The Dissertation Committee for César A. Garza
certifies that this is the approved version of the following dissertation:

A construction of hyperkähler metrics through Riemann-Hilbert problems

Committee:

Andrew Neitzke, Supervisor

Dan Knopf

Dan Freed

Lorenzo Sadun

Jacques Distler

A construction of hyperkähler metrics through Riemann-Hilbert problems

by

César A. Garza, B.S.; M.S.

DISSERTATION

Presented to the Faculty of the Graduate School of

The University of Texas at Austin

in Partial Fulfillment

of the Requirements

for the Degree of

DOCTOR OF PHILOSOPHY

THE UNIVERSITY OF TEXAS AT AUSTIN

August 2015

Dedicated to my wife Rebeca.

Acknowledgments

I am deeply indebted to Dr. Andrew Neitzke, my advisor. With great patience and constant support he contributed to this thesis with brilliant ideas, suggestions and corrections where needed. Throughout my Ph.D., his integrity, commitment and passion for teaching was an inspiring example for me. This thesis could not have come into being without him.

I would also like to thank Dr. Dan Knopf, Dr. Dan Freed, Dr. Lorenzo Sadun and Dr Jacques Distler for being part of my thesis committee. I greatly appreciate their time and effort while reviewing this paper.

Finally, I am very grateful to my beloved wife, Sara Rebeca Tarín for her support, love and patience throughout these years.

A construction of hyperkähler metrics through Riemann-Hilbert problems

Publication No. ________________

César A. Garza, Ph.D.
The University of Texas at Austin, 2015

Supervisor: Andrew Neitzke

In 2009 Gaiotto, Moore and Neitzke presented a new construction of hyperkähler metrics on the total spaces of certain complex integrable systems, represented as a torus fibration $\mathcal{M}$ over a base space $\mathcal{B}$, except for a divisor D in $\mathcal{B}$, in which the torus fiber degenerates into a nodal torus. The hyperkähler metric g is obtained via solutions $\mathcal{X}_\gamma$ of a Riemann-Hilbert problem. We interpret the Kontsevich-Soibelman Wall Crossing Formula as an isomonodromic deformation of a family of RH problems, therefore guaranteeing continuity of $\mathcal{X}_\gamma$ at the walls of marginal stability. The latter functions are obtained through standard Banach contraction principles. By obtaining uniform estimates on arbitrary derivatives of $\mathcal{X}_\gamma$, the smoothness property is obtained. To extend this construction to singular fibers, we use the Ooguri-Vafa case as our model and choose a suitable gauge transformation that allow us to define an integral equation defined at the degenerate fiber, whose solutions are the desired Darboux coordinates $\mathcal{X}_\gamma$.

Table of Contents

List of Figures

Chapter 1

Introduction

1.1 Preliminaries

Hyperkähler manifolds first appeared within the framework of differential geometry as Riemannian manifolds with holonomy group of special restricted group. Nowadays, hyperkähler geometry forms a separate research subject fusing traditional areas of mathematics such as differential and algebraic geometry of complex manifolds, holomorphic symplectic geometry, Hodge theory and many others.

One of the latest links can be found in theoretical physics: In 2009, Gaiotto, Moore and Neitzke [7] proposed a new construction of hyperkähler metrics g on target spaces $\mathcal{M}$ of quantum field theories with $d = 4, \mathcal{N} = 2$ superysmmetry. Such manifolds were already known to be hyperkähler (see [19]), but no known explicit hyperkähler metrics have been constructed.

The manifold $\mathcal{M}$ is a total space of a complex integrable system and it can be expressed as follows. There exists a complex manifold $\mathcal{B}$, a divisor $D \subset \mathcal{B}$ and a subset $\mathcal{M}' \subset \mathcal{M}$ such that $\mathcal{M}'$ is a torus fibration over $\mathcal{B}' := \mathcal{B} \backslash D$. On the divisor D, the torus fibers of $\mathcal{M}$ degenerate, as Figure 1.1 shows.

Moduli spaces $\mathcal{M}$ of Higgs bundles on Riemann surfaces with prescribed

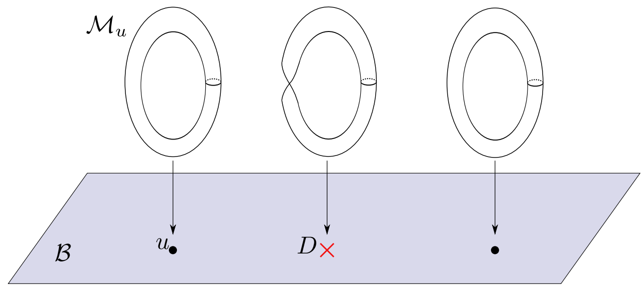

Figure 1.1: Hyperkähler manifolds realized as torus fibrations

singularities at finitely many points are one of the prime examples of this construction. Hyperkähler geometry is useful since we can use Hitchin's twistor space construction [13] and consider all $\mathbb{P}^1$-worth of complex structures at once. In the case of moduli spaces of Higgs bundles, this allows us to consider $\mathcal{M}$ from three distinct viewpoints:

1. (Dolbeault) $\mathcal{M}_{\mathrm{Dol}}$ is the moduli space of Higgs bundles, i.e. pairs (E, Φ), $E \to C$ a rank n degree zero holomorphic vector bundle and $\Phi \in \Gamma(\mathrm{End}(E) \otimes \Omega^1)$ a Higgs field.

2. (De Rham) $\mathcal{M}_{\mathrm{DR}}$ is the moduli space of flat connections on rank n holomorphic vector bundles, consisting of pairs (E, ∇) with $\nabla : E \to \Omega^1 \otimes E$ a holomorphic connection and

3. (Betti) $\mathcal{M}_{\mathrm{B}} = \mathrm{Hom}(\pi_1(C) \to \mathrm{GL}_n(\mathbb{C}))/\mathrm{GL}_n(\mathbb{C})$ of conjugacy classes of representations of the fundamental group of C.

All these algebraic structures form part of the family of complex structures making $\mathcal{M}$ into a hyperkähler manifold.

To prove that the manifolds $\mathcal{M}$ from the integrable systems are indeed hyperkähler, we start with the existence of a simple, explicit hyperkähler metric g^{sf} on $\mathcal{M}'$. Unfortunately, g^{sf} does not extend to $\mathcal{M}$. To construct a complete metric g, it is necessary to do "quantum corrections" to g^{sf}. These are obtained by solving a certain explicit integral equation ((2.9) below). The novelty is that the solutions, acting as Darboux coordinates for the hyperkähler metric g, have discontinuities at a specific locus in $\mathcal{B}$. Such discontinuities cancel the global monodromy around D and is thus feasible to expect that g extends to the entire $\mathcal{M}$.

We start by defining a Riemann-Hilbert problem on the $\mathbb{P}^1$-slice of the twistor space $\mathcal{Z} = \mathcal{M}' \times \mathbb{P}^1$. That is, we look for functions $\mathcal{X}_\gamma$ with prescribed discontinuities and asymptotics. In the language of Riemann-Hilbert theory, this is known as *monodromy data*. Rather than a single Riemann-Hilbert problem, we have a whole family of them parametrized by the $\mathcal{M}'$ manifold. We show that this family constitutes an *isomonodromic deformation* since by the Kontsevich-Soibelman Wall-Crossing Formula, the monodromy data remains invariant.

Although solving Riemann-Hilbert problems in general is not always possible, in this case it can be reduced to an integral equation solved by standard Banach contraction principles. Uniform estimates obtained with saddle-point analysis guarantee that solutions to this Riemann-Hilbert problem not only exist, but they preserve the smooth and holomorphic properties on $\mathcal{M}' \times \mathbb{P}^1$.

The extension of the manifold $\mathcal{M}'$ is obtained by gluing a circle bundle with an appropriate gauge transformation eliminating any monodromy problems near the divisor D. The circle bundle constructs the degenerate tori at the discriminant locus D except for a small neighborhood where the pinch is, since it is not possible to define the integral equation there (see Figure 1.2).

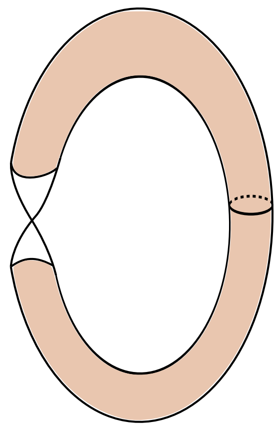

Figure 1.2: Construction of degenerate fibers away from the pinch

On the extended manifold $\mathcal{M}$ we prove that the solutions $\mathcal{X}_\gamma$ of the Riemann-Hilbert problem on $\mathcal{M}'$ extend and the resulting holomorphic symplectic form $\varpi(\zeta)$ gives the desired hyperkähler metric g. On the Appendix we will give a heuristic argument based on numerical evidence as to how extend the metric to the entire degenerate torus in a particular case known as the "Pentagon" (a case of Hitchin systems with gauge group SU(2)).

Although for the most basic examples of this construction such as the moduli space of Higgs bundles it was already known that $\mathcal{M}'$ extends to a hyperkähler manifold $\mathcal{M}$ with degenerate torus fibers, the construction here works for the general case of $\dim_{\mathbb{C}} \mathcal{B} = 1$. Moreover, the functions $\mathcal{X}_\gamma$ here are special coordinates arising in moduli spaces of flat connections, Teichmüller theory and Mirror Symmetry. In particular, these functions are used in [5] for the construction of holomorphic discs with boundary on special Lagrangian torus fibers of mirror manifolds.

We start by presenting the complex integrable systems introduced in [7].

Chapter 2

Complex Integrable Systems

2.1 Integrable Systems Data

As motivation, consider the moduli space $\mathcal{M}$ of Higgs bundles on a complex curve C with Higgs field Φ having prescribed singularities at finitely many points. In [8], it is shown that the space of quadratic differentials u on C with fixed poles and residues is a complex affine space $\mathcal{B}$ and the map $\det : \mathcal{M} \to \mathcal{B}$ is proper with generic fiber $\mathrm{Jac}(\Sigma_u)$, a compact torus obtained from the *spectral curve* $\Sigma_u := \{(z, \phi) \in T^*C : \phi^2 = u\}$, a double-branched cover of C over the zeroes of the quadratic differential u. Σ_u has an involution that flips $\phi \mapsto -\phi$. If we take Γ_u to be the subgroup of $H_1(\Sigma_u, \mathbb{Z})$ odd under this involution, Γ forms a lattice of rank 2 over $\mathcal{B}'$, the space of quadratic differentials with only simple zeroes. This lattice comes with a non-degenerate anti-symmetric pairing $\langle , \rangle$ from the intersection pairing in H_1. It is also proved in [8] that the fiber $\mathrm{Jac}(\Sigma_u)$ can be identified with the set of characters $\mathrm{Hom}(\Gamma_u, \mathbb{R}/2\pi\mathbb{Z})$. If λ denotes the tautological 1-form in T^*C, then for any $\gamma \in \Gamma$,

$$Z_\gamma = \frac{1}{\pi} \oint_\gamma \lambda$$

defines a holomorphic function Z_γ in $\mathcal{B}'$. Let $\{\gamma_1, \gamma_2\}$ be a local basis of Γ with $\{\gamma^1, \gamma^2\}$ the dual basis of Γ^*. Without loss of generality, we also denote by $\langle , \rangle$ the pairing in Γ^*. Let $\langle dZ \wedge dZ \rangle$ be short notation for $\langle \gamma^1, \gamma^2 \rangle \, dZ_{\gamma_1} \wedge dZ_{\gamma_2}$. Since $\dim_{\mathbb{C}} \mathcal{B}' = 1$, $\langle dZ \wedge dZ \rangle = 0$.

This type of data arises very frequently in the construction of hyperkähler manifolds, so we summarize the conditions required:

We start with a complex manifold $\mathcal{B}$ (later shown to be affine) of dimension n and a divisor $D \subset \mathcal{B}$. Let $\mathcal{B}' = \mathcal{B} \backslash D$. Over $\mathcal{B}'$ there is a local system Γ with fiber a rank $2n$ lattice, equipped with a non-degenerate anti-symmetric integer valued pairing $\langle \, , \rangle$.

We will denote by Γ^* the dual of Γ and, by abuse of notation, we'll also use $\langle \, , \rangle$ for the dual pairing (not necessarily integer-valued) in Γ^*. Let u denote a general point of $\mathcal{B}'$. We want to obtain a torus fibration over $\mathcal{B}'$, so let $\mathrm{TChar}_u(\Gamma)$ be the set of twisted unitary characters of ${\Gamma_u}$[1], i.e. maps $\theta : \Gamma_u \to \mathbb{R}/2\pi\mathbb{Z}$ satisfying

$$\theta_\gamma + \theta_{\gamma'} = \theta_{\gamma+\gamma'} + \pi \langle \gamma, \gamma' \rangle \, .$$

Topologically, $\mathrm{TChar}_u(\Gamma)$ is a torus $(S^1)^{2n}$. Letting u vary, the $\mathrm{TChar}_u(\Gamma)$ form a torus bundle $\mathcal{M}'$ over $\mathcal{B}'$. Any local section γ gives a local angular coordinate of $\mathcal{M}'$ by "evaluation on γ", $\theta_\gamma : \mathcal{M}' \to \mathbb{R}/2\pi\mathbb{Z}$.

[1] Although we can also work with the set of unitary characters (no twisting involved) by shifting the θ-coordinates, we choose not to do so, as that results in more complex calculations

We also assume there exists a homomorphism $Z : \Gamma \to \mathbb{C}$ such that the vector $Z(u) \in \Gamma_a^* \otimes \mathbb{C}$ varies holomorphically with u. If we pick a patch $U \subset \mathcal{B}'$ on which Γ admits a basis $\{\gamma_1, \ldots, \gamma_{2n}\}$ of local sections in which $\langle , \rangle$ is the standard symplectic pairing, then (after possibly shrinking U) the functions

$$f_i = \mathrm{Re}(Z_{\gamma_i})$$

are real local coordinates. The transition functions on overlaps $U \cap U'$ are valued on $\mathrm{Sp}(2n, \mathbb{Z})$, as different choices of basis in Γ must fix the symplectic pairing. This gives an affine structure on $\mathcal{B}'$.

By differentiating and evaluating in γ, we get 1-forms $d\theta_\gamma, dZ_\gamma$ on $\mathcal{M}'$ which are linear on Γ. For a local basis $\{\gamma_1, \ldots, \gamma_{2n}\}$ as in the previous paragraph, let $\{\gamma^1, \ldots, \gamma^{2n}\}$ denote its dual basis on Γ^*. We write $\langle dZ \wedge dZ \rangle$ as short notation for

$$\left\langle \gamma^i, \gamma^j \right\rangle dZ_{\gamma_i} \wedge dZ_{\gamma_j}, \tag{2.1}$$

where we sum over repeated indices. Observe that the anti-symmetric pairing $\langle \, , \rangle$ and the anti-symmetric wedge product of 1-forms makes (2.1) symmetric. We require that:

$$\langle dZ \wedge dZ \rangle = 0, \tag{2.2}$$

By (2.2), near u, $\mathcal{B}'$ can be locally identified with a complex Lagrangian submanifold of $\Gamma^* \otimes_{\mathbb{Z}} \mathbb{C}$.

In the example of moduli spaces of Higgs bundles, as u approaches a quadratic differential with non-simple zeros, one homology cycles vanishes (see

Figure 1.1). This cycle γ_0 is primitive in H_1 and its monodromy around the critical quadratic differential is governed by the Picard-Lefschetz formula. In the general case, let D_0 be a component of the divisor $D \subset \mathcal{B}$. We also assume the following:

- $Z_{\gamma_0}(u) \to 0$ as $u \to u_0 \in D_0$ for some $\gamma_0 \in \Gamma$.
- γ_0 is primitive (i.e. there exists some γ' with $\langle \gamma_0, \gamma' \rangle = 1$).
- The monodromy of Γ around D_0 is of "Picard-Lefschetz type", i.e.

$$\gamma \mapsto \gamma + \langle \gamma, \gamma_0 \rangle \gamma_0 \tag{2.3}$$

We assign a complex structure and a holomorphic symplectic form on $\mathcal{M}'$ as follows (see [17] and the references therein for proofs). Take a local basis $\{\gamma_1, \ldots, \gamma_{2n}\}$ of Γ. If $\epsilon^{ij} = \langle \gamma_i, \gamma_j \rangle$ and ϵ_{ij} is its dual, let

$$\omega_+ = \langle dZ \wedge d\theta \rangle = \epsilon_{ij}\, dZ_{\gamma_i} \wedge d\theta_{\gamma_j}. \tag{2.4}$$

By linearity on γ of the 1-forms, ω_+ is independent of the choice of basis. There is a unique complex structure J on $\mathcal{M}'$ for which ω_+ is of type (2,0). The 2-form ω_+ gives a holomorphic symplectic structure on $(\mathcal{M}', J)$. With respect to this structure, the projection $\pi : \mathcal{M}' \to \mathcal{B}'$ is holomorphic, and the torus fibers $\mathcal{M}'_u = \pi^{-1}(u)$ are compact complex Lagrangian submanifolds.

Recall that a positive 2-form ω on a complex manifold is a real 2-form for which $\omega(v, Jv) > 0$ for all real tangent vectors v. From now on, we assume that $\langle dZ \wedge d\overline{Z} \rangle$ is a positive 2-form on $\mathcal{B}'$. Now fix $R > 0$. Then we can define a 2-form on $\mathcal{M}'$ by

$$\omega_3^{\mathrm{sf}} = \frac{R}{4} \langle dZ \wedge d\overline{Z} \rangle - \frac{1}{8\pi^2 R} \langle d\theta \wedge d\theta \rangle .$$

This is a positive form of type (1,1) in the J complex structure. Thus, the triple $(\mathcal{M}', J, \omega_3^{\mathrm{sf}})$ determines a Kähler metric g^{sf} on $\mathcal{M}'$. This metric is in fact hyperkähler (see [6]), so we have a whole $\mathbb{P}^1$-worth of complex structures for $\mathcal{M}'$, parametrized by $\zeta \in \mathbb{P}^1$. The above complex structure J represents $J(\zeta = 0)$, the complex structure at $\zeta = 0$ in $\mathbb{P}^1$. The superscript $^{\mathrm{sf}}$ stands for "semiflat". This is because g^{sf} is flat on the torus fibers $\mathcal{M}'_u$.

Alternatively, it is shown in [7] that if

$$\mathcal{X}_\gamma^{\mathrm{sf}}(\zeta) = \exp\left(\frac{\pi R Z_\gamma}{\zeta} + i\theta_\gamma + \pi R \zeta \overline{Z_\gamma} \right)$$

Then the 2-form

$$\varpi(\zeta) = \frac{1}{8\pi^2 R} \langle d \log \mathcal{X}^{\mathrm{sf}}(\zeta) \wedge d \log \mathcal{X}^{\mathrm{sf}}(\zeta) \rangle$$

(where the DeRham operator d is applied to the $\mathcal{M}'$ part only) can be expressed as

$$-\frac{i}{2\zeta}\omega_+ + \omega_3^{\mathrm{sf}} - \frac{i\zeta}{2}\omega_-,$$

for $\omega_- = \overline{\omega_+} = \langle d\overline{Z} \wedge d\overline{\theta} \rangle$, that is, in the twistor space $\mathcal{Z} = \mathcal{M}' \times \mathbb{P}^1$ of [13], $\varpi(\zeta)$ is a holomorphic section of $\Omega_{\mathcal{Z}/\mathbb{P}^1} \otimes \mathcal{O}(2)$ (the twisting by $\mathcal{O}(2)$ is due to the poles at $\zeta = 0$ and $\zeta = \infty$ in $\mathbb{P}^1$). This is the key step in Hitchin's twistor space construction. By [7, §3], $\mathcal{M}'$ is hyperkähler.

We want to reproduce the same construction of a hyperkähler metric now with corrected Darboux coordinates $\mathcal{X}_\gamma(\zeta)$. For that, we need another piece of data. Namely, a function $\Omega : \Gamma \to \mathbb{Z}$ such that $\Omega(\gamma; u) = \Omega(-\gamma; u)$. For a component of the singular locus D_0 and for γ_0 the primitive element in Γ for which $Z_{\gamma_0} \to 0$ as $u \to u_0 \in D_0$, we also require

$$\Omega(\gamma_0; u) = 1 \text{ for all } u \text{ in a neighborhood of } D_0$$

To see where these invariants arise from, consider the example of moduli spaces of Higgs bundles again. A quadratic differential $u \in \mathcal{B}'$ determines a metric h on C. Namely, if $u = P(z)dz^2$, $h = |P(z)|dzd\bar{z}$. Let C' be the curve obtained after removing the poles and zeroes of u. Consider the finite length inextensible geodesics on C' in the metric h. These come in two types:

1. *Saddle connections*: geodesics running between two zeroes of u. See Figure 2.1.

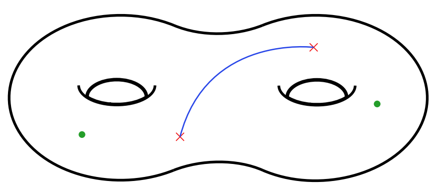

Figure 2.1: Saddle connections on C'

2. *Closed geodesics*: When they exist, they come in 1-parameter families sweeping out annuli in C'. See Figure 2.2.

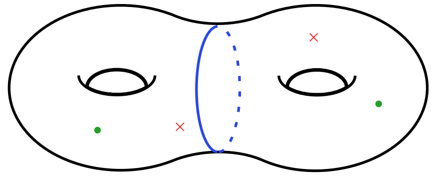

Figure 2.2: Closed geodesics on C' sweeping annuli

On the branched cover $\Sigma_u \to C$, each geodesic can be lifted to a union of closed curves in Σ_u, representing some homology class $\gamma \in H_1(\Sigma_u, \mathbb{Z})$. See Figure 2.3.

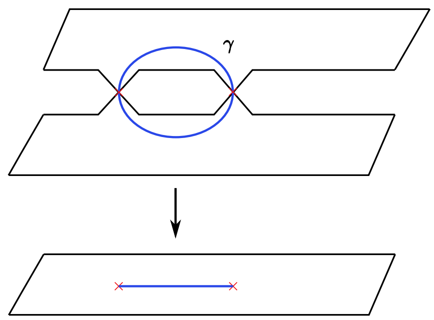

Figure 2.3: Lift of geodesics to Σ_u

In this case, $\Omega(\gamma, u)$ counts these finite length geodesics: every saddle connection with lift γ contributes $+1$ and every closed geodesic with lift γ contributes -2.

Back to the general case, we're ready to formulate a Riemann-Hilbert problem on the $\mathbb{P}^1$-slice of the twistor space $\mathcal{Z} = \mathcal{M}' \times \mathbb{P}^1$. Recall that in a RH problem we have a contour Σ dividing a complex plane (or its compactification) and one tries to obtain functions which are analytic in the regions defined by the contour, with continuous extensions along the boundary and with prescribed discontinuities along Σ and fixed asymptotics at the points where Σ is non-smooth. In our case, the contour is a collection of rays at the origin and the discontinuities can be expressed as symplectomorphisms of a complex torus:

Define a ray associated to each $\gamma \in \Gamma_u$ as:

$$\ell_\gamma(u) = Z_\gamma \mathbb{R}_-.$$

We also define a transformation of the functions $\mathcal{X}_{\gamma'}$ given by each $\gamma \in \Gamma_u$:

$$\mathcal{K}_\gamma \mathcal{X}_{\gamma'} = \mathcal{X}_{\gamma'}(1 - \mathcal{X}_\gamma)^{\langle \gamma', \gamma \rangle} \tag{2.5}$$

Let T_u denote the space of twisted complex characters of Γ_u, i.e. maps $\mathcal{X} : \Gamma_u \to \mathbb{C}^\times$ satisfying

$$\mathcal{X}_\gamma \mathcal{X}_{\gamma'} = (-1)^{\langle \gamma, \gamma' \rangle} \mathcal{X}_{\gamma + \gamma'} \tag{2.6}$$

T_u has a canonical Poisson structure given by

$$\{\mathcal{X}_\gamma, \mathcal{X}_{\gamma'}\} = \langle \gamma, \gamma' \rangle \, \mathcal{X}_{\gamma + \gamma'}$$

The T_u glue together into a bundle over $\mathcal{B}'$ with fiber a complex Poisson torus. Let T be the pullback of this system to $\mathcal{M}'$. We can interpret the transformations $\mathcal{K}_\gamma$ as birational automorphisms of T. To each ray ℓ going from 0 to ∞ in the ζ-plane, we can define a transformation

$$S_\ell = \prod_{\gamma : \ell_\gamma(u) = \ell} \mathcal{K}_\gamma^{\Omega(\gamma; u)} \tag{2.7}$$

Note that all the γ's involved in this product are multiples of each other, so the $\mathcal{K}_\gamma$ commute and it is not necessary to specify an order for the product.

To obtain the corrected $\mathcal{X}_\gamma$, we can formulate a Riemann-Hilbert problem for which the former functions are solutions to it. We seek a map $\mathcal{X} : \mathcal{M}'_u \times \mathbb{C}^\times \to T_u$ with the following properties:

1. $\mathcal{X}$ depends piecewise holomorphically on ζ, with discontinuities only at the rays $\ell_\gamma(u)$ for which $\Omega(\gamma; u) \neq 0$.

2. The limits $\mathcal{X}^\pm$ as ζ approaches any ray ℓ from both sides exist and are related by
$$\mathcal{X}^+ = S_\ell^{-1} \circ \mathcal{X}^- \tag{2.8}$$

3. $\mathcal{X}$ obeys the reality condition
$$\overline{\mathcal{X}_{-\gamma}(-1/\overline{\zeta})} = \mathcal{X}_\gamma(\zeta)$$

4. For any $\gamma \in \Gamma_u$, $\lim_{\zeta\to 0} \mathcal{X}_\gamma(\zeta)/\mathcal{X}^{\mathrm{sf}}_\gamma(\zeta)$ exists and is real.

In [7], this RH problem is formulated as an integral equation:

$$\mathcal{X}_\gamma(u,\zeta) = \mathcal{X}^{\mathrm{sf}}_\gamma(u,\zeta)\exp\left[-\frac{1}{4\pi i}\sum_{\gamma'}\Omega(\gamma';u)\,\langle\gamma,\gamma'\rangle\int_{\ell_{\gamma'(u)}}\frac{d\zeta'}{\zeta'}\frac{\zeta'+\zeta}{\zeta'-\zeta}\log\left(1-\mathcal{X}_{\gamma'}(u,\zeta')\right)\right], \tag{2.9}$$

One can define recursively, setting $\mathcal{X}^{(0)} = \mathcal{X}^{\mathrm{sf}}$:

$$\mathcal{X}^{(n+1)}_\gamma(u,\zeta) = \mathcal{X}^{\mathrm{sf}}_\gamma(u,\zeta)\exp\left[-\frac{1}{4\pi i}\sum_{\gamma'}\Omega(\gamma';u)\,\langle\gamma,\gamma'\rangle\int_{\ell_{\gamma'(u)}}\frac{d\zeta'}{\zeta'}\frac{\zeta'+\zeta}{\zeta'-\zeta}\log\left(1-\mathcal{X}^{(n)}_{\gamma'}(u,\zeta')\right)\right], \tag{2.10}$$

More precisely, we have a family of RH problems, parametrized by $u \in \mathcal{B}'$, as this defines the rays $\ell_\gamma(u)$, the complex torus T_u where the symplectomorphisms are defined and the invariants $\Omega(\gamma; u)$ involved in the definition of the problem.

We still need one more piece of the puzzle, since the latter function Ω may not be continuous. In fact, Ω jumps along a real codimension-1 loci in $\mathcal{B}'$ called the "wall of marginal stability". This is the locus where 2 or more functions Z_γ coincide in phase, so two or more rays $\ell_\gamma(u)$ become one. More precisely:

$$W = \{u \in \mathcal{B}' : \exists \gamma_1, \gamma_2 \text{ with } \Omega(\gamma_1; u) \neq 0, \Omega(\gamma_2; u) \neq 0, \langle \gamma_1, \gamma_2 \rangle \neq 0, Z_{\gamma_1}/Z_{\gamma_2} \in \mathbb{R}_+\}$$

The jumps of Ω are not arbitrary; they are governed by the Kontsevich-Soibelman wall-crossing formula.

To describe this, let V be a strictly convex cone in the ζ-plane with apex at the origin. Then for any $u \notin W$ define

$$A_V(u) = \overset{\curvearrowleft}{\prod_{\gamma : Z_\gamma(u) \in V}} \mathcal{K}_\gamma^{\Omega(\gamma; u)} = \overset{\curvearrowleft}{\prod_{\ell \subset V}} S_\ell{}^2 \tag{2.11}$$

The arrow indicates the order of the rational maps $\mathcal{K}_\gamma$. $A_V(u)$ is a birational Poisson automorphism of T_u. Define a V-*good path* to be a path p in $\mathcal{B}'$ along which there is no point u with $Z_\gamma(u) \in \partial V$ and $\Omega(\gamma; u) \neq 0$. (So

[2]This product may be infinite. One should more precisely think of $A_V(u)$ as living in a certain prounipotent completion of the group generated by $\{\mathcal{K}_\gamma\}_{\gamma : Z_\gamma(u) \in V}$ as explained in [15]

as we travel along a V-good path, no ℓ_γ rays enter or exit V.) If u, u' are the endpoints of a V-good path p, the wall-crossing formula is the condition that $A_V(u), A_V(u')$ are related by parallel transport in T along p. See Figure 2.4.

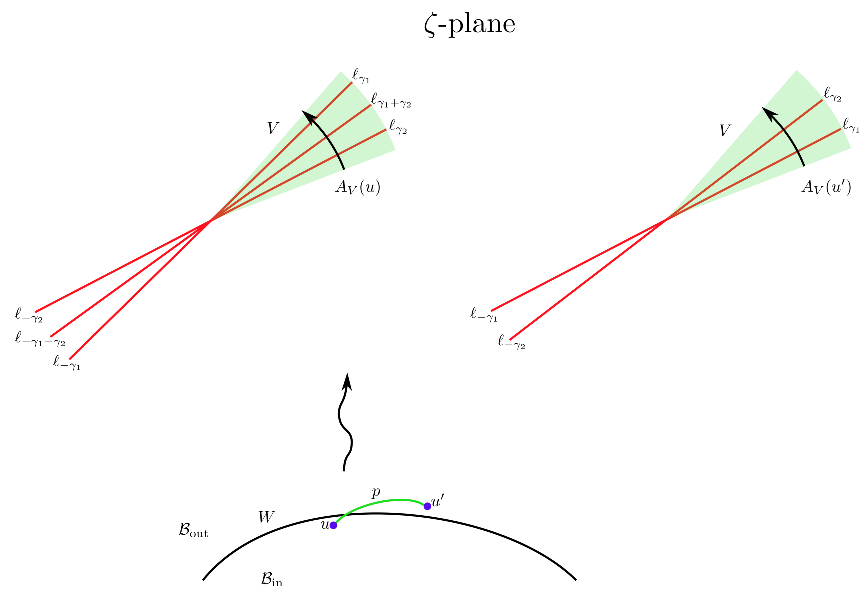

Figure 2.4: For a good path p, the two automorphisms $A_V(u), A_V(u')$ are related by parallel transport

In [7], it is proposed that the iterations in (2.10) converge to a unique solution $\mathcal{X}(u, \zeta)$ for $u \in \mathcal{B}'$. However, this was never proved. It is one of the main purposes of this paper to give a mathematical proof of this convergence, thus proving the construction of the hyperkähler metric g on $\mathcal{M}'$.

2.2 Statement of Results

In the general case, we want to extend the torus fibration $\mathcal{M}'$ to a manifold $\mathcal{M}$ with degenerate torus fibers. For example, the torus bundle $\mathcal{M}'$ is not the moduli space of Higgs bundles yet, as we have to consider quadratic differentials with non-simple zeroes too. The main results of this paper center on the extension of the manifold $\mathcal{M}'$ to a manifold $\mathcal{M}$ with an extended fibration $\mathcal{M} \to \mathcal{B}$ such that the torus fibers $\mathcal{M}'_u$ degenerate to nodal torus (i.e. "singular" or "bad" fibers) for $u \in D$.

We will see that, to give a satisfactory extension, it was necessary to develop the theory of Riemann-Hilbert-Birkhoff problems to suit these infinite-dimensional systems (as the transformations S_ℓ defining the problem can be thought as operators on $C^\infty(T_u)$, rather than matrices). It is not clear that such coordinates can be extended, since we may approach the bad fiber from two different sides of the wall of marginal stability and obtain two different extensions. To overcome this first obstacle, we have to use the theory of isomonodromic deformations as in [3] to reformulate the Riemann-Hilbert problem in [7] independent of the regions determined by the wall.

Having redefined the problem, we want our $\mathcal{X}_\gamma$ to be smooth on the parameters $\theta_{\gamma_1}, \ldots, \theta_{\gamma_{2n}}$ and u, away from where the prescribed jumps are. Even at $\mathcal{M}'$, there was no mathematical proof that such condition must be true. By combining classical Banach contraction methods and Arzela-Ascoli results on uniform convergence on compact sets, we can obtain:

Theorem 2.2.1. *For large parameter R, there exists a unique collection of functions $\mathcal{X}_\gamma$ with the prescribed asymptotics and jumps as in [7]. These functions are smooth on u and the torus coordinates $\theta_1, \ldots, \theta_{2r}$ (even for u at the* *wall of marginal stability), and piecewise holomorphic on ζ, with jumps only at two admissible rays $r, -r$.*

After this is proved, we focus on the case $n = 1$, so Γ is a rank-1 lattice over the Riemann surface $\mathcal{B}'$ and the discriminant locus D where the torus fibers degenerate is a discrete subset of $\mathcal{B}'$. Once the $\{\mathcal{X}_{\gamma_i}\}$ are obtained, and if the invariants $\Omega(\gamma_1; u), \Omega(\gamma_2; u)$ have the correct values, it is necessary to do an analytic continuation along $\mathcal{B}'$ for the particular $\mathcal{X}_{\gamma_i}$ for which $Z_{\gamma_i} \to 0$ as $u \to u_0 \in D$. Without loss of generality, we can assume there is a local basis $\{\gamma_1, \gamma_2\}$ of Γ such that $Z_{\gamma_2} \to 0$ in D. After that, an analysis of the possible divergence of $\mathcal{X}_\gamma$ as $u \to u_0$ shows the necessity of performing a gauge transformation on the torus coordinates of the fibers $\mathcal{M}_u$ that allows us to define an integral equation even at $u_0 \in D$. This transformation is a new result and was not expected in [7]. We will see that in order to even define an integral equation at D, it is necessary that $\mathcal{X}_{\gamma_2}$ stays bounded away from 1. This can be achieved if θ_2 is bounded away from 0 and R is big enough. As in the case of normal fibers, we can run a contraction argument to obtain Darboux coordinates even at the singular fibers and conclude:

Theorem 2.2.2. *Let $\{\gamma_1, \gamma_2\}$ be a local basis for Γ in a small sector centered at $u_0 \in D$ such that $Z_{\gamma_2} \to 0$ as $u \to u_0 \in D$. For a particular value of $\Omega(\gamma_1; u), \Omega(\gamma_2; u)$, the local function $\mathcal{X}_{\gamma_1}$ admits an analytic continuation $\widetilde{\mathcal{X}}_{\gamma_1}$*

to a punctured disk centered at u_0 *in* $\mathcal{B}$*. There exists a gauge transformation* $\theta_1 \mapsto \widetilde{\theta}_1$ *that extends the torus fibration* $\mathcal{M}'$ *to a manifold* $\mathcal{M}$ *that is a (trivial) fibration over an open set of* $\mathcal{B} \times S^1$ *with a small neighborhood of* $(u_0, \theta_2 = 0)$ *removed and with fiber* S^1 *coordinatized by* θ_1*. For* $R > 0$ *big enough, it is possible to extend* $\widetilde{\mathcal{X}}_{\gamma_1}$ *and* $\mathcal{X}_{\gamma_2}$ *to* $\mathcal{M}$*, still preserving the smooth properties as in Theorem 2.2.1.*

Unfortunately, at this point there's no guarantee that we can choose R uniformly giving an extension of these functions to an even bigger fibration $\mathcal{M}$ where only the points $(u_0, \theta_2 = 0)$ are removed (the smaller θ_2, the bigger R must be). We will present a heuristic argument based on numerical evidence for a special integrable system to be defined at the end of this section, extending this construction to the entire degenerate torus.

Once we have the smooth extension of the $\{\mathcal{X}_{\gamma_i}\}$, we can extend the holomorphic symplectic form $\varpi(\zeta)$ labeled by $\zeta \in \mathbb{P}^1$ as in [13] for all points except possibly one at the singular fiber. From $\varpi(\zeta)$ we can obtain the hyperkähler metric g and, after a change of coordinates, we realize g locally as the Taub-NUT metric plus smooth corrections, finishing the construction of $\mathcal{M}$ and its hyperkähler metric. The following is the main theorem of the paper.

Theorem 2.2.3. $\mathcal{M}'$ *admits an extension to a manifold* $\mathcal{M}$ *that is a (trivial) fibration over an open set of* $\mathcal{B} \times S^1$ *with a small neighborhood of* $(u_0, \theta_2 = 0)$

removed and with fiber S^1 coordinatized by θ_1. For R large enough and for specific values of the integers $\Omega(\gamma; u)$, $\mathcal{M}$ admits a hyperkähler metric g obtained by extending the hyperkähler metric on $\mathcal{M}'$ determined by the Darboux coordinates $\{\mathcal{X}_{\gamma_i}\}$.

The construction of the coordinates $\mathcal{X}_\gamma$ are valid for systems of arbitrary rank n, and the extensions obtained work for any general system defined in [17] of rank $n = 1$. We start by fully working out the simplest example known as Ooguri-Vafa [4]. Here we have a fibration over the open unit disk $\mathcal{B} := \{u \in \mathbb{C} : |u| < 1\}$. At the discriminant locus $D := \{u = 0\}$, the fibers degenerate into a nodal torus. The local rank-2 lattice Γ has a basis (γ_m, γ_e) and the skew-symmetric pairing is defined by $\langle \gamma_m, \gamma_e \rangle = 1$. The monodromy of Γ around $u = 0$ is $\gamma_e \mapsto \gamma_e, \gamma_m \mapsto \gamma_m + \gamma_e$. We also have functions $Z_{\gamma_e}(u) = u, Z_{\gamma_m}(u) = \frac{u}{2\pi i}(\log u - 1) + f(u)$, for f holomorphic and admitting an extension to $\mathcal{B}$. Finally, the integer-valued function Ω in Γ is here: $\Omega(\pm\gamma_e; u) = 1$ and $\Omega(\gamma; u) = 0$ for any other $\gamma \in \Gamma_u$. There is no wall of marginal stability in this case. The integral equation (2.9) can be solved after just 1 iteration.

The next nontrivial system fully worked out is the Pentagon case [17]. Here $\mathcal{B} = \mathbb{C}$ with 2 bad fibers which we can assume are at $u = -2, u = 2$ and $\mathcal{B}'$ is the twice-punctured plane. There is a wall of marginal stability where all Z_γ are contained in the same line. This separates $\mathcal{B}$ in two domains $\mathcal{B}_{\text{out}}$ and a simply-connected $\mathcal{B}_{\text{in}}$. See Figure 2.5.

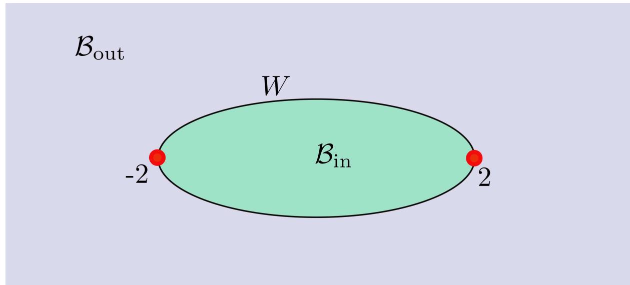

Figure 2.5: The wall W in $\mathcal{B}$ for the Pentagon case

On $\mathcal{B}_{\text{in}}$ we can trivialize Γ and choose a basis $\{\gamma_1, \gamma_2\}$ with pairing $\langle\gamma_1, \gamma_2\rangle = 1$. This basis does not extend to a global basis for Γ since it is not invariant under monodromy. However, the set $\{\gamma_1, \gamma_2, -\gamma_1, -\gamma_2, \gamma_1+\gamma_2, -\gamma_1-\gamma_2\}$ is indeed invariant so the following definition of Ω makes global sense:

$$\text{For } u \in \mathcal{B}_{\text{in}}, \Omega(\gamma; u) = \begin{cases} 1 & \text{for } \gamma \in \{\gamma_1, \gamma_2, -\gamma_1, -\gamma_2\} \\ 0 & \text{otherwise} \end{cases}$$

$$\text{For } u \in \mathcal{B}_{\text{out}}, \Omega(\gamma; u) = \begin{cases} 1 & \text{for } \gamma \in \{\gamma_1, \gamma_2, -\gamma_1, -\gamma_2, \gamma_1+\gamma_2, -\gamma_1-\gamma_2\} \\ 0 & \text{otherwise} \end{cases}$$

In the Appendix we will present evidence in support of the following

Conjecture 2.2.4. *In the Pentagon case, there exists a manifold $\mathcal{M}$ extending the torus fibration $\mathcal{M}'$ such that, locally over each $u_0 \in D$, $\mathcal{M}$ is a trivial torus fibration with a degenerate torus fiber at u_0. For a fixed $R > 0$ big enough, it is possible to define a suitable integral equation in $\mathcal{M}$ whose solutions $\{\mathcal{X}_{\gamma_1}, \mathcal{X}_{\gamma_2}\}$ (after analytically extending the first one around u_0) represent the extension of the Darboux coordinates of [7] to u_0.*

The Pentagon case appears in the study of Hitchin systems with gauge group SU(2). The extension of $\mathcal{M}'$ was previously obtained by hyperkähler quotient methods in [2], but no explicit hyperkähler metric was constructed. For this case it is necessary to redefine the Riemann-Hilbert problems so that the wall W is not an issue for continuity. The techniques used for this extension extends to general integrable systems with possibly infinitely many nonzero coefficients $\Omega(\gamma; u)$.

In the last chapter, we present a heuristic argument supported on numerical evidence as to how extend the Darboux Coordinates uniformly on θ_e, as $\theta_e \to 0$ in the particular case of the Pentagon. The only difficulty is to choose an appropriate branch of the log in order to define the integral equations.

Chapter 3

The Ooguri-Vafa Case

3.1 Classical Case

We start with one of the simplest cases, known as the Ooguri-Vafa case, first treated in [4]. To see where this case comes from, recall that by the SYZ picture of K3 surfaces [10], any K3 surface $\mathcal{M}$ is a hyperkähler manifold. In one of its complex structures (say $J^{(\zeta=0)}$) is elliptically fibered, with base manifold $\mathcal{B} = \mathbb{P}^1$ and generic fiber a compact complex torus. There are a total of 24 singular fibers, although the total space is smooth. See Figure 3.1.

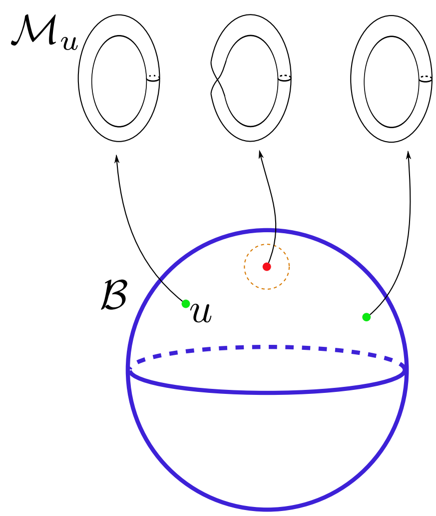

Figure 3.1: A K3 surface $\mathcal{M}$ as an elliptic fibration

Gross and Wilson [11] constructed a hyperkähler metric g on a K3 surface by gluing in the Ooguri-Vafa metric constructed in [18] with a standard metric g^{sf} away from the degenerate fiber. Thus, this simple case can be regarded as a local model for K3 surfaces.

We have a fibration over the open unit disk $\mathcal{B} := \{a \in \mathbb{C} : |a| < 1\}$. At the locus $D := \{a = 0\}$ (in the literature this is also called the *discriminant locus*), the fibers degenerate into a nodal torus. Define $\mathcal{B}'$ as $\mathcal{B}\backslash D$, the punctured unit disk. On $\mathcal{B}'$ there exists a local system Γ of rank-2 lattices with basis (γ_m, γ_e) and skew-symmetric pairing defined by $\langle \gamma_m, \gamma_e \rangle = 1$. The monodromy of Γ around $a = 0$ is $\gamma_e \mapsto \gamma_e, \gamma_m \mapsto \gamma_m + \gamma_e$. We also have functions $Z_{\gamma_e}(a) = a, Z_{\gamma_m}(a) = \frac{a}{2\pi i}(\log a - 1)$. On $\mathcal{B}'$ we have local coordinates (θ_m, θ_e) for the torus fibers with monodromy $\theta_e \mapsto \theta_e, \theta_m \mapsto \theta_m + \theta_e - \pi$. Finally, the integer-valued function Ω in Γ is here: $\Omega(\pm\gamma_e, a) = 1$ and $\Omega(\gamma, a) = 0$ for any other $\gamma \in \Gamma_a$. There is no wall of marginal stability in this case.

We call this the "classical Ooguri-Vafa" case as it is the one appearing in [18] already mentioned at the beginning of this chapter. In the next section, we'll generalize this case by adding a function $f(a)$ to the definition of Z_{γ_m}.

Let

$$\mathcal{X}^{\mathrm{sf}}_\gamma(\zeta, a) := \exp\left(\pi R \zeta^{-1} Z_\gamma(a) + i\theta_\gamma + \pi R \zeta \overline{Z_\gamma(a)}\right) \tag{3.1}$$

These functions receive corrections defined as in [7]. We are only interested in the pair $(\mathcal{X}_m, \mathcal{X}_e)$ which will constitute our desired Darboux coordinates for the holomorphic symplectic form ϖ. The fact that $\Omega(\gamma_m, a) = 0$ gives that

$\mathcal{X}_e = \mathcal{X}_e^{\text{sf}}$. As $a \to 0$, Z_{γ_e} and Z_{γ_m} approach 0. Thus $\mathcal{X}_e|_{a=0} = e^{i\theta_e}$. Since $\mathcal{X}_e = \mathcal{X}_e^{\text{sf}}$ the actual $\mathcal{X}_m$ is obtained after only 1 iteration of (2.10). For each $a \in \mathcal{B}'$, let ℓ_+ be the ray in the ζ-plane defined by $\{\zeta : a/\zeta \in \mathbb{R}_-\}$. Similarly, $\ell_- := \{\zeta : a/\zeta \in \mathbb{R}_+\}$.

Let

$$\mathcal{X}_m = \mathcal{X}_m^{\text{sf}} \exp\left[\frac{i}{4\pi}\int_{\ell_+}\frac{d\zeta'}{\zeta'}\frac{\zeta'+\zeta}{\zeta'-\zeta}\log[1-\mathcal{X}_e(\zeta')] - \frac{i}{4\pi}\int_{\ell_-}\frac{d\zeta'}{\zeta'}\frac{\zeta'+\zeta}{\zeta'-\zeta}\log[1-\mathcal{X}_e(\zeta')^{-1}]\right]. \tag{3.2}$$

For convenience, from this point on we assume a is of the form sb, where s is a positive number, b is fixed and $|b| = 1$. Moreover, in ℓ_+, $\zeta' = -tb$, for $t \in (0, \infty)$, and a similar parametrization holds in ℓ_-.

Lemma 3.1.1. *For fixed b, $\mathcal{X}_m$ as in* (3.2) *has a limit as $|a| \to 0$.*

Proof. Writing $\dfrac{\zeta'+\zeta}{\zeta'(\zeta'-\zeta)} = \dfrac{-1}{\zeta'} + \dfrac{2}{\zeta'-\zeta}$, we want to find the limit as $a \to 0$ of

$$\begin{aligned}&\int_{\ell_+}\left\{\frac{-1}{\zeta'}+\frac{2}{\zeta'-\zeta}\right\}\log[1-\exp(\pi Ra/\zeta'+i\theta_e+\pi R\zeta'\bar{a})]d\zeta'\\ &-\int_{\ell_-}\left\{\frac{-1}{\zeta'}+\frac{2}{\zeta'-\zeta}\right\}\log[1-\exp(-\pi Ra/\zeta'-i\theta_e-\pi R\zeta'\bar{a})]d\zeta'.\end{aligned} \tag{3.3}$$

For simplicity, we'll focus in the first integral only, the second one can be handled similarly. Rewrite:

$$\begin{aligned}&\int_{\ell_+}\left\{\frac{-1}{\zeta'}+\frac{2}{\zeta'-\zeta}\right\}\log[1-\exp(\pi Ra/\zeta'+i\theta_e+\pi R\zeta'\bar{a})]d\zeta'\\ &=\int_0^{-b}\left\{\frac{-1}{\zeta'}+\frac{2}{\zeta'-\zeta}\right\}\log[1-\exp(\pi Ra/\zeta'+i\theta_e+\pi R\zeta'\bar{a})]d\zeta'\end{aligned}$$

$$
\begin{aligned}
&+\int_{-b}^{-b\infty}\left\{\frac{-1}{\zeta'}+\frac{2}{\zeta'-\zeta}\right\}\log[1-\exp(\pi Ra/\zeta'+i\theta_e+\pi R\zeta'\bar{a})]d\zeta' \\
&=\int_{0}^{-b}\left\{\frac{-1}{\zeta'}+\frac{2}{\zeta'-\zeta}\right\}\log[1-\exp(\pi Ra/\zeta'+i\theta_e+\pi R\zeta'\bar{a})]d\zeta' \\
&+\int_{-b}^{-b\infty}\left\{\frac{-1}{\zeta'}+\frac{2}{\zeta'}+\frac{2}{\zeta'-\zeta}-\frac{2}{\zeta'}\right\}\log[1-\exp(\pi Ra/\zeta'+i\theta_e+\pi R\zeta'\bar{a})]d\zeta' \\
&=\int_{0}^{-b}\frac{-1}{\zeta'}\log[1-\exp(\pi Ra/\zeta'+i\theta_e+\pi R\zeta'\bar{a})]d\zeta' \\
&+\int_{-b}^{-b\infty}\frac{1}{\zeta'}\log[1-\exp(\pi Ra/\zeta'+i\theta_e+\pi R\zeta'\bar{a})]d\zeta' \\
&+\int_{0}^{-b}\frac{2}{\zeta'-\zeta}\log[1-\exp(\pi Ra/\zeta'+i\theta_e+\pi R\zeta'\bar{a})]d\zeta' \\
&+\int_{-b}^{-b\infty}\left\{\frac{2}{\zeta'-\zeta}-\frac{2}{\zeta'}\right\}\log[1-\exp(\pi Ra/\zeta'+i\theta_e+\pi R\zeta'\bar{a})]d\zeta' \qquad (3.4)
\end{aligned}
$$

Observe that

$$
\begin{aligned}
&\int_{0}^{-b}\frac{-1}{\zeta'}\log[1-\exp(\pi Ra/\zeta'+i\theta_e+\pi R\zeta'\bar{a})]d\zeta' \\
&=-\int_{0}^{1}\frac{1}{t}\log[1-\exp(-\pi Rs(t+1/t))]dt
\end{aligned}
$$

and after a change of variables $\tilde{t}=1/t$, we get

$$
\begin{aligned}
&=-\int_{1}^{\infty}\frac{1}{\tilde{t}}\log[1-\exp(-\pi Rs(\tilde{t}+1/\tilde{t}))]d\tilde{t} \\
&=-\int_{-b}^{-b\infty}\frac{1}{\zeta'}\log[1-\exp(\pi Ra/\zeta'+i\theta_e+\pi R\zeta'\bar{a})]d\zeta'.
\end{aligned}
$$

Thus, (3.4) reduces to

$$
\begin{aligned}
&\int_{0}^{-b}\frac{2}{\zeta'-\zeta}\log[1-\exp(\pi Ra/\zeta'+i\theta_e+\pi R\zeta'\bar{a})]d\zeta' \\
&+\int_{-b}^{-b\infty}\left\{\frac{2}{\zeta'-\zeta}-\frac{2}{\zeta'}\right\}\log[1-\exp(\pi Ra/\zeta'+i\theta_e+\pi R\zeta'\bar{a})]d\zeta'. \qquad (3.5)
\end{aligned}
$$

If $\theta_e = 0$, (3.3) diverges to $-\infty$, in which case $\mathcal{X}_m = 0$. Otherwise, $\log[1 - \exp(\pi Ra/\zeta' + i\theta_e + \pi R\zeta'\bar{a})]$ is bounded away from 0. Consequently, $|\log[1 - \exp(\pi Ra/\zeta' + i\theta_e + \pi R\zeta'\bar{a})]| < C < \infty$ in ℓ_+. As $a \to 0$, the integrals are dominated by

$$\int_0^{-b} \frac{2C}{|\zeta' - \zeta|}|d\zeta'| + \int_{-b}^{-b\infty} \frac{C|\zeta/b|}{|\zeta'(\zeta' - \zeta)|}|d\zeta'| < \infty$$

if $\theta_e \neq 0$. Hence we can interchange the limit and the integral in (3.5) and obtain that, as $a \to 0$, this reduces to

$$\begin{aligned} &2\log(1 - e^{i\theta_e}) \left[\int_0^{-b} \frac{d\zeta'}{\zeta' - \zeta} + \int_{-b}^{-b\infty} d\zeta' \left\{\frac{1}{\zeta' - \zeta} - \frac{1}{\zeta'}\right\}\right] \\ &= 2\log(1 - e^{i\theta_e})[F(-b) + G(-b)], \end{aligned} \tag{3.6}$$

where

$$F(z) := \log\left(1 - \frac{z}{\zeta}\right), G(z) := \log\left(1 - \frac{\zeta}{z}\right)$$

are the (unique) holomorphic solutions in the simply connected domain $U := \mathbb{C} - \{z : z/\zeta \in \mathbb{R}_+\}$ to the ODEs

$$F'(z) = \frac{1}{z - \zeta}, F(0) = 0 \qquad G'(z) = \frac{1}{z - \zeta} - \frac{1}{z}, \lim_{z\to\infty} G(z) = 0.$$

This forces us to rewrite (3.6) uniquely as

$$2\log(1 - e^{i\theta_e}) \left[\log\left(1 + \frac{b}{\zeta}\right) - \log\left(1 + \frac{\zeta}{b}\right)\right] \tag{3.7}$$

Here log denotes the principal branch of the log in both cases, and the equation makes sense for $\{b \in \mathbb{C} : b \notin \ell_+\}$ (recall that by construction, we have the additional datum $|b| = 1$). We want to conclude that

$$\log(1 + b/\zeta) - \log(1 + \zeta/b) = \log(b/\zeta), \tag{3.8}$$

still using the principal branch of the log. To see this, define $H(z)$ as $F(z) - G(z) - \log(-z/\zeta)$. This is an analytic function on U and clearly $H'(z) \equiv 0$. Thus H is constant in U. It is easy to show that the identity holds for a suitable choice of z (for example, if ζ is not real, choose $z = 1$) and by the above, it holds on all of U; in particular, for $z = -b$.

All the arguments so far can be repeated to the ray ℓ_- to get the final form of (3.3):

$$2\left\{\log\left[\frac{b}{\zeta}\right]\log(1 - e^{i\theta_e}) - \log\left[\frac{-b}{\zeta}\right]\log(1 - e^{-i\theta_e})\right\}, \quad \theta_e \neq 0. \tag{3.9}$$

This yields that (3.2) simplifies to:

$$\begin{aligned}\mathcal{X}_m &= \mathcal{X}_m^{\text{sf}} \exp\left(\frac{i}{2\pi}\left\{\log\left[\frac{b}{\zeta}\right]\log(1 - e^{i\theta_e}) - \log\left[\frac{-b}{\zeta}\right]\log(1 - e^{-i\theta_e})\right\}\right) \\ &= \mathcal{X}_m^{\text{sf}} \exp\left(\frac{i}{2\pi}\left\{\log\left[\frac{a}{|a|\zeta}\right]\log(1 - e^{i\theta_e}) - \log\left[\frac{-a}{|a|\zeta}\right]\log(1 - e^{-i\theta_e})\right\}\right)\end{aligned} \tag{3.10}$$

in the limiting case $a \to 0$. $\square$

To obtain a function that is continuous everywhere and independent of $\arg a$, define regions I, II and III in the a-plane as follows: $\mathcal{X}_m^{\text{sf}}$ has a fixed cut in the negative real axis, both in the ζ-plane and the a-plane. Assuming for the moment that $\arg\zeta \in (0, \pi)$, define region I as the half plane $\{a \in \mathbb{C} : \operatorname{Im}(a/\zeta) < 0\}$. Region II is that enclosed by the ℓ_- ray and the cut in the negative real axis, and region III is the remaining domain so that as we travel counterclockwise we traverse regions I, II and III in this order (see Figure 3.2).

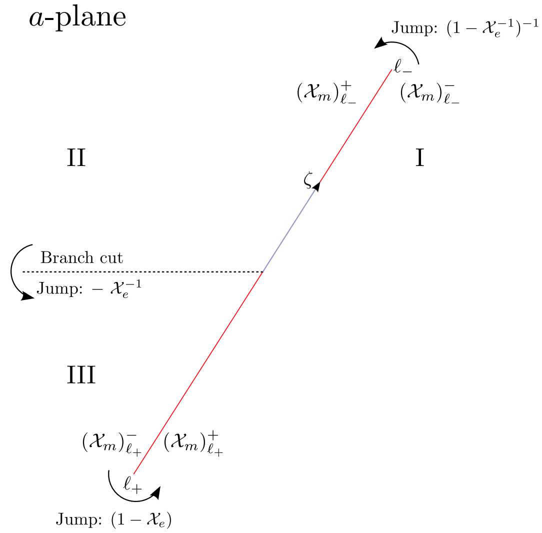

Figure 3.2: The three regions in the a-plane, as we traverse them counterclockwise

For $a \neq 0$, Gaiotto, Moore and Neitzke [7] proved that $\mathcal{X}_m$ has a continuous extension to the punctured disk of the form:

$$\widetilde{\mathcal{X}_m} = \begin{cases} \mathcal{X}_m & \text{in region I} \\ (1-\mathcal{X}_e^{-1})\mathcal{X}_m & \text{in region II} \\ -\mathcal{X}_e(1-\mathcal{X}_e^{-1})\mathcal{X}_m = (1-\mathcal{X}_e)\mathcal{X}_m & \text{in region III} \end{cases} \tag{3.11}$$

Theorem 3.1.2. *$\mathcal{X}_m$ can be extended to $a = 0$, independent of* $\arg a$*.*

Proof. We'll use the following identities:

$$\log(1-e^{i\theta_e}) = \log(1-e^{-i\theta_e}) + i(\theta_e - \pi), \quad \text{for } \theta_e \in (0, 2\pi) \tag{3.12}$$

$$\log\left[\frac{-a}{|a|\zeta}\right] = \begin{cases} \log\left[\frac{a}{|a|\zeta}\right] + i\pi & \text{in region I} \\ \log\left[\frac{a}{|a|\zeta}\right] - i\pi & \text{in regions II and III} \end{cases} \tag{3.13}$$

$$\log[a/\zeta] = \begin{cases} \log a - \log\zeta & \text{in regions I and II} \\ \log a - \log\zeta + 2\pi i & \text{in region III} \end{cases} \tag{3.14}$$

to obtain a formula for $\widetilde{\mathcal{X}_m}$ at $a = 0$ independent of the region. Formula (3.14) can be proved with an argument analogous to that used for the proof of (3.8). Starting with region I, by (3.10), (3.11), (3.12) and (3.13):

$$\widetilde{\mathcal{X}_m} = \exp\left[i\theta_m - \frac{1}{2\pi}(\theta_e - \pi)\log\left[\frac{a}{|a|\zeta}\right] + \frac{1}{2}\log\left(1 - e^{-i\theta_e}\right)\right] \quad \text{in region I.}$$

By (3.14),

$$= \exp\left[i\theta_m - \frac{1}{2\pi}(\theta_e - \pi)\log\left[\frac{a}{|a|}\right] + \frac{\theta_e - \pi}{2\pi}\log\zeta + \frac{1}{2}\log\left(1 - e^{-i\theta_e}\right)\right]$$

In region II, by our formulas above, we get

$$\begin{aligned} \widetilde{\mathcal{X}_m} &= \exp\left[i\theta_m - \frac{1}{2\pi}(\theta_e - \pi)\log\left[\frac{a}{|a|\zeta}\right] - \frac{1}{2}\log\left(1 - e^{-i\theta_e}\right)\right]\left(1 - e^{-i\theta_e}\right) \\ &= \exp\left[i\theta_m - \frac{1}{2\pi}(\theta_e - \pi)\log\left[\frac{a}{|a|\zeta}\right] - \frac{1}{2}\log\left(1 - e^{-i\theta_e}\right) + \log\left(1 - e^{-i\theta_e}\right)\right] \\ &= \exp\left[i\theta_m - \frac{1}{2\pi}(\theta_e - \pi)\log\left[\frac{a}{|a|}\right] + \frac{\theta_e - \pi}{2\pi}\log\zeta + \frac{1}{2}\log\left(1 - e^{-i\theta_e}\right)\right] \text{ in region II.} \end{aligned}$$

Finally, in region III, and making use of (3.12), (3.13), (3.14):

$$\begin{aligned} \widetilde{\mathcal{X}_m} &= \exp\left[i\theta_m - \frac{1}{2\pi}(\theta_e - \pi)\log\left[\frac{a}{|a|\zeta}\right] - \frac{1}{2}\log\left(1 - e^{-i\theta_e}\right)\right]\left(1 - e^{i\theta_e}\right) \\ &= \exp\left[i\theta_m - \frac{1}{2\pi}(\theta_e - \pi)\log\left[\frac{a}{|a|}\right] + \frac{\theta_e - \pi}{2\pi}\log\zeta - i(\theta_e - \pi)\right. \\ &\qquad \left. -\frac{1}{2}\log\left(1 - e^{-i\theta_e}\right) + \log\left(1 - e^{-i\theta_e}\right) + i(\theta_e - \pi)\right] \end{aligned}$$

$$= \exp\left[i\theta_m - \frac{1}{2\pi}(\theta_e - \pi)\log\left[\frac{a}{|a|}\right] + \frac{\theta_e - \pi}{2\pi}\log\zeta + \frac{1}{2}\log\left(1 - e^{-i\theta_e}\right)\right]. \tag{3.15}$$

Observe that, throughout all these calculations, we only had to use the natural branch of the complex logarithm. In summary, (3.15) works for any region in the a-plane, with a cut in the negative real axis.

This also suggest the following gauge transformation

$$\theta_m' = \theta_m + \frac{i(\theta_e - \pi)}{4\pi}\left(\log\frac{a}{\Lambda} - \log\frac{\bar{a}}{\bar{\Lambda}}\right) \tag{3.16}$$

Here Λ is the same cutoff constant as in [7]. Let φ parametrize the phase of $a/|a|$. Then (3.16) simplifies to

$$\theta_m' = \theta_m - \frac{(\theta_e - \pi)\varphi}{2\pi} \tag{3.17}$$

On a coordinate patch around the singular fiber, θ_m' is single-valued. If we regard $\mathcal{M}'$ as a S^1-bundle over $\mathcal{B}' \times S^1$, with the fiber parametrized by θ_m, then the above shows that we can glue to $\mathcal{M}'$ another S^1-bundle over $D \times (0, 2\pi)$, for D a small open disk around $a = 0$, and $\theta_e \in (0, 2\pi)$. The S^1-fiber is parametrized by θ_m' and the transition function is given by (3.17). In this patch, we can extend $\widetilde{\mathcal{X}_m}$ to $a = 0$ as:

$$\left.\widetilde{\mathcal{X}_m}\right|_{a=0} = e^{i\theta_m'}\zeta^{\frac{\theta_e - \pi}{2\pi}}(1 - e^{-i\theta_e})^{\frac{1}{2}} \tag{3.18}$$

where the branch of $\zeta^{\frac{\theta_e-\pi}{2\pi}}$ is determined by the natural branch of the logarithm in the ζ plane.

Now consider the case that $\arg\zeta \in (-\pi, 0)$. Label the regions as one travels counterclockwise, starting with the region bounded by the cut and the ℓ_- (See Figure 3.3). We can do an analytic continuation similar to (3.11) starting in region I, but formulas (3.13), (3.14) become now:

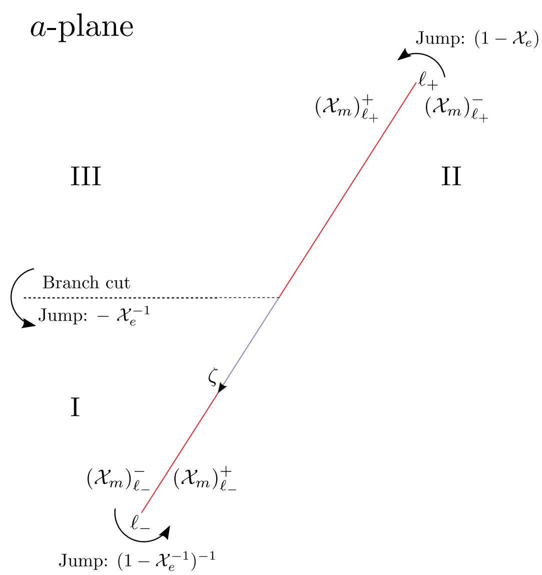

Figure 3.3: The three regions in the case $\arg\zeta < 0$.

$$\log\left[\frac{-a}{|a|\zeta}\right] = \begin{cases} \log\left[\frac{a}{|a|\zeta}\right] - i\pi & \text{in region II} \\ \log\left[\frac{a}{|a|\zeta}\right] + i\pi & \text{in regions I and III} \end{cases}$$

$$\log[a/\zeta] = \begin{cases} \log a - \log\zeta & \text{in regions I and II} \\ \log a - \log\zeta - 2\pi i & \text{in region III} \end{cases}$$

By an argument entirely analogous to the case $\arg\zeta > 0$, we get again:

$$\widetilde{\mathcal{X}_m}\Big|_{a=0} = e^{i\theta'_m}\zeta^{\frac{\theta_e-\pi}{2\pi}}(1-e^{-i\theta_e})^{\frac{1}{2}}$$

The case ζ real and positive is even simpler, as Figure 3.4 shows. Here we have only two regions, and the jumps at the cut and the ℓ_+ ray are combined, since these two lines are the same. Label the lower half-plane as region I and the upper half-plane as region II. Start an analytic continuation of $\mathcal{X}_m$ in region I as before, using the formulas:

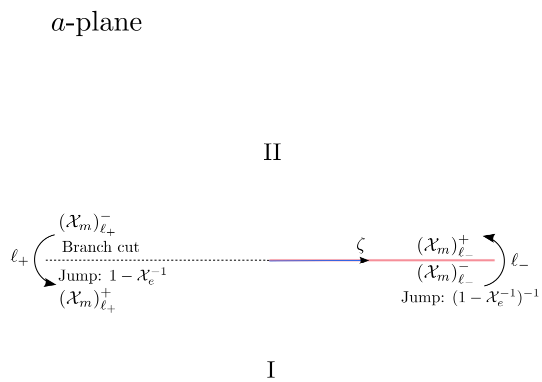

Figure 3.4: Only two regions in the case $\arg\zeta = 0$.

$$\log\left[\frac{-a}{|a|\zeta}\right] = \begin{cases} \log\left[\frac{a}{|a|\zeta}\right] - i\pi & \text{in region II} \\ \log\left[\frac{a}{|a|\zeta}\right] + i\pi & \text{in region I} \end{cases}$$
$$\log[a/\zeta] = \log a - \log\zeta \quad \text{in both regions}$$

The result is equation (3.18) again. The case $\arg\zeta = \pi$ is entirely analogous to this and it yields the same formula, thus proving that (3.18) holds for all ζ and is independent of a. $\square$

3.2 Generalized Ooguri-Vafa coordinates

We can generalize the previous extension to the case $Z_{\gamma_m} := \frac{1}{2\pi i} a \log a + f(a)$, where $f : \mathcal{B}' \to \mathbb{C}$ is holomorphic and admits a holomorphic extension into $\mathcal{B}$. In particular,

$$\mathcal{X}_m^{\mathrm{sf}} = \exp\left(\frac{-iR}{2\zeta} a \log a + \frac{\pi R f(a)}{\zeta} + i\theta_m + \frac{i\zeta R}{2} \overline{a} \log \overline{a} + \pi R \zeta \overline{f(a)}\right) \qquad (3.19)$$

The value at the singular locus $f(0)$ does not have to be 0. All the other data remains the same.

The first thing we observe is that $\mathcal{X}_e$ remains the same. Consequently, the corrections for the generalized $\mathcal{X}_m$ are as before. Using the change of coordinates as in (3.17), we can thus write

$$\widetilde{\mathcal{X}_m}\Big|_{a=0} = \exp\left[\frac{\pi R f(0)}{\zeta} + i\theta'_m + \pi R \zeta f(0)\right] \zeta^{\frac{\theta_e - \pi}{2\pi}} (1 - e^{-i\theta_e})^{\frac{1}{2}} \qquad (3.20)$$

3.3 Turning points

In this section we interpret equations (3.18) and (3.20) as a flat section of a meromorphic connection on the ζ-plane. Take a small open disc with a cut along a fixed ray D centered at 0 in the a-plane such that Γ is a trivial rank-2 lattice over this open submanifold. This gives a trivial complexified torus fibration $\widetilde{T} \to D$ with fibers $\widetilde{T}_a$, the twisted complex characters of Γ_a. Any $\gamma \in \Gamma_a$ defines a canonical $\mathbb{C}^{\times}$-valued function X_γ on $\widetilde{T}_a$ with the property

$$X_\gamma X_{\gamma'} = (-1)^{\langle \gamma, \gamma' \rangle} X_{\gamma + \gamma'}$$

There is a manifold $\mathcal{M}$ and a map $\pi : \mathcal{M} \to D$ which is a torus fibration except at $a = 0$, where the preimage becomes a nodal torus ([7]). Let T denote the pullback $\pi^* \widetilde{T}$. Since Γ is trivial, all fibers T_{a,θ_e,θ_m} of T are isomorphic to a single complexified torus $T_0 \cong (\mathbb{C}^\times)^2$ with Poisson structure specified by

$$\{X_\gamma, X_{\gamma'}\} = \langle \gamma, \gamma' \rangle X_{\gamma+\gamma'}$$

This Poisson structure is, in fact, a symplectic structure ω on T_0. This structure is translation invariant ([15]). Let $G := \mathrm{Ham}(T_0, \omega)$ denote the group of hamiltonian symplectomorphisms of T_0. Recall that $g \in \mathrm{Symp}(T_0, \omega)$ is called hamiltonian if there is a hamiltonian isotopy $g_t \in \mathrm{Symp}(T_0, \omega)$ from $g_0 = \mathrm{id}$ to $g_1 = g$. Let $\mathfrak{g}$ denote the Lie algebra[1] of G consisting of hamiltonian vector fields on T_0 and let $\mathfrak{t}$ be the Lie algebra of T_0. Denote by $\mathscr{H}$ the Lie algebra of regular functions on T_0. Then one has the decomposition ([15], [9]):

$$\begin{aligned} \mathfrak{g} &= \mathfrak{t} \oplus \mathscr{H} \\ &= \mathfrak{t} \oplus \bigoplus_{\gamma \in \Gamma} \mathfrak{g}_\gamma \\ &= \mathfrak{t} \oplus [\mathfrak{t}, \mathfrak{g}], \end{aligned} \tag{3.21}$$

where $\mathfrak{g}_\gamma$ is a 1-dimensional space generated by the function X_γ.

We can now interpret our functions $\mathcal{X}_m, \mathcal{X}_e$ (and, in general, the basis $\{\mathcal{X}_{\gamma_1}, \ldots, \mathcal{X}_{\gamma_{2r}}\}$ in [7]) as a global function

$$\mathcal{X} : \mathcal{M} \times \mathbb{P}^1 \to T$$

[1]By definition, the Lie algebra consists of vector fields of the form $\left.\frac{d}{dt}\right|_{t=0} g(t)$, where $t \mapsto g(t), t \in (-1, 1)$ is a smooth mapping into G with $g(0) = \mathrm{id}$, see [14, §20]

Fixing the parameter $u \in \mathcal{M}$ and identifying the fibers of T, we can write $\mathcal{X}(u) : \mathbb{P}^1 \to G$ (by abuse of notation, an element in T_0 and the symplectomorphism it induces by translation are considered the same). Let P be the trivial principal bundle over $\mathbb{P}^1$ with fiber G. Then $\mathcal{X}(u)$ is a flat section for a family of connections parametrized by $\mathcal{M}$ of the form

$$\nabla(u) = d - \left(\frac{i\pi R Z(u)}{\zeta^2} + \frac{\Lambda(u)}{\zeta} + f(u)\right) d\zeta, \tag{3.22}$$

where ζ is a coordinate for $\mathbb{P}^1$ centered at 0, and Λ, f are in $\mathfrak{g}$. The leading entry Z is the central charge in [7] and can be regarded as an element in $\mathfrak{t}$. By (3.21), every $\gamma \in \Gamma$ is a root $\gamma \in \mathfrak{t}^*$ in the root space decomposition for $\mathfrak{g}$ with pairing $\langle Z(u), \gamma \rangle = Z_\gamma(u)$. For $u \in \mathcal{M}$ away from the singular fiber at $a = 0$, $Z(u)$ is *regular semisimple*, i.e. $Z_\gamma(u) \neq 0$ for all $\gamma \in \Gamma$. In the Ooguri-Vafa case, $Z(0, \theta_e, \theta_m) = 0 \in \mathfrak{t}$, which makes the singular fiber a *turning point* for the family of connections as in (3.22). In general, flat sections around turning points are not continuous, and they are obtained through ad hoc methods. Here the irregular singularities at $\zeta = 0$ and $\zeta = \infty$ degenerate into a regular singularity with monodromy $\widetilde{\mathcal{X}_m} \mapsto -\mathcal{X}_e \widetilde{\mathcal{X}_m}$ according to (3.18). Observe that this monodromy is the limit as u approaches the singular fiber of the topological monodromy of the irregular connections in (3.22), which, by (3.11), is $\widetilde{\mathcal{X}_m} \mapsto -\mathcal{X}_e \widetilde{\mathcal{X}_m}$. Furthermore, if $a = 0$ and $\theta_e = 0$, $\widetilde{\mathcal{X}_m} = 0$ by (3.15).

In the generalized case, $Z(0, \theta_e, \theta_m) \neq 0$ is irregular semisimple, since $\langle Z, \gamma_e \rangle = 0$ there (but the pairing with γ_m is nonzero). Therefore, the irregular singularities do not degenerate at the singular locus. This is reflected in the existence of essential singularities at $\zeta = 0$ in (3.20).

Chapter 4

Extension of the Ooguri-Vafa metric

4.1 Classical Case

4.1.1 A C^1 extension of the coordinates

In section 3.1 we extended $\widetilde{\mathcal{X}_m}$ continuously to the bad fiber at $a = 0$. Now we extend the metric by extending the holomorphic symplectic form $\varpi(\zeta)$. Recall that this is of the form

$$\varpi(\zeta) = -\frac{1}{4\pi^2 R}\frac{d\mathcal{X}_e}{\mathcal{X}_e} \wedge \frac{d\widetilde{\mathcal{X}_m}}{\widetilde{\mathcal{X}_m}}$$

Clearly there are no problems extending $d\log \mathcal{X}_e$, so it remains only to extend $d\log \mathcal{X}_e$.

Lemma 4.1.1. *Let $\widetilde{\mathcal{X}_m}$ denote the analytic continuation around $a = 0$ of the magnetic function, as in the last chapter. The 1-form*

$$d\log \widetilde{\mathcal{X}_m} = \frac{d\widetilde{\mathcal{X}_m}}{\widetilde{\mathcal{X}_m}}, \tag{4.1}$$

(where d denotes the differential of a function on the torus fibration $\mathcal{M}'$ only) has an extension to $a = 0$

Proof. We proceed as in section 3.1 and work in different regions in the a-plane (see Figure 3.2), starting with region I, where $\widetilde{\mathcal{X}_m} = \mathcal{X}_m$. Then observe that we can write the corrections on $\mathcal{X}_m$ as a complex number $\Upsilon_m(\zeta) \in (\mathcal{M}'_a)^{\mathbb{C}}$ such that

$$\mathcal{X}_m = \exp\left(\frac{-iR}{2\zeta}(a\log a - a) + i\Upsilon_m + \frac{i\zeta R}{2}(\overline{a}\log\overline{a} - \overline{a})\right).$$

Thus, by (4.1) and ignoring the i factor, it suffices to obtain an extension of

$$\begin{aligned} & d\left[\frac{-R}{2\zeta}(a\log a - a) + \Upsilon_m + \frac{\zeta R}{2}(\overline{a}\log\overline{a} - \overline{a})\right] \\ &= \frac{-R}{2\zeta}\log a\, da + d\Upsilon_m + \frac{\zeta R}{2}\log\overline{a}\, d\overline{a}. \end{aligned} \tag{4.2}$$

Using (3.2),

$$\begin{aligned} d\Upsilon_m = d\theta_m &- \frac{1}{4\pi}\int_{\ell_+}\frac{d\zeta'}{\zeta'}\frac{\zeta'+\zeta}{\zeta'-\zeta}\frac{\mathcal{X}_e}{1-\mathcal{X}_e}\left(\frac{\pi R}{\zeta'}da + id\theta_e + \pi R\zeta' d\overline{a}\right) \\ &+ \frac{1}{4\pi}\int_{\ell_-}\frac{d\zeta'}{\zeta'}\frac{\zeta'+\zeta}{\zeta'-\zeta}\frac{\mathcal{X}_e^{-1}}{1-\mathcal{X}_e^{-1}}\left(-\frac{\pi R}{\zeta'}da - id\theta_e - \pi R\zeta' d\overline{a}\right). \end{aligned}$$

We have to change our θ_m coordinate into θ'_m according to (3.17) and differentiate to obtain:

$$\begin{aligned} d\Upsilon_m = d\theta'_m &- \frac{i(\theta_e - \pi)}{4\pi}\left(\frac{da}{a} - \frac{d\overline{a}}{\overline{a}}\right) + \frac{\arg a}{2\pi}d\theta_e \\ &- \frac{1}{4\pi}\int_{\ell_+}\frac{d\zeta'}{\zeta'}\frac{\zeta'+\zeta}{\zeta'-\zeta}\frac{\mathcal{X}_e}{1-\mathcal{X}_e}\left(\frac{\pi R}{\zeta'}da + id\theta_e + \pi R\zeta' d\overline{a}\right) \\ &+ \frac{1}{4\pi}\int_{\ell_-}\frac{d\zeta'}{\zeta'}\frac{\zeta'+\zeta}{\zeta'-\zeta}\frac{\mathcal{X}_e^{-1}}{1-\mathcal{X}_e^{-1}}\left(-\frac{\pi R}{\zeta'}da - id\theta_e - \pi R\zeta' d\overline{a}\right) \end{aligned} \tag{4.3}$$

If (4.2) extends to $a = 0$, then every independent 1-form extends individually. Let's consider the form involving $d\theta_e$ first. By (4.3), this part consists of:

$$\frac{\arg a}{2\pi}d\theta_e - \frac{i}{4\pi}\int_{\ell_+}\frac{d\zeta'}{\zeta'}\frac{\zeta'+\zeta}{\zeta'-\zeta}\frac{\mathcal{X}_e}{1-\mathcal{X}_e}d\theta_e - \frac{i}{4\pi}\int_{\ell_-}\frac{d\zeta'}{\zeta'}\frac{\zeta'+\zeta}{\zeta'-\zeta}\frac{\mathcal{X}_e^{-1}}{1-\mathcal{X}_e^{-1}}d\theta_e. \tag{4.4}$$

We can use the exact same technique in section 3.1 to find the limit of (4.4) as $a \to 0$. Namely, split each integral into four parts, use the symmetry of $\dfrac{\mathcal{X}_e}{1-\mathcal{X}_e}$ between 0 and ∞ to cancel two of these integrals and take the limit in the remaining ones. The result is:

$$\begin{aligned} &\frac{\arg a}{2\pi} - \frac{ie^{i\theta_e}}{2\pi(1-e^{i\theta_e})}\log\left[\frac{e^{i\arg a}}{\zeta}\right] - \frac{ie^{-i\theta_e}}{2\pi(1-e^{-i\theta_e})}\log\left[\frac{-e^{i\arg a}}{\zeta}\right] \\ &= \frac{\arg a}{2\pi} - \frac{ie^{i\theta_e}}{2\pi(1-e^{i\theta_e})}\log\left[\frac{e^{i\arg a}}{\zeta}\right] + \frac{i}{2\pi(1-e^{i\theta_e})}\log\left[\frac{-e^{i\arg a}}{\zeta}\right] \end{aligned} \tag{4.5}$$

in region I (we omitted the $d\theta_e$ factor for simplicity). Making use of formulas (3.13) and (3.14), we can simplify the above expression and get rid of the apparent dependence on $\arg a$ until finally getting:

$$-\frac{i\log\zeta}{2\pi} - \frac{1}{2(1-e^{i\theta_e})}, \qquad \theta_e \neq 0.$$

In other regions of the a-plane we have to modify $\widetilde{\mathcal{X}_m}$ as in (3.11). Nonetheless, by (3.13) and (3.14), the result is the same and we conclude that at least the terms involving $d\theta_e$ have an extension to $a=0$ for $\theta_e \neq 0$.

Next we extend the terms involving da. By (4.2) and (4.3), these are:

$$\frac{-R}{2\zeta}\log a\, da - \frac{i(\theta_e-\pi)}{4\pi a}da - \frac{R}{4}\int_{\ell_+}\frac{d\zeta'}{(\zeta')^2}\frac{\zeta'+\zeta}{\zeta'-\zeta}\frac{\mathcal{X}_e}{1-\mathcal{X}_e}da - \frac{R}{4}\int_{\ell_-}\frac{d\zeta'}{(\zeta')^2}\frac{\zeta'+\zeta}{\zeta'-\zeta}\frac{\mathcal{X}_e^{-1}}{1-\mathcal{X}_e^{-1}}da$$

In what follows, we ignore the da part and focus on the coefficients for the extension. The partial fraction decomposition

$$\frac{\zeta'+\zeta}{(\zeta')^2(\zeta'-\zeta)} = \frac{2}{\zeta'(\zeta'-\zeta)} - \frac{1}{(\zeta')^2} \tag{4.6}$$

splits each integral above into two parts. We will consider first the terms

$$-\frac{i(\theta_e-\pi)}{4\pi a} + \frac{R}{4}\int_{\ell_+}\frac{d\zeta'}{(\zeta')^2}\frac{\mathcal{X}_e}{1-\mathcal{X}_e} + \frac{R}{4}\int_{\ell_-}\frac{d\zeta'}{(\zeta')^2}\frac{\mathcal{X}_e^{-1}}{1-\mathcal{X}_e^{-1}}. \tag{4.7}$$

Use the fact that $\mathcal{X}_e$ (resp. $\mathcal{X}_e^{-1}$) has norm less than 1 on ℓ_+ (resp. ℓ_-) and the uniform convergence of the geometric series on ζ' to write (4.7) as:

$$-\frac{i(\theta_e-\pi)}{4\pi a}+\frac{R}{4}\sum_{n=1}^{\infty}\left\{\int_{\ell_+}\frac{d\zeta'}{(\zeta')^2}\exp\left(\frac{\pi Rna}{\zeta'}+in\theta_e+\pi Rn\zeta'\overline{a}\right)+\right.$$
$$\left.\int_{\ell_-}\frac{d\zeta'}{(\zeta')^2}\exp\left(\frac{-\pi Rna}{\zeta'}-in\theta_e-\pi Rn\zeta'\overline{a}\right)\right\},$$

$$\begin{aligned}
&=-\frac{i(\theta_e-\pi)}{4\pi a}+\left(\frac{R}{4}\right)\left(\frac{-2|a|}{a}\right)\sum_{n=1}^{\infty}\left(e^{in\theta_e}-e^{-in\theta_e}\right)K_1(2\pi Rn|a|)\\
&=-\frac{i(\theta_e-\pi)}{4\pi a}-\frac{R|a|}{2a}\sum_{n=1}^{\infty}\left(e^{in\theta_e}-e^{-in\theta_e}\right)K_1(2\pi Rn|a|).
\end{aligned}$$

Since $K_1(x)\sim 1/x$, for x real and $x\to 0$, we obtain, letting $a\to 0$:

$$\begin{aligned}
&-\frac{i(\theta_e-\pi)}{4\pi a}-\frac{R|a|}{2a\cdot 2\pi R|a|}\sum_{n=1}^{\infty}\frac{\left(e^{in\theta_e}-e^{-in\theta_e}\right)}{n}\\
&=-\frac{i(\theta_e-\pi)}{4\pi a}+\frac{1}{4\pi a}[\log(1-e^{i\theta_e})-\log(1-e^{-i\theta_e})]
\end{aligned}$$

and by (3.12),

$$=-\frac{i(\theta_e-\pi)}{4\pi a}+\frac{i(\theta_e-\pi)}{4\pi a}=0.$$

Therefore this part of the da terms extends trivially to 0 in the singular fiber.

It remains to extend the other terms involving da. Recall that by (4.6), these terms are (after getting rid of a factor of $-R/2$):

$$\frac{\log a}{\zeta}+\int_{\ell_+}\frac{d\zeta'}{\zeta'(\zeta'-\zeta)}\frac{\mathcal{X}_e}{1-\mathcal{X}_e}+\int_{\ell_-}\frac{d\zeta'}{\zeta'(\zeta'-\zeta)}\frac{\mathcal{X}_e^{-1}}{1-\mathcal{X}_e^{-1}}. \tag{4.8}$$

We'll focus in the first integral in (4.8). As a starting point, we'll prove that as $a \to 0$, the limiting value of this integral is the same as the limit of

$$\int_{\ell_+} \frac{d\zeta'}{\zeta'(\zeta'-\zeta)} \frac{\exp\left(\frac{\pi Ra}{\zeta'} + i\theta_e\right)}{1-\exp\left(\frac{\pi Ra}{\zeta'} + i\theta_e + \pi R\zeta'\overline{a}\right)}. \tag{4.9}$$

It suffices to show that

$$\int_{\ell_+} \frac{d\zeta'}{\zeta'(\zeta'-\zeta)} \frac{\exp\left(\frac{\pi Ra}{\zeta'}\right)}{1-\exp\left(\frac{\pi Ra}{\zeta'} + i\theta_e + \pi R\zeta'\overline{a}\right)}[1-\exp(\pi R\zeta'\overline{a})] \to 0, \quad \text{as } a \to 0,\ \theta_e \neq 0 \tag{4.10}$$

To see this, we can assume $|a| < 1$. Let $b = a/|a|$. Observe that in the ℓ_+ ray, $|\exp(\pi Ra/\zeta')| < 1$, and since $\theta_e \neq 0$, we can bound (4.10) by

$$\text{const} \int_{\ell_+} \frac{d\zeta'}{\zeta'(\zeta'-\zeta)}[1-\exp(\pi R\zeta'\overline{b})] < \infty.$$

Equation (4.10) now follows from Lebesgue Dominated Convergence and the fact that $1-\exp(\pi R\zeta'\overline{a}) \to 0$ as $a \to 0$. A similar application of Dominated Convergence allows us to reduce the problem to the extension of

$$\int_{\ell_+} \frac{d\zeta'}{\zeta'(\zeta'-\zeta)} \frac{\exp\left(\frac{\pi Ra}{\zeta'} + i\theta_e\right)}{1-\exp\left(\frac{\pi Ra}{\zeta'} + i\theta_e\right)}. \tag{4.11}$$

Introduce the real variable $s = -\pi Ra/\zeta'$. We can write (4.11) as:

$$\begin{aligned} & e^{i\theta_e} \int_0^\infty \frac{ds}{s\left[\frac{-\pi Ra}{s} - \zeta\right]} \frac{e^{-s}}{1-e^{i\theta_e - s}} \\ &= -\frac{1}{\zeta} \int_0^\infty \frac{ds}{s + \frac{\pi Ra}{\zeta}} \cdot \frac{e^{-s}}{e^{-i\theta_e} - e^{-s}} \\ &= \frac{1}{\zeta} \int_0^\infty \frac{ds}{s + \frac{\pi Ra}{\zeta}} \cdot \frac{1}{1-e^{s-i\theta_e}} \end{aligned} \tag{4.12}$$

The integrand of (4.12) has a double zero at ∞, when $a \to 0$, so the only possible non-convergent part in the limit $a = 0$ is the integral

$$\frac{1}{\zeta}\int_0^1 \frac{ds}{s + \frac{\pi R a}{\zeta}} \cdot \frac{1}{1 - e^{s - i\theta_e}}.$$

Since

$$\int_0^1 \frac{ds}{s}\left[\frac{1}{1 - e^{s - i\theta_e}} - \frac{1}{1 - e^{-i\theta_e}}\right] < \infty,$$

we can simplify this analysis even further and focus only on

$$\frac{1}{\zeta(1 - e^{-i\theta_e})}\int_0^1 \frac{ds}{s + \frac{\pi R a}{\zeta}} \tag{4.13}$$

$$= -\frac{\log(\pi R a/\zeta)}{\zeta(1 - e^{-i\theta_e})}. \tag{4.14}$$

We can apply the same technique to obtain a limit for the second integral in (4.8). The result is

$$-\frac{\log(-\pi R a/\zeta)}{\zeta(1 - e^{i\theta_e})},$$

which means that the possibly non-convergent terms in (4.8) are:

$$\frac{\log a}{\zeta} - \frac{\log a}{\zeta(1 - e^{-i\theta_e})} - \frac{\log a}{\zeta(1 - e^{i\theta_e})} = 0. \tag{4.15}$$

Note that the corrections of $\mathcal{X}_m$ in other regions of the a-plane as in (3.11) depend only on $\mathcal{X}_e$, which clearly has a smooth extension to the singular fiber.

The extension of the $d\overline{a}$ part is performed in exactly the same way as with the da forms. We conclude that the 1-form

$$\frac{d\widetilde{\mathcal{X}_m}}{\widetilde{\mathcal{X}_m}}$$

has an extension to the fiber at $a = 0$ in the classical Ooguri-Vafa case. This holds true also in the generalized Ooguri-Vafa case since here we simply add factors of the form $f'(a)da$ and it is assumed that $f(a)$ has a smooth extension to the singular fiber. $\square$

In Chapter 4, we will reinterpret these extension of the derivatives of $\mathcal{X}_m$ if we regard the gauge transformation (3.17) as a contour integral between symmetric contours. It will be then easier to see that the extension can be made smooth.

4.1.2 Extension of the metric

The results of the previous section already show the continuous extension of the holomorphic symplectic form

$$\varpi(\zeta) = -\frac{1}{4\pi^2 R}\frac{d\mathcal{X}_e}{\mathcal{X}_e} \wedge \frac{d\widetilde{\mathcal{X}_m}}{\widetilde{\mathcal{X}_m}}$$

to the limiting case $a = 0$, but we excluded the special case $\theta_e = 0$. Here we obtain $\varpi(\zeta)$ at the singular fiber with a different approach that will allow us to see that such an extension is smooth without testing the extension for each derivative. Although it was already known that $\mathcal{M}'$ extends to the hyperkähler manifold $\mathcal{M}$ constructed here, this approach is new, as it gives an explicit construction of the metric as we will see. Furthermore, the Ooguri-Vafa model can be thought as an elementary model for which more complex integrable systems are modeled locally (see Chapter 5).

Theorem 4.1.2. *The holomorphic symplectic form $\varpi(\zeta)$ extends smoothly to $a = 0$. Near $a = 0$ and $\theta_e = 0$, the hyperkähler metric g looks like a constant multiple of the Taub-NUT metric $g_{\text{Taub-NUT}}$ plus some smooth corrections.*

Proof. By [7], near $a = 0$,

$$\varpi(\zeta) = -\frac{1}{4\pi^2 R}\frac{d\mathcal{X}_e}{\mathcal{X}_e} \wedge \left[id\theta_m + 2\pi i A + \pi i V \left(\frac{1}{\zeta} da - \zeta d\bar{a} \right) \right],$$

where

$$A = \frac{1}{8\pi^2}\left(\log\frac{a}{\Lambda} - \log\frac{\bar{a}}{\bar{\Lambda}} \right) d\theta_e - \frac{R}{4\pi}\left(\frac{da}{a} - \frac{d\bar{a}}{\bar{a}} \right)\sum_{n\neq 0} (\operatorname{sgn} n) e^{in\theta_e}|a|K_1(2\pi R|na|)$$

should be understood as a $U(1)$ connection over the open subset of $\mathbb{C} \times S^1$ parametrized by (a, θ_e) and V is given by Poisson re-summation as

$$V = \frac{R}{4\pi}\left[\frac{1}{\sqrt{R^2|a|^2 + \frac{\theta_e^2}{4\pi^2}}} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left(\frac{1}{\sqrt{R^2|a|^2 + (\frac{\theta_e}{2\pi} + n)^2}} - \frac{1}{n} \right) \right]. \tag{4.16}$$

Observe that the above sum converges even at $a = 0, \theta_e \neq 0$. The curvature F of the unitary connection satisfies

$$dA = *dV. \tag{4.17}$$

Consider now a gauge transformation $\theta_m \mapsto \theta_m + \alpha$ and its induced change in the connection $A \mapsto A' = A - d\alpha/2\pi$ (see [7]). We have $id\theta_m' + 2\pi i A' = id\theta_m + id\alpha + 2\pi i A - id\alpha = id\theta_m + 2\pi i A$. Furthermore, for the particular gauge transformation in (3.16), at $a = 0$ and for $\theta_e \neq 0$:

$$A' = A - \frac{d\alpha}{2\pi}$$

$$
\begin{aligned}
&= \frac{1}{8\pi^2}\left(\log\frac{a}{\Lambda} - \log\frac{\bar{a}}{\overline{\Lambda}}\right)d\theta_e - \frac{1}{8\pi^2}\left(\frac{da}{a} - \frac{d\bar{a}}{\bar{a}}\right)\left[\sum_{n=1}^{\infty}\frac{e^{in\theta_e}}{n} - \sum_{n=1}^{\infty}\frac{e^{-in\theta_e}}{n}\right] \\
&- \frac{1}{8\pi^2}\left(\log\frac{a}{\Lambda} - \log\frac{\bar{a}}{\overline{\Lambda}}\right)d\theta_e - \frac{i(\theta_e-\pi)}{8\pi^2}\left(\frac{da}{a} - \frac{d\bar{a}}{\bar{a}}\right),
\end{aligned}
$$

(here we're using the fact that $K_1(x) \to 1/x$ as $x \to 0$)

$$
= \frac{i(\theta_e-\pi)}{8\pi^2}\left(\frac{da}{a} - \frac{d\bar{a}}{\bar{a}}\right) - \frac{i(\theta_e-\pi)}{8\pi^2}\left(\frac{da}{a} - \frac{d\bar{a}}{\bar{a}}\right) = 0.
$$

since the above sums converge to $-\log(1-e^{i\theta_e}) + \log(1-e^{-i\theta_e}) = -i(\theta_e - \pi)$ for $\theta_e \neq 0$.

Writing V_0 (observe that this only depends on θ_e) for the limit of V as $a \to 0$, we get at $a = 0$

$$
\begin{aligned}
\varpi(\zeta) &= -\frac{1}{4\pi^2 R}\left(\frac{\pi R}{\zeta}da + id\theta_e + \pi R\zeta d\bar{a}\right) \wedge \left(id\theta'_m + \pi i V_0\left(\frac{da}{\zeta} - \zeta d\bar{a}\right)\right) \\
&= \frac{1}{4\pi^2 R}d\theta_e \wedge d\theta'_m + \frac{iV_0}{2}da \wedge d\bar{a} - \frac{i}{4\pi\zeta}da \wedge d\theta'_m - \frac{V_0}{4\pi R\zeta}da \wedge d\theta_e \\
&- \frac{i\zeta}{4\pi}d\bar{a} \wedge d\theta'_m + \frac{V_0\zeta}{4\pi R}d\bar{a} \wedge d\theta_e.
\end{aligned}
$$

This yields that, at the singular fiber,

$$
\omega_3 = \frac{1}{4\pi^2 R}d\theta_e \wedge d\theta'_m + \frac{iV_0}{2}da \wedge d\bar{a} \tag{4.18}
$$

$$
\omega_+ = \frac{1}{2\pi}da \wedge \left(d\theta'_m - \frac{iV_0}{R}d\theta_e\right) \tag{4.19}
$$

$$
\omega_- = \frac{1}{2\pi}d\bar{a} \wedge \left(d\theta'_m + \frac{iV_0}{R}d\theta_e\right) \tag{4.20}
$$

From the last two equations we obtain that $d\theta'_m - iV_0/Rd\theta_e$ and $d\theta'_m + iV_0/Rd\theta_e$ are respectively (1,0) and (0,1) forms under the complex structure J_3. A $(1,0)$ vector field dual to the $(1,0)$ form above is then $\frac{1}{2}\left(\partial_{\theta'_m} + iR/V_0\partial_{\theta_e}\right)$. In particular,

$$J_3(\partial_{\theta'_m}) = -\frac{R}{V_0}\partial_{\theta_e}, \qquad J_3\left(-\frac{R}{V_0}\partial_{\theta_e}\right) = -\partial_{\theta'_m}.$$

With this and (4.18) we can reconstruct the metric at $a = 0$. Observe that

$$\begin{aligned} g(\partial_{\theta_e}, \partial_{\theta_e}) &= \omega_3(\partial_{\theta_e}, J_3(\partial_{\theta_e})) = \omega_3\left(\partial_{\theta_e}, \frac{V_0}{R}\partial_{\theta'_m}\right) = \frac{V_0}{4\pi^2 R^2} \\ g(\partial_{\theta'_m}, \partial_{\theta'_m}) &= \omega_3(\partial_{\theta'_m}, J_3(\partial_{\theta'_m})) = \omega_3\left(\partial_{\theta'_m}, -\frac{R}{V_0}\partial_{\theta_e}\right) = \frac{1}{4\pi^2 V_0} \end{aligned}$$

Consequently,

$$g = \frac{1}{V_0}\left(\frac{d\theta'_m}{2\pi}\right)^2 + V_0 d\vec{x}^2,$$

where $a = x^1 + ix^2, \theta_e = 2\pi R x^3$. Since $V_0(\theta_e)$ is undefined for $\theta_e = 0$, we have to check that g extends to this point. Let (r, ϑ, ϕ) denote spherical coordinates for $\vec{x}$. The formula above is the natural extension of the metric given in [7] for nonzero a:

$$g = \frac{1}{V(\vec{x})}\left(\frac{d\theta'_m}{2\pi} + A'(\vec{x})\right)^2 + V(\vec{x})d\vec{x}^2$$

To see that this extends to $r = 0$, we rewrite

$$\begin{aligned} V &= \frac{R}{4\pi}\left[\frac{1}{\sqrt{R^2|a|^2 + \frac{\theta_e^2}{4\pi^2}}} + \sum_{n\neq 0}\left(\frac{1}{\sqrt{R^2|a|^2 + (\frac{\theta_e}{2\pi} + n)^2}} - \frac{1}{n}\right)\right] \\ &= \frac{1}{4\pi}\left[\frac{1}{\sqrt{|a|^2 + \frac{\theta_e^2}{4R^2\pi^2}}} + R\sum_{n\neq 0}\left(\frac{1}{\sqrt{R^2|a|^2 + (\frac{\theta_e}{2\pi} + n)^2}} - \frac{1}{n}\right)\right] \end{aligned}$$

$$= \frac{1}{4\pi}\left(\frac{1}{r} + C(\vec{x})\right), \tag{4.21}$$

where $C(\vec{x})$ is smooth and bounded in a neighborhood of the origin.

Similarly, we do Poisson re-summation for the unitary connection

$$A' = -\frac{1}{4\pi}\left(\frac{da}{a} - \frac{d\bar{a}}{\bar{a}}\right)\left[\frac{i(\theta_e - \pi)}{2\pi} + R\sum_{n\neq 0}(\operatorname{sgn} n)e^{in\theta_e}|a|K_1(2\pi R|na|)\right].$$

Using the fact that the inverse Fourier transform of $(\operatorname{sgn}\,\xi)e^{i\theta_e\xi}|a|K_1(2\pi R|a\xi|)$ is

$$\frac{i(\frac{\theta_e}{2\pi} + t)}{2R\sqrt{R^2|a|^2 + (\frac{\theta_e}{2\pi} + t)^2}},$$

we obtain

$$\begin{aligned} A' &= -\frac{i}{8\pi}\left(\frac{da}{a} - \frac{d\bar{a}}{\bar{a}}\right)\sum_{n=-\infty}^{\infty}\left(\frac{\frac{\theta_e}{2\pi} + n}{\sqrt{R^2|a|^2 + (\frac{\theta_e}{2\pi} + n)^2}} - \kappa_n\right) \\ &= \frac{1}{4\pi}\left(\frac{da}{a} - \frac{d\bar{a}}{\bar{a}}\right)\left[-\frac{i\theta_e}{4\pi\sqrt{R^2|a|^2 + \left(\frac{\theta_e}{2\pi}\right)^2}} - \frac{i}{2}\sum_{n\neq 0}\left(\frac{\frac{\theta_e}{2\pi} + n}{\sqrt{R^2|a|^2 + (\frac{\theta_e}{2\pi} + n)^2}} - \kappa_n\right)\right] \end{aligned}$$

since $d\phi = d\arg a = -id\log\frac{a}{|a|} = -\frac{i}{2}\left(\frac{da}{a} - \frac{d\bar{a}}{\bar{a}}\right)$ and $\cos\vartheta = \frac{x^3}{r}$, this simplifies to:

$$= \frac{1}{4\pi}(\cos\vartheta + D(\vec{x}))d\phi. \tag{4.22}$$

Here κ_n is a regularization constant that makes the sum converge, and $D(\vec{x})$ is smooth and bounded in a neighborhood of $r = 0$. By (4.21) and (4.22), it follows that near $r = 0$

$$
\begin{aligned}
g &= V^{-1}\left(\frac{d\theta'_m}{2\pi} + A'\right)^2 + V d\vec{x}^2 \\
&= 4\pi\left(\frac{1}{r} + C\right)^{-1}\left(\frac{d\theta'_m}{2\pi} + \frac{1}{4\pi}\cos\vartheta d\phi + D d\phi\right)^2 + \frac{1}{4\pi}\left(\frac{1}{r} + C\right) d\vec{x}^2 \\
&= \frac{1}{4\pi}\left[\left(\frac{1}{r} + C\right)^{-1}\left(2d\theta'_m + \cos\vartheta d\phi + \tilde{D} d\phi\right)^2 + \left(\frac{1}{r} + C\right) d\vec{x}^2\right] \\
&= \frac{1}{4\pi} g_{\text{Taub-NUT}} + \text{smooth corrections.}
\end{aligned}
$$

This shows that our metric extends to $r = 0$ and finishes the construction of the singular fiber. □

4.2 General case

Here we work with the assumption in subsection 3.2. To distinguish this case to the previous one, we will denote by $\varpi_{\text{old}}, g_{\text{old}}$, etc. the forms obtained in the classical case.

Theorem 4.2.1. *In the General Ooguri-Vafa case, the holomorphic symplectic form $\varpi(\zeta)$ and the hyperkähler metric g extend to the singular fiber.*

Proof. By formula (3.19),

$$d\log \mathcal{X}_m^{\text{sf}} = d\log \mathcal{X}_{m,\text{old}}^{\text{sf}} + \frac{R}{\zeta}\left(-\frac{i}{2} + \pi f'(a)\right) da + R\zeta\left(\frac{i}{2} + \pi\overline{f'(a)}\right) d\overline{a} \tag{4.23}$$

Recall that the corrections of $\mathcal{X}_m$ are the same as the classical Ooguri-Vafa case. To simplify the equations, let $C := \lim_{a\to 0}(-i/2 + \pi f'(a))$. Thus, using (4.23), at $a = 0$

$$\varpi(\zeta) = \varpi_{\text{old}}(\zeta) + \frac{iR}{2\pi}\text{Im } C da \wedge d\overline{a} + \frac{iC}{4\pi^2\zeta} da \wedge d\theta_e + \frac{i\zeta\overline{C}}{4\pi^2} d\overline{a} \wedge d\theta_e.$$

Decomposing $\varpi(\zeta) = -i/2\zeta\omega_+ + \omega_3 - i\zeta/2\omega_-$, we obtain:

$$\omega_3 = \omega_{3,\text{old}} + \frac{iR}{2\pi}\text{Im } C da \wedge d\overline{a}, \tag{4.24}$$

$$\omega_+ = \omega_{+,\text{old}} - \frac{C}{2\pi^2} da \wedge d\theta_e \tag{4.25}$$

$$\omega_- = \omega_{-,\text{old}} - \frac{\overline{C}}{2\pi^2} d\overline{a} \wedge d\theta_e \tag{4.26}$$

By (4.25) and (4.26),

$$d\theta'_m - \frac{i}{R}\left(V_0 - \frac{iRC}{\pi}\right) d\theta_e \quad \text{and} \quad d\theta'_m + \frac{i}{R}\left(V_0 + \frac{iR\overline{C}}{\pi}\right) d\theta_e$$

are, respectively, (1,0) and (0,1) forms. It's not hard to see that

$$\frac{-V_0\pi - iR\overline{C}}{R\pi}\partial_{\theta'_m} - i\partial_{\theta_e}$$

or, rearranging real parts,

$$\left(-\frac{V_0}{R} - \frac{\text{Im } C}{\pi}\right)\partial_{\theta'_m} - i\left(\frac{\text{Re } C}{\pi}\partial_{\theta'_m} + \partial_{\theta_e}\right)$$

is a $(1,0)$ vector field. This allow us to obtain

$$J_3\left[\left(-\frac{V_0}{R} - \frac{\text{Im } C}{\pi}\right)\partial_{\theta'_m}\right] = \frac{\text{Re } C}{\pi}\partial_{\theta'_m} + \partial_{\theta_e}$$

$$J_3\left[\frac{\text{Re } C}{\pi}\partial_{\theta'_m} + \partial_{\theta_e}\right] = \left(\frac{V_0}{R} + \frac{\text{Im } C}{\pi}\right)\partial_{\theta'_m}.$$

By linearity,

$$J_3(\partial_{\theta'_m}) = \text{const} \cdot \partial_{\theta'_m} - \frac{R\pi}{V_0\pi + R\text{Im } C}\partial_{\theta_e}$$

$$J_3(\partial_{\theta_e}) = \left(\frac{V_0\pi + R\mathrm{Im}\ C}{\pi R} + \frac{(\mathrm{Re}\ C)^2 R}{\pi(V_0\pi + R\mathrm{Im}\ C)}\right)\partial_{\theta'_m} + \mathrm{const}\cdot\partial_{\theta_e}.$$

With this we can compute

$$\begin{aligned}
g(\partial_{\theta'_m}, \partial_{\theta'_m}) &= \omega_3(\partial_{\theta'_m}, J_3(\partial_{\theta'_m})) \\
&= \frac{1}{4\pi(V_0\pi + R\mathrm{Im}\ C)} \\
g(\partial_{\theta_e}, \partial_{\theta_e}) &= \omega_3(\partial_{\theta_e}, J_3(\partial_{\theta_e})) \\
&= \frac{V_0\pi + R\mathrm{Im}\ C}{4\pi^3 R^2} + \frac{(\mathrm{Re}\ C)^2}{4\pi^3(V_0\pi + R\mathrm{Im}\ C)}.
\end{aligned}$$

Define B_0 as $V_0 + R\mathrm{Im}\ C/\pi$. A condition that $f'(0)$ must satisfy for positive definiteness of g is that $B_0 > 0$. If this is true, then, in this case, the metric at $a = 0$ is

$$g = \frac{1}{B_0}\left(\frac{d\theta'_m}{2\pi}\right)^2 + B_0 d\vec{x}^2 + \left(\frac{R\cdot\mathrm{Re}\ C}{\pi}\right)^2\frac{dx_3^2}{B_0}. \tag{4.27}$$

This metric can be extended to the point $\theta_e = 0$ ($r = 0$ in §4.1) exactly as before, by writing g as the Taub-NUT metric plus smooth corrections and observing that, since $\lim_{\theta_e\to 0} B_0 = \infty$,

$$\lim_{\theta_e\to 0}\left(\frac{R\cdot\mathrm{Re}\ C}{\pi}\right)^2\frac{dx_3^2}{B_0} = 0.$$

□

Chapter 5

The Pentagon and other cases

5.1 Solutions

Now we will extend the results of the Ooguri-Vafa case to the general problem. We will start with the Pentagon example. This example is presented in detail in [17]. By [8], this example represents the moduli space of Higgs bundles with gauge group SU(2) over $\mathbb{P}^1$ with 1 irregular singularity at $z = \infty$.

Here $\mathcal{B} = \mathbb{C}$ with discriminant locus a 2-point set, which we can assume is $\{-2, 2\}$ in the complex plane. Thus $\mathcal{B}'$ is the twice-punctured plane. $\mathcal{B}$ is divided into two domains $\mathcal{B}_{\text{in}}$ and $\mathcal{B}_{\text{out}}$ by the locus

$$W = \{u : Z(\Gamma_u) \text{ is contained in a line in } \mathbb{C}\} \subset \mathcal{B}$$

See Figure 5.1. Since $\mathcal{B}_{\text{in}}$ is simply connected Γ can be trivialized over $\mathcal{B}_{\text{in}}$ by primitive cycles γ_1, γ_2, with $Z_{\gamma_1} = 0$ at $u = -2$, $Z_{\gamma_2} = 0$ at $u = 2$. We can choose them also so that $\langle \gamma_1, \gamma_2 \rangle = 1$.

Take the set $\{\gamma_1, \gamma_2\}$. To compute its monodromy around infinity, take cuts at each point of $D = \{-2, 2\}$ (see Figure 5.2) and move counterclockwise. By (2.3), the jump of γ_2 when you cross the cut at -2 is of the form $\gamma_2 \mapsto \gamma_1 + \gamma_2$. As you return to the original place and cross the cut at 2, the jump of γ_1 is of the type $\gamma_1 \mapsto \gamma_1 - \gamma_2$.

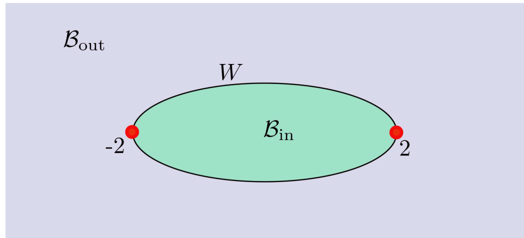

Figure 5.1: The wall W in $\mathcal{B}$ for the Pentagon case

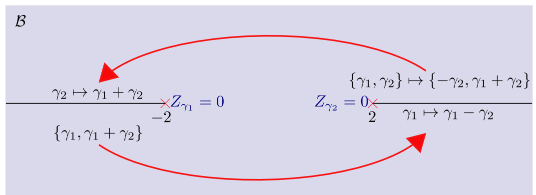

Figure 5.2: The monodromy around infinity of Γ

Thus, around infinity, $\{\gamma_1, \gamma_2\}$ transforms into $\{-\gamma_2, \gamma_1 + \gamma_2\}$. The set $\{\gamma_1, \gamma_2, -\gamma_1, -\gamma_2, \gamma_1 + \gamma_2, -\gamma_1 - \gamma_2\}$ is therefore invariant under monodromy at infinity and it makes global sense to define

$$\begin{aligned} \text{For } u \in \mathcal{B}_{\text{in}}, \quad \Omega(\gamma; u) &= \begin{cases} 1 & \text{for } \gamma \in \{\gamma_1, \gamma_2, -\gamma_1, -\gamma_2\} \\ 0 & \text{otherwise} \end{cases} \\ \text{For } u \in \mathcal{B}_{\text{out}}, \quad \Omega(\gamma; u) &= \begin{cases} 1 & \text{for } \gamma \in \{\gamma_1, \gamma_2, -\gamma_1, -\gamma_2, \gamma_1 + \gamma_2, -\gamma_1 - \gamma_2\} \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{5.1}$$

Let $\mathcal{M}'$ denote the torus fibration over $\mathcal{B}'$ constructed in [17]. Near $u = 2$, we'll denote γ_1 by γ_m and γ_2 by γ_e (the labels will change for $u = -2$). To shorten notation, we'll write ℓ_e, Z_e, etc. instead of $\ell_{\gamma_e}, Z_{\gamma_e}$, etc. Let θ denote the vector of torus coordinates (θ_e, θ_m). With the change of variables $a := Z_e(u)$ we can assume, without loss of generality, that the bad fiber is at

$a = 0$ and

$$\lim_{a \to 0} Z_m(a) = c \neq 0. \tag{5.2}$$

Let T denote the complex torus fibration over $\mathcal{M}'$ constructed in [7]. By the definition of $\Omega(\gamma; a)$, the functions $(\mathcal{X}_e, \mathcal{X}_m)$ both receive corrections. Recall that by (2.10), for each $\nu \in \mathbb{N}$, we get a function $\mathcal{X}_\gamma^{(\nu)}$, which is the ν-th iteration of the function $\mathcal{X}_\gamma$. We can write

$$\mathcal{X}_\gamma^{(\nu)}(a, \zeta, \theta) = \mathcal{X}_\gamma^{\text{sf}}(a, \zeta, \theta) C_\gamma^{(\nu)}(a, \zeta, \theta).$$

It will be convenient to rewrite the above equation as in [7, C.17]. For that, let $\Upsilon^{(\nu)}$ be the map from $\mathcal{M}_a$ to its complexification $\mathcal{M}_a^{\mathbb{C}}$ such that

$$\mathcal{X}_\gamma^{(\nu)}(a, \zeta, \theta) = \mathcal{X}_\gamma^{\text{sf}}(a, \zeta, \Upsilon^{(\nu)}). \tag{5.3}$$

We'll do a modification in the construction of [7] as follows: We'll use the term "BPS ray" for each ray $\{\ell_\gamma : \Omega(\gamma, a) \neq 0\}$ as in [7]. This terminology comes from Physics. In the language of Riemann-Hilbert problems, these are known as "anti-Stokes" rays. That is, they represent the contour Σ where a function has prescribed discontinuities.

The problem is local on $\mathcal{B}$, so instead of defining a Riemann-Hilbert problem using the BPS rays ℓ_γ, we will cover $\mathcal{B}'$ with open sets $\{U_\alpha : \alpha \in \Delta\}$ such that for each α, $\overline{U_\alpha}$ is compact, $\overline{U_\alpha} \subset V_\alpha$, with V_α open and $\mathcal{M}'|_{V_\alpha}$ a trivial fibration. For any ray r in the ζ-plane, define $\mathbb{H}_r$ as the half-plane of vectors making an acute angle with r. Assume that there is a pair of rays $r, -r$ such that for all $a \in U_\alpha$, half of the rays lie inside $\mathbb{H}_r$ and the other half lie in $\mathbb{H}_{-r}$. We call such rays *admissible rays*. If U_α is small enough, there exists admissible rays for such a neighborhood. We are allowing the case that r is a BPS ray ℓ_γ, as long as it satisfies the above condition. As a varies in U_α, some BPS rays (or anti-Stokes rays, in RH terminology) converge into a single ray (wall-crossing phenomenon) (see Figures 5.3 and 5.4).

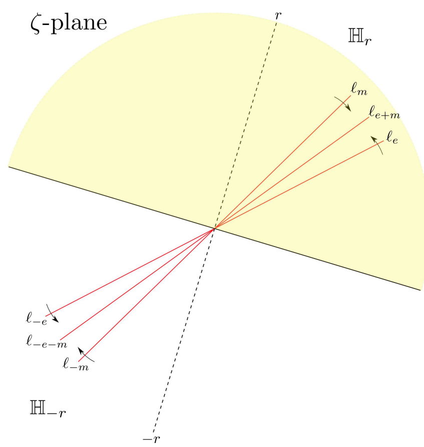

Figure 5.3: 3 anti-Stokes rays before hitting the wall

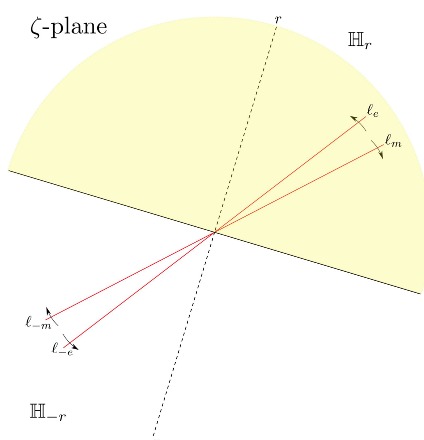

Figure 5.4: At the other side of the wall there are only 2 anti-Stokes rays

For $\gamma \in \Gamma$, we define $\gamma > 0$ (resp. $\gamma < 0$) as $\ell_\gamma \in \mathbb{H}_r$ (resp. $\ell_\gamma \in \mathbb{H}_{-r}$). Our Riemann-Hilbert problem will have only two anti-Stokes rays, namely r and $-r$. The specific discontinuities at the anti-Stokes rays for the function we're trying to obtain are called *Stokes factors* (see [3]). In (2.8), the Stokes factor was given by S_ℓ^{-1}.

In this case, the Stokes factors are the concatenation of all the Stokes factors S_ℓ^{-1} in (2.7) in the counterclockwise direction:

$$S_+ = \prod_{\gamma>0}^{\curvearrowleft} \mathcal{K}_\gamma^{\Omega(\gamma;a)}$$

$$S_- = \prod_{\gamma<0}^{\curvearrowleft} \mathcal{K}_\gamma^{\Omega(\gamma;a)}$$

We will show that such Riemann-Hilbert problem has unique solutions, henceforth denoted by $\mathcal{Y}$. As in (5.3), we can write $\mathcal{Y}$ as

$$\mathcal{Y}_\gamma(a, \zeta, \theta) = \mathcal{X}_\gamma^{\mathrm{sf}}(a, \zeta, \Theta), \tag{5.4}$$

for $\Theta : \mathcal{M}_a \to \mathcal{M}_a^{\mathbb{C}}$.

A different choice of admissible pairs $r', -r'$ gives an equivalent Riemann-Hilbert problem, where the two solutions $\mathcal{Y}, \mathcal{Y}'$ differ only for ζ in the sector defined by the rays r, r', and one can be obtained from the other by analytic continuation. The difference will be made explicit as we solve the Riemann-Hilbert problem.

In the case of the Pentagon, we have two types of wall-crossing phenomenon. Namely, as a varies, ℓ_e moves in the ζ-plane until it coincides with the ℓ_m ray for some value of a in the wall of marginal stability (Fig. 5.3 and 5.4). We'll call this type I of wall-crossing. In this case we have the Pentagon identity

$$\mathcal{K}_e\mathcal{K}_m = \mathcal{K}_m\mathcal{K}_{e+m}\mathcal{K}_e, \tag{5.5}$$

As a goes around 0, the ℓ_e ray will then intersect with the ℓ_{-m} ray now. Because of the monodromy $\gamma_m \mapsto \gamma_{-e+m}$ around 0, ℓ_m becomes ℓ_{-e+m}. This second type (type II) of wall-crossing is illustrated in Fig. 5.5 and 5.6.

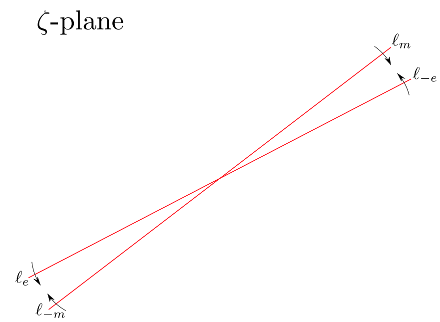

Figure 5.5: 2 anti-Stokes rays before hitting the wall

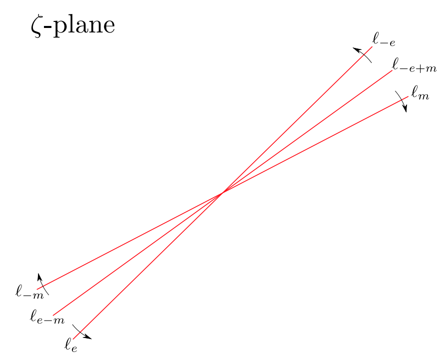

Figure 5.6: At the other side of the wall there are now 3 anti-Stokes rays

This gives a second Pentagon identity

$$\mathcal{K}_e\mathcal{K}_m = \mathcal{K}_m\mathcal{K}_{e+m}\mathcal{K}_e$$

In any case, the Stokes factors above remain the same even if a is in the wall of marginal stability. The way we defined S_+, S_- makes this true for the general case also.

Specifically, in the Pentagon the two Stokes factors for the first type of wall-crossing are given by the maps:

$$\left.\begin{array}{ll} \mathcal{Y}_m & \mapsto \mathcal{Y}_m(1-\mathcal{Y}_e(1-\mathcal{Y}_m))^{-1} \\ \mathcal{Y}_e & \mapsto \mathcal{Y}_e(1-\mathcal{Y}_m) \end{array}\right\} S_+ \tag{5.6}$$

and, similarly

$$\left.\begin{array}{ll} \mathcal{Y}_m & \mapsto \mathcal{Y}_m(1-\mathcal{Y}_e^{-1}(1-\mathcal{Y}_m^{-1})) \\ \mathcal{Y}_e & \mapsto \mathcal{Y}_e(1-\mathcal{Y}_m^{-1})^{-1} \end{array}\right\} S_- \tag{5.7}$$

For the second type:

$$\left.\begin{array}{ll} \mathcal{Y}_m & \mapsto \mathcal{Y}_m(1-\mathcal{Y}_e^{-1}) \\ \mathcal{Y}_e & \mapsto \mathcal{Y}_e(1-\mathcal{Y}_m(1-\mathcal{Y}_e^{-1})) \end{array}\right\} S_+ \tag{5.8}$$

$$\left.\begin{array}{ll} \mathcal{Y}_m & \mapsto \mathcal{Y}_m(1-\mathcal{Y}_e)^{-1} \\ \mathcal{Y}_e & \mapsto \mathcal{Y}_e(1-\mathcal{Y}_m^{-1}(1-\mathcal{Y}_e))^{-1} \end{array}\right\} S_- \tag{5.9}$$

If we take the power series expansion of the above Stokes factors, the integral formula is then:

$$\mathcal{Y}_\gamma^{(\nu+1)}(a,\zeta) = \mathcal{X}_\gamma^{\mathrm{sf}}(a,\zeta)\exp\left\langle \gamma, \frac{1}{4\pi\mathrm{i}}\left\{ \sum_{\gamma'>0} f^{\gamma'}\int_r K(\zeta,\zeta')\mathcal{Y}_{\gamma'}^{(\nu)}(a,\zeta') + \sum_{\gamma'<0} f^{\gamma'}\int_{-r} K(\zeta,\zeta')\mathcal{Y}_{\gamma'}^{(\nu)}(a,\zeta')\right\}\right\rangle, \tag{5.10}$$

where we abbreviated $\dfrac{d\zeta'}{\zeta'}\dfrac{\zeta'+\zeta}{\zeta'-\zeta}$ as $K(\zeta',\zeta)$. The vector $f^{\gamma} \in \Gamma_{\mathbb{Q}}$ is obtained by power series expansion of $\log(S^{+}\mathcal{Y}_{\gamma}), \log(S^{-}\mathcal{Y}_{\gamma})$. Explicitly, for any pair of integers i, j and for the type of wall-crossing in (5.6), (5.7):

$$f^{\gamma_{ie+jm}} = \begin{cases} \dfrac{-1}{j^2}\gamma_{jm} & \text{if } i = 0 \\ \dfrac{(-1)^j}{i^2}\binom{|i|}{|j|}\gamma_{ie} & \text{if } 0 \le j \le i \text{ or } i \le j \le 0 \\ 0 & \text{otherwise.} \end{cases} \tag{5.11}$$

A similar equation holds for the jumps in (5.8), (5.9) with the tags reversed. Formula (5.10) requires an explanation. Assuming $\mathcal{Y}^{(\nu-1)}_{\gamma'}, \gamma' \in \Gamma$ has been constructed, by definition, $\mathcal{Y}^{(\nu)}_{\gamma'}$ has jumps at r and $-r$. By abuse of notation, $\mathcal{Y}^{(\nu)}_{\gamma'}$ in (5.10) denotes the analytic continuation to the ray r (resp. $-r$) along $\mathbb{H}_r$ (resp. $\mathbb{H}_{-r}$) in the case of the first (resp. second) integral. We're looking for a solution of the integral equation

$$e^{i\Theta_\gamma} = e^{i\theta_\gamma} \exp\left\langle \gamma, \frac{1}{4\pi \mathrm{i}}\left\{ \sum_{\gamma'>0} f^{\gamma'} \int_r K(\zeta,\zeta')\mathcal{X}^{\mathrm{sf}}_{\gamma'}(a,\zeta',\Theta) + \sum_{\gamma'<0} f^{\gamma'} \int_{-r} K(\zeta,\zeta')\mathcal{X}^{\mathrm{sf}}_{\gamma'}(a,\zeta',\Theta)\right\}\right\rangle. \tag{5.12}$$

The solution is obtained through iterations. By (5.4), we can write (5.10) equivalently as:

$$\Theta^{(0)}(\zeta,\theta) = \theta, \tag{5.13}$$

$$e^{i\Theta^{(\nu+1)}_\gamma} = e^{i\theta_\gamma} \exp\left\langle \gamma, \frac{1}{4\pi \mathrm{i}}\left\{ \sum_{\gamma'>0} f^{\gamma'} \int_r K(\zeta,\zeta')\mathcal{X}^{\mathrm{sf}}_{\gamma'}(a,\zeta',\Theta^{(\nu)}) + \sum_{\gamma'<0} f^{\gamma'} \int_{-r} K(\zeta,\zeta')\mathcal{X}^{\mathrm{sf}}_{\gamma'}(a,\zeta',\Theta^{(\nu)})\right\}\right\rangle. \tag{5.14}$$

Recall that we are denoting by θ the map $\mathbb{P}^1 \times \mathbb{T} \to \mathbb{C}^{2n}$, which is just the inclusion $\mathbb{T} \hookrightarrow \mathbb{C}^{2n}$ for all $\zeta \in \mathbb{P}^1$. We need to show that $\Theta^{(\nu)}$ converges uniformly in a to a well-defined $\Theta : \mathcal{M}_a \times \mathbb{P}^1 \to \mathcal{M}_a^{\mathbb{C}}$ as $\nu \to \infty$ with the right smooth properties on a and ζ. The following proof will work for the general case, so from now on assume (5.13) and (5.14) hold for arbitrary coefficient vectors $f^{\gamma'}$. Since $\mathcal{M}'$ is trivial in each U_α, we can identify all fibers $\mathcal{M}_a$ in U_α to $\mathbb{T} := (\mathbb{R}/2\pi\mathbb{Z})^{2n}$, where $2n$ is the rank of Γ in the general case. Define $\mathscr{X}$ as the completion of the space of functions $\Phi : \mathbb{P}^1 \times \mathbb{T} \times U_\alpha \to \mathbb{C}^{2n}$ that are smooth on $\mathbb{T} \times U_\alpha$ and bounded in $\mathbb{P}^1 \times \mathbb{T} \times U_\alpha$, under the norm

$$\|\Phi\| = \sup_{\zeta,\theta,a} \|\Phi(\zeta,\theta)\|_{\mathbb{C}^{2n}}, \tag{5.15}$$

where $\mathbb{C}^{2n}$ is assumed to have as norm the maximum of the Euclidean norm of its coordinates. Notice that we have not put any restriction of Φ in the $\mathbb{P}^1$ slice, except that it is bounded. Our strategy will be to solve the Riemann-Hilbert problem in $\mathscr{X}$ and show that for sufficiently big (but finite) R, we can get uniform estimates on the iterations yielding such solutions and any derivative with respect to the parameters a, θ. The Arzela-Ascoli theorem will give us that the solution Φ not only lies in $\mathscr{X}$, but it preserves all the smooth properties. The very nature of the integral equation will guarantee that its solution is piecewise holomorphic on ζ, as desired.

We're assuming as in [7] that Γ has a positive definite norm satisfying the Cauchy-Schwarz property

$$|\langle \gamma, \gamma' \rangle| \leq \|\gamma\| \, \|\gamma'\|$$

as well as the "Support property"

$$\|\gamma\| < \text{const}|Z_\gamma|, \tag{5.16}$$

for all γ such that $\Omega(\gamma; a) \neq 0$. For any $\Phi \in \mathscr{X}$, let Φ_j denote the composition of Φ with the jth projection $\pi_j : \mathbb{C}^{2n} \to \mathbb{C}, j = 1, \ldots, 2n$. Instead of working with the full Banach space $\mathscr{X}$, let $\mathscr{X}^*$ be the collection of $\Phi \in \mathscr{X}$ in the closed ball

$$\|\Phi - \theta\| \leq \epsilon, \tag{5.17}$$

for an $\epsilon > 0$ so small that

$$\sup_{\zeta,\theta,a} \left| e^{i\Phi_j} \right| \leq 2, \tag{5.18}$$

for $j = 1, \ldots, 2n$. In particular, $\mathscr{X}^*$ is closed, hence complete. Note that by (5.18), if $\Phi \in \mathscr{X}^*$, then $e^{i\Phi} \in \mathscr{X}$.

Now we can state the general version of (5.14). Define:

$$\Theta^{(0)}(\zeta, \theta) = \theta, \tag{5.19}$$

$$\begin{aligned} e^{i\Theta_\gamma^{(\nu+1)}} = e^{i\theta_\gamma} \exp \Bigg\langle \gamma, \frac{1}{4\pi \mathrm{i}} \Bigg\{ & \sum_{\gamma'>0} f^{\gamma'} \int_r K(\zeta,\zeta') \mathcal{X}^{\mathrm{sf}}_{\gamma'}(a, \zeta', \Theta^{(\nu)}) + \\ & \sum_{\gamma'<0} f^{\gamma'} \int_{-r} K(\zeta,\zeta') \mathcal{X}^{\mathrm{sf}}_{\gamma'}(a, \zeta', \Theta^{(\nu)}) \Bigg\} \Bigg\rangle . \end{aligned} \tag{5.20}$$

Observe that, by construction, the transformation in ζ is only as an integral transformation, so $\Theta^{(\nu)}$ is holomorphic in either of the half planes $\mathbb{H}_r$ or $\mathbb{H}_{-r}$. We will prove the first of our uniform estimates on $\Theta^{(\nu)}$:

Lemma 5.1.1. $\Theta^{(\nu)} \in \mathscr{X}^*$ *for all* ν.

Proof. We follow [7], using induction on ν. The statement is clearly true for $\nu = 0$ by (5.19). Assuming $\Theta^{(\nu)} \in \mathscr{X}^*$, take the log in both sides of (5.20):

$$\Theta_\gamma^{(\nu+1)} - \theta_\gamma = -\frac{1}{4\pi}\left\langle \gamma,\ \sum_{\gamma'>0} f^{\gamma'} \int_r K(\zeta,\zeta')\mathcal{X}_{\gamma'}^{\mathrm{sf}}(a,\zeta',\Theta^{(\nu)}) + \sum_{\gamma'<0} f^{\gamma'} \int_{-r} K(\zeta,\zeta')\mathcal{X}_{\gamma'}^{\mathrm{sf}}(a,\zeta',\Theta^{(\nu)}) \right\rangle, \tag{5.21}$$

where γ is one of the basis vectors $\{\gamma_1, \ldots, \gamma_{2n}\}$. For general $\Phi \in \mathscr{X}^*$, Φ can be very badly behaved in the $\mathbb{P}^1$ slice, but by our inductive construction, $\Theta^{(\nu+1)}$ is even holomorphic in $\mathbb{H}_r$ and $\mathbb{H}_{-r}$. Consider the integral

$$\int_r K(\zeta,\zeta')\mathcal{X}_{\gamma'}^{\mathrm{sf}}(a,\zeta',\Theta^{(\nu)}) \tag{5.22}$$

The function $\Theta^{(\nu)}$ can be analytically extended along the ray r so that it is holomorphic on the sector V bounded by r and $\ell_{\gamma'}, \gamma' > 0$ (see Figure 5.7). By Cauchy's theorem, we can move (5.22) to one along the ray $\ell_{\gamma'}$, possibly at the expense of a residue of the form

$$4\pi i \exp\left[i\Theta_{\gamma'}^{(\nu)} + \pi R\left(\frac{Z_{\gamma'}}{\zeta} + \overline{Z_{\gamma'}}\zeta \right)\right] \tag{5.23}$$

if ζ lies in V. This residue is in control. Indeed, by the induction hypothesis, $\left| e^{i\Theta_{\gamma'}^{(\nu)}} \right| < 2^{\|\gamma'\|}$, independent of ν. Moreover, we pick a residue only if ζ lies in the sector S bounded by the first and last $\ell_{\gamma_j}, \gamma_j \in \{\gamma_1, \ldots, \gamma_{2n}\}$ included in $\mathbb{H}_r$ traveling in the counterclockwise direction, regardless if they are BPS rays or not. This sector is strictly smaller than $\mathbb{H}_r$ (see Figure 5.8), so $\arg Z_{\gamma'} - \arg \zeta \in (-\pi, \pi)$ and, since r makes an acute angle with all rays $\ell_{\gamma'}, \gamma' > 0$:

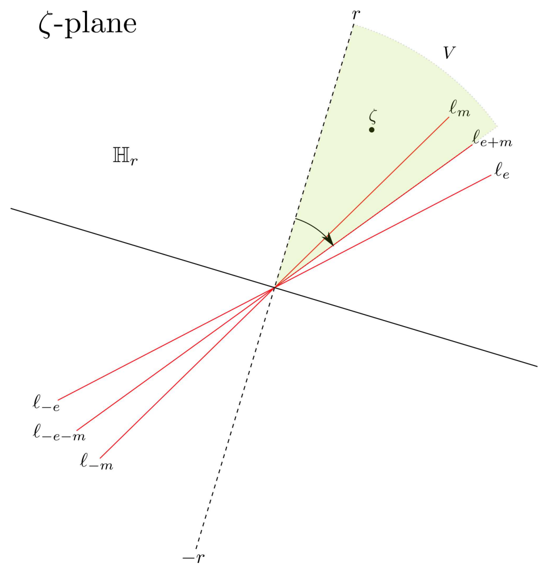

Figure 5.7: Translating the integral to the ray $\ell_{\gamma'}$

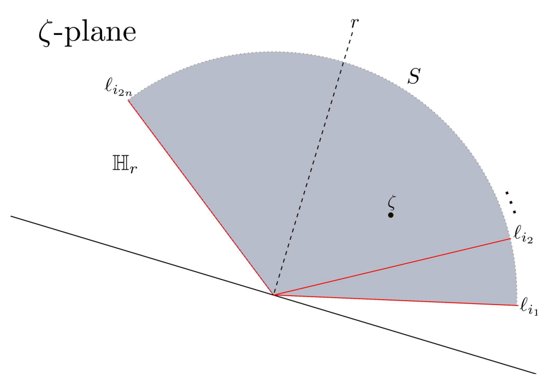

Figure 5.8: A residue appears only if ζ lies in S

$$|\arg Z_{\gamma'} - \arg\zeta| > \text{const} > \frac{\pi}{2} \qquad \text{for all } \gamma' > 0, \zeta \in S.$$

In particular,

$$\cos(\arg Z_{\gamma'} - \arg\zeta) < -\text{const} < 0 \qquad \text{for all } \gamma' > 0, \zeta \in S. \tag{5.24}$$

Using the fact that $\inf(|\zeta| + 1/|\zeta|) = 2$, the sum of residues of the form (5.23) is bounded by:

$$\sum_{\gamma'>0} \left| \left\langle \gamma, f^{\gamma'} \right\rangle \right| 2^{\|\gamma'\|} e^{-\text{const} R |Z_{\gamma'}|} \tag{5.25}$$

Recall that $\|\gamma'\| < \text{const}|Z_{\gamma'}|$, so (5.25) can be simplified to

$$\sum_{\gamma'>0} \left| \left\langle \gamma, f^{\gamma'} \right\rangle \right| e^{(-\text{const} R + \delta)|Z_{\gamma'}|} \tag{5.26}$$

for a constant δ. We're assuming that the $\Omega(\gamma'; a)$ do not grow too quickly with γ' as in [7], so the above sum can be made arbitrarily small if R is big enough. This bound can be chosen to be independent of ν, ζ and the basis element γ (by choosing the maximum among the $\gamma_1, \ldots, \gamma_{2n}$). The exact same argument can be used to show that the residues of the integrals along $-r$ are in control. In fact, let $\epsilon > 0$ be given. Choose $R > 0$ so that the total sum of residues $\text{Res}(\zeta)$ is less than $\epsilon/2$.

Thus, we can assume the integrals are along $\ell_{\gamma'}$ and consider

$$\int_{\ell_{\gamma'}} K(\zeta, \zeta') \mathcal{X}^{\text{sf}}_{\gamma'}(a, \zeta', \Theta^{(\nu)}) \tag{5.27}$$

The next step is to do a saddle point analysis and obtain the asymptotics for large R, following [16]. The integral 5.27 is of the type

$$h(R) = \int_{\ell_{\gamma'}} g(\zeta') e^{\pi R f(\zeta')} \tag{5.28}$$

where

$$g(\zeta') = \frac{\zeta' + \zeta}{\zeta'(\zeta' - \zeta)}, \qquad f(\zeta') = \frac{Z_{\gamma'}}{\zeta'} + \zeta' \overline{Z_{\gamma'}}.$$

The function f has a saddle point $\zeta_0 = -e^{i \arg Z_{\gamma'}}$ at the intersection of the BPS ray $\ell_{\gamma'}$ with the unit circle. Moreover, $f(\zeta_0) = -2|Z_{\gamma'}|$. The ray $\ell_{\gamma'}$ and the unit circle are the locus of $\mathrm{Im}\ f(\zeta') = \mathrm{Im}\ f(\zeta_0) = 0$. It's easy to see that in $\ell_{\gamma'}$ $f(\zeta') < f(\zeta_0)$ if $\zeta' \neq \zeta_0$, so $\ell_{\gamma'}$ is the path of steepest descent (see Figure 5.9).

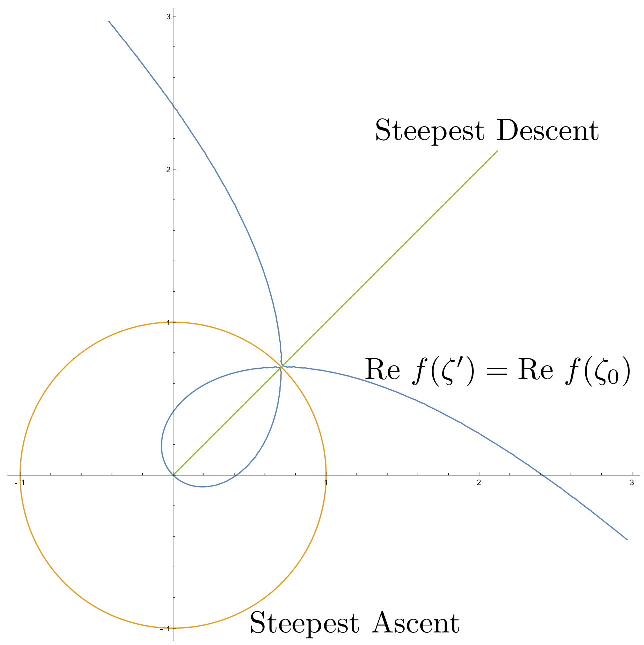

Figure 5.9: Paths of steepest descent and ascent

Introduce τ by

$$\frac{1}{2}(\zeta' - \zeta_0)^2 f''(\zeta_0) + O((\zeta' - \zeta_0)^3) = -\tau^2$$

and so

$$\zeta' - \zeta_0 = \left\{ \frac{-2}{f''(\zeta_0)} \right\}^{\frac{1}{2}} \tau + O(\tau^2) \tag{5.29}$$

for an appropriate branch of $\{f''(\zeta_0)\}^{1/2}$. Let $\alpha = \arg f''(\zeta_0) = -2 \arg Z_{\gamma'} + \pi$. The branch of $\{f''(\zeta_0)\}^{1/2}$ is chosen so that $\tau > 0$ in the part of the steepest descent path outside the unit disk in Figure 5.9. That is, $\tau > 0$ when $\arg(\zeta' - \zeta_0) = \frac{1}{2}\pi - \frac{1}{2}\alpha$, and so $\{f''(\zeta_0)\}^{1/2} = i\sqrt{2|Z_{\gamma'}|}e^{-i \arg Z_{\gamma'}}$. Thus (5.29) simplifies to

$$\zeta' - \zeta_0 = \frac{-\zeta_0}{\sqrt{|Z_{\gamma'}|}} \tau + O(\tau^2)$$

We expand $g(\zeta'(\tau))$ as a power series[1]:

$$g(\zeta'(\tau)) = g(\zeta_0) + g'(\zeta_0) \left\{ \frac{-2}{f''(\zeta_0)} \right\}^{\frac{1}{2}} \tau + O(\tau^2) \tag{5.30}$$

As in [16],

$$h(R) \sim e^{Rf(\zeta_0)} g(\zeta_0) \left\{ \frac{-2}{f''(\zeta_0)} \right\}^{\frac{1}{2}} \int_{-\infty}^{\infty} e^{-R\tau^2} d\tau + \dots$$

and so

$$h(R) = \sqrt{\frac{2\pi}{R|f''(\zeta_0)|}} g(\zeta_0) e^{Rf(\zeta_0) + (i/2)(\pi - \alpha)} + O\left(\frac{e^{Rf(\zeta_0)}}{R} \right)$$

[1]In our case, g depends also on the parameter R, so this is an expansion on ζ'

in our case, and since $\zeta_0 = -e^{i \arg Z_{\gamma'}}$

$$= -\frac{\zeta_0 + \zeta}{\zeta_0 - \zeta} \exp\left(i\Theta^{(\nu)}(\zeta_0)\right) \frac{1}{\sqrt{R|Z_{\gamma'}|}} e^{-2\pi R|Z_{\gamma'}|} + O\left(\frac{e^{-2\pi R|Z_{\gamma'}|}}{R}\right) \quad (5.31)$$

By (5.18), $\left|\exp\left(i\Theta^{(\nu)}_{\gamma'}(\zeta_0)\right)\right| \leq 2^{\|\gamma'\|}$. Thus, for ζ bounded away from the saddle ζ_0, we can bound the contribution from the integral by

$$\text{const } \left|\left\langle \gamma, f^{\gamma'} \right\rangle\right| 2^{\|\gamma'\|} \frac{e^{-2\pi R|Z_{\gamma'}|}}{\sqrt{R|Z_{\gamma'}|}} \quad (5.32)$$

if R is big enough.

If $\zeta \to \zeta_0$, we take a different path of integration, consisting of 3 parts ℓ_1, ℓ_2, ℓ_3 (see Figure 5.10).

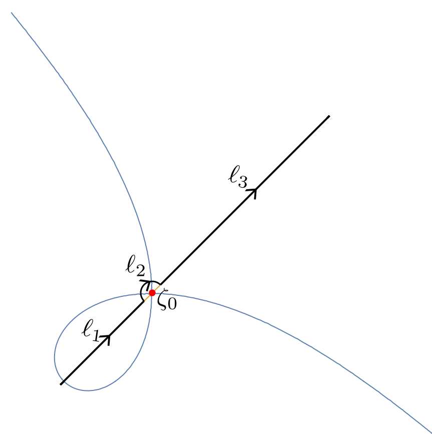

Figure 5.10: If $\zeta \to \zeta_0$, a modification of the path is required

If we parametrize the $\ell_{\gamma'}$ ray as $\zeta' = -e^{t+i\arg Z_{\gamma'}} = -e^t\zeta_0, -\infty < t < \infty$, the ℓ_2 part is a semicircle around $t = -\epsilon$ and $t = \epsilon$, for small ϵ. The contribution from ℓ_2 is clearly (up to a factor of $2\pi i$) half of the residue of the function in (5.27). As in (5.25), this residue is:

$$2\pi i \exp\left(i\Theta^{(\nu)}(\zeta_0) - 2\pi R|Z_{\gamma'}|\right). \tag{5.33}$$

If we denote by $\exp\left(i\Theta^{(\nu)}(t)\right)$ the evaluation $\exp\left(i\Theta^{(\nu)}(-t\zeta_0)\right)$, the contributions from ℓ_1 and ℓ_3 in the integral are of the form

$$\begin{aligned}\lim_{\epsilon\to 0}\Bigg\{ &\int_{-\infty}^{-\epsilon} dt \frac{-e^t+1}{-e^t-1} \exp\left(i\Theta^{(\nu)}(t)\right) \exp\left(\pi R(e^t+e^{-t})\right) \\ &+ \int_{\epsilon}^{\infty} dt \frac{-e^t+1}{-e^t-1} \exp\left(i\Theta^{(\nu)}(t)\right) \exp\left(\pi R(e^t+e^{-t})\right)\Bigg\}\end{aligned} \tag{5.34}$$

If we do the change of variables $t \mapsto -t$ in the first integral, (5.34) simplifies to

$$\int_0^\infty dt \frac{-e^t+1}{-e^t-1}\left[\exp\left(i\Theta^{(\nu)}(t)\right) - \exp\left(i\Theta^{(\nu)}(-t)\right)\right] \exp\left(\pi R(e^t+e^{-t})\right) \tag{5.35}$$

(5.35) is of the type (5.28), with

$$g(\zeta') = \frac{\zeta'+\zeta_0}{\zeta'(\zeta'-\zeta_0)}\left[\exp\left(i\Theta^{(\nu)}(\zeta')\right) - \exp\left(i\Theta^{(\nu)}(1/\zeta')\right)\right]$$

Since $\zeta_0 = 1/\zeta_0$, the apparent pole at ζ_0 of $g(\zeta')$ is removable and the integral can be estimated by the same steepest descent methods as in (5.27). The only difference is that the saddlepoint now lies at one of the endpoints. This only introduces a factor of $1/2$ in the estimates (see [16]). If $g(\zeta_0) \neq 0$ in this case, the integral is just

$$\frac{g(\zeta_0)}{2\sqrt{R|Z_{\gamma'}|}} e^{-2\pi R|Z_{\gamma'}| + i \arg Z_{\gamma'}} + O\left(\frac{e^{-2\pi R|Z_{\gamma'}|}}{R}\right) \tag{5.36}$$

If $g(\zeta_0) = 0$, then the estimate is at least of the order $O\left(\frac{e^{-2\pi R|Z_{\gamma'}|}}{R}\right)$. In any case, since $\exp\left(i\Theta_{\gamma'}^{(\nu)}(\zeta_0)\right) \leq 2^{\|\gamma'\|}$ by (5.18) and by (5.32), (5.33) and (5.36),

$$\left|\sum_{\gamma'} \left\langle \gamma, f^{\gamma'} \right\rangle \int_{\ell_{\gamma'}} K(\zeta, \zeta') \mathcal{X}_{\gamma'}^{\text{sf}}(a, \zeta', \Theta^{(\nu)})\right| < \text{const} \sum_{\gamma'} \left|\left\langle \gamma, f^{\gamma'} \right\rangle\right| e^{(-2\pi R + \delta)|Z_{\gamma'}|}. \tag{5.37}$$

The δ constant is the same appearing in (5.26). This sum is convergent by the tameness condition on the $\Omega(\gamma'; a)$ coefficients, and can be made arbitrarily small if R is big enough. Putting everything together:

$$\begin{aligned}\sup_{\zeta,\theta} \left|\Theta_{\gamma}^{(\nu+1)} - \theta_{\gamma}\right| &= \text{const} \sum_{\gamma'} \left|\left\langle \gamma, f^{\gamma'} \right\rangle\right| e^{(-2\pi R + \delta)|Z_{\gamma'}|} + \text{Res}(\zeta) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

Therefore $\left\|\Theta^{(\nu+1)} - \theta\right\| < \epsilon$. In particular, $\left\|\Theta^{(\nu+1)}\right\| < \infty$, so $\Theta^{(\nu+1)} \in \mathscr{X}^*$. Since ϵ was arbitrary, $\Theta^{(\nu+1)}$ satisfies the side condition (5.18) and thus $\Theta^{(\nu)} \in \mathscr{X}^*$ for all ν if R is big enough. $\square$

Now let $\beta = (\beta_1, \dots, \beta_{2n}, \beta_{2n+1}, \beta_{2n+2})$ be a multi-index in $\mathbb{N}^{2n+2}$, and let D^β be a differential operator acting on the iterations $\Theta^{(\nu)}$:

$$D^\beta \Theta_\gamma^{(\nu)} = \frac{\partial}{\partial \theta_1^{\beta_1} \cdots \theta_{2n}^{\beta_{2n}} \partial a^{\beta_{2n+1}} \partial \overline{a}^{\beta_{2n+2}}} \Theta_\gamma^{(\nu)} \tag{5.38}$$

We need to uniformly bound the partial derivatives of $\Theta^{(\nu)}$ on compact subsets:

Lemma 5.1.2. *Let K be a compact subset of $U_\alpha \times \mathbb{T}$. Then*

$$\sup_{\mathbb{P}^1 \times K} \left\| D^\beta \Theta^{(\nu)} \right\| < C_{\beta,K}$$

for a constant $C_{\beta,K}$ independent of ν.

Proof. Lemma 5.1.1 is the case $|\beta| := \sum \beta_i = 0$, with ϵ as $C_{0,K}$. To simplify notation, we'll drop the K subindex in these constants. Assume by induction we already did this for $|\beta| = k-1$ derivatives and for the first $\nu \geq 0$ iterations, the case $\nu = 0$ being trivial. Take partial derivatives with respect to θ_s, for γ_s one of the basis elements of Γ in (5.21). This introduces a factor of the form

$$i \frac{\partial}{\partial \theta_s} \Theta^{(\nu)}_{\gamma'} \tag{5.39}$$

By (5.17), (5.18) and since no γ' appearing in the integrals for Θ_γ is a multiple of γ, the above can be bounded by $\|\gamma'\| C_0$, $C_0 < 1$. When we take the partial derivatives with respect to a in (5.21), we add a factor of

$$\frac{\pi R}{\zeta'} \frac{\partial}{\partial a} Z_{\gamma'}(a) + i \frac{\partial}{\partial a} \Theta^{(\nu)}_{\gamma'} \tag{5.40}$$

in the integrals (5.22). Similarly, a partial derivative with respect to $\overline{a}$ adds a factor of

$$\pi R \zeta' \frac{\partial}{\partial \overline{a}} \overline{Z_{\gamma'}(a)} + i \frac{\partial}{\partial \overline{a}} \Theta^{(\nu)}_{\gamma'} \tag{5.41}$$

As for (5.39), the second term in (5.40), (5.41) can be bounded by $\|\gamma'\| C_0$. Since $Z_{\gamma'}$ is holomorphic on $U_\alpha \subset \mathcal{B}'$, and since $K \subset U_\alpha \times \mathbb{T}$ is compact,

$$\left| \frac{\partial^k}{\partial a^k} Z_{\gamma'} \right| \leq k! \, \|\gamma'\| \, C \tag{5.42}$$

for all k and some constant C, independent of k and a. Likewise for $\overline{a}$, $\overline{Z'_\gamma}$. Thus if we take $D^\beta \Theta_\gamma^{(\nu+1)}$ in (5.21) for a multi-index β with $|\beta| = k$, the right side of (5.21) becomes:

$$-\frac{1}{4\pi}\left\langle \gamma,\ \sum_{\gamma'>0} f^{\gamma'} \int_r K(\zeta,\zeta')\mathcal{X}^{\mathrm{sf}}_{\gamma'}(a,\zeta',\Theta^{(\nu)})P_{\gamma'}(a,\zeta',\theta)+ \right.$$
$$\left. \sum_{\gamma'<0} f^{\gamma'} \int_{-r} K(\zeta,\zeta')\mathcal{X}^{\mathrm{sf}}_{\gamma'}(a,\zeta',\Theta^{(\nu)})Q_{\gamma'}(a,\zeta',\theta) \right\rangle, \tag{5.43}$$

where each $P_{\gamma'}$ or $Q_{\gamma'}$ is a polynomial obtained as follows:

Each $\mathcal{X}^{\mathrm{sf}}_{\gamma'}(a,\zeta',\Theta^{(\nu)})$ is a function of the type e^g, for some $g(a,\bar{a},\theta_1,\ldots,\theta_{2r})$. If $\{x_1,\ldots,x_k\}$ denotes a choice of k of the variables $a,\bar{a},\theta_1,\ldots,\theta_{2r}$ (possibly with multiplicities), then by the Faà di Bruno Formula:

$$\frac{\partial^k}{\partial x_1\cdots\partial x_k} e^g = e^g \sum_{\pi\in\Pi}\prod_{B\in\pi}\frac{\partial^{|B|}g}{\prod_{j\in B}\partial x_j} := e^g P_{\gamma'} \tag{5.44}$$

where

- π runs through the set Π of all partitions of the set $\{1,\ldots,k\}$.
- $B\in\pi$ means the variable B runs through the list of all of the "blocks" of the partition π, and
- $|B|$ is the size of the block B.

The resulting monomials in $P_{\gamma'}$ (same thing holds for $Q_{\gamma'}$) are products of the variables given by (5.39), (5.40), (5.41) or their subsequent partial derivatives in $\theta, a, \overline{a}$. For each monomial, the sum of powers and total derivatives of terms must add up to k by (5.44). For instance, when computing

$$\frac{\partial^3}{\partial\theta_1 \partial a^2}\mathcal{X}^{\mathrm{sf}}_{\gamma'}(a, \zeta', \Theta^{(\nu)}) = \frac{\partial^3}{\partial\theta_1 \partial a^2}e^{g},$$

a monomial that appears in the expansion is:

$$\frac{\partial g}{\partial\theta_1}\left[\frac{\partial g}{\partial a}\right]^2 = i\frac{\partial}{\partial\theta_1}\Theta^{(\nu)}_{\gamma'}\left[\frac{\pi R}{\zeta'}\frac{\partial}{\partial a}Z_{\gamma'}(a) + i\frac{\partial}{\partial a}\Theta^{(\nu)}_{\gamma'}\right]^2$$

There are a total of (possibly repeated) B_k monomials in $P_{\gamma'}$, where B_k is the Bell number, the total number of partitions of the set $\{1, \ldots, k\}$ and $B_k \leq k!$. We can assume, without loss of generality, that any constant C_β is considerably larger than any of the $C_{\beta'}$ with $|\beta'| < |\beta|$, by a factor that will be made explicit. First notice that since there is only partition of $\{1, \ldots, k\}$ consisting of 1 block, the Faà di Bruno Formula (5.44) shows that $P_{\gamma'}$ contains only one monomial with the factor $D^{\beta}\Theta^{(\nu)}$. The other monomials have factors $D^{\beta'}\Theta^{(\nu)}$ for $|\beta'| < |\beta|$. We can do a saddle point analysis for each integrand of the form

$$\int_r K(\zeta, \zeta')\mathcal{X}^{\mathrm{sf}}_{\gamma'}(a, \zeta', \Theta^{(\nu)})P^{i}_{\gamma'}(a, \zeta', \theta),$$

for $P^i_{\gamma'}$ (or $Q^i_{\gamma'}$) one of the monomials of $P_{\gamma'}$ ($Q_{\gamma'}$). The saddle point analysis and the induction step for the previous $\Theta^{(\nu)}$ give the estimate

$$C_\beta \cdot \mathrm{const}\sum_{\gamma'}\left|\left\langle \gamma, f^{\gamma'}\right\rangle\right| e^{(-2\pi R+\delta)|Z_{\gamma'}|}$$

for the only monomial with $D^{\beta}\Theta^{(\nu)}$ on it. The estimates for the other monomials contain the same exponential decay term, along with powers s of $C_{\beta'}, C$

such that $s \cdot |\beta'| \leq |\beta|$, and constant terms. By making C_β significantly bigger than the previous $C_{\beta'}$, we can bound the entire (5.43) by C_β, completing the induction step

$\square$

5.1.1 Example: The case $|\beta| = 3$

To see better the estimates we obtained in the previous section, let's consider the particular case $k = |\beta| = 3$. In the Pentagon case we have to take derivatives with respect to 4 variables: $a, \bar{a}, \theta_e, \theta_m$. If $k = 3$, there are a total of $\binom{4+3-1}{3} = 20$ different third partial derivatives for each $\Theta^{(\nu+1)}$. There are a total of 5 different partitions of the set $\{1, 2, 3\}$ and correspondingly

$$
\begin{aligned}
&\frac{\partial^3}{\partial x_1 \partial x_2 \partial x_3} e^g = \\
&\qquad e^g \left[\frac{\partial^3}{\partial x_1 \partial x_2 \partial x_3} g + \left(\frac{\partial^2}{\partial x_1 \partial x_2} g \right) \left(\frac{\partial}{\partial x_3} g \right) + \left(\frac{\partial^2}{\partial x_1 \partial x_3} g \right) \left(\frac{\partial}{\partial x_2} g \right) \right. \\
&\qquad \left. + \left(\frac{\partial^2}{\partial x_2 \partial x_3} g \right) \left(\frac{\partial}{\partial x_1} g \right) + \left(\frac{\partial}{\partial x_1} g \right) \left(\frac{\partial}{\partial x_2} g \right) \left(\frac{\partial}{\partial x_3} g \right) \right]
\end{aligned}
$$

If $x_1 = x_2 = x_3 = a$,

$$
\begin{aligned}
\frac{\partial^3}{\partial a^3} \mathcal{X}^{\mathrm{sf}}_{\gamma'}(a, \zeta', \Theta^{(\nu)}) &= \mathcal{X}^{\mathrm{sf}}_{\gamma'}(a, \zeta', \Theta^{(\nu)}) \left[\frac{\pi R}{\zeta'} \frac{\partial^3}{\partial a^3} Z_{\gamma'} + i \frac{\partial^3}{\partial a^3} \Theta^{(\nu)}_{\gamma'} \right. \\
&\quad + 3 \left(\frac{\pi R}{\zeta'} \frac{\partial^2}{\partial a^2} Z_{\gamma'} + i \frac{\partial^2}{\partial a^2} \Theta^{(\nu)}_{\gamma'} \right) \left(\frac{\pi R}{\zeta'} \frac{\partial}{\partial a} Z_{\gamma'} + i \frac{\partial}{\partial a} \Theta^{(\nu)}_{\gamma'} \right) \\
&\quad \left. + \left(\frac{\pi R}{\zeta'} \frac{\partial}{\partial a} Z_{\gamma'} + i \frac{\partial}{\partial a} \Theta^{(\nu)}_{\gamma'} \right)^3 \right] \\
&= \mathcal{X}^{\mathrm{sf}}_{\gamma'}(a, \zeta', \Theta^{(\nu)}) P(\Theta^{(\nu)}_{\gamma'})
\end{aligned}
$$

There is one and only one term containing $\frac{\partial^3}{\partial a^3}\Theta^{(\nu)}_{\gamma'}$. By induction on ν, $|\frac{\partial^3}{\partial a^3}\Theta^{(\nu)}_{\gamma'}| < C_\beta$. For the estimates of

$$i\left\langle \gamma, f^{\gamma'}\right\rangle \int_r K(\zeta,\zeta')\mathcal{X}^{\mathrm{sf}}_{\gamma'}(a,\zeta',\Theta^{(\nu)})\frac{\partial^3}{\partial a^3}\Theta^{(\nu)}_{\gamma'},$$

we do exactly the same as in the proof of Lemma 5.1.1. Namely, move the ray r to the corresponding BPS ray $\ell_{\gamma'}$, possibly at the expense of gaining a residue bounded by

$$C_\beta \cdot \mathrm{const}\left|\left\langle \gamma, f^{\gamma'}\right\rangle\right| e^{(-2\pi R+\delta)|Z_{\gamma'}|} \tag{5.45}$$

The sum of all these residues over those γ' such that $\langle\gamma,\gamma'\rangle \neq 0$ is just a fraction of C_β. After moving the contour we estimate

$$i\left\langle \gamma, f^{\gamma'}\right\rangle \int_{\ell_{\gamma'}} K(\zeta,\zeta')\mathcal{X}^{\mathrm{sf}}_{\gamma'}(a,\zeta',\Theta^{(\nu)})\frac{\partial^3}{\partial a^3}\Theta^{(\nu)}_{\gamma'}.$$

As in (5.37), we run a saddle point analysis and obtain a similar estimate (5.45) as in Lemma 5.1.1. The result is that the estimate for this monomial is an arbitrarily small fraction of C_β.

If we take other monomials, like say

$$P^1_{\gamma'} = 3\left(\frac{\pi R}{\zeta'}\right)^2 \frac{\partial^2}{\partial a^2}Z_{\gamma'}\frac{\partial}{\partial a}Z_{\gamma'}$$

and estimate

$$3\left\langle \gamma, f^{\gamma'}\right\rangle \frac{\partial^2}{\partial a^2}Z_{\gamma'}\frac{\partial}{\partial a}Z_{\gamma'}\int_r \left(\frac{\pi R}{\zeta'}\right)^2 K(\zeta,\zeta')\mathcal{X}^{\mathrm{sf}}_{\gamma'}(a,\zeta',\Theta^{(\nu)}),$$

we do as before, computing residues and doing saddle point analysis. The difference with these terms is that partial derivatives of $Z_{\gamma'}$ are bounded by

(5.42), and at most second derivatives of $\Theta^{(\nu)}$ (for this specific monomial, no such terms appear). The extra powers of $\frac{\pi R}{\zeta'}$ that appear like here don't affect the estimates, since $\mathcal{X}^{\text{sf}}_{\gamma'}$ has exponential decay on $\frac{\pi R}{\zeta'}$. The end result is an estimate of the type

$$C^{s_1}_{\beta'_1} \cdots C^{s_m}_{\beta'_m} C^j \cdot \text{const} \left| \left\langle \gamma, f^{\gamma'} \right\rangle \right| e^{(-2\pi R + \delta)|Z_{\gamma'}|} \tag{5.46}$$

with all $s_i \cdot |\beta'_i|, j \leq |\beta|$. By induction on $|\beta|$, we can make C_β big enough so that (5.46) are just a small fraction of C_β. This completes the proof that $\sup |D^\beta \Theta^{(\nu+1)}| < C_\beta$ on the compact set K.

Now we're ready to prove one of our main theorems in this paper.

Theorem 5.1.3. *The sequence $\{\Theta^{(\nu)}\}$ converges in $\mathscr{X}$. Moreover, its limit Θ is piecewise holomorphic on ζ with jumps along the rays $r, -r$ and continuous on the closed half-planes determined by these rays. Θ is C^∞ on $a, \overline{a}, \theta_1, \ldots, \theta_{2n}$.*

Proof. We first show the contraction of the $\Theta^{(\nu)}$ in the Banach space $\mathscr{X}$ thus proving convergence. We will use the fact that e^x is locally Lipschitz and the $\Theta^{(\nu)}$ are arbitrarily close to θ if R is big. In particular,

$$\sup_{\zeta,\theta,a} \left| e^{i\Theta^{(\nu)}_\gamma} - e^{i\Theta^{(\nu-1)}_\gamma} \right| < \text{const} \cdot \sup_{\zeta,\theta,a} \left| \Theta^{(\nu)}_\gamma - \Theta^{(\nu-1)}_\gamma \right| \leq \text{const} \left\| \Theta^{(\nu)} - \Theta^{(\nu-1)} \right\|,$$

for γ one of the basis elements $\gamma_1, \ldots, \gamma_{2n}$. For arbitrary γ', recall that if $\gamma' = c_1\gamma_1 + \ldots + c_n\gamma_{2n}$, then $\Theta^{(\nu)}_{\gamma'} = c_1\Theta^{(\nu)}_{\gamma_1} + \ldots + c_{2n}\Theta^{(\nu)}_{\gamma_{2n}}$. It follows from the last inequality that

$$\sup_{\zeta,\theta} \left| e^{i\Theta^{(\nu)}_{\gamma'}} - e^{i\Theta^{(\nu-1)}_{\gamma'}} \right| < \text{const}^{\|\gamma'\|} \left\| \Theta^{(\nu)} - \Theta^{(\nu-1)} \right\| \tag{5.47}$$

We estimate

$$\begin{aligned}\left\|\Theta^{(\nu+1)} - \Theta^{(\nu)}\right\| =& \frac{1}{4\pi}\left\|\sum_{\gamma'>0} f^{\gamma'}\int_r K(\zeta,\zeta')\left[\mathcal{X}^{\mathrm{sf}}_{\gamma'}(a,\zeta',\Theta^{(\nu)}) - \mathcal{X}^{\mathrm{sf}}_{\gamma'}(a,\zeta',\Theta^{(\nu-1)})\right]\right.\\ &\left.+\sum_{\gamma'<0} f^{\gamma'}\int_{-r} K(\zeta,\zeta')\left[\mathcal{X}^{\mathrm{sf}}_{\gamma'}(a,\zeta',\Theta^{(\nu)}) - \mathcal{X}^{\mathrm{sf}}_{\gamma'}(a,\zeta',\Theta^{(\nu-1)})\right]\right\|\\ &\leq \frac{1}{4\pi}\left\|\sum_{\gamma'>0} f^{\gamma'}\int_r K(\zeta,\zeta')\left|\mathcal{X}^{\mathrm{sf}}_{\gamma'}(a,\zeta',\theta)\right|\left|e^{i\Theta^{(\nu)}_{\gamma'}} - e^{i\Theta^{(\nu-1)}_{\gamma'}}\right|\right\|\\ &+\frac{1}{4\pi}\left\|\sum_{\gamma'<0} f^{\gamma'}\int_r K(\zeta,\zeta')\left|\mathcal{X}^{\mathrm{sf}}_{\gamma'}(a,\zeta',\theta)\right|\left|e^{i\Theta^{(\nu)}_{\gamma'}} - e^{i\Theta^{(\nu-1)}_{\gamma'}}\right|\right\|\end{aligned}$$

As in the proof of Lemma 5.1.1, we can move the integrals to the rays $\ell_{\gamma'}$ introducing an arbitrary small contribution from the residues. The differences of the form

$$\left|e^{i\Theta^{(\nu)}_{\gamma'}} - e^{i\Theta^{(\nu-1)}_{\gamma'}}\right|$$

can be expressed in terms of $\left\|\Theta^{(\nu)} - \Theta^{(\nu-1)}\right\|$ by (5.47).

The sum of the resulting integrals can be made arbitrarily small if R is big by a saddle point analysis as from (5.28) onwards. By (5.47):

$$\left\|\Theta^{(\nu+1)} - \Theta^{(\nu)}\right\| < \mathrm{const}\left\|\sum_{\gamma'} f^{\gamma'} e^{(-2\pi R+\delta)|Z_{\gamma'}|}\right\|\left\|\Theta^{(\nu)} - \Theta^{(\nu-1)}\right\|,$$

By making R big, we get the desired contraction in $\mathscr{X}$ and the convergence is proved.

The holomorphic properties of Θ on ζ are clear since Θ solves the integral equation (5.12) and the right side of it is piecewise holomorphic, regardless of the integrand.

Finally, by Lemma 5.1.2, $\{D^\beta \Theta^{(\nu)}\}$ is an equicontinuous and uniformly bounded family on compact sets K for any differential operator D^β as in (5.38). By Arzela-Ascoli, a subsequence converges uniformly and hence its limit is of type C^k for any k. Since we just showed that $\Theta^{(\nu)}$ converges, this has to be the limit of any subsequence. Thus such limit Θ must be of type C^∞ on $U_\alpha \times \mathbb{T}$, as claimed. $\square$

Remark 5.1.1. Our construction used integrals along a fixed admissible pair $r, -r$ and our Stokes factors are concatenation of the Stokes factors in [7]. Thus, the coefficients $f^{\gamma'}$ are different here, but they are still obtained by power series expansion of the explicit Stokes factor. In particular, it may not be possible to express

$$f^{\gamma'} = c_{\gamma'} \gamma'$$

for some constant $c_{\gamma'}$. For instance, in the pentagon, wall-crossing type I, we have, for $0 \le j \le i$ and $\gamma' = \gamma_{ie+jm}$:

$$f^{\gamma'} = \frac{(-1)^j \binom{i}{j}}{i^2} \gamma_{ie}.$$

Because of this, we didn't use the Cauchy-Schwarz property of the norm in Γ in the estimates above as in [7]. Nevertheless, the tameness condition on the $\Omega(\gamma', a)$ invariants still give us the desired contraction.

Observe that, since we used admissible rays, the Stokes matrices don't change at the walls of marginal stability and we were able to treat both sides of the wall indistinctly. Thus, the solution to (5.12) Θ is smooth across the wall.

Let's reintroduce the solutions in [7]. Denote by $\mathcal{X}^{(\nu)}$ the iterations in the Riemann-Hilbert problem defined in [7]. That is, $\mathcal{X}^{(\nu)} = \mathcal{X}^{\mathrm{sf}}(\Upsilon^{(\nu)})$, where $\Upsilon^{(0)} = \theta$ and

$$e^{i\Upsilon_\gamma^{(\nu+1)}} = e^{i\theta_\gamma} \exp\left(\frac{1}{4\pi \mathrm{i}} \sum_{\gamma'} c_{\gamma'} \langle \gamma, \gamma' \rangle \int_{\ell_{\gamma'}} K(\zeta, \zeta') \mathcal{X}^{\mathrm{sf}}_{\gamma'}(a, \zeta', \Upsilon^{(\nu)}) \right),$$

for

$$c_{\gamma'} = \sum_n \frac{\Omega(\gamma'/n; a)}{n^2}.$$

In a patch $U_\alpha \subset \mathcal{B}'$ containing the wall of marginal stability, define the admissible ray r as the ray where ℓ_e, ℓ_m (or ℓ_e, ℓ_{-m}) collide. By our proof of Lemma 5.1.1, $\mathcal{X}$ and $\mathcal{Y}$ differ only in a small sector in the ζ-plane bounded by the ℓ_e, ℓ_m (ℓ_e, ℓ_{-m}) rays, for a not in the wall. As a approaches the wall, such a sector converges to the single admissible ray r. Thus, away from the ray where the two BPS rays collide, the solutions $\mathcal{X}$ in [7] are continuous in a.

5.2 Extension to the bad fibers

At points $a \in D$ where the bad fibers are, the generic picture for a fixed value of ζ in $\mathcal{B}$ is a collection of rays $\{\ell_\gamma : \Omega(\gamma; u) \neq 0\}$. We will be working in the case $\dim_{\mathbb{C}} \mathcal{B} = 1$ and Γ is a rank-2 lattice. Consider the case of the Pentagon first. We can assume that the two bad fibers are at $-2, 2$ in the complex u-plane. For almost all $\zeta \in \mathbb{P}^1$, the BPS rays converge in a point of the wall of marginal stability away from any bad fiber:

It is assumed that $\lim_{u \to 2} Z_{\gamma_1}$ exists and it is nonzero. If we denote this

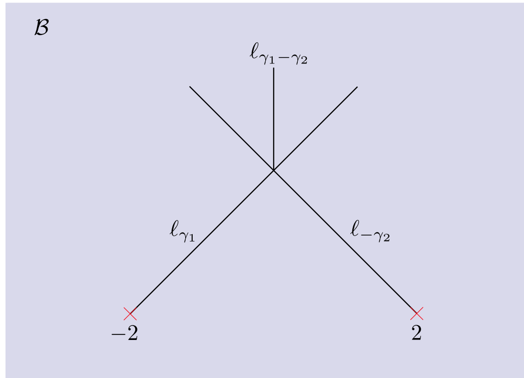

Figure 5.11: For general ζ, there is only 1 pair of rays at each fiber

limit by $c = |c|e^{i\phi}$, then for ζ such that $\arg\zeta \to \phi + \pi$, the ray ℓ_{γ_1} emerging from -2 approaches the other singular point $u = 2$ (see Figure 5.12).

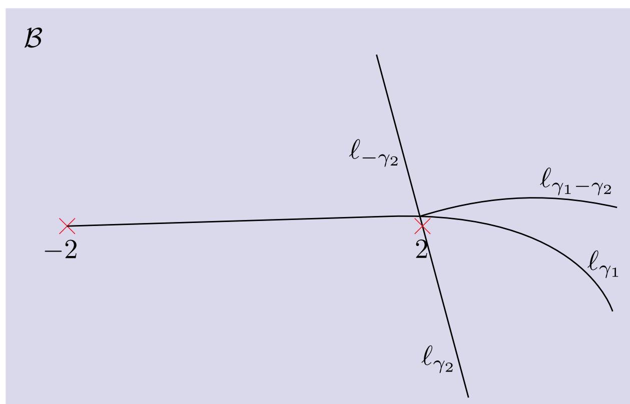

Figure 5.12: The BPS rays in $\mathcal{B}$ nearly coalesce at the singular locus

When $\arg\zeta = \phi + \pi$, the locus $\{u : Z_\gamma(u)/\zeta \in \mathbb{R}_-\}$, for some γ such that $\Omega(\gamma; u) \neq 0$ crosses $u = 2$. See Figure 5.13.

As ζ keeps changing, the rays leave the singular locus, but near $u = 2$, the tags change due to the monodromy of γ_1 around $u = 2$. Despite this

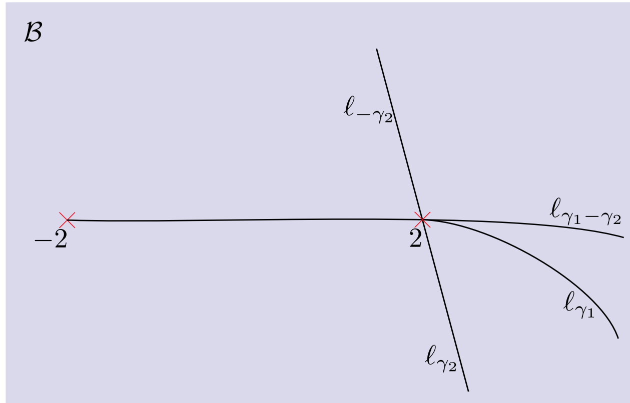

Figure 5.13: For ζ in a special ray, the rays intersect $u = 2$

change of labels, near $u = 2$ only the rays $\ell_{\gamma_2}, \ell_{-\gamma_2}$ pass through this singular point. See Figure 5.14

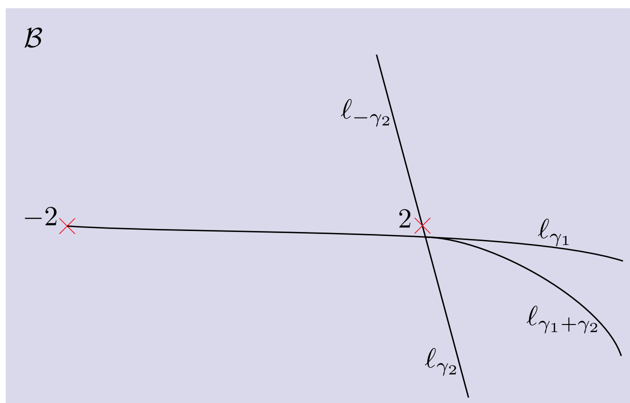

Figure 5.14: After the critical value of ζ, the rays leave $u = 2$ and their tags change

In the general case of Figures 5.11, 5.12 or 5.14, the picture near $u = 2$ is like in the Ooguri-Vafa case, Figure 3.2.

In any case, because of the specific values of the invariants Ω, it is possible to analytically extend the function $\mathcal{X}_{\gamma_1}$ around $u = 2$. The global jump coming from the rays $\ell_{\gamma_2}, \ell_{-\gamma_2}$ is the opposite of the global monodromy coming from the Picard-Lefschetz monodromy of $\gamma_1 \mapsto \gamma_1 - \gamma_2$ (see (2.3)). Thus, it is possible to obtain a function $\widetilde{\mathcal{X}}_{\gamma_1}$ analytic on a punctured disk on $\mathcal{B}'$ near $u = 2$ extending $\mathcal{X}_{\gamma_1}$.

From this point on, we use the original formulation of the Riemann-Hilbert problem using BPS rays as in [7]. We also use $a = Z_{\gamma_2}(u)$ to coordinatize a disk near $u = 2$, and we label $\{\gamma_1, \gamma_2\}$ as $\{\gamma_e, \gamma_m\}$ as in the Ooguri-Vafa case. Recall that, to shorten notation, we write $\ell_e, \mathcal{X}_e$, etc. instead of $\ell_{\gamma_e}, \mathcal{X}_{\gamma_e}$, etc.

By our work in the previous section, solutions $\mathcal{X}_\gamma$ (or, taking logs, Υ_γ) to the Riemann-Hilbert problem are continuous at the wall of marginal stability for all ζ except those in the ray $\ell_m = Z_m/\zeta \in \mathbb{R}_- = \ell_e$ (to be expected by the definition of the RH problem). We want to extend our solutions to the bad fiber located at $a = 0$. We'll see that to achieve this, it is necessary to introduce new θ coordinates.

For convenience, we rewrite the integral formulas for the Pentagon in terms of Υ as in [17]. We will only write the part in $\mathcal{B}_{\text{in}}$, the $\mathcal{B}_{\text{out}}$ part is similar.

$$\begin{aligned}\Upsilon_e(a, \zeta) = \theta_e - \frac{1}{4\pi}\Bigg\{\int_{\ell_m} \frac{d\zeta'}{\zeta'}\frac{\zeta' + \zeta}{\zeta' - \zeta} \log\left[1 - \mathcal{X}^{\text{sf}}_m(a, \zeta', \Upsilon_m)\right] \\ - \int_{\ell_{-m}} \frac{d\zeta'}{\zeta'}\frac{\zeta' + \zeta}{\zeta' - \zeta} \log\left[1 - \mathcal{X}^{\text{sf}}_{-m}(a, \zeta', \Upsilon_{-m})\right]\Bigg\},\end{aligned} \tag{5.48}$$

$$\Upsilon_m(a,\zeta) = \theta_m + \frac{1}{4\pi}\left\{\int_{\ell_e} \frac{d\zeta'}{\zeta'}\frac{\zeta'+\zeta}{\zeta'-\zeta}\log\left[1-\mathcal{X}_e^{\mathrm{sf}}(a,\zeta',\Upsilon_e)\right]\right.$$
$$\left. - \int_{\ell_{-e}} \frac{d\zeta'}{\zeta'}\frac{\zeta'+\zeta}{\zeta'-\zeta}\log\left[1-\mathcal{X}_{-e}^{\mathrm{sf}}(a,\zeta',\Upsilon_{-e})\right]\right\} \tag{5.49}$$

By doing the iteration method:

$$\Upsilon_e^{(\nu+1)}(a,\zeta) = \theta_e - \frac{1}{4\pi}\left\{\int_{\ell_m} \frac{d\zeta'}{\zeta'}\frac{\zeta'+\zeta}{\zeta'-\zeta}\log\left[1-\mathcal{X}_m^{\mathrm{sf}}(a,\zeta',\Upsilon_m^{(\nu)})\right]\right.$$
$$\left. - \int_{\ell_{-m}} \frac{d\zeta'}{\zeta'}\frac{\zeta'+\zeta}{\zeta'-\zeta}\log\left[1-\mathcal{X}_{-m}^{\mathrm{sf}}(a,\zeta',\Upsilon_{-m}^{(\nu)})\right]\right\}, \tag{5.50}$$
$$\Upsilon_m^{(\nu+1)}(a,\zeta) = \theta_m + \frac{1}{4\pi}\left\{\int_{\ell_e} \frac{d\zeta'}{\zeta'}\frac{\zeta'+\zeta}{\zeta'-\zeta}\log\left[1-\mathcal{X}_e^{\mathrm{sf}}(a,\zeta',\Upsilon_e^{(\nu)})\right]\right.$$
$$\left. - \int_{\ell_{-e}} \frac{d\zeta'}{\zeta'}\frac{\zeta'+\zeta}{\zeta'-\zeta}\log\left[1-\mathcal{X}_{-e}^{\mathrm{sf}}(a,\zeta',\Upsilon_{-e}^{(\nu)})\right]\right\} \tag{5.51}$$

We can focus only on the integrals above, so write $\Upsilon_\gamma^{(\nu)}(a,\zeta) = \theta_\gamma + \frac{1}{4\pi}\Phi_\gamma^{(\nu)}(a,\zeta)$, for $\gamma \in \{\gamma_e,\gamma_m\}$. To obtain the right gauge transformation of the torus coordinates θ, we'll split the integrals above into four parts and then we'll show that two of them define the right change of coordinates (in $\mathcal{B}_{\mathrm{in}}$, and a similar transformation for $\mathcal{B}_{\mathrm{out}}$) that simplify the integrals and allow an extension to the singular fiber.

As preparation, we need to check a "reality condition", which expresses a symmetry in the behavior of the complexified coordinates Υ.

Lemma 5.2.1. $\overline{\Upsilon_m^{(\nu)}(a,\zeta)} = \Upsilon_m^{(\nu)}\left(a,-1/\overline{\zeta}\right)$ *(resp.* $\Upsilon_e^{(\nu)}$*) for all* ν *and* $a \neq 0$*. In particular, letting* $\nu \to \infty$*, the same holds for the actual solution* Υ *discussed in the previous section.*

Proof. Let $\zeta = te^{i\varphi}$ not in any of the rays where Υ_m jumps. If $\nu = 0$, $\Upsilon^{(0)} = \theta$, real and independent of ζ, so the statement is true in this case. Assuming this result for ν, we show it for $\nu + 1$. It suffices to prove this for the corrections $\Phi^{(\nu+1)}$. For the proof only, parametrize ℓ_e by $se^{i\rho}$. For the magnetic corrections in the "inside" part of the wall of marginal stability we have:

$$
\begin{aligned}
\overline{\Phi_m^{(\nu+1)}(a,\zeta)} &= \overline{\Phi_m^{(\nu+1)}(a,te^{i\varphi})} \\
&= \overline{\int_{\ell_e} \frac{d\zeta'}{\zeta'}\frac{\zeta'+\zeta}{\zeta'-\zeta}\log\left[1-\mathcal{X}_e^{\mathrm{sf}}(a,\zeta',\Upsilon_e^{(\nu)})\right]} \\
&- \overline{\int_{\ell_{-e}} \frac{d\zeta'}{\zeta'}\frac{\zeta'+\zeta}{\zeta'-\zeta}\log\left[1-\mathcal{X}_{-e}^{\mathrm{sf}}(a,\zeta',\Upsilon_{-e}^{(\nu)})\right]} \\
&= \overline{\int_0^\infty \frac{ds}{s}\frac{se^{i\rho}+te^{i\varphi}}{se^{i\rho}-te^{i\varphi}}\log\left[1-\mathcal{X}_e^{\mathrm{sf}}(a,se^{i\rho},\Upsilon_e^{(\nu)}(a,se^{i\rho}))\right]} \\
&- \overline{\int_0^\infty \frac{ds}{s}\cdot\frac{-se^{i\rho}+te^{i\varphi}}{-se^{i\rho}-te^{i\varphi}}\log\left[1-\mathcal{X}_{-e}^{\mathrm{sf}}(a,-se^{i\rho},\Upsilon_{-e}^{(\nu)}(a,-se^{i\rho}))\right]} \\
&= \int_0^\infty \frac{ds}{s}\frac{se^{-i\rho}+te^{-i\varphi}}{se^{-i\rho}-te^{-i\varphi}}\log\left[1-\overline{\mathcal{X}_e^{\mathrm{sf}}(a,se^{i\rho},\Upsilon_e^{(\nu)}(a,se^{i\rho}))}\right] \\
&- \int_0^\infty \frac{ds}{s}\cdot\frac{-se^{-i\rho}+te^{-i\varphi}}{-se^{-i\rho}-te^{-i\varphi}}\log\left[1-\overline{\mathcal{X}_{-e}^{\mathrm{sf}}(a,-se^{i\rho},\Upsilon_{-e}^{(\nu)}(a,-se^{i\rho}))}\right] \\
&= \int_0^\infty \frac{ds}{s}\frac{se^{i\varphi}+te^{i\rho}}{se^{i\varphi}-te^{i\rho}}\log\left[1-\overline{\mathcal{X}_e^{\mathrm{sf}}(a,se^{i\rho},\Upsilon_e^{(\nu)}(a,se^{i\rho}))}\right] \\
&- \int_0^\infty \frac{ds}{s}\frac{se^{i\varphi}-te^{i\rho}}{se^{i\varphi}+te^{i\rho}}\log\left[1-\overline{\mathcal{X}_{-e}^{\mathrm{sf}}(a,-se^{i\rho},\Upsilon_{-e}^{(\nu)}(a,-se^{i\rho}))}\right] \\
&= \int_0^\infty \frac{ds}{s}\frac{\frac{1}{t}e^{i\varphi}+\frac{1}{s}e^{i\rho}}{\frac{1}{t}e^{i\varphi}-\frac{1}{s}e^{i\rho}}\log\left[1-\overline{\mathcal{X}_e^{\mathrm{sf}}(a,se^{i\rho},\Upsilon_e^{(\nu)}(a,se^{i\rho}))}\right] \\
&- \int_0^\infty \frac{ds}{s}\frac{\frac{1}{t}e^{i\varphi}-\frac{1}{s}e^{i\rho}}{\frac{1}{t}e^{i\varphi}+\frac{1}{s}e^{i\rho}}\log\left[1-\overline{\mathcal{X}_{-e}^{\mathrm{sf}}(a,-se^{i\rho},\Upsilon_{-e}^{(\nu)}(a,-se^{i\rho}))}\right]
\end{aligned}
$$

introducing the change of variables $s \mapsto \frac{1}{s}$:

$$
= \int_0^\infty \frac{ds}{s}\frac{se^{i\rho}+\frac{-1}{t}e^{i\varphi}}{se^{i\rho}-\frac{-1}{t}e^{i\varphi}}\log\left[1-\overline{\mathcal{X}_{-e}^{\mathrm{sf}}\left(a,-\frac{1}{s}e^{i\rho},\Upsilon_{-e}^{(\nu)}\left(a,-\frac{1}{s}e^{i\rho}\right)\right)}\right]
$$

$$- \int_0^\infty \frac{ds}{s} \cdot \frac{-se^{i\rho} + \frac{-1}{t} e^{i\varphi}}{-se^{i\rho} - \frac{-1}{t} e^{i\varphi}} \log \left[1 - \overline{\mathcal{X}_e^{\mathrm{sf}} \left(a, \frac{1}{s} e^{i\rho}, \Upsilon_e^{(\nu)} \left(a, \frac{1}{s} e^{i\rho} \right) \right)} \right]$$

by the definition of $\mathcal{X}^{\mathrm{sf}}$ and since the lemma is assumed true for ν:

$$\begin{aligned}
&= \int_0^\infty \frac{ds}{s} \frac{se^{i\rho} + \frac{-1}{t} e^{i\varphi}}{se^{i\rho} - \frac{-1}{t} e^{i\varphi}} \log \left[1 - \mathcal{X}_e^{\mathrm{sf}} \left(a, se^{i\rho}, \Upsilon_e^{(\nu)} \left(a, se^{i\rho} \right) \right) \right] \\
&- \int_0^\infty \frac{ds}{s} \cdot \frac{-se^{i\rho} + \frac{-1}{t} e^{i\varphi}}{-se^{i\rho} - \frac{-1}{t} e^{i\varphi}} \log \left[1 - \mathcal{X}_{-e}^{\mathrm{sf}} (a, -se^{i\rho}, \Upsilon_{-e}^{(\nu)} (a, -se^{i\rho})) \right] \\
&= \Phi_m^{(\nu+1)} \left(a, -\frac{1}{t} e^{i\varphi} \right) = \Phi_m^{(\nu+1)} \left(a, -1/\overline{\zeta} \right).
\end{aligned}$$

The proof for $\Phi_e^{(\nu+1)}$ is analogous. Since we only used the symmetry between pairs $\{\gamma, -\gamma\}$ and since by assumption $\Omega(\gamma; a) = \Omega(-\gamma; a)$, the theorem also holds in the region $\mathcal{B}_{\mathrm{out}}$. □

If we write as $\Upsilon_0^{(\nu)}$ (resp. $\Upsilon_\infty^{(\nu)}$) the asymptotic of this function as $\zeta \to 0$ (resp. $\zeta \to \infty$) so that

$$\Upsilon_0^{(\nu)} = \theta + \frac{1}{4\pi} \Phi_0^{(\nu)},$$

for a suitable correction $\Phi_0^{(\nu)}$. A similar equation holds for the asymptotic as $\zeta \to \infty$. The following reality condition was stated in [17] without proof:

Lemma 5.2.2. *$\Phi_0^{(\nu)}$ is imaginary for all ν. Consequently, Φ_0 is also imaginary.*

Proof. This follows from Lemma 5.2.1 by letting $\zeta \to 0$. □

Lemma 5.2.1 also shows that $\Phi_0 = -\Phi_\infty$. This and Lemma 5.2.2 gives the reality condition

$$\Upsilon_0 = \overline{\Upsilon_\infty} \tag{5.52}$$

Instead of working with (5.51), we let Υ denote the solution of that integral equation. We showed in section 5.1 that such solutions exist away from $a = 0$. Plugin Υ in (5.51) and split the integrals into 4 parts as in the beginning of this paper. For example, if we denote by $\zeta_e := -a/|a|$, the intersection of the unit circle with the ℓ_e ray, then

$$\begin{aligned}
&\int_{\ell_e} \frac{d\zeta'}{\zeta'} \frac{\zeta' + \zeta}{\zeta' - \zeta} \log\left(1 - \mathcal{X}_e^{\mathrm{sf}}(a, \zeta', \Upsilon_e)\right) = \\
&- \int_0^{\zeta_e} \frac{d\zeta'}{\zeta'} \log\left(1 - \mathcal{X}_e^{\mathrm{sf}}(a, \zeta', \Upsilon_e)\right) + \int_{\zeta_e}^{\zeta_e \infty} \frac{d\zeta'}{\zeta'} \log\left(1 - \mathcal{X}_e^{\mathrm{sf}}(a, \zeta', \Upsilon_e)\right) \\
&+ \int_0^{\zeta_e} \frac{2d\zeta'}{\zeta' - \zeta} \log\left(1 - \mathcal{X}_e^{\mathrm{sf}}(a, \zeta', \Upsilon_e)\right) + \int_{\zeta_e}^{\zeta_e \infty} 2d\zeta' \left\{ \frac{1}{\zeta' - \zeta} - \frac{1}{\zeta'} \right\} \log\left(1 - \mathcal{X}_e^{\mathrm{sf}}(a, \zeta', \Upsilon_e)\right)
\end{aligned} \tag{5.53}$$

We consider the first two integrals apart from the rest. If we take the limit $a \to 0$ the exponential decay in $\mathcal{X}_e^{\mathrm{sf}}$:

$$\exp\left(\frac{\pi R a}{\zeta'} + \pi R \zeta' \overline{a}\right)$$

vanishes and the integrals are no longer convergent.

By combining the two integrals with their analogues in the ℓ_{-e} ray we obtain:

$$\begin{aligned}
&- \int_0^{\zeta_e} \frac{d\zeta'}{\zeta'} \log\left(1 - \mathcal{X}_e^{\mathrm{sf}}(a, \zeta', \Upsilon_e)\right) + \int_{\zeta_e}^{\zeta_e \infty} \frac{d\zeta'}{\zeta'} \log\left(1 - \mathcal{X}_e^{\mathrm{sf}}(a, \zeta', \Upsilon_e)\right) \\
&\int_0^{-\zeta_e} \frac{d\zeta'}{\zeta'} \log\left(1 - \mathcal{X}_e^{\mathrm{sf}^{-1}}(a, \zeta', -\Upsilon_e)\right) - \int_{-\zeta_e}^{-\zeta_e \infty} \frac{d\zeta'}{\zeta'} \log\left(1 - \mathcal{X}_e^{\mathrm{sf}^{-1}}(a, \zeta', -\Upsilon_e)\right)
\end{aligned} \tag{5.54}$$

The parametrization in the first pair of integrals is of the form $\zeta' = t\zeta_e$, and in the second pair $\zeta' = -t\zeta_e$. Making the change of variables $\zeta' \mapsto 1/\zeta'$, we can pair up these integrals in a more explicit way as:

$$
\begin{aligned}
&-\int_0^1 \frac{dt}{t}\left\{\log\left[1-\exp\left(-\pi R|a|\left(\frac{1}{t}+t\right)+i\Upsilon_e(a,-te^{i\arg a})\right)\right]\right.\\
&+\log\left[1-\exp\left(-\pi R|a|\left(\frac{1}{t}+t\right)-i\Upsilon_e(a,\frac{1}{t}e^{i\arg a})\right)\right]\Bigg\}\\
&+\int_0^1 \frac{dt}{t}\left\{\log\left[1-\exp\left(-\pi R|a|\left(\frac{1}{t}+t\right)+i\Upsilon_e(a,-\frac{1}{t}e^{i\arg a})\right)\right]\right.\\
&+\log\left[1-\exp\left(-\pi R|a|\left(\frac{1}{t}+t\right)-i\Upsilon_e(a,te^{i\arg a})\right)\right]\Bigg\}
\end{aligned}
\tag{5.55}
$$

By Lemma 5.2.1, the integrands come in conjugate pairs. Therefore, we can rewrite (5.55) as:

$$
\begin{aligned}
&-2\int_0^1 \frac{dt}{t}\mathrm{Re}\ \Bigg\{\ \log\left[1-\exp\left(-\pi R|a|\left(\frac{1}{t}+t\right)+i\Upsilon_e^{(\nu-1)}(a,-te^{i\arg a})\right)\right]-\\
&\qquad\qquad \log\left[1-\exp\left(-\pi R|a|\left(\frac{1}{t}+t\right)-i\Upsilon_e^{(\nu-1)}(a,te^{i\arg a})\right)\right]\Bigg\}
\end{aligned}
$$

$$
=-2\int_0^1 \frac{dt}{t}\log\left|\frac{1-\exp\left(-\pi R|a|\left(t^{-1}+t\right)+i\Upsilon_e^{(\nu-1)}(a,-te^{i\arg a})\right)}{1-\exp\left(-\pi R|a|\left(t^{-1}+t\right)-i\Upsilon_e^{(\nu-1)}(a,te^{i\arg a})\right)}\right| \tag{5.56}
$$

Observe that (5.56) itself suggest the correct transformation of the θ coordinates that fixes this. Indeed, for a fixed $a \neq 0$ and θ_e, let Q be the map

$$
Q(\theta_m) = \theta_m + \psi(a,\theta),
$$

where

$$
\begin{aligned}
\psi_{\text{in}}(a,\theta) &= \frac{1}{2\pi}\int_0^1 \frac{dt}{t}\log\left|\frac{1-\exp\left(-\pi R|a|\left(t^{-1}+t\right)+i\Upsilon_e(a,-te^{i\arg a})\right)}{1-\exp\left(-\pi R|a|\left(t^{-1}+t\right)-i\Upsilon_e(a,te^{i\arg a})\right)}\right|\\
&= \frac{1}{2\pi}\int_0^1 \frac{dt}{t}\log\left|\frac{1-[\mathcal{X}_e]\left(-te^{i\arg a}\right)}{1-[\mathcal{X}_{-e}]\left(te^{i\arg a}\right)}\right|
\end{aligned}
\tag{5.57}
$$

for $a \in \mathcal{B}_{\text{in}}$. For $a \in \mathcal{B}_{\text{out}}$ where the wall-crossing is of type I, let $\varphi = \arg(Z_{\gamma_e+\gamma_m}(a))$, with $\zeta' = -te^{i\varphi}$ parametrizing the ℓ_{e+m} ray:

$$\begin{aligned}\psi_{\text{out}}(a,\theta) &= \frac{1}{2\pi}\int_0^1 \frac{dt}{t}\left\{\log\left|\frac{1-\exp\left(-\pi R|a|\left(t^{-1}+t\right)+i\Upsilon_e(a,-te^{i\arg a})\right)}{1-\exp\left(-\pi R|a|\left(t^{-1}+t\right)-i\Upsilon_e(a,te^{i\arg a})\right)}\right|\right.\\ &\quad\left.+\log\left|\frac{1-\exp\left(-\pi R|Z_{\gamma_e+\gamma_m}|\left(t^{-1}+t\right)+i\Upsilon_{e+m}(a,-te^{i\arg\varphi})\right)}{1-\exp\left(-\pi R|Z_{\gamma_e+\gamma_m}|\left(t^{-1}+t\right)-i\Upsilon_{e+m}(a,te^{i\arg\varphi})\right)}\right|\right\}\\ &= \frac{1}{2\pi}\int_0^1\frac{dt}{t}\left\{\log\left|\frac{1-[\mathcal{X}_e]\left(-te^{i\arg a}\right)}{1-[\mathcal{X}_{-e}]\left(te^{i\arg a}\right)}\right|+\log\left|\frac{1-[\mathcal{X}_{e+m}]\left(-te^{i\varphi}\right)}{1-[\mathcal{X}_{-e-m}]\left(te^{i\varphi}\right)}\right|\right\}\end{aligned} \tag{5.58}$$

Similarly, for wall-crossing of type II, $\varphi = \arg(Z_{\gamma_{-e}+\gamma_m}(a))$, with $\zeta' = -te^{i\varphi}$ for the ℓ_{-e+m} ray:

$$\begin{aligned}\psi_{\text{out}}(a,\theta) &= \frac{1}{2\pi}\int_0^1 \frac{dt}{t}\left\{\log\left|\frac{1-\exp\left(-\pi R|a|\left(t^{-1}+t\right)+i\Upsilon_e(a,-te^{i\arg a})\right)}{1-\exp\left(-\pi R|a|\left(t^{-1}+t\right)-i\Upsilon_e(a,te^{i\arg a})\right)}\right|\right.\\ &\quad\left.+\log\left|\frac{1-\exp\left(-\pi R|Z_{\gamma_{-e}+\gamma_m}|\left(t^{-1}+t\right)+i\Upsilon_{-e+m}(a,-te^{i\arg\varphi})\right)}{1-\exp\left(-\pi R|Z_{\gamma_{-e}+\gamma_m}|\left(t^{-1}+t\right)-i\Upsilon_{-e+m}(a,te^{i\arg\varphi})\right)}\right|\right\}\\ &= \frac{1}{2\pi}\int_0^1\frac{dt}{t}\left\{\log\left|\frac{1-[\mathcal{X}_e]\left(-te^{i\arg a}\right)}{1-[\mathcal{X}_{-e}]\left(te^{i\arg a}\right)}\right|+\log\left|\frac{1-[\mathcal{X}_{-e+m}]\left(-te^{i\varphi}\right)}{1-[\mathcal{X}_{e-m}]\left(te^{i\varphi}\right)}\right|\right\}\end{aligned} \tag{5.59}$$

As a approaches the wall of marginal stability W, $\arg a \to \varphi$. We need to show the following

Lemma 5.2.3. *The two definitions ψ_{in} and ψ_{out} coincide at the wall of marginal stability.*

Proof. First let a approach W from the "in" region, so we're using definition (5.57). Start with the pair of functions $(\mathcal{X}_e, \mathcal{X}_m)$ in the ζ-plane and let $\widetilde{\mathcal{X}}_e$ denote the analytic continuation of $\mathcal{X}_e$. See Figure 5.15. When they reach the

ℓ_e ray, $\mathcal{X}_e$ jumped to $\mathcal{X}_e(1-\mathcal{X}_m)$ by (5.6). Thus $\mathcal{X}_e = \widetilde{\mathcal{X}_e}(1-\mathcal{X}_m)$ along the ℓ_e ray.

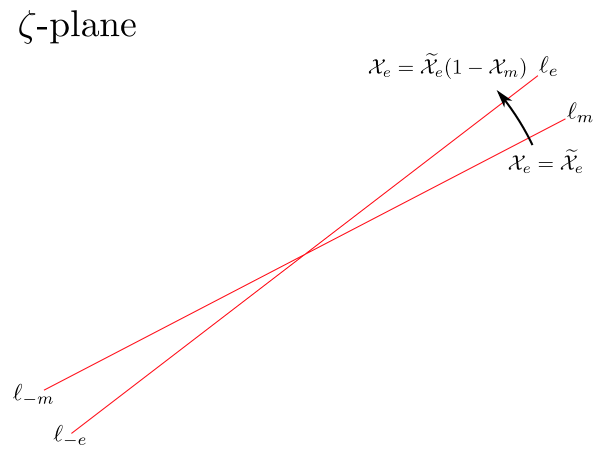

Figure 5.15: Jump of $\mathcal{X}_e$

Therefore,

$$\psi_{\text{in}}(a,\theta) = \frac{1}{2\pi}\int_0^1 \frac{dt}{t}\log\left|\frac{1-[\mathcal{X}_e(1-\mathcal{X}_m)]\,(-te^{i\arg a})}{1-[\mathcal{X}_{-e}(1-\mathcal{X}_m)^{-1}]\,(te^{i\arg a})}\right|$$

Now starting from the "out" region, and focusing on the wall-crossing of type I for the moment, we start with the pair $(\mathcal{X}_e, \mathcal{X}_m)$ as before. This time, $\mathcal{X}_e$ at the ℓ_e ray has not gone to any jump yet. See Figure 5.16. Only $\mathcal{X}_{e+m}$ undergoes a jump at the ℓ_{e+m} ray and it is of the form $\mathcal{X}_{e+m} \mapsto \mathcal{X}_{e+m}(1-\mathcal{X}_e)^{-1}$.

When a hits the wall W, $\varphi = \arg a$ and the integrals are taken over the same ray. Thus, we can combine the logs and obtain:

$$\psi_{\text{out}}(a,\theta) = \frac{1}{2\pi}\int_0^1 \frac{dt}{t}\left\{\log\left|\frac{1-[\mathcal{X}_e]\,(-te^{i\arg a})}{1-[\mathcal{X}_{-e}]\,(te^{i\arg a})}\right| + \log\left|\frac{1-[\mathcal{X}_{e+m}(1-\mathcal{X}_e)^{-1}]\,(-te^{i\arg a})}{1-[\mathcal{X}_{-e-m}(1-\mathcal{X}_e)]\,(te^{i\arg a})}\right|\right\}$$

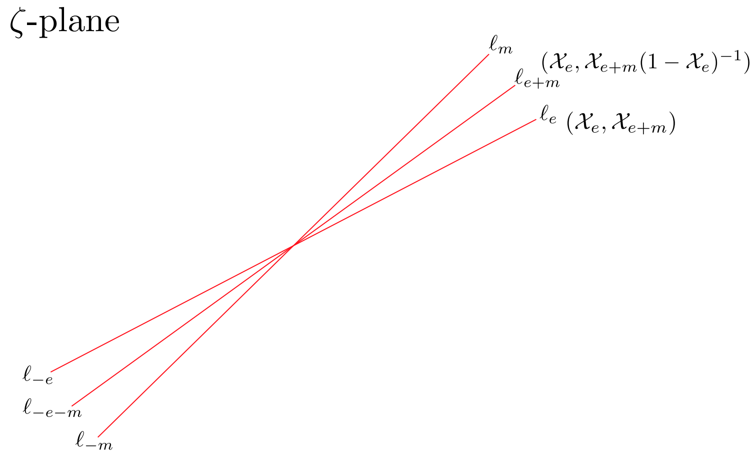

Figure 5.16: Only $\mathcal{X}_{e+m}$ has a jump

$$= \frac{1}{2\pi} \int_0^1 \frac{dt}{t} \log \left| \frac{1 - [\mathcal{X}_e(1-\mathcal{X}_m)]\,(-te^{i \arg a})}{1 - [\mathcal{X}_{-e}(1-\mathcal{X}_m)^{-1}]\,(te^{i \arg a})} \right| \tag{5.60}$$

and the two definitions coincide. For the wall-crossing of type II the proof is entirely analogous.

□

Theorem 5.2.4. *Q is a reparametrization in θ_m; that is, a diffeomorphism of $\mathbb{R}/2\pi\mathbb{Z}$.*

Proof. To show that Q is injective, it suffices to show that $\left|\frac{\partial\psi}{\partial\theta_m}\right| < 1$. We will show this in the $\mathcal{B}_{\text{in}}$ region. The proof for the $\mathcal{B}_{\text{out}}$ region is similar.

To simplify the calculations, write

$$\psi(a,\theta) = 2 \int_0^1 \frac{dt}{t} \log \left| \frac{1 - Cf(\theta_m)}{1 - Cg(\theta_m)} \right| \tag{5.61}$$

for suitable functions f, g (they both depend on other parameters, but they're fixed here) and a factor C of the form

$$C = \exp\left(-\pi R|a|(t^{-1} + t)\right)$$

Now take partials in both sides of (5.61) and bring the derivative inside the integral. After an application of the chain rule we get the estimate

$$\left|\frac{\partial \psi}{\partial \theta_m}\right| \leq 2 \int_0^1 \frac{dt}{t} |C| \left\{ \frac{|f|\left|\frac{\partial \Upsilon_e(t)}{\partial \theta_m}\right|}{|1 - Cf|} + \frac{|g|\left|\frac{\partial \Theta_e(-t)}{\partial \theta_m}\right|}{|1 - Cg|} \right\}$$

Since $\Upsilon \in \mathscr{X}^*$, $\left|\frac{\partial \Upsilon_e^{(\nu)}}{\partial \theta_m}\right| < 1$. By Lemma 5.1.1, we can bound $|f|, |g|$ by 2. The part C has exponential decay so if R is big enough we can bound the above by 1 and injectivity is proved. For surjectivity, just observe that $\psi(\theta_m + 2\pi) = \psi(\theta_m)$, so $Q(\theta_m + 2\pi) = \theta_m + 2\pi$. $\square$

With respect to the new coordinate θ'_m, the functions Υ_e, Υ_m satisfy the equation:

$$\Upsilon_e(a, \zeta) = \theta_e + \frac{1}{4\pi} \sum_{\gamma'} \Omega(\gamma'; a) \langle \gamma_e, \gamma' \rangle \int_{\gamma'} \frac{d\zeta'}{\zeta'} \frac{\zeta' + \zeta}{\zeta' - \zeta} \log\left[1 - \mathcal{X}^{\mathrm{sf}}_{\gamma'}(a, \zeta', \Upsilon_{\gamma'})\right] \tag{5.62}$$

$$\Upsilon_m(a, \zeta) = \theta'_m + \frac{1}{2\pi} \sum_{\gamma'} \Omega(\gamma'; a) \langle \gamma_m, \gamma' \rangle \left\{ \int_0^{b'} \frac{d\zeta'}{\zeta' - \zeta} \log\left[1 - \mathcal{X}^{\mathrm{sf}}_{\gamma'}(a, \zeta', \Upsilon_{\gamma'})\right] + \int_{b'}^{b'\infty} \frac{\zeta d\zeta'}{\zeta'(\zeta' - \zeta)} \log\left[1 - \mathcal{X}^{\mathrm{sf}}_{\gamma'}(a, \zeta', \Upsilon_{\gamma'})\right] \right\}, \tag{5.63}$$

for b' the intersection of the unit circle with the $\ell_{\gamma'}$ ray. The $\Omega(\gamma'; a)$ jump at the wall, but in the Pentagon case, the sum is finite.

In order to show that Υ converges to some function, even at $a = 0$, observe that the integral equations in (5.62) and (5.63) still make sense at the singular fiber, since in the case of (5.62), $\lim_{a\to 0} Z_m = c \neq 0$ and the exponential decay is still present, making the integrals convergent. In the case of (5.63), the exponential decay is gone, but the different kernel makes the integral convergent, at least for $\zeta \in \mathbb{C}^\times$. The limit function $\lim_{a\to 0} \Upsilon$ should be then a solution to the integral equations obtained by recursive iteration, as in Section 5.1.

As we've seen in the Ooguri-Vafa case, we expect our solutions $\lim_{a\to 0} \Upsilon$ to be unbounded in the ζ variable. Define a Banach space $\mathscr{X}$ as the completion under the sup norm of the space of functions $\Phi : \mathbb{C}^\times \times \mathbb{T} \times U \to \mathbb{C}^{2n}$ that are piecewise holomorphic on $\mathbb{C}^\times$, smooth on $\mathbb{T} \times U$, for U an open subset of $\mathcal{B}$ containing 0 and such that (5.62), (5.63) hold.

Like in the Ooguri-Vafa case, let $a \to 0$ fixing $\arg a$. We will later get rid of this dependence on $\arg a$ with another gauge transformation of θ_m. The following estimates on $\Upsilon^{(\nu)}$ will clearly give us that the sequence converges to some limit $\Upsilon^{(\nu)}$.

Lemma 5.2.5. *In the Pentagon case, at the bad fiber $a = 0$:*

$$\Upsilon_e^{(\nu+1)} = \Upsilon_e^{(\nu)} + O\left(e^{-2\pi\nu R|Z_m|}\right), \quad \nu \geq 2 \tag{5.64}$$

$$\Upsilon_m^{(\nu+1)} = \Upsilon_m^{(\nu)} + O\left(e^{-2\pi\nu R|Z_m|}\right), \quad \nu \geq 1 \tag{5.65}$$

Proof. As before, we prove this by induction. Note that $\Upsilon_m^{(1)} = \Upsilon^{\mathrm{OV}}$, the extension of the Ooguri-Vafa case obtained in (3.10), and $\Upsilon_m^{(1)}$ differs considerably from θ_m because of the $\log \zeta$ term. Hence the estimates cannot start at $\nu = 0$. Because of this reason, $\Upsilon_e^{(2)}$ differs considerably from $\Upsilon_e^{(1)}$ since this is the first iteration where $\Upsilon_m^{(1)}$ is considered.

Let $\nu = 1$. Since the integral equations for Υ_e didn't change in this special case, we can still perform a saddle point analysis and obtain as in (5.31) for the general case:

$$\Upsilon_e^{(1)} = \theta_e + \sum_{\gamma'} \Omega(\gamma', a) \langle \gamma_e, \gamma' \rangle \frac{e^{-2\pi R|Z_{\gamma'}|}}{4\pi i \sqrt{R|Z_{\gamma'}|}} \frac{\zeta_{\gamma'} + \zeta}{\zeta_{\gamma'} - \zeta} e^{i\theta_{\gamma'}} + O\left(\frac{e^{-2\pi R|Z_{\gamma'}|}}{R} \right) \tag{5.66}$$

where $\zeta_{\gamma'} = -\frac{Z_{\gamma'}}{|Z_{\gamma'}|}$ is the saddle point for the integrals in (5.62), and ζ is away from a small neighborhood of $\zeta_{\gamma'}$. Note that there is no divergence if $\zeta \to 0$ or $\zeta \to \infty$. As before, if ζ is in such neighborhood, we can deform the paths of integration slightly and obtain similar estimates, except for the $\sqrt{R}$ terms in the denominator (see (5.33)).

In any case, for the Pentagon, the γ' in (5.66) are only $\gamma_{\pm m}, \gamma_{\pm e+m}$, depending on the side of the wall of marginal stability. At $a = 0$, $Z_{e+m} = Z_m$, so (5.66) gives that $\log[1 - e^{i\Upsilon_e^{(1)}}] = \log[1 - e^{i\theta_e}] + O(e^{-2\pi R|Z_m|})$ along the ℓ_e ray, and a similar estimate holds for $\log[1 - e^{-i\Upsilon_e^{(1)}}]$ along the ℓ_{-e} ray. Plugging in this in (5.33), we get (5.65) for $\nu = 1$.

For general ν, a saddle point analysis on $\Upsilon_e^{(\nu)}$ can still be performed and obtain as in (5.66):

$$\Upsilon_e^{(\nu+1)} = \theta_e + \frac{e^{-2\pi R|Z_m|}}{4\pi i\sqrt{R|Z_m|}} \left\{ \frac{\zeta_m+\zeta}{\zeta_m-\zeta} e^{i\Upsilon_m^{(\nu)}(\zeta_m)} - \frac{\zeta_m-\zeta}{\zeta_m+\zeta} e^{-i\Upsilon_m^{(\nu)}(-\zeta_m)} \right\} + O\left(\frac{e^{-2\pi R|Z_{\gamma'}|}}{R}\right), \tag{5.67}$$

from one side of the wall. On the other side (for type I) it will contain the extra terms

$$\frac{e^{-2\pi R|Z_m|}}{4\pi i\sqrt{R|Z_m|}} \left\{ \frac{\zeta_m+\zeta}{\zeta_m-\zeta} e^{i(\Upsilon_m^{(\nu)}(\zeta_m)+\Upsilon_e^{(\nu)}(\zeta_m))} - \frac{\zeta_m-\zeta}{\zeta_m+\zeta} e^{-i(\Upsilon_m^{(\nu)}(-\zeta_m)-\Upsilon_e^{(\nu)}(-\zeta_m))} \right\}. \tag{5.68}$$

Observe that for this approximation we only need $\Upsilon^{(\nu)}$ at the point ζ_m. By the previous part, for $\nu = 2$,

$$e^{i\Upsilon_m^{(2)}(\zeta_m)} = e^{i\Upsilon_m^{(1)}(\zeta_m)} \left(1 + O\left(e^{-2\pi R|Z_m|}\right)\right)$$

Thus, for $\nu = 2$,

$$\begin{aligned} \Upsilon_e^{(3)} &= \theta_e + \frac{e^{-2\pi R|Z_m|}}{4\pi i\sqrt{R|Z_m|}} \left\{ \frac{\zeta_m+\zeta}{\zeta_m-\zeta} e^{i\Upsilon_m^{(1)}(\zeta_m)} \left(1 + O\left(e^{-2\pi R|Z_m|}\right)\right) \right. \\ &\quad \left. - \frac{\zeta_m-\zeta}{\zeta_m+\zeta} e^{-i\Upsilon_m^{(1)}(-\zeta_m)} \left(1 + O\left(e^{-2\pi R|Z_m|}\right)\right) \right\} + O\left(R^{1/2}\right) \\ &= \Upsilon_e^{(2)} + O\left(e^{-4\pi R|Z_m|}\right) \end{aligned} \tag{5.69}$$

and similarly in the other side of the wall. For general ν, the same arguments show that (5.64), (5.65) hold after the appropriate ν. □

These estimates show the following: for fixed R, there is a small neighborhood V of $\theta_e = 0$ such that, away from V, the corrections $\Upsilon_e^{(\nu)}$ to θ_e are sufficiently small so that $\Upsilon_e^{(\nu)}$ does not intersect the negative imaginary axis for any ζ and any a in the neighborhood U where the equations (5.64), (5.65) make sense. Thus it is still possible to define such integral equations since we can choose the principal branch of $\log[1 - \mathcal{X}_e^{(\nu)}]$. In the next chapter we'll give a heuristic argument based on numerical evidence as to how define an appropriate branch of $\log[1 - \mathcal{X}_e^{(\nu)}]$ inside V.

There is still one problem: the limit of $\widetilde{\mathcal{X}}_m$ we obtained as $a \to 0$ for the analytic continuation of $\mathcal{X}_m$ was only along a fixed ray $\arg a =$ constant. To get rid of this dependence, it is necessary to perform another gauge transformation on the torus coordinates θ. For simplicity, we restrict to the Pentagon case. Let $a \to 0$ fixing $\arg a$. Let ζ_γ denote $Z_\gamma/|Z_\gamma|$. In particular, $\zeta_e = a/|a|$ and this remains constant since we're fixing $\arg a$. Also, $\zeta_m = Z_m/|Z_m|$ and this is independent of $\arg a$ since Z_m has a limit as $a \to 0$. The following lemma will allow us to obtain the correct gauge transformation.

Lemma 5.2.6. *For the limit* $\widetilde{\mathcal{X}}_m\Big|_{a=0}$ *obtained above, its imaginary part is independent of the chosen ray* $\arg a = c$ *along which* $a \to 0$.

Proof. Let $\widetilde{\Upsilon}_m$ denote the analytic continuation of Υ_m yielding $\widetilde{\mathcal{X}}_m$. Start with a fixed value $\arg a \equiv \rho_0$, for ρ_0 different from $\arg Z_m(0), \arg(-Z_m(0))$. For another ray $\arg a \equiv \rho$, we compute $\Upsilon_m|_{\substack{a=0\\ \arg a=\rho}} - \Upsilon_m|_{\substack{a=0\\ \arg a=\rho_0}}$ (without analytic continuation for the moment).

The integrals in (5.63) are of two types. One type is of the form

$$\int_0^{\zeta_{\pm e}} \frac{d\zeta'}{\zeta' - \zeta} \log\left[1 - e^{i\Upsilon_{\pm e}(\zeta')}\right] + \int_{\zeta_{\pm e}}^{\zeta_{\pm e}\infty} \frac{\zeta d\zeta'}{\zeta'(\zeta' - \zeta)} \log\left[1 - e^{i\Upsilon_{\pm e}(\zeta')}\right] \quad (5.70)$$

The other type appears only in the outside part of the wall of marginal stability. Since $Z : \Gamma \to \mathbb{C}$ is a homomorphism, $Z_{\gamma_e+\gamma_m} = Z_{\gamma_e} + Z_{\gamma_m}$. At $a = 0$, $Z_e = a = 0$, so $Z_{e+m} = Z_m$. Hence, $\ell_m = \ell_{e+m}$ at the singular fiber. This second type of integral is thus of the form

$$\int_0^{\zeta_{\pm m}} \frac{d\zeta'}{\zeta' - \zeta} \log\left[1 - e^{i\Upsilon_{\pm(e+m)}(\zeta')}\right] + \int_{\zeta_{\pm e}}^{\zeta_{\pm m}\infty} \frac{\zeta d\zeta'}{\zeta'(\zeta' - \zeta)} \log\left[1 - e^{i\Upsilon_{\pm(e+m)}(\zeta')}\right] \tag{5.71}$$

Since the ℓ_m stays fixed at $a = 0$ independently of $\arg a$, (5.71) does not depend of $\arg a$. We should focus then only on integrals of the type (5.70). For a different $\arg a$, ζ_e changes to another point $\widetilde{\zeta}_e$ in the unit circle. See Figure 5.17. The paths of integration change accordingly. We have two possible outcomes: either ζ lies outside the sector determined by the two paths, or ζ lies inside the region.

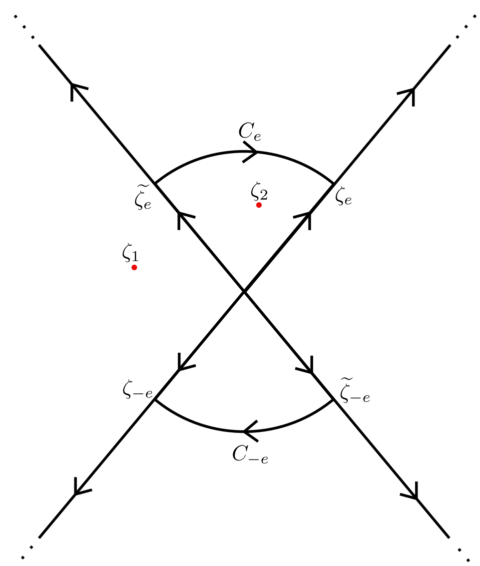

Figure 5.17: As $\arg a$ changes, the paths of integration change

In the first case (ζ_1 on Figure 5.17), the integrands

$$\frac{\log[1 - e^{i\Upsilon_{\pm e}(\zeta')}]}{\zeta' - \zeta}, \qquad \frac{\zeta \log[1 - e^{i\Upsilon_{\pm e}(\zeta')}]}{\zeta'(\zeta' - \zeta)} \tag{5.72}$$

are holomorphic on ζ' in the sector between the two paths. By Cauchy's formula, the difference between the two integrals is just the integration along a path $C_{\pm e}$ between the two endpoints $\zeta_{\pm e}, \widetilde{\zeta}_{\pm e}$. If $f(s)$ parametrizes the path C_e, let $C_{-e} = -1/\overline{f(s)}$. The orientation of C_e in the contour containing ∞ is opposite to the contour containing 0. Similarly for C_{-e}. Thus, the difference of Υ_m for these two values of $\arg a$ is the integral along C_e, C_{-e} of the difference of kernels (5.72), namely:

$$\int_{C_e} \frac{d\zeta'}{\zeta'} \log[1 - e^{i\Upsilon_e(\zeta')}] - \int_{C_{-e}} \frac{d\zeta'}{\zeta'} \log[1 - e^{-i\Upsilon_e(\zeta')}] \tag{5.73}$$

By symmetry of C_e, C_{-e} and the reality condition in Lemma 5.2.1, the second integral is the conjugate of the first one. Thus (5.73) is only real.

When ζ hits one of the contours, ζ coincides with one of the ℓ_e or ℓ_{-e} rays, for some value of $\arg a$. The contour integrals jump since ζ lies now inside the contour (ζ_2 in Figure 5.17). The jump is by the residue of the integrands (5.72). This gives the jump of $\mathcal{X}_m$ that the analytic continuation around $a = 0$ cancels. Therefore, only the real part of Υ_m depends on $\arg a$. □

By the previous lemma, $\widetilde{\Upsilon}_m\Big|_{\substack{a=0\\ \arg a=\rho}} - \widetilde{\Upsilon}_m\Big|_{\substack{a=0\\ \arg a=\rho_0}}$ is real and is given by (5.73). Define then a new gauge transformation:

$$\widetilde{\theta}_m = \theta'_m - \frac{1}{2\pi}\left\{\int_{C_e} \frac{d\zeta'}{\zeta'} \log[1 - e^{i\Upsilon_e(\zeta')}] + \int_{C_{-e}} \frac{d\zeta'}{\zeta'} \log[1 - e^{-i\Upsilon_e(\zeta')}]\right\} \tag{5.74}$$

This eliminates the dependence on $\arg a$ for the limit $\widetilde{\mathcal{X}}_m\Big|_{a=0}$ and we thus obtain a well defined function $\widetilde{\mathcal{X}}_m$ at the singular fiber.

5.3 Extension of the derivatives

So far we were able to extend the functions $\mathcal{X}_e, \widetilde{\mathcal{X}}_m$. Unfortunately, we can no longer bound uniformly on ν the derivatives of $\widetilde{\mathcal{X}}_m$ near $a = 0$, so the Arzela-Ascoli arguments no longer work here. Since there's no difference on the definition of $\mathcal{X}_e$ at $a = 0$ from that of the regular fibers, this function extends smoothly to $a = 0$.

We have to obtain the extension of all derivatives of $\widetilde{\mathcal{X}}_m$ directly from its definition. It suffices to extend the derivatives of $\mathcal{X}_m$ only, as the analytic continuation doesn't affect the symplectic form $\varpi(\zeta)$ (see below).

Lemma 5.3.1. $\log \mathcal{X}_m$ *extends smoothly to* $a = 0$*, for* θ_e *bounded away from 0.*

Proof. For convenience, we rewrite Υ_m with the final magnetic coordinate $\widetilde{\theta_m}$:

$$\Upsilon_m = \widetilde{\theta_m} + \frac{1}{2\pi}\left\{\int_{C_e}\frac{d\zeta'}{\zeta'}\log[1-e^{i\Upsilon_e(\zeta')}] - \int_{C_{-e}}\frac{d\zeta'}{\zeta'}\log[1-e^{-i\Upsilon_e(\zeta')}]\right\}$$
$$+\frac{1}{2\pi}\sum_{\gamma'}\Omega(\gamma';a)\,\langle\gamma_m,\gamma'\rangle\left\{\int_0^{\zeta_{\gamma'}}\frac{d\zeta'}{\zeta'-\zeta}\log\left[1-\mathcal{X}^{\mathrm{sf}}_{\gamma'}(a,\zeta',\Upsilon_{\gamma'})\right]+\right.$$
$$\left.\int_{\zeta_{\gamma'}}^{\zeta_{\gamma'}\infty}\frac{\zeta\, d\zeta'}{\zeta'(\zeta'-\zeta)}\log\left[1-\mathcal{X}^{\mathrm{sf}}_{\gamma'}(a,\zeta',\Upsilon_{\gamma'})\right]\right\}$$

where $e^{i\Upsilon_e(\zeta')}$ is evaluated only at $a = 0$. For γ' of the type $\pm\gamma_e \pm \gamma_m$, $\mathcal{X}_{\gamma'}$ and its derivatives still have exponential decay along the $\ell_{\gamma'}$ ray, so these parts in Υ_m extend to $a = 0$ smoothly. It thus suffices to extend only

$$\Upsilon_m = \widetilde{\theta_m} + \frac{1}{2\pi}\left\{\int_{C_e} \frac{d\zeta'}{\zeta'} \log[1 - e^{i\Upsilon_e(\zeta')}] - \int_{C_{-e}} \frac{d\zeta'}{\zeta'} \log[1 - e^{-i\Upsilon_e(\zeta')}] \right.$$
$$+ \int_0^{\zeta_e} \frac{d\zeta'}{\zeta' - \zeta} \log\left[1 - \mathcal{X}_e^{\text{sf}}(a, \zeta', \Upsilon_e)\right] + \int_{\zeta_e}^{\zeta_e \infty} \frac{\zeta d\zeta'}{\zeta'(\zeta' - \zeta)} \log\left[1 - \mathcal{X}_e^{\text{sf}}(a, \zeta', \Upsilon_e)\right]$$
$$\left. - \int_0^{-\zeta_e} \frac{d\zeta'}{\zeta' - \zeta} \log\left[1 - \mathcal{X}_e^{\text{sf}^{-1}}(a, \zeta', -\Upsilon_e)\right] - \int_{-\zeta_e}^{-\zeta_e \infty} \frac{\zeta d\zeta'}{\zeta'(\zeta' - \zeta)} \log\left[1 - \mathcal{X}_e^{\text{sf}^{-1}}(a, \zeta', -\Upsilon_e)\right]\right\} \tag{5.75}$$

together with the semiflat part $\pi R \frac{Z_m}{\zeta} + \pi R \zeta \overline{Z_m}$, which we assume is as in the Generalized Ooguri-Vafa case, namely:

$$\mathcal{X}_m = \exp\left(\frac{-iR}{2\zeta}(a \log a - a + f(a)) + i\Upsilon_m + \frac{i\zeta R}{2}(\overline{a} \log \overline{a} - \overline{a} + \overline{f(a)})\right) \tag{5.76}$$

for a holomorphic function f near $a = 0$ and such that $f(0) \neq 0$. The derivatives of the terms involving $f(a)$ clearly extend to $a = 0$, so we focus on the rest, as in §4.1.1.

We show first that $\dfrac{\partial \log \mathcal{X}_m}{\partial_{\theta_e}}, \dfrac{\partial \log \mathcal{X}_m}{\partial_{\theta_m}}$ extend to $a = 0$. Since there is no difference in the proof between electric or magnetic coordinates, we'll denote by ∂_θ a derivative with respect to any of these two variables.

We have:

$$\frac{\partial}{\partial \theta} \log \Upsilon_m = \frac{-i}{2\pi}\left\{\int_{C_e} \frac{d\zeta'}{\zeta'} \frac{e^{i\Upsilon_e(\zeta')}}{1 - e^{i\Upsilon_e(\zeta')}} \frac{\partial \Upsilon_e(\zeta')}{\partial \theta} - \int_{C_{-e}} \frac{d\zeta'}{\zeta'} \frac{e^{-i\Upsilon_e(\zeta')}}{1 - e^{-i\Upsilon_e(\zeta')}} \frac{\partial \Upsilon_e(\zeta')}{\partial \theta}\right.$$

$$+\int_0^{\zeta_e}\frac{d\zeta'}{\zeta'-\zeta}\frac{\mathcal{X}_e(\zeta')}{1-\mathcal{X}_e(\zeta')}\frac{\partial\Upsilon_e(\zeta')}{\partial\theta}+\int_{\zeta_e}^{\zeta_e\infty}\frac{\zeta d\zeta'}{\zeta'(\zeta'-\zeta)}\frac{\mathcal{X}_e(\zeta')}{1-\mathcal{X}_e(\zeta')}\frac{\partial\Upsilon_e(\zeta')}{\partial\theta}$$
$$+\int_0^{-\zeta_e}\frac{d\zeta'}{\zeta'-\zeta}\frac{\mathcal{X}_e^{-1}(\zeta')}{1-\mathcal{X}_e^{-1}(\zeta')}\frac{\partial\Upsilon_e(\zeta')}{\partial\theta}+\int_{-\zeta_e}^{\zeta_e\infty}\frac{\zeta d\zeta'}{\zeta'(\zeta'-\zeta)}\frac{\mathcal{X}_e^{-1}(\zeta')}{1-\mathcal{X}_e^{-1}(\zeta')}\frac{\partial\Upsilon_e(\zeta')}{\partial\theta}\Bigg\}$$

when $a\to 0$, $\dfrac{\mathcal{X}_e(\zeta')}{1-\mathcal{X}_e(\zeta')}\to\dfrac{e^{i\Upsilon_e(\zeta')}}{1-e^{i\Upsilon_e(\zeta')}}$. The integrals along C_e and C_{-e} represent a difference of integrals along the contour in the last integrals and a fixed contour, as in Figure 5.17. Thus, when $a=0$,

$$2\pi i\frac{\partial}{\partial\theta}\log\Upsilon_m\bigg|_{a=0}=\int_0^{b}\frac{d\zeta'}{\zeta'-\zeta}\frac{\mathcal{X}_e(\zeta')}{1-\mathcal{X}_e(\zeta')}\frac{\partial\Upsilon_e(\zeta')}{\partial\theta}+\int_{b}^{b\infty}\frac{\zeta d\zeta'}{\zeta'(\zeta'-\zeta)}\frac{\mathcal{X}_e(\zeta')}{1-\mathcal{X}_e(\zeta')}\frac{\partial\Upsilon_e(\zeta')}{\partial\theta}$$
$$+\int_0^{-b}\frac{d\zeta'}{\zeta'-\zeta}\frac{\mathcal{X}_e^{-1}(\zeta')}{1-\mathcal{X}_e^{-1}(\zeta')}\frac{\partial\Upsilon_e(\zeta')}{\partial\theta}+\int_{-b}^{-b\infty}\frac{\zeta d\zeta'}{\zeta'(\zeta'-\zeta)}\frac{\mathcal{X}_e^{-1}(\zeta')}{1-\mathcal{X}_e^{-1}(\zeta')}\frac{\partial\Upsilon_e(\zeta')}{\partial\theta}\Bigg\}$$

for a fixed point b in the unit circle, independent of a. The integrals are well defined and thus the left side has an extension to $a=0$.

Now, for the partials with respect to $a,\overline{a}$, there are two different types of dependence: one is the dependence of the contours, the other is the dependence of the integrands. The former dependence is only present in (5.75), as the contours in Figure 5.17 change with $\arg a$. A simple application of the Fundamental Theorem of Calculus in each integral in (5.75) gives that this change is:

$$-2\pi i\frac{\partial}{\partial\arg a}\log\Upsilon_m\bigg|_{a=0}=\log[1-e^{-i\Upsilon_e(\zeta_e)}]-\log[1-e^{-i\Upsilon_e(\zeta_e)}]$$
$$-\log[1-e^{-i\Upsilon_e(\zeta_e)}]+\log[1-e^{-i\Upsilon_e(\zeta_e)}]=0,$$

where we again used the fact that the integrals along C_e and C_{-e} represent the difference between the integrals in the other pairs with respect to two different rays, one fixed. Compare this with (4.7), where we obtained this explicitly.

Then there is the dependence on $a, \overline{a}$ on the integrands and the semi-flat part. Focusing on a only, we take partials on $\log \mathcal{X}_m$ in (5.76) (ignoring constants and parts that clearly extend to $a = 0$). This is:

$$\frac{\log a}{\zeta} + \int_0^{\zeta_e} \frac{d\zeta'}{\zeta'(\zeta' - \zeta)} \frac{\mathcal{X}_e}{1 - \mathcal{X}_e} + \int_0^{-\zeta_e} \frac{d\zeta'}{\zeta'(\zeta' - \zeta)} \frac{\mathcal{X}_e^{-1}}{1 - \mathcal{X}_e^{-1}} \tag{5.77}$$

This is the equivalent of (4.8) in the general case. In the limit $a \to 0$, we can do an asymptotic expansion of $\dfrac{e^{i\Upsilon_e(\zeta')}}{1 - e^{i\Upsilon_e(\zeta')}} = \dfrac{e^{i\Upsilon_e(0)}}{1 - e^{i\Upsilon_e(0)}} + O(\zeta')$. Clearly when we write this expansion in (5.77), the only divergent term at $a = 0$ is the first degree approximation in the integral. Thus, we can focus on that and assume that the $\dfrac{\mathcal{X}_e}{1 - \mathcal{X}_e}$ (resp. $\dfrac{\mathcal{X}_e^{-1}}{1 - \mathcal{X}_e^{-1}}$) factor is constant. If we do the partial fraction decomposition, we can run the same argument as in Eqs. (4.9) up to (4.15) and obtain that (5.77) is actually 0 at $a = 0$. The only identity needed is

$$\frac{1}{1 - e^{i\Upsilon_e(0)}} + \frac{1}{1 - e^{-i\Upsilon_e(0)}} = 1$$

The argument also works for the derivative with respect to $\overline{a}$, now with an asymptotic expansion around ∞ of Υ_e.

This shows that $\widetilde{\mathcal{X}_m}$ extends in a C^1 way to $a = 0$. For the C^∞ extension, derivatives with respect to any θ coordinate work in the same way, all that was used was the specific form of the contours C_e, C_{-e}. The same thing applies to the dependence on the contours C_e, C_{-e}. For derivatives with respect to $a, \overline{a}$ in the integrands, we can again do an asymptotic expansion of Υ_e at 0 or ∞ and compare it to the asymptotic of the corresponding derivative of $a \log a - a$ as $a \to 0$.

$\square$

Nothing we have done in this chapter is particular of the Pentagon example. We only needed the specific values of $\Omega(\gamma; u)$ given in (5.1) to obtain the Pentagon identities at the wall and to perform the analytic continuation of $\mathcal{X}_m$ around $u = 2$. For any integrable systems data as in Chapter 2 with suitable invariants $\Omega(\gamma; u)$ allowing the wall-crossing formulas and analytic continuation, we can do the same isomonodromic deformation of putting all the jumps at a single admissible ray, perform saddle-point analysis and obtain the same extensions of the Darboux coordinates $\mathcal{X}_\gamma$. This finishes the proof of Theorem 2.2.2.

The extension of the holomorphic symplectic form $\varpi(\zeta)$ is now straightforward. We proceed as in [7] by first writing:

$$\varpi(\zeta) = -\frac{1}{4\pi^2 R} \frac{d\mathcal{X}_e}{\mathcal{X}_e} \wedge \frac{d\mathcal{X}_m}{\mathcal{X}_m}$$

Where we used the fact that the jumps of the functions $\mathcal{X}_\gamma$ are via the symplectomorphisms $\mathcal{K}_{\gamma'}$ of the complex torus T_a (see (2.5)) so $\varpi(\zeta)$ remains the same if we take $\mathcal{X}_m$ or its analytic continuation $\widetilde{\mathcal{X}_m}$.

We need to show that $\varpi(\zeta)$ is of the form

$$-\frac{i}{2\zeta}\omega_+ + \omega_3 - \frac{i\zeta}{2}\varpi_- \tag{5.78}$$

that is, $\varpi(\zeta)$ must have simple poles at $\zeta = 0$ and $\zeta = \infty$, even at the singular fiber where $a = 0$.

By definition, $\mathcal{X}_e = \exp(\frac{\pi R a}{\zeta} + i\Upsilon_e + \pi R\zeta\overline{a})$. Thus

$$\frac{d\mathcal{X}_e(\zeta)}{\mathcal{X}_e(\zeta)} = \frac{\pi R da}{\zeta} + id\Upsilon_e(\zeta) + \pi R\zeta d\overline{a}$$

By (5.62), and since $\lim_{a\to 0} Z_m \neq 0$, $\mathcal{X}_m$ (resp. $\mathcal{X}_{-m}$) of the form $\exp(\frac{\pi R Z_m(a)}{\zeta} + i\Upsilon_m + \pi R\zeta\overline{Z_m(a)})$ still has exponential decay when ζ lies in the ℓ_m ray (resp. ℓ_{-m}), even if $a = 0$. The differential $d\Upsilon_e(\zeta)$ thus exists for any $\zeta \in \mathbb{P}^1$ since the integrals defining it converge for any ζ.

As in [7], we can write

$$\frac{d\mathcal{X}_e}{\mathcal{X}_e} \wedge \frac{d\mathcal{X}_m}{\mathcal{X}_m} = \frac{d\mathcal{X}_e}{\mathcal{X}_e} \wedge \left(\frac{d\mathcal{X}_m^{\mathrm{sf}}}{\mathcal{X}_m^{\mathrm{sf}}} + \mathcal{I}_\pm\right),$$

for $\mathcal{I}_\pm$ denoting the corrections to the semiflat function. By the form of $\mathcal{X}^{\mathrm{sf}} = \exp(\frac{\pi R Z_m(a)}{\zeta} + i\theta_m + \pi R\zeta\overline{Z_m(a)})$, the wedge involving only the semiflat part has only simple poles at $\zeta = 0$ and $\zeta = \infty$, so we can focus on the corrections. These are of the form

$$\begin{aligned}\frac{d\mathcal{X}_e(\zeta)}{\mathcal{X}_e(\zeta)} \wedge \mathcal{I}_\pm = \frac{-i}{2\pi}\Bigg\{ &\int_0^{\zeta_e} \frac{d\zeta'}{\zeta' - \zeta}\frac{\mathcal{X}_e(\zeta')}{1 - \mathcal{X}_e(\zeta')}\frac{d\mathcal{X}_e(\zeta)}{\mathcal{X}_e(\zeta)} \wedge \frac{d\mathcal{X}_e(\zeta')}{\mathcal{X}_e(\zeta')} \\ &+ \int_{\zeta_e}^{\zeta_e\infty} \frac{\zeta d\zeta'}{\zeta'(\zeta' - \zeta)}\frac{\mathcal{X}_e(\zeta')}{1 - \mathcal{X}_e(\zeta')}\frac{d\mathcal{X}_e(\zeta)}{\mathcal{X}_e(\zeta)} \wedge \frac{d\mathcal{X}_e(\zeta')}{\mathcal{X}_e(\zeta')}\end{aligned}$$

$$+\int_0^{-\zeta_e} \frac{d\zeta'}{\zeta'-\zeta}\frac{\mathcal{X}_e^{-1}(\zeta')}{1-\mathcal{X}_e^{-1}(\zeta')}\frac{d\mathcal{X}_e(\zeta)}{\mathcal{X}_e(\zeta)}\wedge\frac{d\mathcal{X}_e(\zeta')}{\mathcal{X}_e(\zeta')}$$
$$+\int_{-\zeta_e}^{-\zeta_e\infty} \frac{\zeta d\zeta'}{\zeta'(\zeta'-\zeta)}\frac{\mathcal{X}_e^{-1}(\zeta')}{1-\mathcal{X}_e^{-1}(\zeta')}\frac{d\mathcal{X}_e(\zeta)}{\mathcal{X}_e(\zeta)}\wedge\frac{d\mathcal{X}_e(\zeta')}{\mathcal{X}_e(\zeta')}\Bigg\}$$

In the "inside" part of the wall of marginal stability. A similar equation holds in the other side. We can simplify the wedge products above by taking instead

$$\frac{d\mathcal{X}_e(\zeta)}{\mathcal{X}_e(\zeta)}\wedge\left(\frac{d\mathcal{X}_e(\zeta)}{\mathcal{X}_e(\zeta)}-\frac{d\mathcal{X}_e(\zeta')}{\mathcal{X}_e(\zeta')}\right)=\pi R\left[\left(\frac{1}{\zeta}-\frac{1}{\zeta'}\right)da+(\zeta-\zeta')d\overline{a}\right]+i\left(d\Phi_e(\zeta)-d\Phi_e(\zeta')\right) \tag{5.79}$$

Recall that Φ_e represents the corrections to θ_e, so $\Upsilon_e=\theta_e+\Phi_e$. By §5.1, Φ_e and $d\Phi_e$ are defined for $\zeta=0$ $\zeta=\infty$ even if $a=0$, since $\lim_{a\to 0} Z_m(a)\neq 0$ and the exponential decay in $\mathcal{X}_m^{\mathrm{sf}}$ still present guarantees convergence of the integrals in 5.62. Hence, the terms involving $d\Phi_e(\zeta)-d\Phi_e(\zeta')$ are holomorphic for any $\zeta\in\mathbb{P}^1$. It thus suffices to consider the other terms. After simplifying the integration kernels, we obtain

$$\frac{\pi R da}{\zeta}\int_0^{\zeta_e}\frac{d\zeta'}{\zeta'}\frac{\mathcal{X}_e(\zeta')}{1-\mathcal{X}_e(\zeta')}+\pi R da\int_{\zeta_e}^{\zeta_e\infty}\frac{d\zeta'}{(\zeta')^2}\frac{\mathcal{X}_e(\zeta')}{1-\mathcal{X}_e(\zeta')}$$
$$\frac{\pi R da}{\zeta}\int_0^{-\zeta_e}\frac{d\zeta'}{\zeta'}\frac{\mathcal{X}_e^{-1}(\zeta')}{1-\mathcal{X}_e^{-1}(\zeta')}+\pi R da\int_{-\zeta_e}^{-\zeta_e\infty}\frac{d\zeta'}{(\zeta')^2}\frac{\mathcal{X}_e^{-1}(\zeta')}{1-\mathcal{X}_e^{-1}(\zeta')}$$
$$-\pi R d\overline{a}\int_0^{\zeta_e}d\zeta'\frac{\mathcal{X}_e(\zeta')}{1-\mathcal{X}_e(\zeta')}-\pi R\zeta d\overline{a}\int_{\zeta_e}^{\zeta_e\infty}\frac{d\zeta'}{\zeta'}\frac{\mathcal{X}_e(\zeta')}{1-\mathcal{X}_e(\zeta')}$$
$$-\pi R d\overline{a}\int_0^{\zeta_e}d\zeta'\frac{\mathcal{X}_e^{-1}(\zeta')}{1-\mathcal{X}_e^{-1}(\zeta')}-\pi R\zeta d\overline{a}\int_{\zeta_e}^{\zeta_e\infty}\frac{d\zeta'}{\zeta'}\frac{\mathcal{X}_e^{-1}(\zeta')}{1-\mathcal{X}_e^{-1}(\zeta')}$$

The only dependence on ζ is in the factors $\zeta, 1/\zeta$. Thus $\varpi(\zeta)$ has only simple poles at $\zeta = 0$ and $\zeta = \infty$. This gives Theorem 2.2.3.

Appendix

Appendix 1

How to define the Darboux coordinates near $\theta_e = 0$

In the last chapter we constructed Darboux coordinates $\mathcal{X}_\gamma$ for the Pentagon case. We saw that such coordinates extend to the singular fibers where one $Z_{\gamma_i} = 0$, for $\{\gamma_1, \gamma_2\}$ a local basis of Γ, at least for θ_e bounded away from 0. In this appendix we'll give a heuristic argument for extending the previous construction to the case θ_e close to 0.

The main obstacle is that for θ_e in a small neighborhood of 0, the quantum corrections $\Upsilon_e^{(\nu)}$ may hit 0 for some ζ, so the term $\log[1 - \mathcal{X}_e]$ in the integral equations is not defined. Even if this is not the case, we have to make sure a branch of $\log[1 - \mathcal{X}_e]$ can be defined uniformly.

First observe that by Lemma 5.2.2, at the singular fiber $a = 0$, $\mathcal{X}_e$ cannot be 1 near $\zeta = 0$ or $\zeta = \infty$, so we won't have issues there. Let $\rho_m = \arg Z_m(0)$. Numerical evidence shows that if $\rho_m < \arg a < \rho_m + \pi$, $1 - \mathcal{X}_e(\zeta') = 1 - \exp(i\Upsilon_e(\zeta')$ has negative imaginary part for ζ' along the path $\ell_e = -te^{i \arg a}$. See Figure 1.1. Thus, for this values of $\arg a$, we can choose any branch of the log with a cut in the closed upper half plane.

For $\rho_m + \pi \leq \arg a <= \rho_m + 2\pi$, $1 - \exp(i\Upsilon_e(\zeta')$ jumps and it may hit 0

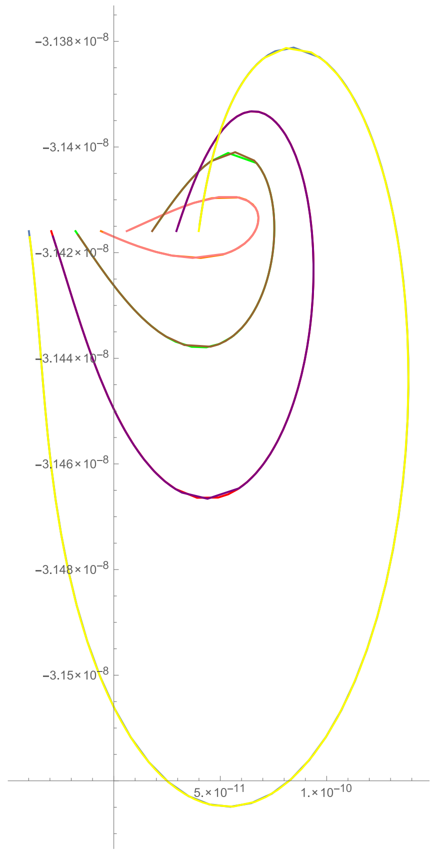

Figure 1.1: $1 - e^{i\Upsilon_e}$ for different values of $\arg a$

along the ℓ_e ray. Fortunately, for these values of $\arg a$, the path where the $\mathcal{X}_e$ jump can be deformed to a ray in a small sector centered at the ray ℓ forming a right angle with ℓ_m as in Figure 1.2.

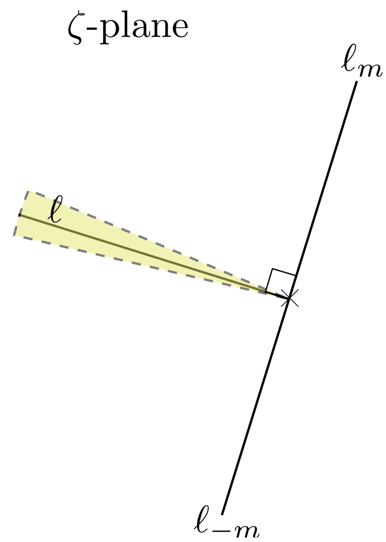

Figure 1.2: A small sector around ℓ

For $\theta_m = 0$, $\Upsilon_e|_{\zeta=0} = \Upsilon_e|_{\zeta=\infty}$ and along ℓ_e, $1-\exp(i\Upsilon_e(\zeta'))$ stays away from 0 as Figure 1.3 shows. As θ_m goes from 0 to π, $\Upsilon_e|_{\zeta=0}$ and $\Upsilon_e|_{\zeta=\infty}$ drift apart and for θ_m in between, $1 - \exp(i\Upsilon_e(\zeta')$ along ℓ may hit 0, as seen in Figure 1.4.

Then as θ_m completes its cycle past $\pi/2$ until 2π, $\Upsilon_e|_{\zeta=0} = \Upsilon_e|_{\zeta=\infty}$ again and the behavior is mirrored in quadrant IV, as Figure 1.5

Note that by deforming the ray ℓ within the shadowed sector, we can make that all the graphs of $1-e^{i\Upsilon_e}$ along this path avoid the positive imaginary axis. Thus, it is now possible to choose a branch of $\log[1 - e^{i\Upsilon_e}]$ that works

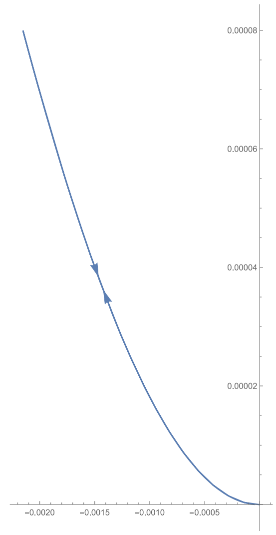

Figure 1.3: $1 - e^{i\Upsilon_e}$ at $\theta_m = 0$

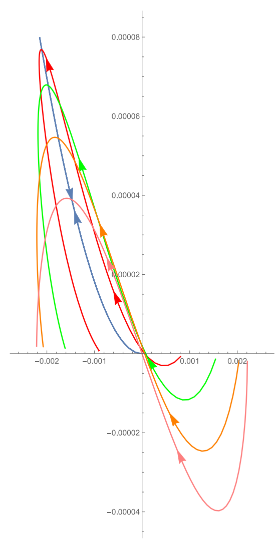

Figure 1.4: $1 - e^{i\Upsilon_e}$ along ℓ for distinct $0 \leq \theta_m \leq \frac{\pi}{2}$

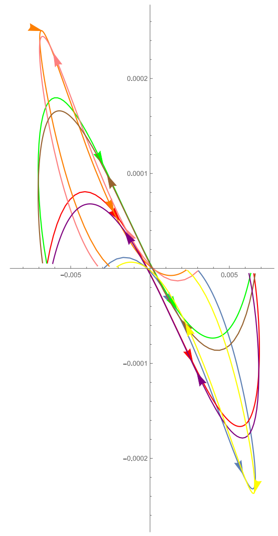

Figure 1.5: $1 - e^{i\Upsilon_e}$ along ℓ for distinct $0 \leq \theta_m \leq 2\pi$

for any value of $\arg a$ and θ_m near $\theta_e = 0$.

By the definition of the integral equations (5.62), (5.63), if $\theta_e = 0$, then automatically $\mathcal{X}_e^{(\nu)} = 1, \mathcal{X}_m^{(\nu)} = 0$ for all ν and any θ_m, ζ at the singular fiber $a = Z_e = 0$. Since $\mathcal{X}$ in the Pentagon are $\mathcal{X}^{\mathrm{OV}}$ plus exponentially small smooth corrections near $a = 0$, this gives solid evidence for Conjecture 2.2.4. Note that, as before, the hyperkähler metric g looks like a factor of the Taub-NUT metric $g_{\text{Taub-NUT}}$ plus smooth corrections.

Bibliography

[1] Werner Balser. *Formal power series and linear systems of meromorphic ordinary differential equations.* Universitext. Springer-Verlag, New York, 2000.

[2] Olivier Biquard and Philip Boalch. Wild non-abelian Hodge theory on curves. *Compos. Math.*, 140(1):179–204, 2004.

[3] Philip Boalch. Symplectic manifolds and isomonodromic deformations. *Adv. Math.*, 163(2):137–205, 2001.

[4] Sergio Cecotti and Cumrun Vafa. On classification of $N = 2$ supersymmetric theories. *Comm. Math. Phys.*, 158(3):569–644, 1993.

[5] Kwokwai Chan. The Ooguri-Vafa metric, holomorphic discs and wall-crossing. *Math. Res. Lett.*, 17(3):401–414, 2010.

[6] Daniel S. Freed. Special Kähler manifolds. *Comm. Math. Phys.*, 203(1):31–52, 1999.

[7] Davide Gaiotto, Gregory Moore, and Andrew Neitzke. Four-dimensional wall-crossing via three-dimensional field theory. *Communications in Mathematical Physics*, 299:163–224, 2010. 10.1007/s00220-010-1071-2.

[8] Davide Gaiotto, Gregory W. Moore, and Andrew Neitzke. Wall-crossing, Hitchin systems, and the WKB approximation. *Adv. Math.*, 234:239–403, 2013.

[9] I. M. Gel′fand and O. Mathieu. On the cohomology of the Lie algebra of Hamiltonian vector fields. *J. Funct. Anal.*, 108(2):347–360, 1992.

[10] Mark Gross. Special Lagrangian fibrations. II. Geometry. A survey of techniques in the study of special Lagrangian fibrations. In *Surveys in differential geometry: differential geometry inspired by string theory*, volume 5 of *Surv. Differ. Geom.*, pages 341–403. Int. Press, Boston, MA, 1999.

[11] Mark Gross and P. M. H. Wilson. Large complex structure limits of $K3$ surfaces. *J. Differential Geom.*, 55(3):475–546, 2000.

[12] N. J. Hitchin. The self-duality equations on a Riemann surface. *Proc. London Math. Soc. (3)*, 55(1):59–126, 1987.

[13] N. J. Hitchin, A. Karlhede, U. Lindstrm, and M. Roek. Hyperkähler metrics and supersymmetry. *Communications in Mathematical Physics*, 108(4):535–589, 1987.

[14] R. S. Ismagilov. *Representations of infinite-dimensional groups*, volume 152 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1996. Translated from the Russian manuscript by D. Deart.

[15] Maxim Kontsevich and Yan Soibelman. Stability structures, motivic Donaldson-Thomas invariants and cluster transformations. 2008.

[16] J. D. Murray. *Asymptotic analysis*, volume 48 of *Applied Mathematical Sciences*. Springer-Verlag, New York, second edition, 1984.

[17] Andrew Neitzke. Notes on a new construction of hyperkahler metrics. 2013.

[18] Hirosi Ooguri and Cumrun Vafa. Summing up Dirichlet instantons. *Phys. Rev. Lett.*, 77(16):3296–3298, 1996.

[19] N. Seiberg and E. Witten. Gauge dynamics and compactification to three dimensions. In *The mathematical beauty of physics (Saclay, 1996)*, volume 24 of *Adv. Ser. Math. Phys.*, pages 333–366. World Sci. Publ., River Edge, NJ, 1997.

Vita

César A. Garza Garza was born in Delicias, Mexico on 9 August 1983. He received the Bachelor of Science degree in Engineering from ITESM Campus Ciudad Juárez, Mexico in 2005. He worked as a software engineer for Delphi Automotive Systems. He left Delphi in 2007 to pursue a Masters in Mathematics at the University of Texas at El Paso. In 2009, he entered in the Ph.D. program at the University of Texas at Austin. For Fall 2015, he'll start a Postdoctoral position at Indiana University, Purdue University Indianapolis.

Permanent address: 7121 Hart Ln.
Austin, Texas 78731

This dissertation was typeset with LaTeX[†] by the author.

[†]LaTeX is a document preparation system developed by Leslie Lamport as a special version of Donald Knuth's TeX Program.