportion held true: the distance from the end marked 1 to the figure 2 was to the distance from 1 to another number as the logarithm of 2 was to the logarithm of that other number; and then he used a pair of compasses to add or subtract the distances along this line, which dis-distances were (obviously, in view of the above proportion) proportional to the logarithms of the numbers with which he was working. To eliminate this use of compasses, Wingate slid two similarly marked rulers against each other—the origin of the name Slide Rule. We now have two scales facing each other, each of which is marked off with digits whose distances from each other were found, theoretically, as follows: First there are laid off on a line equal intervals, 1, 2, 3, 4, 5, etc., corresponding to the Number line of the preceding table. From each of these division points perpendiculars are erected whose lengths *in inches* are shown as the corresponding figure (1, 2, 4, 8, 16, 32, etc.) in the second, or Logarithm, line. A smooth curve is drawn through the upper extremities of these perpendiculars as follows:



The length of the perpendicular from any point, 6 for example, up to the curve, gives the proper relative interval 1–6 on the Slide Rule. Consequently, if we have a 10-inch rule (the usual length) and imagine it to be divided into 100 equal parts, we will find from the properly constructed curve that if the starting point is numbered 1, then at the end of 301 parts we should put the figure 2; at the end of 602 parts, the figure 4; and at the end of 903



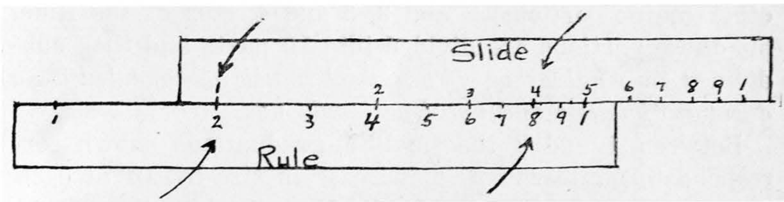
The above is a photograph, slightly enlarged, of the Polyphase Rule sold by The Keuffel & Esser Company.

By cutting around the outline; then down the lines between scales A and B, and between C and D; then pasting the three resulting scales on pieces of thin wood, a Rule can be made which will be of assistance in the learning of the fundamental operations described in the article.

Scales K, S and T and the glass runner carrying the hair-line are not shown.

parts, the figure 8, etc.,—getting a logarithmic scale which, to complete, requires only that the other and intermediary points up to 10 be inserted. This is what is done on the Polyphase and similar Rules. Of course, precision requires that on commercial Rules we use a logarithmic table to find the position of numerous points of subdivisions instead of using the curve similar to the one just constructed to find these points.

On Wingate's Rule one such scale is found, and on the Slide is also a similar scale. The slide is used in order to add distances to given distances on the Rule Scale. For example, if to 2 spaces on the Rule are added 4 spaces on the slide, the *product* is 8 as given on the Rule, which number corresponds to the figure we see we should have by referring to the number line of the first table compiled above. This operation on the Rule would be:—



Hence, a graphical rule for multiplication becomes:—

On Sliding Scale	Set 1	Under the other factor
Rule Scale	Over one factor	Find their product

For example:—

Slide Scale	Set 1	Under 3
Rule Scale	Over 3	Read 9—the product

And the converse for division is:—

On Slide Scale	Set divisor	Under 1
Rule Scale	On dividend	Read quotient

For example:—

Slide Scale	Set 2	Under 1
Rule Scale	On 6	Find 3—the quotient

In the above examples the expressions "Set 1" and "Under 1" refer to *either* the left or the right end of the scales (the *ends* of the scales being known as indices), according as circumstances require. The reason for this lies in the fact that between the left- and right-hand indices there have occurred positions for logarithms referring to all possible sequences of digits composing numbers. Try, therefore, to imagine the scale as being bent to form a circle —its two indices touching and occupying the *same* position.

Now referring to the Polyphase Rule, a cut of which is shown, we see that scales marked at the right by C and D are the only ones with which we have so far dealt. It will be noticed that the portions of both scales between 1 and 2 are subdivided into 10 parts and each part again divided into 10 other parts. On account of their decreasing size all of these subdivisions are not made between each of the other major portions, 2 and 3, 3 and 4, etc., of the Rule, but those portions are divided into 10 parts and then subdivided into *halves* or *fifths which must be considered* as *decimal divisions and not fractional ones.*

Between 1 and 2 the smallest graduations shown correspond to numbers which increase in size by .01 at each step; between 2 and 4 the smallest graduations correspond to numbers increasing in size by .02 at each step; and between 4 and 10 the increase per step is .05.

Since the figures on the scales correspond to logarithms and since the product of any two given numbers contains the same significant figures regardless of the position of the decimal place (the product of 73200 and .00246 containing the same figures as the product of 7.32 and 24.6) we may assign to the left index *any value* which is a decimal part or multiple of 1, such as 10, .1, 100, .01, etc., but the same ratio must then be observed for the other points of division, which become 20, 30, 40; .2, .3, .4; 200, 300, 400, etc. Hence, the point 246 may have the value 2.46; 24.6; or 24,600. Now our examples could have been written:—

Scale C	Set 1	Under 30
Scale D	Over 30	Read 900—the product

or

C	Set .2	Under 1
D	On .6	Read 3—the quotient

The significant figures of the answers remain identical with the previous answers—the position of the decimal point being found (for all elementary calculation) by inspection. One further point with respect to a reading by inspection is this: If 2468 is to be found on the Rule, we can get 246 accurately but we must get 2468 by approximation.

Only after the solution of numerous simple examples will facility in the use of the Rule for multiplication and division be acquired. A good plan to follow is first to try examples the answers of which are known, and thus be certain that the solution is being correctly made.

Now we know another property of logarithms; i. e., to raise a number to a power, we multiply the logarithm of the number by the index of the power. We already have on scale D figures whose *positions* correspond to the logarithms of the figures, and, since the square of a number is often required, we construct a third scale, marked A on the Polyphase Rule, whose figures show the values of the squares of the numbers on scale D. The units in D are twice the units in A. To find the square of 2 (on D) place the hair line of the glass runner over 2 and read the corresponding number on A, the number 4. The square root of a number is found by the converse process. Again we must use common sense to find out the position of the runner, since the square root of 61 has an entirely different sequence of digits from the square root of 610. There are, therefore, *two* scales for any given series of digits, one being for a sequence whose decimal point is different from the other sequence by one place.

The units on Scale K are one-third the units of D. Consequently, numbers on scale K are the cubes of the coinciding numbers on D, and conversely for the cube roots. To extract the fourth root, extract the square root twice.

To extract the sixth root, find the cube root of the square root.

Frequently it happens that in proportion *more* requires *less* and *less* requires *more*. For example, if 80 men can build a bridge in 4 days, how long will it take for 90 men to built it? The C I scale (which is the C scale inverted) used together with the D scale solves such problems.

C I	Set 80	Under 90
D	To 4	Read 3.56—the answer

On the under side of the slide will be found three scales: one (marked S) is for natural sines; one (marked T) for tangents; and a middle one (marked off in equal parts) for logarithms.

To find sines and tangents, remove the slide, turn it upside down, and replace it in the groove, allowing the left and right indices to coincide with the corresponding indices of scales A and D.

Angles on S show natural sines on A:

$$\text{Sine } 4^\circ = .0698$$
$$\text{Sine } 29^\circ\ 30' = .4924$$

(Notice the different positions of the decimal places according to whether the sine is read from the left or right scale on A).

Angles on T show natural tangents on D:

$$\text{Tangent } 10^\circ\ 40' = .1883$$
$$\text{Tangent } 25^\circ = .4663$$

Notice that this scale gives tangents up to 45°; all of which have a characteristic of 0. Those for larger angles are found by using the rule that

$$\tan x = \frac{1}{\tan (90^\circ - x)}.$$

To find logarithms do not change the slide from its normal position with scales B and C uppermost. Place left index of C on the given number read on D, and on the middle scale on the under side of the slide read the logarithm.

Example: To find log. of 25.

C	Set 1	Find on scale of equal parts the mantissa 398
D	To 25	

The characteristic is found by inspection to be 1. Therefore, log 25=1.398.

The mental labor involved in literally thousands of formulas which are in every day use by engineers, merchants, exporters, bankers, etc., are minimized to an astonishing extent by the use of a Slide Rule. Take for example a merchant's problem: He wishes to sell a piece of calico which cost him 65c a yard to realize a profit of 15 per cent on the *cost* price:—

C	Set 100	Below cost
D	To 100 plus percentage of profit	Read Selling price

or

C	Set 100	Below 65
D	Over 115	Read 74.75c—selling price

Now he wants to sell at 15% profit on *selling* price:—

C	Set 100 less % profit (85)	Below cost (65c)
D	Over 100	Read selling price (76.5c)

(Here the indices must be interchanged).

Suppose a banker wishes to know the amount of $150 at 5% compound interest at the end of 10 years:—Set left index of C on 100 plus rate of interest (105) on D; read the corresponding number on Scale of Equal Parts and multiply it (21.2) by the number of years which gives a product (in this example 212) which is then set off on the Scale of Equal Parts to the celluloid index of the under part of the Slide. Coinciding with the principal ($150) on C read the answer ($244.35) on scale D.

To change Centigrade to Fahrenheit:—

C	Set 5	Over Centigrade
D	To 9	Read degrees +32=Fahrenheit

The force of wind, discharge of pumps, horse-power of turbines, discharge of a turbine, horse-power of a steam engine—each from the proper data—may all be found by very simple Slide Rule calculations.

Is it any wonder that those who have made a thorough study of the needs of High School graduates should stress the value of the Slide Rule? Is it not rather a matter to be marveled at that we still use hand-power methods when electrical ones are at hand?

A. E. COOPER.

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EINSTEIN'S RELATIVITY AND GRAVITATION THEORIES

The unsophisticated reader of recent accounts in our popular press of Einstein's theories is likely to get the impression that the world is being turned topsy-turvy; he learns that Euclid and Newton are back numbers, that we are living in four dimensions, that a yardstick is not always a yard long, and many other marvelous "facts." It is true that the theories involve the revision of some of our most fundamental ideas, but the results are not nearly so revolutionary as they are sometimes made to appear. The reason they are still subjects of controversy among physicists is that the practical results differ so little from those of the older theories that it is hard to devise experiments where the difference would be perceptible even to the most delicate instruments we possess.

The theory of relativity gets its name from the fact that it starts with the postulate that all motion is relative. This idea is not a new one; it has long been taught by philosophers. Suppose the earth were completely alone in the midst of infinite space; then not only would we be unable to tell whether it was at rest or in motion, but the very question would be entirely meaningless. Similarly, if there existed just two bodies, and if the distance between them were changing, we could with equal propriety consider either one at rest and the other in motion; the only thing that counts is their *relative* motion. The ancients believed the earth to be motionless, while the sun and stars revolved about it; today we attribute the motion to the earth, yet we continue to say that the sun "rises" and "sets," for only the relative motion affects our daily life.

In spite of the philosophical difficulty of conceiving of absolute motion, modern physicists had adopted theories which led to substantially that idea. In explaining the phenomena of light and electricity, they assumed that all space is filled with a mysterious motionless fluid called the "ether," through which light, wireless telegraph messages, X-rays, etc., are propagated as wave-motions. In these theories, motion with respect to the ether played such a fundamental part that it could properly be called absolute motion. It naturally became an important experimental problem to detect and measure the velocities of bodies with respect to the ether, in particular that of the earth, and a score of experiments were devised for this purpose. Unfortunately for the theories, however, the predicted effects invariably failed to appear; everything happened just as if the earth were at rest, and there were no such thing as absolute motion.

When a theory comes into conflict with the facts, it is of course necessary to revise it, and this was done by Einstein in 1905. By assuming that only relative motions have any significance, he at once accounted for the failures to detect the supposed absolute motion of the earth. (At this time he limited the assumption to uniform motions of translation.) He went much farther than this, however. The whole subject of optical and electrical phenomena in moving bodies had been causing physicists great difficulty, and all the theories explaining them were more or less unsatisfactory. Einstein attacked this problem, and with the aid of a second postulate succeeded in building up a consistent theory which accounted in a simple and natural way for all the known facts.

This second postulate is that the velocity of light appears the same to all observers, regardless of the relative motion of the source of the light and the observer. Stated in this form it sounds innocent enough, but from it (in conjunction with the first postulate) can be deduced some very strange conclusions. These have to do with the measurements of space and time made by two observers who are moving rapidly with respect to each other. Suppose the observers are passing each other at the rate, say, of 10,000 miles a second, carrying clocks which run at the same rate when at rest; then to each one the clock of the other will appear to run more slowly than his own. Furthermore, two clocks far apart from each other which seem to one observer to agree in the time they read will seem to the other to disagree, and the farther apart they are the greater will be their disagreement; so that two events which are simultaneous for one observer are separated by an interval for the other. The situation is like that of a traveler whose watch reads standard time passing through a land where local time is in use. On one particular meridian his watch agrees with the local clocks, but the farther east or west he goes the greater is the discrepancy between them. We can not say that the watch is right and the clocks are wrong, nor that the clocks are right and the watch wrong; both are right, and the difference lies merely in the point of view. So of our two observers above, each naturally considers himself at rest and the other in motion (in accordance with the first postulate), and it is this difference in point of view which results in their disagreement as to measurements of time.

They will differ also in their measurements of lengths. Since a rapidly moving object cannot be measured with a ruler as we would measure a stationary one, the question arises as to what we mean by its length. The most natural definition is that the length is the distance between the position of the front end and that of the rear end of the object *at the same instant.* The italicized words show that the notion of time is involved, and since the clocks of the stationary observer disagree with those of an observer moving with the object, their meausrements of its length will disagree. As a matter of fact, it is easily proved that if two stationary observers mark the positions of the ends of the object at the same instant according to their clocks, the mark at the rear end will appear to the moving observer to be made slightly after that at the front end, and during the interval the rear end moves forward, so that the distance between the marks is less than the length as measured by the moving observer. Thus every rapidly moving object appears to be slightly contracted in the direction of its motion. It will be noted that the change is merely an apparent one, due to the difference in time systems; no physical contraction of the object is implied.

These consequences of the second postulate have led many to reject the theory of relativity, on the ground that they are contrary to common sense, as they undoubtedly are. But to the contemporaries of Copernicus and Galileo the theory that the earth rotates on its axis and revolves around the sun was contrary to common sense; yet the theory prevailed. There is nothing sacred about common sense; in the last analysis its judgments are based on the accumulated experience of the human race. From the beginning of the world up to the present generation no bodies were known whose velocities were not extremely small compared with that of light (186,000 miles a second), and for all such velocities our common notions of space and time and the mechanics based on them are entirely adequate; but the development of modern physics has led to the discovery of very much larger velocities, some of the particles shot out by radium, for example, having velocities as high as 165,000 miles per second. It is not to be wondered at that such an enlargement of our experience requires a corresponding enlargement or generalization of the concepts of space and time, with the aid of which we seek to build up a systematic picture of the external world. Just as the presupposition of primitive man that the earth is flat had to be given up in the light of advancing knowledge, so we are now called upon to give up our presuppositions that space and time are absolute and independent in their nature.

The reader must not expect to understand the theory of relativity in the sense of making it fit in with his previous ideas; for if the theory is right, his ideas of space and time are wrong and must be modified, a process which is apt to be painful. Likewise it is hard to give a satisfactory explanation of the theory in popular language, because the language itself is based on the old concepts; the only language which is really adequate is that of mathematics.

Einstein's theory was put in a beautifully symmetric mathematical form by Minkowski, with the aid of four-dimensional geometry. This introduction of a fourth dimension has led to much misunderstanding; it is not nearly so revolutionary as it sounds. The number of dimensions which any region has means simply the number of quantities it takes to determine a point of it. Thus the surface of the earth has two dimensions, because any point on it is completely determined by two quantities, its latitude and its longitude. Similarly, ordinary space has three dimensions; any point in a room, for example, is determined if we know its distance from the north wall, its distance from the west wall, and its height above the floor. The physicist, however, deals not only with the *positions* of objects, but also with their *motions*, and this involves the use of a fourth quantity, *time*. Hence it is possible to conceive of the history of the universe, past, present, and future, as forming a four-dimensional world, whose elements are not points, but events. This conception is not a new one; the reason it is not common is because the usual assumption that space and time are absolutely independent of each other makes it more natural to think of the three-dimensional space world and the one-dimensional time world as existing separately. According to Einstein, however, space and time are closely bound up with each other, and this close relationship makes it useful to combine them into a four-dimensional "space-time" world. This does not mean, as sometimes stated, that space and time cease to differ from each other; a cook may combine meat with potatoes and call the product hash, but meat and potatoes do not thereby become identical. In short, our views of the dimensionality of the physical world remain exactly the same as they were. Minkowski merely pointed out the great advantages of using the *language* of four-dimensional geometry in describing it mathematically.

The application of Einstein's principles to the various branches of physics has led to many interesting and important results, such as the probable identity of mass with energy, which can not be taken up within the limits of this essay. Suffice it to say that the theory accords perfectly with all known experimental facts.

In the form described above, Einstein's theory was subject to one serious limitation; the first postulate was restricted to uniform motions of translation, although, philosophically, absolute accelerations and absolute rotations are just as inconceivable as absolute translations. After ten years of further work, however, Einstein succeeded in extending and generalizing the theory to include motions of every kind, and in doing so achieved another remarkable triumph, namely a theory of gravitation.

The simple law discovered by Newton, that all material bodies in the universe attract each other with a force proportional to their masses and inversely proportional to the square of the distance between them, had proved remarkably successful in explaining the motions of all the heavenly bodies; in only two or three cases did a slight discrepancy remain between the calculated and the observed motions. The most important of these discrepancies was a slow rotation of the longer axis of the orbit followed by the planet Mercury. In spite of its celestial triumphs, however, gravitation remained persistently aloof from the rest of physics, and the modern scientist knew no more about its real nature than Newton did.

Einstein's view of gravitation is very startling at first sight. Just as the success of his earlier theory was due to his abandonment of our preconceptions about space and time, so the generalized theory gives up another of our cherished notions, that the properties of space are described accurately by the geometry of Euclid. Instead, Einstein assumes that in the neighborhood of every material body space, or rather the "space-time" of Minkowski, is distorted or "curved" in such a way that Euclidean geometry does not apply; and the force of gravitation he regards as our perception of this curvature.

We can not form any mental image of a "curved" space of three or four dimensions, so it is helpful to consider what the curvature means in two dimensions. The surface of the earth looks flat to us, and a surveyor regularly uses Euclidean plane geometry for the figures he lays out on it. How could we prove, without using three dimensions, that it is really not flat, but curved? One way would be to draw a very large triangle, with sides several hundred miles long, and measure the angles accurately; the sum would prove to be greater than 180°, contrary to Euclid's theorem that the sum of the angles of every plane triangle is equal to 180°. Similarly, if we measured the circumference of a very large circle on the earth, its ratio to the diameter would not be equal to π, but a little less. If Einstein is right, similar measurements made in the neighborhood of a large body like the sun would fail to agree with Euclid's theorems, and this is what we mean when we say that space is curved there. Another alternative would be to retain the Euclidean character of space, and merely assume that our measuring rods contract and expand in a peculiar way when near a large gravitating body. This seems rather arbitrary, however, and since we know no logical reason why space should be Euclidean rather than non-Euclidean, it is better to accept the notion of curvature.

The way in which a curvature of space might appear to us as a force is made plainer by an example. Suppose that in a certain room a marble dropped anywhere on the floor always rolled to the center of the room; suppose the same thing happened to a baseball, a billiard ball, and a tennis ball. These results could be explained in two ways; we might assume that a mysterious force of attraction existed at the center of the floor, which affected all kinds of balls alike; or we might assume that the floor was curved. We naturally prefer the latter explanation. But when we find that in the neighborhood of a large material body all other bodies move toward it in exactly the same manner, regardless of their nature or their condition, we are accustomed to postulate a mysterious attractive force (gravitation); Einstein, on the contrary, adopts the other alternative, that the space around the body is curved.

Another way of looking at the matter is this. According to Newton's laws of motion, which form the basis of the classical system of mechanics, a body not acted on by any force remains either at rest or in uniform motion in a straight line. Now in a curved space (the surface of a sphere, for example) motion in a straight line is in general impossible, so the body moves along some curved path; consequently, accepting Newton's laws, we infer that some force must be acting on it.

Even a ray of light follows a path which is not quite straight. Einstein computed how much a light ray from a star would be deviated if it passed close to the sun, and the accurate verification of his result by photographs taken at the total eclipse of the sun on May 29, 1919, has done much to strengthen the belief that the generalized relativity theory is true.

From the mathematical formulation of the theory Einstein derived a new law of gravitation. It is naturally less simple in form than Newton's law, but agrees very closely with the latter in its effects; the motions of most of the members of the solar system are equally well accounted for by either law. There is one significant exception, however; in the case of Mercury, which is the planet nearest to the sun and the one which moves most rapidly, Einstein's law gives an orbit whose longer axis revolves at exactly the rate required to account for the discrepancy with Newton's law already referred to. This furnishes another brilliant confirmation of the validity of Einstein's work.

These new views of space and gravitation have a profound bearing on the grand problem of cosmology—the structure of the universe. One possibility which receives some support from Einstein's theory is that the universe is finite but unbounded, so that a body traveling in a definite direction for a long enough time would ultimately return to its starting point, just as a man travelling in a definite direction on the earth would return after going 25,000 miles. Such questions, however, still remain largely in the realm of speculation.

The generalized theory of relativity represents a long step toward the scientific ideal of a theory which shall bring the whole of physics into one comprehensive scheme; and there is no reason to think that its conquests are yet ended. The extensive field of molecular physics, including such problems as the structure of the atom, still awaits a simple coordinating principle which shall bring order out of the chaos of experimental facts, and where can we look for this principle with better prospects of success? We await the future with hope and confidence.

PAUL M. BATCHELDER.

BROWN MATHEMATICAL PRIZES FOR FRESHMEN

Through the generosity of a graduate of Brown University, three prizes are offered for the best examinations in elementary algebra and plane geometry: a First Prize of fifteen dollars, a Second Prize of ten dollars and a Third Prize of five dollars. Freshmen who offer the minimum requirements in mathematics and are regularly enrolled as students in freshman pure mathematics are eligible.

The second annual competitive examination was held October 18, 1920. The following problems were set and the time allowed was one hour:

1. Given a point P, within the sides of an angle BAC, draw a line through P and terminated in the sides of the angle such that P is the center of this line.

2. Two circles intersect in the points B and E. A line through B intersects the first circle in a point A and the second circle in a point C. A line through E intersects the first circle in a point D and the second circle in a point F. Prove AD parallel to CF.

3. Given:

$$x^2+bx+c=0,$$

find a quadratic equation whose roots are equal and are double a root of the given equation.

4. Which is greater,

$$\sqrt{3}+\sqrt{5}$$

or

$$\sqrt{2}+\sqrt{6}\,?$$

Prove without extracting square roots.

There were twenty-nine contestants, of which sixteen handed in papers. The following high schools had one representative, San Antonio, Kingsville, Temple, Houston, Lyford, Post, Newton, El Paso, Hondo, Oak Cliff, Lufkin; Austin High School and Yorktown High School had two and St. Mary's Academy, Austin, one. The following were the winners:

First Prize, Marcus Evans Mullings, Post High School.

Second Prize, Albert Hughes, Lyford High School.

Third Prize, John H. Weymouth, Main Avenue High School, San Antonio.

The names of the prize winners will be read by President Vinson at the June Commencement and will be published in the University Catalogue.

It may be remarked that this year as well at last year, the first prize winners came from comparatively small schools. A method of stimulating interest in this competition is afforded to the mathematics teachers in the high schools by submitting these problems to their best students. The Editors will be pleased to publish solutions to these problems.

H. J. ETTLINGER.

SOME ELEMENTARY PRINCIPLES OF NON-EUCLIDEAN GEOMETRY

The term non-Euclidean geometry is applied to a system of geometry built up without the use of Euclid's fifth postulate, which reads: "If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two lines, if produced indefinitely, meet on that side on which are the angles less than two right angles."

The idea of a non-Euclidean geometry came into being after many years of futile attempts to prove the truth of the parallel postulate from the other assumptions previously made by Euclid.

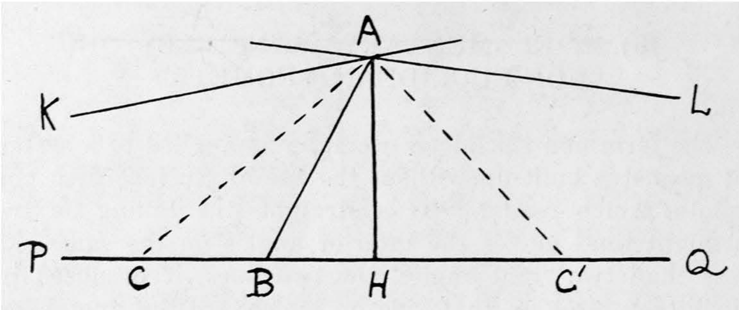
Saccheri (1667-1733), Lambert (1728-1777), and Legendre (1752-1833) each made important contributions to non-Euclidean geometry, but they failed to see the true meaning of their results. Finally Lobachevsky, a Russian, J. Bolyai, a Hungarian, and Gauss, a German, each working independently, reached the conclusion that the parallel postulate could not be proved from the other assumptions, and that a logical system of geometry could be built up without it.

The system built up be these three men is known as Lobachevskian geometry. In 1854 Riemann, working from the standpoint of higher mathematics, discovered a new type of geometry known as Riemannian geometry.

There are three methods of developing the three types of geometry, i. e., Euclidean and the two types of non-Euclidean. The first is by elementary methods similar to those used by Euclid, and was used by Lobachevsky, Bolyai, and Gauss; the second is by a system of projective measurements, introduced by Cayley and used extensively by Klein; and the third is by the use of higher mathematics.

We shall now proceed to give a more general definition of parallel lines than that of Euclid.

Let PQ be any straight line and A any point not on PQ. Through A there passes a set of lines intersecting PQ, since



any point on PQ may be joined to A. It is conceivable that there are lines through A that do not intersect PQ. In that case there will be lines such as AL and AK not intersecting PQ and forming the boundaries of the set of lines which meet PQ. Such lines are said to be parallel to PQ.

Otherwise stated: let AB be any line through A intersecting PQ. The line AL is said to be parallel to PQ at the point A (1) if AL does not intersect PQ no matter how far produced; (2) if any line through A in the angle opening BAL does intersect PQ. The $\angle$HAL is called the angle of parallelism for the distance AH. AH is perpendicular to PQ.

Also, we may now state that:

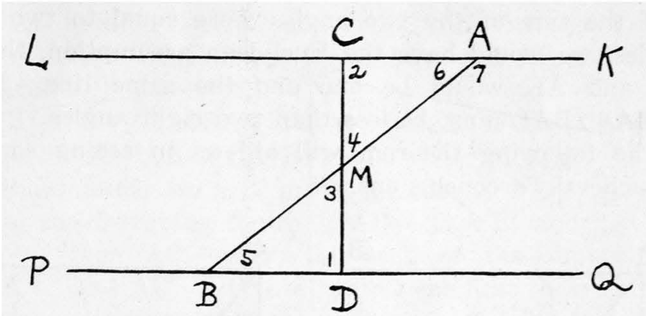
(1) A straight line maintains the property of parallelism at all points.

(2) If a line is parallel to another line, the second is parallel to the first.

(3) If two lines are parallel to a third line, they are parallel to each other.

Euclid, Lobachevsky, and Riemann each made certain assumptions concerning parallel lines from which they drew certain conclusions.

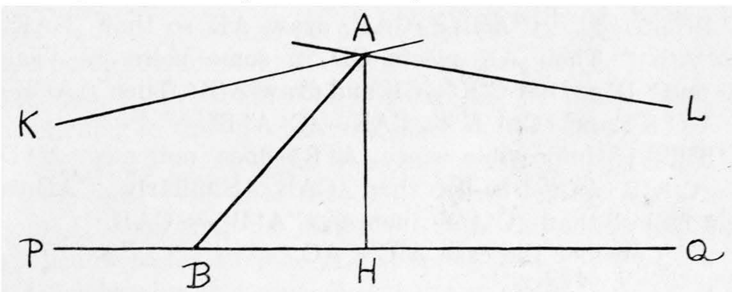
Euclid's postulate may be replaced by the assuption that through any point in the plane there goes one and only one line parallel to a given line. To assume this is to assume that AL and AK of the preceding figure form one and the same straight line. Hence $\angle$HAL and $\angle$HAK are right angles and therefore are equal angles.



Take M as the midpoint of AB and draw MD$\perp$PQ, intersecting AL at C. Then $\angle 2$ is a right angle and the triangles MCA and MBD are right triangles, and are congruent. (Now AM=BM; $\angle 2=\angle 1$, and $\angle 3=\angle 4$); $\therefore \angle 5=\angle 6$, $\therefore \angle 5+\angle 7=2$ rt. $\angle$s.

By our definition, any line through A in the angle BAK meets PQ. This assumption is, we see, equivalent to Euclid's fifth postulate upon which the Euclidean Geometry is built.

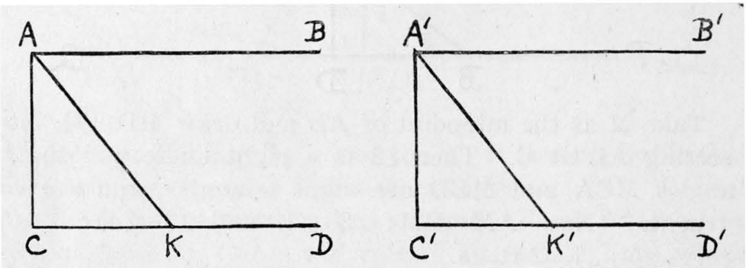
Lobachevsky assumed that through any point in a plane there go two lines parallel to a given line. Let AL be the



limiting position of lines drawn from any point on PQ to the right of H through A and AK the limiting position of those to the left of H, that is, let AL and AK be two lines through A parallel to PQ. It follows that $\angle$QBA+$\angle$BAL is less than two right angles, for if $\angle$QBA+$\angle$BAL were greater than two right angles we could draw through A in the angle opening BAL a line not meeting PQ, and we would have three lines through A parallel to PQ, which is contrary to the assumption.

If the sum of the two angles were equal to two right angles, we would have the Euclidean assumption, that is, AL and AK would be one and the same line. Hence $\angle QBA + \angle BAL$ must be less than two right angles.

The following theorem will aid us in seeing some of Lobachevsky's conclusions.



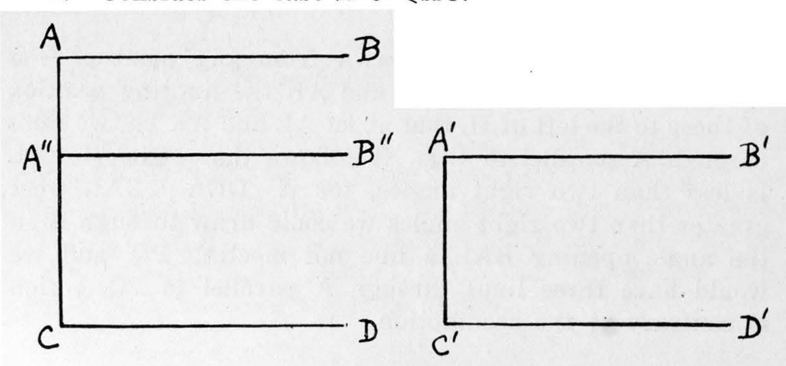
Let AB and CD be two parallel lines cut by a third line AC, and let A′B′ and C′D′ be two other parallel lines cut by a line A′C′, and let $\angle DCA = \angle D'C'A'$. Then:

1. If $A'C' = AC$, $\angle C'A'B' = \angle CAB$.
2. If $A'C' < AC$, $\angle C'A'B' > \angle CAB$.
3. If $A'C' > AC$, $\angle C'A'B' < \angle CAB$.

Proof. 1. $\angle C'A'B' < \angle CAB$, draw AK so that $\angle CAK = \angle C'A'B'$. Then AK meets CD in some point K. Take K′ on C′ D′ so that $C'K' = CK$ and draw A′K′. Then $\triangle ACK = \triangle A'C'K'$, and $\angle C'A'K' = \angle CAK = \angle C'A'B'$.

This is impossible since A′ B′ does not meet C′ D′, $\therefore \angle C'A'B$ can not be less than $\angle CAB$. Similarly $\angle CAB$ can not be less than $\angle C'A'B'$, hence $\angle C'A'B' = \angle CAB$.

2. Consider the case $A'C' < AC$.



On AC take $A''C=A'C'$ and draw $A''B''$ parallel to CD. Then $\angle CA''B''+\angle C'A'B'$, as just shown, and AB and $A''B''$ are parallel.

$\therefore \angle B''A''A+\angle A''AB<2$ rt. $\angle$s. But $\angle B''A''A+\angle CA''B''=$ 2 rt. $\angle$s. $\therefore \angle A''AB<\angle CA''B''$ or $\angle CAB<\angle C'A'B'$.

In like manner we may prove the case $A'C'>AC$.

If in the foregoing figure $\angle CAB=\angle C'A'B'$ and $\angle ACD=\angle A'C'D'$, then $AC=A'C'$, for each of the suppositions $AC<A'C'$ and $AC>A'C'$ contradicts the first theorem.

Riemann assumes that through a given point in a plane no line can be drawn parallel to a given line.

We may now compare certain conclusions based upon the foregoing conclusions.

Euclid: The angle of parallelism is constant.

Lobachevsky: The angle of parallelism is fixed for a fixed distance and decreases as the distance increases.

Euclid: Parallel lines will remain the same distance apart, however, far produced.

Lobachevsky: Two parallel lines approach each other continually and their distance apart becomes less than any assigned value.

Riemann: All lines perpendicular to the same line meet in a point at a constant distance from the straight line. (For example, on the surface of a sphere all meridians are perpendicular to the equator, and meet at the poles.)

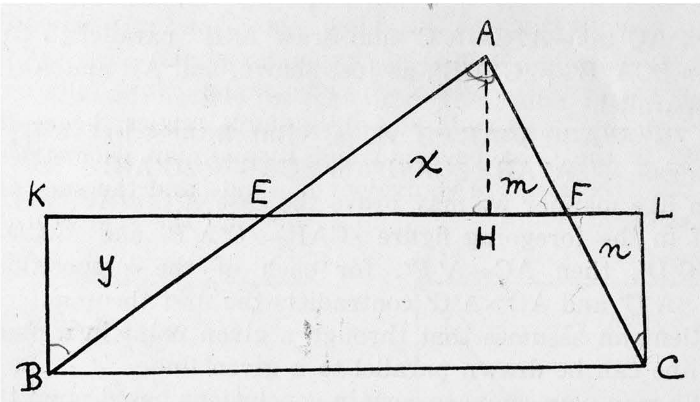
Euclid: Two straight lines intersect in only one point.

Riemann: All lines through a point O meet again in a point O′ such that the distance OO′ is constant. (This is applicable to the surface of a sphere.)

Lobachevsky also concludes that any angle is an angle of parallelism belonging to a certain distance, and that two straight lines which neither intersect nor are parallel have a common perpendicular.

We will now consider the question of the angles of a triangle.

In any $\triangle ABC$ take E and F as the midpoints of the sides AB and AC and draw the straight line KEFL. From A, B, and C, drop perpendiculars to KL. Then $\triangle x=\triangle y$, and



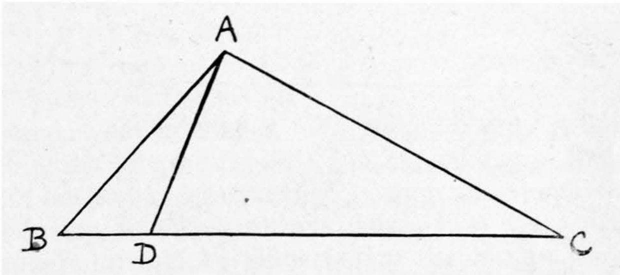
$\triangle m = \triangle n$ (a side and an acute angle of one are equal to a side and an acute angle of the other).

The $\triangle ABC$ is equivalent to the quadrilateral BCLK. Also the sum of the angles of the triangle is equivalent to the base angles of the quadrilateral. BKLC has two right angles and two equal sides adjacent to the right angles, and opposite to each other. Such a figure is called an isosceles birectangular quadrilateral. The study of the angles of a triangle is thus reduced to the study of an equivalent birectangular quadrilateral. Call the lower base the base and the upper one the summit. We can prove easily that the summit angles of an isosceles birectangular quadrilateral are equal, and that each summit angle is equal to, greater than, or less than a right angle according as the summit is equal to, greater than, or less than the base. Also that in Euclidean geometry the summit angles are right angles; in Lobachevskian geometry, less than right angles; and in Riemannian geometry, greater than right angles. (Example: The sum of the angles of a triangle on the surface of a sphere is greater than two right angles.)

Areas

We shall define two equivalent polygons as polygons that can be divided into the same number of triangles which are congruent pairs. It has been proved that a triangle is equivalent to an isosceles birectangular quadrilateral having its summit equal to one side of the △ and each summit angle equal to ½ the sum of the angles of the △.

Now an isosceles birectangular quadrilateral is fully determined by its summit and its summit angles, hence it follows in the Lobachevskian and Riemannian geometries that two triangles are equivalent if a side and the sum of the angles of one are equal to a side and the sum of the angles of the other.



In the △ ABC, draw AD to any point on the base BC; we will say that the triangle is divided by this line transversally. Let S= the sum of the angles of the △ABC, and S_1 and S_2 the sum of the angles of the △s ABD and ADC, respectively. Then $S=S_1+S_2-180°$ and we may write:

$$180°-S=(180°-S_1)+(180°-S_2)$$

or $$S-180°=(S_1-180°)+(S_2-180°).$$

In the Lobachevskian geometry $180°-S$ is positive and is called the defect of the △. In Riemannian geometry $S-180°$ is positive and is called the excess of the △. The following theorem follows:

If a △ is divided transversally the sum of the defects or excesses of the parts is equal to the defect or excess of the triangle. This is true for any system of dividing the △. Also by Lobachevskian geometry the area of a triangle is equal to a constant times its defect, and in the Riemannian, the area of a △ is equal to a constant times the excess. (This is applicable to a spherical triangle.)

The area of a polygon is found by dividing it into triangles.

Reference:

J. W. A. Young: *Monographs on Modern Mathematics*, section on Non-Euclidean Geometry, by F. S. Woods.

ETHEL BURCH.

WHY STUDY MATHEMATICS?

"You are a mathematician, therefore you had better add the score," I said, turning to my neighbor, who leaned back in his easy chair, as soon as our game was over.

"Oh, he can no more add a column of figures than he can fly," replied the pedagogue in our group.

"Well, what is mathematics good for if a professor of mathematics can not even do a simple problem in arithmetic?" inquired the artist.

"Indeed, what is it good for," sighed the professor, who was thinking of grocers' bills and his children's shoes.

"It is not to be expected that an impractical subject such as mathematics, which always moves in a vague, rarefied atmosphere, should be subjected to the most severe criticism nowadays and be eliminated from the curriculum of a really modern school?" asked the pedagogue.

"Yes, and once eliminated from the schools, the study of mathematics will, faute de combatants, be condemned to a languishing existence and gradual extinction in the colleges and universities," said the mathematician.

"Such a prospect is indeed alarming, and, I should think, of a character to set your brotherhood thinking. What do you, mathematicians, propose to do about it?" I asked to get our friend warmed up a little.

"There is not very much to be done about it. The tendency of the times appears to be in that direction and I fear that we must let the pendulum have its swing. Experience has taught at least this lesson, that there is no other school but hers in which the human race will learn anything."

"Do you think then than the pendulum will swing back again, and that experience will prove the current tendency to be a passing one?"

"More than that, I am absolutely convinced that if we really knew what we wanted, we would turn towards mathematics rather than drift away from it. It is the really modern school which needs mathematics; it is the modern college which should stress the study of mathematics and it is the modern university which should give to mathematics the recognition which it deserves."

Our friend raised himself in his chair and became more excited than he had been all evening, quite losing his rather sleepy indifference.

"But, you will have to concede that an education, which claims to be modern, must at least be practical," I ventured to remark.

"Indeed, if not practical," the pedagogue declared, "a subject has no place in a modern school; it must be able to prove its usefulness, and no subject should be included in any school's curriculum nowadays, simply because it has always belonged to the program."

"Practical, practical," the professor said with some irritation, "what does the word mean?"

"Don't you know what practical means? Why, to be practical, a thing must lead somewhere."

"Somewhere? It is sure to do that, as we learned from the Cheshire cat long ago. Even in leading nowhere, it certainly leads us somewhere, and by going to a nowhere we might incidentally get everywhere," he answered with a quizzical smile.

"What sort of talk is this?" the artist queried. "Is that perhaps mathematics?"

"We are indeed," the mathematician replied a little mysteriously, "in the border regions of our domain. The critique and careful delimitation of terms, such as some, any, every and the like, used so slovenly in our everyday language is the concern of one branch of mathematics. And among the words in need of much more careful definition than they ordinarily receive, *practical* takes an important place."

"How do you mean? I believe that that word at least is perfectly understood by every one," I protested.

"Don't you mean by any one," the mathematician interjected smilingly. But, the rest of us looking somewhat puzzled, he continued, "But tell me, is it practical to go to New York?"

"Why! How could I answer this question, without knowing your ultimate object?"

"How then could you expect me to answer the question whether mathematics is practical? For a person in Chicago, it would be practical to go to New York, if he wanted to cross the Atlantic, but quite impractical, if it were his purpose to cross the Pacific. *Practical* needs qualification as to the ultimate end pursued; we must say 'practical for this or that purpose.' There is nothing that is universally useful and there is nothing that is not practical towards some end. The question whether a particular subject in our curriculum is practical depends for its answer therefore upon the aim we are setting for our education, whither we want it to lead, whether to a larger bank account primarily, or to a larger understanding of the universe, or to some other condition equally useful to humanity. When once this question is more or less definitely settled, but not until then, can we determine whether or not a subject is 'practical'; and even then we can do so only if we know enough about the subject to grasp its relation to our ultimate aims."

"*Practical* always needs a qualification then according to your judgment," said the pedagogue, "just like *efficient*."

"No; that word is condemned unqualifiedly," was the reply; and the tone of the reply revealed a bitterness reminiscent of past experiences.

"I can see what you mean and I dare say there is a good deal in what you say. But I do not think there is any doubt in anybody's mind as to the aim he is pursuing nowadays. Everyone knows what he is striving for and there seems to be no need therefore to qualify the word 'practical'."

"This may be true of your adults; but do you think that it is true of our young men? Don't you rather think that there are very few who are conscious of any purpose whatsoever? Usually the young people drift into an acceptance of the aim that is generally pursued in their environment and do not become aware of their having accepted it, if ever, until they are too much loaded with responsibilities to change front. Then all their social interests conspire to make permanent a pursuit which was started by their following the mob. How much fine enthusiasm, how much real power becomes diverted in this way from its natural aim? Would it not be infinitely better to have our young people consider during their educational career what aim they are pursuing,what is to be their ultimate objective? At any rate will you not agree that an educated person should have given some serious thought to this question?"

"Yes, to this I certainly agree," said the artist. "I hold it to be one of the principal tasks of education to lead its objects to a serious examination of what they are capable of doing, and to a more or less conscious choice of their life-work on the basis of such examination. Certainly the colleges should perform this task; the college man should use the inestimable advantage of four years' freedom, to become acquainted with different fields, and to become thereby acquainted with himself. He should, therefore, perhaps with the advice of older men, be able to decide intelligently to what pursuit his life's career is going to be devoted."

"Not only the colleges, but the secondary school as well, which affect a so much greater number of young people, should devote themselves to this task. There the need of guidance is much greater, and much more depends therefore upon the human qualities of the teacher quite apart from his ability in the subject he teaches. If the teachers are to fulfill this function, however, they must have qualifications different from those laid down in Goldsmith's 'Vicar of Wakefield.' They should be men and women of large insight, of culture, energy and intelligence," the mathematician said.

"Will not such men always prefer the more remunerative careers of business to the humble calling of the teacher?" I asked. "But, pray, what has this whole question of practicality got to do with the question under discussion, whether mathematics has a place in a modern school?"

"More than you may think," was the mathematician's reply. "It is probably its apparent impracticability, in the popularly accepted sense of that word, that leads the student of mathematics to ask what it is all good for, and hence to face the important question as to what practicality means. A little right guidance will thence lead him to a serious consideration of his aims and of the ideal which he is to serve. And have we not recognized that this is of the greatest value to him and to the society in which he lives? It is eminently practical, because it is prerequisite to every consideration of practicality."

"This is indeed an interesting claim for mathematics," the pedagogue conceded. "However, it will hardly suffice, I think, for the struggle for recognition which it has to undertake; has it no other claims?"

"And what about your everywhere which is a nowhere?" the artist inquired.

"Ah, that is somewhere and so are the other claims for my subject. But our scores have not yet been added, and the hour is advancing. And what mathematics really is and why the modern world, conscious of its problems, should seek it, had better be left for another time."

"You will agree then at least that it is practical to retire before sunrise."

So we agreed.

It was some time after this before we four met again. Our mathematician was engaged in conversation with the artist, when the pedagogue and I came upon them.

"That is exactly what art considers as its domain," the artist exclaimed in a tone of surprise; and turning to us "our friend here has been trying to convince me that mathematics is an art, and I fear he has pretty nearly succeeded."

"Interesting! but let us hear something of the argument," I requested.

"Well," replied the artist, "to give expression to universal ideas, to emotions common to large groups of men and appearing over long periods of time, rather than to those belonging to a moment or to an individual only, is the ambition, conscious or unconscious, of every artist; it is that which distinguishes art from artistry."

"Very well," said the pedagogue, "but mathematics certainly has nothing to do with the emotions."

"Don't be too sure of that; its development is to a very great extent stimulated by an emotional passion for truth and for beauty. But it is not upon this that I was basing my argument. To really understand what appears to you as a contradiction, you will have to consent to do a little thinking."

"We are ready."

"Well then, in position, and off we go. In the first place, I must observe that your feeling of contradiction between mathematics and the arts arises from the fact that after all you think of mathematics as a decidedly 'practical' subject, as a subject dealing only with hard concrete facts, dry-as-dust, with the purely numerical aspects of life. While nothing is less true in the sweeping sense in which this judgment is generally accepted, it is so nevertheless that mathematics considers many 'practical' questions. On its march towards universals, mathematics passes through various stages, dealing with concepts which range from the most concrete to the most abstract attainable in each domain. For example, to reach the concept of the number 'four,' we may begin, logically speaking, with something as concrete as an apple."

"A real eatable apple?"

"We had better begin with an eatable one, I think; but before we finish we shall consider also less attractive ones. In fact, if we compare different specimina of the genus, we will notice that, although there are no two alike, they all have certain resemblances, as well as many differences. It is the collection of those properties that are common to all the apples within our ken, that constitutes their 'appleness,' and, it is this abstraction which is contained in the concept 'apple.' Successive repetitions of this concept bring us to the notions of two, three, four apples. A similar process leads to notions 'four books,' 'four horses,' 'four objects,' etc., and when this stage is reached, we are ready to take the next step. Just as we abstracted 'appleness' from apples by observing the qualities common to different specimina, so we obtain the concept 'four' by abstracting 'fourness' from the different groups like 'four apples,' 'four books,' 'four horses,' etc. All these different groups possess many differences, but also many likenesses; the collection of their common qualities, difficult to describe explicitly, constitutes 'fourness,' and it is this which gives rise to the abstract concept 'four'."

"There must be a fourness among us four then," said the artist.

"Yes, and you can get a conception of the extent to which the abstraction process has already gone, when you realize that this fourness would not in any way be affected if in place of one of us there were a sponge, or a rock, or a lion."

"That is to say then, that in elementary arithmetic we are already dealing with qualities extremely universal," I observed.

"Yes, but we are only at the beginning of the hierarchy of abstractions with which the mathematician deals and which may ultimately lead him to entities, which possess but few properties. But the fewer the properties of the entities with which we are dealing, the greater is the sweep of any statement concerning them. For every time we take a further step in abstraction, we add new regions to the domain under our sway, and the successive abstractions become representatives of ever vaster domains. Any discovery made concerning these abstract entities becomes therefore immediately applicable to an enormous field, in every section of which it assumes particular significance."

"It is this trend towards, and this interest in universals which makes of mathematics an art," the artist now remarked. "And, indeed," he continued, "it appears that mathematics goes much farther in this direction than any other art, except perhaps music."

"And it is also evident that there will be many who will stop long before the extreme abstractions are reached, and who will be content to deal with fournesses and fivenesses," said the pedagogue.

"Indeed, this is the reason why you will find some mathematicians dealing with things as concrete as projectiles, electric waves, or birth rates, while there are others who never come closer to earth than variables, functions and limits."

"But what connection is there," the pedagogue now asked, "between all this and the claim of mathematics for a position in the modern world?"

"This—that the power to abstract out of a mass of apparently diverse facts the essential kernel, which is determined by their common qualities, that the power to separate, in that sense, the significant from the accidental qualities, the permanent from the ephemeral, the universal from the particular, and furthermore that the power to apply abstract general principles to definite concrete situations, are attributes exceedingly valuable to any man at any time, and particularly at the present time."

"Mathematics looked at from that point of view would indeed be a strong stimulus to the imagination," the artist said, "to a creative imagination, capable of projecting the mind beyond the concrete facts of everyday experience and of seizing fundamental principles."

"Granted," said the pedagogue, "but you will have to concede that these great advantages arising from the study of mathematics can come at best only to those who have advanced far in its study and can have little significance for the question as to the place of mathematics in, let us say, the curriculum of the secondary school."

"On the contrary; while the fullest value of these advantages is necessarily reserved to those who have labored long and hard, there is a great deal to be gained by a few years' study, provided this study takes place under competent teachers who have themselves been in contact with creative mathematics. In order to be more explicit on this point, I shall have to be somewhat technical."

"We don't know much about your subject," I said, "but we shall try to refute your arguments."

"Which a little more knowledge would certainly compel you to accept; but, to the point. The most fundamental abstractions of the mathematics of the last hundred years are the concepts *variable* and *function of a variable*. While the meaning of the former concept is perhaps sufficiently indicated by the word itself as ordinarily understood, the latter requires some explanation. In its broadest sense, a functional relation is said to exist whenever two or more variables are related to each other in some way or other such that the value of one of them follows from the others. Examples of functional relations abound everywhere: the dimensions of an iron bar are functionally related to its temperature; so is the price of a commodity to the amount of it that is available, and to the gold supply, and to the distance between the place of production and the place of sale; so is the growth of a child related to the amount of nourishment it receives, as well as to a number of other elements; and the nutritive value of a particular food to the number of calories which it can produce, and the advertising value of a newspaper to its circulation, and the postage on a letter or package to its weight and value. We can say, in brief, that everything that is capable of quantitative expression is functionally related to one or more other such elements, without having thereby exhausted all the possibilities of the concept *function*. Such a functional relation can be anything from simple proportionality to the most complicated relation, which requires a great deal of mathematical machinery for its study. And elementary mathematics have the almost exclusive role, from the point of view of the science as a whole, of building up the fundamental elements of the machinery necessary for the study of functions."

"It seems to me, that the subject as taught in our schools stops with the building of the machinery, without giving any idea of the uses to which it may be put," said the pedagogue.

"Quite true, but don't you see what tremendous possibilities there are in the subject, if we give to the study of functional relations the central place which it should have, even in elementary mathematics?"

"Yes, but how are you going to do that?"

"Certainly not by eliminating the subject from the schools, nor by curtailing its study. The all-pervasiveness of the function concept, which penetrates even into the simplest phases of our lives, makes it possible to bring every student into intimate contact with it. To attain this end we must undertake a careful revision and reorientation of the subject, so as to organize it around the function as a central concept."

"Is this possible?" the artist asked.

"Unquestionably; a great deal has already been done in this direction and if you could listen to me for a few moments, I should make good this claim for the potential development. But what is more, this reorientation of the subject makes possible, yes, it demands a closer adherence to concreteness, a more organic relation to the pupils' experience than the subject possesses now."

"This then is a most urgent need," I observed, "if mathematics is to survive in the modern school."

"And it is a challenge to the members of your profession," said the pedagogue.

"Very true; and to meet the challenge, the profession needs the confident support of educational authorities and of the public, as well as the inspiration of the artist," the mathematician concluded with a gesture which seemed to invite our cooperation.

"If in that way we should help to bring about the results which you have sketched, our support will be readily granted," I replied.

"Then we shall certainly get somewhere," the artist joined in.

"And I grant that the study of mathematics would then be practical, even if a professor of mathematics could not add a column of figures," the pedagogue added.

Thus we parted.

ARNOLD DRESDEN.

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A MATHEMATICIAN IN LOVE

A mathematician fell madly in love
  With a lady, young, handsome and charming.
By angles and ratios harmonic he strove
Her curves and proportions all faultless to prove,
  As he scrawled hieroglyphics alarming.

He measured with care, from the ends of a base,
  The arcs which her features subtended;
Then he framed transcendental equations, to trace
The flowing outlines of her figure and face,
  And thought the result very splendid.

He studied music (since music hath charms for the fair),
  The theory of fiddles and whistles;
Then composed, by acoustic equations, an air
Which, when 'twas performed, made the lady's long hair
  Stand on end like a porcupine's bristles.

The lady loved dancing, he therefore applied
  To the polka and waltz an equation;
But when to rotate on his axis he tried,
His center of gravity swayed to one side,
  And he fell by the earth's gravitation.

No doubts of the fate of his suit made him pause,
  For he proved to his own satisfaction
That the fair one returned his affection—because
As everyone knows, by mechanical laws,
  Re-action is equal to action.

"Let X denote beauty—Y manners well-bred—
  Z fortune—(this last is essential)—
Let L stand for love," our Philosopher said—
Then L is a function of X, Y, and Z,
  Of the kind which is known as potential.

"Now to integrate L with respect to D T,
  (T standing for time and persuasion)
Then, between proper limits 'tis easy to see,
The definite integral Marriage must be—
  (A very concise demonstration)."

Said he, "If the wandering course of the moon
  By Algebra can be predicted,
The female affections must yield to it soon"—
But the lady ran off with a dashing dragoon,
  And left him amazed and afflicted.

—*Boston Transcript.*