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On some residual and locally virtual properties of groups

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On some residual and locally virtual properties of groups

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Dedicated to my parents, Mike and Yvonne, for their sustained and unwavering support.

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On some residual and locally virtual properties of groups

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We define a strong form of subgroup separability, which we call RS separability, and we use this to combine LERF and Agol's RFRS condition on groups into a property called LVRSS. We show that some infinite classes of groups that are known to be both subgroup separable and virtually RFRS are also LVRSS. We also provide evidence for the naturalness of RS separability and LVRSS by showing that they are preserved under various operations on groups.

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Chapter 1

Introduction

Here we provide motivation for the topics studied in this dissertation as well as a brief summary of background material, and main results. We provide full definitions in Chapter 2.

1.1 Background

Let G be a group. A subgroup $H \leq G$ is **separable in** G if H is equal to the intersection of all subgroups of finite-index in G containing H. A group G is **residually finite** (RF) if the trivial subgroup is separable in G. G is **locally extended residually finite** (LERF) if all finitely generated subgroups are separable in G. Subgroups of RF and LERF groups are RF and LERF, respectively. Also, both RF and LERF are virtual properties: if $H \leq_f G$ is a subgroup of finite index, and K is separable in H, then K is separable in G.

Many groups are known to be RF: for example, Mal'cev's Theorem (sometimes called Selberg's Lemma) asserts that all finitely generated linear groups are RF. LERF is a much stronger property—for example, the linear group $SL_n\mathbb{Z}$ is not LERF for any $n \geq 3$ [20]. However, several important classes of infinite groups are LERF: a celebrated theorem of Marshall Hall Jr. is that free groups are LERF [15]; a well-known theorem of Peter Scott asserts that the fundamental group of a closed surface is LERF [25]; and Henry Wilton has recently shown that fully residually free groups (also known as limit groups; see Section 2.3) are LERF as well [28]. Compact Seifert fiber spaces are LERF ([25] and [26]). An important class of hyperbolic 3-manifold groups that are LERF is the class of Bianchi groups, $PSL(2, \mathcal{O}_d)$; see [20].

The main connection between subgroup separability and topology is Scott's criterion (see [25], and Sections 4.2 and 5.2 below), which allows one to promote π_1-injective immersions to embeddings in finite-sheeted covers. Indeed, some of our motivation for studying these topics comes from hyperbolic 3-manifolds: suppose M is a closed, hyperbolic 3-manifold, so

$$M = \mathbb{H}^3/\Gamma$$

where Γ is a discrete, torsion-free subgroup of $\mathrm{Isom}^+(\mathbb{H}^3) \cong PSL_2(\mathbb{C})$. The Virtual Haken Conjecture states that there is a finite-sheeted cover $M' \to M$ and a closed, homeomorphically embedded, π_1-injective surface $S \hookrightarrow M'$ with $\chi(S) < 0$. There is a famous two-step program to approach this conjecture:

1. **The Surface Subgroup Conjecture**: There exists a closed hyperbolic surface group $\pi_1 S \leq \pi_1 M$. This yields an immersion $S \looparrowright M$.

2. **The LERF conjecture**: $\pi_1 M$ is LERF.

Scott's criterion yields an embedding of S in a finite-sheeted cover $M' \to M$:

$$\begin{tikzcd} & M' \arrow[d] \\ S \arrow[r] \arrow[ur, dashed] & M \end{tikzcd}$$

The surface subgroup conjecture was proven in 2009 by Kahn-Markovich [18], but as of the writing of this thesis, the LERF conjecture remains open.

As mentioned above, RF and LERF are preserved by subgroups and supergroups of finite index. These properties are also preserved under free product ([13], [4]). An important distinction is that RF is preserved under direct product, but LERF is not; for example, F_2, the nonabelian free group on two generators is LERF, but $F_2 \times F_2$ has unsolvable generalized word problem, and so is not LERF [23].

1.2 Main results

In [1], Agol defines a property of groups called RFRS, and he proves that if the fundamental group of an irreducible 3-manifold M has this property, then M is virtually fibered, that is, M has a finite-sheeted cover $M' \to M$ such that M' is a fiber bundle over S^1. He shows that finitely generated right-angled Coxeter groups are virtually RFRS, and it follows that closed surface groups are virtually RFRS as well. Motivated by the theory of subgroup separability and Agol's RFRS condition on groups, we make the following definition in Chapter 2:

Definition 1.2.1. A subgroup $H \leq G$ is **RS-separable** (denoted $H \leq_{RS} G$) if there is a descending tower of subgroups $\cdots \lhd_f G_2 \lhd_f G_1 \lhd_f G_0 := G$ whose intersection is H and

$$G_{i+1} \geq (G_i)_r^{(1)} = \{x \in G_i | x^d \in G_i^{(1)}, \text{ some } 0 \neq d \in \mathbb{Z}\}$$

where $G_i^{(1)}$ denotes the commutator subgroup of G_i. The collection $\{G_i\}$ is called an **RS tower over** H.

A group G is **residually finite and rationally solvable (RFRS)** if $1 \leq_{RS} G$; see [1]. Agol's primary interest is in groups that are **virtually RFRS**, that is, those groups G that contain a subgroup of finite index $K \leq_f G$ such that $1 \leq_{RS} K$. Thus, if $H \leq G$, we will say that H is **virtually RS-separable** in G (denoted $H \leq_{VRS} G$) if there is a finite-index subgroup $K \leq_f G$ such that $H \leq_{RS} K$. In the spirit of [12], we will call a group G **locally virtually RS-separable** (LVRSS) if every finitely generated subgroup $H \leq G$ satisfies $H \leq_{VRS} G$. This is a very strong property; for example, LVRSS implies both virtually RFRS and LERF (see Proposition 2.5.4), so in particular, if a hyperbolic 3-manifold M has LVRSS fundamental group, then, by recent work of Agol [2] using work of Haglund and Wise [14], M is virtually fibered, has large fundamental group, and is cubulated.

As an aside, we mention that the subgroup $G_r^{(1)} \leq G$ appears elsewhere in the literature as the first term in the rational derived series of G; see, for example, [7], [8], and [16].

Agol proves that the trivial subgroup of a surface group is virtually RS-separable [1]. One of the main results of this thesis is that, in fact, all finitely generated subgroups of surface groups are virtually RS-separable:

Theorem A. *Let $\Sigma = \Sigma_{g,n}$ be a compact, orientable surface of genus $g \geq 0$ with $n \geq 0$ boundary components. Then $\pi_1(\Sigma)$ is LVRSS.*

The proof of Theorem A in the case that $n \geq 1$ appears in Chapter 4 and uses a theorem on bases of subgroups of free groups due to Federer and Jónsson [10]. The main technical tool that we use to prove the $n = 0$ case of Theorem A in Chapter 5 is the following geometric result, which is reminiscent of the Federer and Jónsson theorem:

Theorem B. *Let (Σ, x) be an oriented hyperbolic surface with basepoint x. Let A_Σ denote the set of geodesic loops based at x that are not homotopic to a product of shorter loops based at x. Then:*

1. *For every $\alpha \in A_\Sigma$, α is simple.*
2. *For every $\alpha, \beta \in A_\Sigma$, $\alpha \cap \beta = \{x\}$.*
3. *$\langle [A_\Sigma] \rangle = \pi_1(\Sigma, x)$.*

In particular, if $\pi_1(\Sigma, x)$ is finitely generated, then A_Σ is finite.

This collection of simple, minimally intersecting loops based at x gives us just enough control to build certain abelian covers separating subgroups of surface groups. It is well known that every essential (π_1-injective and non-boundary parallel) closed curve on a hyperbolic surface is freely homotopic to a unique closed geodesic. However, this fact is not useful to us in proving Theorem A because infinitely many distinct elements in $\pi_1(\Sigma)$ are freely homotopic to the *same* closed geodesic; indeed, closed geodesics correspond uniquely to conjugacy classes in $\pi_1(\Sigma)$. Thus, we cannot proceed as we did in the free group case, even though there are only finitely many closed geodesics on Σ of length less than a given finite number. The key observation is that every *based* closed loop is homotopic (rel x) to a unique shortest loop based at x, and in fact it is a geodesic loop; this is a special case of a theorem in [5]. And, in addition, there are only finitely many geodesic loops based at x of length less than a given number, and thus, given Theorem B, we may proceed in a manner similar to the one used in the case that $n \geq 1$.

In developing a new property of groups—for example, RF, LERF, LR, or RFRS—it is imperative to understand how it behaves under various natural group operations. In Chapter 3, we show that the relationship $H \leq_{RS} G$ is preserved under several natural group operations:

Theorem C. *Suppose that $H_1 \leq_{RS} G_1, H_2 \leq_{RS} G_2, K_1 \leq G_1$, and $\varphi : G \to G_1$ is surjective. Then $H_1 \cap K_1 \leq_{RS} K_1, H_1 \times H_2 \leq_{RS} G_1 \times G_2$, and $\varphi^{-1}(H_1) \leq_{RS} G$. If, in addition, G_1 is LVRSS, then K_1 is LVRSS, that is, LVRSS behaves well under subgroups.*

Chapter 2

Definitions and Notation

Here we collect some definitions and notation we will use throughout the rest of this thesis.

2.1 Classical group theory

Let G be a group. An **endomorphism** is a self-homomorphism $\varphi : G \to G$, and an **automorphism** of G is an endomorphism $\varphi : G \to G$ that is a bijection on the level of sets. An automorphism $\varphi : G \to G$ is an **inner automorphism** if $\varphi(x) = x^g := g^{-1}xg$ for some fixed $g \in G$. We use the notation $H \leq_f G$ to denote that H is a finite-index subgroup of G. Similarly, for a normal subgroup of finite index, we use the notation $N \lhd_f G$. We will focus on **finitely generated** groups, which are those groups G that contain a finite subset $X \subset G$ such that any element of G is represented by a product of elements in X.

A **commutator** in G is an element of the form $[x, y] := x^{-1}y^{-1}xy$ for elements $x, y \in G$. The **commutator subgroup** $G^{(1)}$ of a group G is the set of all products of commutators in G; the fact that this is a subgroup follows from the observation that $[x, y]^{-1} = [y, x] \in G^{(1)}$. Furthermore, $G^{(1)} \lhd G$, so we may consider the (abelian) quotient group $G/G^{(1)}$, and we remark that if $N \lhd G$, then G/N is abelian if and only if $G^{(1)} \leq N$. Given a subgroup $H \leq G$, we may consider the set

$$H_r^{(1)} := \{h \in H : h^d \in H^{(1)} \text{ for some } d \neq 0\}$$

This is, in fact, a normal subgroup of H; the easiest way to verify this is to see that it is the kernel of the natural map

$$H \to \mathbb{Q} \otimes_{\mathbb{Z}} H/H^{(1)}$$

In fact, given a group G, the normal subgroup $G_r^{(1)}$—which we will refer to as the **radical commutator subgroup** of G—is the first term in the **rational derived series**, which is defined inductively by

$$G_r^{(n+1)} := (G_r^{(n)})_r^{(1)}$$

For some results in low-dimensional topology involving the rational derived series, see [7], [8], and [16].

Given a group G, a **tower of subgroups** $\{H_i\}_{i \geq 0}$ is a collection of subgroups $H_i \leq G$ such that $H_{i+1} \leq H_i$ for all $i \geq 0$. The **limit** of a tower $\{H_i\}$ in G is

$$\lim_i H_i = \bigcap_i H_i \leq G$$

A tower $\{H_i\}$ is a **subnormal tower** if $H_{i+1} \lhd H_i$ for all i, and it is a **normal tower** if, in fact, $H_i \lhd G$ for all i.

2.2 Classical algebraic topology.

A **topological surface** S is a 2-manifold, possibly with boundary, that is, a Hausdorff, second-countable space such that every point $x \in S$ has a neighborhood $x \in U \subset S$ that is homeomorphic with either $\mathbb{R}^2$ or $\{(x, y) \in \mathbb{R}^2 : y \geq 0\}$. A **subsurface** $T \subset S$ is a connected subspace such that T is a surface. An **essential subsurface** $T \subset S$ is a subsurface that is π_1-injective and no component of $S \setminus T$ is an annulus.

Now we recall a few classical facts about covering spaces from [17]. We will only be applying these results to surfaces, and thus, to ease the exposition below we assume that all spaces are path-connected, locally path-connected, and semi-locally simply connected. A **covering space** of a space X is a space X' along with a local homeomorphism $p : X' \to X$. The induced map $p_* : \pi_1(X', x') \to \pi_1(X, x)$ is injective, and the image subgroup $p_*(\pi_1(X', x')) \leq \pi_1(X, x)$ consists of the homotopy classes of loops in X based at x whose lifts to X' starting at x' are loops (Proposition 1.31 in [17]).

We will also use the **lifting criterion** (Proposition 1.33 in [17]): suppose that we have a covering space $p : (X', x') \to (X, x)$ and a map $f : (Y, y) \to (X, x)$. Then a lift $f' : (Y, y) \to (X', x')$ exists if and only if $f_*(\pi_1(Y, y)) \leq p_*(\pi_1(X', x'))$. In particular, a given group element $g \in \pi_1(X, x)$, represented by a loop $\alpha_g : (S^1, y) \to (X, x)$, is contained in $\pi_1(X', x')$ if and only if α_g lifts to a loop (rather than a path) in (X', x').

In Section 5.2, we will use implicitly the following classical results on the existence and uniqueness of covers corresponding to subgroups: given a subgroup $H \leq \pi_1(X, x)$, there is a covering space $p : X_H \to X$ such that $p_*(\pi_1(X_H, x_H)) = H$ for a suitably chosen basepoint $x_H \in X_H$ (Proposition 1.36 in [17]). In fact, this covering $X_H \to X$ is unique up to isomorphism of covering spaces: two covering spaces $p' : (X', x') \to (X, x)$ and $p'' : (X'', x'') \to (X, x)$ are isomorphic via $f : (X', x') \to (X'', x'')$ if and only if $p'_*(\pi_1(X', x')) = p''_*(\pi_1(X'', x''))$ (Proposition 1.37 in [17]).

2.3 Residual properties.

Let P be a property of groups. A group G is **residually** P if for every nontrivial element $g \in G$ there is a homomorphism $\varphi : G \to F_P$ where F_P has property P and $\varphi(g) \neq 1$. Thus, a group G is **residually finite** if, for every nonidentity element $g \in G$, there exists a homomorphism $\varphi : G \to F$ onto a finite group such that $\varphi(g) \neq 1$; this definition is easily seen to be equivalent to the one given previously. Similarly, a group G is **residually free** if, for every nonidentity element $g \in G$, there exists a homomorphism $\varphi : G \to F$ to a nonabelian free group such that $\varphi(g) \neq 1$, and G is called **fully residually free** (or a **limit group**) if, for any finite collection of nontrivial elements $g_1, \ldots, g_n \in G$, there is a homomorphism $\varphi : G \to F$ to a nonabelian free group such that $\varphi(g_i) \neq 1$ for all $1 \leq i \leq n$.

Let $H \leq G$. Recall that H is **separable** in G if H is equal to the intersection of all finite-index subgroups of G containing H. We will borrow some notation from [19]: if $H \leq G$, then define $H^* \leq G$ to be

$$H^* := \bigcap_{H \leq K \leq_f G} K$$

In general, we clearly have $H \leq H^*$, and H is separable in G if $H = H^*$. As mentioned in Chapter 1, a group is **LERF** (locally extended residually finite), or **subgroup separable** if every finitely generated subgroup H is separable in G.

We recall a definition from [1], which is a strong form of residual finiteness:

Definition 2.3.1. A group G is **RFRS** if there is a descending tower of subgroups $\cdots \lhd_f G_2 \lhd_f G_1 \lhd_f G_0 := G$ whose limit is the trivial subgroup and

$$(G_i)_r^{(1)} \leq G_{i+1}$$

for all $i \geq 0$.

2.4 Virtual and locally virtual properties.

Let G be a group, and let P be a property of groups. G is said to be **virtually** P if G contains a finite-index subgroup $K \leq_f G$ such that K has property P. For example, G is **virtually RFRS** if there is a $K \leq_f G$ such that K is RFRS.

Given a property P of subgroups (denoted by $H \leq_p G$)—for example, H could finite-index in G ($H \leq_f G$) or RS-separable in G (H $\leq_{RS}$ G)—G

is said to be **locally virtually P** (or **LVP**) if, for every finitely generated subgroup $H \leq G$, there exists a finite-index subgroup $K \leq_f G$ containing H such that $H \leq_P K$.

Another local virtual property that has garnered attention recently involves retractions in groups. Let $H \leq G$. We say that G **retracts onto** H (and we call H a **retract** of G) if there exists an endomorphism $\varphi : G \to G$ such that $\varphi(G) = H$ and $\varphi|_H \equiv \mathrm{id}_H$. We will often abuse notation and write the retraction as $\varphi : G \to H$ without emphasizing that $H \leq G$. Similarly, we say that G **virtually retracts onto** (and we call H a **virtual retract** of G) if there is a finite-index subgroup $K \leq_f G$ and a retraction $\varphi : K \to H$. A group G admits **locally virtual retractions** (or **local retractions**, or **LR**) if every finitely generated subgroup of G is a virtual retract of G. A group G admits **virtual retractions over** $\mathbb{Z}$ (or **local retractions over** $\mathbb{Z}$, or **LR over** $\mathbb{Z}$) if every infinite cyclic subgroup of G is a virtual retract. These notions were introduced and explored in [20] and [19].

2.5 Some new group theoretical notions

We begin this section with a natural generalization of Agol's RFRS condition as defined in Section 2.3:

Definition 2.5.1. A subgroup $H \leq G$ is **RS-separable** (**rationally solvably separable**), which we denote by $H \leq_{RS} G$, if there is a descending tower of subgroups $\cdots \lhd_f G_2 \lhd_f G_1 \lhd_f G_0 := G$ whose limit is H and

$$(G_i)_r^{(1)} \leq G_{i+1}$$

for all $i \geq 0$. The collection $\{G_i\}$ is called an **RS tower over** H.

Thus, a group G is RFRS if $1 \leq_{RS} G$. And, if $H \leq_{RS} G$ with $\{G_i\}$ an RS tower over H, then since $G_i \leq_f G$ with $\cap G_i = H$, clearly

$$\cap G_i = H \leq H^* \leq \cap G_i$$

and thus $H = H^*$, that is, H is separable in G. We also note that $G \leq_{RS} G$.

In this dissertation, we focus on groups whose subgroups all have torsion-free abelianization, and thus we may use the following proposition to simplify our arguments:

Proposition 2.5.1. *Let $H \leq G$, and suppose that $H/H^{(1)}$ is torsion-free. Then $H^{(1)} = H_r^{(1)}$.*

Proof. Clearly $H^{(1)} \subset H_r^{(1)}$, so it is enough to show that $H^{(1)} \supset H_r^{(1)}$. Suppose this fails, that is, there is an element $g \in H_r^{(1)} \setminus H^{(1)}$. So, $g^d \in H^{(1)}$ for some $|d| \geq 2$ but $g \notin H^{(1)}$. Then $1 \neq \overline{g} \in H/H^{(1)}$, but $1 = \overline{g^d} \in H/H^{(1)}$, contradicting the fact that $H/H^{(1)}$ is torsion-free. ☐

Since both LERF and RFRS are natural extensions of residual finiteness, we would like to combine them, naturally, into a single property. To do this, we recall that for Agol's purposes in [1], it is more important for a group to contain a finite-index subgroup that is RFRS rather than require that the group itself is RFRS. In other words, Agol is interested in those groups G that are **virtually RFRS**, that is, there exists $K \leq_f G$ such that $1 \leq_{RS} K$. Thus, in order to generalize this in a natural way, we borrow some ideas from Section 2.4; for one, we will say that a subgroup $H \leq G$ is **virtually RS separable** (denoted $H \leq_{VRS} G$) if there exists a finite-index subgroup $K \leq_f G$ such that $H \leq_{RS} G$. For example, all finite-index subgroups $H \leq_f G$ are virtually RS separable because $H \leq_{RS} H \leq_f G$. The main focus of this thesis is an investigation of the following property:

Definition 2.5.2. A group G is **LVRSS** (**locally virtually RS-separable**) if, for every finitely generated subgroup $H \leq G$, there exists a finite-index subgroup $K \leq G$ such that $H \leq_{RS} K$.

We hasten to point out that this is not a trivial definition. All finite groups are LVRSS because, if G is a finite group and H is a (finitely generated) subgroup of G, then $H \leq_f G$ and $H \leq_{RS} H$. As a slightly less trivial example, we prove here that the infinite cyclic group is LVRSS:

Proposition 2.5.2. *Suppose that $G = \langle x \rangle$ is infinite cyclic. Then G is LVRSS.*

Proof. Let H be a finitely generated subgroup of G. Then either H is the trivial subgroup, or $H \leq_f G$. By the above observation, $H \leq_f G$ implies $H \leq_{VRS} G$, so it is enough to show that $1 \leq_{RS} G$. But clearly

$$\cdots \lhd_f 8\mathbb{Z} \lhd_f 4\mathbb{Z} \lhd_f 2\mathbb{Z} \lhd_f G$$

is an RS tower with trivial limit, as required. Therefore, since every finitely generated subgroup of G is virtually RS separable, G is LVRSS. $\square$

In fact, it's not hard to see that all finitely generated abelian groups have this property:

Theorem 2.5.3. *Suppose that G is a finitely generated torsion-free abelian group. Then G is LVRSS.*

Proof. Let $H \leq G$ be finitely generated such that $[G : H] = \infty$. By the fundamental theorem of finitely generated abelian groups, $G \cong \mathbb{Z}^n$ for some $n < \infty$. These groups are known to be LERF, and for any descending tower of finite-index subgroups $\{K_i\}$ intersecting in H, the successive quotients are abelian, which, since G is torsion-free and abelian, implies that H is RS separable in G by Proposition 2.5.1. $\square$

Although LVRSS clearly implies virtually RFRS, it is not completely obvious that LVRSS implies LERF, so we provide justification for this fact.

Proposition 2.5.4. *Suppose that G is LVRSS. Then G is LERF.*

Proof. Let $H \leq G$ be a finitely generated subgroup of G. We must show that H is separable in G. Since G is LVRSS, there is a finite-index subgroup $K \leq_f G$ such that $H \leq_{RS} K$, and therefore, by the above observation, H is separable in K. But this implies that H is separable in G because every subgroup of finite index in K also has finite index in G, so the intersection of all subgroups of finite index in K containing H contains the intersection of all subgroups of finite index in G containing H. Thus, H is separable in G. $\square$

This new property of groups, LVRSS, fits into the existing literature in a few ways. On the one hand, it is a very strong form of residual finiteness. On the other hand, since it implies both LERF and virtually RFRS, it may be thought of as a certain combination of those two properties, each of which imply residual finiteness as well. To justify the legitimacy of these new definitions, we prove in Chapter 3 that RS separability is well-behaved under some natural operations on groups, and we compare the behavior to that of RFRS and subgroup separability. Furthermore, we show in Chapters 4 and 5 that free groups and closed (orientable) surface groups are LVRSS, and so LVRSS shares the property of generalizing "freeness" with residual finiteness, LERF, and (virtually) RFRS.

Chapter 3

Behavior of RS Separability

In this chapter, we prove some basic results on the preservation of RS separability and LVRSS under various natural operations on groups. Throughout this chapter, we provide comments on the analogous results for subgroup separability and RFRS.

3.1 Intersection and subgroups

We start with a basic observation, which, since we will use it repeatedly throughout this thesis, we record as a lemma:

Lemma 3.1.1. *Suppose that $G_1, G_2 \leq G$ such that $(G_1)_r^{(1)} \leq G_2$, and let $K \leq G$. Then*

$$(G_1 \cap K)_r^{(1)} \leq G_2 \cap K$$

Proof. If $x \in (G_1 \cap K)_r^{(1)}$, then $x \in G_1 \cap K \leq K$, and $x^d \in (G_1 \cap K)^{(1)} \leq G_1^{(1)}$ for some $d \neq 0$. Thus $x \in (G_1)_r^{(1)} \cap K \leq G_2 \cap K$. □

LERF is well known to be preserved by subgroups; see, for example, Lemma 1.1 in [25]. It is easy to see that RFRS is preserved by subgroups as well; see the remarks following Definition 2.1 in [1]. Now we use Lemma 3.1.1 to show that, perhaps as expected, the same is true of RS separability:

Proposition 3.1.2. *Suppose that $H \leq_{RS} G$ and $K \leq G$. Then $H \cap K \leq_{RS} K$.*

Proof. Let $\{G_i\}$ be an RS tower over H in G, so $(G_i)_r^{(1)} \leq G_{i+1}$ for $i \geq 0$ and

$$\cdots \lhd_f G_2 \lhd_f G_1 \lhd_f G_0 := G$$

such that $H = \cap G_i$. Set $K_i := G_i \cap K$. Then $\cap K_i = H \cap K$, and Lemma 3.1.1 yields $K_{i+1} \geq (K_i)_r^{(1)}$. Therefore $\{K_i\}$ is an RS tower over $H \cap K$, so $H \cap K \leq_{RS} K$. ∎

By setting $H = 1$ in Proposition 3.1.2, we recover the observation that RFRS passes to subgroups. A fortiori, LVRSS passes to subgroups as well; this is the content of the following:

Corollary 3.1.3. *Suppose that G is LVRSS, and let $H \leq G$. Then H is LVRSS as well.*

Proof. Let $L \leq H$ be a finitely generated subgroup of H. Then L is a finitely generated subgroup of G, and thus, since G is LVRSS, there exists a finite-index subgroup $K \leq_f G$ such that $L \leq_{RS} K$. But $H \cap K \leq_f H$, and $L \leq H \cap K$, so by Proposition 3.1.2,

$$L = L \cap (K \cap H) \leq_{RS} K \cap H \leq_f H,$$

that is, L is RS separable in $K \cap H \leq_f H$, and thus $L \leq_{VRS} H$. Since L was an arbitrary finitely generated subgroup of H, it follows that H is LVRSS. ∎

As another application of Lemma 3.1.1, we prove an element-wise characterization of RS separability. This is analogous to the fact that a subgroup H is separable in G if and only if, for every $g \in G \setminus H$ there exists a finite-index subgroup $K \leq_f G$ such that $H \leq K$ and $g \in G \setminus K$.

Proposition 3.1.4. *Let G be finitely generated. Then $H \leq_{RS} G$ if and only if the following condition holds:*

($*$) *For every $g \in G \setminus H$, there exists an $n = n_g \geq 0$ and a finite subnormal tower $G = G_0 \rhd_f G_1 \rhd_f \cdots \rhd_f G_n$ such that $g \notin G_n \leq_f G$, $H \leq G_n$, and $G_{i+1} \geq (G_i)_r^{(1)}$ for all $0 \leq i \leq n-1$.*

Proof. First suppose that $H \leq_{RS} G$, and let G_i be an RS tower over H with $\cap G_i = H$. Then if $g \in G \setminus H$, there is an n such that $g \notin G_n$, so $G_0 \geq \cdots \geq G_n$ is the desired tower.

Now suppose that ($*$) holds, and list the elements of $G \setminus H = \{g_1, g_2, \ldots\}$. By ($*$), there is a subnormal tower $G = G_0 \geq G_1 \geq \cdots \geq G_{n_1}$ that is an RS tower over G_{n_1} such that $g_1 \notin G_{n_1}$ and $H \leq G_{n_1}$. Let i be minimal such that $g_i \in G_{n_1} \setminus H$. By ($*$), there is a tower $G = G'_0 \geq G'_1 \geq \cdots \geq G'_{n_2}$ such that $g \notin G'_{n_2}$ and $H \leq G'_{n_2}$. Then

$$G_0 \geq G_1 \geq \cdots \geq G_n = G_n \cap G'_0 \geq G_n \cap G'_1 \geq \cdots \geq G_n \cap G'_{n_2}$$

is a subnormal RS tower over $G_{n_1} \cap G'_{n_2}$ because $G'_{i+1} \geq (G'_i)_r^{(1)}$ implies $G_n \cap G'_{i+1} \geq (G_n \cap G'_i)_r^{(1)}$ by Lemma 3.1.1. Furthermore, $g_i \notin G_{n_1} \cap G'_{n_2}$, so continuing this process with the first element $g_j \in (G_{n_1} \cap G_{n_2}) \setminus H$, we get an RS tower G_i over H such that $\cap G_i = H$ as needed (because $g_k \notin G_{n_k}$). $\square$

3.2 Direct product

One might hope that a property such as LVRSS would be preserved by such a natural group operation as the direct product; that is, one might expect that if G_1 and G_2 are LVRSS, then $G_1 \times G_2$ is LVRSS as well. Indeed, other related properties—such as residual finiteness, RFRS, LR, and LR over $\mathbb{Z}$—are preserved by direct product. However, things are not quite so simple for LERF (and, consequently, for LVRSS) because, for example, although the free group of rank two F_2 is LVRSS (see Chapter 4), $F_2 \times F_2$ is not even LERF, and hence by Proposition 2.5.4, $F_2 \times F_2$ is not LVRSS. However, we can prove that RS separable subgroups are preserved under direct product:

Proposition 3.2.1. *If $H_1 \leq_{RS} G_1$ and $H_2 \leq_{RS} G_2$, then $H_1 \times H_2 \leq_{RS} G_1 \times G_2$.*

Proof. Let $\cdots \lhd_f G_{1,1} \lhd_f G_{1,0} := G_1$ be an RS tower for $H_1 \leq G_1$, and let $\cdots \lhd_f G_{2,1} \lhd_f G_{2,0} := G_2$ be an RS tower for $H_2 \leq G_2$. As noted in [1],

$$(G_{1,i} \times G_{2,i})_r^{(1)} = (G_{1,i})_r^{(1)} \times (G_{2,i})_r^{(1)} \leq G_{1,i+1} \times G_{2,i+1}$$

Furthermore,

$$\bigcap (G_{1,i} \times G_{2,i}) = \cap G_{1,i} \times \cap G_{2,i} = H_1 \times H_2 \tag{3.2.1}$$

Thus, $G_{1,i} \times G_{2,i}$ is an RS tower over $H_1 \times H_2$. $\square$

Note that Equation 3.2.1 is essentially the proof that if H_1 is separable in G_1 and H_2 is separable in G_2, then $H_1 \times H_2$ is separable in $G_1 \times G_2$.

3.3 Preimages of homomophisms.

We would like to know to what extent RS separability is preserved under preimages of homomorphisms. We begin with an auxiliary lemma that says that the commutator radical is preserved by homomorphisms as well as their inverse images.

Lemma 3.3.1. *Suppose that $\varphi : G \to H$ is a homomorphism. Then $\varphi(G_r^{(1)}) \leq H_r^{(1)}$, so $G_r^{(1)} \leq \varphi^{-1}(H_r^{(1)}) \leq G$.*

Proof. Suppose that $x \in G_r^{(1)}$, so $x^d = \prod_i [y_i, z_i] \in G^{(1)}$ for some $d \neq 0$ and some $y_i, z_i \in G$. Now compute:

$$(\varphi(x))^d = \varphi(x^d) = \varphi\left(\prod [y_i, z_i]\right) = \prod [\varphi(y_i), \varphi(z_i)]$$

Thus, $(\varphi(x))^d \in H^{(1)}$, and hence $\varphi(x) \in H_r^{(1)}$ as needed. The second claim follows since $G_r^{(1)} \leq \varphi^{-1}(\varphi(G_r^{(1)}))$. □

An elementary corollary of this lemma is that RS-separable subgroups are preserved under inverse images:

Corollary 3.3.2. *Suppose that $\varphi : G \to H$ is a homomorphism, and $K \leq_{RS} H$. Then $\varphi^{-1}(K) \leq_{RS} \varphi^{-1}(H)$. In particular, preimages of RS towers over $K \leq H$ are RS towers over $\varphi^{-1}(K) \leq \varphi^{-1}(H)$.*

Proof. Let $\cdots \lhd_f H_1 \lhd_f H_0 := H$ be an RS tower over K, and set $G_i := \varphi^{-1}(H_i)$. Since pre-images preserve intersections, we have

$$\bigcap G_i = \bigcap \varphi^{-1}(H_i) = \varphi^{-1}(\cap H_i) = \varphi^{-1}(K)$$

Furthermore, by applying Lemma 3.3.1 to $\varphi|_{\varphi^{-1}(H_i)} : \varphi^{-1}(H_i) \to H_i$, we find

$$G_{i+1} = \varphi^{-1}(H_{i+1}) \geq \varphi^{-1}((H_i)_r^{(1)}) \geq \left(\varphi^{-1}(H_i)\right)_r^{(1)} = (G_i)_r^{(1)}$$

and therefore $\cdots \lhd_f \varphi^{-1}(H_1) \lhd_f \varphi^{-1}(H_0) = \varphi^{-1}(H)$ is an RS tower over $\varphi^{-1}(K)$. □

It is a fact known to Agol (and communicated to the author by Henry Wilton) that all residually free groups are RFRS; however, no proof exists in the literature. As another application of Lemma 3.3.1, we provide a proof here.

Proposition 3.3.3. *Finitely generated residually free groups are RFRS.*

Proof. Suppose that G is residually free; note that all subgroups of G are residually free as well. List the elements of $G = \{1 = g_0, g_1, g_2, \ldots\}$. Since $G_0 := G$ is residually free, there is a homomorphism $\varphi_0 : G_0 \to F_0$ (where F_0 is free) such that $\varphi_0(g_0) \neq 1$. Free groups are RFRS (see [1]), so there is a RFRS tower $\cdots \lhd_f F_{0,1} \lhd_f F_{0,0} := F_0$. Since $\cap F_{0,i} = \{1\}$, there is a minimal r_0 such that $\varphi_0(g_0) \in F_{0,r_0-1} \setminus F_{0,r_0}$. Now set $G_{0,i} := \varphi^{-1}(F_{0,i})$ for $1 \leq i \leq r_0$, and set $G_1 := G_{0,r_0}$. By Lemma 3.3.1, $G_{0,i+1} \geq \varphi_0^{-1}((F_{0,i})_r^{(1)}) \geq (G_{0,i})_r^{(1)}$. For $k \geq 1$, apply this procedure to the residually free group G_k (and the next $g_i \in G_k$) to get $\varphi_k : G_k \to F_k$. Then $g_k \notin G_{k+1}$ for each k, so $\cap G_k = \{1\}$. □

Chapter 4

Free groups are LVRSS

In Chapter 2, we provided some examples of LVRSS groups. In particular, in Proposition 2.5.2, we showed that the free abelian group $\mathbb{Z}$ is LVRSS. The main result of this chapter is that free nonabelian groups are also LVRSS (Corollary 4.4.2). The main ingredients in the proof are Marshall Hall's Theorem [15], a theorem on bases of subgroups of free groups due to Federer and Jónsson [10], and some techniques inspired by the theory of local retractions [19].

4.1 Free groups and graphs.

Let X be a set. We will denote by F_X the **free group** on X, which means that there is a map $\iota : X \to F_X$ such that, for any set map $f : X \to G$ to any other group G, there exists a unique group homomorphism $\varphi : F_X \to G$ that makes the following diagram commute:

$$\begin{tikzcd} X \arrow[r] \arrow[dr, "f"'] & F_X \arrow[d, dashed, "\varphi"] \\ & G \end{tikzcd}$$

This is known as the **universal property of free groups**. Alternatively, one may define F_X to be the group of equivalence classes of words in X^*, where two words w, v are equivalent if and only if w may be obtained from v by a sequence of insertions and deletions of parts of the form xx^{-1} for $x \in X^{\pm 1}$. We call X a **basis** (or **free basis**) of F_X; more generally, we call a subset $S \subset F_X$ a free basis if $F_S \cong F_X$. This is not to be confused with a **generating set**, which is a subset $S \subset F_X$ such that $\langle S \rangle = F_X$. Another alternative is to define F_X by a **group presentation**, which is a collection of **generators** X and a collection of **relators** R (words in $X^{\pm 1}$ that represent the trivial element in G), and we use the notation

$$\langle X \mid R \rangle$$

The free group F_X is uniquely presented (up to isomorphism) by $\langle X | R \rangle$ where $R = \emptyset$.

Let A and B be groups. We define the **free product** of A and B to be the group $A * B$ that is the coproduct of A and B; that is, for any group G and any homomorphisms $A \to G, B \to G$, there is a unique homomorphism $u : A * B \to G$ such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r] \arrow[dr] & A * B \arrow[d, dashed, "u"] & B \arrow[l] \arrow[dl] \\ & G & \end{tikzcd}$$

This is known as the **universal property of free products**.

It is a standard fact that any element of $A * B$ has a unique so-called "reduced form," and, as a consequence, $A, B \hookrightarrow A * B$. We will require a notion of reduced form just in free groups, and for this, we follow section 1 of chapter 1 of [21]. Let F_X be a free group with basis X. A word in X

is called **reduced** if contains no part xx^{-1} for $x \in X^{\pm 1}$. Each equivalence class of words contains a unique reduced word. We will let $L_X(w)$ denote the X-length of a reduced word $w \in F$ over $X^{\pm 1}$—that is, $L_X(w)$ is the length of the reduced word representing w, and we will write $w = w(1)w(2)\cdots w(n)$ with each $w(i) \in X^{\pm}$. We will sometimes use the notation F_r to denote F_X with $|X| = r$.

We observe that, using the universal property of free groups, it is not difficult to show that $F_{X \sqcup Y} \cong F_X * F_Y$. However, using group presentations is even easier, since each of these is presented $\langle X, Y \mid - \rangle$. For this reason, we may think of $F_X, F_Y \leq F_{X \sqcup Y}$. As the following elementary example shows, retracts occur naturally in free groups:

Proposition 4.1.1. *There exists a canonical retraction $F_{X \sqcup Y} \to F_X$.*

Proof. Let $i : X \to F_X$ be the set inclusion given by the definition of F_X and define a set map $f : X \sqcup Y \to F_X$ by

$$f(a) = \begin{cases} i(a) & a \in X \\ 1 & a \in Y \end{cases}$$

By the universal property of free groups, there is a unique homomorphism $F_{X \sqcup Y} \to F_X$ corresponding to f. The induced map $F_{X \sqcup Y} \to F_X$ is clearly the identity on $F_X \leq F_{X \sqcup Y}$. □

4.2 Marshall Hall's Theorem and retractions.

In this section, we recall the well-known theorem of Marshall Hall Jr. on subgroups of free groups. We begin this section with a useful topological criterion (due to Scott [25]) for testing whether a subgroup is separable:

Proposition 4.2.1. *(Scott's Criterion, [25]) Let X be a Hausdorff topological space and $G = \pi_1(X, x)$. Let $(X', x') \to (X, x)$ be a covering and $H = \pi_1(X', x')$. Then H is separable in G if and only if for any compact subset $\Delta \subset X'$, there exists an intermediate finite-sheeted cover $X' \to \hat{X} \to X$ such that $X' \to \hat{X}$ embeds Δ into $\hat{X}$.*

Proof. See Lemma 1.4 in [25]. $\square$

Scott's Criterion may be applied to covering spaces of graphs, as in Stallings' proof that free groups are LERF (Marshall Hall's Theorem), and in Chapter 5, we will apply Scott's Criterion to covering spaces of closed surfaces. We note that Stallings' proof tells us that $H \leq F_r$ is a free factor of a finite-index subgroup of F_r. By Proposition 4.1.1, this implies that free groups admit local retractions: for any finitely generated $H \leq F_r$, there is a finite-index subgroup $K \leq_f F_r$ such that $K = H * F'$ for some $F' \leq F_r$, and the natural quotient $\varphi : K \to H$ is a retraction.

4.3 Orderings and free bases.

The goal of this section is to make sense of a certain result (recorded as Lemma 4.3.2 below) on free bases of subgroups of free groups due to Federer and Jónsson [10]. We first recall some definitions of orderings on sets.

A binary relation $\leq$ is a **total order** on a set X if the following three conditions are satisfied for all $a, b, c \in X$:

1. if $a \leq b$ and $b \leq a$, then $a = b$ (anti-symmetry);
2. if $a \leq b$ and $b \leq c$, then $a \leq c$ (transitivity); and
3. either $a \leq b$ or $b \leq a$ (totality).

To any total order $\leq$ is associated a **strict total order** $<$, which may be defined as $a < b$ if and only if $a \leq b$ and $a \neq b$. A total order $\leq$ on a set X is a **well-ordering** if every nonempty subset of X has a least element under $\leq$.

If now F is a group and $G \leq F$ with well-ordering $\leq$ and strict total ordering $<$ on a subgroup $G \leq F$ of a group F, define for any element $g \in G$:

$$G_g := \langle \{h \in G \mid h < g\} \rangle \leq G$$

That is, G_g is the subgroup of G generated by all elements of G that are strictly less than g under the well-ordering $\leq$. For example, if G is generated by X, and $x < g$ for all $x \in X$, then $G_g = G$. As another example involving G_g, we state a basic fact which we will use below:

Proposition 4.3.1. *If $H < G$ is a proper subgroup of G and g is the least element in the set $G \setminus H$, then $G_g \leq H$. In particular, $g \notin G_g$.*

Proof. By the choice of g, if $h \in G$ is such that $h < g$, then $h \in H$. Thus $G_g \leq H$. The second assertion clearly follows from the first. $\square$

Recall from Section 4.1 that, for an element $g \in G$, $L_X(g)$ is the length of the unique reduced word in $X^{\pm 1}$ representing g. The definition of G_g is central to the following result of Federer and Jónsson (see Theorem 4.2 in [10], or Proposition 2.10 in [21]), and we will use this to show that free groups are locally virtually RS separable:

Lemma 4.3.2. *Let G be a subgroup of the free group F, and let G be well ordered by any relation $<$ such that $g < h$ implies $L_X(g) \leq L_X(h)$. Then the set*

$$A = A_G = \{g \in G \mid g \notin G_g\}$$

is a free basis for G.

In the situation of Lemma 4.3.2, it is not difficult to see that $\langle A \rangle = G$: if $\langle A \rangle$ is a proper subgroup of G and g is the least element of $G \setminus \langle A \rangle$, then Proposition 4.3.1 implies that $g \notin G_g$, so $g \in A$, a contradiction. Hence, the content of the lemma is that A is a *free* basis for G; however, this set clearly generates its associated subgroup in an arbitrary (that is, not necessarily free) group. We record this observation as a proposition to which we will refer in Chapter 5:

Proposition 4.3.3. *Let $(G, <)$ be a group G with a well-ordering $<$, and let $H \leq G$. For any $g \in H$, define*

$$H_g = \langle \{h \in H : h < g\} \rangle$$

Then the set

$$A_H = \{h \in H : h \notin H_h\}$$

generates H.

We list a basic consequence of Lemma 4.3.2 for convenience.

Corollary 4.3.4. *Let G be a subgroup of the free group F, and let G be well ordered by any relation $<$ such that $g < h$ implies $L_X(g) \leq L_X(h)$, and let $H < G$ be a proper subgroup of G. Then the least element g in $G \setminus H$ is in the basis $A = \{g \in G \mid g \notin G_g\}$ of G.*

Proof. This is an immediate consequence of Proposition 4.3.1 and Lemma 4.3.2. $\square$

We will apply this result by building a well-ordering on $F = F_X$ with the desired property; this is the content of the following basic lemma:

Lemma 4.3.5. *Any strict total order $<$ on the finite set X may be extended to a well-ordering $<$ on the free group F_X such that $u < v$ implies $L_X(u) \leq L_X(v)$ and such that any $x \in X$ is less than any nontrivial element in $F_X \setminus X$.*

Proof. Relabel elements of X so that $x_1 < x_2 < \cdots < x_n \in X$. Extend this to the symmetric set $X^{\pm}$:

$$x_1 < \cdots < x_n < x_1^{-1} < \cdots < x_n^{-1}$$

Define the relation $\leq$ on reduced words in $u, v \in F_X$ by: $u \leq v$ if and only if $L_X(u) < L_X(v)$ or, if $L_X(u) = L_X(v)$, then $u(i) < v(i)$ for the minimal i where u and v differ, or $u = v$.

First, we argue that this is a total order. If $u \leq w$ and $w \leq u$, then $L_X(u) = L_X(w)$ and $u(i) = w(i)$ for all i, so $u = w$. If $u \leq w$ and $w \leq v$, then we may assume $L_X(u) = L_X(v), u \neq w, w \neq v$. But u and w are equal words through the i-th letter (for some i), and w and v are equal words through the j-th letter for some j. So if $i < j$, then u and v are equal through $u(i) < w(i) = v(i)$, hence $u \leq v$. Similarly, if $i > j$ or $i = j$, then $u \leq v$, as needed. To complete the proof that $\leq$ is a total order, suppose $L_X(u) = L_X(w)$. If $u \neq w$, then u and w differ at some letter, hence $u \leq w$ or $w \leq u$.

To show that $\leq$ is a well-ordering, we must show that every nonempty subset $Y \subseteq F_X$ has a least element. But this follows from the definition of $\leq$ and the fact that there are only finitely many elements in F_X of a given (positive, finite) word length.

The last assertion is obvious by the definition of $\leq$ since

$$\{g \in F_X : L_X(g) = 1\} = X^{\pm 1}.$$

$\square$

4.4 RS separating subgroups of free groups.

Now we will apply the results of Section 4.3 to find certain elements of free bases of subgroups of a free group, and we will use these elements to construct an RS tower.

Theorem 4.4.1. *Suppose that X, Y are finite sets. Then $F_X \leq_{RS} F_X * F_Y$.*

Note: It is not enough to consider the composition of the natural maps $F_X * F_Y \to F_Y \to \mathbb{Z}^{|Y|}$ because this kills all conjugates of F_X, so intersecting kernels would yield at least the normal closure of F_X in $F_X * F_Y$, which is much bigger than F_X. A fortiori, for the same reason it is not enough to pull back a RFRS tower via $F_X * F_Y \to F_Y$; its pre-image would intersect in the kernel of the homomorphism, a normal subgroup of $F_X * F_Y$.

Proof. We may assume that X and Y are nonempty. Set $F := F_X * F_Y$, and fix a strict total ordering $<$ on $X \sqcup Y$ such that elements of $X^{\pm}$ are less than those of $Y^{\pm}$. By Lemma 4.3.5, this extends to a well-ordering $\leq$ on $F_{X \sqcup Y} \cong F_X * F_Y$ such that $u < v$ implies $L_{X \sqcup Y}(u) \leq L_{X \sqcup Y}(v)$ and such that elements of $X \sqcup Y$ are less than nontrivial elements in $F \setminus (X \sqcup Y)$. Throughout this proof, given a subgroup $F_i \leq F$, let A_i denote the basis of F_i resulting from Lemma 4.3.2, where word length $L_{X \sqcup Y}(w)$ is calculated over $X \sqcup Y$ in F, and the well-ordering on $F_i \leq F$ is the restriction of the well-ordering $\leq$ on F. Note that if any such subgroup F_i contains X, then the elements of X will be the least elements in F_i, and therefore in this situation, $X \subset A_i$.

Set $F_0 = F$, and let g_0 be a least element of the subset $F_0 \setminus F_X \subset F_0$ of the well-ordered set F_0. Then $g_0 \in A_0$ by Corollary 4.3.4, and thus $X \sqcup \{g_0\} \subset A_0$. Let F_1 be the kernel of the natural map $F_0 \to \langle g_0 \rangle \to \langle g_0 \rangle / \langle g_0^2 \rangle$; in particular, $X \subset F_X \subset F_1$. Thus we have an RS tower $F_0 > F_1$ with $F_1 \geq F_X$.

Assume that we have built an RS tower $F_0 > F_1 > \cdots > F_i$ with $F_i \geq F_X$. Let g_i be a least element in $F_i \setminus F_X$, so $g_i \in A_i$ by Corollary 4.3.4, and let F_{i+1} be the kernel of $F_i \to \langle g_i \rangle \to \langle g_i \rangle / \langle g_i^2 \rangle$. Repeating in this fashion, we get a tower $F_0 > F_1 > F_2 > \cdots$ over F_X. By the construction of the tower, the function

$$f(n) = \min\{L_{X\sqcup Y}(u) : u \in F_n \setminus F_X\}$$

is non-decreasing, and $g_i \notin F_{i+1}$ for each $i \geq 0$. Thus, since there are only finitely many reduced words in F_0 of a given finite length, for any $N \geq 1$ there is a number m_N such that $f(m_N) > N$, so we have $\cap F_i = F_X$. Additionally, since the abelianization of each F_i is torsion-free, the tower $\{F_i\}$ is an RS tower because $F_i/F_{i+1} \cong \mathbb{Z}/2\mathbb{Z}$ for each i. Thus, we conclude that $F_X \leq_{RS} F_X * F_Y$. □

Corollary 4.4.2. *If H is a finitely generated subgroup of a finitely generated free group F, then $H \leq_{VRS} G$. In other words, finitely generated free groups are LVRSS.*

Proof. By Hall's theorem [15], there is a finite-index subgroup $K \leq_f F$ containing H as a free factor, $K = H * F'$. Apply Theorem 4.4.1 to $H \leq K$. □

Thus, finitely generated free groups are LVRSS. In some sense, this result is as strong as possible because not all finitely generated subgroups of free groups are RS-separable. To see this, let $H \leq G_0 = G$ (where G is a free group) with H surjecting G^{ab}, and suppose that G_1 is the first term in a potential RS tower over H:

$$H \leq G_1 \lhd_f G_0 = G$$

with G_0/G_1 abelian. Then we have the following diagram:

$$\begin{tikzcd} H \arrow[r, hook, "i"] \arrow[dr, two heads] & G_0 \arrow[d, two heads] \arrow[dr, two heads, "p"] & \\ & G_0/[G_0,G_0] \arrow[r, two heads] & G_0/G_1 \end{tikzcd}$$

If $\ker(p) = G_1 < G_0$ were a proper subgroup, then $H \cap G_1 = \ker(p \circ i) < H$ would be a proper subgroup of H, contradicting the assumption $H \leq G_1$. Thus, $G_1 = G_0$. Therefore, a proper subgroup that surjects $F/[F,F]$ is not RS-separable; we record this observation in the following:

Proposition 4.4.3. *Let F be a nonabelian free group. If $H < F$ is a proper subgroup such that the composition $H \hookrightarrow F \to F/[F,F]$ is surjective, then H is not RS separable in F.*

In Chapter 5, we will show that closed, orientable surface groups are also locally virtually RS-separable.

Chapter 5

Surface groups are LVRSS

In this chapter, we prove that all finitely generated subgroups of an orientable surface group are virtually RS separable (Theorem 5.4.2). This, along with Corollary 4.4.2 implies that all orientable surfaces of finite type have LVRSS fundamental groups.

This chapter is organized as follows: Section 5.1 contains some topological lemmas that allow us to build cyclic covers to which some curves lift but others do not; Section 5.2 details the way in which we find the nice finite-index subgroup G' and is analogous to Section 4.2 in Chapter 4; in Section 5.3, we record some facts about the geometry of hyperbolic surfaces and geodesic loops, including the fact that the collection A_S of geodesic loops based at x that are not products of shorter loops consists of simple, minimally intersecting loops, and in fact generates $\pi_1(S, x)$. Finally, in Section 5.4, we prove the main theorem.

5.1 Surface groups and topology.

In this section, we accomplish two things: first, we give a sufficient condition for constructing a double cover of a surface that kills a particular non-separating curve. Second, we construct a double cover to which a particular separating curve lifts to a non-separating curve. These results are used in the proof of Theorem 5.4.2 in Section 5.4.

Let $S = S_{g,n}$ be a compact, orientable, connected surface of genus g with n boundary components. Let $\gamma : [0, 1] \to S$ be a **closed curve**, that is, γ is a smooth map such that $\gamma(0) = \gamma(1)$. We will often abuse notation and let $\gamma \subset S$ denote the image of the curve γ. γ is a **simple closed curve** if γ has no self-intersections (that is, $\gamma(x) \neq \gamma(y)$ for all $0 \leq x < y \leq 1$). If γ is a simple closed curve, we call γ a **non-separating curve** if $S \setminus \gamma$ is connected, and otherwise call γ a **separating curve**. A **multicurve** is a collection of disjoint simple closed curves.

Given a closed curve γ, the homology class (with $\mathbb{Z}$ coefficients) of γ is denoted $\overline{\gamma}$. A closed curve γ that is not null-homotopic is called **primitive** if it is simple and $\overline{\gamma}$ is primitive in $H_1(S)$. So, if γ is primitive, then either γ is non-separating or γ is boundary parallel and $n > 1$. If α, β are closed curves on S, then $i(\alpha, \beta)$ is the geometric intersection number of α and β. For multicurves $\mu, \nu \subset S$, we define intersection additively:

$$i(\mu, \nu) = \sum_{\alpha \in \mu, \beta \in \nu} i(\alpha, \beta)$$

We also note that every non-separating curve γ is primitive. This is because there is a dual curve δ (which is easy to find after cutting along γ, for example) such that $T = N(\gamma \cup \delta) \cong S_{1,1}$ with $H_1(T) = \langle \overline{\gamma} \rangle \oplus \langle \overline{\delta} \rangle$. And, in this case $H_1(S) = H_1(T) \oplus H_1(S \setminus T)$, so γ is primitive in $H_1(S)$ since it is obviously primitive in $H_1(T)$. More generally:

Lemma 5.1.1. *Let $K \hookrightarrow S$ be an essential subsurface, and let $\gamma \subset S$ be a non-separating simple closed curve such that $\gamma \subset S \setminus K$, or $i(\gamma, \partial K) = 2$ and γ intersects two distinct components of ∂K. Then there is a map $f : H_1(S) \to \mathbb{Z}/2\mathbb{Z}$ such that the induced $\pi_1(K) \to H_1(S) \to \mathbb{Z}/2\mathbb{Z}$ is trivial but $f(\overline{\gamma})$ is nonzero. In other words, there is a double cover $S' \to S$ to which K lifts but γ does not.*

Proof. If $\gamma \subset S \setminus K$, then the proof follows from the above observation, with f equal to the composition of the natural projection onto $H_1(T)$, followed by projection onto $\langle \overline{\gamma} \rangle$, and finally by the projection onto $\mathbb{Z}/2\mathbb{Z}$; here, we may take $T \subset S \setminus K$.

Now suppose that $i(\gamma, \partial K) = 2$ and γ intersects two distinct components k_1, k_2 of ∂K. Note that since K is essential, k_1 and k_2 are primitive in $H_1(S)$, so there is a map $\phi : H_1(S) \to \mathbb{Z}$ defined by algebraic intersection number with k_1, say. Notice that $\alpha \cdot k_1 = 0$ for any $\alpha \subset K$, so $\phi(K) = 0$. However, $\gamma \cdot k_1 = \pm 1$ (depending on the orientations), so we find the desired map $f : H_1(S) \to \mathbb{Z}/2\mathbb{Z}$ by composing ϕ with $\langle \overline{\gamma} \rangle \to \langle \overline{\gamma} \rangle / \langle \overline{\gamma}^2 \rangle$. The last statement follows by the lifting criterion.

□

In order to apply Lemma 5.1.1, we may need to lift a separating curve outside of an essential subsurface to a nonseparating curve outside the lifted subsurface in a double cover:

Lemma 5.1.2. *Suppose that Σ is a closed surface with a π_1-injective subsurface $K \subset \Sigma$. Let γ be a simple closed curve in a component S of $\Sigma \setminus K$ that is essential and separating on S. Then there exists a double cover $\Sigma' \to \Sigma$ where K and γ lift to K' and γ' such that γ' is not separating on its connected component of $\Sigma' \setminus K'$.*

Proof. We consider two cases depending on the number of components in ∂S. If ∂S has one component, then $S \setminus \gamma \cong S_{k,2} \sqcup S_{g-k,1}$ with $0 < k < g$ (since γ bounds neither a disk nor an annulus in S). Thus there is an essential simple closed curve δ_i in the i-th component of $S \setminus \gamma$ $(i = 1, 2)$, such that neither δ_1 nor δ_2 is separating on Σ. Cut Σ along δ_1, δ_2 and glue two copies of the resulting (connected) surface appropriately to find the desired double-cover; see Figure 5.1.

If ∂S has more than one component, then each component of $S \setminus \gamma$ either has positive genus or has at least three boundary components, and in either case we find nontrivial (but possibly boundary parallel, in the genus 0 case) simple closed curves δ_i in the i-th component of $S \setminus \gamma$, neither of which is separating on Σ, and we cut and glue two copies of the connected surface $\Sigma \setminus (\delta_1 \cup \delta_2)$ to find the desired cover; see Figure 5.2. $\square$

Lemma 5.1.3. *Let (Σ, x) be a based closed surface of genus $g \geq 2$. Let $(K, x) \hookrightarrow (\Sigma, x)$ be a π_1-injective subsurface. Let (β, x) be a simple loop based*

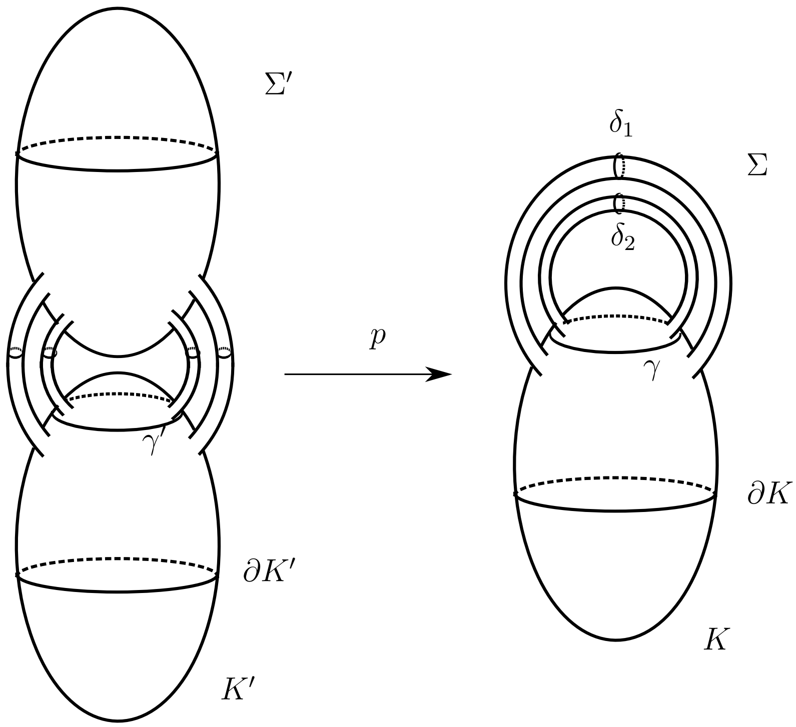

Figure 5.1: Constructing a double-cover $p : \Sigma' \to \Sigma$ where γ' is not separating in $\Sigma' \setminus K'$ and ∂K has one component.

at x such that $\beta \notin \pi_1(K, x)$ and β is freely homotopic into a component S of $\Sigma \setminus K$. Then there is a tower of no more than three double covers such that K lifts to the largest cover but β does not.

Proof. Throughout this proof, let $\gamma \subset S$ denote the simple closed curve in S to which β is freely homotopic.

First assume that γ is non-separating on S. Then applying Lemma 5.1.1 yields the desired double cover.

Next, assume that γ is separating but not essential on S. Then Lemma 5.1.2

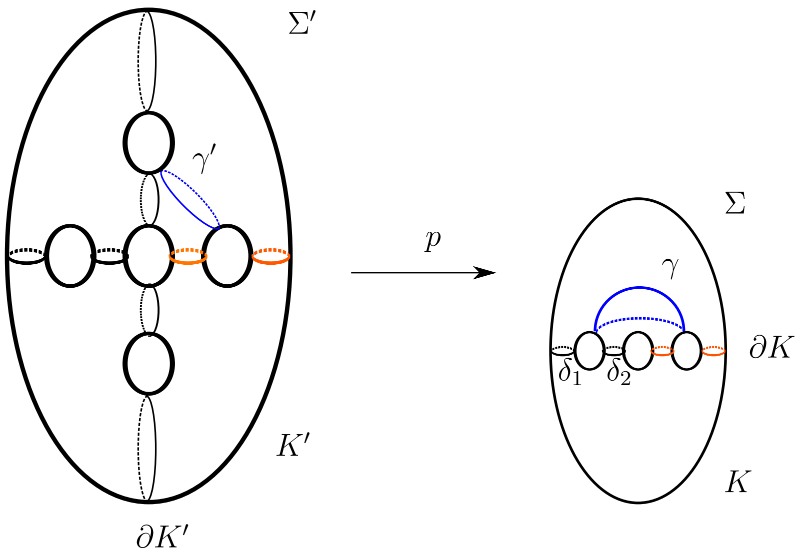

Figure 5.2: Constructing a double-cover $p : \Sigma' \to \Sigma$ where γ' is not separating in $\Sigma' \setminus K'$ and ∂K has at least two components.

yields a double cover to put us in the above case, so we find a tower of two double covers that achieves the goal.

Finally, assume that γ is separating and nonessential on S, that is, γ is freely homotopic to ∂S. In this case, ∂S has at least two components because, by the assumption that $\beta \notin \pi_1(K, x)$, β is not freely homotopic to the component of ∂S that it intersects. Cut along the boundary component to which it is homotopic and glue two copies to find a double cover where β lifts to a separating essential simple closed curve on S; the picture is similar to Figure 5.2. This puts us in the previous case, and thus β does not lift although K does in some tower of three double covers, as claimed. $\square$

5.2 Scott's theorem.

Here we record a theorem due to Peter Scott (Theorem 3.1 in [25]):

Theorem 5.2.1. *Let G be the fundamental group of a closed surface. Then G is LERF.*

In fact, Theorem 2.6 from [19] tells us a bit more:

Theorem 5.2.2. *Suppose that G is the fundamental group of a closed surface of negative Euler characteristic. Then G is LR.*

The main idea from the proof of Theorem 5.2.1 that we will use again in the proof of Theorem 5.4.2 is the notion of a compact core. Let S be a closed surface. Let H be a finitely generated subgroup of G. There is a **compact core** $C \subset S_H$, that is, C is a compact, essential subsurface of S_H.

Theorem 5.2.1, along with Scott's Criterion (Proposition 4.2.1 in Chapter 4) immediately yield the following:

Proposition 5.2.3. *Let S be a closed, hyperbolic surface and $G = \pi_1(S, x)$. Let $H \leq G$ be finitely generated, and let $(S_H, x_H) \to (S, x)$ be the associated covering. Then for any compact subset $C \subset S_H$, there exists an intermediate finite-sheeted cover $S_H \to S' \to S$ such that $S_H \to S'$ isometrically embeds C into S'.*

In Chapter 4, we used Scott's Criterion with the fact that free groups are LERF (Hall's Theorem) to pass to a finite-index subgroup $F' \leq F$ where a given finitely generated $H \leq G$ appears as a free factor and we proved that free factors are RS separable in free groups. In the present chapter, we will use Proposition 5.2.3 to find a finite-sheeted cover $S' \to S$ that contains a large compact set C that carries $H \leq \pi_1(S, x)$.

5.3 Geometry of hyperbolic surfaces.

The hyperbolic plane $\mathbb{H}^2$ is a complete, simply connected Riemannian surface such that $K \equiv -1$, and it is unique up to isometry. A **geodesic** is a curve that is everywhere locally distance minimizing. The hyperbolic plane is also a **unique geodesic metric space**, meaning that between any two points $x, y \in \mathbb{H}^2$, there is a unique geodesic arc from x to y. S is a **hyperbolic surface** if and only if any of the following (equivalent) conditions are true:

- S is a complete, Riemannian surface such that $K \equiv -1$
- $S = G\backslash\mathbb{H}^2$ such that G acts freely, properly discontinuously by isometries on $\mathbb{H}^2$
- $S = G \setminus \mathbb{H}^2$ such that $G \leq \mathrm{Isom}(\mathbb{H}^2)$ is torsion-free and discrete (i.e. $G \cdot x_0$ discrete for any $x_0 \in \mathbb{H}^2$)

In this section, let (S, x) be a hyperbolic surface with basepoint x, and let $p : \mathbb{H}^2 \to S$ denote the universal covering map. A **closed geodesic** is the unique shortest closed curve in the homotopy class of some closed curve, provided that such a thing exists (i.e. γ is not homotopic to a cusp). A

geodesic loop based at x is the image under p of a geodesic arc from $\tilde{x}$ to $g \cdot \tilde{x}$ (for some $\tilde{x} \in p^{-1}(x), g \in \pi_1(S, x)$). Note that a geodesic loop α based at x is not necessarily a closed geodesic. However, in certain special cases α may be a closed geodesic, and in this case α will be the unique closed geodesic in the *free* homotopy class of α.

If (γ, x) is a loop based at x, we use the notation $L(\gamma)$ for the length of γ, and $\ell(\gamma) = \ell_x(\gamma) = \ell_{(S,x)}$ for the length of the (unique) shortest geodesic loop based at x that is homotopic to γ leaving the basepoint fixed—this notation is justified by Lemma 5.3.3 below. In particular, note that $\ell_x(\gamma) \leq L(\gamma)$ with equality holding precisely when γ is a geodesic loop based at x. Let $B_r(x) = \{y \in S : d(x, y) \leq r\}$ denote the closed metric ball of radius r around x; note that this may not be a topological ball.

Recall from [5] that a closed, connected subset $A \subset S$ is **admissible** if either A consists of exactly one point or else A is a compact connected subset of ∂S. For completeness, we record the result from [5] that we apply below:

Theorem 5.3.1. *Let S be a hyperbolic surface, let $A, B \subset S$ be admissible subsets, and let $c : [a, b] \to S$ with $c(a) \in A, c(b) \in B$ be a curve from A to B (A and B need not be different or disjoint). Then the following hold:*

1. *In the homotopy class of c with endpoints gliding on A and B there exists a shortest curve γ. This curve is a geodesic arc.*

2. *If A and B are points, then γ is unique.*

Theorem 5.3.1 is Theorem 1.5.3, parts (i) and (vi) from [5]. The following fact is a direct consequence of Theorem 5.3.1 and says that every element of $\pi_1(S, x)$ may be uniquely represented by a geodesic loop based at x.

Lemma 5.3.2. *Let $g \in \pi_1(S, x)$ be nontrivial. Then up to based homotopy, there is a unique shortest loop α based at x such that $[\alpha] = g$. Furthermore, α is a geodesic loop.*

Proof. Every such $g \in \pi_1(S, x)$ may be represented by some loop $c : [a, b] \to S$ with $c(a) = c(b) = x \in S$. Let $A = B = \{x\}$, so A and B are admissible. Applying Theorem 5.3.1 in this situation yields a unique shortest loop α based at x, and this loop is a geodesic loop. $\square$

Thus, each element of $\pi_1(S, x)$ has a unique shortest representative $(\alpha, x) \subset (S, x)$ and α is a geodesic loop. This allows us to identify group elements with certain curves (not homotopy classes of curves) on S. We save this in the following lemma for future reference.

Lemma 5.3.3. *Every element $g \in \pi_1(S, x)$ has a unique shortest representative $(\alpha_g, x) \subset (S, x)$, and α_g is a geodesic loop based at x. In particular, we may abuse notation and write $\alpha_g = [\alpha_g] = g \in \pi_1(S, x)$.*

Recall that there are only finitely many closed geodesics of length less than N for any given $N < \infty$; see, for example, 1.6.11 in [5]. There is an analogous statement for geodesic loops based at $x \in S$, and we provide the elementary proof here:

Proposition 5.3.4. *Let $N < \infty$ be given. Then the set*

$$\{\alpha \in \pi_1(S, x) : \ell(\alpha) < N\}$$

is finite.

Proof. Although one could mimic the proof of 1.6.11 in [5], we provide our own proof. Fix a preferred lift $\tilde{x} \in \mathbb{H}^2$ of $x \in S$. Let $\Gamma \leq PSL_2(\mathbb{R})$ be a representation of G in $\mathrm{Isom}(\mathbb{H}^2)$, so Γ acts discontinuously on $\mathbb{H}^2$. In particular, every closed metric ball $B_N(\tilde{x})$ of radius N centered at $\tilde{x}$ contains only finitely many lifts of the basepoint $x \in S$. A geodesic loop based at x lifts to a geodesic segment between $\tilde{x}$ and another lift of x. But such a geodesic loop has length $\leq N$ if and only if such a lift is contained in $B_N(\tilde{x}) \subset \mathbb{H}^2$, which contains only finitely many lifts of x. Thus, there are only finitely many geodesic loops of length $\leq N$. $\square$

To prove that subgroups are RS-separable in closed surface groups, we will need the following fact about collections of geodesic loops in S.

Lemma 5.3.5. *Let (S, x) be an oriented hyperbolic surface with basepoint x. Let A_S denote the set of geodesic loops based at x that are not products of shorter loops based at x. Then:*

1. *For every $\alpha \in A_S$, α is simple.*

2. *For every $\alpha, \beta \in A_S$, $\alpha \cap \beta = \{x\}$.*

3. $\langle A_S \rangle = \pi_1(S, x)$.

In particular, if $\pi_1(S, x)$ is finitely generated, then A_S is finite.

Remarks:

1. This result is comparable to Lemma 4.3.2 in Chapter 4 (Theorem 4.3 in [10]), and in particular, condition (3) essentially follows from Proposition 4.3.3.

2. On a hyperbolic surface, the intersection (away from the basepoint x) of two geodesic loops must be transverse because, upon lifting a neighborhood of the intersection to the universal cover $\mathbb{H}^2 \to S$, we see that if two geodesics have parallel tangent vectors at their intersection point then they must coincide.

Proof of Lemma 5.3.5. First we prove (1). Suppose α is not simple and let $c \in \alpha$ be a self-intersection point adjacent on α to x. We will write α as the composition of two shorter loops based at x, which will lead to a contradiction. Let $\eta \subset \alpha$ be one of the two arcs connecting x to c in α, and let β be the other one. Without loss of generality, assume $L(\eta) \leq L(\beta)$. Denote the subloop of α based at c by δ; see Figure 5.3.

Now write

$$
\begin{aligned}
\alpha &= \eta \circ \delta \circ \beta \\
&= \eta \circ \delta \circ (\eta^{-1} \circ \eta) \circ \beta \\
&= \gamma_1 \circ \gamma_2
\end{aligned}
$$

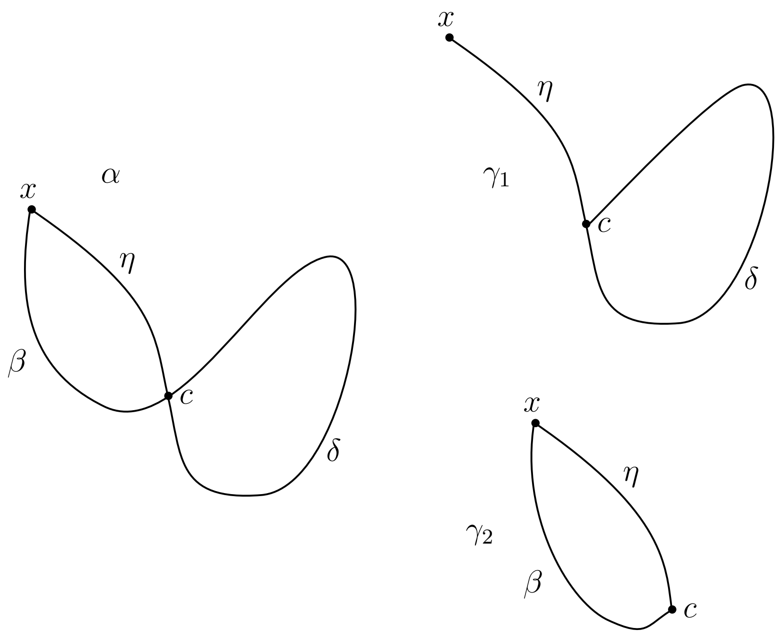

Figure 5.3: Decomposing α into a product of shorter closed loops based at x.

where $\gamma_1 = \eta \circ \delta \circ \eta^{-1}$ and $\gamma_2 = \eta \circ \beta$ are loops based at x. Note that $L(\gamma_1) = 2L(\eta) + L(\delta) \leq L(\alpha)$, although γ_1 is not a geodesic loop based at x since there is a corner at c. Replacing γ_1 with its unique shortest geodesic loop representative, we find that $\ell(\gamma_1) < \ell(\alpha)$. Similarly, $\ell(\gamma_2) < L(\gamma_2) = L(\eta) + L(\beta) < L(\alpha)$. This implies $\alpha \notin A_S$, a contradiction. Therefore α is simple, which proves (1).

Now we prove (2). Assume that $L(\alpha) \leq L(\beta)$. Suppose that α and β intersect somewhere other than x, say at a point $c \in S$, and write $\alpha = \alpha_1 \circ \alpha_2$ and $\beta = \beta_1 \circ \beta_2$ where each arc α_i and β_i has endpoints x and c. Assume further that $L(\alpha_1) \leq L(\alpha_2)$ and $L(\beta_1) \leq L(\beta_2)$. Note that since α and β are geodesic loops, the intersection at c must be transverse. There are two cases:

Case 1: $L(\alpha_1) < L(\beta_1)$. Then $\ell(\beta_1\alpha_1^{-1}) < L(\beta_1\alpha_1^{-1}) < L(\beta) = \ell(\beta)$ and $L(\alpha_1\beta_2) < L(\beta)$. But then $\beta = (\beta_1\alpha_1^{-1})(\alpha_1\beta_2)$ is a product of shorter loops, a contradiction.

Case 2: $L(\beta_1) \leq L(\alpha_1)$. Then $\ell(\beta_1\alpha_2) < L(\beta_1\alpha_2) \leq \ell(\alpha)$ and $\ell(\alpha_1\beta_1^{-1}) < \ell(\alpha)$. So we find that $\alpha = (\alpha_1\beta_1^{-1})(\beta_1\alpha_2)$ is a product of shorter loops, a contradiction.

Finally, we prove (3). If (3) does not hold, let $\beta \in \pi_1(S, x) \setminus \langle A_S \rangle$ be a shortest geodesic loop. But then, since the only curves on S that are shorter than β represent elements of $\langle A_S \rangle$, we find $\beta \in A_S$, a contradiction. If $\pi_1(S, x)$ is finitely generated, say by $X = \{x_1, \ldots, x_n\}$, then each $x_i = w_i([\alpha_{i,j}])$ for some $\alpha_{i,j} \in A_S$, and the collection $\{\alpha_{i,j}\}$ is finite. Let $L = \max_{i,j} L(\alpha_{i,j})$. If $\alpha \in A_S$ with $L(\alpha) > L$, then since $\{[\alpha_{i,j}]\}$ generate, α is a product in $\alpha_{i,j}$, which implies $\alpha \notin A_S$. Therefore A_S is finite by Proposition 5.3.4. $\square$

If $(K, x) \subset (S, x)$ is an essential subsurface, then let A_K denote the geodesic loops $(\alpha, x) \subset (K, x)$ which are not products of shorter loops in (K, x). In certain important cases, $A_K \subset A_S$, and sometimes we may even have $A_K = A_S \cap K$, but this is not true in general; for example, if K is an annular neighborhood of a simple closed loop α not in A_S, then $\alpha \in A_K$ although $\alpha \notin A_S$.

5.4 RS separating subgroups of surface groups.

Here we prove that orientable surface groups are LVRSS (Theorem 5.4.2). The proof is in two main steps: first we use Scott's Theorem to show that, given any finitely generated subgroup H of a surface group, there is a finite-sheeted cover that contains a "large" essential subsurface whose fundamental group is H, and then we show that these large essential subsurfaces are RS separable in that cover.

Theorem 5.4.1. *Let (Σ, x) be a closed, orientable, hyperbolic surface, and let $H \leq \pi_1(\Sigma, x)$ be a finitely generated infinite-index subgroup. Then there exists a finite-sheeted cover $(\Sigma', x') \to (\Sigma, x)$ such that $H \leq_{RS} \pi_1(\Sigma', x')$.*

Proof. Consider the cover $(\Sigma_H, x_H) \to (\Sigma, x)$ corresponding to $H \leq \pi_1(\Sigma, x)$. Note that by Lemma 5.3.5, the collection A_{Σ_H} generates H and is finite. Let L be bigger than the length of any element of A_{Σ_H}, so that $K_H = B_L(x_H)$ contains a compact core of (Σ_H, x_H) and $\pi_1(K_H, x_H) = H$. Then Proposition 5.2.3 implies that there exists a finite-sheeted cover $(\Sigma', x') \to (\Sigma, x)$ where $(K, x_H) \hookrightarrow (K', x') \subset (\Sigma', x')$ embeds isometrically as an essential subsurface of Σ'.

For each $b \in \pi_1(\Sigma', x') \setminus \pi_1(K', x')$, apply Lemma 5.3.2 to fix its unique geodesic loop representative β, and call this collection of geodesic loops $\mathcal{B}$. Let $(\Sigma_0, x_0) := (\Sigma', x')$ and $K_0 = K'$. Our goal is to find an RS tower

$$\cdots \to (\Sigma_n, x_n) \to (\Sigma_{n-1}, x_{n-1}) \to \cdots \to (\Sigma_1, x_1) \to (\Sigma_0, x_0)$$

such that $\cap \pi_1(\Sigma_n, x_n) = \pi_1(K_0, x_0) = H$.

Since there are only finitely many geodesic loops based at x with length less than any given finite number (Proposition 5.3.4), there exists a shortest element $\beta_0 \in \mathcal{B}$. First, we claim that β_0 is simple and intersects each $\alpha \in A_{K_0}$ minimally. Since $\beta_0 \notin \pi_1(K_0, x_0)$, we know that $\beta_0 \in A_{\Sigma_0}$. Also, $\beta_0 \notin A_{K_0}$ because every $\alpha \in A_{K_0}$ has $\alpha \in \pi_1(K_0, x_0)$. The hypothesis implies that $K_0 = B_L(x_0)$ implies $A_{K_0} = A_{\Sigma_0} \cap K_0$, and thus $A_{K_0} \sqcup \{\beta_0\} \subset A_{\Sigma_0}$. Therefore Lemma 5.3.5 implies that β_0 is simple and $\beta_0 \cap \alpha = \{x\}$ for all $\alpha \in A_{K_0}$.

Now β_0 must exit K_0, and thus $i(\beta_0 \cap \partial K_0) = 2n$ for some $n > 0$. If $n > 1$, then, since geodesic arcs from x_0 to $\partial K_0 = \partial B_L(x_0)$ have length L, and general paths (including intersections of K_0 with geodesic loops that exit K_0) from x_0 to ∂K_0 have length at least L, we could write β_0 as the product of loops shorter than β_0 (some exiting K_0 and others remaining inside K_0), a contradiction. Thus, $|\beta_0 \cap \partial K_0| = 2$.

Now we work with β_0. If β_0 intersects two distinct components of ∂K_0, then apply Lemma 5.1.1 to find a double-cover $\Sigma_1 \to \Sigma_0$ to which K_0 lifts but β_0 does not. If β_0 intersects a single component of ∂K_0 twice, then write $\beta_0 = \alpha_0 \gamma_0$ for some simple $\alpha_0 \in \pi_1(K_0, x_0)$ and a simple closed curve (γ_0, x_0) freely isotopic to $\gamma \subset S_0$, a connected component of $\Sigma_0 \setminus K_0$. We apply Lemma 5.1.3 to find a a sequence of at most three double covers $\Sigma_1 \to \Sigma_0'' \to \Sigma_0' \to \Sigma_0$ such that $\beta_0 = \alpha_0 \gamma_0$ does not lift to Σ_1 although K_0 does.

Note that lengths of geodesic loops are preserved under taking covers, i.e. if $(\gamma, x_1) \subset (\Sigma_1, x_1)$ is a geodesic loop, and $p_1 : (\Sigma_1, x_1) \to (\Sigma_0, x_0)$ is the cover, then $\ell_{(\Sigma_1, x_1)}(\gamma) = \ell_{(\Sigma_0, x_0)}(p_1(\gamma))$. Similarly, in any cover (Σ_n, x_n) to which K_0 lifts, it lifts to $K_n = B_L(x_n)$ in the cover. Now for $n \geq 1$, repeat the construction above with the shortest $\beta_n \in \mathcal{B} \cap \Sigma_n$ to get a tower $\cdots \to (\Sigma_{n+1}, x_{n+1}) \to (\Sigma_n, x_n) \to \cdots \to (\Sigma_0, x_0)$. By Proposition 5.3.4, for every $\beta \in \mathcal{B}$ there is a cover in the tower to which β does not lift. Thus, we have $\cap \pi_1(\Sigma_n, x_n) = \pi_1(K, x)$ as needed. Furthermore, this is an RS tower because $\pi_1(\Sigma_n, x_n)/\pi_1(\Sigma_{n+1}, x_{n+1})$ is abelian for every $n \geq 0$. $\square$

Theorem 5.4.2. *Let $\Sigma = \Sigma_{g,n}$ be an orientable surface of genus $g \geq 0$ and $n \geq 0$ boundary components. Then $G = \pi_1(\Sigma)$ is LVRSS.*

Proof. Let $H \leq G$ be finitely generated, where G is the fundamental group of an orientable surface Σ. By Corollary 4.4.2, we may assume that Σ is closed, that is, $n = 0$. By Theorem 2.5.3, we may further assume that $g \geq 2$.

Thus, assume Σ is a closed, hyperbolic surface, set $G := \pi_1(\Sigma, x)$, and let $H \leq G$ be a finitely generated subgroup. Since all finite-index subgroups are virtually RS-separable, we may assume that H is infinite-index in G, and thus Theorem 5.4.1 applies, which finishes the proof. $\square$

Chapter 6

Conclusion

In this chapter, we list a basic application of our main results, and we end with some questions to investigate in the future.

6.1 Applications.

There has been some recent interest in building covers of surfaces and analyzing lifts of certain closed curves to those covers; see, for example, [22] and [24]. In particular, it is interesting to note that, similar to the application of the Federer and Jónsson theorem, the primary methods in [22] are applications of classical group theoretic tools developed by Fox et al. in a series of papers, notably [11] and [6]. One of the main results in [22] is a lower bound on the self-intersection number of closed curves in $\gamma_k(\pi_1(\Sigma_{g,n}))$, the k-th term of the derived series of a punctured surface group, and thus, as an example of an application of Theorem B, we have:

Corollary 6.1.1. *Let (Σ, x) and A_Σ be as in Theorem B, and suppose that $\pi_1(\Sigma, x)$ is free and nonabelian. Then for every $\alpha \in A_\Sigma$, $[\alpha] \notin \gamma_k(G)$ for $k > 4g + n - 1$. In other words, geodesic loops that are not products of shorter loops are not contained in higher terms of the derived series.*

Proof. Theorem 1.1 in [22] states that, for all $k \geq 1$,

$$m_{lcs}(\Sigma, k) \geq \frac{k}{4g+n-1} - 1$$

where

$$m_{lcs}(\Sigma, k) = \min\{i(x) : x \in \gamma_k(\pi_1(\Sigma)), x \neq 1\}$$

and $i(x)$ denotes the self-intersection number. Let $\alpha \in A_\Sigma$, and assume that $[\alpha] \in \gamma_k(G)$. Since α is simple, this implies that $m_{lcs}(\Sigma, k) = 0$, and thus

$$k \leq 4g+n-1$$

Therefore, if $k > 4g+n-1$, then $[\alpha] \notin \gamma_k(G)$. □

6.2 Future research.

Local virtual RS separability is a strong property that implies LERF and virtually RFRS, and therefore LVRSS has important implications in low-dimensional topology. The collection of groups that are known to be both LERF (or LR) and virtually RFRS is rather small, but it includes limit groups. Thus, a natural question to ask is:

Question 1. *Are all fully residually free groups LVRSS?*

Note that, since $F_2 \times F_2$ is residually free (see, for example, [3]), it is not true that all residually free groups are LVRSS.

I would also like to understand better how RS-separable subgroups and LVRSS are preserved under various other group operations. For example:

Question 2. *Suppose that* $H_1 \leq_{RS} G_1$ *and* $H_2 \leq_{RS} G_2$*. Is* $H_1 * H_2 \leq_{RS} G_1 * G_2$*? If so, does* G_1, G_2 *LVRSS imply that* $G_1 * G_2$ *is LVRSS?*

One possible attack on the first part of this problem is via an argument involving Bass-Serre theory (see, for example, [27]) in the spirit of Agol's proof that free products of virtually RFRS groups are virtually RFRS [1]. Then, assuming that the first part of Question 2 is true, one might be able to use methods similar to those in [12] to conclude that the second part of the question is true; more concretely, suppose that P is a property of a subgroup H in G, denoted by $H \leq_P G$. For example, P might be "separable" or "RS-separable"; in [12], the authors are primarily concerned with malnormality: $H \leq G$ is **malnormal** if for every $g \in G \setminus H$, $H \cap H^g = \{1\}$, that is, H is "as far from normal as possible". The authors prove in [12] that the free product of locally virtually malnormal (LVM), LERF groups is again LVM and LERF. In fact, their proof shows that if P is any property that satisfies the following three properties for all LERF groups G, then the free product of two LVP, LERF groups is itself LVP and LERF:

1. $H \leq_P G$ and $H \leq K \leq G$ implies $H \leq_P K$;
2. $H_1 \leq_P G_1$ and $H_2 \leq_P G_2$ implies $H_1 * H_2 \leq_P G_1 * G_2$;
3. $1 \leq_P G$ and $G \leq_P G$.

Suppose that $P = VRS$. Then the first property is contained in the proof of Corollary 3.1.3. Unfortunately, the third property asks for all LERF groups to be virtually RFRS. However, our hope is this: if we knew that the first part of Question 2 was true, then we could develop methods similar to the proof of Theorem 1.5 from [12] to show that the free product of two LVRSS groups is itself LVRSS.

Another question asks exactly how close LVRSS is to LERF and RFRS:

Question 3. *Suppose that G is (virtually) RFRS and LERF. Is G LVRSS?*

The nature of LVRSS suggests that the answer to this question is "no", but I do not know of such an example. Finally, one could ask for some evidence against the LERF conjecture:

Question 4. *Does there exist a closed, hyperbolic 3-manifold M such that $\pi_1(M)$ is not LVRSS?*

Bibliography

[1] Ian Agol. Criteria for virtual fibering. *J. Topol.*, 1(2):269–284, 2008.

[2] Ian Agol. The virtual fibering conjecture and related questions, 2009. slides from a talk given at The University of Georgia; available online.

[3] Martin R. Bridson and Henry Wilton. Subgroup separability in residually free groups. *Math. Z.*, 260(1):25–30, 2008.

[4] R. G. Burns. On finitely generated subgroups of free products. *J. Austral. Math. Soc.*, 12:358–364, 1971.

[5] Peter Buser. *Geometry and spectra of compact Riemann surfaces*, volume 106 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 1992.

[6] K.-T. Chen, R. H. Fox, and R. C. Lyndon. Free differential calculus. IV. The quotient groups of the lower central series. *Ann. of Math. (2)*, 68:81–95, 1958.

[7] Tim D. Cochran. Noncommutative knot theory. *Algebr. Geom. Topol.*, 4:347–398, 2004.

[8] Tim D. Cochran and Peter Teichner. Knot concordance and von Neumann ρ-invariants. *Duke Math. J.*, 137(2):337–379, 2007.

[9] David S. Dummit and Richard M. Foote. *Abstract Algebra.* Wiley, 1999.

[10] Herbert Federer and Bjarni Jónsson. Some properties of free groups. *Trans. Amer. Math. Soc.*, 68:1–27, 1950.

[11] Ralph H. Fox. Free differential calculus. I. Derivation in the free group ring. *Ann. of Math. (2)*, 57:547–560, 1953.

[12] Rita Gitik, Stuart W. Margolis, and Benjamin Steinberg. On the Kurosh theorem and separability properties. *J. Pure Appl. Algebra*, 179(1-2):87–97, 2003.

[13] K. W. Gruenberg. Residual properties of infinite soluble groups. *Proc. London Math. Soc. (3)*, 7:29–62, 1957.

[14] Frédéric Haglund and Daniel T. Wise. Special cube complexes. *Geom. Funct. Anal.*, 17(5):1551–1620, 2008.

[15] Marshall Hall, Jr. Subgroups of finite index in free groups. *Canadian J. Math.*, 1:187–190, 1949.

[16] Shelly L. Harvey. Higher-order polynomial invariants of 3-manifolds giving lower bounds for the Thurston norm. *Topology*, 44(5):895–945, 2005.

[17] Allen Hatcher. *Algebraic topology.* Cambridge University Press, Cambridge, 2002.

[18] Jeremy Kahn and Vladimir Markovic. Immersing almost geodesic surfaces in a closed hyperbolic three manifold. *arXiv:0910.5501*, pages 1–50, 2009.

[19] D. D. Long and A. W. Reid. Subgroup separability and virtual retractions of groups. *Topology*, 47(3):137–159, 2008.

[20] Darren Long and Alan W. Reid. *Surface subgroups and subgroup separability in 3-manifold topology.* Publicações Matemáticas do IMPA. [IMPA Mathematical Publications]. Instituto Nacional de Matemática Pura e Aplicada (IMPA), Rio de Janeiro, 2005. 25º Colóquio Brasileiro de Matemática. [25th Brazilian Mathematics Colloquium].

[21] Roger C. Lyndon and Paul E. Schupp. *Combinatorial group theory.* Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1977 edition.

[22] Justin Malestein and Andrew Putman. On the self-intersections of curves deep in the lower central series of a surface group. *arXiv:0901.2561*, pages 1–12, 2009.

[23] Charles F. Miller, III. *On group-theoretic decision problems and their classification.* Princeton University Press, Princeton, N.J., 1971. Annals of Mathematics Studies, No. 68.

[24] Igor Rivin. Geodesics with one self-intersection, and other stories. *arXiv:0901.2543*, pages 1–17, 2009.

[25] Peter Scott. Subgroups of surface groups are almost geometric. *J. London Math. Soc. (2)*, 17(3):555–565, 1978.

[26] Peter Scott. Correction to: "Subgroups of surface groups are almost geometric" [J. London Math. Soc. (2) **17** (1978), no. 3, 555–565; MR0494062 (58 #12996)]. *J. London Math. Soc. (2)*, 32(2):217–220, 1985.

[27] Peter Scott and Terry Wall. Topological methods in group theory. In *Homological group theory (Proc. Sympos., Durham, 1977)*, volume 36 of *London Math. Soc. Lecture Note Ser.*, pages 137–203. Cambridge Univ. Press, Cambridge, 1979.

[28] Henry Wilton. Hall's theorem for limit groups. *Geom. Funct. Anal.*, 18(1):271–303, 2008.

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