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**Old and New Persectives on Effective Equations: A Study of Quantum Many-Body Systems**

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# Old and New Persectives on Effective Equations: A Study of Quantum Many-Body Systems

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# Old and New Persectives on Effective Equations: A Study of Quantum Many-Body Systems

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This dissertation focuses on the study of nonlinear-Schrödinger-type equations as partial differentiation equations (PDEs) arising as effective descriptions of systems of finitely many interacting bosons. We approach this topic from two perspectives. The *old* perspective consists of proving quantitative convergence in an appropriate function space of solutions to the finite problem to a solution of an effective, limiting PDE, as the number of particles tends to infinity. The *new* perspective consists of proving qualitative convergence of geometric structure, such as the properties of being an integrable and Hamiltonian system. Through these two complementary perspectives, focusing on both quantitative and qualitative convergence, we gain a deeper understanding of how field theories, both classical and quantum, may be deformed to a new field theory, and of how this deformation may be reversed.

# Table of Contents

# List of Tables

# Chapter 1

# Introduction

## 1.1 The Cubic Nonlinear Schrödinger Equation

Hamiltonian partial differential equations (PDEs) are a ubiquitous class of equations which arise as models of physical systems exhibiting at least one, and often several, conservation laws. While the framework of finite-dimensional Hamiltonian systems was initially introduced to formalize Newtonian mechanics, infinite-dimensional Hamiltonian systems have since become a vast area of study, comprising an important class of models in diverse areas such as fluid mechanics, plasma physics, and quantum many-body systems. Establishing a comprehensive mathematical theory of infinite-dimensional Hamiltonian systems which is rich enough to accommodate all the physical problems of interest seems beyond reach; however, one can make mathematically rigorous sense of infinite-dimensional Hamiltonian systems in many interesting cases, see for instance [16] and [2].

Integrable PDE are a special class of Hamiltonian PDE which, broadly speaking, can be solved explicitly,[1] for instance by the inverse scattering transform (IST) discovered by Gardner, Greene, Kruskal and Miura [34] and its subsequent reformulation by Lax [51]. In the years since these (and many other) landmark works, there has been much activity on

[1]Originally, the typical method employed to solve such systems was by method of "quadratures," or, in other words, integration.

determining which equations, and more generally, systems, are or should be integrable and the mathematical consequences of being integrable. The reader may acquire a sense for the scope of this activity in the very nice survey [19] of Deift. Despite the lively, ongoing debate [100] over the defining features of integrability, consensus holds that certain equations, such as the *Korteweg-de Vries (KdV)* or *one-dimensional cubic nonlinear Schrödinger equation (NLS)*, should be integrable under any reasonable definition of the term.

Thus, a compelling example of an integrable, Hamiltonian PDE is the *cubic nonlinear Schrödinger equation (NLS)* in one spatial dimension, which, together with its $d$-spatial-dimensional analogues, is the subject of this dissertation:

$$i\partial_t \phi + \Delta\phi = 2\kappa|\phi|^2\phi, \qquad (t,x) \in \mathbb{R}\times\mathbb{R}, \ \ \kappa \in \{\pm 1\}. \tag{1.1.1}$$

The NLS is a ubiquitous model in physics for approximately describing propagation in *dispersive* media, which have the property that wave packets of different frequencies travel at different velocities. The NLS arises in a myriad of contexts, ranging from Bose-Einstein condensates to water waves to fiber optics. In this dissertation, we are interested in the physical setting of a quantum-mechanical system of bosons, which corresponds to Bose-Einstein condensates.

## 1.2 Old Perspective: Derivation via Dynamics

Over recent years, many authors have sought how to understand the manner in which the dynamics of the NLS arise as an *effective equation*. By effective equation, we mean that solutions of the NLS equation approximate solutions to an underlying physical equation in some topology in a particular asymptotic regime.

In the field of quantum many-body systems, the traditional understanding of a derivation of the NLS from the dynamics of the system of bosons has been as follows. For simplicity, we shall sketch the derivation starting from the *Lieb-Liniger (LL) model*, which describes a finite number of bosons in one dimension with two-body interactions governed by the $\delta$ potential. Formally, the Hamiltonian for $N$ bosons is given by

$$\sum_{i=1}^{N} -\Delta_i + c \sum_{1 \leq i < j \leq N} \delta(X_i - X_j), \tag{1.2.1}$$

where $-\Delta_i$ denotes the Laplacian in the $i$-th particle variable $x_i \in \mathbb{R}$, $\delta(X_i - X_j)$ denotes multiplication by the distribution $\delta(x_i - x_j)$, and $c \in \mathbb{R}$ is the coupling constant determining the strength of the interaction and whether it is repulsive ($c > 0$) or attractive ($c < 0$). The LL model is named for Lieb and Liniger, who showed in the seminal works [54, 53] that when considered on a finite interval $[0, L]$ with periodic boundary conditions, the model is exactly solvable by Bethe ansatz.[2] While it was originally introduced as a toy quantum many-body system, the LL model has since attracted interest from both the physics community [70, 75, 23, 43, 55, 71, 22] and the mathematics community [56, 87] in modeling quasi-one-dimensional dilute Bose gases which have been realized in laboratory settings [21, 81, 96, 27].

In applications, the number of particles $N$ is large, ranging upwards from $N \approx 10^3$ in the case of very dilute Bose-Einstein condensates. For large $N$, it is computationally expensive to extract useful information about the time evolution of the system directly from its wave function. Thus, one seeks to find an evolution equation, for which one can more

[2] Bethe ansatz refers to a method in the study of exactly solvable models originally introduced by Hans Bethe to find exact eigenvalues and eigenvectors of the antiferromagnetic Heisenberg spin chain [11]. For more on this technique and its applications, we refer the reader to the monograph [35].

efficiently extract information, that provides an effective description of the $N$-body system for large values of $N$.

Accordingly, the goal of Chapter 2 of this dissertation is to rigorously obtain an effective description of the dynamics of the LL model in the limit as the number of particles tends to infinity. To obtain nontrivial dynamics in the limit, we consider the mean-field scaling regime, where the coupling constant $c$ in (1.2.1) is taken to be equal to $2\kappa/N$, for some $\kappa \in \mathbb{R} \setminus \{0\}$, so that the Hamiltonian becomes

$$H_N = \sum_{i=1}^{N} -\Delta_i + \frac{2\kappa}{N} \sum_{1 \leq i < j \leq N} \delta(X_i - X_j). \tag{1.2.2}$$

Note that the mean-field scaling is such that the free and interacting components of the Hamiltonian $H_N$ are of the same order in $N$. By means of quadratic forms (see Section 2.3), the expression (1.2.2) can be realized as a self-adjoint operator on the Hilbert space $L^2_{sym}(\mathbb{R}^N)$ consisting of wave functions $\Phi_N \in L^2(\mathbb{R}^N)$ satisfying

$$\Phi_N(x_{\pi(1)}, \ldots, x_{\pi(N)}) = \Phi_N(x_1, \ldots, x_N) \text{ almost everywhere}, \qquad \forall \pi \in \mathbb{S}_N. \tag{1.2.3}$$

By Stone's theorem, the corresponding Schrödinger problem

$$\begin{cases} i\partial_t \Phi_N = H_N \Phi_N \\ \Phi_N(0) = \Phi_{N,0} \in L^2_{sym}(\mathbb{R}^N) \end{cases} \tag{1.2.4}$$

has a unique global solution $\Phi_N(t) = e^{-itH_N}\Phi_{N,0}$. Of particular interest are *factorized* initial data $\Phi_{N,0} = \phi_0^{\otimes N}$, for $\phi_0 \in L^2(\mathbb{R})$ satisfying $\|\phi_0\|_{L^2(\mathbb{R})} = 1$, which correspond to a system where the $N$ particles are all in the same initial state $\phi_0$. By rescaling spacetime, it suffices to consider the case $\kappa \in \{\pm 1\}$.

In general, factorization of the wave function $\Phi_N$ is not preserved under the time evolution due to the interaction between particles. However, it is reasonable to expect from the factor of $\frac{1}{N}$ in the potential term in (1.2.2) that the total potential experienced by each particle is approximately described by an effective *mean-field* potential in the limit as $N \to \infty$. Formally, we may expect that

$$\Phi_N \approx \phi^{\otimes N} \quad \text{as } N \to \infty, \tag{1.2.5}$$

for some $\phi : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$, in some sense to be made precise momentarily.

To find an equation satisfied by $\phi$ and to give rigorous meaning to the approximation (1.2.5), we argue as follows. Let $\Phi_N$ be the solution to the Schrödinger equation (1.2.4), and consider the *density matrix*

$$\Psi_N \coloneqq |\Phi_N\rangle \langle \Phi_N| \tag{1.2.6}$$

associated to $\Phi_N$.[3] This density matrix is the rank-one projection onto the state $\Phi_N$ with integral kernel

$$\Psi_N(t, \underline{x}_N; \underline{x}_N') = \Phi_N(t, \underline{x}_N)\overline{\Phi(t, \underline{x}_N')}, \qquad \forall \underline{x}_N, \underline{x}_N' \in \mathbb{R}^N, \; t \in \mathbb{R}. \tag{1.2.7}$$

For $k \in \{1, \ldots, N\}$, we define the *$k$-particle reduced density matrix* $\gamma_N^{(k)}$ associated to $\Phi_N$ by

$$\gamma_N^{(k)} \coloneqq \mathrm{Tr}_{k+1,\ldots,N} \Psi_N, \tag{1.2.8}$$

where $\mathrm{Tr}_{k+1,\ldots,N}$ denotes the partial trace over the coordinates $(x_{k+1}, \ldots, x_N)$. By conservation of mass for (1.2.4) (i.e. $\|\Phi_N(t)\|_{L^2(\mathbb{R}^N)} = \|\Phi_{N,0}\|_{L^2(\mathbb{R}^N)} = 1$), it follows that $\mathrm{Tr}_{1,\ldots,k}(\gamma_N^{(k)}(t)) = 1$ for every $N \in \mathbb{N}$, $k \in \{1, \ldots, N\}$, and $t \in \mathbb{R}$. Using equation (1.2.4),

[3] Here and in the sequel, we use Dirac's bra-ket notation: for $f, g, h \in L^2(\mathbb{R}^d)$, the operator $|f\rangle \langle g| : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is defined by $(|f\rangle \langle g|)h = \langle g|h\rangle_{L^2} f$. The integral kernel of $|f\rangle \langle g|$ is $f(x)\overline{g(x')}$.

one can show that $\{\gamma_N^{(k)}\}_{k=1}^N$ solve the coupled system of equations known as the *Bogoliubov-Born-Green-Kirkwood-Yvon hierarchy (BBGKY) hierarchy*:

$$\begin{aligned} i\partial_t \gamma_N^{(k)} &= \left[-\underline{\Delta}_k, \gamma_N^{(k)}\right] + \frac{2\kappa}{N} \sum_{1\leq \ell<j\leq k} \left[\delta(X_\ell - X_j), \gamma_N^{(k)}\right] \\ &\quad + \frac{2(N-k)\kappa}{N} \sum_{j=1}^{k} \mathrm{Tr}_{k+1}\Big(\left[\delta(X_j - X_{k+1}), \gamma_N^{(k+1)}\right]\Big), \end{aligned} \tag{1.2.9}$$

where we have introduced the notation $\underline{\Delta}_k := \sum_{i=1}^k \Delta_i$ and $[\cdot,\cdot]$ denotes the usual commutator bracket. As $N \to \infty$, the sequence $\{\gamma_N^{(k)}\}_{k\in\mathbb{N}}$, where by convention $\gamma_N^{(k)} := 0$ for $k > N$, formally converges to a solution $\{\gamma_k\}_{k\in\mathbb{N}}$ of the *Gross-Pitaevskii (GP) hierarchy*:

$$i\partial_t \gamma^{(k)} = \left[-\underline{\Delta}_k, \gamma^{(k)}\right] + 2\kappa \sum_{j=1}^{k} \mathrm{Tr}_{k+1}\big(\left[\delta(X_j - X_{k+1}), \gamma^{(k+1)}\right]\big). \tag{1.2.10}$$

If there is some function $\phi : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$, such that the GP solution takes the form $\gamma^{(k)} = |\phi^{\otimes k}\rangle \langle \phi^{\otimes k}|$ for every $k \in \mathbb{N}$, it is an easy computation from (1.2.10) that $\phi$ solves the *one-dimensional (1D) cubic nonlinear Schrödinger (NLS) equation*

$$(i\partial_t + \Delta)\phi = 2\kappa|\phi|^2\phi, \qquad \phi(0) = \phi_0. \tag{1.2.11}$$

Thus, we formally refer to the 1D cubic NLS as the *mean-field limit* of the LL model. It is quite interesting that just as the LL model is exactly solvable by Bethe ansatz, as we commented above, the 1D cubic NLS is exactly solvable by the inverse scattering transform [101, 28]. In Chapter 4, we consider the relationship between $N$-body exact solvability and limiting exact solvability. See also the remarks at the end of Section 1.3.2.

Establishing the validity of the mean-field approximation to the Schrödinger problem (1.2.4) consists of showing convergence of the $k$-particle reduced density matrices $\gamma_N^{(k)}$ to

$|\phi^{\otimes k}\rangle\,\langle\phi^{\otimes k}|$, as $N \to \infty$, in trace norm:

$$\forall k \in \mathbb{N}, \qquad \lim_{N\to\infty} \mathrm{Tr}_{1,\ldots,k} \left| \gamma_N^{(k)} - |\phi^{\otimes k}\rangle\,\langle\phi^{\otimes k}| \right| = 0. \tag{1.2.12}$$

We refer to (1.2.12) as *convergence to the mean-field limit* or, following terminology in the kinetic theory literature, as *propagation of chaos.*

The mathematical investigation of the validity of the mean-field approximation for the LL model was initiated by Adami, Bardos, Golse, and Teta [3]. The authors proceed by the so-called BBGKY method, which was pioneered by Spohn [90] for the study of quantum many-body systems. Namely, Adami et al. show that for each $k \in \mathbb{N}$ fixed, the sequence $\{\gamma_N^{(k)}\}_{N\in\mathbb{N}}$ has a limit point $\gamma^{(k)}$ with respect to a topology weaker than trace norm. They then show that the sequence $\{\gamma^{(k)}\}_{k\in\mathbb{N}}$ is a solution to the GP hierarchy (1.2.10) with initial datum $(\,|\phi_0^{\otimes k}\rangle\,\langle\phi_0^{\otimes k}|)_{k\in\mathbb{N}}$ in a certain class akin to the Sobolev space $H^1$. In order to conclude their proof, they need to show that there can only be one such solution (i.e. prove uniqueness for the GP hierarchy in the class under consideration), from which propagation of chaos (1.2.12) follows. However, they could not prove this uniqueness, and to our knowledge, their argument has yet to be completed. We remark that the BBGKY approach does not yield a rate of convergence in (1.2.12) as $N \to \infty$ and $|t| \to \infty$.

Several years later, Ammari and Breteaux [6] revisited the mean-field approximation to the LL model from the perspective of quantum field theory. Inspired by the approach of Rodnianski and Schlein [82], which in turn builds on earlier ideas of Hepp [39] and Ginibre and Velo [36], the authors use the framework of second quantization and reformulate the problem of mean-field limit for the Hamiltonian (1.2.2) in terms of the semiclassical limit for a related Hamiltonian on the Fock space. Through a very technical argument involving abstract non-autonomous Schrödinger equations, they construct a time-dependent quadratic Hamiltonian which provides a semiclasical approximation for the evolution of coherent states. Borrowing an argument from [82], they are able to show the convergence (1.2.12) from their approximation result for coherent states. We note that the authors do not provide a quantitative rate for the convergence (1.2.12) in terms of $N$ and $t$.

In Chapter 2, we give a simple, quantitative proof of the validity of the mean-field convergence (1.2.12). We defer a precise statement of our result (see Theorem 2.1.1) until Section 2.1, so as to maintain the accessibility of the introduction. Our proof is inspired by the method of Pickl [76, 77, 78] and Knowles and Pickl [46] and is based on an energy-type estimate for a time-dependent functional which gives a weighted count of the number of particles in the system at time $t$ which are not in the state $\phi(t)$. To overcome difficulties stemming from the singularity of the $\delta$-potential, we introduce a new short-range approximation argument that exploits the Hölder continuity of the $N$-body wave function in a single particle variable. In contrast to the previous work of Ammari and Breteaux [6], our simple proof makes no use of second quantization and provides an explicit rate of convergence to the mean-field limit.

## 1.3 New Perspective: Derivation via Geometry

### 1.3.1 Hamiltonian Structure

In contrast to the vast amounts of activity on the derivation of the dynamics of the NLS, to the best of our knowledge, questions about the origins of the Hamiltonian structure of the NLS have remained unexplored. Indeed, continuing with our example from the previous section, the $N$-body Schrödinger problem is well-known to admit a description as an infinite-dimensional Hamiltonian system, but we are unaware of work which mathematically demonstrates whether, and if so the manner in which, the Hamiltonian structure of the NLS can be interpreted as a limit of the Hamiltonian structure of the $N$-body Schrödinger problem.

This line of inquiry is not merely aesthetically pleasing. Since the Hamiltonian structure completely determines an equation's behavior as a dynamical system, understanding how the geometry arises from the underlying physical system is foundational for understanding how complex behavior is a limiting effect of the system in a specified scaling regime. Furthermore, from the physics' perspective of connecting field theories, both classical and quantum, one often obtains a new field theory by deformation (e.g. first and second quantization) of one Hamiltonian structure to another. Ideally, one would like to know that this process is reversible, in the sense that a certain scaling limit recovers the initial structure. See Remark 1.3.3 for further elaboration on this point.

The Hamiltonian formulation for the NLS has two components: the Hamiltonian functional itself and an underlying phase space geometry provided by a weak Poisson manifold.[4] More precisely, to give the Hamiltonian formulation of the NLS, we endow the $d$-dimensional Schwartz space $\mathcal{S}(\mathbb{R}^d)$ with the standard weak symplectic structure

$$\omega_{L^2}(f,g) = 2\,\mathrm{Im}\left\{\int_{\mathbb{R}^d} dx \overline{f(x)} g(x)\right\}, \quad \forall f,g \in \mathcal{S}(\mathbb{R}^d). \tag{1.3.1}$$

Letting $\boldsymbol{\nabla}_s$ denote the symplectic $L^2$ gradient, see Remark 3.3.12, the symplectic form $\omega_{L^2}$

[4]We refer to Definition 3.3.1 and Definition 3.3.5 for definitions of a weak Poisson and weak symplectic manifold, respectively.

induces the canonical Poisson structure

$$\{F, G\}_{L^2}(\cdot) \coloneqq \omega_{L^2}(\boldsymbol{\nabla}_s F(\cdot), \boldsymbol{\nabla}_s G(\cdot)), \tag{1.3.2}$$

defined for $F, G$ belonging to a certain sub-algebra $A_{\mathcal{S}} \subset C^\infty(\mathcal{S}(\mathbb{R}^d); \mathbb{R})$, the precise description of which we postpone to Proposition 3.3.13. The solution of the NLS (1.3.7) is then the flow associated to a Hamiltonian equation of motion on the infinite-dimensional weak Poisson manifold $(\mathcal{S}(\mathbb{R}^d), \mathcal{A}_{\mathcal{S}}, \{\cdot, \cdot\}_{L^2})$. More precisely, (1.3.7) is equivalent to

$$\left(\frac{d}{dt}\phi\right)(t) = \boldsymbol{\nabla}_s \mathcal{H}_{NLS}(\phi(t)), \tag{1.3.3}$$

where

$$\mathcal{H}_{NLS}(\phi(t)) \coloneqq \int_{\mathbb{R}^d} dx \big(|\nabla\phi(t,x)|^2 + \kappa|\phi(t,x)|^4\big). \tag{1.3.4}$$

The goal of Chapter 3 of this dissertation is to derive both the weak Poisson structure and Hamiltonian functional constituting the Hamiltonian formulation of the NLS. Providing a rigorous definition and derivation of the geometry will pose the bulk of the difficulty in this work.

The methods we adopt are guided by the extensive research activity in recent years on the derivation of NLS-type equations from the dynamics of interacting bosons, as discussed in Section 1.2. There are a number of different approaches to this derivation problem beginning with the aforementioned influential work of Hepp [39], later generalized by Ginibre and Velo [36]. But the one which informs our strategy involves the BBGKY hierarchy introduced in equation (1.2.9) (see also (3.1.4) below for the precise version considered in Chapter 3). This approach was pioneered by Spohn [90] in the quantum context of the derivation of the Hartree equation in the mean field scaling regime.[5] We mention the works of Adami, Bardos, Golse, and Teta and Adami, Golse, and Teta [3, 4], who provided a derivation of the one-dimensional cubic NLS via the BBGKY approach in an intermediate scaling regime between the mean field and Gross-Pitaevskii regimes. We also mention in particular the works of Erdös, Schlein, and Yau [24, 25, 26], who provided the first rigorous derivation of the three-dimensional cubic NLS in the Gross-Pitaevskii scaling regime via the BBGKY hierarchy, resolving what was a significant open problem. There is by now an extensive body of work, spanning many years, on deriving the dynamics of the NLS from many-body quantum systems. A thorough account of this history would take us too far afield from our current goals, and consequently we are not mentioning many important contributions in our very brief account. We instead refer the reader to [85] for a general survey and more extensive review on the history of the derivation problem and to the more recent lecture notes [84].

To appreciate some of the difficulties involved in our pursuit, it is important to note that while the dynamics of a system of $N$-bosons is described by the linear Schrödinger evolution of a wave function, such an equation is not amenable to taking the infinite-particle limit directly since the wave functions for different particle numbers do not live in a common topological space. Consequently, in order to take an infinite-particle limit, one performs a non-linear transformation of the $N$-body wave functions and considers sequences of $k$-particle marginal density matrices whose evolution is governed by the BBGKY hierarchy. In particular, there is no clear link between the evolution of the $N$-particle wave function and the NLS each as Hamiltonian dynamical systems. To complicate matters further, the

[5] See also the influential works of Lanford [49, 50] on the derivation of the Boltzmann equation.

BBGKY hierarchy is no longer an evidently Hamiltonian flow.

At the cost of the added complication of working with the BBGKY hierarchy, the aforementioned works on the derivation of the one-particle dynamics actually yield the following stronger result: the full dynamics of the interacting boson system governed by the BBGKY hierarchy converges to dynamics described by the cubic GP hierarchy, which is an infinite coupled system of partial differential equations for kernels[6] $(\gamma^{(k)})_{k=1}^{\infty}$ of $k$-particle density matrices, defined in (1.2.10) above. The connection to the NLS is then as follows: the GP hierarchy admits a special class of factorized solutions given by

$$\gamma^{(k)} := |\phi^{\otimes k}\rangle \langle \phi^{\otimes k}|, \qquad k \in \mathbb{N}, \tag{1.3.5}$$

where $\phi : I \times \mathbb{R}^d \to \mathbb{C}$ solves the cubic NLS

$$i\partial_t \phi + \Delta\phi = 2\kappa|\phi|^2\phi, \qquad (t,x) \in \mathbb{R} \times \mathbb{R}^d. \tag{1.3.6}$$

One might conjecture that the BBGKY and GP hierarchies provide the required link to understand the derivation of the geometry associated to the Hamiltonian formulation of (1.3.7). In particular, it is natural to wonder whether the BBGKY and GP hierarchies are Hamiltonian evolution equations posed on underlying weak Poisson manifolds of density matrices,[7] and whether the Poisson structure for the infinite-particle setting arises in the infinite-particle limit from the Poisson structure for the $N$-body problem. To summarize, one can pose the following questions:

[6]In this work, we follow the widespread convention of using the same notation for both the kernel and the operator.

[7]We will in fact work on a Poisson manifold of density matrix *hierarchies*.

**Question 1.3.1.** Can we connect the Hamiltonian structure of the many-body system with that of the infinite-particle system in the following sense: can the GP hierarchy be realized as a Hamiltonian equation of motion with associated functional $\mathcal{H}_{GP}$ on some weak Poisson manifold? Can the Poisson structure and Hamiltonian functional for the GP hierarchy be derived in a suitable sense from a Poisson structure and Hamiltonian functional at $N$-particle level?

In the current work, we answer these questions affirmatively and establish, for the first time, a Hamiltonian formulation for the BBGKY and GP hierarchies, see Theorem 3.1.3 and Theorem 3.1.10 below, and a link between the underlying weak Poisson geometry and Hamiltonian functionals in the finite- and infinite-particle settings, see Proposition 3.1.4.

Our geometric constructions will rely on a special type of weak Poisson structure, namely a Lie-Poisson structure, on a space of density matrix $\infty$-hierarchies, see Section 3.1.2 below. These constructions are motivated by the work of Marsden, Morrison, and Weinstein [59] on the Hamiltonian structure of the classical BBGKY hierarchy, which relates to the earlier works on the Hamiltonian structure for plasma systems discovered in Morrison and Green [68], Morrison [66, 67], Marsden and Weinstein [61], Spencer and Kaufman [89], and Spencer [88]. We refer to [57] for more discussion on the Hamiltonian formulation of equations of motion for systems arising in plasma physics. Our geometric perspective for the $N$-body Schrödinger equation is inspired by taking a “quantized” version of the work of [59]. By adapting their work to the quantum setting, we obtain the formulae for the Poisson structure for the (quantum) BBGKY hierarchy. Taking the infinite-particle limit, which was not considered in [59], we obtain the formula for the Poisson structure we use in the infinite-particle setting. We expect that our proofs can serve as a blueprint for deriving the Hamiltonian structure of more general infinite-particle equations arising from systems of interacting classical and quantum particles.

Returning to the setting of the NLS, the fact that the GP hierarchy admits the factorized solutions given by (1.3.5) tells us that the dynamics of the NLS are embedded in those of the GP hierarchy. Given that the NLS is a Hamiltonian system and, with our affirmative answer to Question 1.3.1, so is the GP hierarchy, one might ask if there exists an embedding of the Hamiltonian structure such that the pullback of this embedding yields the NLS Hamiltonian and phase space geometry from that of the GP. In other words, one can pose the following question:

**Question 1.3.2.** Given our affirmative answer to the previous question, is there then a natural way to connect the Hamiltonian formulation of the GP hierarchy with the Hamiltonian formulation of the NLS in such a manner so as to respect the geometric structure?

We provide an affirmative answer to this second question by showing, in Theorem 3.1.12 below, that the natural embedding map taking one-particle functions to factorized density matrices described in (1.3.5) is a Poisson morphism between the weak symplectic manifold constituting the NLS phase space and the weak Poisson manifold[8] constituting the GP phase space. Moreover, the NLS Hamiltonian, see (1.3.4) below, is just the pullback of the GP Hamiltonian under this embedding, see (3.1.30) below. In summary, the factorization embedding pulls back the GP Hamiltonian structure to that of the NLS.

We claim that our work provides a new perspective on what it means to "derive" an equation from an underlying physical problem. Indeed, to justify this assertion, we highlight

[8]We refer to Section 3.3 for definitions of Poisson morphism and weak Poisson manifold.

some parallels between our results and the aforementioned works of Erdös et al. on the derivation of solutions to the NLS equation from the $N$-body problem. In [24, 25, 26], solutions to the BBGKY hierarchy with factorized or asymptotically factorized initial data are shown to converge to solutions of the GP hierarchy as the number of particles tends to infinity. The authors then show that solutions to the GP hierarchy in a certain Sobolev-type space are unique.[9] Thus, the solution to the NLS equation provides the unique solution to the GP hierarchy starting from factorized initial data, thereby providing a rigorous derivation of the dynamics of the NLS from (3.1.2). In the current work, we establish the existence of both the underlying Lie algebra and Poisson structure associated to a Hamiltonian formulation of the BBGKY hierarchy and prove that in the infinite-particle limit, these converge to a (previously unobserved) Hamiltonian structure for the GP hierarchy. Moreover, the BBGKY Hamiltonian, defined in (3.1.16), converges to the GP Hamiltonian. Finally, we demonstrate that the Hamiltonian functional and phase space of the NLS can be obtained via the pullback of the canonical embedding (3.1.38), thereby providing a derivation of the Hamiltonian structure of the NLS.

**Remark 1.3.1.** We note that our work does not address any derivation of the *dynamics* of the nonlinear Schrödinger equation from many-body quantum systems in the vein of the aforementioned works by Erdös et al. [24, 25, 26]. Our current work is complementary to those in the sense that it addresses geometric aspects of the connection of the NLS with quantum many-body systems, answering questions which are of a different nature than those about the dynamics.

[9] A new proof of this uniqueness result was later given by Chen et al. in [14].

**Remark 1.3.2.** We view this work as part of broader program of understanding how qualitative properties of PDE arise from underlying physical problems, in particular the importance of the Hamiltonian formalism. Related to this program, we mention the works of Fröhlich, Tsai, and Yau [32]; Fröhlich, Knowles, and Pizzo [29]; and Fröhlich, Knowles, and Schwarz [31]. While these works concern quantization, mean field theory, and the dynamics of the Hartree and Vlasov equations, the interpretation of these equations as infinite-dimensional Hamiltonian systems and more generally the Hamiltonian perspective figures prominently in these very interesting works. We also mention the works of Lewin, Nam, and Rougerie [52] and Fröhlich, Knowles, Schlein, and Sohinger [30], which derive invariant Gibbs measures for the NLS from many-body quantum systems, as we believe they are related in spirit to this program.

**Remark 1.3.3.** As a final inspirational thought for this subsection, we share the suggestion of Moshe Flato, which we learned of from [29], that new physical theories obtained in the early 20th century developments of Quantum Mechanics, Special Relativity, and General Relativity arise from "deformations of precursor theories". Based on the results of Chapter 3, we tentatively supplement Flato's suggestion with the idea that the precursor theory should be recoverable from the new physical theory through a limiting procedure.

### 1.3.2 Integrability

Even with the vast research on the implications of an equation's integrability, such as conserved quantities, solitons, or hidden symmetries, it remains unclear *why* equations which are so physically relevant also happen to be integrable. Mathematical insight into this line of inquiry would certainly deepen our understanding of the important models that comprise the extensive catalog of known integrable systems. In an article [12] on this very question, Calogero advances his thesis that equations are integrable because they are scaling limits of integrable (or conjecturally integrable) systems, which we refer to as *progenitor models* in this discussion.

Inspired by Calogero's suggestion, Chapter 4 of this dissertation considers aforementioned 1D cubic NLS

$$i\partial_t\phi + \Delta\phi = 2\kappa|\phi|^2\phi, \qquad \phi : \mathbb{R} \times \mathbb{R} \to \mathbb{C}, \quad \kappa \in \{\pm 1\}, \tag{1.3.7}$$

which was shown by Zakharov and Shabat [101] to be exactly solvable by the IST (see also [1, 99, 28]). We consider equation (1.3.7) from the viewpoint that it arises as a *mean field* scaling limit from the progenitor LL model (recall (1.2.2) and (1.2.4)), which we discussed in Section 1.2. Keeping with Calogero's thesis, we conjecture that integrability of the NLS is a consequence of the exact solvability of the underlying LL model, leading us to the expectation of some manifestation of integrability intrinsically at the level of the GP hierarchy (1.2.10), for which we saw in (1.3.5) that the NLS corresponds to a special case. Accordingly, Chapter 4 of this dissertation focuses on providing evidence for the GP hierarchy as a new integrable system.

Given the aforementioned debate over the precise definition of an integrable PDE, this work focuses on a particular type of integrability known as *Liouville integrability*. The notion of a Liouville integrable Hamiltonian system was originally introduced in the 19th century and refers to a finite-dimensional Hamiltonian system where there is a maximal (in the sense of degrees of freedom) independent set of Poisson commuting integrals. In the finite-dimensional setting, a Liouville completely integrable system, which satisfies some technical conditions, can be solved by so-called action angle variables, which allow for explicit integration of the system.

The exact solvability of the one-dimensional cubic NLS by the IST was formally shown in the aforementioned work [101] and was mathematically revisited by Beals and Coifman [8, 9, 7, 10], Terng and Uhlenbeck [93, 94], Deift and Zhou [103, 102, 20], among others. Liouville integrability is a particular consequence of this exact solvability, which asserts that the Hamiltonian is one element of a countable sequence of functionals in nontrivial[10] mutual involution. More precisely, one recursively defines (see Appendix 1.2) a sequence of operators

$$w_n : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R}), \qquad \begin{cases} w_1[\phi] & := \phi \\ w_{n+1}[\phi] & := -i\partial_x w_n[\phi] + \kappa\bar{\phi}\sum_{k=1}^{n-1} w_k[\phi]w_{n-k}[\phi]. \end{cases} \tag{1.3.8}$$

Each $w_n$ generates a functional $I_n : \mathcal{S}(\mathbb{R}) \to \mathbb{C}$ by

$$I_n(\phi) := \int_{\mathbb{R}} dx\overline{\phi(x)}w_n[\phi](x), \qquad \forall \phi \in \mathcal{S}(\mathbb{R}), \tag{1.3.9}$$

which is, in fact, real-valued (see Lemma 1.2.2). One can verify (see Appendix 1.3) that

$$\{I_n, I_m\}_{L^2}(\phi) = 0, \qquad \forall \phi \in \mathcal{S}(\mathbb{R}), \ \forall n, m \in \mathbb{N}, \tag{1.3.10}$$

where the reader will recall from (1.3.1) the definition of the $L^2$ Poisson bracket $\{\cdot,\cdot\}_{L^2}$.

Furthermore, the solution to the NLS (1.3.7) is the integral curve to the Hamiltonian equation of motion associated to the third functional $I_3$. That is,

$$\left(\frac{d}{dt}\phi\right)(t) = \boldsymbol{\nabla}_s I_3(\phi(t)). \tag{1.3.11}$$

[10]By nontrivial, we mean that these functionals are not all Casimirs for the Poisson structure (i.e. they Poisson commute with any functional).

In particular, if $\phi \in C^\infty([t_0, t_1]; \mathcal{S}(\mathbb{R}))$ is a classical solution to (1.3.7), then $I_n(\phi)$ is conserved on the lifespan $[t_0, t_1]$ of $\phi$ for every $n \in \mathbb{N}$. Furthermore, each of the functionals $I_n$ has an associated equation of motion

$$\left(\frac{d}{dt}\phi\right)(t) = \boldsymbol{\nabla}_s I_n(\phi(t)). \tag{1.3.12}$$

Following the terminology of Faddeev and Takhtajan [28], we call (1.3.12) the *n-th nonlinear Schrödinger equation (nNLS)*. The $n = 1, 2$ equations are trivial, the $n = 3$ equation is the NLS (1.3.7), and the $n = 4$ equation is the complex mKdV equation

$$\partial_t \phi = \partial_x^3 \phi - 6\kappa |\phi|^2 \partial_x \phi, \qquad \kappa \in \{\pm 1\}. \tag{1.3.13}$$

To our knowledge, the $n$-th nonlinear Schrödinger equations do not have specific names for $n \geq 5$. Together, the family of $n$-th nonlinear Schrödinger equations constitutes the *nonlinear Schrödinger hierarchy*, as termed by Palais [74].

To set the stage for Chapter 4 of this dissertation, we begin by recalling from Section 1.2 the progenitor LL model and its relation to the NLS. As we previously saw, the LL model is the many-body problem

$$i\partial_t \Phi_N = H_N \Phi_N, \qquad H_N = \sum_{j=1}^{N} -\Delta_{x_j} + \frac{2\kappa}{(N-1)} \sum_{1 \leq j < k \leq N} \delta(X_j - X_k), \tag{1.3.14}$$

where $\Phi_N \in L^2_{sym}(\mathbb{R}^N)$, the coupling constant has been taken to be proportional to $1/N$ so that we are in the mean field scaling regime. The value of $\kappa \in \{\pm 1\}$ determines whether the system is repulsive ($\kappa = 1$) or attractive ($\kappa = -1$). Mathematical and physical interest in (1.3.14) stems in large part from its remarkable property of being *exactly solvable*, meaning we have explicit formulae for the eigenfunctions and spectrum of the Hamiltonian $H_N$.

Analogous to the free Schrödinger equation, one has an explicit distorted Fourier transform associated to $H_N$, which by solving an ordinary differential equation in the distorted Fourier domain yields a formula for the solution to (1.3.14).

As we previously saw, the connection between the LL model and the NLS is via an infinite particle limit by way of the GP hierarchy (1.2.10). In light of our previous discussion on Liouville integrability of the NLS, we turn to our search for evidence of integrability at the infinite-particle level. We note that this search necessitates a Hamiltonian formulation of the GP hierarchy, for which we rely on the recent work of the authors [63, Theorem 2.10] that shows that the GP hierarchy is the equation of motion on a weak Poisson manifold for a Hamiltonian $\mathcal{H}_{GP}$. We formulate the following question:

**Question 1.3.3.** Does the one-dimensional cubic GP hierarchy possess an infinite sequence of functionals $\{\mathcal{H}_n\}_{n\in\mathbb{N}}$ containing the Hamiltonian $\mathcal{H}_{GP}$ for the GP hierarchy, which are in nontrivial involution?

We provide an affirmative answer to Question 1.3.3 with our Theorem 4.1.7, evidencing Liouville integrability of the GP hierarchy. Note that an immediate consequence of the affirmative answer to Question 1.3.3 is that the functionals $\mathcal{H}_n$ are conserved along the flow of the GP hierarchy.

The functionals $\mathcal{H}_n$ which we construct are trace functionals associated to the family of observable $\infty$-hierarchies $\{-i\mathbf{W}_n\}_{n\in\mathbb{N}}$ which belong to the Lie algebra $\mathfrak{G}_\infty$ defined in [63], the definition of which we review in Proposition 3.1.7 below. Heuristically speaking, our definition of these observable hierarchies proceeds by a quantization of the recursive formula (1.3.8) for the one-particle nonlinear operators $\{w_n\}_{n\in\mathbb{N}}$. More precisely, we observe that the functionals $I_n$ defined in (1.3.9) are finite sums of multilinear forms whose arguments are restricted to a single function $\phi \in \mathcal{S}(\mathbb{R})$ and its complex conjugate $\overline{\phi} \in \mathcal{S}(\mathbb{R})$:

$$I_n(\phi) = \sum_{k=1}^{N(n)} I_n^{(k)}[\underbrace{\phi, \ldots, \phi}_{k}; \underbrace{\overline{\phi}, \ldots, \overline{\phi}}_{k}], \qquad N(n) \in \mathbb{N}. \tag{1.3.15}$$

A posteriori of our construction, we show that the $k$-particle component $\mathbf{W}_n^{(k)}$ of $\mathbf{W}_n = (\mathbf{W}_n^{(j)})_{j\in\mathbb{N}}$ is the Schwartz kernel of each $I_n^{(k)}$.

To prove the Poisson commutativity of the functionals $\mathcal{H}_n$ with respect to the Poisson structure underlying the GP hierarchy from [63], we simultaneously proceed at the level of the GP hierarchy and at the level of the NLS equation. We combine a good understanding of the multilinear structure of the $I_n$ with a knowledge of the structure of bosonic density matrices to show that Poisson commutativity of the $\mathcal{H}_n$ is equivalent to that of certain functionals $I_{b,n}$ defined in (4.1.40), which are associated to an integrable system generalizing the NLS.[11] We rewrite the NLS (1.3.7) as the system

$$\begin{cases} i\partial_t\phi = -\Delta\phi + 2\kappa\phi^2\overline{\phi} \\ i\partial_t\overline{\phi} = \Delta\overline{\phi} - 2\kappa\overline{\phi}^2\phi \end{cases}, \tag{1.3.16}$$

and relax the requirement that $\overline{\phi}$ denotes the complex conjugate of $\phi$ (i.e. $\phi$ and $\overline{\phi}$ are independent coordinates on $\mathcal{S}(\mathbb{R})$). We then show that the family $\{I_{b,n}\}_{n\in\mathbb{N}}$ is mutually involutive (see Proposition 1.3.7). By also showing that there is a Poisson morphism from the phase space of (1.3.16)[12] to the phase space of the GP hierarchy, we obtain the desired

[11] The inspiration for considering this system comes from a remark of Faddeev and Takhtajan [28, Remark 13, pg. 181].

[12] Strictly speaking, the domain of the morphism is a quotient space of the phase space of (1.3.16) with the property that the elements are "self-adjoint".

conclusion. This equivalence we prove, recorded in (4.1.48) below, is quite interesting in its own right and was not expected by the authors at the onset of this project.

**Remark 1.3.4.** In [64], the author's four co-authors of the article [62], which is the basis of Chapter 4 of this dissertation, identified an infinite sequence of conserved quantities for the GP hierarchy, which agreed with the $I_n$ defined in (1.3.9) when evaluated on factorized states. At the time of [64], a Hamiltonian structure for the GP hierarchy had not been identified, so it was premature to ask if the conservation of these quantities was a consequence of their Poisson commuting with the GP Hamiltonian, let alone their being in mutual involution, as is the case with the functionals $I_n$. The current work also provides a substantial generalization of the previous work [64], in that the definition of the functionals $\mathcal{H}_n$ in [64] used the quantum de Finetti theorems [42, 91, 52]. Indeed, these functionals are initially defined on factorized states of the form in (1.3.5), and then their domain of definition is extended to statistical averages of such factorized states by means of quantum de Finetti. In contrast, we now establish that these functionals are defined on the entire GP phase space. In particular, we construct $\mathcal{H}_n$ without any considerations of admissibility[13] and without any recourse to representation theorems, such as the quantum de Finetti theorems. In fact, admissibility plays no role in this dissertation.

Following our affirmative answer to Question 1.3.3, one may wonder from a more dynamical perspective, if there is a natural connection between the flows generated by the Poisson commuting functionals $\mathcal{H}_n$ and other well-known one-particle equations. We are thus motivated to address the following question:

[13]An infinite sequence of trace-class density matrices $\{\gamma^{(k)}\}_{k\in\mathbb{N}}$ is said to be *admissible* if $\gamma^{(k)} = \mathrm{Tr}_{k+1}(\gamma^{(k+1)})$.

**Question 1.3.4.** Does each of the functionals $\mathcal{H}_n$ generate a Hamiltonian equation of motion related to the $n$-th nonlinear Schrödinger equation (1.3.12) via factorized solutions in the spirit of (1.3.5)?

Our Theorem 4.1.10 below provides an affirmative answer to Question 1.3.4, proving that factorized solutions of the equation of motion with Hamiltonian $\mathcal{H}_n$ are of the form (1.3.5), where now each factor solves the $n$-th NLS equation. In this sense, we establish that the family comprised of the $n$-th GP hierarchies is the appropriate infinite-particle generalization of the nonlinear Schrödinger hierarchy. As with the proof of our involution result, our proof of this factorization connection relies on a good understanding of the multilinear structure underlying the $I_n$. We then use this understanding to find a formula for the symplectic gradients $\boldsymbol{\nabla}_s I_n$, which together with a general formula for Hamiltonian vector fields on the GP phase space allows us to arrive at the desired conclusion. We also include an explicit computation of the fourth GP hierarchy in Section 4.7.3, which corresponds to the complex mKdV equation (1.3.13).

We close this section by returning to the aforementioned thesis of Calogero with an eye towards future work. As we previously commented, if Calogero's thesis is correct for the NLS, as we believe it is, then there should be some evidence of integrability at the level of the GP hierarchy. Our work provides such evidence by showing that there is a family of Poisson commuting functionals which encode the nonlinear Schrödinger hierarchy. Given that our work in Chapter 2 mathematically demonstrates that the NLS (1.3.16) is the mean field limit of the LL model (1.3.14), it is natural to ask if there exists a connection between our functionals $\mathcal{H}_n$ together with the family of $n$-th GP hierarchies–and by implication the functionals $I_n$ together with the nonlinear Schrödinger hierarchy–and the LL model. Establishing this connection in rigorous mathematical terms seems a difficult but worthwhile task. We believe that the core difficulty lies in understanding the connection between classical and quantum field theories via the processes of quantization and mean field limit. This connection figures prominently in the work of Fröhlich, Tsai, and Yau [33] and Frölich, Knowles, and Pizzo [29] and references therein. We also mention the work [95], in which Thacker posits a conjecture related to this line of inquiry, and the work [18], in which Davies discusses the issues with naive quantization of classical approaches to integrability. We hope that the work of our dissertation will inspire others to join us in elucidating these fascinating connections.

## 1.4 Organization of the Dissertation

To conclude the introduction, we make some comments on the organization of the dissertation. This dissertation is organized into four chapters, including the introduction, drawing from three articles by the author [83, 63, 62], the latter two of which are co-authored with Mendelson, Nahmod, Pavlović, and Staffilani. Chapter 2 focuses on a new proof of the mean-field convergence of the Lieb-Liniger model to the 1D cubic NLS, as described in Section 1.2. Chapter 3 focuses on the rigorous derivation of the Hamiltonian structure of the cubic NLS in all dimensions from the Hamiltonian structure of the Schrödinger problem for finitely many interacting bosons, as described in Section 1.3.1. Chapter 4 focuses on the search for mathematical evidence of integrability of the cubic GP hierarchy in one dimension, in particular the construction of infinitely many Poisson commuting functionals, as described in Section 1.3.2. For more detailed comments on the organization of each of

Chapter 2, Chapter 3, and Chapter 4, we refer the reader to Section 2.1.2, Section 3.1.4, and Section 4.1.3, respectively.

We have also included several appendices to make this dissertation as self-contained as possible. The appendices are primarily intended to aid in the reading of Chapters 3 and 4. Appendix 1 revisits the treatment in Faddeev and Takhtajan's monograph [28] of the involution of the functionals $I_n$ in the more general setting of the system (4.1.37). We were unable to find a reference covering this generalization. Therefore, we provide a fairly thorough presentation at the expense of a lengthy appendix. Appendix 2 contains some background material on locally convex spaces, specifying certain choices which we make in the current work, which in infinite dimensions can lead to non-equivalent definitions. Appendix 3 is devoted to technical facts about distribution-valued operators and topological tensor products, which justify the manipulations used extensively in Chapters 3 and 4. Furthermore, this appendix includes an elaboration on the good mapping property, in particular, some technical consequences of it which are used in the body of Chapters 3 and 4. Appendix 4 contains technical material on products of distributions, specifically on when the product of two distributions can be rigorously defined. Appendix 5 contains a quick review of some facts from multilinear algebra on symmetric tensors, which we use to establish approximation results for bosonic Schwartz functions and density matrices.

# Chapter 2

# The Mean-Field Limit of the Lieb-Liniger Model

## 2.1 Statement of Main Result and Overview of Proof

### 2.1.1 Main Result and Its Proof

Having introduced the LL model and the problem of establishing the mean-field approximation and having reviewed prior work on this problem in Section 1.2, we are now prepared to state our main result. For notational convenience in this chapter, we change units so that the parameter $\kappa$ in Section 1.2 is replaced by $\frac{1}{2}\kappa$.

**Theorem 2.1.1** (Main result)**.** *Let $\kappa \in \{\pm 1\}$, and let $\phi_0 \in H^2(\mathbb{R})$ with $\|\phi_0\|_{L^2(\mathbb{R})} = 1$. Then there exists an absolute constant $C > 0$ such that for every $N \in \mathbb{N}$ and $k \in \{1, \ldots, N\}$,*

$$\mathrm{Tr}_{1,\ldots,k}\left|\gamma_N^{(k)}(t) - |\phi(t)^{\otimes k}\rangle\,\langle\phi(t)^{\otimes k}|\right| \leq C\sqrt{|t|k}\left(\frac{\|\phi_0\|_{H^1(\mathbb{R})}^2}{N^{1/3}} + \frac{\|\phi_0\|_{H^2(\mathbb{R})}^2}{N^{1/2}}\right)^{1/2} e^{C\|\phi_0\|_{H^2(\mathbb{R})}^2|t|}, \qquad \forall t \in \mathbb{R}, \tag{2.1.1}$$

*where $\gamma_N^{(k)}$ is the $k$-particle reduced density matrix defined in* (1.2.8) *and $\phi$ is the unique solution to the cubic NLS* (1.2.11) *in $C_t^0(\mathbb{R}; H_x^2(\mathbb{R}))$.*[1]

Our Theorem 2.1.1 establishes the convergence to the mean-field limit (1.2.12) for the LL model with an explicit rate of convergence which holds for arbitrary lengths of time

[1]It is textbook that the 1D cubic NLS is globally well-posed in the class $C_t^0 H_x^2$ of functions which are continuous in time values in $H^2(\mathbb{R})$. For instance, see [13] and [92].

in both the repulsive and attractive settings. The $H^2$ regularity assumption on the initial datum $\phi_0$ is consistent with the assumption of Ammari and Breteaux [6]. Additionally, an examination of the argument in Section 2.4 and Section 2.5 shows that if we replace the Hamiltonian $H_N$ in (1.2.2) with the “regularized Hamiltonian”

$$H_{N,\sigma} \coloneqq \sum_{i=1}^{N} -\Delta_i + \frac{\kappa}{N} \sum_{1 \leq i < j \leq N} V_N(X_i - X_j), \qquad \kappa \in \{\pm 1\}, \tag{2.1.2}$$

where $V$ is a short-range potential satisfying certain regularity conditions and $V_N \coloneqq N^{\sigma} V(N^{\sigma} \cdot)$, for some fixed $\sigma \in (0, \infty)$, then for any $T > 0$ fixed,

$$\forall k \in \mathbb{N}, \qquad \lim_{N \to \infty} \sup_{0 \leq |t| \leq T} \operatorname{Tr} \left| \gamma_{N,\sigma}^{(k)}(t) - |\phi(t)^{\otimes k}\rangle \langle \phi(t)^{\otimes k}| \right| = 0, \tag{2.1.3}$$

where $\gamma_{N,\sigma}^{(k)}$ is the $k$-particle reduced density matrix associated to the Schrödinger problem obtained by replacing $H_N$ in (1.2.4) with $H_{N,\sigma}$. One can extract a rate of convergence for (2.1.3) which tends to the rate (2.1.1) as $\sigma \to \infty$.

We now comment on the proof of Theorem 2.1.1 and highlight the major difficulties and differences from existing work. Inspired by the method of Pickl [76, 77, 78] and the refinement of this method developed by Knowles and Pickl [46] for derivation of the Hartree equation[2] in the mean-field limit, our argument is based on an energy-type estimate for a functional $\beta_N$ of the solution $\Phi_N$ to equation (1.2.4), which gives a weighted count of the number of “bad particles” in the system at time $t$ which are not in the state $\phi(t)$, where $\phi$

[2]A function $\phi : \mathbb{R} \times \mathbb{R}^d$ satisfies the Hartree equation if $(i\partial_t + \Delta)\phi = (V * |\phi|^2)\phi$, where $V$ is a chosen locally integrable function. The cubic NLS (1.2.11) may be viewed as the special case of the Hartree equation with $V = \delta$.

solves the cubic NLS (1.2.11). The functional $\beta_N$ takes the form

$$\beta_N(t) := \left\langle \Phi_N(t) \middle| \widehat{n_N(t)} \Phi_N(t) \right\rangle_{L^2(\mathbb{R}^N)} = \sum_{k=0}^{N} \sqrt{\frac{k}{N}} \left\langle \Phi_N(t) | P_k(t) \Phi_N(t) \right\rangle_{L^2(\mathbb{R}^N)}, \qquad \forall t \in \mathbb{R}, \tag{2.1.4}$$

where $\Phi_N$ is the solution to (1.2.4) and $P_k(t)$ is the projector mapping a wave function onto the subspace of $L^2_{sym}(\mathbb{R}^N)$ of functions corresponding to $k$ of the particles not being in the state $\phi(t)$. See (2.4.4) and more generally Section 2.4.1 for the precise definition and properties of these projectors. The main estimate for $\beta_N$ is given by Proposition 2.1.2 below. To state the proposition, we first introduce some notation. Let $E_N^\Phi$ denote the *energy per particle* of the $N$-body system (1.2.4), which is defined by

$$E_N^\Phi(t) := \frac{1}{N} \left\langle \Phi_N(t) | H_N \Phi_N(t) \right\rangle_{L^2(\mathbb{R}^N)} = \|\nabla_1 \Phi_N(t)\|^2_{L^2(\mathbb{R}^N)} + \frac{\kappa(N-1)}{2N} \| \operatorname{tr}_{1=2} \Phi_N(t)\|^2_{L^2(\mathbb{R}^{N-1})}, \tag{2.1.5}$$

where the ultimate equality follows from integration by parts and the symmetry (1.2.3). Let $E^\phi$ denote the *cubic NLS energy*, which is defined by

$$E^\phi(t) := \|\nabla \phi(t)\|^2_{L^2(\mathbb{R})} + \frac{\kappa}{2} \|\phi(t)\|^4_{L^4(\mathbb{R})}. \tag{2.1.6}$$

Above, we have used the notation $\operatorname{tr}_{i=j}$ to denote the trace to the hyperplane $\{\underline{x}_N \in \mathbb{R}^N : x_i = x_j\}$. Note that both $E_N^\Phi$ and $E^\phi$ are independent of time by conservation of energy for equations (1.2.4) and (1.2.11):

$$E_N^\Phi(t) = \|\nabla_1 \Phi_{N,0}\|^2_{L^2(\mathbb{R}^N)} + \frac{\kappa(N-1)}{2N} \| \operatorname{tr}_{1=2} \Phi_{N,0}\|^2_{L^2(\mathbb{R}^{N-1})}, \tag{2.1.7}$$

$$E^\phi(t) = \|\nabla \phi_0\|^2_{L^2(\mathbb{R})} + \frac{\kappa}{2} \|\phi_0\|^4_{L^4(\mathbb{R})}. \tag{2.1.8}$$

**Proposition 2.1.2** (Evolution of $\beta_N$)**.** *Let $\kappa \in \{\pm 1\}$. Then there exists an absolute constant $C > 0$, such that for every $N \in \mathbb{N}$, there exists a continuous function $\mathfrak{A}_N : [0, \infty) \to [0, \infty)$ such that*

$$\beta_N(t) \leq \mathfrak{A}_N(|t|) e^{C\|\phi_0\|_{H^2(\mathbb{R})}^2 |t|}, \qquad \forall t \in \mathbb{R}, \tag{2.1.9}$$

*where $\mathfrak{A}_N$ satisfies the bound*

$$\mathfrak{A}_N(t) \leq \beta_N(0) + C|t| \left( \frac{\|\phi_0\|_{H^1(\mathbb{R})}^2}{N^{1/3}} + \frac{\|\phi_0\|_{H^2(\mathbb{R})}^2}{N^{1/2}} + (E_N^\Phi - E^\phi) \|\phi_0\|_{H^1(\mathbb{R})}^2 \right), \qquad \forall t \in \mathbb{R}. \tag{2.1.10}$$

**Remark 2.1.3.** An examination of the argument in Section 2.5 for obtaining Theorem 2.1.1 from Proposition 2.1.2 shows that we have propagation of chaos for any sequence of initial wave functions $\Phi_{N,0} \in L^2_{sym}(\mathbb{R}^N)$ such that

$$\lim_{N \to \infty} \beta_N(0) = 0 \quad \text{and} \quad \lim_{N \to \infty} E_N^\Phi - E^\phi = 0. \tag{2.1.11}$$

To prove Proposition 2.1.2, we proceed by a Gronwall-type argument. Differentiating $\beta_N$ with respect to time and performing some simplications, we find that we need to estimate the following three terms:

$$\text{Term}_1 := \left\langle \Phi_N \middle| p_1 p_2 \left[ V_1^\phi, \widehat{n_N} \right] q_1 p_2 \Phi_N \right\rangle_{L^2_{\underline{x}_N}(\mathbb{R}^N)}, \tag{2.1.12}$$

$$\text{Term}_2 := \left\langle \Phi_N \middle| q_1 p_2 \left[ (N-1) V_{12} - N V_2^\phi, \widehat{n_N} \right] q_1 q_2 \Phi_N \right\rangle_{L^2_{\underline{x}_N}(\mathbb{R}^N)}, \tag{2.1.13}$$

$$\text{Term}_3 := \langle \Phi_N | p_1 p_2 [(N-1) V_{12}, \widehat{n_N}] q_1 q_2 \Phi_N \rangle_{L^2_{\underline{x}_N}(\mathbb{R}^N)}, \tag{2.1.14}$$

where we have used the notation $V_{12} := \delta(X_1 - X_2)$ and $V_j^\phi := |\phi(X_j)|^2$ and we remind the reader that $[\cdot, \cdot]$ denotes the commutator. $V_{12}(q_1 q_2 \Phi_N)$ and $V_{12}(\widehat{n_N} q_1 q_2 \Phi_N)$, similarly for the other terms, should be interpreted as elements of $H^{-1}(\mathbb{R}^N)$ and the inner product as a duality pairing. Here, $p_j$ is the rank-one projector $|\phi\rangle\langle\phi|$ acting in the $x_j$-variable, and $q_j = \mathbf{1}_N - p_j$, where $\mathbf{1}_N$ is the identity operator on $L^2(\mathbb{R}^N)$ (see Section 2.4.1 for more details). As $\mathrm{Term}_3$ is the most difficult case in the analysis and where the existing arguments in the literature break down, we focus on it.

By expanding the commutator in the definition of $\mathrm{Term}_3$ and using Lemma 2.4.7 to shift the projectors $P_k$ in the definition of $\widehat{n_N}$ (see Definition 2.4.4), we reduce to bounding the expression

$$\left| \langle \Phi_N | p_1 p_2 V_{12} q_1 q_2 \widehat{\nu_N} \Phi_N \rangle_{L^2_{\underline{x}_N}(\mathbb{R}^N)} \right|, \tag{2.1.15}$$

where $\widehat{\nu_N} = \sum_{k=0}^{N} \nu_N(k) P_k$ is a time-dependent operator on $L^2_{sym}(\mathbb{R}^N)$ such that the coefficients satisfy $\nu_N(k) \lesssim n_N^{-1}(k)$. See (2.4.68) for the precise definition of $\nu_N$ and $\widehat{\nu_N}$. To obtain an acceptable bound for our Gronwall argument, we need to produce an operator $\widehat{n_N}^2$, so that

$$\widehat{n_N}^2 \widehat{\nu_N} \lesssim \widehat{n_N}. \tag{2.1.16}$$

In [46], Knowles and Pickl had to contend with an expression similar to $\mathrm{Term}_3$ but with a much more regular potential $V$, which satisfies certain integrability assumptions of the form $V \in L^{p_0} + L^\infty$. In order to simplify the comparison, we assume that $V \in L^{p_0}$. To deal with their analogue of (2.1.15), they split the potential into its "regular" and "singular" parts by making an $N$-dependent decomposition of the form

$$V_{reg} := V 1_{\{|V| \leq N^\sigma\}}, \qquad V_{sing} := V 1_{\{V| > N^\sigma\}}, \tag{2.1.17}$$

where $1_{\{\cdot\}}$ denotes the indicator function for the set $\{\cdot\}$ and $\sigma \in (0,1)$ is a parameter to be optimized at the end. For the singular part, they express the potential as the divergence of a vector field,

$$V = \nabla \cdot \xi, \tag{2.1.18}$$

and integrate by parts . Crucially, their integrability assumption implies that $\xi \in L^2(\mathbb{R}^N)$ with $L^2$ norm $O(N^{-\delta})$, for some $\delta > 0$, which is necessary to close their estimate. For the regular part, the important idea is to exploit the symmetry (1.2.3) of the wave function, since the operator norm of $p_1p_2V_{12}q_1q_2$ is much smaller on the bosonic subspace $L^2_{sym}(\mathbb{R}^N)$ than on the full space $L^2(\mathbb{R}^N)$. As the argument is a bit involved, we only comment that it importantly requires $V_{reg}^2$ to be integrable.

For $V = \delta(x)$, Knowles and Pickl's argument described above breaks down. While we have the identity

$$\delta(x) = \frac{1}{2}\nabla\,\mathrm{sgn}(x), \tag{2.1.19}$$

the signum function is only in $L^\infty$, not in $L^2$ as their singular-part argument requires. Additionally, since $\delta$ is only a distribution, we cannot assign meaning to $\delta^2$ in the regular part of their argument. In fact, the regular part of their argument is formally vacuous for the $\delta$ potential.

To overcome the difficulties stemming from the lack of integrability of the $\delta$ potential, we introduce a new short-range approximation argument as follows. We make an $N$-dependent mollification of the potential by setting

$$V_\sigma(x) := N^\sigma \tilde{V}(N^\sigma x), \qquad \forall x \in \mathbb{R}, \tag{2.1.20}$$

where $\sigma \in (0,1)$, $0 \leq \tilde{V} \leq 1$, $\tilde{V} \in C_c^\infty(\mathbb{R})$ is even, and $\int_{\mathbb{R}} dx\tilde{V}(x) = 1$. By the triangle inequality, we have

$$\begin{aligned}\left|\langle\Phi_N|p_1p_2V_{12}q_1q_2\widehat{\nu_N}\Phi_N\rangle_{L^2_{\underline{x}_N}(\mathbb{R}^N)}\right| \leq \left|\langle\Phi_N|p_1p_2(V_{12}-V_{\sigma,12})q_1q_2\widehat{\nu_N}\Phi_N\rangle_{L^2_{\underline{x}_N}(\mathbb{R}^N)}\right| \\ + \left|\langle\Phi_N|p_1p_2V_{\sigma,12}q_1q_2\widehat{\nu_N}\Phi_N\rangle_{L^2_{\underline{x}_N}(\mathbb{R}^N)}\right|.\end{aligned} \tag{2.1.21}$$

Combining the scaling relation

$$\int_{\mathbb{R}} dx|x|^{1/2}V_\sigma(x) \sim N^{-\sigma/2} \tag{2.1.22}$$

with fact that the wave function $\Phi_N$ is $\frac{1}{2}$-Hölder-continuous in a single particle variable by conservation of mass and energy together with Sobolev embedding (see Lemma 2.2.3), we can estimate

$$\begin{aligned}\left|\langle\Phi_N|p_1p_2(V_{12}-V_{\sigma,12})q_1q_2\widehat{\nu_N}\Phi_N\rangle_{L^2_{\underline{x}_N}(\mathbb{R}^N)}\right| \lesssim N^{-\sigma} + \|\phi\|^2_{C_x^{1/2}(\mathbb{R})}\|\phi\|^2_{H_x^1(\mathbb{R})}\beta_N \\ + \|\phi\|^2_{C_x^{1/2}(\mathbb{R})}\|\nabla_1q_1\Phi_N\|^2_{L^2_{\underline{x}_N}(\mathbb{R}^N)}.\end{aligned} \tag{2.1.23}$$

Note that by the Sobolev embedding $H^1(\mathbb{R}) \subset C^{1/2}(\mathbb{R})$ together with conservation of mass and energy for the cubic NLS (1.2.11), we have that $\|\phi\|_{L_t^\infty(\mathbb{R};C_x^{1/2}(\mathbb{R}))} \lesssim \|\phi_0\|_{H^1(\mathbb{R})}$. We can estimate the second term in the right-hand side of (2.1.21) by proceeding similarly as to the aforementioned Knowles-Pickl argument for the regular part $V_{reg}$ of the potential. While $\|V_\sigma\|_{L^2(\mathbb{R})} \sim N^{\sigma/2}$, we are able to extract sufficient decay in $N$ from other factors to absorb this growth in $N$, provided we appropriately choose $\sigma$.

To close the proof of Proposition 2.1.2, we need to control the auxiliary quantity $\|\nabla_1q_1\Phi_N\|_{L^2_{\underline{x}_N}(\mathbb{R}^N)}$ in terms of $\beta_N$ and other quantities which tend to zero as $N \to \infty$. The desired control is given by Proposition 2.4.10. Our argument exploits the conservation of mass and energy together with the identity (2.1.19) and integration by parts (cf. [46, Lemma 4.6]). Crucially, $\mathrm{sgn} \in L^\infty$ so that the multiplication operator $\mathrm{sgn}(X_1 - X_2)$ is bounded on $L^2(\mathbb{R}^N)$.

Strictly speaking, we do not work in Section 2.4 directly with the wave function $\Phi_N$ and with the functional $\beta_N$ but rather with an approximation obtained by replacing the Hamiltonian $H_N$ in the Schrödinger problem (1.2.4) with the mollified Hamiltonian

$$H_{N,\varepsilon} := \sum_{i=1}^{N} -\Delta_i + \frac{\kappa}{N} \sum_{1 \leq i < j \leq N} V_\varepsilon(X_i - X_j), \qquad \kappa \in \{\pm 1\}, \tag{2.1.24}$$

where $V_\varepsilon := \varepsilon^{-1}\tilde{V}(\cdot/\varepsilon)$, for $\varepsilon > 0$ and $\tilde{V}$ as above. This step is purely technical to deal with issues of operator domains involved in differentiating the functional $\beta_N$ and to avoid awkward notation involving distributions. Since $H_{N,\varepsilon} \to H_N$, as $\varepsilon \to 0^+$, in norm-resolvent sense (see Section 2.3.3), we are able to obtain Proposition 2.1.2 from an analogous estimate for the mollified version of $\beta_N$ (see (2.4.19) and Proposition 2.4.9 and Proposition 2.4.10).

### 2.1.2 Organization of the Chapter

We now comment on the organization of the chapter. Section 2.2 is devoted to basic notation and preliminary facts from functional analysis used extensively in the chapter. We begin the section with an index (see Table 2.1) of the frequently used notation in the chapter. Section 2.2.1 introduces the spaces of functions and distributions used in the body of the chapter, and Section 2.2.2 contains some basic estimates for the traces of Sobolev functions, which we use in Section 2.3 and Section 2.4.

Section 2.3 gives the rigorous construction of the self-adjoint operator $H_N$ corresponding to the expression (1.2.2). The main result is Proposition 2.3.4. As the construction proceeds by means of quadratic forms, we first review such forms in Section 2.3.1 and then prove Proposition 2.3.4 in Section 2.3.2. We close the section by establishing a short-range approximation to $H_N$ in Section 2.3.3, which is used in Section 2.4. While most of the results of Section 2.2 seem to be folklore in the math physics community and have appeared in other forms elsewhere in the literature (for instance, see [6, Proposition 3.3] for a presentation in terms of the Fock space formalism), we believe that our presentation is new.

In Section 2.4, we prove Proposition 2.1.2, which is the main estimate for the functional $\beta_N$ and the main ingredient for the proof of Theorem 2.1.1. As this section constitutes the bulk of the paper, we have divided it into several subsections corresponding to the steps in the proof of Proposition 2.1.2. In Section 2.4.1, we introduce the time-dependent projectors which underlie the definition of the functional $\beta_N$. In Section 2.4.2, we approximate the functional $\beta_N$ with a functional $\beta_{N,\varepsilon}$ obtained by regularizing the Hamiltonian $H_N$ (see (2.4.19)) and prove a preliminary estimate for $\beta_{N,\varepsilon}$, which is Proposition 2.4.9. In Section 2.4.3, we prove Proposition 2.4.10, which gives an estimate in terms of $\beta_{N,\varepsilon}$ for an auxiliary quantity appearing in Proposition 2.4.9. In Section 2.4.4, we send the regularization parameter $\varepsilon$ to zero and obtain Proposition 2.1.2 from Proposition 2.4.9 and Proposition 2.4.10.

Lastly, in Section 2.5, we show how to obtain Theorem 2.1.1 from Proposition 2.1.2. As the arguments used in this step are by now well-known, we only sketch the details.

### 2.1.3 Acknowledgments

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## 2.2 Preliminaries

We include Section 2.2, located at the end of the chapter, as a table of the notation frequently used in the chapter with an explanation for the notation and/or a reference to where the definition is given.

### 2.2.1 Function Spaces

Fix $N \in \mathbb{N}$. We denote the Schwartz space on $\mathbb{R}^N$ by $\mathcal{S}(\mathbb{R}^N)$ and the dual space of tempered distributions on $\mathbb{R}^N$ by $\mathcal{S}'(\mathbb{R}^N)$. The subspace of $\mathcal{S}(\mathbb{R}^N)$ consisting of functions with compact support is denoted by $C_c^\infty(\mathbb{R}^N)$. Given a Schwartz function $\Phi \in \mathcal{S}(\mathbb{R}^N)$ and a tempered distribution $\Upsilon \in \mathcal{S}'(\mathbb{R}^N)$, we denote their duality pairing by

$$\langle \Phi, \Upsilon \rangle_{\mathcal{S}(\mathbb{R}^N)-\mathcal{S}'(\mathbb{R}^N)} := \Upsilon(\Phi). \tag{2.2.1}$$

For $1 \leq p \leq \infty$, we define $L^p(\mathbb{R}^N)$ to be the usual Banach space of equivalence classes of measurable functions $\Phi : \mathbb{R}^N \to \mathbb{C}$ with respect to the norm

$$\|\Phi\|_{L^p(\mathbb{R}^N)} := \left( \int_{\mathbb{R}^N} d\underline{x}_N |\Phi(\underline{x}_N)|^p \right)^{1/p} \tag{2.2.2}$$

with obvious modification when $p = \infty$. We denote the inner product on $L^2(\mathbb{R}^N)$ by

$$\langle \Phi | \Psi \rangle_{L^2(\mathbb{R}^N)} := \int_{\mathbb{R}^N} d\underline{x}_N \overline{\Phi(\underline{x}_N)} \Psi(\underline{x}_N). \tag{2.2.3}$$

Note that we use the physicist's convention that the inner product is complex linear in the second entry. For $s \in \mathbb{R}$, we define the Sobolev space $H^s(\mathbb{R}^N)$ to be the completion of the space $\mathcal{S}(\mathbb{R}^N)$ with respect to the norm

$$\|\Phi\|_{H^s(\mathbb{R}^N)} := \left( \int_{\mathbb{R}^N} d\underline{\xi}_N |\mathcal{F}(\Phi)(\underline{\xi}_N)|^2 \right)^{1/2}, \tag{2.2.4}$$

where $\mathcal{F}$ denotes the Fourier transform defined via the convention

$$\mathcal{F}(\Phi)(\underline{\xi}_N) := \int_{\mathbb{R}^N} d\underline{x}_N \Phi(\underline{x}_N) e^{-i\underline{x}_N \cdot \underline{\xi}_N}. \tag{2.2.5}$$

We can anti-isomorphically identify $H^{-s}(\mathbb{R}^N)$ with the dual of $(H^s(\mathbb{R}^N))^*$ by

$$\langle \Phi, \Upsilon \rangle_{H^s(\mathbb{R}^N)-H^{-s}(\mathbb{R}^N)} := \left\langle \langle \underline{\nabla}_N \rangle^{-s} \Upsilon \middle| \langle \underline{\nabla}_N \rangle^s \Phi \right\rangle_{L^2(\mathbb{R}^N)}, \tag{2.2.6}$$

where $\langle x \rangle := (1 + |x|^2)^{1/2}$ is the Japanese bracket and $\langle \underline{\nabla}_N \rangle$ is the Fourier multiplier with symbol $\langle \underline{\xi}_N \rangle$. For $\gamma \in (0, 1)$, we denote the Hölder norm on $\mathbb{R}^N$ of exponent $\gamma$ by

$$\|\Phi\|_{\dot{C}^\gamma(\mathbb{R}^N)} := \sup_{\substack{x,y \in \mathbb{R}^N \\ x \neq y}} \frac{|\Phi(x) - \Phi(y)|}{|x - y|^\gamma}, \qquad \|\Phi\|_{C^\gamma(\mathbb{R}^N)} := \|\Phi\|_{L^\infty(\mathbb{R}^N)} + \|\Phi\|_{\dot{C}^\gamma(\mathbb{R}^N)}. \tag{2.2.7}$$

**Remark 2.2.1.** In the sequel, we generally omit the underlying domain for norms (e.g. we write $\|\cdot\|_{L^p}$ instead of $\|\cdot\|_{L^p(\mathbb{R}^N)}$), as the domain will be clear from context. Similarly, we omit the underlying domain for the inner product $\langle \cdot | \cdot \rangle$ and for the duality pairing $\langle \cdot, \cdot \rangle$. To avoid any confusion, we generally reserve upper-case Greek letters (e.g. $\Phi, \Psi$) for functions or distributions $\mathbb{R}^N \to \mathbb{C}$ and lower-case Greek letters (e.g. $\varphi, \psi$) for functions or distributions $\mathbb{R} \to \mathbb{C}$. To emphasize the variable with respect to which a norm is taken, we use a subscript (e.g. $C_t^0$, $L_x^2$, or $L_{\underline{x}_N}^2$).

### 2.2.2 Some Trace Estimates

In this subsection, we establish some basic estimates pertaining to the trace of a Sobolev function. We use these trace estimates for the rigorous construction of the LL Hamiltonian (recall expression (1.2.2)) in Section 2.3 and in the proof of Proposition 2.1.2 in Section 2.4.

For a Schwartz function $\Phi \in \mathcal{S}(\mathbb{R}^N)$ and indices $1 \leq i < j \leq N$, we let $\Phi_{i=j}$ denote the restriction of $\Phi$ to the hyperplane $\{\underline{x}_N \in \mathbb{R}^N : x_i = x_j\}$. We recall from elementary functional analysis that for any $s > 1/2$, there is a unique bounded linear map

$$\operatorname{tr}_{i=j} : H^s(\mathbb{R}^N) \to H^{s-\frac{1}{2}}(\mathbb{R}^{N-1}), \qquad \|\operatorname{tr}_{i=j}\Phi\|_{H^{s-\frac{1}{2}}(\mathbb{R}^{N-1})} \lesssim \|\Phi\|_{H^s(\mathbb{R}^N)} \tag{2.2.8}$$

with the property that for any $\Phi \in \mathcal{S}(\mathbb{R}^N)$,

$$\operatorname{tr}_{i=j}(\Phi) = \Phi_{i=j}. \tag{2.2.9}$$

For the next lemma, we first recall the elementary distributional identity

$$\delta(x) = \frac{1}{2}\nabla \operatorname{sgn}(x), \qquad \forall x \in \mathbb{R}. \tag{2.2.10}$$

**Lemma 2.2.2** ($H^1$ Trace estimate)**.** *Let $N \in \mathbb{N}$. For any $1 \leq i < j \leq N$,*

$$\left|\langle \operatorname{tr}_{i=j}\Phi | \operatorname{tr}_{i=j}\Psi\rangle_{L^2(\mathbb{R}^{N-1})}\right| \leq \frac{1}{2}\big(\|\nabla_i\Phi\|_{L^2(\mathbb{R}^N)}\|\Psi\|_{L^2(\mathbb{R}^N)} + \|\Phi\|_{L^2(\mathbb{R}^N)}\|\nabla_i\Psi\|_{L^2(\mathbb{R}^N)}\big), \qquad \forall \Phi,\Psi \in H^1(\mathbb{R}^N). \tag{2.2.11}$$

*Consequently, if $\Phi \in H^1(\mathbb{R}^N)$, then we can define $\delta(X_i - X_j)\Phi := \Phi\delta(X_i - X_j) \in H^{-1}(\mathbb{R}^N)$ by*

$$\langle \Psi, \delta(X_i - X_j)\Phi\rangle_{H^1(\mathbb{R}^N)-H^{-1}(\mathbb{R}^N)} := \langle \operatorname{tr}_{i=j}\Psi, \operatorname{tr}_{i=j}\Phi\rangle_{L^2(\mathbb{R}^{N-1})-L^2(\mathbb{R}^{N-1})}, \tag{2.2.12}$$

*and*

$$\|\delta(X_i - X_j)\Phi\|_{H^{-1}(\mathbb{R}^N)} \leq \|\Phi\|_{H^1(\mathbb{R}^N)}. \tag{2.2.13}$$

*Proof.* By considerations of symmetry, it suffices to consider $(i,j) = (1,2)$. Let $\Phi,\Psi \in \mathcal{S}(\mathbb{R}^N)$. Then by definition of the product distribution $\delta(X_1 - X_2)\Phi \in \mathcal{S}'(\mathbb{R}^N)$, we have that

$$\langle \Psi, \delta(X_1 - X_2)\Phi\rangle_{\mathcal{S}-\mathcal{S}'} = \langle \Psi_{1=2}, \Phi_{1=2}\rangle_{L^2-L^2}. \tag{2.2.14}$$

Substituting the distributional identity (2.2.10) into the left-hand side of the preceding equality and applying the definition of the distributional derivative together with the product rule, we obtain that

$$\langle \Psi_{1=2}, \Phi_{1=2}\rangle_{L^2-L^2} = -\frac{1}{2}\big(\langle \nabla_1\Psi, \mathrm{sgn}(X_1 - X_2)\Psi\rangle_{L^2-L^2} + \langle \Psi, \mathrm{sgn}(X_1 - X_2)\nabla_1\Phi\rangle_{L^2-L^2}\big). \tag{2.2.15}$$

Taking absolute values of both sides, applying the triangle inequality, followed by Cauchy-Schwarz, we obtain that

$$|\langle \Psi_{1=2}, \Phi_{1=2}\rangle_{L^2-L^2}| \leq \frac{1}{2}(\|\nabla_1\Psi\|_{L^2}\|\Phi\|_{L^2} + \|\Psi\|_{L^2}\|\nabla_1\Phi\|_{L^2}). \tag{2.2.16}$$

The conclusion (2.2.11) then follows from density of $\mathcal{S}(\mathbb{R}^N) \subset H^1(\mathbb{R}^N)$ and the continuity of the map $\mathrm{tr}_{1=2} : H^1(\mathbb{R}^N) \to H^{1/2}(\mathbb{R}^{N-1})$.

Next, given $\Phi \in H^1(\mathbb{R}^N)$, we define the linear functional $\delta(X_1 - X_2)\Phi$ on $H^1(\mathbb{R}^N)$ by extending the definition of the product distribution for $\Phi \in \mathcal{S}(\mathbb{R}^N)$. Then by Cauchy-Schwarz and the estimate (2.2.16),

$$\begin{aligned}\sup_{\|\Psi\|_{H^1}=1} |\langle \Psi, \delta(X_1 - X_2)\Phi\rangle_{H^1-H^{-1}}| &= \sup_{\|\Psi\|_{H^1}=1} |\langle \mathrm{tr}_{1=2}\,\Psi, \mathrm{tr}_{1=2}\,\Phi\rangle_{L^2-L^2}| \\ &\leq \sup_{\|\Psi\|_{H^1}=1} \frac{1}{2}(\|\nabla_1\Psi\|_{L^2}\|\Phi\|_{L^2} + \|\Psi\|_{L^2}\|\nabla_1\Phi\|_{L^2}) \\ &\leq \|\Phi\|_{H^1}, \end{aligned} \tag{2.2.17}$$

which by duality, implies the desired conclusion. □

We also record here a partial Hölder continuity result for functions in $H^1(\mathbb{R}^N)$ used in Section 2.4.

**Lemma 2.2.3** (Partial Hölder continuity). *Let $N \in \mathbb{N}$. For any $i \in \{1, \ldots, N\}$, we have the estimate*

$$\|\Phi\|_{L^2_{(\underline{x}_{1;i-1}, \underline{x}_{i+1;N})}(\mathbb{R}^{N-1}; \dot{C}^{1/2}_{x_i}(\mathbb{R}))} \leq \|\nabla_i \Phi\|_{L^2(\mathbb{R}^N)}, \qquad \forall \Phi \in \mathcal{S}(\mathbb{R}^N). \tag{2.2.18}$$

*Consequently, every element of $H^1(\mathbb{R}^N)$ has a modification belonging to $L^2_{(\underline{x}_{1;i-1}, \underline{x}_{i+1;N})}(\mathbb{R}^{N-1}; C^{1/2}_{x_i}(\mathbb{R}))$.*

*Proof.* By considerations of symmetry, it suffices to consider $i = 1$. Let $\Phi \in \mathcal{S}(\mathbb{R}^N)$, and fix $\underline{x}_{2;N} \in \mathbb{R}^{N-1}$. Define the function

$$\phi_{\underline{x}_{2;N}} : \mathbb{R} \to \mathbb{C}, \qquad \phi_{\underline{x}_{2;N}}(x) \coloneqq \Phi(x, \underline{x}_{2;N}), \;\; \forall x \in \mathbb{R}. \tag{2.2.19}$$

Applying the fundamental theorem of calculus to $\phi_{\underline{x}_{2;N}}$ followed by Cauchy-Schwarz, we obtain that

$$|\phi_{\underline{x}_{2;N}}(x) - \phi_{\underline{x}_{2;N}}(y)| \leq |x - y|^{1/2} \|\nabla \phi_{\underline{x}_{2;N}}\|_{L^2(\mathbb{R})}, \qquad \forall x, y \in \mathbb{R}, \tag{2.2.20}$$

which implies that $\|\phi_{\underline{x}_{2;N}}\|_{\dot{C}^{1/2}(\mathbb{R})} \leq \|\nabla \phi_{\underline{x}_{2;N}}\|_{L^2(\mathbb{R})}$. Therefore, we see from the Fubini-Tonelli theorem that

$$\int_{\mathbb{R}^{N-1}} d\underline{x}_{2;N} \|\phi_{\underline{x}_{2;N}}\|^2_{\dot{C}^{1/2}(\mathbb{R})} \leq \int_{\mathbb{R}^{N-1}} d\underline{x}_{2;N} \|\nabla \phi_{\underline{x}_{2;N}}\|^2_{L^2(\mathbb{R})} = \|\nabla_1 \Phi\|^2_{L^2(\mathbb{R}^N)}. \tag{2.2.21}$$

The conclusion of the proof then follows from the density of $\mathcal{S}(\mathbb{R}^N) \subset H^1(\mathbb{R}^N)$. $\square$

## 2.3 Construction of the Hamiltonian $H_N$

In this section, we give the rigorous construction of the Hamiltonian $H_N$, which we recall from (1.2.2) corresponds to the expression

$$\sum_{i=1}^{N} -\Delta_i + \frac{\kappa}{N} \sum_{1 \leq i < j \leq N} \delta(X_i - X_j), \qquad \kappa \in \{\pm 1\}. \tag{2.3.1}$$

The construction requires some care due to the presence of the $\delta$ pair potential. The main ingredients in the construction are the KLMN theorem, which we recall in Proposition 2.3.3 below, and the trace estimate of Lemma 2.2.2. Before proceeding to the construction, we need to introduce some terminology from the theory of unbounded operators on Hilbert spaces. Our presentation follows that of Reed and Simon [80, 79]. In what follows, $(\mathcal{H}, \langle\cdot|\cdot\rangle_{\mathcal{H}})$ is a separable complex Hilbert space.

### 2.3.1 Quadratic Forms

We begin with the definition of and basic facts about quadratic forms.

**Definition 2.3.1** (Quadratic form)**.** A *quadratic form* is a sesquilinear map $q : \mathcal{Q}(q) \times \mathcal{Q}(q) \to \mathbb{C}$, where $\mathcal{Q}(q)$ is a dense subset of $\mathcal{H}$ called the *form domain.* If $q(\varphi, \psi) = \overline{q(\psi, \varphi)}$ for all $\varphi, \psi \in \mathcal{Q}(q)$, then we say that $q$ is *symmetric.* If $q(\varphi, \varphi) \geq 0$ for every $\varphi \in \mathcal{H}$, then we say that $q$ is *positive*, and if there exists a constant $M > 0$ such that $q(\varphi, \varphi) \geq -M\|\varphi\|_{\mathcal{H}}^2$, then we say that $q$ is *semibounded.*[3]

**Definition 2.3.2** (Closed quadratic forms)**.** Let $q : \mathcal{Q}(q) \times \mathcal{Q}(q) \to \mathbb{C}$ be a semibounded quadratic form with constant $M > 0$ such that

$$q(\psi, \psi) \geq -M\|\psi\|_{\mathcal{H}}^2, \qquad \forall \psi \in \mathcal{Q}(q). \tag{2.3.2}$$

We say that $q$ is *closed* if $\mathcal{Q}(q)$ is complete under the norm

$$\|\psi\|_q \coloneqq \sqrt{q(\psi, \psi) + (M+1)\|\psi\|_{\mathcal{H}}^2}, \qquad \forall \psi \in \mathcal{Q}(q). \tag{2.3.3}$$

[3]If the quadratic form $q$ is semibounded, then it is in fact symmetric.

If $q$ is closed and $D \subset \mathcal{Q}(q)$ is dense in $\mathcal{Q}(q)$ with respect to the norm $\|\cdot\|_q$, then we call $D$ a *form core* for $q$.

Let $A$ be a self-adjoint operator on $\mathcal{H}$. We define a subset of $\mathcal{H}$ by

$$\mathcal{Q}(A) \coloneqq \{\psi \in \mathcal{H} : \||A|^{1/2}\psi\|_{\mathcal{H}} < \infty\}. \tag{2.3.4}$$

We can then define the quadratic form $q$ *associated to* $A$ by setting $\mathcal{Q}(q) \coloneqq \mathcal{Q}(A)$ and

$$q : \mathcal{Q}(q) \times \mathcal{Q}(q) \to \mathbb{C}, \qquad q(\varphi, \psi) \coloneqq \left\langle |A|^{1/2} U^* \varphi \middle| |A|^{1/2}\psi \right\rangle_{\mathcal{H}}, \ \ \forall \varphi, \psi \in \mathcal{H}, \tag{2.3.5}$$

where $A = U|A|$ is the polar decomposition for $A$ (see [80, Theorem VIII.32]). In the sequel, we agree to write $\langle \varphi | A\psi \rangle_{\mathcal{H}}$ for the quadratic form associated to $A$, even though $\psi \in \mathcal{Q}(A)$ may not belong to $\mathrm{Dom}(A)$. We hope this abuse of notation causes no confusion for the reader.

We now are prepared to state the KLMN theorem.

**Proposition 2.3.3** (KLMN theorem, [79, Theorem X.17])**.** *Let $A$ be a positive self-adjoint operator on $\mathcal{H}$ with domain $D(A)$. Suppose that $\beta : \mathcal{Q}(A) \times \mathcal{Q}(A) \to \mathbb{C}$ is a symmetric quadratic form such that there exist constants $a < 1$ and $b \in \mathbb{R}$ so that*

$$|\beta(\psi, \psi)| \leq a \langle \psi | A\psi \rangle_{\mathcal{H}} + b \langle \psi | \psi \rangle_{\mathcal{H}}, \qquad \forall \psi \in D(A). \tag{2.3.6}$$

*Then there exists a unique self-adjoint operator $C$ on $\mathcal{H}$ with $\mathcal{Q}(C) = \mathcal{Q}(A)$ and*

$$\langle \varphi | C\psi \rangle_{\mathcal{H}} = \langle \varphi | A\psi \rangle_{\mathcal{H}} + \beta(\varphi, \psi), \qquad \forall \varphi, \psi \in \mathcal{Q}(C). \tag{2.3.7}$$

*Moreover, $C$ is bounded below by $-b$, and any domain of essential self-adjointness for $A$ is a form core for $C$.*

### 2.3.2 Existence of $H_N$

We now use Proposition 2.3.3 and Lemma 2.2.2 to realize $H_N$ as a self-adjoint operator on $L^2_{sym}(\mathbb{R}^N)$. Let $\underline{\Delta}_N := \sum_{i=1}^N \Delta_i$ denote the Laplacian on $\mathbb{R}^N$. It is easy to check that $-\underline{\Delta}_N$ is a positive, self-adjoint operator on $H^2_{sym}(\mathbb{R}^N)$ and that $\mathcal{Q}(-\underline{\Delta}_N) = H^1_{sym}(\mathbb{R}^N)$. We then have the following proposition.

**Proposition 2.3.4** (Existence of $H_N$)**.** *Let $N \in \mathbb{N}$, and let $\kappa \in \{\pm 1\}$. Then there exists a unique self-adjoint operator $H_N$ on $L^2_{sym}(\mathbb{R}^N)$ with form domain $\mathcal{Q}(H_N) = H^1_{sym}(\mathbb{R}^N)$ and such that*

$$\langle \Phi | H_N \Psi \rangle_{L^2(\mathbb{R}^N)} = \langle \Phi | -\underline{\Delta}_N \Psi \rangle_{L^2(\mathbb{R}^N)} + \frac{\kappa}{N} \sum_{1 \leq i < j \leq N} \langle \operatorname{tr}_{i=j} \Phi | \operatorname{tr}_{i=j} \Psi \rangle_{L^2(\mathbb{R}^{N-1})}, \qquad \forall \Phi, \Psi \in H^1_{sym}(\mathbb{R}^N). \tag{2.3.8}$$

*Moreover, $H_N$ is bounded from below by $0$, if $\kappa = 1$, and $-\frac{(N-1)}{2}$, if $\kappa = -1$, and any domain of essential self-adjointness for $\underline{\Delta}_N$ is a form core for $H_N$.*

*Proof.* We want to use Proposition 2.3.3. To this end, we let

$$A := -\underline{\Delta}_N : H^2_{sym}(\mathbb{R}^N) \to L^2_{sym}(\mathbb{R}^N), \tag{2.3.9}$$

and we define the quadratic form

$$\beta : \mathcal{Q}(A) \times \mathcal{Q}(A) \to \mathbb{C}, \qquad \beta(\Phi, \Psi) := \frac{\kappa}{N} \sum_{1 \leq i < j \leq N} \langle \operatorname{tr}_{i=j} \Phi | \operatorname{tr}_{i=j} \Psi \rangle, \tag{2.3.10}$$

which is evidently symmetric. Using the symmetry of $\Phi, \Psi$ under exchange of particle labels, we see that

$$\beta(\Phi, \Psi) = \frac{\kappa(N-1)}{2} \langle \operatorname{tr}_{1=2} \Phi | \operatorname{tr}_{1=2} \Psi \rangle. \tag{2.3.11}$$

By Lemma 2.2.2 and Young's inequality for products, we have that

$$|\langle \mathrm{tr}_{1=2}\,\Phi | \mathrm{tr}_{1=2}\,\Phi\rangle| \leq \|\nabla_1\Phi\|_{L^2}\|\Phi\|_{L^2} \leq \frac{1}{2}\big(\|\nabla_1\Phi\|_{L^2}^2 + \|\Phi\|_{L^2}^2\big). \tag{2.3.12}$$

Since by another application of the symmetry of $\Phi$,

$$\langle \Phi | -\underline{\Delta}_N \Phi\rangle = N\|\nabla_1\Phi\|_{L^2}^2, \tag{2.3.13}$$

we obtain that

$$|\beta(\Phi,\Phi)| \leq \frac{1}{2}\left\langle \Phi | -\underline{\Delta}_N\Phi\right\rangle + \frac{(N-1)}{2}\left\langle \Phi|\Phi\right\rangle. \tag{2.3.14}$$

The desired conclusion then follows from application of Proposition 2.3.3. □

**Remark 2.3.5.** An examination of the proof of the KLMN theorem in [79] shows that the domain of $H_N$ consists of all $\Phi \in H^1_{sym}(\mathbb{R}^N)$ such that the distribution

$$\left(-\underline{\Delta}_N + \frac{\kappa}{N}\sum_{1\leq i<j\leq N}\delta(X_i - X_j)\right)\Phi \in H^{-1}(\mathbb{R}^N) \tag{2.3.15}$$

may be (uniquely) identified with an element $\Psi \in L^2_{sym}(\mathbb{R}^N)$, which we denote by $H_N\Phi$. With a little more work, one can show that $\mathrm{Dom}(H_N)$ consists of all functions

$$\Phi \in H^1_{sym}(\mathbb{R}^N) \cap H^2_{sym}(\mathbb{R}^N \setminus \bigcup_{1\leq i<j\leq N}\{\underline{x}_N \in \mathbb{R}^N : x_i = x_j\}) \tag{2.3.16}$$

such that

$$\lim_{x_i - x_j = 0^+}(\nabla_i - \nabla_j)\Phi - \lim_{x_i - x_j = 0^-}(\nabla_i - \nabla_j)\Phi = \frac{\kappa}{2N}\,\mathrm{tr}_{i=j}\,\Phi. \tag{2.3.17}$$

Note for $1 \leq i < j \leq N$ and almost every $(\underline{x}_{1;i-1}, \underline{x}_{i+1;j-1}, \underline{x}_{j+1;N}) \in \mathbb{R}^{N-2}$ fixed, $\nabla_i\Phi$ and $\nabla_j\Phi$ are continuous away from the hyperplane $\{\underline{x}_N \in \mathbb{R}^N : x_i = x_j\}$ by Sobolev embedding.

### 2.3.3 Approximation of $H_N$

We close this section with some approximation results obtained from mollifying the $\delta$ pair potential in the expression (1.2.2) for $H_N$. These approximation results are used extensively in Section 2.4.

More precisely, let $\tilde{V} \in C_c^\infty(\mathbb{R})$ be an even function such that $0 \leq \tilde{V} \leq 1$, $\int_{\mathbb{R}} dx \tilde{V}(x) = 1$, and

$$\tilde{V}(x) = \begin{cases} 1, & |x| \leq \frac{1}{4} \\ 0, & |x| \geq \frac{1}{2} \end{cases}. \tag{2.3.18}$$

For $\varepsilon > 0$, set $V_\varepsilon(x) \coloneqq \varepsilon^{-1} V(x/\varepsilon)$. It is straightforward to check that the operator

$$H_{N,\varepsilon} \coloneqq -\underline{\Delta}_N + \frac{\kappa}{N} \sum_{1 \leq i < j \leq N} V_\varepsilon(X_i - X_j), \qquad \kappa \in \{\pm 1\} \tag{2.3.19}$$

is self-adjoint on $H^2_{sym}(\mathbb{R}^N)$. So by Stone's theorem, $H_{N,\varepsilon}$ generates a strongly continuous one-parameter unitary group $\{e^{itH_{N,\varepsilon}}\}_{t \in \mathbb{R}}$. We set $\Phi_N^\varepsilon \coloneqq e^{-itH_{N,\varepsilon}} \Phi_{N,0}$, where $\Phi_{N,0}$ is the same initial datum as in the Cauchy problem (1.2.4), so that $\Phi_N^\varepsilon$ is the unique global solution in $C_t^0(\mathbb{R}; L^2_{\underline{x}_N}(\mathbb{R}^N))$ to the Schrödinger equation

$$\begin{cases} i\partial_t \Phi_N^\varepsilon = H_{N,\varepsilon} \Phi_N^\varepsilon, \\ \Phi_N^\varepsilon(0) = \Phi_{N,0} \end{cases}. \tag{2.3.20}$$

Given that $V_\varepsilon \to \delta$ in distribution, as $\varepsilon \to 0$, we expect that $H_{N,\varepsilon} \to H_N$ in some sense. The sense in which this convergence holds is that of norm-resolvent convergence.

**Definition 2.3.6** (Norm-resolvent convergence)**.** Let $\{A_n\}_{n=1}^\infty$ be a sequence of self-adjoint operators on $\mathcal{H}$. Then we say that $A_n$ converges to $A$ in *norm-resolvent sense* if $R_\lambda(A_n) \to R_\lambda(A)$ in norm, for every $\lambda$ with $\operatorname{Im} \lambda \neq 0$, where $R_\lambda$ denotes the resolvent.

**Lemma 2.3.7.** *Fix $N \in \mathbb{N}$. We have that $H_{N,\varepsilon} \to H_N$ in norm-resolvent sense, as $\varepsilon \to 0^+$. Consequently, $e^{itH_{N,\varepsilon}} \to e^{itH_N}$ strongly, as $\varepsilon \to 0^+$, uniformly on compact intervals of time.*

*Proof.* Fix $\kappa \in \{\pm 1\}$. The second assertion regarding convergence of unitary groups follows from [44, Chapter 9, Theorem 2.16], so we focus on the first assertion. To show that $H_{N,\varepsilon} \to H_N$ in norm-resolvent sense, it suffices by [79, Theorem VII.25] to show that

$$\lim_{\varepsilon \to 0^+} \|H_{N,\varepsilon} - H_N\|_{H^1 \to H^{-1}} = 0, \tag{2.3.21}$$

where $\|\cdot\|_{H^1 \to H^{-1}}$ denotes the operator norm for maps $H^1(\mathbb{R}^N) \to H^{-1}(\mathbb{R}^N)$. To see that (2.3.21) holds, we observe that for any $\Phi \in H^1(\mathbb{R}^N)$,

$$(H_N - H_{N,\varepsilon})\Phi = \frac{\kappa}{N} \sum_{1 \le i < j \le N} (\delta(X_i - X_j) - V_\varepsilon(X_i - X_j))\Phi \in H^{-1}(\mathbb{R}^N). \tag{2.3.22}$$

Since $H^{-1}(\mathbb{R}^N)$ is isomorphic to $(H^1(\mathbb{R}^N))^*$ and by considerations of symmetry, it suffices to estimate

$$\begin{aligned} &\left|\langle \Psi, (\delta(X_1 - X_2) - V_\varepsilon(X_1 - X_2))\Phi\rangle_{H^1 - H^{-1}}\right| \\ &= \left| \int_{\mathbb{R}^{N-1}} d\underline{x}_{2;N} (\mathrm{tr}_{1=2}\,\Psi)(\underline{x}_{2;N})(\mathrm{tr}_{1=2}\,\Phi)(\underline{x}_{2;N}) - \int_{\mathbb{R}^N} d\underline{x}_N V_\varepsilon(x_1 - x_2)\Psi(\underline{x}_N)\Phi(\underline{x}_N) \right|, \end{aligned}$$

for every $\Psi \in H^1(\mathbb{R}^N)$ with $\|\Psi\|_{H^1} \le 1$. By the density of $\mathcal{S}(\mathbb{R}^N)$ in $H^1(\mathbb{R}^N)$, we may assume without loss of generality that $\Phi, \Psi$ are Schwartz. By Fubini-Tonelli,

$$\int_{\mathbb{R}^N} d\underline{x}_N V_\varepsilon(x_1 - x_2)\Psi(\underline{x}_N)\Phi(\underline{x}_N) = \int_{\mathbb{R}^{N-1}} d\underline{x}_{2;N} \int_{\mathbb{R}} dx_1 V_\varepsilon(x_1 - x_2)\Psi(\underline{x}_N)\Phi(\underline{x}_N), \tag{2.3.23}$$

and since $\int_{\mathbb{R}} dx V_\varepsilon(x) = 1$, it follows from translation invariance of Lebesgue measure that

$$\begin{aligned} &\int_{\mathbb{R}^{N-1}} d\underline{x}_{2;N} (\mathrm{tr}_{1=2}\,\Psi)(\underline{x}_{2;N})(\mathrm{tr}_{1=2}\,\Phi)(\underline{x}_{2;N}) \\ &= \int_{\mathbb{R}^{N-1}} d\underline{x}_{2;N} \int_{\mathbb{R}} dx_1 V_\varepsilon(x_1 - x_2)\Psi(x_2, \underline{x}_{2;N})\Phi(x_2, \underline{x}_{2;N}). \end{aligned} \tag{2.3.24}$$

Using the algebra property of Hölder norms followed by the dilation invariance of Lebesgue measure, we see that

$$\begin{aligned}&\left|\int_{\mathbb{R}} dx_1 V_\varepsilon(x_1-x_2)\big(\Psi(x_2,\underline{x}_{2;N})\Phi(x_2,\underline{x}_{2;N})-\Psi(x_1,\underline{x}_{2;N})\Phi(x_1,\underline{x}_{2;N})\big)\right|\\&\lesssim \varepsilon^{1/2}\|\Psi(\cdot,\underline{x}_{2;N})\|_{C^{1/2}}\|\Phi(\cdot,\underline{x}_{2;N})\|_{C^{1/2}}.\end{aligned}\tag{2.3.25}$$

Integrating both sides of the preceding inequality over $\mathbb{R}^{N-1}$ with respect to $\underline{x}_{2;N}$ then applying Cauchy-Schwarz, we obtain that

$$\begin{aligned}&\int_{\mathbb{R}^{N-1}} d\underline{x}_{2;N}\left|\int_{\mathbb{R}} dx_1 V_\varepsilon(x_1-x_2)\big(\Psi(x_2,\underline{x}_{2;N})\Phi(x_2,\underline{x}_{2;N})-\Psi(x_1,\underline{x}_{2;N})\Phi(x_1,\underline{x}_{2;N})\big)\right|\\&\lesssim \varepsilon^{1/2}\|\Psi\|_{L^2_{\underline{x}_{2;N}}C^{1/2}_{x_1}}\|\Phi\|_{L^2_{\underline{x}_{2;N}}C^{1/2}_{x_1}}\\&\lesssim \varepsilon^{1/2}\|\Phi\|_{H^1},\end{aligned}\tag{2.3.26}$$

where the ultimate inequality follows from Lemma 2.2.3 and the assumption that $\|\Psi\|_{H^1}\leq 1$. We therefore conclude that

$$\left|\langle\Psi,(\delta(X_1-X_2)-V_\varepsilon(X_1-X_2))\Phi\rangle_{H^1-H^{-1}}\right|\lesssim \varepsilon^{1/2}\|\Phi\|_{H^1},\tag{2.3.27}$$

which implies that $\|\delta(X_1-X_2)-V_\varepsilon(X_1-X_2)\|_{H^1\to H^{-1}}\lesssim\varepsilon^{1/2}$. It then follows from symmetry that

$$\limsup_{\varepsilon\to0^+}\|H_{N,\varepsilon}-H_N\|_{H^1\to H^{-1}}\lesssim\limsup_{\varepsilon\to0^+}N\varepsilon^{1/2}=0,\tag{2.3.28}$$

which completes the proof of the lemma.

We remark that one can also prove the desired norm-resolvent convergence by modifying the argument from [5, Subsubsection I.3.2]. □

## 2.4 Control of $\beta_N$

### 2.4.1 Projectors

As the goal of Section 2.4 is to prove Proposition 2.1.2, we first define the projectors underlying the definition of the functional $\beta_N$ in the statement of the proposition. Recall that $\phi \in C_t^0(\mathbb{R}; H_x^2(\mathbb{R}))$ is the unique solution to the cubic NLS (1.2.11) with initial datum $\phi_0 \in H^2(\mathbb{R})$. We define the projectors

$$p(t) \coloneqq |\phi(t)\rangle \langle\phi(t)|, \qquad q(t) \coloneqq \mathbf{1} - p(t), \qquad \forall t \in \mathbb{R}, \tag{2.4.1}$$

where $\mathbf{1}$ denotes the identity operator on $L^2(\mathbb{R})$. For $N \in \mathbb{N}$ and $j \in \{1, \ldots, N\}$, we define

$$p_j \coloneqq \mathbf{1}^{\otimes j-1} \otimes p \otimes \mathbf{1}^{\otimes N-j}, \qquad q_j \coloneqq \mathbf{1}_N - p_j = \mathbf{1}^{\otimes j-1} \otimes q \otimes \mathbf{1}^{\otimes N-j}, \tag{2.4.2}$$

where $\mathbf{1}_N = \mathbf{1}^{\otimes N}$ denotes the identity operator on $L^2(\mathbb{R}^N)$. Since $\mathbf{1} = p + q$, it follows that

$$\mathbf{1}_N = (p_1 + q_1) \cdots (p_N + q_N), \tag{2.4.3}$$

and therefore

$$\mathbf{1}_N = \sum_{k=0}^{N} P_k, \qquad P_k \coloneqq \sum_{\substack{\underline{\alpha}_N \in \{0,1\}^N \\ |\underline{\alpha}_N| = k}} \prod_{j=1}^{N} p_j^{1-\alpha_j} q_j^{\alpha_j}. \tag{2.4.4}$$

We define $P_k$ to be the zero operator on $L^2(\mathbb{R}^N)$ for $k \in \mathbb{Z} \setminus \{0, \ldots, N\}$. Important properties of the operators $P_k$ are the following:

(i) $P_k$ is an orthogonal projector on $L^2(\mathbb{R}^N)$;

(ii) $P_k(L^2_{sym}(\mathbb{R}^N)) \subset L^2_{sym}(\mathbb{R}^N)$;

(iii) $P_k P_l = \delta_{kl} P_k$, where $\delta_{kl}$ is the Kronecker delta function;

(iv) $p_j, q_j$ commute with $P_k$, for any $j \in \{1, \ldots, N\}$ and $k \in \mathbb{Z}$.

**Remark 2.4.1.** Since the function $\phi : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ underlying the definition of the projectors $p_j, q_j$ is time-dependent, the projector $P_k$ is also time-dependent (i.e. $P_k(t)$ is a projector on $L^2_{sym}(\mathbb{R}^N)$ for each $t \in \mathbb{R}$). For convenience, we do not emphasize the dependence on time with our notation in the sequel.

**Remark 2.4.2.** In the sequel, we frequently use without comment the elementary fact that $p_j, q_j$ are self-adjoint and that we have the operator norm identities

$$\|p_j\|_{L^2_{\underline{x}_N}(\mathbb{R}^N) \to L^2_{\underline{x}_N}(\mathbb{R}^N)} = \|q_j\|_{L^2_{\underline{x}_N}(\mathbb{R}^N) \to L^2_{\underline{x}_N}(\mathbb{R}^N)} = 1. \tag{2.4.5}$$

Given a function $f : \mathbb{Z} \to \mathbb{C}$, we define the operator

$$\widehat{f} \coloneqq \sum_{k \in \mathbb{Z}} f(k) P_k = \sum_{k=0}^{N} f(k) P_k. \tag{2.4.6}$$

The reader may check that for $f, g : \mathbb{Z} \to \mathbb{C}$, we have that $\widehat{fg} = \widehat{f}\widehat{g}$. Furthermore, since $p_j, q_j, P_k$ commute, it follows that $\widehat{f}$ commutes with $p_j, q_j, P_k$. Additionally, if $f, g$ are such that $f \geq g$. Then $\widehat{f} \geq \widehat{g}$. Indeed, since $P_k$ is an orthogonal projector,

$$\langle \Phi | \widehat{(f-g)} \Phi \rangle = \sum_{k=0}^{N} \langle P_k \Phi | \widehat{(f-g)} P_k \Phi \rangle \geq 0, \qquad \forall \Phi \in L^2(\mathbb{R}^N). \tag{2.4.7}$$

If $f \geq 0$, then we agree to abuse notation by writing

$$f^{-1}(k) \coloneqq \frac{1}{f(k)} 1_{>0}(k) \quad \text{and} \quad \widehat{f}^{-1} \coloneqq \sum_{k \in \mathbb{Z}} f^{-1}(k) P_k \tag{2.4.8}$$

with the convention that $0 \cdot \infty = 0$.

**Remark 2.4.3.** Since each $P_k$ is time-dependent, as commented in Remark 2.4.1, the operator $\widehat{f}$ is also time-dependent. Out of convenience, we do not emphasize the dependence on time with our notation in the sequel.

**Definition 2.4.4.** Given $N \in \mathbb{N}$, we define the functions $m_N, n_N : \mathbb{Z} \to [0,\infty)$ by

$$m_N(k) \coloneqq \frac{k}{N} 1_{\geq 0}(k) \quad \text{and} \quad n_N(k) \coloneqq \sqrt{\frac{k}{N}} 1_{\geq 0}(k), \qquad \forall k \in \mathbb{Z}. \tag{2.4.9}$$

Letting $\Phi_N$ denote the solution to the Schrödinger equation (1.2.4) and with the notation introduced in (2.4.6), we define the time-dependent quantities

$$\alpha_N(t) \coloneqq \langle \Phi_N(t) | \widehat{m_N}(t) \Phi_N(t) \rangle_{L^2(\mathbb{R}^N)} \quad \text{and} \quad \beta_N(t) \coloneqq \langle \Phi_N(t) | \widehat{n_N}(t) \Phi_N(t) \rangle_{L^2(\mathbb{R}^N)}, \qquad \forall t \in \mathbb{R}. \tag{2.4.10}$$

**Remark 2.4.5.** Since $\sum_{k=0}^{N} P_k = \mathbf{1}_N$, we have that

$$\frac{1}{N} \sum_{j=1}^{N} q_j = \frac{1}{N} \sum_{k \in \mathbb{Z}} \sum_{j=1}^{N} q_j P_k. \tag{2.4.11}$$

By unpacking the definition of $P_k$ in (2.4.4), the reader can check that $\sum_{j=1}^{N} q_j P_k = k P_k$, which implies that

$$\frac{1}{N} \sum_{j=1}^{N} q_j = \sum_{k \in \mathbb{Z}} \frac{k}{N} P_k = \widehat{m_N}. \tag{2.4.12}$$

It then follows from the symmetry of the wave function $\Phi_N$ under exchange of particle labels that

$$\alpha_N(t) = \langle \Phi_N(t) | \widehat{m_N}(t) \Phi_N(t) \rangle = \frac{1}{N} \sum_{i=1}^{N} \langle \Phi_N(t) | q_i(t) \Phi_N(t) \rangle = \langle \Phi_N(t) | q_1(t) \Phi_N(t) \rangle, \qquad \forall t \in \mathbb{R}. \tag{2.4.13}$$

We now record two technical lemmas from [46] pertaining to the operator $\widehat{m_N}$, which we frequently use in Section 2.4.

**Lemma 2.4.6** ([46, Lemma 3.9])**.** *For any function $f : \mathbb{Z} \to [0, \infty)$, the following hold:*

*(i)*

$$\|\widehat{f}^{1/2} q_1 \Phi_N\|^2_{L^2_{\underline{x}_N}} = \left\langle \Phi_N \middle| \widehat{f} q_1 \Phi_N \right\rangle_{L^2_{\underline{x}_N}} = \left\langle \Phi_N \middle| \widehat{f} \widehat{m_N} \Phi_N \right\rangle_{L^2_{\underline{x}_N}}, \tag{2.4.14}$$

*(ii)*

$$\|\widehat{f}^{1/2} q_1 q_2 \Phi_N\|^2_{L^2_{\underline{x}_N}} = \left\langle \Phi_N \middle| \widehat{f} q_1 q_2 \Phi_N \right\rangle_{L^2_{\underline{x}_N}} \leq \frac{N}{N-1} \left\langle \Phi_N \middle| \widehat{f} \widehat{m_N}^2 \Phi_N \right\rangle_{L^2_{\underline{x}_N}}. \tag{2.4.15}$$

Given $n \in \mathbb{N}$, we define the shift operator

$$\tau_n : \mathbb{C}^{\mathbb{Z}} \to \mathbb{C}^{\mathbb{Z}}, \qquad (\tau_n f)(k) \coloneqq f(k+n), \quad \forall k \in \mathbb{Z}, \ f \in \mathbb{C}^{\mathbb{Z}}. \tag{2.4.16}$$

**Lemma 2.4.7** ([46, Lemma 3.10])**.** *Let $r \in \mathbb{N}$, and let $A^{(r)}$ be a linear operator on $L^2_{sym}(\mathbb{R}^r)$. For $i \in \{1, 2\}$, let $Q_i$ be a projector of the form*

$$Q_i = \#_1 \cdots \#_r, \tag{2.4.17}$$

*where each $\#$ stands for either $p$ or $q$. Define the linear operator $A^{(r)}_{1\cdots r} \coloneqq A^{(r)} \otimes \mathbf{1}^{N-r}$. Then for any function $f : \mathbb{Z} \to \mathbb{C}$, we have that*

$$Q_1 A^{(r)}_{1\cdots r} \widehat{f} Q_2 = Q_1 \widehat{(\tau_n f)} A^{(r)}_{1\cdots r} Q_2, \tag{2.4.18}$$

*where $n \coloneqq n_2 - n_1$ and $n_i$ is the number of factors $q$ in $Q_i$, for $i \in \{1, 2\}$.*

### 2.4.2 Evolution of $\beta_{N,\varepsilon}$

In this subsection, we would like to control the evolution of the quantity $\beta_N$ introduced in Definition 2.4.4, thereby proving Proposition 2.1.2. As commented in Section 2.1.1 of the introduction, rather than work directly with $\beta_N$, we work with the approximation $\beta_{N,\varepsilon}$ defined in (2.4.19) below, which is obtained by replacing the $N$-body Hamiltonian $H_N$ constructed in Proposition 2.3.4 with the mollified Hamiltonian $H_{N,\varepsilon}$ from Section 2.3.3. The motivation is to justify some computations involving questions of operator domains and to avoid awkward notation involving distributions.

Similarly to $\alpha_N$ and $\beta_N$, we define the time-dependent quantities $\alpha_{N,\varepsilon}$ and $\beta_{N,\varepsilon}$ by

$$\alpha_{N,\varepsilon}(t) := \langle \Phi_N^\varepsilon(t) | \widehat{m_N}(t) \Phi_N^\varepsilon(t) \rangle \quad \text{and} \quad \beta_{N,\varepsilon}(t) := \langle \Phi_N^\varepsilon(t) | \widehat{n_N}(t) \Phi_N^\varepsilon(t) \rangle , \qquad \forall t \in \mathbb{R}, \tag{2.4.19}$$

where $\Phi_N^\varepsilon$ is the solution to the regularized Schrödinger equation (2.3.20). As a corollary of Lemma 2.3.7, we obtain that $\alpha_{N,\varepsilon} \to \alpha_N$ and $\beta_{N,\varepsilon} \to \beta_N$ uniformly on compact intervals on time. This result is a consequence of the following more general lemma.

**Lemma 2.4.8.** *Let $T > 0$, and let $f : \mathbb{Z} \to \mathbb{C}$ be bounded. For $N \in \mathbb{N}$ and $\varepsilon > 0$, define the functions $\vartheta_N, \vartheta_{N,\varepsilon} : \mathbb{R} \to \mathbb{C}$ by*

$$\vartheta_N(t) := \left\langle \Phi_N^\varepsilon(t) \middle| \widehat{f}(t) \Phi_N^\varepsilon(t) \right\rangle \quad \textit{and} \quad \vartheta_{N,\varepsilon}(t) := \left\langle \Phi_N(t) \middle| \widehat{f}(t) \Phi_N(t) \right\rangle , \qquad \forall t \in \mathbb{R}. \tag{2.4.20}$$

*Then for $N$ fixed,*

$$\lim_{\varepsilon \to 0^+} \sup_{|t| \le T} |\vartheta_{N,\varepsilon}(t) - \vartheta(t)| = 0. \tag{2.4.21}$$

*Proof.* First, observe from the definition (2.4.6) for $\widehat{f}$ that for any $\Psi_N \in L^2(\mathbb{R}^N)$,

$$\|\widehat{f}\Psi_N\|_{L^2_{\underline{x}_N}}^2 = \sum_{k=0}^N \|f(k) P_k \Psi_N\|_{L^2_{\underline{x}_N}}^2 \le \|f\|_{\ell^\infty}^2 \sum_{k=0}^N \|P_k \Psi_N\|_{L^2_{\underline{x}_N}}^2 = \|f\|_{\ell^\infty}^2 \|\Psi_N\|_{L^2_{\underline{x}_N}}^2 , \tag{2.4.22}$$

which implies that $\|\widehat{f}\|_{L^2_{\underline{x}_N}\to L^2_{\underline{x}_N}} \leq \|f\|_{\ell^\infty}$. Now by definition, $\Phi_N = e^{-itH_N}\Phi_{N,0}$ and $\Phi_N^\varepsilon = e^{-itH_{N,\varepsilon}}\Phi_{N,0}$, so that by the triangle inequality and Cauchy-Schwarz,

$$\begin{aligned}|\vartheta_N(t)-\vartheta_{N,\varepsilon}(t)| &= \left|\left\langle e^{-itH_N}\Phi_{N,0}\middle|\widehat{f}(t)e^{-itH_N}\Phi_{N,0}\right\rangle - \left\langle e^{-itH_{N,\varepsilon}}\Phi_{N,0}\middle|\widehat{f}(t)e^{-itH_{N,\varepsilon}}\Phi_{N,0}\right\rangle\right| \\ &\leq \left(\|e^{-itH_N}\Phi_{N,0}\|_{L^2} + \|e^{-itH_{N,\varepsilon}}\Phi_{N,0}\|_{L^2}\right)\|\widehat{f}(t)(e^{-itH_N}-e^{-itH_{N,\varepsilon}})\Phi_{N,0}\|_{L^2} \\ &\leq 2\|f\|_{\ell^\infty}\|(e^{-itH_N}-e^{-itH_{N,\varepsilon}})\Phi_{N,0}\|_{L^2}, \end{aligned} \tag{2.4.23}$$

where the ultimate inequality follows from the operator norm bound for $\widehat{f}$, unitarity of $e^{-itH_N}$ and $e^{-itH_{N,\varepsilon}}$, and $\|\Phi_{N,0}\|_{L^2} = 1$. The desired conclusion is then immediate from Lemma 2.3.7. □

The goal of this subsection is to prove the following proposition. The reader will recall that $\phi$ is the solution to the cubic NLS (1.2.11).

**Proposition 2.4.9.** *Let $\kappa \in \{\pm 1\}$. Then we have the estimate*

$$\begin{aligned}\dot{\beta}_{N,\varepsilon}(t) \lesssim\ & \frac{\|\phi(t)\|^2_{L^\infty(\mathbb{R})}}{N} + \frac{1}{N^\sigma} + \frac{\|\phi(t)\|^2_{L^4(\mathbb{R})}}{N^{(1-\sigma)/2}} + \frac{\|\phi(t)\|^2_{L^\infty(\mathbb{R})}}{N^{\delta/2}} + N^{\frac{2(\sigma-1)+\delta}{2}} + \varepsilon^{1/2}\|\phi(t)\|^2_{C^{1/2}(\mathbb{R})} \\ &+ \|\phi(t)\|^2_{C^{1/2}(\mathbb{R})}\|\phi(t)\|^2_{H^1(\mathbb{R})}\beta_{N,\varepsilon}(t) + \left(1+\|\phi(t)\|^2_{C^{1/2}(\mathbb{R})}\right)\|\nabla_1 q_1(t)\Phi_N^\varepsilon(t)\|^2_{L^2(\mathbb{R}^N)}, \end{aligned} \tag{2.4.24}$$

*for every $t \in \mathbb{R}$, uniformly in $(\varepsilon,\sigma,\delta) \in (0,1)^3$ and $N \in \mathbb{N}$.*

*Proof.* By time-reversal symmetry, it is enough to consider $t \geq 0$. Using that

$$\phi \in C_t^0(\mathbb{R}; H_x^2(\mathbb{R})) \cap C_t^1(\mathbb{R}; L_x^2(\mathbb{R})) \quad \text{and} \quad \Phi_{N,0} \in H^2(\mathbb{R}^N) = \mathrm{Dom}(H_{N,\varepsilon}) \tag{2.4.25}$$

together with following the argument in [46, Subsubsection 3.3.2, pg. 113], we see that $\beta_{N,\varepsilon}$ is differentiable with respect to $t$ and its derivative $\dot{\beta}_{N,\varepsilon}$ is given by

$$\dot{\beta}_{N,\varepsilon} = i\kappa\left\langle \Phi_N^\varepsilon \middle| \left[\frac{1}{N}\sum_{1\leq i<j\leq N} V_{\varepsilon,ij} - \sum_{i=1}^N V_i^\phi, \widehat{n_N}\right]\Phi_N^\varepsilon\right\rangle_{L^2_{\underline{x}_N}}, \tag{2.4.26}$$

where we have introduced the notation

$$V_{\varepsilon,ij} \coloneqq V_\varepsilon(X_i - X_j) \quad \text{and} \quad V_i^\phi \coloneqq |\phi(X_i)|^2. \tag{2.4.27}$$

Using the symmetry of $\Phi_N^\varepsilon$ and $\widehat{n_N}$ with respect to exchange of particle labels and the decomposition $\mathbf{1}_N = (p_1 + q_1)(p_2 + q_2)$, then examining which terms cancel, we see that

$$\begin{aligned}\dot{\beta}_{N,\varepsilon} &= \frac{i\kappa}{2}\left\langle \Phi_N^\varepsilon \middle| \left[(N-1)V_{\varepsilon,12} - NV_1^\phi - NV_2^\phi, \widehat{n_N}\right]\Phi_N^\varepsilon \right\rangle_{L^2_{\underline{x}_N}} \\ &= \mathrm{Term}_1 + \mathrm{Term}_2 + \mathrm{Term}_3,\end{aligned} \tag{2.4.28}$$

where

$$\mathrm{Term}_1 \coloneqq 2\,\mathrm{Re}\left\{ i\kappa \left\langle \Phi_N^\varepsilon \middle| p_1p_2\left[(N-1)V_{\varepsilon,12} - NV_1^\phi - NV_2^\phi, \widehat{n_N}\right]q_1p_2\Phi_N^\varepsilon \right\rangle_{L^2_{\underline{x}_N}} \right\}, \tag{2.4.29}$$

$$\mathrm{Term}_2 \coloneqq 2\,\mathrm{Re}\left\{ i\kappa \left\langle \Phi_N^\varepsilon \middle| q_1p_2\left[(N-1)V_{\varepsilon,12} - NV_1^\phi - NV_2^\phi, \widehat{n_N}\right]q_1q_2\Phi_N^\varepsilon \right\rangle_{L^2_{\underline{x}_N}} \right\}, \tag{2.4.30}$$

$$\mathrm{Term}_3 \coloneqq \mathrm{Re}\left\{ i\kappa \left\langle \Phi_N^\varepsilon \middle| p_1p_2\left[(N-1)V_{\varepsilon,12} - NV_1^\phi - NV_2^\phi, \widehat{n_N}\right]q_1q_2\Phi_N^\varepsilon \right\rangle_{L^2_{\underline{x}_N}} \right\}. \tag{2.4.31}$$

We proceed to estimate $\mathrm{Term}_1$, $\mathrm{Term}_2$, and $\mathrm{Term}_3$ individually. In the sequel, we drop the subscript $N$, as the number of particles is fixed. For convenience, we also introduce the notation

$$V_\varepsilon^\phi(x) \coloneqq (V_\varepsilon * |\phi|^2)(x) \ \text{ and } \ V_{\varepsilon,j}^\phi \coloneqq (V_\varepsilon * |\phi|^2)(X_j), \qquad \forall j \in \{1, \ldots, N\}. \tag{2.4.32}$$

Note that by Young's inequality and $\|V_\varepsilon\|_{L^1} = 1$, we have the operator norm estimate

$$\|V_{\varepsilon,j}^\phi\|_{L^2_{\underline{x}_N} \to L^2_{\underline{x}_N}} \le \|\phi\|_{L^\infty_x}^2, \qquad \forall \varepsilon > 0, \ j \in \{1, \ldots, N\}. \tag{2.4.33}$$

**Estimate for** $\mathrm{Term}_1$ We first observe that since $q_1$ commutes with $V_2^\phi, \widehat{n}$ and $p_1, q_1$ are orthogonal,

$$\left\langle \Phi^\varepsilon \middle| p_1p_2\left[NV_2^\phi, \widehat{n}\right]q_1p_2\Phi^\varepsilon \right\rangle_{L^2_{\underline{x}_N}} = \left\langle \Phi^\varepsilon \middle| \underbrace{p_1q_1}_{=0}\, p_2\left[NV_2^\phi, \widehat{n}\right]p_2\Phi^\varepsilon \right\rangle_{L^2_{\underline{x}_N}} = 0. \tag{2.4.34}$$

Since $p_2 V_{\varepsilon,12} p_2 = V_{\varepsilon,1}^{\phi} p_2$, it follows that

$$\begin{aligned}|\mathrm{Term}_1| &\lesssim \left| \left\langle \Phi^{\varepsilon} \middle| p_1 p_2 \left[ (N-1) V_{\varepsilon,1}^{\phi} - N V_1^{\phi}, \widehat{n} \right] q_1 p_2 \Phi^{\varepsilon} \right\rangle_{L^2_{\underline{x}_N}} \right| \\ &= \left| \left\langle \Phi^{\varepsilon} \middle| p_1 p_2 \Big( (N-1) V_{\varepsilon,1}^{\phi} - N V_1^{\phi} \Big) (\widehat{n} - \widehat{(\tau_{-1} n)}) q_1 p_2 \Phi^{\varepsilon} \right\rangle_{L^2_{\underline{x}_N}} \right|, \end{aligned} \tag{2.4.35}$$

where the ultimate equality follows from an application of Lemma 2.4.7. Define the function

$$\mu : \mathbb{Z} \to \mathbb{R}, \qquad \mu(k) \coloneqq N(n(k) - (\tau_{-1} n)(k)), \qquad \forall k \in \mathbb{Z}, \tag{2.4.36}$$

and observe that

$$\mu(k) = \frac{\sqrt{N}}{\sqrt{k} + 1_{\geq 0}(k-1)\sqrt{k-1}} 1_{\geq 0}(k) \leq n^{-1}(k), \qquad \forall k \in \mathbb{Z}. \tag{2.4.37}$$

So by the triangle inequality,

$$\begin{aligned}|\mathrm{Term}_1| &\lesssim \frac{1}{N} \left| \left\langle \Phi^{\varepsilon} \middle| p_1 p_2 V_{\varepsilon,1}^{\phi} \widehat{\mu} q_1 p_2 \Phi^{\varepsilon} \right\rangle_{L^2_{\underline{x}_N}} \right| + \left| \left\langle \Phi^{\varepsilon} \middle| p_1 p_2 (V_{\varepsilon,1}^{\phi} - V_1^{\phi}) \widehat{\mu} q_1 p_2 \Phi^{\varepsilon} \right\rangle_{L^2_{\underline{x}_N}} \right| \\ &\leq \frac{1}{N} \| V_{\varepsilon,1}^{\phi} \widehat{\mu} q_1 \Phi^{\varepsilon} \|_{L^2_{\underline{x}_N}} + \| (V_{\varepsilon,1}^{\phi} - V_1^{\phi}) \widehat{\mu} q_1 \Phi^{\varepsilon} \|_{L^2_{\underline{x}_N}}, \end{aligned} \tag{2.4.38}$$

where the ultimate inequality follows from Cauchy-Schwarz and $\|\Phi^{\varepsilon}\|_{L^2_{\underline{x}_N}} = 1$. By translation invariance of Lebesgue measure and $\int_{\mathbb{R}} dy V_{\varepsilon}(y) = 1$, for any $x \in \mathbb{R}$,

$$\left| (V_{\varepsilon} * |\phi|^2)(x) - |\phi(x)|^2 \right| = \left| \int_{\mathbb{R}} dy V_{\varepsilon}(y) \big( |\phi(x-y)|^2 - |\phi(x)|^2 \big) \right| \leq \int_{\mathbb{R}} dy V_{\varepsilon}(y) |y|^{1/2} \| |\phi|^2 \|_{\dot{C}_x^{1/2}} \leq \varepsilon^{1/2} \|\phi\|_{C_x^{1/2}}^2 \tag{2.4.39}$$

where the ultimate inequality follows from dilation invariance of Lebesgue measure and the algebra property of $C_x^{1/2}$. Hence,

$$\| (V_{\varepsilon} * |\phi|^2) - |\phi|^2 \|_{L_x^{\infty}} \leq \varepsilon^{1/2} \|\phi\|_{C_x^{1/2}}^2 \Longrightarrow \| V_{\varepsilon,1}^{\phi} - V_1^{\phi} \|_{L^2_{\underline{x}_N} \to L^2_{\underline{x}_N}} \leq \varepsilon^{1/2} \|\phi\|_{C_x^{1/2}}^2. \tag{2.4.40}$$

Using the preceding operator norm estimate together with (2.4.33), we obtain that

$$(2.4.38) \leq \left(\frac{\|\phi\|_{L_x^\infty}^2}{N} + \varepsilon^{1/2}\|\phi\|_{C_x^{1/2}}^2\right)\|\widehat{\mu}q_1\Phi^\varepsilon\|_{L_{\underline{x}_N}^2} \lesssim \frac{\|\phi\|_{L_x^\infty}^2}{N} + \varepsilon^{1/2}\|\phi\|_{C_x^{1/2}}^2, \tag{2.4.41}$$

where the ultimate inequality follows from the bound (2.4.37) for $\mu$ and an application of Lemma 2.4.6(i) together with recalling that $\widehat{n}^2 = \widehat{m}$. Thus, we conclude that

$$|\mathrm{Term}_1| \lesssim \frac{\|\phi\|_{L_x^\infty}^2}{N} + \varepsilon^{1/2}\|\phi\|_{C_x^{1/2}}^2. \tag{2.4.42}$$

**Estimate for** $\mathrm{Term}_2$ Arguing similarly as in (2.4.34), we see that

$$\left\langle \Phi^\varepsilon \middle| q_1p_2\left[V_1^\phi, \widehat{n}\right]q_1q_2\Phi^\varepsilon \right\rangle_{L_{\underline{x}_N}^2} = 0. \tag{2.4.43}$$

Therefore,

$$\begin{aligned}
2\,|\mathrm{Term}_2| &= \left|\left\langle \Phi^\varepsilon \middle| q_1p_2\left[(N-1)V_{\varepsilon,12} - NV_2^\phi, \widehat{n}\right]q_1q_2\Phi^\varepsilon \right\rangle_{L_{\underline{x}_N}^2}\right| \\
&= \left|\left\langle \Phi^\varepsilon \middle| q_1p_2\left(\frac{(N-1)}{N}V_{\varepsilon,12} - V_2^\phi\right)\widehat{\mu}q_1q_2\Phi^\varepsilon \right\rangle_{L_{\underline{x}_N}^2}\right| \\
&\leq \underbrace{\left|\langle \Phi^\varepsilon | q_1p_2V_{\varepsilon,12}\widehat{\mu}q_1q_2\Phi^\varepsilon\rangle_{L_{\underline{x}_N}^2}\right|}_{=:\mathrm{Term}_{2,1}} + \underbrace{\left|\left\langle \Phi^\varepsilon \middle| q_1p_2V_2^\phi\widehat{\mu}q_1q_2\Phi^\varepsilon \right\rangle_{L_{\underline{x}_N}^2}\right|}_{=:\mathrm{Term}_{2,2}},
\end{aligned} \tag{2.4.44}$$

where to obtain the penultimate equality have used Lemma 2.4.7 and introduced the notation $\mu$ from (2.4.36) and to obtain the ultimate equality we have used the triangle inequality.

We first consider $\mathrm{Term}_{2,2}$. By Cauchy-Schwarz together with the estimate (2.4.33),

$$\mathrm{Term}_{2,2} \leq \|q_1\Phi^\varepsilon\|_{L_{\underline{x}_N}^2}\|p_2V_2^\phi\widehat{\mu}q_1q_2\Phi^\varepsilon\|_{L_{\underline{x}_N}^2} \leq \|q_1\Phi^\varepsilon\|_{L_{\underline{x}_N}^2}\|\phi\|_{L_x^\infty}^2\|\widehat{\mu}q_1q_2\Phi^\varepsilon\|_{L_{\underline{x}_N}^2}. \tag{2.4.45}$$

By Remark 2.4.5 and Lemma 2.4.6(ii), respectively, together with the $\mu$ bound (2.4.37), we have that

$$\|q_1\Phi^\varepsilon\|_{L_{\underline{x}_N}^2} \leq \sqrt{\alpha_\varepsilon} \leq \sqrt{\beta_\varepsilon} \quad \text{and} \quad \|\widehat{\mu}q_1q_2\Phi^\varepsilon\|_{L_{\underline{x}_N}^2} \lesssim \sqrt{\beta_\varepsilon}. \tag{2.4.46}$$

Therefore,

$$\mathrm{Term}_{2,2} \lesssim \|\phi\|_{L_x^\infty}^2 \beta_\varepsilon. \tag{2.4.47}$$

We now consider $\mathrm{Term}_{2,1}$. It follows from the distributional identity (2.2.10) and the fact that $\delta * V_\varepsilon = V_\varepsilon$ that

$$V_\varepsilon = \frac{1}{2}(\nabla \operatorname{sgn} * V_\varepsilon) = \frac{1}{2}\nabla(\operatorname{sgn} * V_\varepsilon). \tag{2.4.48}$$

We introduce the notation $X_{\varepsilon,12} \coloneqq \frac{1}{2}(\operatorname{sgn} * V_\varepsilon)(X_1 - X_2)$. By Young's inequality, $\|V_\varepsilon\|_{L^1} = \|\operatorname{sgn}\|_{L^\infty} = 1$, so that

$$\|X_{\varepsilon,12}\|_{L^2_{\underline{x}_N} \to L^2_{\underline{x}_N}} \leq \frac{1}{2}. \tag{2.4.49}$$

Therefore, we find from integrating by parts and applying the product rule and triangle inequality that

$$\mathrm{Term}_{2,1} \leq \left| \langle \nabla_1 q_1 p_2 \Phi^\varepsilon | X_{\varepsilon,12} \widehat{\mu} q_1 q_2 \Phi^\varepsilon \rangle_{L^2_{\underline{x}_N}} \right| + \left| \langle \Phi^\varepsilon | q_1 p_2 X_{\varepsilon,12} \nabla_1 \widehat{\mu} q_1 q_2 \Phi^\varepsilon \rangle_{L^2_{\underline{x}_N}} \right| =: \mathrm{Term}_{2,1,1} + \mathrm{Term}_{2,1,2}. \tag{2.4.50}$$

By Cauchy-Schwarz and the estimate (2.4.49),

$$\mathrm{Term}_{2,1,1} \leq \|\nabla_1 q_1 p_2 \Phi^\varepsilon\|_{L^2_{\underline{x}_N}} \|\widehat{\mu} q_1 q_2 \Phi^\varepsilon\|_{L^2_{\underline{x}_N}}, \tag{2.4.51}$$

so by application of the second estimate of (2.4.46) and $\|p_2\|_{L^2_{\underline{x}_N} \to L^2_{\underline{x}_N}} = 1$,

$$\mathrm{Term}_{2,1,1} \lesssim \|\nabla_1 q_1 \Phi^\varepsilon\|_{L^2_{\underline{x}_N}} \sqrt{\beta_\varepsilon}. \tag{2.4.52}$$

Next, we write $\mathbf{1} = p_1 + q_1$ and use the triangle inequality to obtain

$$\mathrm{Term}_{2,1,2} \leq \left| \langle p_2 q_1 \Phi^\varepsilon | X_{\varepsilon,12} p_1 \nabla_1 \widehat{\mu} q_1 q_2 \Phi^\varepsilon \rangle_{L^2_{\underline{x}_N}} \right| + \left| \langle p_2 q_1 \Phi^\varepsilon | X_{\varepsilon,12} q_1 \nabla_1 \widehat{\mu} q_1 q_2 \Phi^\varepsilon \rangle_{L^2_{\underline{x}_N}} \right|. \tag{2.4.53}$$

By Lemma 2.4.7, we have the operator identity

$$p_1 \nabla_1 \widehat{\mu} q_1 = p_1 \widehat{(\tau_1 \mu)} \nabla_1 q_1. \tag{2.4.54}$$

Hence,

$$\begin{aligned}\left|\langle p_2 q_1 \Phi^\varepsilon | X_{\varepsilon,12} p_1 \nabla_1 \widehat{\mu} q_1 q_2 \Phi^\varepsilon \rangle_{L^2_{\underline{x}_N}}\right| &\leq \|X_{\varepsilon,12} p_2 q_1 \Phi^\varepsilon\|_{L^2_{\underline{x}_N}} \|p_1 \widehat{(\tau_1\mu)} \nabla_1 q_1 q_2 \Phi^\varepsilon\|_{L^2_{\underline{x}_N}} \\ &\leq \|q_1 \Phi^\varepsilon\|_{L^2_{\underline{x}_N}} \|\widehat{(\tau_1\mu)} \nabla_1 q_1 q_2 \Phi^\varepsilon\|_{L^2_{\underline{x}_N}}. \end{aligned} \tag{2.4.55}$$

By Remark 2.4.5, $\|q_1 \Phi^\varepsilon\|_{L^2_{\underline{x}_N}} \leq \sqrt{\beta_\varepsilon}$. Now using the $\mu$ bound (2.4.37), we have that

$$(\tau_1\mu)(k) \lesssim n^{-1}(k+1) \lesssim n^{-1}(k), \qquad \forall k \in \mathbb{Z}. \tag{2.4.56}$$

Combining this estimate with the symmetry of $\Phi^\varepsilon$ under permutation of particle labels, we find that

$$\begin{aligned}\|\widehat{(\tau_1\mu)} \nabla_1 q_1 q_2 \Phi^\varepsilon\|_{L^2_{\underline{x}_N}} &\lesssim \sqrt{\langle \nabla_1 q_1 \Phi^\varepsilon | \widehat{n}^{-2} \nabla_1 q_1 q_2 \Phi^\varepsilon \rangle_{L^2_{\underline{x}_N}}} \\ &= \sqrt{\frac{1}{N-1} \sum_{i=2}^N \langle \nabla_1 q_1 \Phi^\varepsilon | q_i \widehat{n}^{-2} \nabla_1 q_1 \Phi^\varepsilon \rangle_{L^2_{\underline{x}_N}}}. \end{aligned} \tag{2.4.57}$$

Since the projector $q_1$ commutes with $\widehat{n}^{-2}$ and $\widehat{n}^{-2} \geq 0$, we have that

$$\left\langle \nabla_1 q_1 \Phi^\varepsilon \middle| q_1 \widehat{n}^{-2} \nabla_1 q_1 \right\rangle_{L^2_{\underline{x}_N}} = \left\langle q_1 \nabla_1 q_1 \Phi^\varepsilon \middle| \widehat{n}^{-2} q_1 \nabla_1 q_1 \Phi^\varepsilon \right\rangle_{L^2_{\underline{x}_N}} \geq 0, \tag{2.4.58}$$

so that by Remark 2.4.5 and the identity $n^2 = m$,

$$\begin{aligned}\sqrt{\frac{1}{N-1} \sum_{i=2}^N \langle \nabla_1 q_1 \Phi^\varepsilon | q_i \widehat{n}^{-2} \nabla_1 q_1 \Phi^\varepsilon \rangle_{L^2_{\underline{x}_N}}} &\lesssim \sqrt{\frac{1}{N} \sum_{i=1}^N \langle \nabla_1 q_1 \Phi^\varepsilon | q_i \widehat{n}^{-2} \nabla_1 q_1 \Phi^\varepsilon \rangle_{L^2_{\underline{x}_N}}} \\ &= \sqrt{\langle \nabla_1 q_1 \Phi^\varepsilon | \widehat{n}^{-2} \widehat{n}^2 \nabla_1 q_1 \Phi^\varepsilon \rangle_{L^2_{\underline{x}_N}}} \\ &= \|\nabla_1 q_1 \Phi^\varepsilon\|_{L^2_{\underline{x}_N}}. \end{aligned} \tag{2.4.59}$$

After a little bookkeeping, we find that

$$\left|\langle p_2 q_1 \Phi^\varepsilon | X_{\varepsilon,12} p_1 \nabla_1 \widehat{\mu} q_1 q_2 \Phi^\varepsilon \rangle_{L^2_{\underline{x}_N}}\right| \lesssim \sqrt{\beta_\varepsilon} \|\nabla_1 q_1 \Phi^\varepsilon\|_{L^2_{\underline{x}_N}}. \tag{2.4.60}$$

Again by Lemma 2.4.7, we have the operator identity

$$q_1 \nabla_1 \widehat{\mu} q_1 = q_1 \widehat{\mu} \nabla_1 q_1, \tag{2.4.61}$$

and proceeding similarly as immediately above, we find that

$$\left| \langle p_2 q_1 \Phi^\varepsilon | X_{\varepsilon,12} q_1 \nabla_1 \widehat{\mu} q_1 q_2 \Phi^\varepsilon \rangle_{L^2_{\underline{x}_N}} \right| \lesssim \sqrt{\beta_\varepsilon} \|\nabla_1 q_1 \Phi^\varepsilon\|_{L^2_{\underline{x}_N}}, \tag{2.4.62}$$

and therefore

$$\mathrm{Term}_{2,1,2} \lesssim \|\nabla_1 q_1 \Phi^\varepsilon\|_{L^2_{\underline{x}_N}} \sqrt{\beta_\varepsilon}. \tag{2.4.63}$$

Together the estimate (2.4.52) for $\mathrm{Term}_{2,1,1}$, we obtain that

$$\mathrm{Term}_{2,1} \lesssim \|\nabla_1 q_1 \Phi^\varepsilon\|_{L^2_{\underline{x}_N}} \sqrt{\beta_\varepsilon}. \tag{2.4.64}$$

Collecting the estimates (2.4.64) for $\mathrm{Term}_{2,1}$ and (2.4.47) for $\mathrm{Term}_{2,2}$, we conclude that

$$\mathrm{Term}_2 \lesssim \|\phi\|^2_{L^\infty_x} \beta_\varepsilon + \|\nabla_1 q_1 \Phi^\varepsilon\|_{L^2_{\underline{x}_N}} \sqrt{\beta_\varepsilon}. \tag{2.4.65}$$

**Estimate for** $\mathrm{Term}_3$ We now consider $\mathrm{Term}_3$, which is the most difficult portion of the analysis. We first note that by arguing similarly as in (2.4.34), we see that

$$p_1 p_2 \left[ V_1^\phi, \widehat{n} \right] q_1 q_2 = 0 = p_1 p_2 \left[ V_2^\phi, \widehat{n} \right] q_1 q_2, \tag{2.4.66}$$

where the reader will recall the notation $V_j^\phi$ introduced in (2.4.27). Therefore,

$$\begin{aligned} |\mathrm{Term}_3| &\lesssim \left| \langle \Phi^\varepsilon | p_1 p_2 [(N-1) V_{\varepsilon,12}, \widehat{n}] q_1 q_2 \Phi^\varepsilon \rangle_{L^2_{\underline{x}_N}} \right| \\ &= \frac{N-1}{N} \left| \left\langle \Phi^\varepsilon \middle| p_1 p_2 N V_{\varepsilon,12} \left( \widehat{n} - \widehat{(\tau_{-2} n)} \right) q_1 q_2 \Phi^\varepsilon \right\rangle_{L^2_{\underline{x}_N}} \right|, \end{aligned} \tag{2.4.67}$$

where the ultimate equality follows from unpacking the commutator and applying Lemma 2.4.7. Analogously to the function $\mu$ defined in (2.4.36), we define the function

$$\nu : \mathbb{Z} \to \mathbb{R}, \qquad \nu(k) \coloneqq N(n(k) - (\tau_{-2}n)(k)), \qquad \forall k \in \mathbb{Z}. \tag{2.4.68}$$

It is a straightforward computation from the definition of $n$ in Definition 2.4.4 that

$$\nu(k) = \frac{2\sqrt{N}}{\sqrt{k} + 1_{\geq 2}(k)\sqrt{k-2}} 1_{\geq 0}(k), \qquad \forall k \in \mathbb{Z}, \tag{2.4.69}$$

which implies that

$$\nu(k) \lesssim n^{-1}(k), \qquad \forall k \in \mathbb{Z}. \tag{2.4.70}$$

We now introduce an approximation of the pair potential $V_\epsilon$ as follows. Define $V_\sigma(x) \coloneqq N^\sigma \tilde{V}(N^\sigma x)$, where $\sigma \in (0,1)$ is a parameter to be specified momentarily and $\tilde{V}$ is as in Section 2.3.3. We convolve $V_\varepsilon$ with $V_\sigma$ to define

$$V_{\varepsilon,\sigma} \coloneqq V_\varepsilon * V_\sigma \quad \text{and} \quad V_{\varepsilon,\sigma,ij} \coloneqq V_{\varepsilon,\sigma}(X_i - X_j), \;\; \forall 1 \leq i < j \leq N. \tag{2.4.71}$$

By the triangle inequality,

$$\left|\langle \Phi^\varepsilon | p_1 p_2 V_{\varepsilon,12} \widehat{\nu} q_1 q_2 \Phi^\varepsilon \rangle_{L^2_{\underline{x}_N}}\right| \leq \underbrace{\left|\langle \Phi^\varepsilon | p_1 p_2 (V_{\varepsilon,12} - V_{\varepsilon,\sigma,12}) \widehat{\nu} q_1 q_2 \Phi^\varepsilon \rangle_{L^2_{\underline{x}_N}}\right|}_{=:\mathrm{Term}_{3,1}} + \underbrace{\left|\langle \Phi^\varepsilon | p_1 p_2 V_{\varepsilon,\sigma,12} \widehat{\nu} q_1 q_2 \Phi^\varepsilon \rangle_{L^2_{\underline{x}_N}}\right|}_{=:\mathrm{Term}_{3,2}}. \tag{2.4.72}$$

Observe that by moving $p_1 p_2$ over to the first entry of the inner product, writing out the convolution implicit in $V_{\varepsilon,\sigma,12}$, and using the Fubini-Tonelli theorem, we have that

$$\begin{aligned}
&\langle\Phi^\varepsilon|p_1p_2V_{\varepsilon,\sigma,12}\widehat{\nu}q_1q_2\Phi^\varepsilon\rangle_{L^2_{\underline{x}_N}}\\
&=\int_{\mathbb{R}}dyV_\sigma(y)\int_{\mathbb{R}^2}d\underline{x}_{1;2}V_\varepsilon(x_1-x_2-y)\int_{\mathbb{R}^{N-2}}d\underline{x}_{3;N}\Big(\overline{(p_1p_2\Phi^\varepsilon)}(\widehat{\nu}q_1q_2\Phi^\varepsilon)\Big)(x_1,x_2,\underline{x}_{3;N})\\
&=\int_{\mathbb{R}}dyV_\sigma(y)\int_{\mathbb{R}^2}d\underline{x}_{1;2}V_\varepsilon(x_1-x_2-y)\int_{\mathbb{R}^{N-2}}d\underline{x}_{3;N}\,\Big(\Big(\overline{(p_1p_2\Phi^\varepsilon)}(\widehat{\nu}q_1q_2\Phi^\varepsilon)\Big)(x_1,x_2,\underline{x}_{3;N})\\
&\qquad\qquad-\Big(\overline{(p_1p_2\Phi^\varepsilon)}(\widehat{\nu}q_1q_2\Phi^\varepsilon)\Big)(x_1,x_2+y,\underline{x}_{3;N})\Big)\\
&\quad+\int_{\mathbb{R}}dyV_\sigma(y)\int_{\mathbb{R}^2}d\underline{x}_{1;2}V_\varepsilon(x_1-x_2-y)\int_{\mathbb{R}^{N-2}}d\underline{x}_{3;N}\Big(\overline{(p_1p_2\Phi^\varepsilon)}(\widehat{\nu}q_1q_2\Phi^\varepsilon)\Big)(x_1,x_2+y,\underline{x}_{3;N}).
\end{aligned}\tag{2.4.73}$$

By translation invariance of Lebesgue measure applied in the $x_2$-coordinate, we have that for any $y\in\mathbb{R}$,

$$\begin{aligned}
&\int_{\mathbb{R}^2}d\underline{x}_{1;2}V_\varepsilon(x_1-x_2-y)\int_{\mathbb{R}^{N-2}}d\underline{x}_{3;N}\Big(\overline{(p_1p_2\Phi^\varepsilon)}(\widehat{\nu}q_1q_2\Phi^\varepsilon)\Big)(x_1,x_2+y,\underline{x}_{3;N})\\
&=\int_{\mathbb{R}^2}d\underline{x}_{1;2}V_\varepsilon(x_1-x_2)\int_{\mathbb{R}^{N-2}}d\underline{x}_{3;N}\Big(\overline{(p_1p_2\Phi^\varepsilon)}(\widehat{\nu}q_1q_2\Phi^\varepsilon)\Big)(x_1,x_2,\underline{x}_{3;N})\\
&=\langle\Phi^\varepsilon|p_1p_2V_{\varepsilon,12}\widehat{\nu}q_1q_2\Phi^\varepsilon\rangle_{L^2_{\underline{x}_N}},
\end{aligned}\tag{2.4.74}$$

where the ultimate equality follows from using the Fubini-Tonelli theorem and the self-adjointness of $p_1p_2$. Since $\int_{\mathbb{R}}dyV_\sigma(y)=1$, we conclude that

$$\begin{aligned}
&\int_{\mathbb{R}}dyV_\sigma(y)\int_{\mathbb{R}^2}d\underline{x}_{1;2}V_\varepsilon(x_1-x_2-y)\int_{\mathbb{R}^{N-2}}d\underline{x}_{3;N}\Big(\overline{(p_1p_2\Phi^\varepsilon)}(\widehat{\nu}q_1q_2\Phi^\varepsilon)\Big)(x_1,x_2+y,\underline{x}_{3;N})\\
&=\langle\Phi^\varepsilon|p_1p_2V_{\varepsilon,12}\widehat{\nu}q_1q_2\Phi^\varepsilon\rangle_{L^2_{\underline{x}_N}}.
\end{aligned}\tag{2.4.75}$$

Next, we have by definition of the Hölder norm in the $x_2$-coordinate that

$$\begin{aligned}
&\sup_{x_2\in\mathbb{R}}\left|\Big(\overline{(p_1p_2\Phi^\varepsilon)}(\widehat{\nu}q_1q_2\Phi^\varepsilon)\Big)(x_1,x_2,\underline{x}_{3;N})-\Big(\overline{(p_1p_2\Phi^\varepsilon)}(\widehat{\nu}q_1q_2\Phi^\varepsilon)\Big)(x_1,x_2+y,\underline{x}_{3;N})\right|\\
&\leq\|\Big(\overline{(p_1p_2\Phi^\varepsilon)}(\widehat{\nu}q_1q_2\Phi^\varepsilon)\Big)(x_1,\cdot,\underline{x}_{3;N})\|_{\dot{C}^{1/2}_{x_2}}|y|^{1/2}\\
&\leq\|(p_1p_2\Phi^\varepsilon)(x_1,\cdot,\underline{x}_{3;N})\|_{C^{1/2}_{x_2}}\|(\widehat{\nu}q_1q_2\Phi^\varepsilon)(x_1,\cdot,\underline{x}_{3;N})\|_{C^{1/2}_{x_2}}|y|^{1/2},
\end{aligned}\tag{2.4.76}$$

for every $y \in \mathbb{R}$ and almost every $(x_1, \underline{x}_{3;N}) \in \mathbb{R}^{N-1}$, where the ultimate inequality follows from the fact $C^{1/2}$ is an algebra. So by the Fubini-Tonelli theorem, followed by using the translation and dilation invariance of Lebesgue measure and then Cauchy-Schwarz, we find that

$$\begin{aligned}
&\int_{\mathbb{R}} dy V_\sigma(y) \int_{\mathbb{R}^2} d\underline{x}_{1;2} V_\varepsilon(x_1 - x_2 - y) \int_{\mathbb{R}^{N-2}} d\underline{x}_{3;N} \Big| \Big(\overline{(p_1 p_2 \Phi^\varepsilon)}(\widehat{\nu} q_1 q_2 \Phi^\varepsilon)\Big)(x_1, x_2, \underline{x}_{3;N}) \\
&\qquad\qquad\qquad\qquad\qquad\qquad - \Big(\overline{(p_1 p_2 \Phi^\varepsilon)}(\widehat{\nu} q_1 q_2 \Phi^\varepsilon)\Big)(x_1, x_2 + y, \underline{x}_{3;N}) \Big| \\
&\leq \int_{\mathbb{R}^{N-1}} dx_1 d\underline{x}_{3;N} \Bigg( \|(p_1 p_2 \Phi^\varepsilon)(x_1, \cdot, \underline{x}_{3;N})\|_{C_{x_2}^{1/2}} \|(\widehat{\nu} q_1 q_2 \Phi^\varepsilon)(x_1, \cdot, \underline{x}_{3;N})\|_{C_{x_2}^{1/2}} \\
&\qquad\qquad \times \underbrace{\left( \int_{\mathbb{R}} dy |y|^{1/2} V_\sigma(y) \int_{\mathbb{R}} dx_2 V_\varepsilon(x_1 - x_2 - y) \right)}_{\lesssim N^{-\sigma/2}} \Bigg) \\
&\lesssim N^{-\sigma/2} \|p_1 p_2 \Phi^\varepsilon\|_{L^2_{\underline{x}_{2;N}} C^{1/2}_{x_1}} \|\widehat{\nu} q_1 q_2 \Phi^\varepsilon\|_{L^2_{\underline{x}_{2;N}} C^{1/2}_{x_1}},
\end{aligned} \tag{2.4.77}$$

where in the ultimate inequality we use the symmetry of $\Phi_\varepsilon$ to swap $x_1$ and $x_2$ in order to ease the burden of notation. By Fubini-Tonelli, Cauchy-Schwarz, and the normalization $\|\phi\|_{L^2_x} = 1$, we have the estimate

$$\|p_1 p_2 \Phi^\varepsilon\|_{L^2_{\underline{x}_{2;N}} C^{1/2}_{x_1}} \leq \|\phi\|_{C_x^{1/2}} \|p_2 \Phi^\varepsilon\|_{L^2_{\underline{x}_N}} \leq \|\phi\|_{C_x^{1/2}}, \tag{2.4.78}$$

where the ultimate inequality follows from the normalization $\|\Phi^\varepsilon\|_{L^2_{\underline{x}_N}} = 1$. By Lemma 2.2.3 and the $H^{1/2+} \subset L^\infty$ Sobolev embedding,

$$\|\widehat{\nu} q_1 q_2 \Phi^\varepsilon\|_{L^2_{\underline{x}_{2;N}} C^{1/2}_{x_1}} \lesssim \|\widehat{\nu} q_1 q_2 \Phi^\varepsilon\|_{L^2_{\underline{x}_{2;N}} H^1_{x_1}} \lesssim \|\widehat{\nu} q_1 q_2 \Phi^\varepsilon\|_{L^2_{\underline{x}_N}} + \|\nabla_1 \widehat{\nu} q_1 q_2 \Phi^\varepsilon\|_{L^2_{\underline{x}_N}}, \tag{2.4.79}$$

where the ultimate inequality follows from splitting the $H^1_x$ norm and Fubini-Tonelli. Using the $\nu$ estimate (2.4.70), Lemma 2.4.6(ii), and the identity $\widehat{m} = \widehat{n}^2$, we see that

$$\|\widehat{\nu} q_1 q_2 \Phi^\varepsilon\|_{L^2_{\underline{x}_N}} \lesssim \sqrt{\langle \Phi^\varepsilon | \widehat{n}^{-2} \widehat{m}^2 \Phi^\varepsilon \rangle_{L^2_{\underline{x}_N}}} = \sqrt{\langle \Phi^\varepsilon | \widehat{m} \Phi^\varepsilon \rangle_{L^2_{\underline{x}_N}}} = \sqrt{\alpha_\varepsilon} \leq \sqrt{\beta_\varepsilon}. \tag{2.4.80}$$

Next, inserting the decomposition $\nabla_1 = p_1\nabla_1 + q_1\nabla_1$ and applying the triangle inequality,

$$\|\nabla_1\widehat{\nu}q_1q_2\Phi^\varepsilon\|_{L^2_{\underline{x}_N}} \leq \|p_1\nabla_1\widehat{\nu}q_1q_2\Phi^\varepsilon\|_{L^2_{\underline{x}_N}} + \|q_1\nabla_1\widehat{\nu}q_1q_2\Phi^\varepsilon\|_{L^2_{\underline{x}_N}}. \tag{2.4.81}$$

Since $p_1\nabla_1 = -(\,|\phi\rangle\,\langle\nabla\phi|)_1$,

$$\|p_1\nabla_1\widehat{\nu}q_1q_2\Phi^\varepsilon\|_{L^2_{\underline{x}_N}} \leq \|\nabla\phi\|_{L^2_x}\|\widehat{\nu}q_1q_2\Phi^\varepsilon\|_{L^2_{\underline{x}_N}} \lesssim \|\nabla\phi\|_{L^2_x}\sqrt{\beta_\varepsilon}, \tag{2.4.82}$$

where the ultimate inequality follows from the estimate (2.4.80). By Lemma 2.4.7 followed by using the $\nu$ estimate (2.4.70),

$$\|q_1\nabla_1\widehat{\nu}q_1q_2\Phi^\varepsilon\|_{L^2_{\underline{x}_N}} = \|q_1\widehat{\nu}\nabla_1q_1q_2\Phi^\varepsilon\|_{L^2_{\underline{x}_N}} \lesssim \sqrt{\langle\nabla_1q_1\Phi^\varepsilon|q_2\widehat{n}^{-2}\nabla_1q_1\Phi^\varepsilon\rangle_{L^2_{\underline{x}_N}}}, \tag{2.4.83}$$

and arguing as for the estimate (2.4.59), we find that the right-hand side is $\lesssim \|\nabla_1q_1\Phi^\varepsilon\|_{L^2_{\underline{x}_N}}$. Therefore,

$$\|\widehat{\nu}q_1q_2\Phi^\varepsilon\|_{L^2_{\underline{x}_{2;N}}C^{1/2}_{x_1}} \lesssim \left(1 + \|\nabla\phi\|_{L^2_x}\right)\sqrt{\beta_\varepsilon} + \|\nabla_1q_1\Phi^\varepsilon\|_{L^2_{\underline{x}_N}} \lesssim \|\phi\|_{H^1_x}\sqrt{\beta_\varepsilon} + \|\nabla_1q_1\Phi^\varepsilon\|_{L^2_{\underline{x}_N}}. \tag{2.4.84}$$

Collecting the estimates (2.4.78) and (2.4.84), we see that

$$\begin{aligned} N^{-\sigma/2}\|p_1p_2\Phi^\varepsilon\|_{L^2_{\underline{x}_{2;N}}C^{1/2}_{x_1}}\|\widehat{\nu}q_1q_2\Phi^\varepsilon\|_{L^2_{\underline{x}_{2;N}}C^{1/2}_{x_1}} &\lesssim N^{-\sigma/2}\|\phi\|_{C^{1/2}_x}\Big(\|\phi\|_{H^1_x}\sqrt{\beta_\varepsilon} + \|\nabla_1q_1\Phi^\varepsilon\|_{L^2_{\underline{x}_N}}\Big) \\ &\lesssim N^{-\sigma} + \|\phi\|^2_{C^{1/2}_x}\|\phi\|^2_{H^1_x}\beta_\varepsilon + \|\phi\|^2_{C^{1/2}_x}\|\nabla_1q_1\Phi^\varepsilon\|^2_{L^2_{\underline{x}_N}}, \end{aligned} \tag{2.4.85}$$

where the ultimate line follows from Young's inequality for products.

After a little bookkeeping, we conclude that

$$|\mathrm{Term}_{3,1}| \lesssim N^{-\sigma} + \|\phi\|^2_{C^{1/2}_x}\|\phi\|^2_{H^1_x}\beta_\varepsilon + \|\phi\|^2_{C^{1/2}_x}\|\nabla_1q_1\Phi^\varepsilon\|^2_{L^2_{\underline{x}_N}}, \tag{2.4.86}$$

leaving us with $\mathrm{Term}_{3,2}$.

For $\text{Term}_{3,2}$, we borrow an idea from [46] and introduce a partition of unity as follows. Let $\chi^{(1)}, \chi^{(2)} : \mathbb{Z} \to [0,\infty)$ be the two functions respectively defined by

$$\chi^{(1)}(k) \coloneqq 1_{\leq N^{1-\delta}}(k), \quad \chi^{(2)}(k) \coloneqq 1 - \chi^{(1)}(k) = 1_{>N^{1-\delta}}(k), \qquad \forall k \in \mathbb{Z}. \tag{2.4.87}$$

where $\delta \in (0,1)$ will be optimized at the end. Trivially, we have that $\chi^{(j)} \in \{0,1\}^{\mathbb{Z}}$, so that $(\chi^{(j)}(k))^2 = \chi^{(j)}(k)$, and $\chi^{(1)}(k) + \chi^{(2)}(k) = 1$. We insert this decomposition into the expression for $\text{Term}_{3,2}$ and use the triangle inequality to obtain

$$|\text{Term}_{3,2}| \leq \underbrace{\left|\left\langle \Phi^{\varepsilon} \middle| p_1 p_2 V_{\varepsilon,\sigma,12} \widehat{\nu} \widehat{\chi^{(1)}} q_1 q_2 \Phi^{\varepsilon} \right\rangle_{L^2_{\underline{x}_N}}\right|}_{=:\text{Term}_{3,2,1}} + \underbrace{\left|\left\langle \Phi^{\varepsilon} \middle| p_1 p_2 V_{\varepsilon,\sigma,12} \widehat{\nu} \widehat{\chi^{(2)}} q_1 q_2 \Phi^{\varepsilon} \right\rangle_{L^2_{\underline{x}_N}}\right|}_{=:\text{Term}_{3,2,2}}. \tag{2.4.88}$$

We consider $\text{Term}_{3,2,1}$ and $\text{Term}_{3,2,2}$ separately.

For $\text{Term}_{3,2,1}$, we want to use the fact that the operator norm of $p_1 p_2 V_{\varepsilon,\sigma,12} q_1 q_2$ is much smaller on the bosonic subspace $L^2_{sym}(\mathbb{R}^N)$ than on the full space $L^2(\mathbb{R}^N)$. Accordingly, we symmetrize the expression $p_2 V_{\varepsilon,\sigma,12} q_2$ to write

$$\begin{aligned}\text{Term}_{3,2,1} &= \frac{1}{N-1}\left|\left\langle \Phi^{\varepsilon} \middle| \sum_{i=2}^{N} p_1 p_i V_{\varepsilon,\sigma,1i} q_i q_1 \widehat{\chi^{(1)}} \widehat{\nu} q_1 \Phi^{\varepsilon} \right\rangle_{L^2_{\underline{x}_N}}\right| \\ &\leq \frac{1}{N-1} \| \sum_{i=2}^{N} \widehat{\chi^{(1)}} q_i q_1 V_{\varepsilon,\sigma,1i} p_i p_1 \Phi^{\varepsilon} \|_{L^2_{\underline{x}_N}} \| \widehat{\nu} q_1 \Phi^{\varepsilon} \|_{L^2_{\underline{x}_N}}. \end{aligned} \tag{2.4.89}$$

where the ultimate line follows from Cauchy-Schwarz. We claim that $\|\widehat{\nu} q_1 \Phi^{\varepsilon}\|_{L^2_{\underline{x}_N}} \lesssim 1$. Indeed, by the $\nu$ bound (2.4.70) and Lemma 2.4.6(i),

$$\|\widehat{\nu} q_1 \Phi^{\varepsilon}\|_{L^2_{\underline{x}_N}} = \sqrt{\langle \Phi^{\varepsilon} | \widehat{\nu}^2 q_1 \Phi^{\varepsilon} \rangle_{L^2_{\underline{x}_N}}} \lesssim \sqrt{\langle \Phi^{\varepsilon} | \widehat{n}^{-2} \widehat{m} \Phi^{\varepsilon} \rangle_{L^2_{\underline{x}_N}}} = 1, \tag{2.4.90}$$

since $\widehat{n}^2 = \widehat{m}$ and $\|\Phi^\varepsilon\|_{L^2_{\underline{x}_N}} = 1$. Now expanding the $L^2_{\underline{x}_N}$ norm and using that $\widehat{\chi^{(1)}}^2 = \widehat{\chi^{(1)}}$, we see that

$$\begin{aligned}
\|\sum_{i=2}^{N}\widehat{\chi^{(1)}}q_iq_1V_{\varepsilon,\sigma,1i}p_ip_1\Phi^\varepsilon\|_{L^2_{\underline{x}_N}} &= \sqrt{\sum_{i,j=2}^{N}\left\langle\Phi^\varepsilon\middle|p_1p_iV_{\varepsilon,\sigma,1i}q_1q_i\widehat{\chi^{(1)}}q_1q_jV_{\varepsilon,\sigma,1j}p_jp_1\Phi^\varepsilon\right\rangle_{L^2_{\underline{x}_N}}} \\
&\leq \underbrace{\sqrt{\sum_{i=2}^{N}\left\langle\Phi^\varepsilon\middle|p_1p_iV_{\varepsilon,\sigma,1i}q_1q_i\widehat{\chi^{(1)}}q_1q_iV_{\varepsilon,\sigma,1i}p_ip_1\Phi^\varepsilon\right\rangle_{L^2_{\underline{x}_N}}}}_{=:\sqrt{B}} \\
&\quad + \underbrace{\sqrt{\sum_{2\leq i\neq j\leq N}\left\langle\Phi^\varepsilon\middle|p_1p_iV_{\varepsilon,\sigma,1i}q_1q_i\widehat{\chi^{(1)}}q_1q_jV_{\varepsilon,\sigma,1j}p_jp_1\Phi^\varepsilon\right\rangle_{L^2_{\underline{x}_N}}}}_{=:\sqrt{A}},
\end{aligned} \tag{2.4.91}$$

where the ultimate inequality follows from the embedding $\ell^{1/2} \subset \ell^1$. Therefore,

$$\mathrm{Term}_{3,2,1} \lesssim \frac{1}{N-1}\Big(\sqrt{B} + \sqrt{A}\Big). \tag{2.4.92}$$

We first consider $B$, which is the easy term. Since $\|q_1q_i\widehat{\chi^{(1)}}q_1q_i\|_{L^2_{\underline{x}_N}\rightarrow L^2_{\underline{x}_N}} \leq 1$,

$$B \leq \sum_{i=2}^{N}\|V_{\varepsilon,\sigma,1i}p_1p_i\Phi^\varepsilon\|^2_{L^2_{\underline{x}_N}} = \sum_{i=2}^{N}\left\langle\Phi^\varepsilon\middle|p_1p_iV^2_{\varepsilon,\sigma,1i}p_1p_i\Phi^\varepsilon\right\rangle_{L^2_{\underline{x}_N}}. \tag{2.4.93}$$

Now by examination of the integral kernel of $p_1p_iV^2_{\varepsilon,\sigma,1i}p_1p_i$,

$$p_1p_iV^2_{\varepsilon,\sigma,1i}p_1p_i = \left(\int_{\mathbb{R}^2}dy_1dy_iV^2_{\varepsilon,\sigma}(y_1-y_i)|\phi(y_1)|^2|\phi(y_i)|^2\right)p_1p_i = \||\phi|^2(V^2_{\varepsilon,\sigma}*|\phi|^2)\|_{L^1_x}p_1p_i, \tag{2.4.94}$$

and by Cauchy-Schwarz followed by Young's inequality,

$$\||\phi|^2(V^2_{\varepsilon,\sigma}*|\phi|^2)\|_{L^1_x} \leq \|\phi\|^2_{L^4_x}\|V^2_{\varepsilon,\sigma}*|\phi|^2\|_{L^2_x} \leq \underbrace{\|V_{\varepsilon,\sigma}\|^2_{L^2}}_{\leq N^\sigma}\|\phi\|^4_{L^4_x}. \tag{2.4.95}$$

It then follows from $\|\Phi^\varepsilon\|_{L^2_{\underline{x}_N}} = 1$ that

$$B \leq (N-1)N^\sigma \|\phi\|_{L^4_x}^4. \tag{2.4.96}$$

We proceed to consider $A$. We first make a further decomposition of $A$ by using that $(\chi^{(1)})^2 = \chi^{(1)}$ and then applying Lemma 2.4.7 in order to obtain

$$\begin{aligned}
A &= \sum_{2\leq i\neq j\leq N} \left\langle \Phi^\varepsilon \middle| p_1 p_i V_{\varepsilon,\sigma,1i} q_1 q_i \widehat{\chi^{(1)}}\widehat{\chi^{(1)}} q_j q_1 V_{\varepsilon,\sigma,1j} p_j p_1 \Phi^\varepsilon \right\rangle_{L^2_{\underline{x}_N}} \\
&= \sum_{2\leq i\neq j\leq N} \left\langle \Phi^\varepsilon \middle| p_1 p_i q_j (\widehat{\tau_2\chi^{(1)}}) V_{\varepsilon,\sigma,1i} q_1 V_{\varepsilon,\sigma,1j} (\widehat{\tau_2\chi^{(1)}}) q_i p_j p_1 \Phi^\varepsilon \right\rangle_{L^2_{\underline{x}_N}} \\
&= \underbrace{\sum_{2\leq i\neq j\leq N} \left\langle \Phi^\varepsilon \middle| p_1 p_i q_j (\widehat{\tau_2\chi^{(1)}}) V_{\varepsilon,\sigma,1,i} V_{\varepsilon,\sigma,1j} (\widehat{\tau_2\chi^{(1)}}) q_i p_j p_1 \Phi^\varepsilon \right\rangle_{L^2_{\underline{x}_N}}}_{=:A_1} \\
&\quad \underbrace{- \sum_{2\leq i\neq j\leq N} \left\langle \Phi^\varepsilon \middle| p_1 p_i q_j (\widehat{\tau_2\chi^{(1)}}) V_{\varepsilon,\sigma,1i} p_1 V_{\varepsilon,\sigma,1j} (\widehat{\tau_2\chi^{(1)}}) q_i p_j p_1 \Phi^\varepsilon \right\rangle_{L^2_{\underline{x}_N}}}_{=:A_2},
\end{aligned} \tag{2.4.97}$$

where the ultimate equality follows from writing $q_1 = \mathbf{1} - p_1$.

For $A_1$, we have by the triangle inequality and self-adjointness of $(\widehat{\tau_2\chi^{(1)}})q_j$ that

$$|A_1| \leq \sum_{2\leq i\neq j\leq N} \left| \left\langle (\widehat{\tau_2\chi^{(1)}}) q_j \Phi^\varepsilon \middle| p_1 p_i V_{\varepsilon,\sigma,1i} V_{\varepsilon,\sigma,1j} p_j p_1 (\widehat{\tau_2\chi^{(1)}}) q_i \Phi^\varepsilon \right\rangle_{L^2_{\underline{x}_N}} \right|. \tag{2.4.98}$$

Using that $V_{\varepsilon,\sigma} \geq 0$ and commutativity of point-wise multiplication operators, we can write

$$V_{\varepsilon,\sigma,1i} V_{\varepsilon,\sigma,1j} = (V_{\varepsilon,\sigma,1i} V_{\varepsilon,\sigma,1j})^{1/2} (V_{\varepsilon,\sigma,1i} V_{\varepsilon,\sigma,1j})^{1/2} \tag{2.4.99}$$

and then use Cauchy-Schwarz to obtain

$$\begin{aligned}
\left| \left\langle (\widehat{\tau_2\chi^{(1)}}) q_j \Phi^\varepsilon \middle| p_1 p_i V_{\varepsilon,\sigma,1i} V_{\varepsilon,\sigma,1j} p_j p_1 (\widehat{\tau_2\chi^{(1)}}) q_i \Phi^\varepsilon \right\rangle_{L^2_{\underline{x}_N}} \right| &\leq \|(V_{\varepsilon,\sigma,1i} V_{\varepsilon,\sigma,1j})^{1/2} p_1 p_i (\widehat{\tau_2\chi^{(1)}}) q_j \Phi^\varepsilon\|_{L^2_{\underline{x}_N}} \\
&\quad \times \|(V_{\varepsilon,\sigma,1i} V_{\varepsilon,\sigma,1j})^{1/2} p_j p_1 (\widehat{\tau_2\chi^{(1)}}) q_i \Phi^\varepsilon\|_{L^2_{\underline{x}_N}}.
\end{aligned} \tag{2.4.100}$$

From Young's inequality for products and the symmetry of $\Phi^{\varepsilon}$ under permutation of particle labels, we then find that

$$(2.4.98) \leq \sum_{2\leq i\neq j\leq N} \left\langle \Phi^{\varepsilon} \middle| \widehat{(\tau_2\chi^{(1)})} q_j p_1 p_i V_{\varepsilon,\sigma,1i} V_{\varepsilon,\sigma,1j} p_1 p_i q_j \widehat{(\tau_2\chi^{(1)})} \Phi^{\varepsilon} \right\rangle_{L^2_{\underline{x}_N}}. \tag{2.4.101}$$

Next, by computation of its integral kernel, we see that

$$p_i V_{\varepsilon,\sigma,1i} V_{\varepsilon,\sigma,1j} p_i = p_i (V_{\varepsilon,\sigma} * |\phi|^2)_1 V_{\varepsilon,\sigma,1j}, \tag{2.4.102}$$

and

$$(p_1 (V_{\varepsilon,\sigma} * |\phi|^2)_1 V_{\varepsilon,\sigma,1j} p_1) = p_1 \big(V_{\varepsilon,\sigma} * (|\phi|^2 (V_{\varepsilon,\sigma} * |\phi|^2))\big)_j. \tag{2.4.103}$$

By Young's inequality with $\|V_{\varepsilon,\sigma}\|_{L^1} = 1$, followed by Hölder's inequality, and then another application of Young's, we have that

$$\|\big(V_{\varepsilon,\sigma} * (|\phi|^2 (V_{\varepsilon,\sigma} * |\phi|^2))\big)\|_{L^\infty_x} \leq \|\phi\|^2_{L^\infty_x} \|V_{\varepsilon,\sigma} * |\phi|^2\|_{L^\infty_x} \leq \|\phi\|^4_{L^\infty_x}, \tag{2.4.104}$$

which implies that

$$\|p_1 p_i V_{\varepsilon,\sigma,1i} V_{\varepsilon,\sigma,1j} p_1 p_i\|_{L^2_{\underline{x}_N} \to L^2_{\underline{x}_N}} \leq \|\phi\|^4_{L^\infty_x}. \tag{2.4.105}$$

Applying this last estimate to the right-hand side of (2.4.101) and the symmetry of $\Phi^{\varepsilon}$, we obtain that

$$|A_1| \lesssim \|\phi\|^4_{L^\infty_x} \sum_{2\leq i\neq j\leq N} \|\widehat{(\tau_2\chi^{(1)})} q_j \Phi^{\varepsilon}\|^2_{L^2_{\underline{x}_N}} \leq N^2 \|\phi\|^4_{L^\infty_x} \|\widehat{(\tau_2\chi^{(1)})} q_1 \Phi^{\varepsilon}\|^2_{L^2_{\underline{x}_N}} \leq N^2 \|\phi\|^4_{L^\infty_x} \|\widehat{(\tau_2\chi^{(1)})} \widehat{n} \Phi^{\varepsilon}\|^2_{L^2_{\underline{x}_N}}, \tag{2.4.106}$$

where the ultimate inequality follows by application of Lemma 2.4.6(i) to the factor $\|\widehat{(\tau_2\chi^{(1)})} q_1 \Phi^{\varepsilon}\|_{L^2_{\underline{x}_N}}$. In order to estimate the last expression, we claim that

$$(\tau_2\chi^{(1)})(k) n(k) \leq N^{-\delta/2}, \qquad \forall k \in \{0, \ldots, N\}. \tag{2.4.107}$$

Indeed, recalling from (2.4.87) that $\chi^{(1)} = 1_{\leq N^{1-\delta}}$, where $\delta \in (0,1)$, we see that

$$(\tau_2\chi^{(1)})(k)n(k) = 1_{\leq N^{1-\delta}}(k+2)1_{\geq 0}(k)\sqrt{\frac{(k+2)-2}{N}} \leq 1_{\leq N^{1-\delta}}(k)\sqrt{\frac{N^{1-\delta}}{N} - \frac{2}{N}}, \quad (2.4.108)$$

from which the claim follows. Applying this estimate to the right-hand side of (2.4.106) leads to the conclusion

$$|A_1| \lesssim N^{2-\delta}\|\phi\|_{L_x^\infty}^4. \quad (2.4.109)$$

Now using the identity

$$p_1 V_{\varepsilon,\sigma,1i} p_1 V_{\varepsilon,\sigma,1j} p_1 = p_1 (V_{\varepsilon,\sigma} * |\phi|^2)_i (V_{\varepsilon,\sigma} * |\phi|^2)_j, \quad (2.4.110)$$

which follows from examination of the integral kernel, and arguing similarly as for $A_1$, we find that

$$|A_2| \leq \|V_{\varepsilon,\sigma} * |\phi|^2\|_{L_x^\infty}^2 \sum_{2\leq i\neq j\leq N} \|q_j \widehat{(\tau_2\chi^{(1)})} \Phi^\varepsilon\|_{L_{\underline{x}_N}^2} \|q_i \widehat{(\tau_2\chi^{(1)})} \Phi^\varepsilon\|_{L_{\underline{x}_N}^2} \lesssim N^{2-\delta}\|\phi\|_{L_x^\infty}^4.$$
$$(2.4.111)$$

Thus, we conclude from (2.4.109) and (2.4.111) that

$$|A| \lesssim N^{2-\delta}\|\phi\|_{L_x^\infty}^4. \quad (2.4.112)$$

To conclude the estimate for $\mathrm{Term}_{3,2,1}$ defined in (2.4.88) above, we insert the estimate (2.4.96) for $B$ and the estimate (2.4.112) for $A$ into the right-hand side of (2.4.92), obtaining

$$\mathrm{Term}_{3,2,1} \lesssim \frac{1}{N-1}\Big(\sqrt{(N-1)N^\sigma\|\phi\|_{L_x^4}^4} + \sqrt{N^{2-\delta}\|\phi\|_{L_x^\infty}^4}\Big) \lesssim \frac{\|\phi\|_{L_x^4}^2}{N^{(1-\sigma)/2}} + \frac{\|\phi\|_{L_x^\infty}^2}{N^{\delta/2}}. \quad (2.4.113)$$

It remains for us to estimate $\mathrm{Term}_{3,2,2}$, which we recall from (2.4.88) is defined by

$$\mathrm{Term}_{3,2,2} = \left|\left\langle \Phi^\varepsilon \middle| p_1 p_2 V_{\varepsilon,\sigma,12} \widehat{\nu} \widehat{\chi^{(2)}} q_1 q_2 \Phi^\varepsilon \right\rangle_{L_{\underline{x}_N}^2}\right|. \quad (2.4.114)$$

Writing $\widehat{\nu} = \widehat{\nu}^{1/2}\widehat{\nu}^{1/2}$ and using the same symmetrization trick as above, we find that

$$\begin{aligned}\mathrm{Term}_{3,2,2} &= \frac{1}{N-1}\left|\left\langle \Phi^\varepsilon \middle| \sum_{i=2}^N p_1 p_i V_{\varepsilon,\sigma,1i} q_i q_1 \widehat{\chi^{(2)}} \widehat{\nu}^{1/2}\widehat{\nu}^{1/2}\Phi^\varepsilon \right\rangle_{L^2_{\underline{x}_N}}\right| \\ &\leq \frac{1}{N-1}\|\widehat{\nu}^{1/2} q_1\Phi^\varepsilon\|_{L^2_{\underline{x}_N}} \sqrt{\sum_{i,j=2}^N \left\langle \Phi^\varepsilon \middle| p_1 p_i V_{\varepsilon,\sigma,1i} q_1 q_i \widehat{\chi^{(2)}}\widehat{\nu} q_1 q_j V_{\varepsilon,\sigma,1j} p_j p_1 \Phi^\varepsilon\right\rangle_{L^2_{\underline{x}_N}}},\end{aligned} \tag{2.4.115}$$

where the ultimate inequality follows by Cauchy-Schwarz and expanding the $L^2_{\underline{x}_N}$ norm of the second factor. By the $\nu$ estimate (2.4.70) together with Lemma 2.4.6(i),

$$\|\widehat{\nu}^{1/2} q_1 \Phi^\varepsilon\|_{L^2_{\underline{x}_N}} = \sqrt{\langle \Phi^\varepsilon | \widehat{\nu} q_1 \Phi^\varepsilon\rangle_{L^2_{\underline{x}_N}}} \lesssim \sqrt{\langle \Phi^\varepsilon | \widehat{n}^{-1} q_1 \Phi^\varepsilon\rangle_{L^2_{\underline{x}_N}}} \lesssim \sqrt{\beta_\varepsilon}. \tag{2.4.116}$$

Thus, splitting the sum $\sum_{i,j} = \sum_i + \sum_{i\neq j}$ in the second factor of (2.4.115) and applying the embedding $\ell^{1/2} \subset \ell^1$, we obtain that

$$\mathrm{Term}_{3,2,2} \leq \frac{\sqrt{\beta_\varepsilon}}{N-1}\Big(\sqrt{A} + \sqrt{B}\Big), \tag{2.4.117}$$

where

$$B \coloneqq \sum_{i=2}^N \left\langle \Phi^\varepsilon \middle| p_1 p_i V_{\varepsilon,\sigma,1i} q_1 q_i \widehat{\chi^{(2)}} \widehat{\nu} V_{\varepsilon,\sigma,1i} p_i p_1 \Phi^\varepsilon \right\rangle_{L^2_{\underline{x}_N}}, \tag{2.4.118}$$

$$A \coloneqq \sum_{2\leq i\neq j\leq N} \left\langle \Phi^\varepsilon \middle| p_1 p_i V_{\varepsilon,\sigma,1i} q_1 q_i \widehat{\chi^{(2)}} \widehat{\nu} q_j V_{\varepsilon,\sigma,1j} p_j p_1 \Phi^\varepsilon \right\rangle_{L^2_{\underline{x}_N}}. \tag{2.4.119}$$

Note that in contrast to the inequality (2.4.92) for $\mathrm{Term}_{3,2,1}$, we have a factor of $\sqrt{\beta_\varepsilon}$ in the right-hand side of inequality (2.4.117).

We first dispense with the easy case $B$. We recall from (2.4.87) that $\chi^{(2)} = 1_{>N^{1-\delta}}$, which together with the $\nu$ bound (2.4.70) implies the estimate

$$\chi^{(2)}(k)\nu(k) \lesssim 1_{>N^{1-\delta}}(k) n^{-1}(k) = 1_{>N^{1-\delta}}(k)\sqrt{\frac{N}{k}} < N^{\delta/2}, \qquad \forall k \in \mathbb{Z}. \tag{2.4.120}$$

Therefore, we have the $L^2_{\underline{x}_N}$ operator norm estimate

$$\|q_1 q_i \widehat{\chi^{(2)}}\widehat{\nu}\|_{L^2_{\underline{x}_N}\to L^2_{\underline{x}_N}} \lesssim N^{\delta/2}, \qquad \forall i\in\{1,\ldots,N\}, \tag{2.4.121}$$

which implies that

$$B \lesssim N^{\delta/2}\sum_{i=2}^{N}\|V_{\varepsilon,\sigma,1i}p_1p_i\Phi^\varepsilon\|^2_{L^2_{\underline{x}_N}} = (N-1)N^{\delta/2}\|V_{\varepsilon,\sigma,12}p_1p_2\Phi^\varepsilon\|^2_{L^2_{\underline{x}_N}}, \tag{2.4.122}$$

where the ultimate identity follows from the symmetry of $\Phi^\varepsilon$. Since by Cauchy-Schwarz and Young's inequality,

$$p_1p_2V^2_{\varepsilon,\sigma,12}p_1p_2 = \||\phi|^2(V^2_{\varepsilon,\sigma}*|\phi|^2)\|_{L^1_x}p_1p_2 \lesssim N^\sigma\|\phi\|^4_{L^4_x}p_1p_2, \tag{2.4.123}$$

where we also use $\|V_{\varepsilon,\sigma}\|^2_{L^2}\lesssim N^\sigma$, we conclude that

$$B \lesssim N^{1+\frac{\delta}{2}+\sigma}\|\phi\|^4_{L^4_x}. \tag{2.4.124}$$

For the hard case $A$, we again use Lemma 2.4.7 as in (2.4.97) to write $A = A_1 + A_2$, where

$$A_1 := \sum_{2\le i\neq j\le N}\left\langle \Phi^\varepsilon\middle| p_1p_iq_j\widehat{(\tau_2\chi^{(2)})}\widehat{(\tau_2\nu)}^{1/2}V_{\varepsilon,\sigma,1i}V_{\varepsilon,\sigma,1j}\widehat{(\tau_2\chi^{(2)})}\widehat{(\tau_2\nu)}^{1/2}q_ip_jp_1\Phi^\varepsilon\right\rangle_{L^2_{\underline{x}_N}}, \tag{2.4.125}$$

$$A_2 := -\sum_{2\le i\neq j\le N}\left\langle \Phi^\varepsilon\middle| p_1p_iq_j\widehat{(\tau_2\chi^{(2)})}\widehat{(\tau_2\nu)}^{1/2}V_{\varepsilon,\sigma,1i}p_1V_{\varepsilon,\sigma,1j}\widehat{(\tau_2\chi^{(2)})}^{1/2}\widehat{(\tau_2\nu)}^{1/2}q_ip_jp_1\Phi^\varepsilon\right\rangle_{L^2_{\underline{x}_N}}. \tag{2.4.126}$$

For $A_1$, we use that $V_{\varepsilon,\sigma}\geq 0$ to apply Cauchy-Schwarz and exploit the symmetry of $\Phi_\varepsilon$ under exchange of particle labels in order to obtain

$$|A_1| \le \sum_{2\le i\neq j\le N}\left|\left\langle \Phi^\varepsilon\middle| q_j\widehat{(\tau_2\chi^{(2)})}\widehat{(\tau_2\nu)}^{1/2}p_1p_iV_{\varepsilon,\sigma,1i}V_{\varepsilon,\sigma,1j}p_ip_1\widehat{(\tau_2\chi^{(2)})}\widehat{(\tau_2\nu)}^{1/2}q_j\Phi^\varepsilon\right\rangle_{L^2_{\underline{x}_N}}\right|. \tag{2.4.127}$$

Using the $L^2_{\underline{x}_N}$ operator norm estimate (2.4.105), we conclude that

$$|A_1| \lesssim \|\phi\|^4_{L^\infty_x} \sum_{2\leq i\neq j\leq N} \underbrace{\|\widehat{(\tau_2\chi^{(2)})}\widehat{(\tau_2\nu)}^{1/2} q_1\Phi^\varepsilon\|^2_{L^2_{\underline{x}_N}}}_{\leq\left\langle\Phi^\varepsilon\middle|\widehat{(\tau_2\nu)}q_1\Phi^\varepsilon\right\rangle_{L^2_{\underline{x}_N}}} \lesssim N^2\|\phi\|^4_{L^\infty_x}\left\langle\Phi^\varepsilon|\widehat{n}\Phi^\varepsilon\right\rangle_{L^2_{\underline{x}_N}} = N^2\|\phi\|^4_{L^\infty_x}\beta_\varepsilon, \tag{2.4.128}$$

where the penultimate inequality follows from the $\nu$ estimate (2.4.70) together with Lemma 2.4.6(i) and the ultimate equality is by definition of $\beta_\varepsilon$ (recall (2.4.19)). Next, using the operator identity (2.4.110) and arguing similarly as for $A_2$ in the case of $\chi^{(1)}$, we also obtain the estimate

$$|A_2| \lesssim N^2\|\phi\|^4_{L^\infty_x}\beta_\varepsilon, \tag{2.4.129}$$

leading us to conclude that

$$|A| \lesssim N^2\|\phi\|^4_{L^\infty_x}\beta_\varepsilon. \tag{2.4.130}$$

Inserting the estimates (2.4.124) for $B$ and (2.4.130) for $A$ into the right-hand side of (2.4.117), we find from the normalization $\|\phi\|_{L^2_x} = 1$ and Young's inequality for products that

$$\mathrm{Term}_{3,2,2} \lesssim \frac{\sqrt{\beta_\varepsilon}}{N-1}\left(N\|\phi\|^2_{L^\infty_x}\sqrt{\beta_\varepsilon} + N^{\frac{1+\sigma}{2}+\frac{\delta}{4}}\|\phi\|^2_{L^4_x}\right) \lesssim \|\phi\|^2_{L^\infty_x}\beta_\varepsilon + N^{\frac{2(\sigma-1)+\delta}{2}}. \tag{2.4.131}$$

Collecting the estimates (2.4.113) for $\mathrm{Term}_{3,2,1}$ and (2.4.131) for $\mathrm{Term}_{3,2,2}$, we find that

$$|\mathrm{Term}_{3,2}| \lesssim N^{\frac{\sigma-1}{2}}\|\phi\|^2_{L^4_x} + N^{-\frac{\delta}{2}}\|\phi\|^2_{L^\infty_x} + \|\phi\|^2_{L^\infty_x}\beta_\varepsilon + N^{\frac{2(\sigma-1)+\delta}{2}}. \tag{2.4.132}$$

Now inserting the estimates (2.4.86) for $\mathrm{Term}_{3,1}$ and (2.4.132) for $\mathrm{Term}_{3,2}$ into the right-hand side of (2.4.72), we conclude that

$$
\begin{aligned}
|\mathrm{Term}_3| &\lesssim N^{-\sigma} + \|\phi\|_{C_x^{1/2}}^2 \|\phi\|_{H_x^1}^2 \beta_\varepsilon + \|\phi\|_{C_x^{1/2}}^2 \|\nabla_1 q_1 \Phi^\varepsilon\|_{L^2_{\underline{x}_N}}^2 \\
&\quad + N^{\frac{\sigma-1}{2}} \|\phi\|_{L_x^4}^2 + N^{-\frac{\delta}{2}} \|\phi\|_{L_x^\infty}^2 + \|\phi\|_{L_x^\infty}^2 \beta_\varepsilon + N^{\frac{2(\sigma-1)+\delta}{2}} \\
&\lesssim N^{-\sigma} + \|\phi\|_{C_x^{1/2}}^2 \|\phi\|_{H_x^1}^2 \beta_\varepsilon + \|\phi\|_{C_x^{1/2}}^2 \|\nabla_1 q_1 \Phi^\varepsilon\|_{L^2_{\underline{x}_N}}^2 + N^{\frac{\sigma-1}{2}} \|\phi\|_{L_x^4}^2 + N^{-\frac{\delta}{2}} \|\phi\|_{L_x^\infty}^2 + N^{\frac{2(\sigma-1)+\delta}{2}}.
\end{aligned}
\tag{2.4.133}
$$

where the ultimate line follows from the trivial $C_x^{1/2} \subset L_x^\infty$ embedding and the fact $\|\phi\|_{H_x^1}^2 \geq 1$.

We are now prepared to conclude the proof of the proposition. After a bookkeeping of the estimates (2.4.42) for $\mathrm{Term}_1$, (2.4.65) for $\mathrm{Term}_2$, and (2.4.133) for $\mathrm{Term}_3$, we find that

$$
\begin{aligned}
\dot{\beta}_\varepsilon &\lesssim \frac{\|\phi\|_{L_x^\infty}^2}{N} + \varepsilon^{1/2} \|\phi\|_{C_x^{1/2}}^2 + \|\phi\|_{L_x^\infty}^2 \beta_\varepsilon + \|\nabla_1 q_1 \Phi^\varepsilon\|_{L^2_{\underline{x}_N}} \sqrt{\beta_\varepsilon} + \frac{1}{N^\sigma} + \|\phi\|_{C_x^{1/2}}^2 \|\phi\|_{H_x^1}^2 \beta_\varepsilon \\
&\quad + \|\phi\|_{C_x^{1/2}}^2 \|\nabla_1 q_1 \Phi^\varepsilon\|_{L^2_{\underline{x}_N}}^2 + \frac{\|\phi\|_{L_x^4}^2}{N^{(1-\sigma)/2}} + \frac{\|\phi\|_{L_x^\infty}^2}{N^{\delta/2}} + N^{\frac{2(\sigma-1)+\delta}{2}}.
\end{aligned}
\tag{2.4.134}
$$

The desired conclusion now follows from Young's inequality for products, $\|\phi\|_{L_x^2} = 1$, and some algebra. $\square$

### 2.4.3 Control of $\|\nabla_1 q_1 \Phi_N\|_{L^2}$

Before we can pass to the limit $\varepsilon \to 0^+$ to remove the regularization of the LL Hamiltonian, we need to control the auxiliary quantity $\|\nabla_1 q_1 \Phi_N^\varepsilon\|_{L^2_{\underline{x}_N}}$ appearing in the right-hand side of (2.4.134). To this end, we first introduce the energy per particle of the solution $\Phi_N^\varepsilon$ to equation (2.3.20):

$$
E_N^{\Phi^\varepsilon} := \frac{1}{N} \langle \Phi_N^\varepsilon | H_{N,\varepsilon} \Phi_N^\varepsilon \rangle_{L^2_{\underline{x}_N}(\mathbb{R}^N)} = \|\nabla_1 \Phi_{N,0}\|_{L^2(\mathbb{R}^N)}^2 + \frac{\kappa(N-1)}{2N} \langle \Phi_{N,0} | V_{\varepsilon,12} \Phi_{N,0} \rangle_{L^2(\mathbb{R}^N)},
\tag{2.4.135}
$$

where the ultimate equality follows from conservation of energy, unpacking the definition (2.3.19) of $H_{N,\varepsilon}$, and exploiting the symmetry of $\Phi^{\varepsilon}$. We recall from (2.1.8) that the energy of the solution $\phi$ to the cubic NLS (1.2.11) is given by

$$E^{\phi} = \|\nabla\phi\|_{L_x^2(\mathbb{R})}^2 + \frac{\kappa}{2}\|\phi\|_{L_x^4(\mathbb{R})}^4 = \|\nabla\phi_0\|_{L^2(\mathbb{R})}^2 + \frac{\kappa}{2}\|\phi_0\|_{L^4(\mathbb{R})}^4. \tag{2.4.136}$$

The reader will remember that $\kappa \in \{\pm 1\}$ denotes the sign of the interaction (i.e. repulsive or attractive). The goal of this subsection is to prove the following proposition, which controls $\|\nabla_1 q_1 \Phi_N^{\varepsilon}\|_{L_{\underline{x}_N}^2}^2$ in terms of $\beta_{\varepsilon}$, $N$, and $(E_N^{\Phi^{\varepsilon}} - E^{\phi})$.

**Proposition 2.4.10** (Control of $\|\nabla_1 q_1 \Phi_N\|_{L^2}^2$)**.** *Let $\kappa \in \{\pm 1\}$. Then we have the estimate*

$$\|\nabla_1 q_1(t)\Phi_N^{\varepsilon}(t)\|_{L^2(\mathbb{R}^N)}^2 \lesssim E_N^{\Phi^{\varepsilon}} - E^{\phi} + \varepsilon^{1/2}\|\phi(t)\|_{C^{1/2}(\mathbb{R})}^2 + \|\phi(t)\|_{H^2(\mathbb{R})}\beta_{\varepsilon}(t) + \frac{\|\phi(t)\|_{H^2(\mathbb{R})}}{\sqrt{N}}, \tag{2.4.137}$$

*for every $t \in \mathbb{R}$, uniformly in $\varepsilon > 0$ and $N \in \mathbb{N}$.*

*Proof.* As before, we drop the subscript $N$, as the number of particles is fixed throughout the proof. We introduce two parameters $\kappa_1 \in (0,1)$ and $\kappa_2 > 0$, the precise values of which we shall specify momentarily. Using the decomposition $\mathbf{1} = p_1p_2 + (\mathbf{1} - p_1p_2)$ and the normalizations $\|\Phi^{\varepsilon}\|_{L_{\underline{x}_N}^2} = 1 = \|\phi\|_{L_x^2}$, together with some algebraic manipulation of the quantities (2.4.135) and (2.4.136), we arrive at the identity

$$(1-\kappa_1)\|\nabla_1(\mathbf{1} - p_1p_2)\Phi^{\varepsilon}\|_{L_{\underline{x}_N}^2}^2 = E^{\Phi^{\varepsilon}} - E^{\phi} + \sum_{i=1}^{6} \mathrm{Term}_i, \tag{2.4.138}$$

where

$$\mathrm{Term}_1 := -\|\nabla_1 p_1 p_2 \Phi^\varepsilon\|_{L^2_{\underline{x}_N}}^2 + \|\nabla\phi\|_{L^2_x}^2, \tag{2.4.139}$$

$$\mathrm{Term}_2 := -\kappa_2 \left\langle \Phi^\varepsilon | p_1 p_2 \Phi^\varepsilon \right\rangle_{L^2_{\underline{x}_N}} + \kappa_2, \tag{2.4.140}$$

$$\mathrm{Term}_3 := -\frac{\kappa(N-1)}{2N} \left\langle \Phi^\varepsilon | p_1 p_2 V_{\varepsilon,12} p_1 p_2 \Phi^\varepsilon \right\rangle_{L^2_{\underline{x}_N}} + \frac{\kappa}{2}\|\phi\|_{L^4_x}^4, \tag{2.4.141}$$

$$\mathrm{Term}_4 := -2\,\mathrm{Re}\Big\{ \left\langle \nabla_1(\mathbf{1}-p_1p_2)\Phi^\varepsilon | \nabla_1 p_1 p_2 \Phi^\varepsilon \right\rangle_{L^2_{\underline{x}_N}} \Big\}, \tag{2.4.142}$$

$$\mathrm{Term}_5 := -\frac{\kappa(N-1)}{N}\,\mathrm{Re}\Big\{ \left\langle \Phi^\varepsilon | (\mathbf{1}-p_1p_2) V_{\varepsilon,12} p_1 p_2 \Phi^\varepsilon \right\rangle_{L^2_{\underline{x}_N}} \Big\}, \tag{2.4.143}$$

$$\mathrm{Term}_6 := -\frac{\kappa(N-1)}{2N}\|V_{\varepsilon,12}^{1/2}(\mathbf{1}-p_1p_2)\Phi^\varepsilon\|_{L^2_{\underline{x}_N}}^2 - \kappa_1\|\nabla_1(\mathbf{1}-p_1p_2)\Phi^\varepsilon\|_{L^2_{\underline{x}_N}}^2 - \kappa_2\|(\mathbf{1}-p_1p_2)\Phi^\varepsilon\|_{L^2_{\underline{x}_N}}^2. \tag{2.4.144}$$

We keep the term $E^{\Phi^\varepsilon} - E^\phi$. We want to obtain upper bounds for the moduli of $\mathrm{Term}_1, \ldots, \mathrm{Term}_5$, and we want to show that $\mathrm{Term}_6 \leq 0$ provided that we appropriately choose $\kappa_1, \kappa_2$ depending on $\kappa$.

**Estimate for** $\mathrm{Term}_1$ Since $\nabla_1 p_1 = (\,|\nabla\phi\rangle\ \langle\phi|)_1$, it follows from $1 = \|\Phi^\varepsilon\|_{L^2_{\underline{x}_N}}$ that

$$\mathrm{Term}_1 = \|\nabla\phi\|_{L^2_x}^2\Big(1 - \left\langle \Phi^\varepsilon | p_1 p_2 \Phi^\varepsilon \right\rangle_{L^2_{\underline{x}_N}}\Big) = \left\langle \Phi^\varepsilon | (\mathbf{1}-p_1p_2)\Phi^\varepsilon \right\rangle_{L^2_{\underline{x}_N}}. \tag{2.4.145}$$

Since $\mathbf{1} - p_1p_2 = q_1p_2 + q_2p_1 + q_1q_2$, it follows from Remark 2.4.5 and the triangle inequality that

$$\left\langle \Phi^\varepsilon | (\mathbf{1}-p_1p_2)\Phi^\varepsilon \right\rangle_{L^2_{\underline{x}_N}} \leq 3\alpha_\varepsilon \lesssim \beta_\varepsilon, \tag{2.4.146}$$

leading us to conclude that

$$\mathrm{Term}_1 \lesssim \|\nabla\phi\|_{L^2_x}^2 \beta_\varepsilon. \tag{2.4.147}$$

**Estimate for** $\mathrm{Term}_2$ Using the identity $\kappa_2\|\Phi^\varepsilon\|_{L^2_{\underline{x}_N}}^2 = \kappa_2$ and the estimate (2.4.146), we find that

$$\mathrm{Term}_2 = \kappa_2 \left\langle \Phi^\varepsilon | (\mathbf{1}-p_1p_2)\Phi^\varepsilon \right\rangle_{L^2_{\underline{x}_N}} \lesssim \kappa_2 \beta_\varepsilon. \tag{2.4.148}$$

**Estimate for** $\mathrm{Term}_3$ First, observe that

$$p_1p_2V_{12}p_1p_2 = \|\phi\|_{L_x^4}^4 p_1p_2 \quad \text{and} \quad p_1p_2V_{\varepsilon,12}p_1p_2 = \||\phi|^2(V_\varepsilon * |\phi|^2)\|_{L_x^1} p_1p_2. \tag{2.4.149}$$

So by the triangle inequality,

$$|\mathrm{Term}_3| \leq \frac{1}{2}\left|\langle \Phi^\varepsilon | p_1p_2(V_{\varepsilon,12} - V_{12})p_1p_2\Phi^\varepsilon\rangle_{L^2_{\underline{x}_N}}\right| + \frac{\|\phi\|_{L_x^4}^4}{2}\left| -\frac{(N-1)}{N}\langle \Phi^\varepsilon | p_1p_2\Phi^\varepsilon\rangle_{L^2_{\underline{x}_N}} + 1\right|. \tag{2.4.150}$$

Since $\|\Phi^\varepsilon\|_{L^2_{\underline{x}_N}}^2 = 1$, the second term in the right-hand side equals

$$\frac{\|\phi\|_{L_x^4}^4}{2}\left|\frac{1}{N}\langle \Phi^\varepsilon | p_1p_2\Phi^\varepsilon\rangle_{L^2_{\underline{x}_N}} + \langle \Phi^\varepsilon | (\mathbf{1} - p_1p_2)\Phi^\varepsilon\rangle_{L^2_{\underline{x}_N}}\right| \lesssim \|\phi\|_{L_x^4}^4\left(\frac{1}{N} + \beta_\varepsilon\right), \tag{2.4.151}$$

where the ultimate inequality follows from the triangle inequality, $\langle \Phi^\varepsilon | p_1p_2\Phi^\varepsilon\rangle \leq \|\Phi^\varepsilon\|_{L^2_{\underline{x}_N}}^2 = 1$, and the estimate (2.4.146). Again using that $\|\Phi^\varepsilon\|_{L^2_{\underline{x}_N}} = 1$, we see that the first term in the right-hand side of (2.4.150) is bounded by

$$\frac{1}{2}\||\phi|^2\big((V_\varepsilon * |\phi|^2) - |\phi|^2\big)\|_{L_x^1} \lesssim \|\phi\|_{C_x^{1/2}}^2 \varepsilon^{1/2}, \tag{2.4.152}$$

which follows from the estimate (2.4.40) and $\|\phi\|_{L_x^2} = 1$. Therefore,

$$\mathrm{Term}_3 \lesssim \varepsilon^{1/2}\|\phi\|_{C_x^{1/2}}^2 + \|\phi\|_{L_x^4}^4\left(\frac{1}{N} + \beta_\varepsilon\right). \tag{2.4.153}$$

**Estimate for** $\mathrm{Term}_4$ By using the decomposition $\mathbf{1} - p_1p_2 = q_1p_2 + q_2p_1 + q_1q_2$, the triangle inequality, and the fact that $[q_2, \nabla_1] = 0 = q_2p_2$, we see that

$$\begin{aligned} |\mathrm{Term}_4| &\lesssim \left| \langle \nabla_1 q_1p_2\Phi^\varepsilon | \nabla_1 p_1p_2\Phi^\varepsilon\rangle_{L^2_{\underline{x}_N}} + \underbrace{\langle \nabla_1 q_2p_1\Phi^\varepsilon | \nabla_1 p_1p_2\Phi^\varepsilon\rangle_{L^2_{\underline{x}_N}}}_{=0} + \underbrace{\langle \nabla_1 q_1q_2\Phi^\varepsilon | \nabla_1 p_1p_2\Phi^\varepsilon\rangle_{L^2_{\underline{x}_N}}}_{=0} \right| \\ &= \left|\langle q_1\Phi^\varepsilon | (-\Delta_1)p_1p_2\Phi^\varepsilon\rangle_{L^2_{\underline{x}_N}}\right| \\ &= \left|\left\langle \widehat{n}^{-1/2}q_1\Phi^\varepsilon \middle| \widehat{n}^{1/2}(-\Delta_1)p_1p_2\Phi^\varepsilon\right\rangle_{L^2_{\underline{x}_N}}\right|, \end{aligned} \tag{2.4.154}$$

where the penultimate equality follows from integration by parts and the ultimate equality from writing $\mathbf{1} = \widehat{n}^{-1/2}\widehat{n}^{1/2}$. The reader will recall the definitions of $n$ and $\widehat{n}$ from Definition 2.4.4. By Cauchy-Schwarz and $q_1^2 = q_1$,

$$\begin{aligned}\left|\left\langle \widehat{n}^{-1/2} q_1 \Phi^\varepsilon \middle| \widehat{n}^{1/2}(-\Delta_1)p_1p_2\Phi^\varepsilon\right\rangle_{L^2_{\underline{x}_N}}\right| &\leq \|\widehat{n}^{-1/2}q_1\Phi^\varepsilon\|_{L^2_{\underline{x}_N}} \|q_1\widehat{n}^{1/2}(-\Delta_1)p_1p_2\Phi^\varepsilon\|_{L^2_{\underline{x}_N}} \\ &\leq \sqrt{\beta_\varepsilon}\|q_1\widehat{n}^{1/2}(-\Delta_1)p_1p_2\Phi^\varepsilon\|_{L^2_{\underline{x}_N}},\end{aligned} \tag{2.4.155}$$

where the ultimate line follows from applying Lemma 2.4.6(i) to the first factor in the right-hand side of the first line. By Lemma 2.4.7, we have the operator identity

$$q_1\widehat{n}^{1/2}(-\Delta_1)p_1 = q_1(-\Delta_1)\widehat{(\tau_1 n)}^{1/2}p_1 = q_1(-\Delta_1)p_1\widehat{(\tau_1 n)}^{1/2}. \tag{2.4.156}$$

So writing $q_1 = \mathbf{1} - p_1$ and using the triangle inequality together with the operator norm estimates

$$\|(-\Delta_1)p_1\|_{L^2_{\underline{x}_N}\to L^2_{\underline{x}_N}} \leq \|\Delta\phi\|_{L^2_x} \quad \text{and} \quad \|p_1(-\Delta_1)p_1\|_{L^2_{\underline{x}_N}\to L^2_{\underline{x}_N}} \leq \|\nabla\phi\|_{L^2_x}^2,^4 \tag{2.4.157}$$

we find that

$$\begin{aligned}\|q_1\widehat{n}^{1/2}(-\Delta_1)p_1p_2\Phi^\varepsilon\|_{L^2_{\underline{x}_N}} &\leq \|(-\Delta_1)p_1\widehat{(\tau_1 n)}^{1/2}p_2\Phi^\varepsilon\|_{L^2_{\underline{x}_N}} + \|p_1(-\Delta_1)p_1\widehat{(\tau_1 n)}^{1/2}p_2\Phi^\varepsilon\|_{L^2_{\underline{x}_N}} \\ &\leq \left(\|\Delta\phi\|_{L^2_x} + \|\nabla\phi\|_{L^2_x}^2\right)\|\widehat{(\tau_1 n)}^{1/2}\Phi^\varepsilon\|_{L^2_{\underline{x}_N}},\end{aligned} \tag{2.4.158}$$

where we eliminate $p_2$ using $\|p_2\|_{L^2_{\underline{x}_N}\to L^2_{\underline{x}_N}} = 1$. Using the embedding $\ell^{1/2} \subset \ell^1$, we see that

$$(\tau_1 n)(k) = \sqrt{\frac{k+1}{N}}1_{\geq 0}(k+1) \leq \sqrt{\frac{k}{N}}1_{\geq 0}(k) + \frac{1}{\sqrt{N}} = n(k) + \frac{1}{\sqrt{N}}, \qquad \forall k \in \mathbb{Z}. \tag{2.4.159}$$

---

[4] This is the only place in this work where the $H^2$ regularity assumption is strictly needed.

By another application of $\ell^{1/2} \subset \ell^1$ together with $\|\Phi^\varepsilon\|_{L^2_{\underline{x}_N}} = 1$,

$$\|\widehat{(\tau_1 n)}^{1/2}\Phi^\varepsilon\|_{L^2_{\underline{x}_N}} \leq \sqrt{\beta_\varepsilon} + N^{-1/4}. \tag{2.4.160}$$

Using Young's inequality for products and interpolation of $H^s$ spaces with $\|\phi\|_{L^2_x} = 1$, we obtain that

$$|\mathrm{Term}_4| \lesssim \Big(\|\Delta\phi\|_{L^2_x} + \|\nabla\phi\|^2_{L^2_x}\Big)\sqrt{\beta_\varepsilon}\Big(\sqrt{\beta_\varepsilon} + N^{-1/4}\Big) \lesssim \|\phi\|_{H^2_x}\big(\beta_\varepsilon + N^{-1/2}\big). \tag{2.4.161}$$

**Estimate for** $\mathrm{Term}_5$ Using the decomposition $\mathbf{1} - p_1p_2 = p_1q_2 + p_2q_1 + q_1q_2$ together with the triangle inequality and the symmetry of $\Phi^\varepsilon$ under exchange of particle labels, we have that

$$\begin{aligned}|\mathrm{Term}_5| &\lesssim \Big|\langle\Phi^\varepsilon|p_1p_2V_{\varepsilon,12}q_1p_2\Phi^\varepsilon\rangle_{L^2_{\underline{x}_N}} + \langle\Phi^\varepsilon|p_1p_2V_{\varepsilon,12}q_2p_1\Phi^\varepsilon\rangle_{L^2_{\underline{x}_N}} + \langle\Phi^\varepsilon|p_1p_2V_{\varepsilon,12}q_1q_2\Phi^\varepsilon\rangle_{L^2_{\underline{x}_N}}\Big| \\ &\lesssim \underbrace{\Big|\langle\Phi^\varepsilon|p_1p_2V_{\varepsilon,12}q_1p_2\Phi^\varepsilon\rangle_{L^2_{\underline{x}_N}}\Big|}_{=:\mathrm{Term}_{5,1}} + \underbrace{\Big|\langle\Phi^\varepsilon|p_1p_2V_{\varepsilon,12}q_1q_2\Phi^\varepsilon\rangle_{L^2_{\underline{x}_N}}\Big|}_{=:\mathrm{Term}_{5,2}},\end{aligned} \tag{2.4.162}$$

For $\mathrm{Term}_{5,1}$, we note from an examination of its integral kernel that

$$p_1p_2V_{\varepsilon,12}q_1p_2 = p_1p_2V^{\phi}_{\varepsilon,1}q_1, \tag{2.4.163}$$

where we use the notation $V^{\phi}_{\varepsilon,1}$ introduced in (2.4.32). Now writing $\mathbf{1} = \widehat{n}^{-1/2}\widehat{n}^{1/2}$, we find that

$$\begin{aligned}\mathrm{Term}_{5,1} &= \left|\left\langle\Phi^\varepsilon\middle|p_1p_2V^{\phi}_{\varepsilon,1}\widehat{n}^{1/2}\widehat{n}^{-1/2}q_1\Phi^\varepsilon\right\rangle_{L^2_{\underline{x}_N}}\right| \\ &= \left|\left\langle\Phi^\varepsilon\middle|p_1p_2\widehat{(\tau_1 n)}^{1/2}V^{\phi}_{\varepsilon,1}\widehat{n}^{-1/2}q_1\Phi^\varepsilon\right\rangle_{L^2_{\underline{x}_N}}\right| \\ &\leq \|p_1p_2\widehat{(\tau_1 n)}^{1/2}\Phi^\varepsilon\|_{L^2_{\underline{x}_N}}\|V^{\phi}_{\varepsilon,1}\widehat{n}^{-1/2}q_1\Phi^\varepsilon\|_{L^2_{\underline{x}_N}},\end{aligned} \tag{2.4.164}$$

where the penultimate line follows from an application of Lemma 2.4.7 and the ultimate line follows from Cauchy-Schwarz. Applying the operator norm identity $\|p_j\|_{L^2\to L^2} = 1$ together with the estimate (2.4.160) to the first factor in (2.4.164), we obtain that

$$\mathrm{Term}_{5,1} \lesssim \left(\sqrt{\beta_\varepsilon} + N^{-1/4}\right)\|V_{\varepsilon,1}^{\phi}\widehat{n}^{-1/2}q_1\Phi^\varepsilon\|_{L^2_{\underline{x}_N}}. \tag{2.4.165}$$

Now since $\|V_{\varepsilon,1}^{\phi}\|_{L^2_{\underline{x}_N}\to L^2_{\underline{x}_N}} \leq \|\phi\|_{L^\infty_x}^2$, we find that

$$\|V_{\varepsilon,1}^{\phi}\widehat{n}^{-1/2}q_1\Phi^\varepsilon\|_{L^2_{\underline{x}_N}} \leq \|\phi\|_{L^\infty_x}^2\|\widehat{n}^{-1/2}q_1\Phi^\varepsilon\|_{L^2_{\underline{x}_N}} \leq \|\phi\|_{L^\infty_x}^2\sqrt{\beta_\varepsilon}. \tag{2.4.166}$$

where the ultimate equality follows from Lemma 2.4.6(i) and the trivial fact that $\widehat{n}^2 = \widehat{m}$. Using the embedding $\ell^{1/2} \subset \ell^1$, we conclude that

$$\mathrm{Term}_{5,1} \lesssim \|\phi\|_{L^\infty_x}^2\sqrt{\beta_\varepsilon}\left(\sqrt{\beta_\varepsilon} + N^{-1/4}\right) \lesssim \|\phi\|_{L^\infty_x}^2\left(\beta_\varepsilon + N^{-1/2}\right). \tag{2.4.167}$$

For $\mathrm{Term}_{5,2}$, we use, as in the proof of Proposition 2.4.9, the distributional identity (2.2.10) to write $V_{\varepsilon,12} = (\nabla_1 X_{\varepsilon,12})$, where $X_{\varepsilon,12} := \frac{1}{2}(V_\varepsilon * \mathrm{sgn})(X_1 - X_2)$. Thus,

$$\begin{aligned}
\mathrm{Term}_{5,2} &= \left|\langle\Phi^\varepsilon|p_1p_2(\nabla_1 X_{\varepsilon,12})q_1q_2\Phi^\varepsilon\rangle_{L^2_{\underline{x}_N}}\right| \\
&= \left|\left\langle\Phi^\varepsilon\middle|p_1p_2(\nabla_1 X_{\varepsilon,12})\widehat{n}\widehat{n}^{-1}q_1q_2\Phi^\varepsilon\right\rangle_{L^2_{\underline{x}_N}}\right| \\
&= \left|\left\langle\Phi^\varepsilon\middle|p_1p_2\widehat{(\tau_2 n)}(\nabla_1 X_{\varepsilon,12})\widehat{n}^{-1}q_1q_2\Phi^\varepsilon\right\rangle_{L^2_{\underline{x}_N}}\right| \\
&= \left|\left\langle\widehat{(\tau_2 n)}p_1p_2\Phi^\varepsilon\middle|(\nabla_1 X_{\varepsilon,12})\widehat{n}^{-1}q_1q_2\Phi^\varepsilon\right\rangle_{L^2_{\underline{x}_N}}\right|,
\end{aligned} \tag{2.4.168}$$

where the penultimate line follows from an application of Lemma 2.4.7. Now integrating by parts and then applying the product rule and triangle inequality, we obtain that

$$\begin{aligned}
\left|\left\langle\widehat{(\tau_2 n)}p_1p_2\Phi^\varepsilon\middle|(\nabla_1 X_{\varepsilon,12})\widehat{n}^{-1}q_1q_2\Phi^\varepsilon\right\rangle_{L^2_{\underline{x}_N}}\right| &\leq \left|\left\langle\nabla_1\widehat{(\tau_2 n)}p_1p_2\Phi^\varepsilon\middle|X_{\varepsilon,12}\widehat{n}^{-1}q_1q_2\Phi^\varepsilon\right\rangle_{L^2_{\underline{x}_N}}\right| \\
&\quad + \left|\left\langle\widehat{(\tau_2 n)}p_1p_2\Phi^\varepsilon\middle|X_{\varepsilon,12}\nabla_1\widehat{n}^{-1}q_1q_2\Phi^\varepsilon\right\rangle_{L^2_{\underline{x}_N}}\right| \\
&=: \mathrm{Term}_{5,2,1} + \mathrm{Term}_{5,2,2}.
\end{aligned} \tag{2.4.169}$$

We first dispense with the easy case $\mathrm{Term}_{5,2,1}$. By Cauchy-Schwarz and using the operator norm estimates

$$\|\nabla_1 p_1\|_{L^2_{\underline{x}_N} \to L^2_{\underline{x}_N}} \leq \|\nabla \phi\|_{L^2_x} \quad \text{and} \quad \|X_{\varepsilon,12}\|_{L^2_{\underline{x}_N} \to L^2_{\underline{x}_N}} \leq \frac{1}{2}, \tag{2.4.170}$$

we obtain that

$$\mathrm{Term}_{5,2,1} \leq \|\nabla \phi\|_{L^2_x} \|\widehat{(\tau_2 n)} \Phi^\varepsilon\|_{L^2_{\underline{x}_N}} \|\widehat{n}^{-1} q_1 q_2 \Phi^\varepsilon\|_{L^2_{\underline{x}_N}}. \tag{2.4.171}$$

By arguing similarly as for the estimates (2.4.159) and (2.4.160), we find that

$$\|\widehat{(\tau_2 n)} \Phi^\varepsilon\|_{L^2_{\underline{x}_N}} \lesssim \sqrt{\beta_\varepsilon} + \frac{1}{\sqrt{N}}, \tag{2.4.172}$$

and by applying Lemma 2.4.6(ii), we have that

$$\|\widehat{n}^{-1} q_1 q_2 \Phi^\varepsilon\|_{L^2_{\underline{x}_N}} \lesssim \sqrt{\beta_\varepsilon}. \tag{2.4.173}$$

Thus, we conclude that

$$\mathrm{Term}_{5,2,1} \lesssim \|\nabla \phi\|_{L^2_x} \left(\beta_\varepsilon + \frac{1}{N}\right). \tag{2.4.174}$$

For the hard case $\mathrm{Term}_{5,2,2}$, we first use Cauchy-Schwarz and (2.4.170) to obtain

$$\begin{aligned} \mathrm{Term}_{5,2,2} &\leq \|\widehat{(\tau_2 n)} p_1 p_2 \Phi^\varepsilon\|_{L^2_{\underline{x}_N}} \|\nabla_1 \widehat{n}^{-1} q_1 q_2 \Phi^\varepsilon\|_{L^2_{\underline{x}_N}} \\ &\lesssim \left(\sqrt{\beta_\varepsilon} + N^{-1/2}\right) \|\nabla_1 \widehat{n}^{-1} q_1 q_2 \Phi^\varepsilon\|_{L^2_{\underline{x}_N}}, \end{aligned} \tag{2.4.175}$$

where the second line follows from applying the estimate (2.4.172) to the first factor in the right-hand side of the first line. For the remaining factor $\|\nabla_1 \widehat{n}^{-1} q_1 q_2 \Phi^\varepsilon\|_{L^2_{\underline{x}_N}}$, we write $\mathbf{1} = p_1 + q_1$ and use the triangle inequality to obtain

$$\|\nabla_1 \widehat{n}^{-1} q_1 q_2 \Phi^\varepsilon\|_{L^2_{\underline{x}_N}} \leq \|p_1 \nabla_1 \widehat{n}^{-1} q_1 q_2 \Phi^\varepsilon\|_{L^2_{\underline{x}_N}} + \|q_1 \nabla_1 \widehat{n}^{-1} q_1 q_2 \Phi^\varepsilon\|_{L^2_{\underline{x}_N}}. \tag{2.4.176}$$

Since $\|p_1\nabla_1\|_{L^2_{\underline{x}_N}\to L^2_{\underline{x}_N}} \leq \|\nabla\phi\|_{L^2_x}$, it follows that

$$\|p_1\nabla_1\widehat{n}^{-1}q_1q_2\Phi^\varepsilon\|_{L^2_{\underline{x}_N}} \leq \|\nabla\phi\|_{L^2_x}\|\widehat{n}^{-1}q_1q_2\Phi^\varepsilon\|_{L^2_{\underline{x}_N}} \lesssim \|\nabla\phi\|_{L^2_x}\sqrt{\beta_\varepsilon}, \tag{2.4.177}$$

where the ultimate inequality follows from applying Lemma 2.4.6(ii) and $\widehat{n}^2 = \widehat{m}$. Next, observe that by Lemma 2.4.7, $q_1\nabla_1\widehat{n}^{-1}q_1 = q_1\widehat{n}^{-1}\nabla_1 q_1$, which implies that

$$\|q_1\nabla_1\widehat{n}^{-1}q_1q_2\Phi^\varepsilon\|_{L^2_{\underline{x}_N}} \leq \|\widehat{n}^{-1}\nabla_1q_1q_2\Phi^\varepsilon\|_{L^2_{\underline{x}_N}} = \sqrt{\langle\nabla_1q_1\Phi^\varepsilon|q_2\widehat{n}^{-2}\nabla_1q_1\Phi^\varepsilon\rangle_{L^2_{\underline{x}_N}}}, \tag{2.4.178}$$

where the ultimate equality follows from the fact that $q_2$ commutes with $\widehat{n}^{-2}\nabla_1q_1$ and $q_2^2 = q_2$. By the symmetry of $\Phi^\varepsilon$ with respect to permutation of particle labels and the operator identity

$$\frac{1}{N-1}\sum_{i=2}^{N} q_i\widehat{n}^{-2} \leq \left(\frac{N}{N-1}\right)\widehat{m}\widehat{n}^{-2} \lesssim \mathbf{1}, \tag{2.4.179}$$

which follows from Remark 2.4.5, we see that

$$\left\langle\nabla_1q_1\Phi^\varepsilon\middle|q_2\widehat{n}^{-2}\nabla_1q_1\Phi^\varepsilon\right\rangle_{L^2_{\underline{x}_N}} = \frac{1}{N-1}\sum_{i=2}^{N}\left\langle\nabla_1q_1\Phi^\varepsilon\middle|q_i\widehat{n}^{-2}\nabla_1q_1\Phi^\varepsilon\right\rangle_{L^2_{\underline{x}_N}} \lesssim \|\nabla_1q_1\Phi^\varepsilon\|^2_{L^2_{\underline{x}_N}}. \tag{2.4.180}$$

Hence,

$$\|q_1\nabla_1\widehat{n}^{-1}q_1q_2\Phi^\varepsilon\|_{L^2_{\underline{x}_N}} \lesssim \|\nabla_1q_1\Phi^\varepsilon\|_{L^2_{\underline{x}_N}}. \tag{2.4.181}$$

We therefore conclude from another application of Young's inequality that

$$\mathrm{Term}_{5,2,2} \lesssim \|\nabla\phi\|_{L^2_x}\left(\beta_\varepsilon + N^{-1}\right) + \left(\sqrt{\beta_\varepsilon} + N^{-1/2}\right)\|\nabla_1q_1\Phi^\varepsilon\|_{L^2_{\underline{x}_N}}. \tag{2.4.182}$$

Collecting the estimate (2.4.174) for $\mathrm{Term}_{5,2,1}$ and the estimate (2.4.182) for $\mathrm{Term}_{5,2,2}$, we find that

$$\mathrm{Term}_{5,2} \lesssim \|\nabla\phi\|_{L^2_x}\left(\beta_\varepsilon + N^{-1}\right) + \left(\sqrt{\beta_\varepsilon} + N^{-1/2}\right)\|\nabla_1q_1\Phi^\varepsilon\|_{L^2_{\underline{x}_N}}. \tag{2.4.183}$$

Together with the estimate (2.4.167) for $\mathrm{Term}_{5,1}$, we conclude that

$$|\mathrm{Term}_5| \lesssim \|\phi\|_{L_x^\infty}^2 \left(\beta_\varepsilon + N^{-1/2}\right) + \|\nabla\phi\|_{L_x^2}\left(\beta_\varepsilon + N^{-1}\right) + \left(\sqrt{\beta_\varepsilon} + N^{-1/2}\right)\|\nabla_1 q_1 \Phi^\varepsilon\|_{L^2_{\underline{x}_N}}. \tag{2.4.184}$$

**Estimate for** $\mathrm{Term}_6$ We want to show that $\mathrm{Term}_6 \leq 0$. We assume here that $\kappa = -1$; otherwise, it is trivial that $\mathrm{Term}_6 \leq 0$ and we can take $\kappa_2 = 0$. By the same argument used to prove Lemma 2.2.2,

$$\|V_{\varepsilon,12}^{1/2}(\mathbf{1} - p_1p_2)\Phi^\varepsilon\|_{L^2_{\underline{x}_N}}^2 \leq \|\nabla_1(\mathbf{1} - p_1p_2)\Phi^\varepsilon\|_{L^2_{\underline{x}_N}}\|(\mathbf{1} - p_1p_2)\Phi^\varepsilon\|_{L^2_{\underline{x}_N}}, \tag{2.4.185}$$

and by Young's inequality for products,

$$\frac{(N-1)}{2N}\|\nabla_1(\mathbf{1}-p_1p_2)\Phi^\varepsilon\|_{L^2_{\underline{x}_N}}\|(\mathbf{1}-p_1p_2)\Phi^\varepsilon\|_{L^2_{\underline{x}_N}} \leq \kappa_1\|\nabla_1(\mathbf{1}-p_1p_2)\Phi^\varepsilon\|_{L^2_{\underline{x}_N}}^2 + \frac{(N-1)^2}{4N^2\kappa_1}\|(\mathbf{1}-p_1p_2)\Phi^\varepsilon\|_{L^2_{\underline{x}_N}}^2. \tag{2.4.186}$$

We choose $\kappa_2 > 1/(2\kappa_1)$. Then,

$$\begin{aligned}
\mathrm{Term}_6 &= \frac{(N-1)}{2N}\|V_{\varepsilon,12}^{1/2}(\mathbf{1} - p_1p_2)\Phi^\varepsilon\|_{L^2_{\underline{x}_N}}^2 - \kappa_1\|\nabla_1(\mathbf{1} - p_1p_2)\Phi^\varepsilon\|_{L^2_{\underline{x}_N}}^2 - \kappa_2\|(\mathbf{1} - p_1p_2)\Phi^\varepsilon\|_{L^2_{\underline{x}_N}}^2 \\
&\leq \left(\frac{(N-1)^2}{4N^2\kappa_1} - \kappa_2\right)\|(\mathbf{1} - p_1p_2)\Phi^\varepsilon\|_{L^2_{\underline{x}_N}}^2 \\
&\leq 0,
\end{aligned} \tag{2.4.187}$$

as desired.

Having estimated the terms $\mathrm{Term}_1, \ldots, \mathrm{Term}_6$, we can now complete the proof of the proposition. Combining estimate (2.4.147) for $\mathrm{Term}_1$, (2.4.148) for $\mathrm{Term}_2$, (2.4.153) for $\mathrm{Term}_3$, (2.4.161) for $\mathrm{Term}_4$, and (2.4.184) for $\mathrm{Term}_5$, we see that there exists an absolute constant $C>0$ such that

$$(1-\kappa_1)\|\nabla_1(\mathbf{1}-p_1p_2)\Phi^\varepsilon\|_{L^2_{\underline{x}_N}}^2 \leq \left(E^{\Phi^\varepsilon}-E^\phi\right)+C\Big(\varepsilon^{1/2}\|\phi\|_{C_x^{1/2}}^2+\Big(\sqrt{\beta_\varepsilon}+N^{-1/2}\Big)\|\nabla_1 q_1\Phi^\varepsilon\|_{L^2_{\underline{x}_N}}\Big)$$
$$+C\Big(\big(\|\phi\|_{L_x^\infty}^2+\|\phi\|_{H_x^2}\big)N^{-1/2}+\Big(\|\nabla\phi\|_{L_x^2}+\|\phi\|_{L_x^4}^4\Big)N^{-1}\Big)$$
$$+C\beta_\varepsilon\Big(\|\nabla\phi\|_{L_x^2}^2+\kappa_2 1_{\{-1\}}(\kappa)+\|\phi\|_{H_x^2}+\|\phi\|_{L_x^\infty}^2+\|\nabla\phi\|_{L_x^2}+\|\phi\|_{L_x^4}^4\Big). \tag{2.4.188}$$

Note that by using Sobolev embedding, the interpolation property of $H^s$ norms, and the normalization $\|\phi\|_{L_x^2}=1$, we can simplify the right-hand side of (2.4.188) to

$$(1-\kappa_1)\|\nabla_1(\mathbf{1}-p_1p_2)\Phi^\varepsilon\|_{L^2_{\underline{x}_N}}^2 \leq \left(E^{\Phi^\varepsilon}-E^\phi\right)+C\|\phi\|_{H_x^2}\left(N^{-1/2}+\beta_\varepsilon\right)$$
$$+C\Big(\varepsilon^{1/2}\|\phi\|_{C_x^{1/2}}^2+\Big(\sqrt{\beta_\varepsilon}+N^{-1/2}\Big)\|\nabla_1 q_1\Phi^\varepsilon\|_{L^2_{\underline{x}_N}}\Big), \tag{2.4.189}$$

for some larger absolute constant $C>0$. To close the proof of the lemma, we want to obtain a lower bound for the left-hand side of (2.4.189) in terms $\|\nabla_1 q_1\Phi^\varepsilon\|_{L^2_{\underline{x}_N}}^2$. To this end, we note that

$$\mathbf{1}-p_1p_2=p_1+q_1-p_1p_2=p_1q_2+q_1, \tag{2.4.190}$$

so that by the triangle inequality and the fact that $q_2$ commutes with $\nabla_1$,

$$\|\nabla_1 q_1\Phi^\varepsilon\|_{L^2_{\underline{x}_N}} \leq \|\nabla_1(\mathbf{1}-p_1p_2)\Phi^\varepsilon\|_{L^2_{\underline{x}_N}}+\|\nabla_1 p_1q_2\Phi^\varepsilon\|_{L^2_{\underline{x}_N}}. \tag{2.4.191}$$

Since $\|\nabla_1 p_1\|_{L^2_{\underline{x}_N}\to L^2_{\underline{x}_N}} \leq \|\nabla\phi\|_{L_x^2}$, it follows that

$$\|\nabla_1 p_1 q_2\Phi^\varepsilon\|_{L^2_{\underline{x}_N}} \leq \|\nabla\phi\|_{L_x^2}\|q_2\Phi^\varepsilon\|_{L^2_{\underline{x}_N}} \leq \|\nabla\phi\|_{L_x^2}\sqrt{\beta_\varepsilon}, \tag{2.4.192}$$

where the ultimate inequality follows from Remark 2.4.5 and $\alpha_\varepsilon\leq\beta_\varepsilon$. Therefore,

$$\|\nabla_1(\mathbf{1}-p_1p_2)\Phi^\varepsilon\|_{L^2_{\underline{x}_N}}^2 \geq \Big(\|\nabla_1 q_1\Phi^\varepsilon\|_{L^2_{\underline{x}_N}}-\|\nabla\phi\|_{L_x^2}\sqrt{\beta_\varepsilon}\Big)^2 \geq \frac{3\|\nabla_1 q_1\Phi^\varepsilon\|_{L^2_{\underline{x}_N}}^2}{4}-15\|\nabla\phi\|_{L_x^2}^2\beta_\varepsilon, \tag{2.4.193}$$

where the ultimate inequality follows from application of Young's inequality for products. Inserting the preceding lower bound into the inequality (2.4.189) and rearranging, we find that

$$\begin{aligned}\frac{3}{4}\|\nabla_1 q_1 \Phi\|^2_{L^2_{\underline{x}_N}} \leq \frac{E^{\Phi^\varepsilon} - E^{\phi}}{1-\kappa_1} + \frac{C}{1-\kappa_1}\Big(\varepsilon^{1/2}\|\phi\|^2_{C^{1/2}_x} + \Big(\sqrt{\beta_\varepsilon} + N^{-1/2}\Big)\|\nabla_1 q_1 \Phi^\varepsilon\|_{L^2_{\underline{x}_N}}\Big) \\ + \frac{C\|\phi\|_{H^2_x}}{1-\kappa_1}\big(N^{-1/2} + \beta_\varepsilon\big) + 15\|\nabla\phi\|^2_{L^2_x}\beta_\varepsilon.\end{aligned} \tag{2.4.194}$$

By Young's inequality for products,

$$\frac{C}{1-\kappa_1}\|\nabla_1 q_1 \Phi^\varepsilon\|_{L^2_{\underline{x}_N}}\left(\sqrt{\beta_\varepsilon} + N^{-1/2}\right) \leq \frac{4C^2}{(1-\kappa_1)^2}\left(\beta_\varepsilon + \frac{1}{N}\right) + \frac{1}{4}\|\nabla_1 q_1 \Phi^\varepsilon\|^2_{L^2_{\underline{x}_N}}, \tag{2.4.195}$$

The desired conclusion now follows after some algebra. □

### 2.4.4 Proof of Proposition 2.1.2

We now use the results of the previous subsections to pass to the limit $\varepsilon \to 0^+$ and obtain an inequality for $\beta_N$, thereby proving Proposition 2.1.2.

*Proof of Proposition 2.1.2.* Applying Proposition 2.4.10 to factors $\|\nabla_1 q_1 \Phi_N^\varepsilon\|_{L^2_{\underline{x}_N}}$ appearing in the right-hand side of the inequality given by Proposition 2.4.9 and using the majorization $\|\phi\|^2_{H^1_x} \leq \|\phi\|_{H^2_x}$ together with a bit of algebra, we obtain the point-wise estimate

$$\begin{aligned}\dot{\beta}_{N,\varepsilon} \lesssim \frac{\|\phi\|^2_{L^\infty_x}}{N} + \varepsilon^{1/2}\|\phi\|^2_{C^{1/2}_x} + \frac{(1+\|\phi\|^2_{C^{1/2}_x})\|\phi\|_{H^2_x}}{\sqrt{N}} + \frac{1}{N^\sigma} + \frac{\|\phi\|^4_{L^4_x}}{N^{(1-\sigma)/2}} + \frac{\|\phi\|^2_{L^\infty_x}}{N^{\delta/2}} + N^{\frac{2(\sigma-1)+\delta}{2}} \\ + \Big(1 + \|\phi\|^2_{C^{1/2}_x}\Big)\|\phi\|_{H^2_x}\beta_{N,\varepsilon} + \Big(1+\|\phi\|^2_{C^{1/2}_x}\Big)\Big(E_N^{\Phi^\varepsilon} - E^{\phi} + \varepsilon^{1/2}\|\phi\|^2_{C^{1/2}_x}\Big).\end{aligned} \tag{2.4.196}$$

We now optimize the choice of $\delta, \sigma \in (0,1)$. We choose $\delta, \sigma \in (0,1)$ such that

$$1 - \sigma = \delta \quad \text{and} \quad \sigma = \frac{1-\sigma}{2}, \tag{2.4.197}$$

which, after some algebra, implies that $(\delta,\sigma)=(2/3,1/3)$. Inserting this choice of $(\delta,\sigma)$ into the right-hand side of inequality (2.4.196) and using Sobolev embedding together with the interpolation property of the $H^s$ norm, we obtain

$$\dot{\beta}_{N,\varepsilon} \lesssim \frac{\|\phi\|_{H_x^2}^2}{\sqrt{N}} + \frac{\|\phi\|_{H_x^1}^2}{N^{1/3}} + \|\phi\|_{H_x^2}^2\beta_{N,\varepsilon} + \left(1+\|\phi\|_{C_x^{1/2}}^2\right)\left(E_N^{\Phi^\varepsilon} - E^\phi + \varepsilon^{1/2}\|\phi\|_{C_x^{1/2}}^2\right). \tag{2.4.198}$$

Integrating both sides of the preceding inequality over the interval $[0,t]$ and applying the fundamental theorem of calculus, we obtain that

$$\begin{aligned}\beta_{N,\varepsilon}(t) &\leq \beta_{N,\varepsilon}(0) + C\int_0^t ds\|\phi(s)\|_{H^2}^2\beta_{N,\varepsilon}(s)\\ &\quad + C\int_0^t ds\left(\frac{\|\phi(s)\|_{H^2}^2}{\sqrt{N}} + \frac{\|\phi(s)\|_{H^1}^2}{N^{1/3}} + \left(1+\|\phi(s)\|_{C^{1/2}}^2\right)\left(E_N^{\Phi^\varepsilon} - E^\phi + \varepsilon^{1/2}\|\phi(s)\|_{C^{1/2}}^2\right)\right),\end{aligned} \tag{2.4.199}$$

where $C>0$ is an absolute constant. So applying the Gronwall-Bellman inequality, specifically [73, Theorem 1.3.1], we find that

$$\beta_{N,\varepsilon}(t) \leq \mathfrak{A}_{N,\varepsilon}(t)\exp\left(C\int_0^t ds\|\phi(s)\|_{H^2}^2\right), \qquad \forall t\geq 0, \tag{2.4.200}$$

where $\mathfrak{A}_{N,\varepsilon}:[0,\infty)\rightarrow[0,\infty)$ is the function defined by

$$\mathfrak{A}_{N,\varepsilon}(t) \coloneqq \beta_{N,\varepsilon}(0) + C\int_0^t ds\left(\frac{\|\phi(s)\|_{H^2}^2}{\sqrt{N}} + \frac{\|\phi(s)\|_{H_x^1}^2}{N^{1/3}} + \left(1+\|\phi(s)\|_{C^{1/2}}^2\right)\left(E_N^{\Phi^\varepsilon} - E^\phi + \varepsilon^{1/2}\|\phi(s)\|_{C^{1/2}}^2\right)\right), \tag{2.4.201}$$

for every $t\geq 0$.

We now send $\varepsilon\rightarrow 0^+$. By Lemma 2.4.8, we have that $\beta_{\varepsilon,N}(t)\rightarrow\beta_N(t)$, as $\varepsilon\rightarrow 0^+$, uniformly on compact intervals of time. Recalling the definition of the energy per particle $E_N^{\Phi^\varepsilon}$ and the cubic NLS energy $E^\phi$ from (2.4.135) and (2.4.136), respectively, we see that

$$E_N^{\Phi^\varepsilon} - E^\phi = \|\nabla_1\Phi_{N,0}\|_{L^2(\mathbb{R}^N)}^2 + \frac{\kappa(N-1)}{2N}\left\langle\Phi_{N,0}|V_{\varepsilon,12}\Phi_{N,0}\right\rangle_{L^2(\mathbb{R}^N)} - \|\nabla\phi_0\|_{L^2(\mathbb{R})}^2 - \|\phi_0\|_{L^4(\mathbb{R})}^4. \tag{2.4.202}$$

It follows from the proof of Lemma 2.3.7 that $V_{\varepsilon,12}\Phi_{N,0} \to V_{12}\Phi_{N,0}$ in $H^{-1}(\mathbb{R}^N)$ as $\varepsilon \to 0^+$. Therefore,

$$\lim_{\varepsilon\to 0^+} E_N^{\Phi^\varepsilon} - E^\phi = E_N^\Phi - E^\phi, \tag{2.4.203}$$

where $E_N^\Phi$ is the energy per particle of the solution $\Phi_N$ to equation (1.2.4) introduced in (2.1.5), so that

$$\mathfrak{A}_{N,\varepsilon}(t) \to \beta_N(0)+C\int_0^t ds\left(\frac{\|\phi(s)\|_{H^2}^2}{\sqrt{N}} + \frac{\|\phi(s)\|_{H_x^1}^2}{N^{1/3}} + \left(E_N^\Phi - E^\phi\right)\left(1+\|\phi(s)\|_{C^{1/2}}^2\right)\right) =: \mathfrak{A}_N(t), \tag{2.4.204}$$

as $\varepsilon \to 0^+$, locally uniformly. Using the higher conservation laws of the 1D cubic NLS (see [28, Chapter I]),[5] we have that the $H^k$ norms of $\phi$ are bounded (up to an absolute constant) by $\|\phi_0\|_{H^k}$, for any $k \in \mathbb{N}_0$. Thus, there exists an absolute constant $C' \geq C$ such that

$$\mathfrak{A}_N(t) \leq \beta_N(0) + C't\left(\frac{\|\phi_0\|_{H^2}^2}{\sqrt{N}} + \frac{\|\phi_0\|_{H^1}^2}{N^{1/3}} + \|\phi_0\|_{H^1}^2\left(E_N^\Phi - E^\phi\right)\right), \qquad \forall t \geq 0. \tag{2.4.205}$$

Now taking the limit as $\varepsilon \to 0^+$ of the inequality (2.4.200) and using Lemma 2.4.8 once more, we obtain that

$$\beta_N(t) \leq \mathfrak{A}_N(t)\exp\left(C'\|\phi_0\|_{H^2}^2 t\right), \qquad \forall t \geq 0. \tag{2.4.206}$$

Comparison with the statement of Proposition 2.1.2 completes the proof of the proposition.

$\square$

[5] In fact, Koch and Tataru [47] have shown that there exist conserved quantities for the 1D cubic NLS corresponding to the $H^s$ norm, for any $s > -\frac{1}{2}$. See also the work [45] of Killip, Visan, and Zhang for a similar result for the case $-\frac{1}{2} < s < 0$.

## 2.5 Proof of Theorem 2.1.1

In this last section, we show how Proposition 2.1.2 implies Theorem 2.1.1. As the implication is well-known, we only sketch the details. We first recall two technical lemmas from [46].

**Lemma 2.5.1** ([46, Lemma 2.1]). *Let $k \in \mathbb{N}$, and let $\{\gamma^{(j)}\}_{j=1}^{k}$ be a sequence of nonnegative, trace-class operators on $L^2_{sym}(\mathbb{R}^j)$, for $j \in \{1, \ldots, k\}$, with unit trace and such that*

$$\mathrm{Tr}_{j+1}\, \gamma^{(j+1)} = \gamma^{(j)}, \qquad \forall j \in \{1, \ldots, k-1\}. \tag{2.5.1}$$

*Let $\varphi \in L^2(\mathbb{R})$ satisfy $\|\varphi\|_{L^2} = 1$. Then*

$$1 - \left\langle \varphi^{\otimes k} \middle| \gamma^{(k)} \varphi^{\otimes k} \right\rangle \leq k\big(1 - \left\langle \varphi \middle| \gamma^{(1)} \varphi \right\rangle\big). \tag{2.5.2}$$

**Lemma 2.5.2** ([46, Lemma 2.3]). *Let $k \in \mathbb{N}$, and let $\gamma^{(k)}$ be a nonnegative self-adjoint trace-class operator on $L^2_{sym}(\mathbb{R}^k)$ with unit trace (i.e. a density matrix). Let $\varphi \in L^2(\mathbb{R})$ with $\|\varphi\|_{L^2} = 1$. Then*

$$1 - \left\langle \varphi^{\otimes k} \middle| \gamma^{(k)} \varphi^{\otimes k} \right\rangle \leq \mathrm{Tr}_{1,\ldots,k} \left| \gamma^{(k)} - \left|\varphi^{\otimes k}\right\rangle \left\langle \varphi^{\otimes k}\right| \right| \leq \sqrt{8(1 - \langle \varphi^{\otimes k} | \gamma^{(k)} \varphi^{\otimes k} \rangle)}. \tag{2.5.3}$$

*Proof of Theorem 2.1.1.* For $k \in \{1, \ldots, N\}$, let $\gamma_N^{(k)} = \mathrm{Tr}_{1,\ldots,k}(\left|\Phi_N\right\rangle \left\langle \Phi_N\right|)$ denote the $k$-particle reduced density matrix of the $N$-body system, where $\Phi_N$ is the solution to the Schrödinger equation (1.2.4). Let $\phi$ be the solution to the 1D cubic NLS (1.2.11). It is straightforward from the definition of partial trace that

$$\left\langle \phi \middle| \gamma_N^{(1)} \phi \right\rangle_{L^2_x} = \left\langle \Phi_N \middle| \big( (\left|\phi\right\rangle \left\langle \phi\right|) \otimes \mathbf{1}^{\otimes N-1} \big) \Phi_N \right\rangle_{L^2_{\underline{x}_N}} = \langle \Phi_N | p_1 \Phi_N \rangle_{L^2_{\underline{x}_N}}, \tag{2.5.4}$$

which implies by Remark 2.4.5 that

$$1-\left\langle \phi \middle| \gamma_N^{(1)}\phi\right\rangle_{L_x^2} = \langle \Phi_N | q_1 \Phi_N\rangle_{L^2_{\underline{x}_N}} = \alpha_N. \tag{2.5.5}$$

Since $\alpha_N \leq \beta_N$, Proposition 2.1.2 implies that there is an absolute constant $C>0$ such that

$$1-\left\langle \phi(t) \middle| \gamma_N^{(1)}(t)\phi(t)\right\rangle \leq \left(\beta_N(0) + C|t|\left(\frac{\|\phi_0\|_{H^1}^2}{N^{1/3}} + \frac{\|\phi_0\|_{H^2}^2}{N^{1/2}} + \|\phi_0\|_{H^1}^2\left(E_N^\Phi - E^\phi\right)\right)\right)e^{C\|\phi_0\|_{H^2}^2|t|}, \tag{2.5.6}$$

for every $t\in\mathbb{R}$. Since $\Phi_{N,0}=\phi_0^{\otimes N}$, we see from unpacking Definition 2.4.4 for $\beta_N$ that

$$\beta_N(0) = \left\langle \phi_0^{\otimes N} \middle| \widehat{n_N(0)}\phi_0^{\otimes N}\right\rangle = \sum_{k=0}^N \sqrt{\frac{k}{N}}\left\langle \phi_0^{\otimes N} \middle| P_k(0)\phi_0^{\otimes N}\right\rangle, \tag{2.5.7}$$

where the reader will recall the definition of the projector $P_k$ from (2.4.4). For $k\in\{1,\ldots,N\}$, the terms in the definition of $P_k(0)$ contain a projector $q_j(0) = (1-|\phi_0\rangle\langle\phi_0|)_j$, for some $j\in\{1,\ldots,N\}$, which is orthogonal to the state $\phi_0^{\otimes N}$. Thus,

$$P_k(0)\phi_0^{\otimes N} = 0, \qquad \forall k\in\{1,\ldots,N\}, \tag{2.5.8}$$

which together with the identity (2.5.7) implies that $\beta_N(0)=0$. Additionally, using the normalization $\|\phi_0\|_{L^2}=1$ and Fubini-Tonelli, we have that

$$\begin{aligned} E_N^\Phi - E^\phi &= \|\nabla_1\phi_0^{\otimes N}\|_{L^2(\mathbb{R}^N)}^2 + \frac{(N-1)\kappa}{2N}\||\phi_0|\phi_0\otimes\phi_0^{\otimes(N-2)}\|_{L^2(\mathbb{R}^{N-1})}^2 - \|\nabla\phi_0\|_{L^2(\mathbb{R})}^2 - \frac{\kappa}{2}\|\phi_0\|_{L^4(\mathbb{R})}^4 \\ &= -\frac{\kappa}{2N}\|\phi_0\|_{L^4(\mathbb{R})}^4. \end{aligned} \tag{2.5.9}$$

Now by application of Lemma 2.5.1, Lemma 2.5.2, and the $\dot{H}_x^{1/4}\subset L_x^4$ Sobolev embedding, the inequality (2.5.6) implies that there is an absolute constant $C'\geq C$, such that for any $k\in\mathbb{N}$ fixed,

$$\mathrm{Tr}_{1,\ldots,k}\left|\gamma_N^{(k)}(t) - |\phi(t)^{\otimes k}\rangle\langle\phi(t)^{\otimes k}|\right| \leq \left(8kC'|t|\left(\frac{\|\phi_0\|_{H^1}^2}{N^{1/3}} + \frac{\|\phi_0\|_{H^2}^2}{N^{1/2}}\right)e^{C\|\phi_0\|_{H^2}^2|t|}\right)^{1/2}, \qquad \forall t\in\mathbb{R}. \tag{2.5.10}$$

Thus, the proof of Theorem 2.1.1 is complete. □

| Symbol | Definition |
| --- | --- |
| $A \lesssim B,\ A \sim B$ | There are absolute constants $C_1, C_2 > 0$ such that $A \leq C_1 B$ or $C_2 B \leq A \leq C_1 B$ |
| $\underline{x}_k,\ \underline{x}_{i;i+k}$ | $(x_1, \ldots, x_k),\ (x_i, \ldots, x_{i+k})$, where $x_j \in \mathbb{R}$ for $j \in \{1, \ldots, k\}$ or $j \in \{i, \ldots, i+k\}$ |
| $d\underline{x}_k,\ d\underline{x}_{i;i+k}$ | $dx_1 \cdots dx_k,\ dx_i \cdots dx_{i+k}$ |
| $\mathbb{N},\ \mathbb{N}_0$ | natural numbers, natural numbers inclusive of zero |
| $\mathbb{S}_N$ | symmetric group on $N$ elements |
| $C_c^\infty(\mathbb{R}^N)$ | smooth, compactly supported functions on $\mathbb{R}^N$ |
| $\mathcal{S}(\mathbb{R}^N)$ | Schwartz space on $\mathbb{R}^N$ |
| $\mathcal{S}'(\mathbb{R}^N)$ | tempered distributions on $\mathbb{R}^N$ |
| $L^p(\mathbb{R}^N),\ \|\cdot\|_{L^p}$ | standard $p$-integrable function space: see (2.2.2) |
| $H^s(\mathbb{R}^N),\ \|\cdot\|_{H^s}$ | standard $L^2$-based Sobolev function space: see (2.2.4) |
| $C^\gamma(\mathbb{R}^N),\ \|\cdot\|_{C^\gamma}$ | standard Hölder-continuous function space: see (2.2.7) |
| $sym$ | subscript which denotes functions symmetric under permutation of coordinates |
| $\langle \cdot \vert \cdot \rangle$ | $L^2(\mathbb{R}^N)$ inner product with physicist's convention: $\langle f \vert g \rangle \coloneqq \int_{\mathbb{R}^N} d\underline{x}_N \overline{f(\underline{x}_N)} g(\underline{x}_N)$ |
| $\langle \cdot, \cdot \rangle$ | duality pairing |
| $\langle \cdot \vert\ \ \vert \cdot \rangle$ | Dirac's bra-ket notation: see footnote 3 |
| $A^{(k)}_{i_1 \cdots i_k}$ | subscript denotes that the operator on $L^2(\mathbb{R}^N)$ acts in the variables $(x_{i_1}, \ldots, x_{i_k})$ |
| $\phi^{\otimes k}$ | $k$-fold tensor product of $\phi$ with itself realized as $\phi^{\otimes k}(\underline{x}_k) = \prod_{i=1}^k \phi(x_i),\ \underline{x}_k \in \mathbb{R}^k$ |
| $\mathrm{Tr}_{1,\ldots,N}$ | trace on $L^2(\mathbb{R}^N)$ |
| $\mathrm{Tr}_{k+1,\ldots,N}$ | partial trace on $L^2(\mathbb{R}^N)$ over $x_{k+1}, \ldots, x_N$ coordinates |
| $\mathbf{1},\ \mathbf{1}_N$ | identity operator on $L^2(\mathbb{R})$ and on $L^2(\mathbb{R}^N)$ |
| $\Phi_N,\ \Phi_N^\varepsilon$ | solution to Schrödinger problem (1.2.4) and to regularized problem (2.3.20) |
| $\phi$ | solution to cubic NLS (1.2.11) |
| $H_N,\ H_{N,\varepsilon}$ | LL Hamiltonian and regularized LL Hamiltonian: see (1.2.2) and (2.3.19) |
| $p(t),\ q(t)$ | rank-one projector $\vert\phi(t)\rangle \langle\phi(t)\vert$ and $\mathbf{1} - \vert\phi(t)\rangle \langle\phi(t)\vert$: see (2.4.1) |
| $p_j,\ q_j$ | projectors $\mathbf{1}^{\otimes j-1} \otimes p \otimes \mathbf{1}^{N-j},\ \mathbf{1}^{\otimes j-1} \otimes q \otimes \mathbf{1}^{N-j}$: see (2.4.2) |
| $P_k$ | projector onto subspace of $k$ particles not in the state $\phi(t)$: see (2.4.4) |
| $\widehat{f},\ \widehat{f}^{-1}$ | operator $L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ defined by $\widehat{f} \coloneqq \sum_{k=0}^N f(k) P_k$, for $f : \mathbb{Z} \to \mathbb{C}$: see (2.4.6) |
| $n_N, m_N\ \ \widehat{n_N}, \widehat{m_N}$ | functions $\mathbb{Z} \to \mathbb{C}$ and operators $L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$: see Definition 2.4.4 |
| $\mu, \nu\ \ \widehat{\mu}, \widehat{\nu}$ | functions $\mathbb{Z} \to \mathbb{C}$ and operators $L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$: see (2.4.36) and (2.4.68) |
| $\alpha_N,\ \beta_N$ | time-dependent functionals of solution $\phi$ to (1.2.11) and $\Phi_N$ to (1.2.4): see Definition 2.4.4 |
| $\tau_n$ | shift operator on $\mathbb{C}^{\mathbb{Z}}$: see (2.4.16) |
| $\mathrm{tr}_{i=j}$ | trace of a function to hyperplane $\{\underline{x}_N \in \mathbb{R}^N : x_i = x_j\}$: see (2.2.8) |
| $\underline{\Delta}_k$ | Laplacian on $\mathbb{R}^k$: $\underline{\Delta}_k \coloneqq \sum_{i=1}^k \Delta_i$ |
| $[\cdot, \cdot]$ | commutator bracket: $[A, B] \coloneqq AB - BA$ |

Table 2.1: Notation

# Chapter 3

# A Rigorous Derivation of the Hamiltonian Structure of the Nonlinear Schrödinger Equation[1]

## 3.1 Statements of Main Results and Blueprint of Proofs

We now state precisely and outline the proofs of our three main results: Theorem 3.1.3, Theorem 3.1.10, and Theorem 3.1.12. The first two results provide the affirmative answer to Question 1.3.1, establishing the BBGKY hierarchy and GP hierarchy, respectively, as Hamiltonian flows. Theorem 3.1.12 provides the link between the Hamiltonian structure for the GP hierarchy and the Hamiltonian structure for the nonlinear Schrödinger equation, answering Question 1.3.2. Our approach to answering these questions is to meticulously build a formalism, step-by-step, which renders the desired conclusions quite intuitive in hindsight.

We recall the $N$-body Schrödinger equation, BBGKY hierarchy, and limiting GP hierarchy to set the stage for our discussion of the geometry below. It will be useful going forward to fix the following notation: for $d \geq 1$, we denote the point $(x_1, \ldots, x_N) \in \mathbb{R}^{dN}$ by $\underline{x}_N$. We let $\mathcal{S}_s(\mathbb{R}^{dN})$ be the subspace of $\mathcal{S}(\mathbb{R}^{dN})$ of Schwartz functions which are symmetric

[1]This chapter is based on an article published prior to the final submission of this dissertation (see reference [63] for the bibliographic information). The article is an equal collaboration with D. Mendelson, A.R. Nahmod, N. Pavlović, and G. Staffilani.

in their arguments, that is, for any $\pi \in \mathbb{S}_N$[2] we have

$$\Phi(x_{\pi(1)}, \dots, x_{\pi(N)}) = \Phi(x_1, \dots, x_N), \qquad \underline{x}_N \in \mathbb{R}^{dN}. \tag{3.1.1}$$

We call $\mathcal{S}_s(\mathbb{R}^{dN})$ the bosonic Schwartz space, see Definition 3.3.24 for more details.

Consider the $N$-body Schrödinger equation

$$i\partial_t \Phi_N = H_N \Phi_N, \qquad \Phi_N \in \mathcal{S}_s(\mathbb{R}^{dN}) \tag{3.1.2}$$

where $H_N$ is the $N$-body Hamiltonian

$$H_N \coloneqq \sum_{j=1}^{N} (-\Delta_{x_j}) + \frac{2\kappa}{N-1} \sum_{1 \le i < j \le N} V_N(X_i - X_j), \qquad \kappa \in \{\pm 1\}. \tag{3.1.3}$$

The pair interaction potential has the form $V_N = N^{d\beta} V(N^{d\beta} \cdot)$, where $\beta \in (0,1)$, $V$ is an even nonnegative function in $C_c^\infty(\mathbb{R}^d)$ with $\int_{\mathbb{R}} dx V(x) = 1$, and $V_N(X_i - X_j)$ denotes the operator which is multiplication by $V_N(x_i - x_j)$.

The $N$-body density matrix, associated to the wave function $\Phi_N \in \mathcal{S}_s(\mathbb{R}^{dN})$ is given by

$$\Psi_N \coloneqq |\Phi_N\rangle \langle \Phi_N| \in \mathcal{L}(\mathcal{S}_s'(\mathbb{R}^{dN}), \mathcal{S}_s(\mathbb{R}^{dN})),\text{[3]}$$

and the reduced density matrix hierarchy

$$(\gamma_N^{(k)})_{k=1}^N \coloneqq (\mathrm{Tr}_{k+1,\dots,N}(\Psi_N))_{k=1}^N$$

[2] $\mathbb{S}_N$ is the symmetric group of order $N$.

[3] $\mathcal{L}(\mathcal{S}_s'(\mathbb{R}^{dN}), \mathcal{S}_s(\mathbb{R}^{dN}))$ denotes the space of continuous linear maps from symmetric tempered distributions to symmetric Schwartz functions.

solves the *quantum BBGKY hierarchy*

$$\begin{aligned}
i\partial_t \gamma_N^{(k)} &= \Big[-\Delta_{\underline{x}_k}, \gamma_N^{(k)}\Big] + \frac{2\kappa}{N-1} \sum_{1\leq i<j\leq k} \Big[V_N(X_i - X_j), \gamma_N^{(k)}\Big] \\
&\quad + \frac{2\kappa(N-k)}{N-1} \sum_{i=1}^{k} \mathrm{Tr}_{k+1}\Big(\Big[V_N(X_i - X_{k+1}), \gamma_N^{(k+1)}\Big]\Big), \qquad 1 \leq k \leq N-1 \qquad (3.1.4) \\
&= \Big[-\Delta_{\underline{x}_k}, \gamma_N^{(k)}\Big] + \frac{2\kappa}{N-1} \sum_{1\leq i<j\leq k} \Big[V_N(X_i - X_j), \gamma_N^{(k)}\Big], \qquad k = N,
\end{aligned}$$

where we have introduced the notation $\Delta_{\underline{x}_k} \coloneqq \sum_{j=1}^{k} \Delta_{x_j}$.

The GP hierarchy is formally obtained from the BBGKY hierarchy (3.1.4) by letting $N \to \infty$. More precisely, a time-dependent family of density matrix $\infty$-hierarchies $\Gamma(t) = (\gamma(t)^{(k)})_{k=1}^{\infty}$ solves the GP hierarchy if

$$i\partial_t \gamma^{(k)} = -\big[\Delta_{\underline{x}_k}, \gamma^{(k)}\big] + 2\kappa B_{k+1}\gamma^{(k+1)}, \qquad \forall k \in \mathbb{N} \tag{3.1.5}$$

with $\kappa \in \{\pm 1\}$ and

$$B_{k+1}\gamma^{(k+1)} \coloneqq \sum_{j=1}^{k} \big(B_{j;k+1}^{+} - B_{j;k+1}^{-}\big)\gamma^{(k+1)}, \tag{3.1.6}$$

where

$$\big(B_{j;k+1}^{+}\gamma^{(k+1)}\big)(t, \underline{x}_k; \underline{x}_k') \coloneqq \int_{\mathbb{R}^{2d}} dx_{k+1} dx_{k+1}' \delta(x_{k+1} - x_{k+1}')\delta(x_j - x_{k+1})\gamma^{(k+1)}(t, \underline{x}_{k+1}; \underline{x}_{k+1}') \tag{3.1.7}$$

with an analogous definition for $B_{j;k+1}^{-}$ with $\delta(x_j - x_{k+1})$ replaced by $\delta(x_j' - x_{k+1})$. When $\kappa = 1$, we say that the hierarchy is *defocusing* and for $\kappa = -1$, we say that the hierarchy is *focusing* (in analogy with the defocusing and focusing NLS, respectively).

As we outlined in the introduction, our first main results establish that the BBGKY hierarchy (3.1.4) and the GP hierarchy (3.1.5) are Hamiltonian flows on appropriate weak

Lie-Poisson manifolds. To do this, we need to define a suitable phase space for the Hamiltonian evolution in both the finite- and infinite-particle settings. In particular, we need to construct certain Lie-Poisson manifolds of density matrix hierarchies, and we outline this construction in the next subsection. We will also establish that the procedure described above for obtaining the BBGKY hierarchy from the $N$-body Schrödinger equation can be given by the composition of several natural Poisson maps, thereby establishing the existence of a natural Poisson morphism which maps the $N$-body Schrödinger equation to the BBGKY hierarchy.

### 3.1.1 Construction of the Lie algebra $\mathfrak{G}_N$ and Lie-Poisson manifold $\mathfrak{G}_N^*$

For each $k \in \mathbb{N}$, we let

$$\mathfrak{g}_k \coloneqq \{A^{(k)} \in \mathcal{L}(\mathcal{S}_s(\mathbb{R}^k), \mathcal{S}_s(\mathbb{R}^k)) : (A^{(k)})^* = -A^{(k)}\},$$

endowed with the subspace topology of $\mathcal{L}(\mathcal{S}_s(\mathbb{R}^k), \mathcal{S}_s'(\mathbb{R}^k))$. We define a Lie algebra $(\mathfrak{g}_k, [\cdot,\cdot]_{\mathfrak{g}_k})$, with Lie bracket defined by

$$\left[A^{(k)}, B^{(k)}\right]_{\mathfrak{g}_k} \coloneqq k\left[A^{(k)}, B^{(k)}\right], \tag{3.1.8}$$

where the right-hand side denotes the usual commutator bracket. We refer to elements of $\mathfrak{g}_k$ as *k-particle bosonic observables*. For $N \in \mathbb{N}$, we then define the locally convex direct sum

$$\mathfrak{G}_N \coloneqq \bigoplus_{k=1}^{N} \mathfrak{g}_k, \tag{3.1.9}$$

and we refer to elements of $\mathfrak{G}_N$ as *observable $N$-hierarchies*.

To define a Lie bracket on the space $\mathfrak{G}_N$, we consider the following natural embedding maps. For $N \in \mathbb{N}$ and $k \in \mathbb{N}_{\leq N}$, there exists a smooth map

$$\epsilon_{k,N} : \mathfrak{g}_k \to \mathfrak{g}_N, \tag{3.1.10}$$

which embeds a $k$-particle bosonic observable in the space of $N$-particle bosonic operators so as to have the filtration property

$$\left[\epsilon_{\ell,N}(\mathfrak{g}_\ell), \epsilon_{j,N}(\mathfrak{g}_j)\right]_{\mathfrak{g}_N} \subset \epsilon_{\min\{\ell+j-1,N\},N}\big(\mathfrak{g}_{\min\{\ell+j-1,N\}}\big) \subset \mathfrak{g}_N. \tag{3.1.11}$$

Using this filtration property and the injectivity of the maps $\epsilon_{k,N}$, we can now endow $\mathfrak{G}_N$ with a Lie algebra structure by defining the bracket

$$[A,B]_{\mathfrak{G}_N}^{(k)} \coloneqq \sum_{\substack{1\leq \ell,j\leq N \\ \min\{\ell+j-1,N\}=k}} \epsilon_{k,N}^{-1}\Big(\left[\epsilon_{\ell,N}\big(A^{(\ell)}\big), \epsilon_{j,N}\big(B^{(j)}\big)\right]_{\mathfrak{g}_N}\Big), \qquad k \in \{1,\ldots,N\}. \tag{3.1.12}$$

Furthermore, the maps $\{\epsilon_{k,N}\}_{k=1}^N$ induce a Lie algebra homomorphism

$$\iota_{\epsilon,N} : \mathfrak{G}_N \to \mathfrak{g}_N, \qquad \iota_{\epsilon,N}(A_N) \coloneqq \sum_{k=1}^N \epsilon_{k,N}(A_N^{(k)}), \qquad \forall A_N = (A_N^{(k)})_{k\in\mathbb{N}_{\leq N}}. \tag{3.1.13}$$

In other words, $\iota_{\epsilon,N}$ maps an observable $N$-hierarchy to an $N$-body bosonic observable. In Section 3.4, we will establish several properties of the embedding map, which ultimately enable us to prove the following result.

**Proposition 3.1.1.** $(\mathfrak{G}_N, [\cdot,\cdot]_{\mathfrak{G}_N})$ *is a Lie algebra in the sense of Definition 3.3.14.*

Next, we define the real topological vector space

$$\mathfrak{G}_N^* \coloneqq \Big\{\Gamma_N = (\gamma_N^{(k)})_{k=1}^N \in \prod_{k=1}^N \mathcal{L}(\mathcal{S}_s'(\mathbb{R}^{dk}), \mathcal{S}_s(\mathbb{R}^{dk})) : (\gamma_N^{(k)})^* = \gamma_N^{(k)}\Big\}, \tag{3.1.14}$$

and we refer to elements of $\mathfrak{G}_N^*$ as *density matrix $N$-hierarchies.* Let $\mathcal{A}_{H,N}$ be the algebra with respect to point-wise product generated by the functionals in the set

$$\{F \in C^\infty(\mathfrak{G}_N^*; \mathbb{R}) : F(\cdot) = i\,\mathrm{Tr}(A_N \cdot),\ \ A_N \in \mathfrak{G}_N\} \cup \{F \in C^\infty(\mathfrak{G}_N^*; \mathbb{R}) : F(\cdot) \equiv C \in \mathbb{R}\}.$$

We can define a Lie-Poisson structure on $\mathfrak{G}_N^*$, given by

$$\{F, G\}_{\mathfrak{G}_N^*}(\Gamma_N) \coloneqq i\,\mathrm{Tr}\big([dF[\Gamma_N], dG[\Gamma_N]]_{\mathfrak{G}_N} \cdot \Gamma_N\big), \qquad \forall \Gamma_N \in \mathfrak{G}_N^*, \tag{3.1.15}$$

where $F, G \in \mathcal{A}_{H,N}$.

To construct the weak Lie-Poisson manifold $\mathfrak{G}_N^*$, a good heuristic to keep in mind is that density matrices are dual to skew-adjoint operators. The superscript $*$, however, does not denote the literal functional analytic dual, but rather denotes a space in weakly non-degenerate pairing with $\mathfrak{G}_N$. The fact that we only have weak non-degeneracy means that we will be unable to appeal to classical results on Lie-Poisson structures, see for instance Proposition 3.3.20 below, and instead we will proceed by direct proof to establish the following result.

**Proposition 3.1.2.** *$(\mathfrak{G}_N^*, \mathcal{A}_{H,N}, \{\cdot,\cdot\}_{\mathfrak{G}_N^*})$ is a weak Poisson manifold.*

To establish that the BBGKY hierarchy is a Hamiltonian flow on this weak Poisson manifold, we need to prescribe the *BBGKY Hamiltonian functional*

$$\mathcal{H}_{BBGKY,N}(\Gamma_N) \coloneqq \mathrm{Tr}(\mathbf{W}_{BBGKY,N} \cdot \Gamma_N), \tag{3.1.16}$$

where $-i\mathbf{W}_{BBGKY,N}$ is the observable 2-hierarchy defined by

$$\mathbf{W}_{BBGKY,N} \coloneqq (-\Delta_x, \kappa V_N(X_1 - X_2), 0, \ldots). \tag{3.1.17}$$

We can now state the following theorem, which establishes that the BBGKY hierarchy admits a Hamiltonian formulation and lays the groundwork for our answering of Question 1.3.1.

**Theorem 3.1.3.** *Let $I \subset \mathbb{R}$ be a compact interval. Then $\Gamma_N = (\gamma_N^{(k)})_{k=1}^N \in C^\infty(I; \mathfrak{G}_N^*)$ is a solution to the BBGKY hierarchy* (3.1.4) *if and only if*

$$\frac{d}{dt}\Gamma_N = X_{\mathcal{H}_{BBGKY,N}}(\Gamma_N), \tag{3.1.18}$$

*where $X_{\mathcal{H}_{BBGKY,N}}$ is the unique vector field defined by $\mathcal{H}_{BBGKY,N}$ (see Definition 3.3.1) with respect to the weak Poisson structure $(\mathfrak{G}_N^*, \mathcal{A}_{H,N}, \{\cdot,\cdot\}_{\mathfrak{G}_N^*})$.*

### 3.1.2 Derivation of the Lie algebra $\mathfrak{G}_\infty$ and Lie-Poisson manifold $\mathfrak{G}_\infty^*$

Having established the necessary framework at the $N$-body level, we are now prepared to address the infinite-particle limit of our constructions. Via the natural inclusion map, one has $\mathfrak{G}_N \subset \mathfrak{G}_M$ for $M \geq N$. Hence, one has a natural limiting algebra[4] given by

$$\mathfrak{F}_\infty \coloneqq \bigcup_{N=1}^\infty \mathfrak{G}_N = \bigoplus_{k=1}^\infty \mathfrak{g}_k. \tag{3.1.19}$$

By embedding $\mathfrak{G}_N$ into this limiting algebra, the rather complicated Lie bracket $[\cdot,\cdot]_{\mathfrak{G}_N}$ converges pointwise to a much simpler Lie bracket.

We let $\mathrm{Sym}_k$ denote the $k$-particle bosonic symmetrization operator, see Definition 3.3.30, and we let $[\cdot,\cdot]_1$ be a certain separately continuous, bilinear map, the precise definition of which we defer to Section 3.4. We establish the following result.

[4]This discussion could be formulated more precisely in terms of co-limits of topological spaces ordered by inclusion.

**Proposition 3.1.4.** *Let $N_0 \in \mathbb{N}$. For $A = (A^{(k)})_{k\in\mathbb{N}}, B = (B^{(k)})_{k\in\mathbb{N}} \in \mathfrak{G}_{N_0}$, we have that*

$$\lim_{N\to\infty} [A, B]_{\mathfrak{G}_N} = C = (C^{(k)})_{k\in\mathbb{N}}, \tag{3.1.20}$$

*where*

$$C^{(k)} := \sum_{\substack{\ell,j\geq 1 \\ \ell+j-1=k}} \mathrm{Sym}_k\big(\big[A^{(\ell)}, B^{(j)}\big]_1\big), \tag{3.1.21}$$

*in the topology of $\mathfrak{F}_\infty$.*

The topological vector space given in (3.1.19) is too small to capture the generator of the GP Hamiltonian, defined in (3.1.29) below. Indeed, the 2-particle component $V_N(X_1 - X_2)$ of the $N$-body Hamiltonian $H_N$ given in (3.1.3) converges to the distribution-valued operator[5] $\delta(X_1 - X_2)$ as $N \to \infty$. The operator $-i\delta(X_1 - X_2)$ does not belong to $\mathfrak{g}_2$ since it does not map $\mathcal{S}_s(\mathbb{R}^{2d})$ to itself.

Since we will need our Lie algebra $\mathfrak{G}_\infty$ to contain the generator of the GP Hamiltonian functional, this necessitates an underlying topological vector space which includes distribution-valued operators (DVOs). The inclusion of DVOs introduces technical difficulties in the definition of the bracket $[\cdot,\cdot]_1$. As we will see, the definition of the bracket $[\cdot,\cdot]_1$, involves compositions of distribution-valued operators in one coordinate, which in general is not possible. Consequently, we need to find a setting in which we can give meaning to such a composition, thus motivating our introduction of the *good mapping property*:

**Definition 3.1.5** (Good mapping property)**.** Let $i \in \mathbb{N}$. We say that an operator $A^{(i)} \in \mathcal{L}(\mathcal{S}(\mathbb{R}^{di}), \mathcal{S}'(\mathbb{R}^{di}))$ has the *good mapping property* if for any $\alpha \in \mathbb{N}_{\leq i}$, the continuous bilinear

[5]Not to be confused with operator-valued distribution.

map

$$\mathcal{S}(\mathbb{R}^{di}) \times \mathcal{S}(\mathbb{R}^{di}) \to \mathcal{S}'(\mathbb{R}^{d})\hat{\otimes}\mathcal{S}(\mathbb{R}^{d})$$
$$(f^{(i)}, g^{(i)}) \mapsto \int_{\mathbb{R}^{i-1}} dx_1 \ldots dx_{\alpha-1} dx_{\alpha+1} \ldots dx_i A^{(i)}(f^{(i)})(x_1, \ldots, x_i) g^{(i)}(x_1, \ldots, x_{\alpha-1}, x'_\alpha, x_{\alpha+1}, \ldots, x_i),$$

may be identified with a continuous bilinear map $\mathcal{S}(\mathbb{R}^{di}) \times \mathcal{S}(\mathbb{R}^{di}) \to \mathcal{S}(\mathbb{R}^{2d})$.[6]

Here and throughout this chapter, an integral should be interpreted as a distributional pairing, unless specified otherwise. We will denote by $\mathcal{L}_{gmp}(\mathcal{S}(\mathbb{R}^{di}), \mathcal{S}'(\mathbb{R}^{di}))$ the subset of $\mathcal{L}(\mathcal{S}(\mathbb{R}^{di}), \mathcal{S}'(\mathbb{R}^{di}))$ of operators with the good mapping property.

**Remark 3.1.6.** It is evident that $\mathcal{L}_{gmp}(\mathcal{S}(\mathbb{R}^{di}), \mathcal{S}'(\mathbb{R}^{di}))$ is closed under linear combinations and therefore a subspace. Note that here and throughout we endow $\mathcal{L}(\mathcal{S}(\mathbb{R}^{di}), \mathcal{S}'(\mathbb{R}^{di}))$ with the topology of uniform convergence on bounded sets, and we endow $\mathcal{L}_{gmp}$ with the subspace topology. To see that $\mathcal{L}_{gmp}$ is a proper subspace of $\mathcal{L}$, consider the multiplication operator $\delta(\underline{X}_2) \in \mathcal{L}(\mathcal{S}(\mathbb{R}^{2d}), \mathcal{S}'(\mathbb{R}^{2d}))$.

The formula for the limiting Lie bracket given in Proposition 3.1.4 has a greatly simplified form compared to the $N$-body bracket $[\cdot,\cdot]_{\mathfrak{G}_N}$ due to the vanishing of the higher "contraction commutators". Moreover, as we prove in Appendix 3.3, the good mapping property gives an appropriate definition to the bracket $\left[A^{(i)}, B^{(j)}\right]_1$ as a well-defined element of $\mathcal{L}_{gmp}(\mathcal{S}(\mathbb{R}^{dk}), \mathcal{S}'(\mathbb{R}^{dk}))$. Hence, we can take advantage of the good mapping property and extend the limiting formula from Proposition 3.1.4 to a map on a much larger real topological

[6] We use $\hat{\otimes}$ to denote the completion of the tensor product in either the projective or injective topology (which coincide). See Section 3.3.3 for furhter discussion.

vector space $\mathfrak{G}_\infty$ given by the locally convex direct sum

$$\mathfrak{G}_\infty := \bigoplus_{k=1}^{\infty} \mathfrak{g}_{k,gmp}, \quad \mathfrak{g}_{k,gmp} := \{A^{(k)} \in \mathcal{L}_{gmp}(\mathcal{S}_s(\mathbb{R}^{dk}), \mathcal{S}_s'(\mathbb{R}^{dk})) : A^{(k)} = -(A^{(k)})^*\}. \quad (3.1.22)$$

We refer to the elements of $\mathfrak{G}_\infty$ as *observable* $\infty$*-hierarchies*, and the elements of $\mathfrak{g}_{k,gmp}$ as *$k$-particle bosonic observables*. The verification of the Lie algebra axioms then proceeds by direct computation, and we are able to establish the following result.

**Proposition 3.1.7.** *$(\mathfrak{G}_\infty, [\cdot,\cdot]_{\mathfrak{G}_\infty})$ is a Lie algebra in the sense of Definition 3.3.14.*

Analogously to the $N$-body setting, our second step is the dual problem of building a weak Lie-Poisson manifold $(\mathfrak{G}_\infty^*, \mathcal{A}_\infty, \{\cdot,\cdot\}_{\mathfrak{G}_\infty^*})$. If we were in the finite-dimensional setting or a "nice" infinite-dimensional setting, such as $\mathfrak{G}_\infty^*$ being a Fréchet space and $\mathfrak{G}_\infty$ being its predual, then this step would follow from standard results (see Section 3.3.2). While $\mathfrak{G}_\infty^*$ is Fréchet, the predual of $\mathfrak{G}_\infty^*$ is

$$\left\{A = (A^{(k)})_{k\in\mathbb{N}} \in \bigoplus_{k=1}^{\infty} \mathcal{L}(\mathcal{S}_s(\mathbb{R}^{dk}), \mathcal{S}_s'(\mathbb{R}^{dk}) : (A^{(k)})^* = -A^{(k)}\right\}, \quad (3.1.23)$$

which is too large a space for the Lie bracket $[\cdot,\cdot]_{\mathfrak{G}_\infty}$ to be well-defined. Therefore, the standard procedure for obtaining a Lie-Poisson manifold from a Lie algebra can only serve as inspiration.

We define the real topological vector space

$$\mathfrak{G}_\infty^* := \left\{\Gamma = (\gamma^{(k)})_{k\in\mathbb{N}} \in \prod_{k=1}^{\infty} \mathcal{L}(\mathcal{S}_s'(\mathbb{R}^{dk}), \mathcal{S}_s(\mathbb{R}^{dk})) : \gamma^{(k)} = (\gamma^{(k)})^* \ \ \forall k \in \mathbb{N}\right\}, \quad (3.1.24)$$

where the topology is the product topology. Using the isomorphism

$$\mathcal{L}(\mathcal{S}_s'(\mathbb{R}^{dk}), \mathcal{S}_s(\mathbb{R}^{dk})) \cong \mathcal{S}_{s,s}(\mathbb{R}^{dk} \times \mathbb{R}^{dk}), \quad (3.1.25)$$

the elements of $\mathfrak{G}_\infty^*$, which we call *density matrix $\infty$-hierarchies*, are infinite sequences of $k$-particle integral operators with Schwartz class kernels $K(\underline{x}_k;\underline{x}'_k)$, which are separately invariant under permutation in the $\underline{x}_k$ and $\underline{x}'_k$ coordinates.

Let $\mathcal{A}_\infty$ be the algebra with respect to point-wise product generated by functionals in the set

$$\{F \in C^\infty(\mathfrak{G}_\infty^*;\mathbb{R}) : F(\cdot) = i\,\mathrm{Tr}(A\cdot),\ A \in \mathfrak{G}_\infty\} \cup \{F \in C^\infty(\mathfrak{G}_\infty^*;\mathbb{R}) : F(\cdot) \equiv C \in \mathbb{R}\}. \tag{3.1.26}$$

We will observe later that, importantly, our choice of $\mathcal{A}_\infty$ contains the observable $\infty$-hierarchy $-i\mathbf{W}_{GP}$, which generates the GP Hamiltonian.

As in the finite-particle setting, the Lie algebra structure on $\mathfrak{G}_\infty$ canonically induces a Poisson structure on $\mathfrak{G}_\infty^*$. This canonical Poisson structure, which is called a Lie-Poisson structure, is defined by the Poisson bracket

$$\{F,G\}_{\mathfrak{G}_\infty^*}(\Gamma) := i\,\mathrm{Tr}\big([dF[\Gamma],dG[\Gamma]]_{\mathfrak{G}_\infty}\cdot\Gamma\big), \qquad \forall\Gamma \in \mathfrak{G}_\infty^*, \tag{3.1.27}$$

where $F,G \in C^\infty(\mathfrak{G}_\infty^*;\mathbb{R})$ are functionals in the unital[7] sub-algebra $\mathcal{A}_\infty$ and we identify the Gâteaux derivatives $dF[\Gamma], dG[\Gamma]$ as observable $\infty$-hierarchies via the trace pairing $i\,\mathrm{Tr}(\cdot)$. We will ultimately establish the following result, which provides the underlying geometric structure required to address Question 1.3.1.

**Proposition 3.1.8.** $(\mathfrak{G}_\infty^*, \mathcal{A}_\infty, \{\cdot,\cdot\}_{\mathfrak{G}_\infty^*})$ *is a weak Poisson manifold.*

Define the *Gross-Pitaevskii Hamiltonian functional*

$$\mathcal{H}_{GP} : \mathfrak{G}_\infty^* \to \mathbb{R} \tag{3.1.28}$$

[7]i.e. containing a multiplicative identity

by

$$\mathcal{H}_{GP}(\Gamma) := -\operatorname{Tr}_1\bigl(\Delta_{x_1}\gamma^{(1)}\bigr) + \operatorname{Tr}_{1,2}\bigl(\delta(X_1 - X_2)\gamma^{(2)}\bigr), \qquad \Gamma = (\gamma^{(k)})_{k\in\mathbb{N}} \in \mathfrak{G}_\infty^*, \tag{3.1.29}$$

where $\operatorname{Tr}_{1,\ldots,j}$ denotes the $j$-particle generalized trace, see Appendix 3.2 for definition and discussion. Then we can rewrite $\mathcal{H}_{GP}$ as

$$\mathcal{H}_{GP}(\Gamma) = \operatorname{Tr}(\mathbf{W}_{GP} \cdot \Gamma), \qquad \mathbf{W}_{GP} := (-\Delta_{x_1}, \delta(X_1 - X_2), 0, \ldots), \tag{3.1.30}$$

which one should compare with (3.1.16).

**Remark 3.1.9.** Note that $-i\mathbf{W}_{GP}$ is an observable $\infty$-hierarchy, that is, an element of $\mathfrak{G}_\infty$. Since we have the convergence $-i\mathbf{W}_{BBGKY,N} \to -i\mathbf{W}_{GP}$ in $\mathfrak{G}_\infty$, as $N \to \infty$, it follows that $\mathcal{H}_{BBGKY,N} \to \mathcal{H}_{GP}$ in $C^\infty(\mathfrak{G}_\infty^*; \mathbb{R})$ endowed with the topology of uniform convergence on bounded sets.

We now state our next main result, which addresses the final component of Question 1.3.1:

**Theorem 3.1.10** (Hamiltonian structure for GP)**.** *Let $I \subset \mathbb{R}$ be a compact interval. Then $\Gamma \in C^\infty(I; \mathfrak{G}_\infty^*)$ is a solution to the GP hierarchy* (3.1.5) *if and only if*

$$\left(\frac{d}{dt}\Gamma\right)(t) = X_{\mathcal{H}_{GP}}(\Gamma(t)), \qquad \forall t \in I, \tag{3.1.31}$$

*where $X_{\mathcal{H}_{GP}}$ is the unique Hamiltonian vector field defined by $\mathcal{H}_{GP}$ with respect to the weak Poisson structure $(\mathfrak{G}_\infty^*, \mathcal{A}_\infty, \{\cdot,\cdot\}_{\mathfrak{G}_\infty^*})$.*

**Remark 3.1.11.** The result of Theorem 3.1.10 extends, with an almost identical proof, to the Hartree hierarchy, and it seems likely that this result should also extend to the quintic

GP hierarchy [15] and other variants which account for higher-order particle interactions [98].

We now give a geometric formulation of the procedure by which one obtains the BBGKY hierarchy from the $N$-body Schrödinger equation. The results described below will be proved in Section 3.4.3. To record the Hamiltonian structure for the $N$-body Schrödinger equation, we equip the bosonic Schwartz space $\mathcal{S}_s(\mathbb{R}^{dN})$ with the standard symplectic structure and define the Hamiltonian functional

$$\mathcal{H}_N(\Phi_N) \coloneqq \frac{1}{N}\int_{\mathbb{R}^{dN}} d\underline{x}_N \overline{\Phi_N(\underline{x}_N)}(H_N\Phi_N)(\underline{x}_N), \qquad \forall \Phi_N \in \mathcal{S}_s(\mathbb{R}^{dN}). \tag{3.1.32}$$

Then the Schrödinger equation (3.1.2) can be viewed as a Hamiltonian flow on this weak symplectic manifold. We can endow the space $\mathcal{L}(\mathcal{S}_s'(\mathbb{R}^{dN}), \mathcal{S}_s(\mathbb{R}^{dN}))$ of bosonic density matrices with a weak Poisson structure by defining

$$\{F, G\}_N \coloneqq i \operatorname{Tr}_{1,\ldots,N}\Big([dF[\Psi_N], dG[\Psi_N]]_{\mathfrak{g}_N}\Psi_N\Big), \qquad \forall \Psi_N \in \mathcal{L}(\mathcal{S}_s'(\mathbb{R}^{dN}), \mathcal{S}_s(\mathbb{R}^{dN})), \tag{3.1.33}$$

where $dF$ and $dG$ denote the Gâteaux derivatives, see Definition 2.1.4, of $F$ and $G$, which are smooth real-valued functionals with suitably regular Gâteaux derivatives. Then the Poisson bracket $\{\cdot,\cdot\}_N$ is a Lie-Poisson bracket induced by the Lie algebra of $N$-body bosonic observables with Lie bracket given by $[\cdot,\cdot]_{\mathfrak{g}_N}$.

There is a canonical map from $N$-body wave functions to $N$-body density matrices given by

$$\iota_{DM,N} : \mathcal{S}_s(\mathbb{R}^{dN}) \to \mathcal{L}(\mathcal{S}_s'(\mathbb{R}^{dN}), \mathcal{S}_s(\mathbb{R}^{dN})), \qquad \iota_{DM,N}(\Phi_N) \coloneqq |\Phi_N\rangle \langle\Phi_N|\,. \tag{3.1.34}$$

We will show in Proposition 3.4.27 that

$$\iota_{DM,N} : (\mathcal{S}_s(\mathbb{R}^{dN}), \{\cdot,\cdot\}_{L^2,N}) \to (\mathcal{L}(\mathcal{S}_s'(\mathbb{R}^{dN}), \mathcal{S}_s(\mathbb{R}^{dN})), \{\cdot,\cdot\}_N),$$

is a Poisson morphism[8] and consequently maps solutions of the Schrödinger equation (3.1.2) to solutions of the von Neumann equation

$$i\partial_t \Psi_N = [H_N, \Psi_N], \tag{3.1.35}$$

where the right-hand side denotes the usual commutator. Defining the Hamiltonian functional

$$\mathcal{H}_N(\Psi_N) := \frac{1}{N} \operatorname{Tr}_{1,\ldots,N}(H_N \Psi_N), \qquad \forall \Psi_N \in \mathcal{L}(\mathcal{S}_s'(\mathbb{R}^{dN}), \mathcal{S}_s(\mathbb{R}^{dN})), \tag{3.1.36}$$

the von Neumann equation (3.1.35) can be viewed as a Hamiltonian equation of motion on the weak Poisson manifold $(\mathcal{L}(\mathcal{S}_s'(\mathbb{R}^{dN}), \mathcal{S}_s(\mathbb{R}^{dN})), \{\cdot,\cdot\}_N)$. We will prove in Proposition 3.4.29 that the dual of the map $\iota_{\epsilon,N}$ given in (3.1.13) induces a canonical morphism of Poisson manifolds, which is precisely the *reduced density matrix map*, given by

$$\iota_{RDM,N} = \iota_{\epsilon,N}^* : \mathfrak{g}_N^* \to \mathfrak{G}_N^*, \qquad \iota_{RDM,N}(\Psi_N) := (\operatorname{Tr}_{k+1,\ldots,N}(\Psi_N))_{k=1}^N =: (\gamma_N^{(k)})_{k=1}^N, \tag{3.1.37}$$

which maps solutions of the von Neumann equation to solutions of the quantum BBGKY hierarchy.

### 3.1.3 The Connection with the NLS

We will now tie together our main results and state the result which provides an affirmative answer to Question 1.3.2. We connect the GP hierarchy to the cubic NLS, each as infinite-dimensional Hamiltonian systems, through the canonical embedding

$$\iota : \mathcal{S}(\mathbb{R}^d) \to \mathfrak{G}_\infty^*, \qquad \phi \mapsto (\, |\phi^{\otimes k}\rangle \, \langle \phi^{\otimes k}|)_{k\in\mathbb{N}}. \tag{3.1.38}$$

[8] We recall $\{\cdot,\cdot\}_{L^2,N} = N\{\cdot,\cdot\}_{L^2}$, and see (1.3.2) for a definition of $\{\cdot,\cdot\}_{L^2}$. We also note that the co-domain of this map will be replaced by the appropriate space of $N$-body density matrices.

Although $\iota$ is rather trivial in terms of the simplicity of its definition, and for this reason we sometimes refer to $\iota$ as the trivial embedding, it has the important property of being a Poisson morphism (see Definition 3.3.7 below).

**Theorem 3.1.12.** *The map $\iota$ is a Poisson morphism of $(\mathcal{S}(\mathbb{R}^d), \mathcal{A}_{\mathcal{S}}, \{\cdot,\cdot\}_{L^2})$ into $(\mathfrak{G}_\infty^*, \mathcal{A}_\infty, \{\cdot,\cdot\}_{\mathfrak{G}_\infty^*})$, i.e. it is a smooth map such that*

$$\{F \circ \iota, G \circ \iota\}_{L^2}(\phi) = \{F, G\}_{\mathfrak{G}_\infty^*}(\iota(\phi)), \qquad \forall \phi \in \mathcal{S}(\mathbb{R}^d), \tag{3.1.39}$$

*for all functionals $F, G \in \mathcal{A}_\infty$.*

We conclude by discussing why the results described in this section provide "a rigorous derivation of the Hamiltonian structure of the NLS". It is a quick computation to show that the pullback of the GP Hamiltonian (3.1.30) under the map $\iota$, denoted by $\iota^*\mathcal{H}_{GP}$, equals the NLS Hamiltonian (1.3.4),[9] that is

$$\iota^*\mathcal{H}_{GP} = \mathcal{H}_{NLS}. \tag{3.1.40}$$

Hence, Theorem 3.1.12, Theorem 3.1.10 and (3.1.40) ultimately demonstrate that the Hamiltonian functional and phase space of the NLS can be obtained via the pullback of the canonical embedding (3.1.38). Together with the results of Section 3.4.3, which provide a geometric correspondence between the $N$-body Schrodinger equation and the BBGKY hierarchy, and Proposition 3.1.4, which enables us to take the infinite-particle limit of our geometric constructions at the $N$-body level, this provides a rigorous derivation of the Hamiltonian structure of the NLS from the Hamiltonian formulation of the $N$-body Schrödinger equation.

[9] In particular, as a corollary of Theorem 3.1.10 and Theorem 3.1.12, we obtain the well-known fact that if $\phi(t)$ is a solution to the cubic NLS (1.3.7), then $\Gamma(t) := \iota(\phi(t))$ is a solution to the GP hierarchy (3.1.5).

### 3.1.4 Organization of the Chapter

Section 3.3 is devoted to preliminary material on weak Poisson manifolds modeled on locally convex spaces, Lie algebras, and tensor products. The reader familiar with infinite-dimensional Poisson manifolds and Lie algebras may wish to skip the first two subsections upon first reading and instead consult them as necessary during the reading of Section 3.4 and Section 3.5.

In Section 3.4, we build the requisite Lie algebra structure for $\mathfrak{G}_N$ and weak Lie-Poisson structure for $\mathfrak{G}_N^*$, thereby proving Proposition 3.1.1 and Proposition 3.1.2. Section 3.4.1 contains the Lie algebra construction, and Section 3.4.2 contains the dual Lie-Poisson construction. Lastly, in Section 3.4.3, we show that the familiar maps of forming a density matrix from a wave function and taking the sequence of reduced density matrices of a density matrix have geometric content. Namely, we prove Proposition 3.4.27 and Proposition 3.4.29, which assert that these maps are Poisson morphisms.

In Section 3.5, we build the requisite Lie algebra structure for $\mathfrak{G}_\infty$ and weak Lie-Poisson structure for $\mathfrak{G}_\infty^*$, thereby proving Proposition 3.1.7 and Proposition 3.1.8. The section is broken up into several subsections. Section 3.5.2 is devoted the Lie algebra construction, and Section 3.5.3 is devoted to the dual Lie-Poisson construction. Finally, we will prove Theorem 3.1.12 in Section 3.5.4.

Lastly, in Section 3.6, we prove our Hamiltonian flows results Theorem 3.1.3 and Theorem 3.1.10, which assert that the BBGKY and GP hierarchies, respectively, are Hamiltonian flows on the weak Lie-Poisson manifolds constructed in the previous sections.

**Remark 3.1.13.** In Section 3.4, Section 3.5, and Section 3.6, we will fix the dimension to be one for simplicity, but we emphasize that our results hold *independently* of the dimension.

## 3.2 Notation

### 3.2.1 Index of Notation

At the end of the chapter, we include Table 3.1 as a notational guide for the various symbols which appear in this chapter. In this table, we either provide a definition of the notation or a reference for where the symbol is defined. When definitions for these objects may have appeared in the introduction, we will give references to where they first appear in subsequent sections.

## 3.3 Preliminaries

### 3.3.1 Weak Poisson Structures and Hamiltonian Systems

The classical notion of Poisson structure, as can be found in [60], is ill-suited outside the Hilbert or Banach manifold setting due to the fact that for a given smooth, locally convex manifold $M$, not every functional in $C^\infty(M, \mathbb{R})$, the space of smooth, real-valued functionals on $M$, need admit a Hamiltonian vector field. Since we will need to work with Fréchet manifolds, an alternative theory is needed. We opt for the notion of a *weak Poisson structure* due to Neeb et al. [69].

We recall that a unital subalgebra $\mathcal{A} \subseteq C^\infty(M; \mathbb{R})$ contains constant functions and is closed under pointwise multiplication.

**Definition 3.3.1** (Weak Poisson manifold)**.** A *weak Poisson structure* on $M$ is a unital subalgebra $\mathcal{A} \subset C^\infty(M; \mathbb{R})$ and a bilinear map $\{\cdot, \cdot\} : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ satisfying the following properties:

(P1) The bilinear map $\{\cdot,\cdot\}$, is a Lie bracket and satisfies the Leibnitz rule

$$\{F, GH\} = \{F, G\}H + G\{F, H\}, \qquad \forall F, G, H \in \mathcal{A}. \tag{3.3.1}$$

We call $\{\cdot,\cdot\}$ a *Poisson bracket.*

(P2) For all $m \in M$ and $v \in T_m M$ satisfying $dF[m](v) = 0$ for all $F \in \mathcal{A}$, we have that $v = 0$.

(P3) For every $H \in \mathcal{A}$, there exists a smooth vector field $X_H$ on $M$ satisfying

$$X_H F = \{F, H\}, \qquad \forall F \in \mathcal{A}.^{10} \tag{3.3.2}$$

We call $X_H$ the *Hamiltonian vector field* associated to $H$.

If properties (P1) - (P3) are satisfied, then we call the triple $(M, \mathcal{A}, \{\cdot,\cdot\})$ a *weak Poisson manifold.*

We now record some observations from [69] about the definition of a weak Poisson structure.

**Remark 3.3.2.** (P2) implies that the Hamiltonian vector field $X_H$ associated to some $H \in \mathcal{A}$ is uniquely determined by the relation

$$\{F, H\}(m) = (X_H F)(m) = dF[m](X_H(m)), \qquad \forall F \in \mathcal{A}. \tag{3.3.3}$$

[10] In the left-hand side of identity (3.3.2), we use the notation $X_H$ to denote the vector field identified as a derivation.

Indeed, if $X_{H,1}$ and $X_{H,2}$ are two smooth vector fields satisfying the preceding relation, then the smooth vector $\widetilde{X}_H \coloneqq X_{H,1} - X_{H,2}$ satisfies

$$dF[m](\widetilde{X}_H(m)) = 0, \qquad \forall F \in \mathcal{A}, \tag{3.3.4}$$

for all $m \in M$, which by (P2) implies that $\widetilde{X}_H \equiv 0$.

**Remark 3.3.3.** For all $F, G, H \in \mathcal{A}$, we have that

$$\begin{aligned}
[X_F, X_G]H &= \{\{H, G\}, F\} - \{\{H, F\}, G\} \\
&= \{H, \{G, F\}\} \\
&= X_{\{G,F\}}H.
\end{aligned} \tag{3.3.5}$$

Hence, by Remark 3.3.2, $[X_F, X_G] = X_{\{G,F\}}$ for $F, G \in \mathcal{A}$. Additionally, the Leibnitz rule for $\{\cdot, \cdot\}$ implies the identity

$$X_{FG} = FX_G + GX_F, \qquad \forall F, G \in \mathcal{A}. \tag{3.3.6}$$

**Remark 3.3.4.** If $\mathcal{A} \subset C^\infty(M; \mathbb{R})$ is a unital sub-algebra which satisfies properties (P1) and (P2) of Definition 3.3.1, then (3.3.6) implies that the subspace

$$\{H \in \mathcal{A} : X_H \text{ exists as in (P3)}\} \tag{3.3.7}$$

is a sub-algebra of $\mathcal{A}$ with respect to pointwise product. Hence, it suffices to verify property (P3) for a generating subset $\mathcal{A}_0 \subset \mathcal{A}$.

We note that unlike in the finite-dimensional setting, a symplectic form $\omega : V \times V \to \mathbb{R}$ on an infinite-dimensional locally convex space $V$ need not represent every continuous linear functional via $\omega(\cdot, v)$, for some $v \in V$. If the form does satisfy such a Riesz-representation-type condition, we call a symplectic form $\omega$ *strong*, otherwise, we call $\omega$ *weak*. Analogously, a 2-form $\omega$ on a smooth locally convex manifold $M$ is strong (resp. weak) if all forms $\omega_p : T_pM \times T_pM \to \mathbb{R}$, for $p \in M$, are strong (resp. weak).

**Definition 3.3.5** (Weak symplectic manifold)**.** Let $M$ be a smooth locally convex manifold, and let $\mathcal{X}(M)$ denote smooth vector fields on $M$. A *weak symplectic manifold* is a pair $(M, \omega)$ consisting of a smooth manifold $M$ and a closed non-degenerate 2-form $\omega$ on $M$.

Given a weak symplectic manifold, we denote the Lie algebra of Hamiltonian vector fields on $M$ by

$$\operatorname{ham}(M, \omega) := \{X \in \mathcal{X}(M) : \exists H \in C^\infty(M; \mathbb{R}) \text{ s.t. } \omega(X, \cdot) = dH\}. \tag{3.3.8}$$

Similarly, we denote the larger Lie algebra of symplectic vector fields on $M$ by

$$\operatorname{sp}(M, \omega) := \{X \in \mathcal{M} : \mathcal{L}_X \omega = 0\}, \tag{3.3.9}$$

where $\mathcal{L}_X$ denotes the Lie derivative with respect to the vector field $X$.

With this definition in hand, we see that one has the desired implication analogous to the finite dimensional setting, namely that weak symplectic manifolds canonically lead to weak Poisson manifolds.

**Remark 3.3.6** (Weak symplectic $\Rightarrow$ weak Poisson)**.** Let $(M, \omega)$ be a weak symplectic manifold. Let

$$\mathcal{A} := \{H \in C^\infty(M; \mathbb{R}) : \exists X_H \in \mathcal{X}(M) \text{ s.t. } \omega(X_H, \cdot) = dH\}, \tag{3.3.10}$$

then

$$\{\cdot,\cdot\} : \mathcal{A} \times \mathcal{A} \to \mathcal{A}, \qquad \{F, G\} \coloneqq \omega(X_F, X_G) = dF[X_G] = X_G F \tag{3.3.11}$$

defines a Poisson bracket on $\mathcal{A}$ satisfying properties (P1) and (P3). If we additionally have that for each $m \in \mathcal{M}$ and all $v \in T_m M$, the condition

$$\omega(X(m), v) = 0, \qquad \forall X \in \mathrm{ham}(M, \omega) \tag{3.3.12}$$

implies that $v = 0$, then property (P2) is also satisfied. Consequently, the triple $(M, \mathcal{A}, \{\cdot,\cdot\})$ is a weak Poisson manifold.

We now turn to mappings between weak Poisson manifolds which preserve the Poisson structures. This leads to the notion of a Poisson mapping, alternatively Poisson morphism.

**Definition 3.3.7** (Poisson map)**.** Let $(M_j, \mathcal{A}_j, \{\cdot,\cdot\}_j)$, for $j = 1, 2$, be weak Poisson manifolds. We say that a smooth map $\varphi : M_1 \to M_2$ is a *Poisson map*, or *morphism of Poisson manifolds*, if $\varphi^* \mathcal{A}_2 \subset \mathcal{A}_1$ and

$$\varphi^*\{F, G\}_2 = \{\varphi^* F, \varphi^* G\}_1, \qquad \forall F, G \in \mathcal{A}_2. \tag{3.3.13}$$

**Remark 3.3.8.** In [69], the authors define a Poisson morphism

$$\varphi : (M_1, \mathcal{A}_1, \{\cdot,\cdot\}_1) \to (M_2, \mathcal{A}_2, \{\cdot,\cdot\}_2)$$

with the requirement that $\varphi^* \mathcal{A}_2 = \mathcal{A}_1$. We drop this requirement in our Definition 3.3.7.

As an example, we demonstrate that the Schwartz space $\mathcal{S}(\mathbb{R}^k)$ is a weak, but not strong, symplectic manifold. The following analysis also holds for the bosonic Schwartz space $\mathcal{S}_s(\mathbb{R}^k)$ *mutatis mutandis*, which will be important for our applications in the sequel.

We equip the space $\mathcal{S}(\mathbb{R}^k)$ with a real pre-Hilbert inner product by defining

$$\langle f|g\rangle_{\mathrm{Re}} \coloneqq 2\,\mathrm{Re}\bigg\{\int_{\mathbb{R}^k} d\underline{x}_k \overline{f(\underline{x}_k)} g(\underline{x}_k)\bigg\}. \tag{3.3.14}$$

The operator $J : \mathcal{S}(\mathbb{R}^k) \to \mathcal{S}(\mathbb{R}^k)$ defined by $J(f) \coloneqq if$ defines an almost complex structure on $(\mathcal{S}(\mathbb{R}^k), \langle\cdot|\cdot\rangle_{\mathrm{Re}})$, leading to the *standard $L^2$ symplectic form*

$$\omega_{L^2}(f,g) \coloneqq \langle Jf|g\rangle_{\mathrm{Re}} = 2\,\mathrm{Im}\bigg\{\int_{\mathbb{R}^k} d\underline{x}_k \overline{f(\underline{x}_k)} g(\underline{x}_k)\bigg\}, \qquad \forall f,g \in \mathcal{S}(\mathbb{R}^k). \tag{3.3.15}$$

**Proposition 3.3.9.** $(\mathcal{S}(\mathbb{R}^k), \omega_{L^2})$ *is a weak symplectic manifold.*

*Proof.* $\mathcal{S}(\mathbb{R}^k)$ is trivially a smooth manifold modeled on itself. Moreover, it is evident from its definition that $\omega_{L^2}$ is bilinear, alternating, and closed. To see that $\omega_{L^2}$ is non-degenerate, let $f \in \mathcal{S}(\mathbb{R}^k)$ and suppose that

$$\omega_{L^2}(f,g) = 0 \qquad \forall g \in \mathcal{S}(\mathbb{R}^k). \tag{3.3.16}$$

It then follows tautologically that $\mathrm{Im}\{\langle f|g\rangle\} = 0$. Replacing $g$ by $ig$, we obtain that $\mathrm{Re}\{\langle f|g\rangle\} = 0$, which implies that $\langle f|f\rangle = 0$, hence $f = 0$. $\square$

Now given a functional $F \in C^\infty(\mathcal{S}(\mathbb{R}^k);\mathbb{R})$, the Gâteaux derivative $dF[f]$ at the point $f \in \mathcal{S}(\mathbb{R}^k)$ defines a tempered distribution. We consider the case when $dF[f]$ can be identified with a Schwartz function via the inner product $\langle\cdot|\cdot\rangle_{\mathrm{Re}}$. The next lemma follows by the Lebesgue lemma[11] and the same argument used to prove non-degeneracy in Proposition 3.3.9.

[11] We use the name Lebesgue lemma to refer to the result that if $u_1, u_2$ are two locally integrable functions such that $u_1 = u_2$ in distribution, then $u_1 = u_2$ point-wise almost everywhere.

**Lemma 3.3.10** (Uniqueness of gradient)**.** *Let $F \in C^\infty(\mathcal{S}(\mathbb{R}^k);\mathbb{R})$ and $f \in \mathcal{S}(\mathbb{R}^k)$. Suppose that there exist $g_1, g_2 \in \mathcal{S}(\mathbb{R}^k)$ such that*

$$\langle g_1 | \delta f \rangle_{\mathrm{Re}} = dF[f](\delta f) = \langle g_2 | \delta f \rangle_{\mathrm{Re}}, \qquad \forall \delta f \in \mathcal{S}(\mathbb{R}^k). \tag{3.3.17}$$

*Then $g_1 = g_2$.*

**Definition 3.3.11** (Real $L^2$ gradient)**.** We define the *real $L^2$ gradient* of $F \in C^\infty(\mathcal{S}(\mathbb{R}^k);\mathbb{R})$ at the point $f \in \mathcal{S}(\mathbb{R}^k)$, denoted by $\boldsymbol{\nabla} F(f)$, to be the unique element of $\mathcal{S}(\mathbb{R}^k)$ (if it exists) such that

$$dF[f](\delta f) = \langle \boldsymbol{\nabla} F(f) | \delta f \rangle_{\mathrm{Re}}, \qquad \forall \delta f \in \mathcal{S}(\mathbb{R}^k). \tag{3.3.18}$$

We say that $F$ has a real $L^2$ gradient if $\boldsymbol{\nabla} F : \mathcal{S}(\mathbb{R}^k) \to \mathcal{S}(\mathbb{R}^k)$ is a smooth map.

**Remark 3.3.12.** Since the Hamiltonian vector field of $X_F$, if it exists, is defined by the relation

$$dF[f](\delta f) = \omega_{L^2}(X_F(f), \delta f), \tag{3.3.19}$$

and since $X_F$ is unique by the fact that $\mathcal{S}(\mathbb{R}^k)$ is dense in $\mathcal{S}'(\mathbb{R}^k)$, we see that $X_F(f) = -i\boldsymbol{\nabla} F(f)$. In the sequel, we will use the notation $\boldsymbol{\nabla}_s F \coloneqq X_F$, which we refer to as the *symplectic $L^2$ gradient.*

We now use Remark 3.3.6 to show that the symplectic form $\omega_{L^2}$, which we recall is defined in (1.3.1), canonically induces an $L^2$ Poisson structure on $\mathcal{S}(\mathbb{R}^k)$.

**Proposition 3.3.13.** *Define a subset $A_{\mathcal{S}} \subset C^\infty(\mathcal{S}(\mathbb{R}^k);\mathbb{R})$ by*

$$\mathcal{A}_{\mathcal{S}} \coloneqq \left\{ H \,:\, \boldsymbol{\nabla}_s H \in C^\infty(\mathcal{S}(\mathbb{R}^k); \mathcal{S}(\mathbb{R}^k)) \right\}, \tag{3.3.20}$$

*and define a bracket $\{\cdot,\cdot\}_{L^2}$ on $\mathcal{A}_{\mathcal{S}} \times \mathcal{A}_{\mathcal{S}}$ by*

$$\{F, G\}_{L^2} \coloneqq \omega_{L^2}(\boldsymbol{\nabla}_s F, \boldsymbol{\nabla}_s G). \tag{3.3.21}$$

*Then $(\mathcal{S}(\mathbb{R}^k), \mathcal{A}_{\mathcal{S}}, \{\cdot,\cdot\}_{L^2})$ is a weak Poisson manifold.*

*Proof.* By Remark 3.3.6, we only need to check that for every fixed $g \in \mathcal{S}(\mathbb{R}^k)$, the condition

$$\omega_{L^2}(X(f), g) = 0, \qquad \forall X \in \mathrm{ham}(\mathcal{S}(\mathbb{R}^k), \omega_{L^2}) \tag{3.3.22}$$

implies that $g = 0 \in \mathcal{S}(\mathbb{R}^k)$. Since $\mathrm{ham}(\mathcal{S}(\mathbb{R}^k), \omega_{L^2})$ contains the constant vector fields $X(\cdot) \equiv f_0$, for any fixed $f_0 \in \mathcal{S}(\mathbb{R}^k)$, we see that by taking $X(f) \coloneqq ig$ for all $f \in \mathcal{S}(\mathbb{R}^k)$, that the condition (3.3.22) implies that

$$0 = \omega(ig, g) = -2\,\mathrm{Im}\left\{\int_{\mathbb{R}^k} d\underline{x}_k \overline{(ig)}(\underline{x}_k) g(\underline{x}_k)\right\} = 2\|g\|_{L^2(\mathbb{R}^k)}^2. \tag{3.3.23}$$

Hence, $g = 0$, completing the proof. $\square$

### 3.3.2 Some Lie Algebra Facts

In this subsection, we collect some facts about Lie algebras for easy referencing. We outline a canonical construction of a Poisson structure on the dual of a Lie algebra, which is known as a *Lie-Poisson structure.* Furthermore, we will outline a construction of hierarchies of Lie algebras which will serve as an inspiration for our construction of the Lie algebra $\mathfrak{G}_\infty$. We refer the reader to [60, 59] for more background and details.

We begin by recording the definition of a Lie algebra for subsequent reference in our proofs.

**Definition 3.3.14** (Lie algebra)**.** A *Lie algebra* is a locally convex space $\mathfrak{g}$ over the field $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ together with a separately continuous binary operation $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ called the *Lie bracket*, which satisfies the following properties:

(L1) $[\cdot,\cdot]$ is bilinear.

(L2) $[x,x] = 0$ for all $x \in \mathfrak{g}$.

(L3) $[\cdot,\cdot]$ satisfies the *Jacobi identity*

$$[x,[y,z]] + [z,[x,y]] + [y,[z,x]] = 0 \tag{3.3.24}$$

for all $x, y, z \in \mathfrak{g}$.

**Remark 3.3.15.** Usually (see, for instance, [72]), a Lie bracket is required to be continuous, as opposed to separately continuous. We drop this requirement in this work, due to functional analytic difficulties.

**Definition 3.3.16** (Nondegenerate pairings)**.** Let $V$ and $W$ be topological vector spaces over the field $\mathbb{F}$, and let

$$\langle\cdot|\cdot\rangle : V \times W \to \mathbb{F}$$

be a bilinear pairing between $V$ and $W$. We say that the pairing is *$V$-nondegenerate* (respectively, *$W$-nondegenerate*) if the map $V \to W^*, x \mapsto \langle x|\cdot\rangle$ (respectively, $W \to V^*, y \mapsto \langle\cdot|y\rangle$) is an isomorphism. If the pairing is both $V$- and $W$-nondegenerate, then we say that the pairing is *nondegenerate*.

**Definition 3.3.17** (dual space $\mathfrak{g}^*$)**.** Let $(\mathfrak{g}, [\cdot,\cdot])$ be a Lie algebra. We say that a topological vector $\mathfrak{g}^*$ is a *dual space* to $\mathfrak{g}$ if there exists a pairing $\langle\cdot|\cdot\rangle : \mathfrak{g}\times\mathfrak{g}^* \to \mathbb{F}$ which is nondegenerate.

**Example 3.3.18.** If $\mathfrak{g}$ is a reflexive Fréchet space, for instance the Schwartz space $\mathcal{S}(\mathbb{R}^d)$, then taking $\mathfrak{g}^*$ to be the topological dual of $\mathfrak{g}$ equipped with the strong dual topology, the standard duality pairing

$$\mathfrak{g} \times \mathfrak{g}^* \to \mathbb{F} : \langle x|\varphi\rangle = \varphi(x)$$

is nondegenerate.

A consequence of the existence of a dual space $\mathfrak{g}^*$ for a Lie algebra $\mathfrak{g}$ is the existence of functional derivatives, which is crucial to proving that the Lie-Poisson bracket in Proposition 3.3.20 below is well-defined.

**Lemma 3.3.19** (Existence of functional derivatives)**.** *Let* $\mathfrak{g}$ *be a Lie algebra, and let* $\mathfrak{g}^*$ *be dual to* $\mathfrak{g}$ *with respect to the nondegenerate pairing* $\langle\cdot|\cdot\rangle_{\mathfrak{g}-\mathfrak{g}^*}$. *For any functional* $F \in C^1(\mathfrak{g}^*;\mathbb{F})$, *there exists a unique element* $\frac{\delta F}{\delta \mu} \in \mathfrak{g}$ *such that*

$$\left\langle \frac{\delta F}{\delta \mu} \middle| \delta\mu \right\rangle_{\mathfrak{g}-\mathfrak{g}^*} = dF[\mu](\delta\mu), \qquad \mu, \delta\mu \in \mathfrak{g}^*. \tag{3.3.25}$$

*Proof.* Let $\mu \in \mathfrak{g}^*$. The Gâteaux derivative of $F$ at $\mu$ denoted $dF[\mu]$ and defined in Definition 2.1.4 is a continuous linear functional on $\mathfrak{g}^*$. Hence by the nondegeneracy of the pairing, there exists a unique element $\frac{\delta F}{\delta \mu} \in \mathfrak{g}$ such that

$$\left\langle \frac{\delta F}{\delta \mu} \middle| \delta\mu \right\rangle_{\mathfrak{g}-\mathfrak{g}^*} = dF[\mu][\delta\mu], \qquad \delta\mu \in \mathfrak{g}^*. \qquad \square$$

We now have the necessary ingredients to define the canonical Poisson structure on the dual space $\mathfrak{g}^*$, which we call the *Lie-Poisson* structure, following Marsden and Weinstein [58].

**Proposition 3.3.20** (Lie-Poisson structure)**.** *Let* $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ *be a Lie algebra, such that the Lie bracket is continuous, and let* $\mathfrak{g}^*$ *be dual to* $\mathfrak{g}$ *with respect to the non-degenerate pairing* $\langle\cdot|\cdot\rangle_{\mathfrak{g}-\mathfrak{g}^*}$. *Define the* Lie-Poisson bracket

$$\{\cdot,\cdot\} : C^\infty(\mathfrak{g}^*;\mathbb{F}) \times C^\infty(\mathfrak{g}^*;\mathbb{F}) \to C^\infty(\mathfrak{g}^*;\mathbb{F}) \tag{3.3.26}$$

*by*

$$\{F, G\}(\mu) \coloneqq \left\langle \left[\frac{\delta F}{\delta \mu}, \frac{\delta G}{\delta \mu}\right]_{\mathfrak{g}} \middle| \mu \right\rangle_{\mathfrak{g}-\mathfrak{g}^*}, \qquad \mu \in \mathfrak{g}^*. \tag{3.3.27}$$

*Then* $(C^\infty(\mathfrak{g}^*;\mathbb{F}), \{\cdot,\cdot\})$ *is a Lie algebra.*

**Remark 3.3.21.** Note that in the statement of Proposition 3.3.20, we require that the Lie bracket $[\cdot,\cdot]_{\mathfrak{g}}$ be continuous, not merely separately continuous as in Definition 3.3.14. Since the Lie brackets we consider in Section 3.4 and Section 3.5 are only separately continuous, we do not use Proposition 3.3.20 directly, and therefore we have omitted the proof of it. We emphasize, though, that the construction of the proposition inspires our constructions in the sequel.

### 3.3.3 Bosonic Functions, Operators and Tensor Products

We denote the symmetric group on $k$ letters by $\mathbb{S}_k$. For a permutation $\pi \in \mathbb{S}_k$, we define the map $\pi : \mathbb{R}^k \to \mathbb{R}^k$ by

$$\pi(\underline{x}_k) \coloneqq (x_{\pi(1)}, \ldots, x_{\pi(k)}). \tag{3.3.28}$$

For a complex-valued, measurable function $f : \mathbb{R}^k \to \mathbb{C}$, we define the map

$$(\pi f)(\underline{x}_k) \coloneqq (f \circ \pi)(\underline{x}_k) = f(x_{\pi(1)}, \ldots, x_{\pi(k)}). \tag{3.3.29}$$

We denote the pairing of a tempered distribution $u \in \mathcal{S}'(\mathbb{R}^k)$ with a Schwartz function $f \in \mathcal{S}(\mathbb{R}^k)$ by

$$\langle u, f \rangle_{\mathcal{S}'(\mathbb{R}^k)-\mathcal{S}(\mathbb{R}^k)}. \tag{3.3.30}$$

Throughout, we will use an integral to represent the pairing of a distribution and a test function. For $1 \leq p \leq \infty$, we use the notation $L^p(\mathbb{R}^k)$ to denote Banach space of $p$-integrable functions with norm $\|\cdot\|_{L^p(\mathbb{R}^k)}$. In particular, when $p = 2$, we denote the $L^2$ inner product by

$$\langle f|g \rangle := \int_{\mathbb{R}^k} d\underline{x}_k \overline{f(\underline{x}_k)} g(\underline{x}_k). \tag{3.3.31}$$

Note that we use the physicist's convention that the inner product is complex linear in the second entry. Similarly, for $u \in \mathcal{S}'(\mathbb{R}^k)$ and $f \in \mathcal{S}(\mathbb{R}^k)$, we use the notation $\langle u|f \rangle$ to denote

$$\langle u|f \rangle := \overline{\langle u, \bar{f} \rangle_{\mathcal{S}'(\mathbb{R}^k)-\mathcal{S}(\mathbb{R}^k)}}. \tag{3.3.32}$$

Alternatively, the right-hand side may be taken as the definition of the tempered distribution $\bar{u}$.

**Definition 3.3.22.** We say that a measurable function $f : \mathbb{R}^k \to \mathbb{C}$ is *symmetric* or *bosonic* if

$$\pi(f) = f \tag{3.3.33}$$

for all permutations $\pi \in \mathbb{S}_k$.

**Definition 3.3.23.** We define the *symmetrization operator* $\mathrm{Sym}_k$ on the space of measurable complex-valued functions by

$$\mathrm{Sym}_k(f)(\underline{x}_k) := \frac{1}{k!} \sum_{\pi \in \mathbb{S}_k} (\pi f)(\underline{x}_k). \tag{3.3.34}$$

By duality, we can extend the symmetrization operator to $\mathcal{S}'(\mathbb{R}^k)$.

**Definition 3.3.24** (Symmetric Schwartz space)**.** For $k \in \mathbb{N}$, let $\mathcal{S}_s(\mathbb{R}^k)$ denote the subspace of $\mathcal{S}(\mathbb{R}^k)$ consisting of Schwartz functions $f$ with the property that

$$f(x_{\pi(1)}, \ldots, x_{\pi(k)}) = f(\underline{x}_k), \qquad (\underline{x}_k) \in \mathbb{R}^k \tag{3.3.35}$$

for all permutations $\pi \in \mathbb{S}_k$.

**Definition 3.3.25** (Symmetric tempered distribution)**.** We say that a tempered distribution $u \in \mathcal{S}'(\mathbb{R}^k)$ is *symmetric* or *bosonic* if for all permutations $\pi \in \mathbb{S}_k$,

$$\langle u, \pi g \rangle_{\mathcal{S}'(\mathbb{R}^k) - \mathcal{S}(\mathbb{R}^k)} = \langle u, g \rangle_{\mathcal{S}'(\mathbb{R}^k), \mathcal{S}(\mathbb{R}^k)}, \tag{3.3.36}$$

for all $g \in \mathcal{S}(\mathbb{R}^k)$. We denote the subspace of symmetric tempered distributions by $\mathcal{S}_s'(\mathbb{R}^k)$.

**Remark 3.3.26.** It is straightforward to check that $\mathrm{Sym}_k$ is a continuous operator $\mathcal{S}(\mathbb{R}^k) \to \mathcal{S}_s(\mathbb{R}^k)$ and $\mathcal{S}'(\mathbb{R}^k) \to \mathcal{S}_s'(\mathbb{R}^k)$. Furthermore, a measurable function $f$ is bosonic if and only if $f = \mathrm{Sym}_k(f)$.

**Lemma 3.3.27.** *We have the identification*

$$\mathcal{S}_s'(\mathbb{R}^k) \cong (\mathcal{S}_s(\mathbb{R}^k))'. \tag{3.3.37}$$

*Proof.* Let $\ell \in (\mathcal{S}_s(\mathbb{R}^k))'$. For all $f \in \mathcal{S}_s(\mathbb{R}^k)$, we have that

$$\ell(f) = \ell(\pi(f)), \qquad \pi \in \mathbb{S}_k. \tag{3.3.38}$$

Hence,

$$\ell(f) = \frac{1}{k!} \sum_{\pi \in \mathbb{S}_k} \ell(\pi(f)) = \ell(\mathrm{Sym}_k(f)). \tag{3.3.39}$$

Since $\mathrm{Sym}_k$ is a continuous linear operator on $\mathcal{S}(\mathbb{R}^k)$, it follows that $\ell \circ \mathrm{Sym}_k \in \mathcal{S}'(\mathbb{R}^k)$. Since $\mathrm{Sym}_k(\pi(f)) = \mathrm{Sym}_k(f)$ for any permutation $\pi \in \mathbb{S}_k$, it follows that $\ell \circ \mathrm{Sym}_k$ is permutation invariant, hence an element of $\mathcal{S}_s'(\mathbb{R}^k)$. $\square$

Given two locally convex spaces $E$ and $F$, we denote the space of continuous linear maps $E \to F$ by $\mathcal{L}(E; F)$. We topologize $\mathcal{L}(E; F)$ with the topology of bounded convergence. For our purposes, we will typically have $E, F \in \{\mathcal{S}(\mathbb{R}^k), \mathcal{S}_s(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k), \mathcal{S}_s'(\mathbb{R}^k)\}$.

**Remark 3.3.28.** In the special case where $E = F = \mathcal{S}(\mathbb{R}^k)$, we will write $\tilde{\mathcal{L}}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}(\mathbb{R}^k))$ to denote the vector space $\mathcal{L}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}(\mathbb{R}^k))$ equipped with the subspace topology induced by $\mathcal{L}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k))$. The same statement holds with the Schwartz space replaced by the bosonic Schwartz space.

In the case that $E = \mathcal{S}(\mathbb{R}^d)$ and $F = \mathcal{S}'(\mathbb{R}^d)$, the bounded topology is generated by the seminorms

$$\|A\|_{\mathfrak{R}} := \sup_{f,g \in \mathfrak{R}} |\langle Af, g \rangle_{\mathcal{S}'(\mathbb{R}^d)-\mathcal{S}(\mathbb{R}^d)}|, \qquad \forall A \in \mathcal{L}(\mathcal{S}(\mathbb{R}^d), \mathcal{S}'(\mathbb{R}^d)), \tag{3.3.40}$$

where $\mathfrak{R}$ ranges over the bounded subsets of $\mathcal{S}(\mathbb{R}^d)$. An identical statement holds with all spaces replaced by their symmetric counterparts. We topologize $\mathcal{S}'(\mathbb{R}^N)$ with the *strong dual topology*, which is the locally convex topology generated by the seminorms of the form

$$\|f\|_B := \sup_{\varphi \in B} \left| \int_{\mathbb{R}^N} d\underline{x}_N f(\underline{x}_N) \varphi(\underline{x}_N) \right|, \tag{3.3.41}$$

where $B$ ranges over the family of all bounded subsets of $\mathcal{S}(\mathbb{R}^N)$. Note that since $\mathcal{S}(\mathbb{R}^N)$ is a Montel space, bounded subsets are precompact. An identical statement holds with all spaces replaced by their symmetric counterparts.

**Definition 3.3.29** (Symmetric wave functions)**.** For $k \in \mathbb{N}$, let $L_s^2(\mathbb{R}^k)$ denote the subspace of $L^2(\mathbb{R}^k)$ consisting of functions $f$ which are bosonic a.e.

For $A \in \mathcal{L}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k))$ and $\tau \in \mathbb{S}_k$, we define

$$A_{(\tau(1),\ldots,\tau(k))} \coloneqq \tau \circ A \circ \tau^{-1}. \tag{3.3.42}$$

**Definition 3.3.30.** Given $A \in \mathcal{L}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k))$, we define its *bosonic symmetrization* $\mathrm{Sym}_k(A)$ by

$$\mathrm{Sym}_k(A) \coloneqq \frac{1}{k!} \sum_{\pi \in \mathbb{S}_k} A_{(\pi(1),\ldots,\pi(k))}. \tag{3.3.43}$$

**Definition 3.3.31** (Bosonic operators)**.** Let $k \in \mathbb{N}$. We say that an operator $A : \mathcal{S}(\mathbb{R}^k) \to \mathcal{S}'(\mathbb{R}^k)$ is *bosonic* or *permutation invariant* if $A$ maps $\mathcal{S}_s(\mathbb{R}^k)$ into $\mathcal{S}_s'(\mathbb{R}^k)$.

The analogue of Remark 3.3.26 holds for the symmetrization of operators in that symmetrized operators are indeed operators on the bosonic Schwartz space.

**Lemma 3.3.32.** *Let $k \in \mathbb{N}$. If $A^{(k)} \in \mathcal{L}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k))$, then*

$$\mathrm{Sym}_k(A^{(k)}) \in \mathcal{L}(\mathcal{S}_s(\mathbb{R}^k), \mathcal{S}_s'(\mathbb{R}^k)). \tag{3.3.44}$$

*Proof.* It suffices to show that for any $k$-particle operator $A^{(k)} \in \mathcal{L}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k))$ and any permutation $\sigma \in \mathbb{S}_k$, it holds that

$$\int_{\mathbb{R}^k} d\underline{x}_k \big(\mathrm{Sym}_k(A^{(k)})f\big)(\underline{x}_k) g(\sigma^{-1}(\underline{x}_k)) = \int_{\mathbb{R}^k} d\underline{x}_k \big(\mathrm{Sym}_k(A^{(k)})f\big)(\underline{x}_k) g(\underline{x}_k) \tag{3.3.45}$$

for all $f \in \mathcal{S}_s(\mathbb{R}^k)$ and for all $g \in \mathcal{S}(\mathbb{R}^k)$. To this end, observe that

$$\begin{aligned}
&\int_{\mathbb{R}^k} d\underline{x}_k \big(\mathrm{Sym}_k(A^{(k)})f\big)(\underline{x}_k) g(x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(k)}) \\
&= \int_{\mathbb{R}^k} d\underline{x}_k \left(\frac{1}{k!} \sum_{\pi \in \mathbb{S}_k} \left(A^{(k)}_{(\pi(1),\ldots,\pi(k))} f\right)(\underline{x}_k)\right) g(x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(k)}).
\end{aligned} \tag{3.3.46}$$

By definition (3.3.42), we have

$$A^{(k)}_{(\pi(1),\dots,\pi(k))} f = \pi A^{(k)}(\pi^{-1} f). \tag{3.3.47}$$

Therefore,

$$\begin{aligned}
&\frac{1}{k!}\sum_{\pi\in\mathbb{S}_k}\int_{\mathbb{R}^k} d\underline{x}_k \Big(A^{(k)}_{(1,\dots,k)}(\pi^{-1}f)\Big)(x_{\pi(1)},\dots,x_{\pi(k)})g(x_{\sigma^{-1}(1)},\dots,x_{\sigma^{-1}(k)})\\
&=\frac{1}{k!}\sum_{\pi\in\mathbb{S}_k}\int_{\mathbb{R}^k} d\underline{x}_k \big(A^{(k)}(\pi^{-1}f)\big)(\underline{x}_k)g(x_{\pi^{-1}\sigma^{-1}(1)},\dots,x_{\pi^{-1}\sigma^{-1}(k)})\\
&=\frac{1}{k!}\sum_{\pi\in\mathbb{S}_k}\int_{\mathbb{R}^k} d\underline{x}_k \big(A^{(k)}f\big)(\underline{x}_k)g(x_{\pi^{-1}\sigma^{-1}(1)},\dots,x_{\pi^{-1}\sigma^{-1}(k)}),
\end{aligned} \tag{3.3.48}$$

where, recalling (3.3.29), the second line follows from a change of variable and the third line follows from the assumption that $f$ is symmetric with respect to permutation of the coordinates. Since for any fixed $\sigma \in \mathbb{S}_k$, $\pi \mapsto \pi^{-1}\sigma^{-1}$ defines a bijection of the group $\mathbb{S}_k$, it follows from a change of summation index that

$$\begin{aligned}
&\frac{1}{k!}\sum_{\pi\in\mathbb{S}_k}\int_{\mathbb{R}^k} d\underline{x}_k \big(A^{(k)}f\big)(\underline{x}_k)g(x_{\pi^{-1}\sigma^{-1}(1)},\dots,x_{\pi^{-1}\sigma^{-1}(k)})\\
&=\frac{1}{k!}\sum_{\tilde{\pi}\in\mathbb{S}_k}\int_{\mathbb{R}^k} d\underline{x}_k \big(A^{(k)}f\big)(\underline{x}_k)g(x_{\tilde{\pi}(1)},\dots,x_{\tilde{\pi}(k)})\\
&=\frac{1}{k!}\sum_{\tilde{\pi}\in\mathbb{S}_k}\int_{\mathbb{R}^k} d\underline{x}_k \big(A^{(k)}(\tilde{\pi}f)\big)(x_{\tilde{\pi}^{-1}(1)},\dots,x_{\tilde{\pi}^{-1}(k)})g(\underline{x}_k)\\
&=\int_{\mathbb{R}^k} d\underline{x}_k \big(\mathrm{Sym}_k(A^{(k)})f\big)(\underline{x}_k)g(\underline{x}_k),
\end{aligned} \tag{3.3.49}$$

where the penultimate line follows from the assumption that $f$ is symmetric and a change of variable. This concludes the proof. $\square$

The following technical lemma will be useful in the sequel. For definitions and discussion of the generalized trace, see Definition 3.2.1.

**Lemma 3.3.33.** *Let $k \in \mathbb{N}$, and let $\gamma^{(k)} \in \mathcal{L}(\mathcal{S}'_s(\mathbb{R}^k), \mathcal{S}_s(\mathbb{R}^k))$ and $A^{(k)} \in \mathcal{L}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k))$. Then for any permutation $\tau \in \mathbb{S}_k$, we have that*

$$\mathrm{Tr}_{1,\ldots,k}\Big(A^{(k)}_{(\tau(1),\ldots,\tau(k))}\gamma^{(k)}\Big) = \mathrm{Tr}_{1,\ldots,k}\big(A^{(k)}\gamma^{(k)}\big). \tag{3.3.50}$$

*Proof.* Let $\tau \in \mathbb{S}_k$. Now let

$$\gamma^{(k)} = \sum_{j=1}^{\infty} \lambda_j \left|f_j\right\rangle \left\langle g_j\right| \tag{3.3.51}$$

be a decomposition for $\gamma^{(k)}$, where $\sum_{j=1}^{\infty} |\lambda_j| \leq 1$, and $\{f_j\}_{j=1}^{\infty}, \{g_j\}_{j=1}^{\infty}$ are sequences tending to zero in $\mathcal{S}_s(\mathbb{R}^k)$. In particular, the partial sums

$$\sum_{j=1}^{N} \lambda_j \left|f_j\right\rangle \left\langle g_j\right| \xrightarrow[N\to\infty]{} \gamma^{(k)} \text{ in } \mathcal{L}(\mathcal{S}'_s(\mathbb{R}^k), \mathcal{S}_s(\mathbb{R}^k)). \tag{3.3.52}$$

Since the map

$$\mathrm{Tr}_{1,\ldots,k}\Big(A^{(k)}_{(\tau(1),\ldots,\tau(k)}\cdot\Big) : \mathcal{L}(\mathcal{S}'(\mathbb{R}^k), \mathcal{S}(\mathbb{R}^k)) \to \mathbb{C}, \tag{3.3.53}$$

is continuous and the inclusion $\mathcal{S}_s(\mathbb{R}^k) \subset \mathcal{S}(\mathbb{R}^k)$ is trivially continuous, it follows that

$$\begin{aligned}
\mathrm{Tr}_{1,\ldots,k}\Big(A^{(k)}_{(\tau(1),\ldots,\tau(k))}\gamma^{(k)}\Big) &= \lim_{N\to\infty} \mathrm{Tr}_{1,\ldots,k}\left(A^{(k)}_{(\tau(1),\ldots,\tau(k))}\left(\sum_{j=1}^{N} \lambda_j \left|f_j\right\rangle \left\langle g_j\right|\right)\right) \\
&= \lim_{N\to\infty} \sum_{j=1}^{N} \lambda_j \, \mathrm{Tr}_{1,\ldots,k}\Big(A^{(k)}_{(\tau(1),\ldots,\tau(k))}(\left|f_j\right\rangle \left\langle g_j\right|)\Big) \\
&= \lim_{N\to\infty} \sum_{j=1}^{N} \lambda_j \left\langle g_j \middle| A^{(k)}_{(\tau(1),\ldots,\tau(k))} f_j \right\rangle.
\end{aligned} \tag{3.3.54}$$

Since $f_j$ and $g_j$ are both bosonic, we have by definition of the notation $A^{(k)}_{(\tau(1),\ldots,\tau(k))}$ in (3.3.42) that

$$\left\langle g_j \middle| A^{(k)}_{(\tau(1),\ldots,\tau(k))} f_j \right\rangle = \left\langle \tau^{-1}(g_j) \middle| A^{(k)}(\tau^{-1}(f_j)) \right\rangle = \left\langle g_j \middle| A^{(k)} f_j \right\rangle, \qquad \forall j \in \mathbb{N}. \tag{3.3.55}$$

Therefore,

$$\begin{aligned}
\lim_{N\to\infty}\sum_{j=1}^{N}\lambda_j\left\langle g_j\middle|A^{(k)}_{(\tau(1),\ldots,\tau(k))}f_j\right\rangle &= \lim_{N\to\infty}\sum_{j=1}^{N}\lambda_j\left\langle g_j\middle|A^{(k)}f_j\right\rangle \\
&= \lim_{N\to\infty}\mathrm{Tr}_{1,\ldots,k}\Bigg(A^{(k)}\bigg(\sum_{j=1}^{N}\lambda_j\left|f_j\right\rangle\left\langle g_j\right|\bigg)\Bigg) \\
&= \mathrm{Tr}_{1,\ldots,k}\big(A^{(k)}\gamma^{(k)}\big), \qquad (3.3.56)
\end{aligned}$$

where in order to obtain the ultimate equality, we again use the continuity of the functional $\mathrm{Tr}_{1,\ldots,k}\big(A^{(k)}\cdot\big)$ and the convergence of the partial sums. $\square$

We define the usual contraction operator $B_{i;j}$ appearing in the literature on derivation of quantum many-body systems.

**Definition 3.3.34** (The contractions operator $B_{i;j}$)**.** Let $k\in\mathbb{N}$. For integers $1\leq i,j\leq k$ with $i\neq j$, we define the continuous linear operators operators

$$B^{\pm}_{i;j}:\mathcal{L}(\mathcal{S}'(\mathbb{R}^{k+1}),\mathcal{S}(\mathbb{R}^{k+1}))\to\mathcal{L}(\mathcal{S}'(\mathbb{R}^{k}),\mathcal{S}(\mathbb{R}^{k})) \qquad (3.3.57)$$

by defining the Schwartz kernel of $B^{+}_{i;j}(\gamma^{(k+1)})$ by the formula

$$B^{+}_{i;j}(\gamma^{(k+1)})(\underline{x}_k;\underline{x}'_k)\coloneqq\int_{\mathbb{R}}dy\delta(x_i-y)\gamma^{(k+1)}(\underline{x}_{1;j-1},y,\underline{x}_{j;k};\underline{x}'_{1;j-1},y,\underline{x}'_{j;k}),$$

for all $(\underline{x}_k,\underline{x}'_k)\in\mathbb{R}^{2k}$. Similarly, we define the Schwartz kernel of $B^{-}_{i;j}(\gamma^{(k+1)})$ by the formula

$$B^{-}_{i;j}(\gamma^{(k+1)})(\underline{x}_k;\underline{x}'_k)\coloneqq\int_{\mathbb{R}}dy\delta(x'_i-y)\gamma^{(k+1)}(\underline{x}_{1;j-1},y,\underline{x}_{j;k};\underline{x}'_{1;j-1},y,\underline{x}'_{j;k}),$$

for all $(\underline{x}_k,\underline{x}'_k)\in\mathbb{R}^{2k}$ We define the continuous linear operator

$$B_{i;j}:\mathcal{L}(\mathcal{S}'_s(\mathbb{R}^{k+1}),\mathcal{S}_s(\mathbb{R}^{k+1}))\to\mathcal{L}(\mathcal{S}'_s(\mathbb{R}^{k}),\mathcal{S}_s(\mathbb{R}^{k}))$$

by

$$B_{i;j} \coloneqq B^{+}_{i;j} - B^{-}_{i;j}. \tag{3.3.58}$$

Given two locally convex spaces $E$ and $F$, we denote an[12] algebraic tensor product of $E$ and $F$ consisting of finite linear combinations

$$\sum_{j=1}^{n} \lambda_j e_j \otimes f_j, \qquad e_j \in E, \;\; f_j \in F \tag{3.3.59}$$

by $E \otimes F$. We note that since the spaces we deal with in this chapter are nuclear, the topologies of the injective and projective tensor products coincide. Hence, we can unambiguously write $E \hat{\otimes} F$ to denote the completion of $E \otimes F$ under either of the aforementioned topologies.

Given locally convex spaces $E_j$ and $F_j$ for $j = 1, 2$ and linear maps $T : E_1 \to E_2$ and $S : F_1 \to F_2$, and a tensor product

$$B : E_1 \times E_2 \to E_1 \otimes E_2, \tag{3.3.60}$$

the notation $T \otimes S$ denotes the unique linear map $T \otimes S : E_1 \otimes F_1 \to E_2 \times F_2$ such that

$$(T \otimes S) \circ B = T \times S. \tag{3.3.61}$$

Note that the existence of such a unique map is guaranteed by the universal property of the tensor product.

When $E$ and $F$ are subspaces of measurable functions on $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, and $e \in E$ and $f \in F$, we let $e \otimes f$ denote the function

$$e \otimes f : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{C}, \qquad (e \otimes f)(\underline{x}_m; \underline{x}'_n) \coloneqq e(\underline{x}_m) f(\underline{x}'_n), \tag{3.3.62}$$

[12]The reader will recall that the algebraic tensor product is only defined up to unique isomorphism.

which induces a bilinear map $E \times F \to E \otimes F$. Similarly, if $E'$ and $F'$ are the duals of spaces of test functions $E$ and $F$, for instance $E' = \Delta'(\mathbb{R}^m)$ and $F' = \Delta'(\mathbb{R}^n)$, we let $u \otimes v$ denote the unique distribution satisfying

$$(u \otimes v)(e \otimes f) = u(e) \cdot v(f). \tag{3.3.63}$$

Finally, if $\phi : \mathbb{R}^m \to \mathbb{C}$ is a measurable function, we use the notation $\phi^{\otimes k}$, for $k \in \mathbb{N}$, to denote the measurable function $\phi^{\otimes k} : \mathbb{R}^{mk} \to \mathbb{C}$ defined by

$$\phi^{\otimes k}(\underline{x}_{m,1}, \ldots, \underline{x}_{m,k}) \coloneqq \prod_{\ell=1}^{k} \phi(\underline{x}_{m,\ell}), \tag{3.3.64}$$

and we use the notation $\phi^{\times k}$ to denote the measurable function $\phi^{\times k} : \mathbb{R}^m \to \mathbb{C}^k$

$$\phi^{\times k}(\underline{x}_m) \coloneqq (\phi(\underline{x}_m), \ldots, \phi(\underline{x}_m)). \tag{3.3.65}$$

## 3.4 Geometric Structure for the $N$-Body Problem

In this section we establish proofs of the results stated in Section 3.1.1.

### 3.4.1 Lie Algebra $\mathfrak{G}_N$ of Finite Hierarchies of Quantum Observables

We begin by defining a Lie algebra $\mathfrak{g}_k$ of $k$-body observables. We have some freedom to choose our definition of this Lie algebra, provided that our choice is large enough to include the Hamiltonian of the $N$-body problem yet small enough so that operations such as composition and taking adjoints are well-defined. We find that continuous linear maps from the bosonic Schwartz space to itself forms a convenient choice.

For $k \in \mathbb{N}$, define

$$\mathfrak{g}_k \coloneqq \{A^{(k)} \in \tilde{\mathcal{L}}(\mathcal{S}_s(\mathbb{R}^k), \mathcal{S}_s(\mathbb{R}^k)) : (A^{(k)})^* = -A^{(k)}\}, \tag{3.4.1}$$

where we recall that $\tilde{\mathcal{L}}(\mathcal{S}_s(\mathbb{R}^k), \mathcal{S}_s'(\mathbb{R}^k))$ is defined in Remark 3.3.28. Let

$$[\cdot,\cdot]_{\mathfrak{g}_k} : \mathfrak{g}_k \times \mathfrak{g}_k \to \mathfrak{g}_k$$

be the usual commutator bracket scaled by a factor of $k$:

$$[A,B]_{\mathfrak{g}_k} := k[A,B] = k(AB - BA). \tag{3.4.2}$$

Note that the commutator is well-defined since the space $\mathcal{L}(\mathcal{S}_s(\mathbb{R}^k), \mathcal{S}_s(\mathbb{R}^k))$ is closed under composition. We refer to the elements of $\mathfrak{g}_k$ as *$k$-body observables.*

The first goal of this subsection is to verify that $(\mathfrak{g}_k, [\cdot,\cdot]_{\mathfrak{g}_k})$ is a Lie algebra in the sense of Definition 3.3.14. Namely, we prove the following proposition.

**Proposition 3.4.1.** $(\mathfrak{g}_k, [\cdot,\cdot]_{\mathfrak{g}_k})$ *is a Lie algebra in the sense of Definition 3.3.14*

*Proof.* That $[\cdot,\cdot]_{\mathfrak{g}_k}$ is algebraically a Lie bracket is immediate from the fact that the commutator satisfies properties (L1), (L2), and (L3). Therefore, it remains to verify that the commutator is separately continuous with respect to the topology on $\mathfrak{g}_k$. By symmetry, it suffices to show that for fixed $A^{(k)} \in \mathfrak{g}_k$, the map $B^{(k)} \mapsto A^{(k)}B^{(k)}$ is continuous on $\tilde{\mathcal{L}}(\mathcal{S}_s(\mathbb{R}^k), \mathcal{S}_s(\mathbb{R}^k))$, which amounts to showing that for any bounded subset $\mathfrak{R} \subset \mathcal{S}_s(\mathbb{R}^k)$, there exists a bounded subset $\tilde{\mathfrak{R}} \subset \mathcal{S}_s(\mathbb{R}^k)$, such that

$$\sup_{f,g\in\mathfrak{R}} \left|\left\langle g \middle| A^{(k)}B^{(k)}f \right\rangle\right| \lesssim \sup_{f,g\in\tilde{\mathfrak{R}}} \left|\left\langle g \middle| B^{(k)}f \right\rangle\right|. \tag{3.4.3}$$

Now note that $\left\langle g \middle| A^{(k)}B^{(k)}f \right\rangle = \left\langle (A^{(k)})^* g \middle| B^{(k)}f \right\rangle$. Since $(A^{(k)})^* = -A^{(k)}$, it follows from the continuity of $A^{(k)}$ that $(A^{(k)})^*(\mathfrak{R})$ it a bounded subset of $\mathcal{S}_s(\mathbb{R}^k)$. Choosing $\tilde{\mathfrak{R}} = \mathfrak{R} \cup (A^{(k)})^*(\mathfrak{R})$ completes the proof. $\square$

We next introduce some combinatorial notation used frequently in the sequel. For $N \in \mathbb{N}$ and $k \in \mathbb{N}_{\leq N}$, let $P_k^N$ denote the collection of $k$-tuples $(j_1, \ldots, j_k)$ with $k$ distinct elements drawn from the set $\mathbb{N}_{\leq N}$. Given an element $(j_1, \ldots, j_k) \in P_k^N$, let $(m_1, \ldots, m_{N-k})$ denote the increasing arrangement of $\mathbb{N}_{\leq N} \setminus \{j_1, \ldots, j_k\}$. We denote by $\pi_{j_1 \cdots j_k} \in \mathbb{S}_N$ the permutation

$$\pi(a) \coloneqq \begin{cases} i, & a = j_i \text{ for } i \in \mathbb{N}_{\leq k} \\ k+i, & a = m_i \text{ for } i \in \mathbb{N}_{\leq N-k} \end{cases}. \tag{3.4.4}$$

Our first lemma defines a continuous linear map $\epsilon_{k,N}$ which allows us to regard a $k$-particle observable as an $N$-particle observable. This map $\epsilon_{k,N}$ is crucial to the definition of the Lie bracket between two observable $N$-hierarchies and by duality, to the Poisson bracket of two density matrix $N$-hierarchies.

For $A^{(k)} \in \mathcal{L}(\mathcal{S}_s(\mathbb{R}^k), \mathcal{S}_s(\mathbb{R}^k))$, $N \in \mathbb{N}$ with $1 \leq k \leq N$, and $(j_1, \ldots, j_k) \in P_k^N$ we can define the operator

$$A_{(j_1, \ldots, j_k)}^{(k)} \in \mathcal{L}(\mathcal{S}_s(\mathbb{R}^N), \mathcal{S}(\mathbb{R}^N)) \tag{3.4.5}$$

which acts only on the variables $\{j_1, \ldots, j_k\}$ by defining

$$A_{(1, \ldots, k)}^{(k)} = A^{(k)} \otimes Id_{N-k}$$

and setting

$$A_{(j_1, \ldots, j_k)}^{(k)} = \pi_{j_1 \cdots j_k}^{-1} \circ A_{(1, \ldots, k)}^{(k)} \circ \pi_{j_1 \cdots j_k}. \tag{3.4.6}$$

We establish some properties of such operators, which we call $k$-particle extensions, in Proposition 3.3.1. These $k$-particle extensions are used to define a map $\epsilon_{k,N}$. We will show first, in the following lemma, that $\varepsilon_{k,N}$ have the desired mapping properties, and then subsequently that the $\epsilon_{k,N}$ are injective, and hence they are proper embeddings of the space $\mathfrak{g}_k$ into $\mathfrak{g}_N$.

**Remark 3.4.2.** Although $A^{(k)}$ is a priori only defined on the proper subspace $\mathcal{S}_s(\mathbb{R}^k) \subset \mathcal{S}(\mathbb{R}^k)$, this operator admits an extension to the space $\mathcal{S}(\mathbb{R}^k)$ since we may always consider $A^{(k)} \circ \mathrm{Sym}_k$. We agree going forward to abuse notation by identifying $A^{(k)}$ with this extension. Consequently, we may regard $A^{(k)}_{(j_1,\ldots,j_k)} \in \mathcal{L}(\mathcal{S}(\mathbb{R}^N), \mathcal{S}(\mathbb{R}^N))$. As the reader will see, though, all our constructions are independent of the choice of extension.

**Lemma 3.4.3.** *For integers $1 \leq k \leq N$, there is a continuous linear map*

$$\epsilon_{k,N} : \mathcal{L}(\mathcal{S}_s(\mathbb{R}^k), \mathcal{S}_s'(\mathbb{R}^k)) \to \mathcal{L}(\mathcal{S}_s(\mathbb{R}^N), \mathcal{S}_s'(\mathbb{R}^N)) \tag{3.4.7}$$

*defined by*

$$\epsilon_{k,N}(A^{(k)}) \coloneqq C_{k,N} \sum_{(j_1,\ldots,j_k) \in P_k^N} A^{(k)}_{(j_1,\ldots,j_k)}, \tag{3.4.8}$$

*where*

$$C_{k,N} \coloneqq \left(k! \binom{N}{k}\right)^{-1} = \frac{1}{N \cdots (N-k+1)}.^{13} \tag{3.4.9}$$

*Moreover, if $A^{(k)} \in \mathcal{L}(\mathcal{S}_s(\mathbb{R}^k), \mathcal{S}_s(\mathbb{R}^k))$, then $\epsilon_{k,N}(A^{(k)}) \in \mathcal{L}(\mathcal{S}_s(\mathbb{R}^N), \mathcal{S}_s(\mathbb{R}^N))$, and if $A^{(k)}$ is skew-adjoint, then $\epsilon_{k,N}(A^{(k)})$ is skew-adjoint. In particular, $\epsilon_{k,N}(\mathfrak{g}_k) \subset \mathfrak{g}_N$.*

*Proof.* Fix $1 \leq k \leq N$. From Proposition 3.3.1, it follows that if $A^{(k)} \in \mathcal{L}(\mathcal{S}_s(\mathbb{R}^k), \mathcal{S}_s'(\mathbb{R}^k))$, then $\epsilon_{k,N}(A^{(k)})$ as given in (3.4.8) is a well-defined element of $\mathcal{L}(\mathcal{S}_s(\mathbb{R}^N), \mathcal{S}_s'(\mathbb{R}^N))$ and the map $\epsilon_{k,N}$ is linear. Furthermore, it follows from Lemma 3.3.2 that skew-adjointness is preserved. So it remains for us to show that

$$\epsilon_{k,N}(\mathcal{L}(\mathcal{S}_s(\mathbb{R}^k), \mathcal{S}_s(\mathbb{R}^k))) \subset \mathcal{L}(\mathcal{S}_s(\mathbb{R}^N), \mathcal{S}_s(\mathbb{R}^N)) \tag{3.4.10}$$

and that $\epsilon_{k,N}$ is continuous.

[13] Note that $C_{k,N} = 1/|P_k^N|$.

- Consider the assertion (3.4.10). By properties of tensor product and the continuity of $A^{(k)}$, it follows that $A^{(k)}_{(1,\ldots,k)} = A^{(k)}\hat{\otimes} Id_{N-k}$ is a continuous map of $\mathcal{S}_s(\mathbb{R}^k) \otimes \mathcal{S}(\mathbb{R}^{N-k})$ to itself, and hence that

$$A^{(k)}_{(j_1,\ldots,j_k)} : \mathcal{S}_s(\mathbb{R}^N) \rightarrow \mathcal{S}(\mathbb{R}^N)$$

is a continuous map follows directly from (3.4.6). We thus need to show that $\epsilon_{k,N}(A^{(k)})(f)$ is bosonic.

Let $\pi \in \mathbb{S}_N$. It is straightforward from the definition of $A^{(k)}_{(j_1,\ldots,j_k)}$ and (3.3.29) that, for any test function $f \in \mathcal{S}_s(\mathbb{R}^N)$, we have

$$\pi A^{(k)}_{(j_1,\ldots,j_k)}(f) = A^{(k)}_{(\pi(j_1),\ldots,\pi(j_k))}(\pi f) = A^{(k)}_{(\pi(j_1),\ldots,\pi(j_k))}(f), \tag{3.4.11}$$

where the ultimate equality follows from $f$ being bosonic. Since $\mathbb{S}_N$ induces a left group action on $P^N_k$, it follows that

$$\sum_{(j_1,\ldots,j_k)\in P^N_k} A^{(k)}_{(j_1,\ldots,j_k)} = \sum_{(j_1,\ldots,j_k)\in P^N_k} A^{(k)}_{(\pi(j_1),\ldots,\pi(j_k))} \tag{3.4.12}$$

on $\mathcal{S}_s(\mathbb{R}^k)$, which implies together with (3.4.11) that

$$\pi\epsilon_{k,N}(A^{(k)})(f) = C_{k,N} \sum_{(j_1,\ldots,j_k)\in P^N_k} \pi A^{(k)}_{(j_1,\ldots,j_k)}(f) = \epsilon_{k,N}(A^{(k)})(f), \tag{3.4.13}$$

as desired.

- Now we will prove the assertion that $\epsilon_{k,N}$ is continuous. Let $\mathfrak{R}_N$ be a bounded subset of $\mathcal{S}_s(\mathbb{R}^N)$. We need to show that there exists a bounded subset $\mathfrak{R}_k \subset \mathcal{S}_s(\mathbb{R}^k)$ such that

$$\sup_{f^{(N)},g^{(N)}\in\mathfrak{R}_N} \left|\left\langle g^{(N)}\middle|\epsilon_{k,N}(A^{(k)})f^{(N)}\right\rangle\right| \lesssim \sup_{f^{(k)},g^{(k)}\in\mathfrak{R}_k} \left|\left\langle g^{(k)}\middle|A^{(k)}f^{(k)}\right\rangle\right|. \tag{3.4.14}$$

Using the fact that there are finitely many terms in the definition of $\epsilon_{k,N}$ and that the finite union of bounded subsets is again a bounded subset, it suffices to show that, for $\mathfrak{R}_N$ as above and any tuple $(j_1, \ldots, j_k) \in P_k^N$, there exists a bounded subset $\mathfrak{R}_{(j_1,\ldots,j_k)} \subset \mathcal{S}(\mathbb{R}^k)$, such that

$$\sup_{f^{(N)}, g^{(N)} \in \mathfrak{R}_N} \left| \left\langle g^{(N)} \middle| A_{(j_1,\ldots,j_k)}^{(k)} f^{(N)} \right\rangle \right| \lesssim \sup_{f^{(k)}, g^{(k)} \in \mathfrak{R}_{(j_1,\ldots,j_k)}} \left| \left\langle g^{(k)} \middle| A^{(k)} f^{(k)} \right\rangle \right|, \tag{3.4.15}$$

since then the desired bounded subset $\mathfrak{R}_k \subset \mathcal{S}_s(\mathbb{R}^k)$ is obtained by taking

$$\mathfrak{R}_k \coloneqq \mathrm{Sym}_k \left( \bigcup_{\underline{j}_k \in P_k^N} \mathfrak{R}_{(j_1,\ldots,j_k)} \right).$$

Now (3.4.15) is a consequence of the fact that

$$\mathcal{L}(\mathcal{S}_s(\mathbb{R}^k), \mathcal{S}_s'(\mathbb{R}^k)) \mapsto \mathcal{L}(\mathcal{S}_s(\mathbb{R}^k) \hat{\otimes} \mathcal{S}(\mathbb{R}^{N-k}), \mathcal{S}'(\mathbb{R}^N)), \qquad A^{(k)} \mapsto A^{(k)} \otimes Id_{N-k} \tag{3.4.16}$$

is continuous, (3.4.6), and the fact that for any $\underline{j}_k \in P_k^N$, the map $\pi_{j_1 \cdots j_k}$ defined by (3.4.4) and duality is a continuous endomorphism of $\mathcal{S}'(\mathbb{R}^N)$.

□

We next show that the maps $\epsilon_{k,N}$ are injective. This property is crucial as we will ultimately construct our Lie bracket on the hierarchy algebra by embedding elements of the sequence into the ambient algebra $\mathfrak{g}_N$, taking the bracket in $\mathfrak{g}_N$, and then identifying the output as an embedded element of $\mathfrak{g}_k$, for some $k \in \mathbb{N}_{\leq N}$.

**Lemma 3.4.4** (Injectivity of $\epsilon_{k,N}$). *For integers $1 \leq k \leq N$, the map $\epsilon_{k,N} : \mathfrak{g}_k \to \mathfrak{g}_N$ is injective. Consequently, $\epsilon_{k,N}$ has a well-defined inverse on its image, which we denote by $\epsilon_{k,N}^{-1}$.*

*Proof.* Fix $1 \leq k \leq N$. We will show the contrapositive statement: if $A^{(k)} \neq 0$, then $\epsilon_{k,N}(A^{(k)}) \neq 0$.

We introduce a parameter $n \in \mathbb{N}_0$, with $n < k$. We say that $A^{(k)}$ has *property* $\mathbf{P}_n$ if the following holds: there exists $f, g_1, \ldots, g_{k-n} \in \mathcal{S}(\mathbb{R})$ such that

$$A^{(k)}\left(\mathrm{Sym}_k\left(f^{\otimes k-n} \otimes \bigotimes_{a=1}^{n} g_a\right)\right) \neq 0, \tag{3.4.17}$$

where the tensor product is understood as vacuous when $n = 0$. We define the integer $n_{\min}$ by

$$n_{\min} \coloneqq \max\{\min\{n \in \mathbb{N}_{<k} : A^{(k)} \text{ has property } P_n\}, k\}.^{14} \tag{3.4.18}$$

Note that we must have $n_{\min} < k$, else, by definition of property $\mathbf{P}_n$, we would then have that for all $g_1, \ldots, g_k \in \mathcal{S}(\mathbb{R})$,

$$A^{(k)}(\mathrm{Sym}_k(g_1 \otimes \cdots \otimes g_k)) = 0. \tag{3.4.19}$$

By linearity and continuity of $A^{(k)}$ together with density of finite linear combinations of symmetric pure tensors in $\mathcal{S}_s(\mathbb{R}^k)$, (3.4.19) implies that $A^{(k)} \equiv 0$, which is a contradiction.

To avoid notation confusion, we first dispense with the trivial case $n_{\min} = 0$. The definition of property $\mathbf{P}_0$ implies that there exists an element $f \in \mathcal{S}(\mathbb{R})$ such that $A^{(k)}(f^{\otimes k}) \neq 0$. It then follows trivially from the definition of each summand $A^{(k)}_{(j_1,\ldots,j_k)}$ in the definition of $\epsilon_{k,N}(A^{(k)})$ that

$$\epsilon_{k,N}(A^{(k)})(f^{\otimes N}) \neq 0 \in \mathcal{S}_s'(\mathbb{R}^N). \tag{3.4.20}$$

[14] We adopt the convention that the minimum of the empty set is $\infty$, and therefore we take the maximum with $k$ to ensure that $n_{\min}$ is finite.

We now consider the case $1 \leq n_{\min} < k$. The definition of property $\mathbf{P}_{n_{\min}}$ implies that there exist elements $f, g_1, \ldots, g_{n_{\min}} \in \mathcal{S}(\mathbb{R})$ such that

$$A^{(k)}\left(\mathrm{Sym}_k\left(f^{\otimes k-n_{\min}} \otimes \bigotimes_{a=1}^{n_{\min}} g_a\right)\right) \neq 0 \in \mathcal{S}_s'(\mathbb{R}^k). \tag{3.4.21}$$

Define an element $h^{(N)} \in \mathcal{S}_s(\mathbb{R}^N)$ by

$$h^{(N)} := \mathrm{Sym}_N\left(f^{\otimes k-n_{\min}} \otimes (\bigotimes_{a=1}^{n_{\min}} g_a) \otimes f^{\otimes N-k}\right). \tag{3.4.22}$$

We claim that $\epsilon_{k,N}(A^{(k)})(h^{(N)}) \neq 0 \in \mathcal{S}_s'(\mathbb{R}^N)$. Indeed, unpacking the definition of $\epsilon_{k,N}(A^{(k)})$ and $\mathrm{Sym}_N$, we have

$$\epsilon_{k,N}(A^{(k)})(h^{(N)}) = C_{k,N} \sum_{\underline{j}_k \in P_k^N} A^{(k)}_{(j_1,\ldots,j_k)}\left(\sum_{\pi \in \mathbb{S}_N} \pi(f^{\otimes k-n_{\min}} \otimes (\bigotimes_{a=1}^{n_{\min}} g_a) \otimes f^{\otimes N-k})\right). \tag{3.4.23}$$

We first examine the interior sum. For each $\underline{j}_k \in P_k^N$, we can partition $\mathbb{S}_N$ into the sets

$$\mathbb{S}_{\underline{j}_k,r} := \{\pi \in \mathbb{S}_N : |\{\pi(k-n_{\min}+1), \ldots, \pi(k)\} \cap \{j_1, \ldots, j_k\}| = r\} \tag{3.4.24}$$

for $r = 0, \ldots, n_{\min}$. We write

$$\sum_{\pi \in \mathbb{S}_N} \pi(f^{\otimes k-n_{\min}} \otimes (\bigotimes_{a=1}^{n_{\min}} g_a) \otimes f^{\otimes N-k}) = \sum_{r=0}^{n_{\min}} \sum_{\pi \in \mathbb{S}_{\underline{j}_k,r}} \pi(f^{\otimes k-n_{\min}} \otimes (\bigotimes_{a=1}^{n_{\min}} g_a) \otimes f^{\otimes N-k}). \tag{3.4.25}$$

By symmetry considerations, we may suppose that $(j_1, \ldots, j_k) = (1, \ldots, k)$. It is a short counting argument that for each $r \in \{0, \ldots, n_{\min}\}$, we have that

$$\begin{aligned}&\sum_{\pi \in \mathbb{S}_{(1,\ldots,k),r}} \pi(f^{\otimes k-n_{\min}} \otimes (\bigotimes_{a=1}^{n_{\min}} g_a) \otimes f^{\otimes N-k}) \\ &= C(k,n_{\min},r,N) \sum_{\underline{\ell}_{n_{\min}} \in P_{n_{\min}}^{n_{\min}}} \mathrm{Sym}_k\left(f^{\otimes k-r} \otimes \bigotimes_{a=1}^{r} g_{\ell_a}\right) \otimes \mathrm{Sym}_{N-k}\left((\bigotimes_{a=r+1}^{n_{\min}} g_{\ell_a}) \otimes f^{\otimes N-n_{\min}-k+r}\right),\end{aligned} \tag{3.4.26}$$

where $C(k, n_{\min}, r, N)$ is another combinatorial factor depending on the data $(k, n_{\min}, r, N)$. Each term

$$\mathrm{Sym}_k\left(f^{\otimes k-r} \otimes \bigotimes_{a=1}^{r} g_{\ell_a}\right) \otimes \mathrm{Sym}_{N-k}\left( (\bigotimes_{a=r+1}^{n_{\min}} g_{\ell_a}) \otimes f^{\otimes N-n_{\min}-k+r}\right) \tag{3.4.27}$$

is an element of $\mathcal{S}_s(\mathbb{R}^k)\hat{\otimes}\mathcal{S}_s(\mathbb{R}^{N-k})$, and therefore (3.4.27) belongs to the domain of $A^{(k)}_{(1,\ldots,k)}$. Now by definition of $n_{\min}$, we have that for each $r \in \{0, \ldots, n_{\min}-1\}$ that

$$\begin{aligned}
&A^{(k)}_{(1,\ldots,k)}\left(\mathrm{Sym}_k\left(f^{\otimes k-r} \otimes \bigotimes_{a=1}^{r} g_{\ell_a}\right) \otimes \mathrm{Sym}_{N-k}\left( (\bigotimes_{a=r+1}^{n_{\min}} g_{\ell_a}) \otimes f^{\otimes N-n_{\min}-k+r}\right)\right) \\
&= A^{(k)}\left(\mathrm{Sym}_k(f^{\otimes k-r} \otimes \bigotimes_{a=1}^{r} g_{\ell_a})\right) \otimes \mathrm{Sym}_{N-k}\left( (\bigotimes_{a=r+1}^{n_{\min}} g_{\ell_a}) \otimes f^{\otimes N-n_{\min}-k+r}\right) \\
&= 0 \in \mathcal{S}'_s(\mathbb{R}^k)\hat{\otimes}\mathcal{S}_s(\mathbb{R}^{N-k}).
\end{aligned}$$

When $r = n_{\min}$, we have that

$$\begin{aligned}
&A^{(k)}_{(1,\ldots,k)}\left(\mathrm{Sym}_k(f^{\otimes k-n_{\min}} \otimes \bigotimes_{a=1}^{n_{\min}} g_{\ell_a}) \otimes f^{\otimes N-k})\right) \\
&= A^{(k)}\left(\mathrm{Sym}_k(f^{\otimes k-n_{\min}} \otimes \bigotimes_{a=1}^{n_{\min}} g_{a})\right) \otimes f^{\otimes N-k}
\end{aligned}$$

is a non-zero element of $\mathcal{S}'_s(\mathbb{R}^k)\hat{\otimes}\mathcal{S}_s(\mathbb{R}^{N-k})$ by choice of the elements $f, g_1, \ldots, g_{n_{\min}} \in \mathcal{S}(\mathbb{R})$. Consequently, for a possibly different combinatorial factor $C'(k, N)$, we conclude that

$$\epsilon_{k,N}(A^{(k)})(h^{(N)}) = C(k,N)' \,\mathrm{Sym}_N\left(A^{(k)}\left(\mathrm{Sym}_k(f^{\otimes k-n_{\min}} \otimes \bigotimes_{a=1}^{n_{\min}} g_a)\right) \otimes f^{\otimes N-k}\right) \tag{3.4.28}$$

is a nonzero element of $\mathcal{S}'_s(\mathbb{R}^N)$, completing the proof of the lemma. $\square$

We next show that the bracket $[\cdot,\cdot]_{\mathfrak{g}_N}$ respects the hierarchy in the sense that

$$[\epsilon_{\ell,N}(\mathfrak{g}_\ell), \epsilon_{j,N}(\mathfrak{g}_j)]_{\mathfrak{g}_N} \subset \epsilon_{\min\{\ell+j-1,N\},N}(\mathfrak{g}_{\min\{\ell+j-1,N\}}) \subset \mathfrak{g}_N. \tag{3.4.29}$$

This filtration or gradation property is crucial to our definition of the hierarchy Lie bracket in the sequel.

Before proving Lemma 3.4.7 below, we introduce some contraction and commutator-type notation used in the proof and in the sequel. Consider integers $N \in \mathbb{N}$, $\ell, j \in N_{\leq N}$, $k := \min\{\ell + j - 1, N\}$ and $r \geq 1$ satisfying appropriate conditions. Let $A^{(\ell)} \in \mathcal{L}(\mathcal{S}_s(\mathbb{R}^\ell), \mathcal{S}_s(\mathbb{R}^\ell))$ and $B^{(j)} \in \mathcal{L}(\mathcal{S}_s(\mathbb{R}^j), \mathcal{S}_s(\mathbb{R}^j))$. We define the *r-fold contractions*

$$A^{(\ell)} \circ_r B^{(j)} := A^{(\ell)}_{(1,\ldots,\ell)} \Bigg( \sum_{\underline{\alpha}_r \in P_r^\ell} B^{(j)}_{(\underline{\alpha}_r, \ell+1, \ldots, \ell+j-r)} \Bigg) \in \mathcal{L}(\mathcal{S}_s(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k)) \tag{3.4.30}$$

$$B^{(j)} \circ_r A^{(\ell)} := B^{(j)}_{(1,\ldots,j)} \Bigg( \sum_{\underline{\alpha}_r \in P_r^j} A^{(\ell)}_{(\underline{\alpha}_r, j+1, \ldots, j+\ell-r)} \Bigg) \in \mathcal{L}(\mathcal{S}_s(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k)). \tag{3.4.31}$$

Note that the compositions are well-defined since

$$\sum_{\underline{\alpha}_r \in P_r^\ell} B^{(j)}_{(\underline{\alpha}_r, \ell+1, \ldots, \ell+j-r)} \text{ and } \sum_{\underline{\alpha}_r \in P_r^j} A^{(\ell)}_{(\underline{\alpha}_r, j+1, \ldots, j+\ell-r)} \tag{3.4.32}$$

have targets which are symmetric under permutation in the first $\ell$ and $j$ coordinates, respectively. We then set

$$\left[A^{(\ell)}, B^{(j)}\right]_r := \binom{j}{r} A^{(\ell)} \circ_r B^{(j)} - \binom{\ell}{r} B^{(j)} \circ_r A^{(\ell)}. \tag{3.4.33}$$

The motivation for the combinatorial factors in (3.4.33) will become clear from the proof of Lemma 3.4.7 below.

**Remark 3.4.5.** We may also proceed term-by-term to define (3.4.30) and (3.4.31) by considering an extensions of $A^{(\ell)}$ and $B^{(j)}$ to $\mathcal{L}(\mathcal{S}(\mathbb{R}^\ell), \mathcal{S}(\mathbb{R}^\ell))$ and $\mathcal{L}(\mathcal{S}(\mathbb{R}^j), \mathcal{S}(\mathbb{R}^j))$, so that $A^{(\ell)}_{(1,\ldots,\ell)}$ and $B^{(j)}_{(1,\ldots,j)}$ are then elements of $\mathcal{L}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}(\mathbb{R}^k))$. The choice of extensions is immaterial by the target symmetry of operators with which the extensions are right-composed.

In the sequel, we will need a technical lemma concerning the separate continuity of the binary operation $\circ_r$. The proof of this result is quite similar to that of (the more general) Lemma 3.5.1 below, so we omit the proof.

**Lemma 3.4.6.** *Let $\ell, j, k, N \geq 1$ be integers such that $\ell, j \leq N$ and $\min\{\ell + j - 1, N\} = k$. Let $r$ be an integer such that $r_0 \leq r \leq \min\{\ell, j\}$, where*

$$r_0 \coloneqq \max\{1, \min\{\ell, j\} - (N - \max\{\ell, j\})\}. \tag{3.4.34}$$

*Then the bilinear map*

$$(\cdot) \circ_r (\cdot) : \tilde{\mathcal{L}}(\mathcal{S}(\mathbb{R}^\ell), \mathcal{S}(\mathbb{R}^\ell)) \times \tilde{\mathcal{L}}(\mathcal{S}(\mathbb{R}^j), \mathcal{S}(\mathbb{R}^j)) \to \tilde{\mathcal{L}}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}(\mathbb{R}^k)) \tag{3.4.35}$$

*is separately continuous.*[15]

**Lemma 3.4.7** (Filtration of hierarchy)**.** *Let $N \in \mathbb{N}$ and let $1 \leq \ell, j \leq N$. Then for any $A^{(\ell)} \in \mathfrak{g}_\ell$ and $B^{(j)} \in \mathfrak{g}_j$, there exists a unique $C^{(k)} \in \mathfrak{g}_k$, for $k \coloneqq \min\{\ell + j - 1, N\}$, such that*

$$\left[\epsilon_{\ell,N}(A^{(\ell)}), \epsilon_{j,N}(B^{(j)})\right]_{\mathfrak{g}_N} = \epsilon_{k,N}(C^{(k)}). \tag{3.4.36}$$

[15]We recall that $\tilde{\mathcal{L}}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}(\mathbb{R}^k)$ denotes the space $\mathcal{L}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}(\mathbb{R}^k))$ of continuous linear maps from Schwartz space to itself equipped with the subspace topology induced by $\mathcal{L}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k))$.

*Proof.* By definition,

$$\begin{aligned}
&\left[\epsilon_{\ell,N}(A^{(\ell)}),\epsilon_{j,N}(B^{(j)})\right]_{\mathfrak{g}_N} \\
&= NC_{\ell,N}C_{j,N}\left(\sum_{\underline{m}_\ell\in P_\ell^N} A^{(\ell)}_{(m_1,\ldots,m_\ell)}\left(\sum_{\underline{n}_j\in P_j^N} B^{(j)}_{(n_1,\ldots,n_j)}\right) - \sum_{\underline{n}_j\in P_j^N} B^{(j)}_{(n_1,\ldots,n_j)}\left(\sum_{\underline{m}_\ell\in P_\ell^N} A^{(\ell)}_{(m_1,\ldots,m_\ell)}\right)\right) \\
&= NC_{\ell,N}C_{j,N}\sum_{r=1}^{\min\{\ell,j\}}\left(\sum_{\underline{m}_\ell\in P_\ell^N} A^{(\ell)}_{(m_1,\ldots,m_\ell)}\left(\sum_{\substack{\underline{n}_j\in P_j^N \\ |\{m_1,\ldots,m_\ell\}\cap\{n_1,\ldots,n_j\}|=r}} B^{(j)}_{(n_1,\ldots,n_j)}\right)\right. \\
&\qquad\qquad \left. - \sum_{\underline{n}_j\in P_j^N} B^{(j)}_{(n_1,\ldots,n_j)}\left(\sum_{\substack{\underline{m}_\ell\in P_\ell^N \\ |\{m_1,\ldots,m_\ell\}\cap\{n_1,\ldots,n_j\}|=r}} A^{(\ell)}_{(m_1,\ldots,m_\ell)}\right)\right).
\end{aligned} \tag{3.4.37}$$

Without loss of generality, suppose that $\ell \geq j$. We consider the case $\ell + j - 1 \leq N$. For each integer $1 \leq r \leq j$, we have by the $\mathbb{S}_j$-invariance of the operator $B^{(j)}$ that

$$\sum_{\substack{\underline{n}_j\in P_j^N \\ |\{m_1,\ldots,m_\ell\}\cap\{n_1,\ldots,n_j\}|=r}} B^{(j)}_{(n_1,\ldots,n_j)} = \binom{j}{r}\sum_{\substack{\underline{n}_j\in P_j^N \\ \{n_1,\ldots,n_r\}\subset\{m_1,\ldots,m_\ell\} \\ \{n_{r+1},\ldots,n_j\}\cap\{m_1,\ldots,m_\ell\}=\emptyset}} B^{(j)}_{(n_1,\ldots,n_j)}. \tag{3.4.38}$$

Similarly, by the $\mathbb{S}_\ell$-invariance of the operator $A^{(\ell)}$, we have that

$$\sum_{\substack{\underline{m}_\ell\in P_\ell^N \\ |\{n_1,\ldots,n_j\}\cap\{m_1,\ldots,m_\ell\}|=r}} A^{(\ell)}_{(m_1,\ldots,m_\ell)} = \binom{\ell}{r}\sum_{\substack{\underline{m}_\ell\in P_\ell^N \\ \{m_1,\ldots,m_r\}\subset\{n_1,\ldots,n_j\} \\ \{m_{r+1},\ldots,m_\ell\}\cap\{n_1,\ldots,n_j\}=\emptyset}} A^{(\ell)}_{(m_1,\ldots,m_\ell)}. \tag{3.4.39}$$

Upon relabeling the summation, we see that

$$\begin{aligned}
(3.4.37) = NC_{\ell,N}C_{j,N}\sum_{r=1}^{\min\{\ell,j\}}\sum_{\underline{p}_{\ell+j-r}\in P_{\ell+j-r}^N}&\left(\binom{j}{r}A^{(l)}_{(p_1,\ldots,p_l)}\left(\sum_{\substack{1\leq \ell_1,\ldots,\ell_r\leq \ell \\ |\{\ell_1,\ldots,\ell_r\}|=r}} B^{(j)}_{(p_{\ell_1},\ldots,p_{\ell_r},p_{\ell+1},\ldots,p_{\ell+j-r})}\right)\right. \\
&\left. - \binom{\ell}{r}B^{(j)}_{(p_1,\ldots,p_j)}\left(\sum_{\substack{1\leq j_1,\ldots,j_r\leq j \\ |\{j_1,\ldots,j_r\}|=r}} A^{(\ell)}_{(p_{j_1},\ldots,p_{j_r},p_{j+1},\ldots,p_{j+\ell-r})}\right)\right).
\end{aligned} \tag{3.4.40}$$

If $r = 1$, then the summand of (3.4.40) equals

$$\begin{aligned}
&NC_{\ell,N}C_{j,N} \sum_{\underline{p}_k \in P_k^N} jA^{(\ell)}_{(p_1,\ldots,p_\ell)}\left(\sum_{\alpha=1}^{\ell} B^{(j)}_{(p_\alpha,p_{\ell+1},\ldots,p_k)}\right) - \ell B^{(\ell)}_{(p_1,\ldots,p_\ell)}\left(\sum_{\alpha=1}^{j} A^{(\ell)}_{(p_\alpha,p_{j+1},\ldots,p_k)}\right) \\
&= NC_{\ell,N}C_{j,N} \sum_{\underline{p}_k \in P_k^N} j(A^{(\ell)} \circ_1 B^{(j)})_{(p_1,\ldots,p_k)} - \ell(B^{(j)} \circ_1 A^{(\ell)})_{(p_1,\ldots,p_k)} \\
&= \epsilon_{k,N}\left(\frac{NC_{\ell,N}C_{j,N}}{C_{k,N}} \operatorname{Sym}_k\big(j(A^{(\ell)} \circ_1 B^{(j)}) - \ell(B^{(j)} \circ_1 A^{(\ell)})\big)\right). \qquad (3.4.41)
\end{aligned}$$

Now suppose that $r > 1$. Observe that

$$\begin{aligned}
\sum_{\underline{p}_{\ell+j-r} \in P^N_{\ell+j-r}} &\left(\binom{j}{r} A^{(\ell)}_{(p_1,\ldots,p_\ell)}\left(\sum_{\substack{1 \le \ell_1,\ldots,\ell_r \le \ell \\ |\{\ell_1,\ldots,\ell_r\}|=r}} B^{(j)}_{(p_{\ell_1},\ldots,p_{\ell_r},p_{\ell+1},\ldots,p_{\ell+j-r})}\right)\right. \\
&\left. - \binom{\ell}{r} B^{(j)}_{(p_1,\ldots,p_j)}\left(\sum_{\substack{1 \le j_1,\ldots,j_r \le j \\ |\{j_1,\ldots,j_r\}|=r}} A^{(\ell)}_{(p_{j_1},\ldots,p_{j_r},p_{j+1},\ldots,p_{j+\ell-r})}\right)\right) \qquad (3.4.42)
\end{aligned}$$

cannot be immediately identified as an embedded element of $\mathfrak{g}_k$ because the summation is not over tuples $\underline{p}_k \in P_k^N$. Indeed, we are missing $k - (\ell + j - r) = r - 1$ coordinates. To address this issue, we observe that we can write $\underline{p}_k \in P_k^N$ as $\underline{p}_k = (\underline{p}_{\ell+j-r}, \underline{q}_{r-1})$, where $\underline{p}_{\ell+j-r} \in P^N_{\ell+j-r}$ and

$$\underline{q}_{r-1} \in (\mathbb{N}_{\le N} \setminus \{p_1,\ldots,p_{\ell+j-r}\})^{r-1}, \text{ with } |\{q_1,\ldots,q_{r-1}\}| = r - 1. \qquad (3.4.43)$$

For each $\underline{p}_{\ell+j-r} \in P^N_{\ell+j-r}$, the number of $(r-1)$-cardinality subsets of $\mathbb{N}_{\le N} \setminus \{p_1,\ldots,p_{\ell+j-r}\}$ is

$$\binom{N-\ell-j+r}{r-1}.$$

Since there are $(r-1)!$ ways of permuting $r-1$ distinct elements, we conclude that for

$\underline{p}_{\ell+j-r} \in P^N_{\ell+j-r}$,

$$\begin{aligned}|\{\underline{q}_{r-1} \in (\mathbb{N}_{\leq N} \setminus \{p_1,\ldots,p_{\ell+j-r}\})^{r-1} : |\{q_1,\ldots,q_{r-1}\}| = r-1\}| &= \binom{N-\ell-j+r}{r-1}(r-1)! \\ &= \prod_{m=1}^{r-1}(N-k+m),\end{aligned} \tag{3.4.44}$$

where we use that $\ell + j - 1 = k$. Hence, the summand of (3.4.40) equals

$$\begin{aligned}\frac{NC_{\ell,N}C_{j,N}}{\prod_{m=1}^{r-1}(N-k+m)} \sum_{\underline{p}_k \in P^N_k} &\Bigg(\binom{j}{r} A^{(\ell)}_{(p_1,\ldots,p_\ell)}\Bigg(\sum_{\underline{l}_r \in P^\ell_r} B^{(j)}_{(p_{\ell_1},\ldots,p_{\ell_r},p_{r+1},\ldots,p_{\ell+j-r})}\Bigg) \\ &- \binom{\ell}{r} B^{(j)}_{(p_1,\ldots,p_j)}\Bigg(\sum_{\underline{j}_r \in P^j_r} A^{(\ell)}_{(p_{j_1},\ldots,p_{j_r},p_{j+1},\ldots,p_{j+\ell-r})}\Bigg)\Bigg),\end{aligned} \tag{3.4.45}$$

and by definition, we obtain that this expression equals

$$\epsilon_{k,N}\Bigg(\frac{NC_{\ell,N}C_{j,N}}{C_{k,N}\prod_{m=1}^{r-1}(N-k+m)} \mathrm{Sym}_k\Bigg(\binom{j}{r} A^{(\ell)} \circ_r B^{(j)} - \binom{\ell}{r} B^{(j)} \circ_r A^{(\ell)}\Bigg)\Bigg). \tag{3.4.46}$$

Now suppose that $\ell + j - 1 > N$. Then proceeding as above, we see that $r \geq 1$ must in fact satisfy the lower bound

$$r \geq \min\{\ell, j\} - (N - \max\{\ell, j\}) =: r_0. \tag{3.4.47}$$

Combining these results, we conclude that

$$\begin{aligned}&\left[\epsilon_{\ell,N}(A^{(\ell)}), \epsilon_{j,N}(B^{(j)})\right]_{\mathfrak{g}_N} \\ &= \epsilon_{k,N}\Bigg(\mathrm{Sym}_k\Bigg(\sum_{r=r_0}^{N} \frac{NC_{\ell,N}C_{j,N}}{C_{k,N}\prod_{m=1}^{r-1}(N-k+m)}\Bigg(\binom{j}{r} A^{(\ell)} \circ_r B^{(j)} - \binom{\ell}{r} B^{(j)} \circ_r A^{(\ell)}\Bigg)\Bigg),\end{aligned} \tag{3.4.48}$$

which concludes the proof of the lemma. □

We now have all the technical lemmas needed to define the Lie algebra $\mathfrak{G}_N$ of observable $N$-hierarchies. For $N \in \mathbb{N}$, let $\mathfrak{G}_N$ denote the locally convex direct sum

$$\mathfrak{G}_N := \bigoplus_{k=1}^{N} \mathfrak{g}_k, \tag{3.4.49}$$

where we recall that

$$\mathfrak{g}_k = \{A^{(k)} \in \tilde{\mathcal{L}}(\mathcal{S}_s(\mathbb{R}^k), \mathcal{S}_s(\mathbb{R}^k)) : (A^{(k)})^* = -A^{(k)}\}. \tag{3.4.50}$$

We define a bracket on $A_N = (A_N^{(k)})_{k\in\mathbb{N}_{\leq N}}, B_N = (B_N^{(k)})_{k\in\mathbb{N}_{\leq N}} \in \mathfrak{G}_N$ by

$$[A_N, B_N]_{\mathfrak{G}_N} := C_N = (C_N^{(k)})_{k\in\mathbb{N}_{\leq N}}, \tag{3.4.51}$$

where

$$C_N^{(k)} := \sum_{\substack{1\leq \ell,j\leq N \\ \min\{\ell+j-1,N\}=k}} \epsilon_{k,N}^{-1}\left(\left[\epsilon_{\ell,N}(A_N^{(\ell)}), \epsilon_{j,N}(B_N^{(j)})\right]_{\mathfrak{g}_N}\right). \tag{3.4.52}$$

It remains for us to check that $\mathfrak{G}_N$ together with its bracket is actually a Lie algebra in the sense of Definition 3.3.14, as we have so claimed above. Before doing so, we collect a result which will be useful in the sequel. Namely, that as a byproduct of the proof Lemma 3.4.7, we have the following explicit formula for the Lie bracket $[A_N, B_N]_{\mathfrak{G}_N}$ for two observable $N$-hierarchies, which is quite useful for computations.

**Proposition 3.4.8** (Formula for $[A_N, B_N]_{\mathfrak{G}_N}^{(k)}$). *Let $N \in \mathbb{N}$, and let $A_N = (A_N^{(k)})_{k\in\mathbb{N}_{\leq N}}, B_N = (B_N^{(k)})_{k\in\mathbb{N}_{\leq N}}$ be observable $N$-hierarchies. Then for integers $1 \leq k \leq N$, we have that*

$$[A_N, B_N]_{\mathfrak{G}_N}^{(k)} = \sum_{\substack{1\leq \ell,j\leq N \\ \min\{\ell+j-1,N\}=k}} \mathrm{Sym}_k\left(\sum_{r=r_0}^{\min\{\ell,j\}} C_{\ell jkrN}\left[A_N^{(\ell)}, B_N^{(j)}\right]_r\right), \tag{3.4.53}$$

*where*

$$C_{\ell jkrN} \coloneqq \frac{NC_{\ell,N}C_{j,N}}{C_{k,N}\prod_{m=1}^{r-1}(N-k+m)},^{16} \qquad r_0 \coloneqq \max\{1, \min\{\ell,j\} - (N - \max\{\ell,j\})\}, \tag{3.4.54}$$

*and where* $[\cdot,\cdot]_r$ *is defined in* (3.4.33).

We now establish Proposition 3.1.1, which is our first main result of this section.

**Proposition 3.1.1.** $(\mathfrak{G}_N, [\cdot,\cdot]_{\mathfrak{G}_N})$ *is a Lie algebra in the sense of Definition 3.3.14.*

*Proof of Proposition 3.1.1.* There are two parts to the verification: an algebraic part and an analytic part.

- We first consider the algebraic part, which amounts to checking bilinearity, anti-symmetry, and the Jacobi identity. The first two properties are obvious from the definition of $\mathfrak{G}_N$. For the third property, let $A_N, B_N, C_N \in \mathfrak{G}_N$. We need to show that

$$\left[A_N, [B_N, C_N]_{\mathfrak{G}_N}\right]_{\mathfrak{G}_N} + \left[C_N, [A_N, B_N]_{\mathfrak{G}_N}\right]_{\mathfrak{G}_N} + \left[B_N, [C_N, A_N]_{\mathfrak{G}_N}\right]_{\mathfrak{G}_N} = 0. \tag{3.4.55}$$

  Since $\epsilon_{k,N}$ is injective, it suffices to show that $\epsilon_{k,N}$ applied to the left-hand side of the preceding identity equals the zero element of $\mathfrak{g}_N$. We only present the details when the component index satisfies $1 \leq k < N$ and leave verification of the remaining $k = N$ case as an exercise to the reader. Using the definition of the Lie bracket and bilinearity, we

[16] Recall that $C_{\ell,N} = 1/|P_\ell^N|$.

have the identities

$$\begin{aligned}
\epsilon_{k,N}\Big(\big[A_N,[B_N,C_N]_{\mathfrak{G}_N}\big]^{(k)}_{\mathfrak{G}_N}\Big) &= \sum_{j_1+j_2-1=k}\Big[\epsilon_{j_1,N}(A_N^{(j_1)}),\epsilon_{j_2,N}([B_N,C_N]^{(j_2)}_{\mathfrak{G}_N})\Big]_{\mathfrak{g}_N} \\
&= \sum_{j_1+j_2-1=k}\ \sum_{j_3+j_4-1=j_2}\left[\epsilon_{j_1,N}(A_N^{(j_1)}),\Big[\epsilon_{j_3,N}(B_N^{(j_3)}),\epsilon_{j_4,N}(C_N^{(j_4)})\Big]_{\mathfrak{g}_N}\right]_{\mathfrak{g}_N} \\
&= \sum_{\ell_1+\ell_2+\ell_3=k+2}\left[\epsilon_{\ell_1,N}(A_N^{(\ell_1)}),\Big[\epsilon_{\ell_2,N}(B_N^{(\ell_2)}),\epsilon_{\ell_3,N}(C_N^{(\ell_3)})\Big]_{\mathfrak{g}_N}\right]_{\mathfrak{g}_N},
\end{aligned}$$

$$\begin{aligned}
\epsilon_{k,N}\Big(\big[C_N,[A_N,B_N]_{\mathfrak{G}_N}\big]^{(k)}_{\mathfrak{G}_N}\Big) &= \sum_{j_1+j_2-1=k}\Big[\epsilon_{j_1,N}(C_N^{(j_1)}),\epsilon_{j_2,N}([A_N,B_N]^{(j_2)}_{\mathfrak{G}_N})\Big]_{\mathfrak{g}_N} \\
&= \sum_{j_1+j_2-1=k}\ \sum_{j_3+j_4-1=j_2}\left[\epsilon_{j_1,N}(C_N^{(j_1)}),\Big[\epsilon_{j_3,N}(A_N^{(j_3)}),\epsilon_{j_4,N}(B_N^{(j_4)})\Big]_{\mathfrak{g}_N}\right]_{\mathfrak{g}_N} \\
&= \sum_{\ell_1+\ell_2+\ell_3=k+2}\left[\epsilon_{\ell_3,N}(C_N^{(\ell_3)}),\Big[\epsilon_{\ell_1,N}(A_N^{(\ell_1)}),\epsilon_{\ell_2,N}(B_N^{(\ell_2)})\Big]_{\mathfrak{g}_N}\right]_{\mathfrak{g}_N},
\end{aligned}$$

$$\begin{aligned}
\epsilon_{k,N}\Big(\big[B_N,[C_N,A_N]_{\mathfrak{G}_N}\big]^{(k)}_{\mathfrak{G}_N}\Big) &= \sum_{j_1+j_2-1=k}\Big[\epsilon_{j_1,N}(B_N^{(j_1)}),\epsilon_{j_2,N}([C_N,A_N]^{(j_2)}_{\mathfrak{G}_N})\Big]_{\mathfrak{g}_N} \\
&= \sum_{j_1+j_2-1=k}\ \sum_{j_3+j_4-1=j_2}\left[\epsilon_{j_1,N}(B_N^{(j_1)}),\Big[\epsilon_{j_3,N}(C_N^{(j_3)}),\epsilon_{j_4,N}(A_N^{(j_4)})\Big]_{\mathfrak{g}_N}\right]_{\mathfrak{g}_N} \\
&= \sum_{\ell_1+\ell_2+\ell_3=k+2}\left[\epsilon_{\ell_2,N}(B_N^{(\ell_2)}),\Big[\epsilon_{\ell_3,N}(C_N^{(\ell_3)}),\epsilon_{\ell_1,N}(A_N^{(\ell_1)})\Big]_{\mathfrak{g}_N}\right]_{\mathfrak{g}_N}.
\end{aligned}$$

Since $[\cdot,\cdot]_{\mathfrak{g}_N}$ is a Lie bracket and therefore satisfies the Jacobi identity, it follows that for fixed integers $1 \le \ell_1, \ell_2, \ell_3 \le N$,

$$\begin{aligned}
0 &= \left[\epsilon_{\ell_1,N}(A_N^{(\ell_1)}),\Big[\epsilon_{\ell_2,N}(B_N^{(\ell_2)}),\epsilon_{\ell_3,N}(C_N^{(\ell_3)})\Big]_{\mathfrak{g}_N}\right]_{\mathfrak{g}_N} \\
&\quad + \left[\epsilon_{\ell_3,N}(C_N^{(\ell_3)}),\Big[\epsilon_{\ell_1,N}(A_N^{(\ell_1)}),\epsilon_{\ell_2,N}(B_N^{(\ell_2)})\Big]_{\mathfrak{g}_N}\right]_{\mathfrak{g}_N} \\
&\quad + \left[\epsilon_{\ell_2,N}(B_N^{(\ell_2)}),\Big[\epsilon_{\ell_3,N}(C_N^{(\ell_3)}),\epsilon_{\ell_1,N}(A_N^{(\ell_1)})\Big]_{\mathfrak{g}_N}\right]_{\mathfrak{g}_N}.
\end{aligned} \tag{3.4.56}$$

Hence,

$$\epsilon_{k,N}\Big(\big[A_N,[B_N,C_N]_{\mathfrak{G}_N}\big]^{(k)}_{\mathfrak{G}_N}+\big[C_N,[A_N,B_N]_{\mathfrak{G}_N}\big]^{(k)}_{\mathfrak{G}_N}+\big[B_N,[C_N,A_N]_{\mathfrak{G}_N}\big]^{(k)}_{\mathfrak{G}_N}\Big)=0\in\mathfrak{g}_N. \tag{3.4.57}$$

- We now consider the analytic part, which amounts to checking the separate continuity of $[\cdot,\cdot]_{\mathfrak{G}_N}$. Using the anti-symmetry of the bracket, it suffices to show that for $A_N\in\mathfrak{G}_N$ fixed, the map

$$\mathfrak{G}_N\rightarrow\mathfrak{G}_N,\qquad B_N\mapsto[A_N,B_N]_{\mathfrak{G}_N} \tag{3.4.58}$$

is continuous. Moreover, it suffices to show that for each $k\in\mathbb{N}_{\leq N}$, the map

$$\mathfrak{G}_N\rightarrow\mathfrak{g}_k,\qquad B_N\mapsto[A_N,B_N]^{(k)}_{\mathfrak{G}_N}$$

is continuous.

Let $(B_{N,a})_{a\in\mathsf{A}}$, where $B_{N,a}=(B^{(k)}_{N,a})_{k\in\mathbb{N}_{\leq N}}$, be a net in $\mathfrak{G}_N$ converging to $B_N=(B^{(k)}_N)_{k\in\mathbb{N}_{\leq N}}\in\mathfrak{G}_N$. By the continuity of the projection maps $\mathfrak{G}_N\rightarrow\mathfrak{g}_k$ for each $k\in\mathbb{N}_{\leq N}$, we have that $(B^{(k)}_{N,a})_{a\in\mathsf{A}}$ is a net in $\mathfrak{g}_k$ converging to $B^{(k)}_N\in\mathfrak{g}_k$.

Unpacking the definition of $[A_N,B_{N,a}]^{(k)}_{\mathfrak{G}_N}$ and using the continuity of the $\mathrm{Sym}_k$ operator and the operations of addition and scalar multiplication, together with the fact there are only finitely many terms, it suffices to show that for any integers $1\leq\ell,j\leq N$ satisfying $\min\{\ell+j-1,N\}=k$, any integer $r_0\leq r\leq\min\{\ell,j\}$, we have the net convergence

$$\Big[A^{(\ell)}_N,B^{(j)}_{N,a}\Big]_r\rightarrow\Big[A^{(\ell)}_N,B^{(j)}_N\Big]_r \tag{3.4.59}$$

in $\tilde{\mathcal{L}}(\mathcal{S}_s(\mathbb{R}^k),\mathcal{S}(\mathbb{R}^k))$. But this convergence is a consequence of Lemma 3.4.6, thus completing the proof.

$\square$

### 3.4.2 Lie-Poisson Manifold $\mathfrak{G}_N^*$ of Finite Hierachies of Density Matrices

In this subsection, we define the Lie-Poisson manifold $\mathfrak{g}_N^*$ of $N$-body density matrices and the Lie-Poisson manifold $\mathfrak{G}_N^*$ of density matrix $N$-hierarchies. A good heuristic to keep in mind is that density matrices are dual to skew-adjoint operators. We remind the reader that the superscript $*$ does not denote the literal functional analytic dual of $\mathfrak{g}_N$ (respectively, $\mathfrak{G}_N$) as a topological vector space, but rather a space in weakly non-degenerate pairing with $\mathfrak{g}_N$ (respectively, $\mathfrak{G}_N$).

To begin with, we define the real topological vector space

$$\mathfrak{g}_N^* := \{\Psi_N \in \mathcal{L}(\mathcal{S}_s'(\mathbb{R}^N), \mathcal{S}_s(\mathbb{R}^N)) : \Psi_N^* = \Psi_N\} \tag{3.4.60}$$

endowed with the subspace topology.

**Remark 3.4.9.** Our definition of $\mathfrak{g}_N^*$ is quite natural as it is isomorphic to the strong dual of $\mathfrak{g}_N$. The proof of this fact is quite similar to that of Lemma 3.5.8 shown below.

We now define a suitable unital sub-algebra $\mathcal{A}_{DM,N} \subset C^\infty(\mathfrak{g}_N^*;\mathbb{R})$ of admissible functionals to build a weak Poisson structure for $\mathfrak{g}_N^*$.

**Definition 3.4.10.** Let $\mathcal{A}_{DM,N}$ be the algebra with respect to point-wise product generated by the functionals in

$$\{F \in C^\infty(\mathfrak{g}_N^*;\mathbb{R}) : F(\cdot) = i\,\mathrm{Tr}_{1,\ldots,N}(A^{(N)}\cdot),\ \ A^{(N)} \in \mathfrak{g}_N\} \cup \{F \in C^\infty(\mathfrak{g}_N^*;\mathbb{R}) : F(\cdot) = C \in \mathbb{R}\}. \tag{3.4.61}$$

In words, $\mathcal{A}_{DM,N}$ is the algebra (under point-wise product) generated by the constants and the image of $\mathfrak{g}_N$ under the canonical embedding into $(\mathfrak{g}_N^*)^*$.

We record the following result, whose proof we omit since it is similar to and simpler than that of Proposition 3.1.8, which will be used in Section 3.4.3 below.

**Proposition 3.4.11.** $(\mathfrak{g}_N^*, \mathcal{A}_{DM,N}, \{\cdot,\cdot\}_{\mathfrak{g}_N^*})$ *is a weak Poisson manifold.*

Before proceeding, it will be useful to record the following lemma regarding the dual of $\mathfrak{g}_N^*$. In particular, we note that the dual of $\mathfrak{g}_N^*$ is *not* isomorphic to $\mathfrak{g}_N$.

**Lemma 3.4.12** (Dual of $\mathfrak{g}_N^*$)**.** *The topological dual of $\mathfrak{g}_N^*$, denoted by $(\mathfrak{g}_N^*)^*$ and endowed with the strong dual topology, is isomorphic to*

$$\{A^{(N)} \in \mathcal{L}(\mathcal{S}_s(\mathbb{R}^N), \mathcal{S}_s'(\mathbb{R}^N)) : (A^{(N)})^* = -A^{(N)}\}, \tag{3.4.62}$$

*equipped with the subspace topology induced by $\mathcal{L}(\mathcal{S}_s(\mathbb{R}^N), \mathcal{S}_s'(\mathbb{R}^N))$, via the canonical bilinear form*

$$i\,\mathrm{Tr}_{1,\ldots,N}(A^{(N)}\Psi_N), \qquad \Psi_N \in \mathfrak{g}_N^*. \tag{3.4.63}$$

*Proof.* The proof follows from the duality $\mathcal{L}(\mathcal{S}_s(\mathbb{R}^N), \mathcal{S}_s'(\mathbb{R}^N))) \cong \mathcal{L}(\mathcal{S}_s'(\mathbb{R}^N), \mathcal{S}_s(\mathbb{R}^N))^*$ together with a polarization-type argument. We leave the details to the reader. □

**Remark 3.4.13.** The previous lemma implies that, given a functional $F \in C^\infty(\mathfrak{g}_N^*; \mathbb{R})$ and a point $\Psi_N \in \mathfrak{g}_N^*$, we may identify the continuous linear functional $dF[\Psi_N]$, given by the Gâteaux derivative of $F$ at the point $\Psi_N$, as a skew-adjoint element of $\mathcal{L}(\mathcal{S}_s(\mathbb{R}^N), \mathcal{S}_s'(\mathbb{R}^N))$. We will abuse notation and denote this element by $dF[\Psi_N]$. Moreover, as we will see below, it is a small computation using the generating structure of $\mathcal{A}_{DM,N}$ that $dF[\Psi_N] \in \mathfrak{g}_N$.

We next define the Lie-Poisson manifold of density matrix $N$-hierarchies. To begin, define the real topological vector space

$$\mathfrak{G}_N^* := \left\{\Gamma_N = (\Gamma_N^{(k)})_{k\in\mathbb{N}_{\leq N}} \in \prod_{k=1}^{N} \mathcal{L}(\mathcal{S}_s'(\mathbb{R}^k), \mathcal{S}_s(\mathbb{R}^k)) : \gamma_N^{(k)} = (\gamma_N^{(k)})^* \;\; \forall k \in \mathbb{N}\right\} \tag{3.4.64}$$

endowed with the subspace product topology. We first note that our definition of $\mathfrak{G}_N^*$ is quite natural, as it is isomorphic to the topological dual of $\mathfrak{G}_N$, a fact we prove in the next lemma.

**Lemma 3.4.14** (Dual of $\mathfrak{G}_N$)**.** *The topological dual of $\mathfrak{G}_N$, denoted by $(\mathfrak{G}_N)^*$ and endowed with the strong dual topology, is isomorphic to $\mathfrak{G}_N^*$.*

*Proof.* Using the isomorphism

$$\left(\tilde{\mathcal{L}}(\mathcal{S}_s(\mathbb{R}^k), \mathcal{S}_s(\mathbb{R}^k))\right)^* \cong \left(\mathcal{L}(\mathcal{S}_s(\mathbb{R}^k), \mathcal{S}_s'(\mathbb{R}^k))\right)^* = \mathcal{L}(\mathcal{S}_s'(\mathbb{R}^k), \mathcal{S}_s(\mathbb{R}^k)), \qquad \forall k \in \mathbb{N}, \tag{3.4.65}$$

which follows from the proof of Lemma 3.4.2 together with the duality of direct sums and direct products, see for instance [41, Proposition 2 in §14, Chapter 3], we have that

$$\left(\bigoplus_{k=1}^{N} \tilde{\mathcal{L}}(\mathcal{S}_s(\mathbb{R}^k), \mathcal{S}_s(\mathbb{R}^k))\right)^* \underbrace{\cong}_{=:\Phi'} \prod_{k=1}^{N} \mathcal{L}(\mathcal{S}_s'(\mathbb{R}^k), \mathcal{S}_s(\mathbb{R}^k)), \tag{3.4.66}$$

via the canonical trace pairing

$$(A_N, \Gamma_N) \mapsto i\,\mathrm{Tr}(A_N \cdot \Gamma_N).$$

Thus elements of $(\mathfrak{G}_N)^*$ may be identified with functionals $i\,\mathrm{Tr}(\cdot\Gamma_N)$, and so to prove the lemma, we will show that the map

$$\Phi : \mathfrak{G}_N^* \to (\mathfrak{G}_N)^*, \qquad \Gamma_N \mapsto i\,\mathrm{Tr}(\cdot\Gamma_N), \tag{3.4.67}$$

is bijective and that both $\Phi$ and $\Phi^{-1}$ are continuous.

First, we show surjectivity of $\Phi$. Given any functional $F \in (\mathfrak{G}_N)^*$, we need to find some density matrix $N$-hierarchy $\Gamma_N \in \mathfrak{G}_N^*$ such that

$$F(A_N) = i\,\mathrm{Tr}(A_N \cdot \Gamma_N). \tag{3.4.68}$$

To accomplish this task, we define a functional

$$\widetilde{F} \in \left( \bigoplus_{k=1}^{N} \tilde{\mathcal{L}}(\mathcal{S}_s(\mathbb{R}^k), \mathcal{S}_s(\mathbb{R}^k)) \right)^* \tag{3.4.69}$$

by the formula

$$\widetilde{F}(A_N) := \frac{1}{2}F(A_N - A_N^*) - \frac{i}{2}F((A_N - A_N^*)) + \frac{1}{2}F(i(A_N + A_N^*)) - \frac{i}{2}F(i(A_N + A_N^*)). \tag{3.4.70}$$

By the canonical dual trace pairing, there exists a unique

$$\Gamma_N \in \prod_{k=1}^{N} \mathcal{L}(\mathcal{S}_s'(\mathbb{R}^k), \mathcal{S}_s(\mathbb{R}^k))$$

such that

$$\widetilde{F}(A_N) = i\,\mathrm{Tr}(A_N \cdot \Gamma_N), \qquad \forall A_N \in \bigoplus_{k=1}^{N} \tilde{\mathcal{L}}(\mathcal{S}_s(\mathbb{R}^k), \mathcal{S}_s(\mathbb{R}^k)). \tag{3.4.71}$$

Evaluating $\widetilde{F}$ on $A_N \in \mathfrak{G}_N$, that is assuming $A_N = -A_N^*$, we obtain from (3.4.70) that

$$(1-i)F(A_N) = i\,\mathrm{Tr}(A_N \cdot \Gamma_N), \tag{3.4.72}$$

and adding this expression to its conjugate implies that

$$2F(A_N) = i\Big(\mathrm{Tr}(A_N \cdot \Gamma_N) - \overline{\mathrm{Tr}(A_N \cdot \Gamma_N)}\Big).$$

Since

$$(A_N \cdot \Gamma_N)^{(k)} = A_N^{(k)} \gamma_N^{(k)} \in \mathcal{L}(\mathcal{S}_s'(\mathbb{R}^k), \mathcal{S}_s(\mathbb{R}^k)), \qquad \forall k \in \mathbb{N}_{\leq N},$$

its trace exists in the usual sense of an operator on a separable Hilbert space. Furthermore, the adjoint of $A_N^{(k)}\gamma_N^{(k)}$ as a bounded linear operator on $L_s^2(\mathbb{R}^k)$, denoted by $(A_N^{(k)}\gamma_N^{(k)})^*$, belongs to $\mathcal{L}(\mathcal{S}_s'(\mathbb{R}^k), \mathcal{S}(\mathbb{R}^k))$. A short computation using the skew- and self-adjointness of $A_N^{(k)}$ and $\gamma_N^{(k)}$, respectively, shows that

$$(A_N^{(k)}\gamma_N^{(k)})^* = -\gamma_N^{(k)} A_N^{(k)},$$

where we abuse notation by letting $A_N^{(k)}$ also denote the extension to an element of $\mathcal{L}(\mathcal{S}_s'(\mathbb{R}^k), \mathcal{S}_s'(\mathbb{R}^k))$. Consequently, we are justified in writing

$$\overline{\mathrm{Tr}_{1,\ldots,k}\Big(A_N^{(k)}\gamma_N^{(k)}\Big)} = \mathrm{Tr}_{1,\ldots,k}\Big((A_N^{(k)}\gamma_N^{(k)})^*\Big) = -\,\mathrm{Tr}_{1,\ldots,k}\Big(\gamma_N^{(k)} A_N^{(k)}\Big) = -\,\mathrm{Tr}_{1,\ldots,k}\Big(A_N^{(k)}\gamma_N^{(k)}\Big),$$

where the ultimate equality follows from an approximation of $A_N^{(k)}$ and the cyclicity of trace. Therefore,

$$\widetilde{\Gamma}_N = \frac{1}{2}(\Gamma_N + \Gamma_N^*) \tag{3.4.73}$$

is the desired density matrix $N$-hierarchy. Injectivity of $\Phi$ follows from the polarization identity by considering elements of $\mathfrak{G}_N$ of the form

$$A_{N,k_0}^{(k)} = \begin{cases} i\,|f^{(k_0)}\rangle\,\langle f^{(k_0)}|, & k = k_0 \\ 0, & \text{otherwise} \end{cases}, \tag{3.4.74}$$

where $k_0 \in \mathbb{N}_{\leq N}$ and $f^{(k_0)} \in \mathcal{S}_s(\mathbb{R}^{k_0})$. Hence $\Phi$ is bijective.

Next, we claim that both $\Phi$ and $\Phi^{-1}$ are continuous. Since $\mathfrak{G}_N^*$ is a Fréchet space, it suffices by the open mapping theorem to show that $\Phi$ is continuous. Let $\iota_{\mathfrak{G}_N}$ denote the canonical inclusion map

$$\mathfrak{G}_N \subset \bigoplus_{k=1}^{N} \tilde{\mathcal{L}}(\mathcal{S}_s(\mathbb{R}^k), \mathcal{S}_s(\mathbb{R}^k)), \tag{3.4.75}$$

which is continuous by definition of the subspace topology, with adjoint

$$\iota_{\mathfrak{G}_N}^* : \left( \bigoplus_{k=1}^{N} \tilde{\mathcal{L}}(\mathcal{S}_s(\mathbb{R}^k), \mathcal{S}_s(\mathbb{R}^k)) \right)^* \to (\mathfrak{G}_N)^*, \tag{3.4.76}$$

and let $\iota_{\mathfrak{G}_N^*}$ denote the canonical inclusion map

$$\mathfrak{G}_N^* \subset \prod_{k=1}^{N} \mathcal{L}(\mathcal{S}_s'(\mathbb{R}^k), \mathcal{S}_s(\mathbb{R}^k)), \tag{3.4.77}$$

which is also continuous by definition of the subspace topology. Then we can write

$$\Phi = \iota_{\mathfrak{G}_N}^* \circ (\Phi')^{-1} \circ \iota_{\mathfrak{G}_N^*}, \tag{3.4.78}$$

where $\Phi'$ is the canonical isomorphism described in (3.4.66). Since $\iota_{\mathfrak{G}_N}^*$ is continuous, as can be checked directly or by appealing to the corollary of Proposition 19.5 in [97], it follows that $\Phi$ is the composition of continuous maps, completing the proof of the claim. $\square$

We now need to establish the existence of a Poisson structure for $\mathfrak{G}_N^*$. As before, we choose a unital sub-algebra $\mathcal{A}_{H,N} \subset C^\infty(\mathfrak{G}_N^*; \mathbb{R})$, generated by trace functionals and constant functionals, to be the algebra of admissible functionals.

**Definition 3.4.15.** Let $\mathcal{A}_{H,N}$ be the algebra with respect to point-wise product generated by the functionals in

$$\{F \in C^\infty(\mathfrak{G}_N^*; \mathbb{R}) : F(\cdot) = i\,\mathrm{Tr}(A_N \cdot), \;\; A_N \in \mathfrak{G}_N\} \cup \{F \in C^\infty(\mathfrak{G}_N^*; \mathbb{R}) : F(\cdot) \equiv C \in \mathbb{R}\}. \tag{3.4.79}$$

**Remark 3.4.16.** Our definition of $\mathcal{A}_{H,N}$ is not canonical in the sense that one may include additional functionals in it. However, since we are really only interested in trace functionals, we will not do so in this work.

**Remark 3.4.17.** The structure of $\mathcal{A}_{H,N}$ will be frequently used in the following way: it will suffice to verify various identities for finite products of trace functionals and constant functionals. Moreover, by Remark 3.4.18 below and the Leibnitz rule for the Gâteaux derivative, it will often suffice to check identities on trace functionals.

**Remark 3.4.18.** By the linearity of the trace and the definition of the Gâteaux derivative, a trace functional has constant Gâteaux derivative. Similarly, a constant functional has zero Gâteaux derivative.

To define the Lie-Poisson bracket on $\mathcal{A}_{H,N} \times \mathcal{A}_{H,N}$ using the Lie bracket $[\cdot,\cdot]_{\mathfrak{G}_N}$ constructed in Section 3.4.1, we need the following identification of continuous linear functionals with skew-adjoint operators, given via the canonical trace pairing. We note, in particular, that $(\mathfrak{G}_N^*)^*$ is not isomorphic to $\mathfrak{G}_N$.

**Lemma 3.4.19** (Dual of $\mathfrak{G}_N^*$)**.** *The topological dual of $\mathfrak{G}_N^*$, denoted by $(\mathfrak{G}_N^*)^*$ and endowed with the strong dual topology, is isomorphic to*

$$\widetilde{\mathfrak{G}}_N \coloneqq \left\{A_N \in \bigoplus_{k=1}^{N} \mathcal{L}(\mathcal{S}_s(\mathbb{R}^k), \mathcal{S}_s'(\mathbb{R}^k)) : (A_N^{(k)})^* = -A_N^{(k)}\right\}. \tag{3.4.80}$$

*Proof.* We omit the proof as it proceeds quite similarly to that of Lemma 3.4.14. $\square$

We continue to abuse notation by using $dF[\Gamma_N]$ to denote both the continuous linear functional and the element of $\widetilde{\mathfrak{G}}_N$. We are now prepared to introduce the Lie-Poisson bracket $\{\cdot,\cdot\}_{\mathfrak{G}_N^*}$ on $\mathcal{A}_{H,N} \times \mathcal{A}_{H,N}$.

**Definition 3.4.20.** Let $N \in \mathbb{N}$. For $F, G \in \mathcal{A}_{H,N}$, we define

$$\{F, G\}_{\mathfrak{G}_N^*}(\Gamma_N) := i\operatorname{Tr}\big([dF[\Gamma_N], dG[\Gamma_N]]_{\mathfrak{G}_N} \cdot \Gamma_N\big) = \sum_{k=1}^{N} i\operatorname{Tr}_{1,\ldots,k}\Big([dF[\Gamma_N], dG[\Gamma_N]]_{\mathfrak{G}_N}^{(k)} \gamma_N^{(k)}\Big), \tag{3.4.81}$$

for $\Gamma_N = (\gamma_N^{(k)})_{k \in \mathbb{N}_{\leq N}} \in \mathfrak{G}_N^*$.

We now turn to the second main goal of this subsection, that is, proving Proposition 3.1.2, the statement of which we repeat here for the reader's convenience.

**Proposition 3.1.2.** *$(\mathfrak{G}_N^*, \mathcal{A}_{H,N}, \{\cdot,\cdot\}_{\mathfrak{G}_N^*})$ is a weak Poisson manifold.*

We begin with the following technical lemma for the functional derivative of $\{\cdot,\cdot\}_{\mathfrak{G}_N^*}$.

**Lemma 3.4.21.** *Suppose that $G_j \in \mathcal{A}_{H,N}$ is a trace functional $G_j(\Gamma_N) = i\operatorname{Tr}(dG_j[0] \cdot \Gamma_N)$ for $j = 1, 2$. Then for all $\Gamma_N \in \mathfrak{G}_N^*$, the Gâteaux derivative $d\{G_1, G_2\}_{\mathfrak{G}_N^*}[\Gamma_N]$ at the point $\Gamma_N$ may be identifed with the element*

$$[dG_1[0], dG_2[0]]_{\mathfrak{G}_N} \in \mathfrak{G}_N \tag{3.4.82}$$

*via the canonical trace pairing. If $G_1$ is a trace functional and $G_2 = G_{2,1}G_{2,2}$ is the product of two trace functionals in $\mathcal{A}_{H,N}$, then $d\{G_1, G_2\}_{\mathfrak{G}_N^*}[\Gamma_N]$ may be identified with*

$$G_{2,1}(\Gamma_N)[dG_1[0], dG_{2,2}[0]]_{\mathfrak{G}_N} + G_{2,2}(\Gamma_N)[dG_1[0], dG_{2,1}[0]]_{\mathfrak{G}_N} \tag{3.4.83}$$

*for all $\Gamma_N \in \mathfrak{G}_N^*$ via the canonical trace pairing.*

*Proof.* The first assertion follows readily from the definition of $\{G_1, G_2\}_{\mathfrak{G}_N^*}$. To see the second assertion, observe that by the Leibniz rule for the Gâteaux derivative and the bilinearity of the bracket $[\cdot,\cdot]_r$,

$$\big[dG_1[\Gamma_N]^{(\ell)}, dG_2[\Gamma_N]^{(j)}\big]_r = G_{2,1}(\Gamma_N)\big[dG_1[0]^{(\ell)}, dG_{2,2}[0]^{(j)}\big]_r + G_{2,2}(\Gamma_N)\big[dG_1[0]^{(\ell)}, dG_{2,1}[0]^{(j)}\big]_r.$$

Hence using Proposition 3.4.8 and introducing the notation

$$C_{\ell jkrN} \coloneqq \frac{NC_{\ell,N}C_{j,N}}{C_{k,N}\prod_{m=1}^{r-1}(N-k+m)}, \qquad r_0 \coloneqq \max\{1, \min\{\ell,j\}-(N-\max\{\ell,j\})\}, \tag{3.4.84}$$

we obtain that

$$\begin{aligned}
&[dG_1[\Gamma_N], dG_2[\Gamma_N]]_{\mathfrak{G}_N}^{(k)} \\
&= \sum_{\substack{1\leq \ell,j\leq N \\ \min\{\ell+j-1,N\}=k}} \mathrm{Sym}_k\Bigg(\sum_{r=r_0}^{\min\{\ell,j\}} C_{\ell jkrN}\big[dG_1[\Gamma_N]^{(\ell)}, dG_2[\Gamma_N]^{(j)}\big]_r\Bigg) \\
&= G_{2,1}(\Gamma_N) \sum_{\substack{1\leq \ell,j\leq N \\ \min\{\ell+j-1,N\}=k}} \mathrm{Sym}_k\Bigg(\sum_{r=r_0}^{\min\{\ell,j\}} C_{\ell jkrN}\big[dG_1[0]^{(\ell)}, dG_{2,2}[0]^{(j)}\big]_r\Bigg) \\
&\quad G_{2,2}(\Gamma_N) \sum_{\substack{1\leq \ell,j\leq N \\ \min\{\ell+j-1,N\}=k}} \mathrm{Sym}_k\Bigg(\sum_{r=r_0}^{\min\{\ell,j\}} C_{\ell jkrN}\big[dG_1[0]^{(\ell)}, dG_{2,1}[0]^{(j)}\big]_r\Bigg) \\
&= G_{2,1}(\Gamma_N)[dG_1[0], dG_{2,2}[0]]_{\mathfrak{G}_N}^{(k)} + G_{2,2}(\Gamma_N)[dG_1[0], dG_{2,1}[0]]_{\mathfrak{G}_N}^{(k)},
\end{aligned} \tag{3.4.85}$$

where the ultimate equality follows from another application of Proposition 3.4.8. □

We divide our proof of Proposition 3.1.2 into several lemmas. We first show that $\{\cdot,\cdot\}_{\mathfrak{G}_N^*}$ is well-defined and is a Lie bracket satisfying the Leibnitz rule.

**Lemma 3.4.22.** *The formula*

$$\{F,G\}_{\mathfrak{G}_N^*}(\Gamma_N) \coloneqq i\,\mathrm{Tr}\big([dF[\Gamma_N], dG[\Gamma_N]]_{\mathfrak{G}_N}\cdot\Gamma_N\big), \qquad \forall \Gamma_N \in \mathfrak{G}_N^* \tag{3.4.86}$$

*defines a map $\mathcal{A}_{H,N}\times\mathcal{A}_{H,N}\to\mathcal{A}_{H,N}$ which satisfies property (P1) in Definition 3.3.1.*

*Proof.* We first show that for $F,G\in\mathcal{A}_{H,N}$, one has $\{F,G\}_{\mathfrak{G}_N^*}\in\mathcal{A}_{H,N}$. Recall that $\mathcal{A}_{H,N}$ is generated by constant functionals and trace functionals, hence using the Leibnitz rule, bilinearity of $[\cdot,\cdot]_{\mathfrak{G}_N}$, and the linearity of the trace, it suffices to consider the case where $F, G$ are both trace functionals. Indeed, elements of $\mathcal{A}_{H,N}$ are finite linear combinations of finite products of trace functionals and constant functionals, hence using that the derivative of constant functionals is zero, upon applying the Leibnitz rule, the elements of the product which are not differentiated can be treated as scalars when evaluated at a point $\Gamma_N$ and hence can be pulled out of the Lie bracket and then out of the trace by bilinearity.

When $F, G$ are both trace functionals, $dF[\Gamma_N]$ and $dG[\Gamma_N]$ are constant in $\Gamma_N$ by Remark 3.4.18, hence

$$\{F,G\}_{\mathfrak{G}_N^*}(\Gamma_N) = i\,\mathrm{Tr}\big([dF[0], dG[0]]_{\mathfrak{G}_N} \cdot \Gamma_N\big), \qquad \forall\, \Gamma_N \in \mathfrak{G}_N^*. \tag{3.4.87}$$

So, we only need to show that the right-hand side defines an element of $\mathcal{A}_{H,N}$. Since $dF[0]$ and $dG[0]$ both belong to $\mathfrak{G}_N$, it follows from Proposition 3.1.1 that $[dF[0], dG[0]]_{\mathfrak{G}_N} \in \mathfrak{G}_N$. Hence, $\{F,G\}_{\mathfrak{G}_N^*} \in \mathcal{A}_{H,N}$, which completes the proof of the claim.

Bilinearity and anti-symmetry of $\{\cdot,\cdot\}_{\mathfrak{G}_N^*}$ are immediate from the bilinearity and anti-symmetry of $[\cdot,\cdot]_{\mathfrak{G}_N}$, so it remains to verify the Jacobi identity. Let $F, G, H \in \mathcal{A}_{H,N}$. As we argued above, it suffices to consider the case where $G$ and $H$ are trace functionals and $F$ is a product of two trace functionals, that is, $F = F_1 F_2$, where $F_1, F_2 \in \mathcal{A}_{H,N}$ are such that

$$F_j(\Gamma_N) = i\,\mathrm{Tr}(dF_j[0] \cdot \Gamma_N), \qquad \forall \Gamma_N \in \mathfrak{G}_N^*, \;\; j = 1, 2. \tag{3.4.88}$$

Thus, we need to show that for all $\Gamma_N \in G_N^*$,

$$
\begin{aligned}
0 &= \Big\{F, \{G, H\}_{\mathfrak{G}_N^*}\Big\}_{\mathfrak{G}_N^*}(\Gamma_N) + \Big\{G, \{H, F\}_{\mathfrak{G}_N^*}\Big\}_{\mathfrak{G}_N^*}(\Gamma_N) + \Big\{H, \{F, G\}_{\mathfrak{G}_N^*}\Big\}_{\mathfrak{G}_N^*}(\Gamma_N) \\
&= i\operatorname{Tr}\bigg(\Big[dF[\Gamma_N], d\{G, H\}_{\mathfrak{G}_N^*}[\Gamma_N]\Big]_{\mathfrak{G}_N} \cdot \Gamma_N\bigg) + i\operatorname{Tr}\bigg(\Big[dG[\Gamma_N], d\{H, F\}_{\mathfrak{G}_N^*}[\Gamma_N]\Big]_{\mathfrak{G}_N} \cdot \Gamma_N\bigg) \\
&\quad + i\operatorname{Tr}\bigg(\Big[dH[\Gamma_N], d\{F, G\}_{\mathfrak{G}_N^*}[\Gamma_N]\Big]_{\mathfrak{G}_N} \cdot \Gamma_N\bigg).
\end{aligned} \tag{3.4.89}
$$

We show the desired equality by direct computation:

First, since $dF[\Gamma_N] = F_1(\Gamma_N)dF_2[0] + F_2(\Gamma_N)dF_1[0]$, where we use that $F_1$ and $F_2$ have constant Gâteaux derivatives by Remark 3.4.18, it follows from the linearity of the trace that

$$
\begin{aligned}
i\operatorname{Tr}\bigg(\Big[dF[\Gamma_N], d\{G, H\}_{\mathfrak{G}_N^*}[\Gamma_N]\Big]_{\mathfrak{G}_N} \cdot \Gamma_N\bigg) &= iF_1(\Gamma_N)\operatorname{Tr}\bigg(\Big[dF_2[0], d\{G, H\}_{\mathfrak{G}_N^*}[\Gamma_N]\Big]_{\mathfrak{G}_N} \cdot \Gamma_N\bigg) \\
&\quad + iF_2(\Gamma_N)\operatorname{Tr}\bigg(\Big[dF_1[0], d\{G, H\}_{\mathfrak{G}_N^*}[\Gamma_N]\Big]_{\mathfrak{G}_N} \cdot \Gamma_N\bigg) \\
&= iF_1(\Gamma_N)\operatorname{Tr}\Big(\big[dF_2[0], [dG[0], dH[0]]_{\mathfrak{G}_N}\big]_{\mathfrak{G}_N} \cdot \Gamma_N\Big) \\
&\quad + iF_2(\Gamma_N)\operatorname{Tr}\Big(\big[dF_1[0], [dG[0], dH[0]]_{\mathfrak{G}_N}\big]_{\mathfrak{G}_N} \cdot \Gamma_N\Big),
\end{aligned} \tag{3.4.90}
$$

where we use Lemma 3.4.21 to obtain the ultimate equality.

Next, since $F$ is a product of two trace functionals, we have by Lemma 3.4.21 that

$$
d\{H, F\}_{\mathfrak{G}_N^*}[\Gamma_N] = F_1(\Gamma_N)[dH[0], dF_2[0]]_{\mathfrak{G}_N} + F_2(\Gamma_N)[dH[0], dF_1[0]]_{\mathfrak{G}_N}, \qquad \forall \Gamma_N \in \mathfrak{G}_N^*. \tag{3.4.91}
$$

Hence by bilinearity of the Lie bracket and linearity of the trace,

$$
\begin{aligned}
i\operatorname{Tr}\left(\left[dG[\Gamma_N], d\{H,F\}_{\mathfrak{G}_N^*}[\Gamma_N]\right]_{\mathfrak{G}_N}\cdot\Gamma_N\right) &= iF_1(\Gamma_N)\operatorname{Tr}\Big(\big[dG[0],[dH[0],dF_2[0]]_{\mathfrak{G}_N}\big]_{\mathfrak{G}_N}\cdot\Gamma_N\Big) \\
&\quad + iF_2(\Gamma_N)\operatorname{Tr}\Big(\big[dG[0],[dH[0],dF_1[0]]_{\mathfrak{G}_N}\big]_{\mathfrak{G}_N}\cdot\Gamma_N\Big).
\end{aligned}
\tag{3.4.92}
$$

Finally, similarly to the preceding case,

$$
d\{F,G\}_{\mathfrak{G}_N^*}[\Gamma_N] = F_1(\Gamma_N)[dF_2[0],dG[0]]_{\mathfrak{G}_N} + F_2(\Gamma_N)[dF_1[0],dG[0]]_{\mathfrak{G}_N}, \tag{3.4.93}
$$

and therefore,

$$
\begin{aligned}
i\operatorname{Tr}\left(\left[dH[\Gamma_N], d\{F,G\}_{\mathfrak{G}_N^*}[\Gamma_N]\right]_{\mathfrak{G}_N}\cdot\Gamma_N\right) &= iF_1(\Gamma_N)\operatorname{Tr}\Big(\big[dH[0],[dF_2[0],dG[0]]_{\mathfrak{G}_N}\big]_{\mathfrak{G}_N}\cdot\Gamma_N\Big) \\
&\quad + iF_2(\Gamma_N)\operatorname{Tr}\Big(\big[dH[0],[dF_1[0],dG[0]]_{\mathfrak{G}_N}\big]_{\mathfrak{G}_N}\cdot\Gamma_N\Big).
\end{aligned}
\tag{3.4.94}
$$

Combining the preceding identities, we obtain that

$$
\begin{aligned}
&i\operatorname{Tr}\left(\left[dF[\Gamma_N], d\{G,H\}_{\mathfrak{G}_N^*}[\Gamma_N]\right]_{\mathfrak{G}_N}\cdot\Gamma_N\right) + i\operatorname{Tr}\left(\left[dG[\Gamma_N], d\{H,F\}_{\mathfrak{G}_N^*}[\Gamma_N]\right]_{\mathfrak{G}_N}\cdot\Gamma_N\right) \\
&\qquad + i\operatorname{Tr}\left(\left[dH[\Gamma_N], d\{F,G\}_{\mathfrak{G}_N^*}[\Gamma_N]\right]_{\mathfrak{G}_N}\cdot\Gamma_N\right) \\
&\quad = iF_1(\Gamma_N)\operatorname{Tr}\Big(\Big(\big[dF_2[0],[dG[0],dH[0]]_{\mathfrak{G}_N}\big]_{\mathfrak{G}_N} + \big[dG[0],[dH[0],dF_2[0]]_{\mathfrak{G}_N}\big]_{\mathfrak{G}_N} \\
&\qquad\qquad + \big[dH[0],[dF_2[0],dG[0]]_{\mathfrak{G}_N}\big]_{\mathfrak{G}_N}\Big)\cdot\Gamma_N\Big) \\
&\qquad + iF_2(\Gamma_N)\operatorname{Tr}\Big(\Big(\big[dF_1[0],[dG[0],dH[0]]_{\mathfrak{G}_N}\big]_{\mathfrak{G}_N} + \big[dG[0],[dH[0],dF_1[0]]_{\mathfrak{G}_N}\big]_{\mathfrak{G}_N} \\
&\qquad\qquad + \big[dH[0],[dF_1[0],dG[0]]_{\mathfrak{G}_N}\big]_{\mathfrak{G}_N}\Big)\cdot\Gamma_N\Big) \\
&\quad = 0,
\end{aligned}
\tag{3.4.95}
$$

where the ultimate equality follows from the fact that both lines in the penultimate equality vanish by virtue of the Jacobi identity of the Lie bracket $[\cdot,\cdot]_{\mathfrak{G}_N}$.

Finally, we claim that $\{\cdot,\cdot\}_{\mathfrak{G}_N^*}$ satisfies the Leibnitz rule:

$$\{FG,H\}_{\mathfrak{G}_N^*}(\Gamma_N) = G(\Gamma_N)\{F,H\}_{\mathfrak{G}_N^*}(\Gamma_N) + F(\Gamma_N)\{G,H\}_{\mathfrak{G}_N^*}(\Gamma_N), \qquad \forall \Gamma_N \in \mathfrak{G}_N^*. \tag{3.4.96}$$

Since $d(FG)[\Gamma_N] = F(\Gamma_N)dG[\Gamma_N] + G(\Gamma_N)dF[\Gamma_N]$ by the Leibnitz rule for the Gâteaux derivative, we see that

$$\begin{aligned}
\{FG,H\}_{\mathfrak{G}_N^*}(\Gamma_N) &= i\,\mathrm{Tr}\Big([d(FG)[\Gamma_N], dH[\Gamma_N]]_{\mathfrak{G}_N}\cdot\Gamma_N\Big) \\
&= iF(\Gamma_N)\,\mathrm{Tr}\Big([dG[\Gamma_N], dH[\Gamma_N]]_{\mathfrak{G}_N}\cdot\Gamma_N\Big) + iG(\Gamma_N)\,\mathrm{Tr}\Big([dF[\Gamma_N], dH[\Gamma_N]]_{\mathfrak{G}_N}\cdot\Gamma_N\Big) \\
&= F(\Gamma_N)\{G,H\}_{\mathfrak{G}_N^*}(\Gamma_N) + G(\Gamma_N)\{F,H\}_{\mathfrak{G}_N^*}(\Gamma_N),
\end{aligned} \tag{3.4.97}$$

where the penultimate equality follows by bilineariy of the Lie bracket and linearity of the trace and the ultimate equality follows from the definition of the Poisson bracket. □

We next verify that $\mathcal{A}_{H,N}$ satisfies the non-degeneracy property (P2).

**Lemma 3.4.23.** *$\mathcal{A}_{H,N}$ satisfies property (P2) in Definition 3.3.1.*

*Proof.* Let $\Gamma_N \in \mathfrak{G}_N^*$ and $v \in T_{\Gamma_N}\mathfrak{G}_N^*$, and note that $T_{\Gamma_N}\mathfrak{G}_N^* = \mathfrak{G}_N^*$. Suppose that $dF[\Gamma_N](v) = 0$ for all $F \in \mathcal{A}_{H,N}$. We will show that $v = 0$.

Consider functionals of the form $F_{f,k_0}(\cdot) := i\,\mathrm{Tr}(A_{N,k_0}\cdot)$,

$$A_{N,k_0}^{(k)} := \begin{cases} -i\left|f^{(k_0)}\right\rangle\left\langle f^{(k_0)}\right|, & k = k_0 \\ 0, & \text{otherwise} \end{cases}, \tag{3.4.98}$$

for $k_0 \in \mathbb{N}_{\leq N}$ and $f^{(k_0)} \in \mathcal{S}_s(\mathbb{R}^{k_0})$. By Remark 3.4.18, we have $dF_{f,k_0}[\Gamma_N](\cdot) = F_{f,k_0}(\cdot)$, so if $v = (v^{(k)})_{k\in\mathbb{N}_{\leq N}} \in \mathfrak{G}_N^*$ is as above, we have by definition of the trace that

$$F_{f,k_0}(v) = \left\langle v^{(k_0)} f^{(k_0)} \middle| f^{(k_0)} \right\rangle = 0. \tag{3.4.99}$$

Since $v^{(k)}$ extends uniquely to a bounded operator on $L^2_s(\mathbb{R}^k)$ and $\mathcal{S}_s(\mathbb{R}^k)$ is dense in $L^2_s(\mathbb{R}^k)$, it follows from a standard polarization argument that $v^{(k)} = 0$ for all $k \in \mathbb{N}_{\leq N}$, which completes the proof. $\square$

Lastly, we show the existence of a unique Hamiltonian vector $X_H$ for $H \in \mathcal{A}_{H,N}$ with respect to the Poisson structure $\{\cdot,\cdot\}_{\mathfrak{G}_N^*}$. With this last (most difficult) step, the proof of Proposition 3.1.2 will be complete.

**Lemma 3.4.24.** $(\mathfrak{G}_N^*, \mathcal{A}_{H,N}, \{\cdot,\cdot\}_{\mathfrak{G}_N^*})$ *satisfies property (P3) in Definition 3.3.1. Furthermore, if $H \in \mathcal{A}_{H,N}$, then we have the following formula for the Hamiltonian vector field $X_H$:*

$$X_H(\Gamma_N)^{(\ell)} = \sum_{j=1}^{N} \sum_{r=r_0}^{\min\{\ell,j\}} C'_{\ell jkrN} \operatorname{Tr}_{\ell+1,\ldots,k}\left(\left[\sum_{\underline{\alpha}_r \in P_r^\ell} dH[\Gamma_N]^{(j)}_{(\underline{\alpha}_r,\ell+1,\ldots,\min\{\ell+j-r,k\})}, \gamma_N^{(k)}\right]\right), \tag{3.4.100}$$

*where*

$$k := \min\{\ell+j-1, N\}, \qquad r_0 := \max\{1, \min\{\ell,j\} - (N - \max\{\ell,j\})\}$$

*and where*

$$C'_{\ell jkrN} := \binom{j}{r} \frac{N C_{\ell,N} C_{j,N}}{C_{k,N} \prod_{m=1}^{r-1}(N-k+m)},$$

*for $C_{\ell,N}, C_{k,N}$ as in* (3.4.9).

*Proof.* Given $F, H \in \mathcal{A}_{H,N}$, we first identify a candidate vector field $X_H$ by directly computing $\{F,H\}_{\mathfrak{G}_N^*}$. Once we have found the candidate and verified its smoothness as a map $\mathfrak{G}_N^* \to \mathfrak{G}_N^*$, the proof is complete by the uniqueness guaranteed by Remark 3.3.2.

By definition of the Poisson bracket on $\mathfrak{G}_N^*$, we have that

$$\begin{aligned}\{F,H\}_{\mathfrak{G}_N^*}(\Gamma_N) &= i\,\mathrm{Tr}\Big([dF[\Gamma_N], dH[\Gamma_N]]_{\mathfrak{G}_N}\cdot\Gamma_N\Big)\\ &= i\sum_{k=1}^N \mathrm{Tr}_{1,\ldots,k}\Big([dF[\Gamma_N], dH[\Gamma_N]]_{\mathfrak{G}_N}^{(k)}\gamma_N^{(k)}\Big),\end{aligned} \tag{3.4.101}$$

for $\Gamma_N=(\gamma_N^{(k)})_{k=1}^N\in\mathfrak{G}_N^*$. Using the linearity of the $\mathrm{Sym}_k$ operator, we have by the formula from Proposition 3.4.8 that

$$[dF[\Gamma_N], dH[\Gamma_N]]_{\mathfrak{G}_N}^{(k)} = \sum_{\substack{1\leq \ell,j\leq N\\ \min\{\ell+j-1,N\}=k}}\sum_{r=r_0}^{\min\{\ell,j\}} C_{\ell jkrN}\,\mathrm{Sym}_k\big(\big[dF[\Gamma_N]^{(\ell)}, dH[\Gamma_N]^{(j)}\big]_r\big),$$

and

$$\begin{aligned}\mathrm{Sym}_k\big(\big[dF[\Gamma_N]^{(\ell)}, dH[\Gamma_N]^{(j)}\big]_r\big) &= \mathrm{Sym}_k\left(\binom{j}{r} dF[\Gamma_N]^{(\ell)}_{(1,\ldots,\ell)}\left(\sum_{\underline{\alpha}_r\in P_r^\ell} dH[\Gamma_N]^{(j)}_{(\underline{\alpha}_r,\ell+1,\ldots,\ell+j-r)}\right)\right)\\ &\quad-\mathrm{Sym}_k\left(\binom{\ell}{r} dH[\Gamma_N]^{(j)}_{(1,\ldots,j)}\left(\sum_{\underline{\alpha}_r\in P_r^j} dF[\Gamma_N]^{(\ell)}_{(\underline{\alpha}_r,j+1,\ldots,j+\ell-r)}\right)\right),\end{aligned}$$

where we have used the combinatorial notation $C_{\ell jkrN}$ defined in (3.4.84). Recall from Remark 3.4.5 that we are justified in writing

$$dH[\Gamma_N]^{(j)}_{(1,\ldots,j)}\left(\sum_{\underline{\alpha}_r\in P_r^j} dF[\Gamma_N]^{(\ell)}_{(\underline{\alpha}_r,j+1,\ldots,j+\ell-r)}\right) = \sum_{\underline{\alpha}_r\in P_r^j} dH[\Gamma_N]^{(j)}_{(1,\ldots,j)} dF[\Gamma_N]^{(\ell)}_{(\underline{\alpha}_r,j+1,\ldots,j+\ell-r)}. \tag{3.4.102}$$

Let $(m_1,\ldots,m_{j-r})$ be the increasing arrangement of the set $\mathbb{N}_{\leq j}\setminus\{\alpha_1,\ldots,\alpha_r\}$. Defining the permutation $\tau\in\mathbb{S}_k$ by the formula

$$\tau(a) \coloneqq \begin{cases} i, & a=\alpha_i \text{ for } 1\leq i\leq r\\ a-j+r, & j+1\leq a\leq j+\ell-r\\ \ell+i, & a=m_i \text{ for } 1\leq i\leq j-r\\ a, & \text{otherwise}\end{cases}, \tag{3.4.103}$$

we find that for each $\underline{\alpha}_r \in P_r^j$ fixed,

$$\Big(dH[\Gamma_N]^{(j)}_{(1,\ldots,j)} dF[\Gamma_N]^{(\ell)}_{(\underline{\alpha}_r,j+1,\ldots,j+\ell-r)}\Big)_{(\tau(1),\ldots,\tau(k))} = dH[\Gamma_N]^{(j)}_{(1,\ldots,r,\ell+1,\ldots,\ell+j-r)} dF[\Gamma_N]^{(\ell)}_{(1,\ldots,\ell)}. \tag{3.4.104}$$

Since the $\mathrm{Sym}_k$ operator is $\mathbb{S}_k$-invariant, it then follows that

$$\mathrm{Sym}_k\Big(dH[\Gamma_N]^{(j)}_{(1,\ldots,j)} dF[\Gamma_N]^{(\ell)}_{(\underline{\alpha}_r,\ell+1,\ldots,\ell+j-r)}\Big) = \mathrm{Sym}_k\Big(dH[\Gamma_N]^{(j)}_{(1,\ldots,r,\ell+1,\ldots,\ell+j-r)} dF[\Gamma_N]^{(\ell)}_{(1,\ldots,\ell)}\Big). \tag{3.4.105}$$

Consequently, using that $|P_r^j| = \binom{j}{r} r!$, we obtain that

$$\begin{aligned} &\mathrm{Sym}_k\left(\binom{\ell}{r} dH[\Gamma_N]^{(j)}_{(1,\ldots,j)}\left(\sum_{\underline{\alpha}_r\in P_r^j} dF[\Gamma_N]^{(\ell)}_{(\underline{\alpha}_r,j+1,\ldots,j+\ell-r)}\right)\right) \\ &= \binom{\ell}{r}\binom{j}{r} r!\,\mathrm{Sym}_k\Big(dH[\Gamma_N]^{(j)}_{(1,\ldots,r,\ell+1,\ldots,\ell+j-r)} dF[\Gamma_N]^{(\ell)}_{(1,\ldots,\ell)}\Big). \end{aligned} \tag{3.4.106}$$

Now given $\underline{\alpha}_r \in P_r^\ell$, let $(m_1, \ldots, m_{\ell-r})$ be the increasing arrangement of the set $\mathbb{N}_{\leq \ell} \setminus \{\alpha_1, \ldots, \alpha_r\}$. We recycle notation to define a new permutation $\tau \in \mathbb{S}_k$ by

$$\tau(i) \coloneqq \begin{cases} \alpha_i, & 1 \leq i \leq r \\ m_{i-r}, & r+1 \leq i \leq \ell \\ i, & \text{otherwise} \end{cases}. \tag{3.4.107}$$

Then

$$\begin{aligned} &\mathrm{Sym}_k\left(\Big(dH[\Gamma_N]^{(j)}_{(1,\ldots,r,\ell+1,\ldots,\ell+j-r)} dF[\Gamma_N]^{(\ell)}_{(1,\ldots,\ell)}\Big)_{(\tau(1),\ldots,\tau(k))}\right) \\ &= \mathrm{Sym}_k\Big(dH[\Gamma_N]^{(j)}_{(\underline{\alpha}_r,\ell+1,\ldots,\ell+j-r)} dF[\Gamma_N]^{(\ell)}_{(1,\ldots,\ell)}\Big), \end{aligned} \tag{3.4.108}$$

where we can "undo" the permutation $\tau$'s effect on $dF[\Gamma_N]^{(\ell)}_{(1,\ldots,\ell)}$ by its $\mathbb{S}_\ell$-invariance. Using that $|P_r^\ell| = \binom{\ell}{r} r!$, we obtain that

$$\begin{aligned} &\binom{\ell}{r}\binom{j}{r} r!\,\mathrm{Sym}_k\Big(dH[\Gamma_N]^{(j)}_{(1,\ldots,r,\ell+1,\ldots,\ell+j-r)} dF[\Gamma_N]^{(\ell)}_{(1,\ldots,\ell)}\Big) \\ &= \binom{j}{r}\sum_{\underline{\alpha}_r\in P_r^\ell} \mathrm{Sym}_k\Big(dH[\Gamma_N]^{(j)}_{(\underline{\alpha}_r,\ell+1,\ldots,\ell+j-r)} dF[\Gamma_N]^{(\ell)}_{(1,\ldots,\ell)}\Big). \end{aligned} \tag{3.4.109}$$

Substituting the preceding identity into the expression $\operatorname{Tr}_{1,\ldots,k}([dF[\Gamma_N], dH[\Gamma_N]]_{\mathfrak{G}_N}^{(k)}\gamma_N^{(k)})$ and using Lemma 3.3.33 to eliminate the $\operatorname{Sym}_k$ operator, we obtain that

$$\begin{aligned}
&i\operatorname{Tr}_{1,\ldots,k}\Big([dF[\Gamma_N], dH[\Gamma_N]]_{\mathfrak{G}_N}^{(k)}\gamma_N^{(k)}\Big)\\
&= i\sum_{\min\{\ell+j-1,N\}=k}\sum_{r=r_0}^{\min\{\ell,j\}} C_{\ell jkrN}\binom{j}{r}\sum_{\underline{\alpha}_r\in P_r^\ell}\Bigg(\operatorname{Tr}_{1,\ldots,k}\Big(dF[\Gamma_N]_{(1,\ldots,\ell)}^{(\ell)}dH[\Gamma_N]_{(\underline{\alpha}_r,\ell+1,\ldots,\ell+j-r)}^{(j)}\gamma_N^{(k)}\Big)\\
&\qquad\qquad - \operatorname{Tr}_{1,\ldots,k}\Big(dH[\Gamma_N]_{(\underline{\alpha}_r,\ell+1,\ldots,\ell+j-r)}^{(j)}dF[\Gamma_N]_{(1,\ldots,\ell)}^{(\ell)}\gamma_N^{(k)}\Big)\Bigg).
\end{aligned} \tag{3.4.110}$$

Since $dH[\Gamma_N]_{(\underline{\alpha}_r,\ell+1,\ldots,\ell+j-r)}^{(j)}$ is skew-adjoint and therefore by duality extends to an element in $\mathcal{L}(\mathcal{S}_s'(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k))$, it follows from the cyclicity property of Proposition 3.2.3(iii) that

$$\begin{aligned}
&\operatorname{Tr}_{1,\ldots,k}\Big(dH[\Gamma_N]_{(\underline{\alpha}_r,\ell+1,\ldots,\ell+j-r)}^{(j)}dF[\Gamma_N]_{(1,\ldots,\ell)}^{(\ell)}\gamma_N^{(k)}\Big)\\
&= \operatorname{Tr}_{1,\ldots,k}\Big(dF[\Gamma_N]_{(1,\ldots,\ell)}^{(\ell)}(\gamma_N^{(k)}dH[\Gamma_N]_{(\underline{\alpha}_r,\ell+1,\ldots,\ell+j-r)}^{(j)})\Big).
\end{aligned} \tag{3.4.111}$$

Since

$$dH[\Gamma_N]_{(\underline{\alpha}_r,\ell+1,\ldots,\ell+j-r)}^{(j)}\gamma_N^{(k)},\;\; \gamma_N^{(k)}dH[\Gamma_N]_{(\underline{\alpha}_r,\ell+1,\ldots,\ell+j-r)}^{(j)} \in \mathcal{L}(\mathcal{S}_s'(\mathbb{R}^k), \mathcal{S}(\mathbb{R}^k)), \tag{3.4.112}$$

the usual partial trace $\operatorname{Tr}_{\ell+1,\ldots,k}$ of each of these operators exists and defines an element of $\mathcal{L}(\mathcal{S}_s'(\mathbb{R}^\ell), \mathcal{S}(\mathbb{R}^\ell))$. Moreover, since $dH[\Gamma_N]^{(j)}$ and $\gamma_N^{(k)}$ are skew- and self-adjoint, respectively, these partial traces are self-adjoint.

Returning to the expression $i\operatorname{Tr}\big([dF[\Gamma_N], dH[\Gamma_N]]_{\mathfrak{G}_N}\cdot\Gamma_N\big)$ and interchanging the order of the $k$ and $\ell$ summations, we see that

$$\begin{aligned}
&\sum_{k=1}^{N} i\,\mathrm{Tr}_{1,\ldots,k}\Big([dF[\Gamma_N], dH[\Gamma_N]]_{\mathfrak{G}_N}^{(k)}\gamma_N^{(k)}\Big)\\
&= i\sum_{\ell=1}^{N}\sum_{j=1}^{N}\sum_{r=r_0}^{\min\{\ell,j\}} C'_{lj\tilde{k}rN}\Bigg(\mathrm{Tr}_{1,\ldots,\ell}\Bigg(dF[\Gamma_N]^{(\ell)}\Bigg(\sum_{\underline{\alpha}_r\in P_r^\ell}\mathrm{Tr}_{\ell+1,\ldots,\tilde{k}}\Big(dH[\Gamma_N]^{(j)}_{(\underline{\alpha}_r,\ell+1,\ldots,\min\{\ell+j-r,\tilde{k}\})}\gamma_N^{(\tilde{k})}\Big)\Bigg)\Bigg)\\
&\qquad - \mathrm{Tr}_{1,\ldots,\ell}\Bigg(dF[\Gamma_N]^{(\ell)}\Bigg(\sum_{\underline{\alpha}_r\in P_r^\ell}\mathrm{Tr}_{\ell+1,\ldots,\tilde{k}}\Big(\gamma_N^{(\tilde{k})}dH[\Gamma_N]^{(j)}_{(\underline{\alpha}_r,\ell+1,\ldots,\min\{\ell+j-r,\tilde{k}\})}\Big)\Bigg)\Bigg)\Bigg),
\end{aligned}$$

where

$$\tilde{k} \coloneqq \min\{\ell+j-1, N\}, \tag{3.4.113}$$

$$C'_{\ell j\tilde{k}rN} \coloneqq \frac{NC_{\ell,N}C_{j,N}}{C_{\tilde{k},N}\prod_{m=1}^{r-1}(N-\tilde{k}+m)}\binom{j}{r}. \tag{3.4.114}$$

Note that since $\gamma_N^{(\tilde{k})}$ admits a decomposition

$$\gamma_N^{(\tilde{k})} = \sum_{m=1}^{\infty}\lambda_m\,|f_m^{(\tilde{k})}\rangle\,\langle f_m^{(\tilde{k})}|\,, \tag{3.4.115}$$

where $\sum_{m=1}^{\infty}|\lambda_m| \leq 1$ and $f_m^{(\tilde{k})}, g_m^{(\tilde{k})}$ converge to zero in $\mathcal{S}_s(\mathbb{R}^{\tilde{k}})$, we see that

$$\begin{aligned}
&\mathrm{Tr}_{\ell+1,\ldots,\tilde{k}}\Big(\gamma_N^{\tilde{k}}dH[\Gamma_N]^{(j)}_{(\alpha_r,\ell+1,\ldots,\min\{\ell+j-r,\tilde{k}\})}\Big)\\
&= \sum_{m=1}^{\infty}\lambda_m\left\langle f_m^{(\tilde{k})}\middle| dH[\Gamma_N]^{(j)}_{(\underline{\alpha}_r,\ell+1,\ldots,\min\{\ell+j-r,\tilde{k}\})}f_m^{(\tilde{k})}\right\rangle,
\end{aligned} \tag{3.4.116}$$

which is independent of the choice of extension of $dH[\Gamma_N]^{(j)}$ to domain $\mathcal{S}(\mathbb{R}^j)$ by the permutation invariance of each $f_m^{(\tilde{k})}$. Furthermore, the operator

$$\sum_{\underline{\alpha}_r\in P_r^\ell}\mathrm{Tr}_{\ell+1,\ldots,\tilde{k}}\Big(\gamma_N^{(\tilde{k})}dH[\Gamma_N]^{(j)}_{(\underline{\alpha}_r,\ell+1,\ldots,\min\{\ell+j-r,\tilde{k}\})}\Big) \tag{3.4.117}$$

is invariant under the $\mathbb{S}_\ell$ action, since $P_r^\ell$ is invariant under the $\mathbb{S}_\ell$ group action. Hence, it maps into $\mathcal{S}_s(\mathbb{R}^\ell)$, and its left-composition with $dF[\Gamma_N]^{(\ell)}$ is well-defined.

Using the bilinearity of the generalized trace, we obtain the candidate Hamiltonian vector field

$$X_H(\Gamma_N)^{(\ell)} := \sum_{j=1}^{N} \sum_{r=r_0}^{\min\{\ell,j\}} C'_{\ell j\tilde{k} r N} \sum_{\underline{\alpha}_r \in P_r^\ell} \Bigg( \mathrm{Tr}_{\ell+1,\ldots,\tilde{k}}\Big( dH[\Gamma_N]^{(j)}_{(\underline{\alpha}_r,\ell+1,\ldots,\min\{\ell+j-r,\tilde{k}\})} \gamma_N^{(\tilde{k})} \Big) - \mathrm{Tr}_{\ell+1,\ldots,\tilde{k}}\Big( \gamma_N^{(\tilde{k})} dH[\Gamma_N]^{(j)}_{(\underline{\alpha}_r,\ell+1,\ldots,\min\{\ell+j-r,\tilde{k}\})} \Big) \Bigg). \tag{3.4.118}$$

We now verify that $X_H$, as defined above, is a smooth map $\mathfrak{G}_N^* \to \mathfrak{G}_N^*$, so that we may conclude the proof by Remark 3.4.18. We claim that the right-hand side of the preceding identity defines a continuous linear (hence, smooth) map

$$\mathfrak{G}_N^* \to \bigoplus_{k=1}^{N} \mathcal{L}(\mathcal{S}'_s(\mathbb{R}^k), \mathcal{S}_s(\mathbb{R}^k)). \tag{3.4.119}$$

Linearity is obvious, and the map is continuous from

$$\mathfrak{G}_N^* \to \bigoplus_{k=1}^{N} \mathcal{L}(\mathcal{S}'_s(\mathbb{R}^k), \mathcal{S}(\mathbb{R}^k))$$

by Proposition 3.2.4. That we may replace the target $\mathcal{S}(\mathbb{R}^k)$ by the bosonic subspace $\mathcal{S}_s(\mathbb{R}^k)$ is a consequence of the following facts: $P_r^\ell$ is invariant under the $\mathbb{S}_\ell$ group action, $dH[\Gamma_N]^{(j)}$ is $\mathbb{S}_j$-invariant, and $\gamma_N^{(\tilde{k})}$ is a fortiori $\mathbb{S}_\ell$-invariant. The self-adjointness of $X_H(\Gamma_N)^{(\ell)}$ follows from the skew- and self-adjointness of $dH[\Gamma_N]^{(j)}$ and $\gamma_N^{(\tilde{k})}$, respectively, and the adjoint properties of the generalized partial trace. $\square$

### 3.4.3 Density Matrix Maps as Poisson Morphisms

We close this section with the observations that the well-known operations of forming a density matrix out of a wave function and forming an $N$-hierarchy of reduced density matrices from an $N$-body density matrix respect the geometric structure we have developed, in the sense that these operations define Poisson morphisms.

We first define the *density matrix map* or *ket-bra map* from $N$-body bosonic wave functions to $N$-body bosonic density matrices.

**Definition 3.4.25** (Density matrix map)**.** We define the *density matrix map* or *ket-bra map* by

$$\iota_{DM,N} : \mathcal{S}_s(\mathbb{R}^N) \to \mathfrak{g}_N^* \qquad \iota_{DM,N}(\Phi_N) := |\Phi_N\rangle \langle\Phi_N| = \Phi_N \otimes \overline{\Phi_N}. \tag{3.4.120}$$

It is easy to verify that $\iota_{DM,N}$ is a smooth map from $\mathcal{S}_s(\mathbb{R}^N)$ to $\mathfrak{g}_N^*$. We now show that the density matrix map is a Poisson map. To prove this property, we recall from Definition 3.3.7 the requirement that $\iota_{DM,N}^* \mathcal{A}_{DM,N} \subset \mathcal{A}_{\mathcal{S}}$. If $F$ is smooth, then the smoothness of $\iota_{DM,N}$ implies by the chain rule that $f = F \circ \iota_{DM,N} \in C^\infty(\mathcal{S}_s(\mathbb{R}^N);\mathbb{R})$. However, it is not a priori clear that $f \in \mathcal{A}_{\mathcal{S}}$, where we recall that $A_{\mathcal{S}} \subset C^\infty(\mathcal{S}(\mathbb{R}^N);\mathbb{R})$ is defined by

$$\mathcal{A}_{\mathcal{S}} := \left\{H \,:\, \boldsymbol{\nabla}_s H \in C^\infty(\mathcal{S}(\mathbb{R}^N);\mathcal{S}(\mathbb{R}^N))\right\}, \tag{3.4.121}$$

In the sequel, we will use the notation $\mathcal{A}_{\mathcal{S},N}$ to make the dependence on $N$ explicit.

**Lemma 3.4.26.** *Let $N \in \mathbb{N}$. For any $F \in \mathcal{A}_{DM,N}$, the functional $f := F \circ \iota_{DM,N} \in C^\infty(\mathcal{S}_s(\mathbb{R}^N);\mathbb{R})$ belongs to $\mathcal{A}_{\mathcal{S},N}$. Furthermore,*

$$\boldsymbol{\nabla}_s f(\Phi_N) = dF[\iota_{DM,N}(\Phi_N)](\Phi_N), \qquad \forall \Phi_N \in \mathcal{S}_s(\mathbb{R}^N), \tag{3.4.122}$$

*where we identify $dF[\iota_{DM,N}(\Phi_N)]$ as a skew-adjoint operator by Remark 3.4.13.*

*Proof.* Observe from the chain rule that for $\Phi_N, \delta\Phi_N \in \mathcal{S}_s(\mathbb{R}^N)$,

$$\begin{aligned} df[\Phi_N](\delta\Phi_N) &= dF[\iota_{DM,N}(\Phi_N)](d\iota_{DM,N}[\Phi_N](\delta\Phi_N)) \\ &= dF[\iota_{DM,N}(\Phi_N)](|\Phi_N\rangle\langle\delta\Phi_N| + |\delta\Phi_N\rangle\langle\Phi_N|), \end{aligned} \tag{3.4.123}$$

where we use the elementary computation

$$d\iota_{DM,N}[\Phi_N](\delta\Phi_N) = |\Phi_N\rangle\ \langle\delta\Phi_N| + |\delta\Phi_N\rangle\ \langle\Phi_N|\,. \tag{3.4.124}$$

Identifying the functional $dF[\iota_{DM,N}(\Phi_N)](\cdot)$ with a skew-adjoint DVO given by $dF[\iota_{DM,N}(\Phi_N)]$ as in Remark 3.4.13, we have that

$$\begin{aligned} dF[\iota_{DM,N}(\Phi_N)](|\Phi_N\rangle\langle\delta\Phi_N| + |\delta\Phi_N\rangle\langle\Phi_N|) &= i\operatorname{Tr}_{1,\ldots,N}(dF[\iota_{DM,N}(\Phi_N)](|\Phi_N\rangle\langle\delta\Phi_N| + |\delta\Phi_N\rangle\langle\Phi_N|)) \\ &= i\langle\delta\Phi_N|dF[\iota_{DM,N}(\Phi_N)]\Phi_N\rangle + i\langle\Phi_N|dF[\iota_{DM,N}[\Phi_N]\delta\Phi_N\rangle\,. \end{aligned}$$

Since $dF[\iota_{DM,N}(\Phi_N)]$ is skew-adjoint, the preceding expression equals

$$\begin{aligned} i\langle\delta\Phi_N|dF[\iota_{DM,N}(\Phi_N)]\Phi_N\rangle - i\langle dF[\iota_{DM,N}(\Phi_N)]\Phi_N|\delta\Phi_N\rangle &= -2\operatorname{Im}\langle\delta\Phi_N|dF[\iota_{DM,N}(\Phi_N)]\Phi_N\rangle \\ &= \omega_{L^2}(dF[\iota_{DM,N}(\Phi_N)]\Phi_N, \delta\Phi_N). \end{aligned}$$

We claim that the map $\Phi_N \mapsto dF[\iota_{DM,N}(\Phi_N)]\Phi_N$ is a smooth map of $\mathcal{S}_s(\mathbb{R}^N)$ to itself, which justifies our preceding manipulations. Indeed, suppose first that $F \in \mathcal{A}_{DM,N}$ is a trace functional. Then $dF[\iota_{DM,N}(\Phi_N)] = dF[0]$, and therefore the claim follows since $dF[0]$ is a continuous linear map of $\mathcal{S}_s(\mathbb{R}^N)$ to itself by definition of $\mathcal{A}_{DM,N}$. The general case then follows by the Leibnitz rule for the Gâteaux derivative. Therefore, the functional $f$ has symplectic $L^2$ gradient

$$\boldsymbol{\nabla}_s f(\Phi_N) = dF[\iota_{DM,N}(\Phi_N)]\Phi_N,$$

and $\boldsymbol{\nabla}_s f$ is a smooth map of $\mathcal{S}_s(\mathbb{R}^N)$ to itself, which implies that $f \in \mathcal{A}_{\mathcal{S},N}$. □

We recall from (1.3.2) the definition for $\{\cdot,\cdot\}_{L^2}$, and we consider the rescaled Poisson bracket

$$\{\cdot,\cdot\}_{L^2,N} \coloneqq N\{\cdot,\cdot\}_{L^2}. \tag{3.4.125}$$

**Proposition 3.4.27.** *Let $N \in \mathbb{N}$. Then*

$$\iota_{DM,N} : (\mathcal{S}_s(\mathbb{R}^N), \mathcal{A}_{\mathcal{S},N}, \{\cdot,\cdot\}_{L^2,N}) \to (\mathfrak{g}_N^*, \mathcal{A}_{DM,N}, \{\cdot,\cdot\}_{\mathfrak{g}_N^*}) \tag{3.4.126}$$

*is a Poisson map.*

*Proof.* As observed above, the smoothness of $\iota_{DM,N}$ is evident, and by Lemma 3.4.26, $F \circ \iota_{DM,N} \in \mathcal{A}_{\mathcal{S},N}$ for any $F \in \mathcal{A}_{DM,N}$. Hence, it remains for us to show that for all $F, G \in \mathcal{A}_{DM,N}$,

$$\{F \circ \iota_{DM,N}, G \circ \iota_{DM,N}\}_{L^2,N}(\Phi_N) = \{F,G\}_{\mathfrak{g}_N^*} \circ \iota_{DM,N}(\Phi_N), \qquad \forall\, \Phi_N \in \mathcal{S}_s(\mathbb{R}^N). \tag{3.4.127}$$

For convenience, we introduce the notation $f \coloneqq F \circ \iota_{DM,N}$ and $g \coloneqq G \circ \iota_{DM,N}$. We first consider the expression $\{f,g\}_{L^2,N}(\Phi_N)$. Observe that by definition of the Poisson bracket $\{\cdot,\cdot\}_{L^2,N}$,

$$\begin{aligned}\{f,g\}_{L^2,N}(\Phi_N) &= N\omega_{L^2}(\boldsymbol{\nabla}_s f(\Phi_N), \boldsymbol{\nabla}_s g(\Phi_N)) \\ &= 2N \operatorname{Im} \langle dF[\iota_{DM,N}(\Phi_N)]\Phi_N | dG[\iota_{DM,N}(\Phi_N)]\Phi_N \rangle .\end{aligned} \tag{3.4.128}$$

Now using the skew-adjointness of $dG[\iota_{DM,N}(\Phi_N)]$ and $dF[\iota_{DM,N}(\Phi_N)]$, we conclude that the last expression equals

$$\begin{aligned}&iN(\langle \Phi_N | dF[\iota_{DM,N}(\Phi_N) dG[\iota_{DM,N}(\Phi_N)]\Phi_N\rangle - \langle \Phi_N | dG[\iota_{DM,N}(\Phi_N)] dF[\iota_{DM,N}(\Phi_N)]\Phi_N\rangle) \\ &= i \operatorname{Tr}_{1,\ldots,N}\Big( [dF[\iota_{DM,N}(\Phi_N)], dG[\iota_{DM,N}(\Phi_N)]]_{\mathfrak{g}_N} |\Phi_N\rangle \, \langle \Phi_N| \Big) \\ &= \{F,G\}_{\mathfrak{g}_N^*} \circ \iota_{DM,N}(\Phi_N),\end{aligned} \tag{3.4.129}$$

which is exactly what we wanted to show. $\square$

We next show that there is a linear homomorphism of Lie algebras $\mathfrak{G}_N \to \mathfrak{g}_N$ induced by the embeddings $\{\epsilon_{k,N}\}_{k\in\mathbb{N}_{\leq N}}$. We will then combine this fact with a duality argument to prove that the reduced density matrix operation is a Poisson mapping

$$(\mathfrak{g}_N^*, \mathcal{A}_{DM,N}, \{\cdot,\cdot\}_{\mathfrak{g}_N^*}) \to (\mathfrak{G}_N^*, \mathcal{A}_{H,N}, \{\cdot,\cdot\}_{\mathfrak{G}_N^*}). \tag{3.4.130}$$

**Proposition 3.4.28.** *For any $N \in \mathbb{N}$, the map*

$$\iota_{\epsilon,N} : \mathfrak{G}_N \to \mathfrak{g}_N, \qquad \iota_{\epsilon,N}(A_N) := \sum_{k=1}^{N} \epsilon_{k,N}(A_N^{(k)}), \tag{3.4.131}$$

*is a continuous linear homomorphism of Lie algebras.*

*Proof.* Continuity and linearity are evident from the continuity and linearity of the maps $\epsilon_{k,N}$ (recall Lemma 3.4.3). To show that $\iota_{sum,N}$ is a homomorphism of Lie algebras, we need to show that for any

$$A_N = (A_N^{(k)})_{k\in\mathbb{N}_{\leq N}}, \ B_N = (B_N^{(k)})_{k\in\mathbb{N}_{\leq N}} \in \mathfrak{G}_N, \tag{3.4.132}$$

we have that

$$\iota_{\epsilon,N}\big([A_N, B_N]_{\mathfrak{G}_N}\big) = [\iota_{\epsilon,N}(A_N), \iota_{\epsilon,N}(B_N)]_{\mathfrak{g}_N}. \tag{3.4.133}$$

Consider the left-hand side expression. By the definition of the map $\iota_{\epsilon,N}$, the definition of the Lie bracket $[\cdot,\cdot]_{\mathfrak{G}_N}$ from (3.4.52), and Lemma 3.4.7, we obtain that

$$\begin{aligned}\iota_{\epsilon,N}\big([A_N,B_N]_{\mathfrak{G}_N}\big) &= \sum_{k=1}^N \epsilon_{k,N}\Big([A_N,B_N]_{\mathfrak{G}_N}^{(k)}\Big) \\ &= \sum_{k=1}^N \epsilon_{k,N}(C_N^{(k)}) \\ &= \sum_{k=1}^N \sum_{\substack{1\leq \ell,j\leq N \\ \min\{\ell+j-1,N\}=k}} \Big[\epsilon_{\ell,N}(A_N^{(\ell)}),\epsilon_{j,N}(B_N^{(j)})\Big]_{\mathfrak{g}_N}.\end{aligned}$$

Using the partition

$$\{(\ell,j)\in(\mathbb{N}_{\leq N})^2\} = \bigcup_{k=1}^N \{(\ell,j)\in(\mathbb{N}_{\leq N})^2 : \min\{\ell+j-1,N\}=k\}, \tag{3.4.134}$$

we see that

$$\sum_{k=1}^N \sum_{\substack{1\leq \ell,j\leq N \\ \min\{\ell+j-1,N\}=k}} \Big[\epsilon_{\ell,N}(A_N^{(\ell)}),\epsilon_{j,N}(B_N^{(j)})\Big]_{\mathfrak{g}_N} = \sum_{\ell=1}^N\sum_{j=1}^N \Big[\epsilon_{\ell,N}(A_N^{(\ell)}),\epsilon_{j,N}(B_N^{(j)})\Big]_{\mathfrak{g}_N}. \tag{3.4.135}$$

By the definition of the map $\iota_{\epsilon,N}$ and the bilinearity of Lie brackets, we observe that

$$\sum_{\ell=1}^N\sum_{j=1}^N \Big[\epsilon_{\ell,N}(A_N^{(\ell)}),\epsilon_{j,N}(B_N^{(j)})\Big]_{\mathfrak{g}_N} = [\iota_{\epsilon,N}(A_N),\iota_{\epsilon,N}(B_N)]_{\mathfrak{g}_N}, \tag{3.4.136}$$

which completes the proof. □

Finally, we show that there is a canonical Poisson mapping of $\mathfrak{g}_N^* \to \mathfrak{G}_N^*$ given by taking the sequence of reduced density matrices.

**Proposition 3.4.29** (RDM Map is Poisson)**.** *The map $\iota_{RDM,N} : \mathfrak{g}_N^* \to \mathfrak{G}_N^*$ given by*

$$\iota_{RDM,N}(\Psi_N) := \Gamma_N = (\gamma_N^{(k)})_{k\in\mathbb{N}_{\leq N}}, \qquad \gamma_N^{(k)} := \mathrm{Tr}_{k+1,\ldots,N}(\Psi_N) \tag{3.4.137}$$

*is a Poisson map.*

To prove Proposition 3.4.29, we will show that $\iota_{RDM,N}$ is the dual of the map $\iota_{sum,N}$, which, by Proposition 3.4.28, we know is a continuous linear homomorphism of Lie algebras. We then appeal to the following general result, the statement of which we have taken from [60, Proposition 10.7.2].

**Lemma 3.4.30.** *Let $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ and $(\mathfrak{h}, [\cdot,\cdot]_{\mathfrak{h}})$ be Lie algebras. Let $\alpha : \mathfrak{g} \to \mathfrak{h}$ be a linear map. Then the map $\alpha$ is a homomorphism of Lie algebras if and only if its dual map $\alpha^* : \mathfrak{h}^* \to \mathfrak{g}^*$ is a (linear) Poisson map.*

*Proof of Proposition 3.4.29.* As stated above, we want to show that the reduced density matrix $\iota_{RDM,N}$ is the dual of the map

$$\iota_{\epsilon,N} : \mathfrak{G}_N \to \mathfrak{g}_N, \qquad A_N = (A_N^{(1)}, \ldots, A_N^{(N)}) \mapsto \sum_{k=1}^{N} \epsilon_{k,N}(A_N^{(k)}). \tag{3.4.138}$$

Indeed, observe that for $\Psi_N \in \mathfrak{g}_N^*$ and $A_N = (A_N^{(k)})_{k\in\mathbb{N}_{\leq N}} \in \mathfrak{G}_N$, we see from unpacking the definition of $\iota_{\epsilon,N}$ and using the bilinearity of the generalized trace that

$$\iota_{\epsilon,N}^*(\Psi_N)(A_N) = i\,\mathrm{Tr}_{1,\ldots,N}(\iota_{\epsilon,N}(A_N)\Psi_N) = \sum_{k=1}^{N} i\,\mathrm{Tr}_{1,\ldots,N}\Big(\epsilon_{k,N}(A_N^{(k)})\Psi_N\Big). \tag{3.4.139}$$

Unpacking the definition (3.4.8) of the map $\epsilon_{k,N}(A_N^{(k)})$ and using the bilinearity of the generalized trace again, we see that

$$\sum_{k=1}^{N} i\,\mathrm{Tr}_{1,\ldots,N}\Big(\epsilon_{k,N}(A_N^{(k)})\Psi_N\Big) = \sum_{k=1}^{N} \sum_{\underline{p}_k \in P_k^N} iC_{k,N}\,\mathrm{Tr}_{1,\ldots,N}\Big(A_{N,(p_1,\ldots,p_k)}^{(k)}\Psi_N\Big). \tag{3.4.140}$$

Hence using that $\Psi_N$ is bosonic and Lemma 3.3.33, we have that

$$\begin{aligned} \mathrm{Tr}_{1,\ldots,N}\Big(A_{N,(p_1,\ldots,p_k)}^{(k)}\Psi_N\Big) = \mathrm{Tr}_{1,\ldots,N}\Big(A_{N,(1,\ldots,k)}^{(k)}\Psi_N\Big) &= \mathrm{Tr}_{1,\ldots,k}\Big(A_N^{(k)}\,\mathrm{Tr}_{k+1,\ldots,N}(\Psi_N)\Big) \\ &= \mathrm{Tr}_{1,\ldots,k}\Big(A_N^{(k)}\gamma_N^{(k)}\Big), \end{aligned} \tag{3.4.141}$$

where the ultimate equality follows by definition of $\gamma_N^{(k)}$. Since $|P_k^N| = 1/C_{k,N}$, we conclude that

$$\iota_{\epsilon,N}^*(\Psi_N)(A_N) = \sum_{k=1}^N i \operatorname{Tr}_{1,\ldots,k}\left(A_N^{(k)}\gamma_N^{(k)}\right) = i \operatorname{Tr}(A_N \cdot \iota_{RDM,N}(\Psi_N)), \tag{3.4.142}$$

which completes the proof of the proposition. □

## 3.5 Geometric Structure for Infinity Hierarchies

In this section, we compute the limit of the $N$-body Lie algebra $(\mathfrak{G}_N, [\cdot,\cdot]_{\mathfrak{G}_N})$ as $N \to \infty$. We then show that in this limit, the higher-order contractions appearing in formula (3.4.53) vanish. Consequently, the domain of definition of the Lie bracket may be enlarged, for which we construct the Lie algebra $(\mathfrak{G}_\infty, [\cdot,\cdot]_{\mathfrak{G}_\infty})$ of observable $\infty$-hierarchies and dually, the weak Lie-Poisson manifold $(\mathfrak{G}_\infty^*, \mathcal{A}_\infty, \{\cdot,\cdot\}_{\mathfrak{G}_\infty^*})$ of density matrix $\infty$-hierarchies.

### 3.5.1 The Limit of $\mathfrak{G}_N$ as $N \to \infty$

In order to pass from the $N$-particle setting to the $\infty$-particle setting, we first study the limit of the Lie algebra $(\mathfrak{G}_N, [\cdot,\cdot]_{\mathfrak{G}_N})$ as $N \to \infty$.

Via the natural inclusion map, we can identify $\mathfrak{G}_N$ as the subspace of the locally convex direct sum

$$\mathfrak{F}_\infty \coloneqq \bigcup_{N=1}^\infty \mathfrak{G}_N = \bigoplus_{k=1}^\infty \mathfrak{g}_k \tag{3.5.1}$$

consisting of elements $A = (A^{(k)})_{k\in\mathbb{N}}$, where $A^{(k)} = 0$ for $k \geq N+1$. In our next result, Proposition 3.1.4, we establish a formula for the limiting bracket structure for $\mathfrak{G}_\infty$.

**Proposition 3.1.4.** *Let $N_0 \in \mathbb{N}$. For $A = (A^{(k)})_{k\in\mathbb{N}}, B = (B^{(k)})_{k\in\mathbb{N}} \in \mathfrak{G}_{N_0}$, we have that*

$$\lim_{N\to\infty} [A,B]_{\mathfrak{G}_N} = C = (C^{(k)})_{k\in\mathbb{N}}, \tag{3.1.20}$$

*where*

$$C^{(k)} := \sum_{\substack{\ell,j\geq 1\\ \ell+j-1=k}} \mathrm{Sym}_k\big(\big[A^{(\ell)}, B^{(j)}\big]_1\big), \tag{3.1.21}$$

*in the topology of* $\mathfrak{F}_\infty$.

*Proof.* Let $k \in \mathbb{N}$. For $M \gg k$, we have by Proposition 3.4.8 and the linearity of the map $\epsilon_{k,N}$ that

$$\begin{aligned}
&\sum_{\substack{\ell,j\geq 1\\ \ell+j-1=k}} \epsilon_{k,M}^{-1}\Big(\big[\epsilon_{\ell,M}(A^{(\ell)}), \epsilon_{j,M}(B^{(j)})\big]_{\mathfrak{g}_M}\Big)\\
&= \sum_{\substack{\ell,j\geq 1\\ \ell+j-1=k}} \mathrm{Sym}_k\left(\sum_{r=1}^{\min\{\ell,j\}} \frac{MC_{\ell,M}C_{j,M}}{C_{k,M}\prod_{a=1}^{r-1}(M-k+a)}\big[A^{(\ell)}, B^{(j)}\big]_r\right)\\
&= \sum_{\substack{\ell,j\geq 1\\ \ell+j-1=k}} \mathrm{Sym}_k\left(\frac{MC_{\ell,M}C_{j,M}}{C_{k,M}}\big[A^{(\ell)}, B^{(j)}\big]_1\right)\\
&\qquad\qquad + \sum_{\substack{\ell,j\geq 1\\ \ell+j-1=k}} \mathrm{Sym}_k\left(\sum_{r=2}^{\min\{\ell,j\}} \frac{MC_{\ell,M}C_{j,M}}{C_{k,M}\prod_{a=1}^{r-1}(M-k+a)}\big[A^{(\ell)}, B^{(j)}\big]_r\right)\\
&=: \mathrm{Term}_{1,M} + \mathrm{Term}_{2,M}.
\end{aligned} \tag{3.5.2}$$

We first consider $\mathrm{Term}_{1,M}$. Since

$$\lim_{M\to\infty} \frac{MC_{\ell,M}C_{j,M}}{C_{k,M}} = \lim_{M\to\infty} \frac{M\prod_{a=1}^{k}(M+1-a)}{(\prod_{a=1}^{\ell}(M+1-a))(\prod_{a=1}^{j}(M+1-a))} = \lim_{M\to\infty} \frac{M^{k+1}}{M^{\ell+j}} = 1,$$

we see that

$$\mathrm{Term}_{1,M} \to \sum_{\ell,j\geq 1;\ell+j-1=k} \mathrm{Sym}_k\big(\big[A^{(\ell)}, B^{(j)}\big]_1\big), \tag{3.5.3}$$

as $M \to \infty$, in $\mathfrak{g}_k$.

We next consider $\mathrm{Term}_{2,M}$. Let $2 \le r \le \min\{\ell, j\}$. Since

$$\begin{aligned}
\lim_{M\to\infty} \frac{MC_{\ell,N}C_{j,M}}{C_{k,M}\prod_{a=1}^{r-1}(M-k+a)} &= \lim_{M\to\infty} \frac{M\prod_{a=1}^{k}(M+1-a)}{(\prod_{a=1}^{\ell}(M+1-a))(\prod_{a=1}^{j}(M+1-a))(\prod_{a=1}^{r-1}(M-k+a))} \\
&= \lim_{M\to\infty} \frac{M^{k+1}}{M^{\ell+j+r-1}} \\
&= \lim_{M\to\infty} M^{1-r} \\
&= 0, \qquad (3.5.4)
\end{aligned}$$

we see that

$$\mathrm{Sym}_k\left(\frac{MC_{\ell,M}C_{j,M}}{C_{k,M}\prod_{a=1}^{r-1}(M-k+a)}\left[A^{(\ell)}, B^{(j)}\right]_r\right) \to 0, \qquad (3.5.5)$$

as $M \to \infty$, in $\mathfrak{g}_k$. Summing over the ranges $2 \le r \le \min\{\ell, j\}$ and $\ell + j - 1 = k$, for a total of finitely many terms, we conclude that

$$\mathrm{Term}_{2,M} \to 0, \qquad (3.5.6)$$

as $M \to \infty$, in $\mathfrak{g}_k$, proving the result. $\square$

### 3.5.2 The Lie Algebra $\mathfrak{G}_\infty$ of Observable $\infty$-Hierarchies

As mentioned in the introduction, the simplified form of $[\cdot,\cdot]_{\mathfrak{G}_\infty}$ allows us to take advantage of the good mapping property and extend this bracket to a map on a much larger real topological vector space, which we redefine $\mathfrak{G}_\infty$ to be, to obtain a Lie algebra of observable $\infty$-hierarchies. We rigorously construct this extension now.

We define $\mathfrak{g}_{k,gmp}$ to be

$$\mathfrak{g}_{k,gmp} \coloneqq \{A^{(k)} \in \mathcal{L}_{gmp}(\mathcal{S}_s(\mathbb{R}^k), \mathcal{S}_s'(\mathbb{R}^k)) : A^{(k)} = -(A^{(k)})^*\}. \qquad (3.5.7)$$

In words, $\mathfrak{g}_{k,gmp}$ is the real, locally convex space consisting of skew-adjoint elements of $\mathcal{L}_{gmp}(\mathcal{S}_s(\mathbb{R}^k), \mathcal{S}_s'(\mathbb{R}^k))$. We will hereafter refer to the elements of $\mathfrak{g}_{k,gmp}$ as *k-particle or k-body observables*. We define the locally convex direct sum

$$\mathfrak{G}_\infty := \bigoplus_{k=1}^{\infty} \mathfrak{g}_{k,gmp}. \tag{3.5.8}$$

We refer to the elements of $\mathfrak{G}_\infty$ as *observable $\infty$-hierarchies*. For

$$A = (A^{(k)})_{k\in\mathbb{N}}, \ B = (B^{(k)})_{k\in\mathbb{N}} \in \mathfrak{G}_\infty,$$

we define

$$\begin{aligned} &[A, B]_{\mathfrak{G}_\infty} := C = (C^{(k)})_{k\in\mathbb{N}}, \\ &C^{(k)} := \mathrm{Sym}_k\Bigg(\sum_{\substack{\ell,j\geq 1 \\ \ell+j-1=k}} \left[A^{(\ell)}, B^{(j)}\right]_1\Bigg), \end{aligned} \tag{3.5.9}$$

where $\mathrm{Sym}_k$ denotes the bosonic symmetrization operator defined in Section 3.3, which we recall is given by

$$\mathrm{Sym}_k(A^{(k)}) := \frac{1}{k!}\sum_{\pi\in\mathbb{S}_k} A^{(k)}_{(\pi(1),\ldots,\pi(k))}, \quad A^{(k)}_{(\pi(1),\ldots,\pi(k))} = \pi \circ A^{(k)}_{1,\ldots,k} \circ \pi^{-1} \tag{3.5.10}$$

and where $\left[A^{(\ell)}, B^{(j)}\right]_1$ is given according to (3.4.33) by

$$\begin{aligned} \left[A^{(\ell)}, B^{(j)}\right]_1 &= jA^{(\ell)} \circ_1 B^{(j)} - \ell B^{(j)} \circ_1 A^{(\ell)} \\ &= jA^{(\ell)}_{(1,\ldots,\ell)}\left(\sum_{\alpha=1}^{\ell} B^{(j)}_{(\alpha,\ell+1,\ldots,\ell+j-1)}\right) - \ell B^{(j)}\left(\sum_{\alpha=1}^{j} A^{(\ell)}_{(\alpha,j+1,\ldots,j+\ell-1)}\right). \end{aligned} \tag{3.5.11}$$

The main goal of this section is to establish the existence of a Lie algebra of observable $\infty$-hierarchies, namely, to prove Proposition 3.1.7:

**Proposition 3.1.7.** $(\mathfrak{G}_\infty, [\cdot,\cdot]_{\mathfrak{G}_\infty})$ *is a Lie algebra in the sense of Definition 3.3.14.*

The construction follows closely our $N$-body approach in Section 3.4; however, there are new technical difficulties that have to be considered. Indeed, $\mathfrak{G}_\infty$ contains more singular objects than $\mathfrak{G}_N$, and we have to heavily exploit the good mapping property in order to handle this issue. We remind the reader the enlarged definition of $\mathfrak{G}_\infty$, as opposed to simply the union of the $\mathfrak{G}_N$, is necessary to accommodate the observable $\infty$-hierarchy $-i\mathbf{W}_{GP}$ which generates the GP Hamiltonian functional.

We first need to establish that the Lie bracket given by (3.5.9) is well-defined on $\mathfrak{G}_\infty$. To this end, we must begin by giving meaning to the composition

$$A^{(\ell)}_{(1,\ldots,\ell)}\left(\sum_{\alpha=1}^{\ell} B^{(j)}_{(\alpha,\ell+1,\ldots,\ell+j-1)}\right) \tag{3.5.12}$$

as an operator in $\mathcal{L}(\mathcal{S}(\mathbb{R}^k),\mathcal{S}'(\mathbb{R}^k))$, for which it will be convenient to proceed term-wise by extending $A^{(\ell)}$ and $B^{(j)}$ to operators defined on the entire space $\mathcal{S}(\mathbb{R}^\ell)$ and $\mathcal{S}(\mathbb{R}^j)$, respectively, as described in Remark 3.4.5.[17] For general $A^{(\ell)} \in \mathcal{L}(\mathcal{S}(\mathbb{R}^\ell),\mathcal{S}'(\mathbb{R}^\ell))$ and $B^{(j)} \in \mathcal{L}(\mathcal{S}(\mathbb{R}^j),\mathcal{S}'(\mathbb{R}^j))$, such a composition may not be well-defined, see Remark 3.3.3, and hence we appeal to the good mapping property of Definition 3.1.5 to give meaning to (3.5.12). It will be useful in the sequel to observe that the definition of the good mapping property says the following: let $A^{(\ell)} \in \mathcal{L}(\mathcal{S}(\mathbb{R}^\ell),\mathcal{S}'(\mathbb{R}^\ell))$ and $(f^{(\ell)},g^{(\ell)}) \in \mathcal{S}(\mathbb{R}^\ell)\times\mathcal{S}(\mathbb{R}^\ell)$, and for fixed $x'_\alpha \in \mathbb{R}$, consider the distribution in $\mathcal{S}'(\mathbb{R})$ defined by

$$\phi \mapsto \left\langle A^{(\ell)}f^{(\ell)}, \left(\phi\otimes_\alpha g^{(\ell)}(\cdot,x'_\alpha,\cdot)\right)\right\rangle_{\mathcal{S}'(\mathbb{R}^\ell)-\mathcal{S}(\mathbb{R}^\ell)}, \tag{3.5.13}$$

where

$$\left(\phi\otimes_\alpha g^{(\ell)}(\cdot,x'_\alpha,\cdot)\right)(\underline{y}_\ell) := \phi(y_\alpha)g^{(\ell)}(\underline{y}_{1;\alpha-1},x'_\alpha,\underline{y}_{\alpha+1;\ell}), \qquad \underline{y}_\ell \in \mathbb{R}^\ell. \tag{3.5.14}$$

[17] We will see later that the choice of extension is immaterial.

Then $A^{(\ell)} \in \mathcal{L}_{gmp}(\mathcal{S}(\mathbb{R}^\ell), \mathcal{S}'(\mathbb{R}^\ell))$ if the element of $\mathcal{S}(\mathbb{R}; \mathcal{S}'(\mathbb{R}))$[18] defined by

$$x'_\alpha \mapsto \left\langle A^{(\ell)} f^{(\ell)}, (\cdot) \otimes_\alpha g^{(\ell)}(\cdot, x'_\alpha, \cdot) \right\rangle_{\mathcal{S}'(\mathbb{R}^\ell) - \mathcal{S}(\mathbb{R}^\ell)}, \tag{3.5.15}$$

may be identified with a (necessarily unique) Schwartz function $\Phi(f^{(\ell)}, g^{(\ell)})$ in $\mathcal{S}(\mathbb{R}^2)$ by

$$\left\langle A^{(\ell)} f^{(\ell)}, \phi \otimes_\alpha g^{(\ell)}(\cdot, x'_\alpha, \cdot) \right\rangle_{\mathcal{S}'(\mathbb{R}^\ell) - \mathcal{S}(\mathbb{R}^\ell)} = \int_{\mathbb{R}} dx_\alpha \Phi(f, g)(x_\alpha, x'_\alpha) \phi(x_\alpha), \qquad x'_\alpha \in \mathbb{R}, \tag{3.5.16}$$

and the assignment $\Phi : \mathcal{S}(\mathbb{R}^\ell) \times \mathcal{S}(\mathbb{R}^\ell) \to \mathcal{S}(\mathbb{R}^2)$ is continuous.

**Lemma 3.5.1** ($\circ_\alpha^\beta$ contraction)**.** *Let $i, j \in \mathbb{N}$, let $k := i + j - 1$, and let $(\alpha, \beta) \in \mathbb{N}_{\leq i} \times \mathbb{N}_{\leq j}$. Then there exists a bilinear map, continuous in the first entry,*

$$\circ_\alpha^\beta : \mathcal{L}(\mathcal{S}(\mathbb{R}^i), \mathcal{S}'(\mathbb{R}^i)) \times \mathcal{L}_{gmp}(\mathcal{S}(\mathbb{R}^j), \mathcal{S}'(\mathbb{R}^j)) \to \mathcal{L}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k)), \tag{3.5.17}$$

*such that $A^{(i)} \circ_\alpha^\beta B^{(j)}$ corresponds to*

$$A^{(i)} \circ_\alpha^\beta B^{(j)} = A^{(i)}_{(1,\ldots,i)} B^{(j)}_{(i+1,\ldots,i+\beta-1,\alpha,i+\beta,\ldots,k)}, \tag{3.5.18}$$

*when $A^{(i)} \in \mathcal{L}(\mathcal{S}(\mathbb{R}^i), \mathcal{S}(\mathbb{R}^i))$ and $B^{(j)} \in \mathcal{L}(\mathcal{S}(\mathbb{R}^j), \mathcal{S}(\mathbb{R}^j))$ or $A^{(i)} \in \mathcal{L}(\mathcal{S}(\mathbb{R}^i), \mathcal{S}'(\mathbb{R}^i))$ and $B^{(j)} \in \mathcal{L}(\mathcal{S}'(\mathbb{R}^j), \mathcal{S}'(\mathbb{R}^j))$. If we replace the domain space $\mathcal{L}(\mathcal{S}(\mathbb{R}^i), \mathcal{S}'(\mathbb{R}^i))$ for the first entry by $\mathcal{L}_{gmp}(\mathcal{S}(\mathbb{R}^i), \mathcal{S}'(\mathbb{R}^i))$, then the bilinear map*

$$\circ_\alpha^\beta : \mathcal{L}_{gmp}(\mathcal{S}(\mathbb{R}^i), \mathcal{S}'(\mathbb{R}^i)) \times \mathcal{L}_{gmp}(\mathcal{S}(\mathbb{R}^j), \mathcal{S}'(\mathbb{R}^j)) \to \mathcal{L}_{gmp}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k)) \tag{3.5.19}$$

*is continuous in the first entry.*

---

[18] Given a Hausdorff locally convex space $E$, we let $\mathcal{S}(\mathbb{R}^d; E)$ denote the space of functions $f \in C^\infty(\mathbb{R}^d; E)$ such that for each pair of $d$-dimensional polynomials $P$ and $Q$ with complex coefficients, the union $\bigcup_{x \in \mathbb{R}^d} \{P(x) Q(\partial_x) f(x)\}$ is contained in a bounded subset of $E$. We endow $\mathcal{S}(\mathbb{R}^d; E)$ with the topology of uniform convergence of the functions $P(x)Q(\partial_x)f(x)$, for all $P$ and $Q$.

**Remark 3.5.2.** Using this lemma and bosonic symmetry, we note that we can rewrite our definition of $[\cdot,\cdot]_1$ from (3.4.33) using the contractions $\circ_\alpha^\beta$ as follows: Let $i, j \in \mathbb{N}$ and set $k := i + j - 1$. We extend $[\cdot,\cdot]_1$ to be the bilinear, continuous in the first entry, map

$$\begin{aligned} [\cdot,\cdot]_1 : \mathcal{L}_{gmp}(\mathcal{S}(\mathbb{R}^i), \mathcal{S}'(\mathbb{R}^i)) \times \mathcal{L}_{gmp}(\mathcal{S}(\mathbb{R}^j), \mathcal{S}'(\mathbb{R}^j)) &\to \mathcal{L}_{gmp}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k)) \\ (A^{(i)}, B^{(j)}) &\mapsto \sum_{\alpha=1}^{i} \sum_{\beta=1}^{j} A^{(i)} \circ_\alpha^\beta B^{(j)} - B^{(j)} \circ_\beta^\alpha A^{(i)}, \end{aligned} \tag{3.5.20}$$

for $\circ_\alpha^\beta$ and $\circ_\beta^\alpha$ as in Lemma 3.5.1.

*Proof of Lemma 3.5.1.* We first show that for fixed $f \in \mathcal{S}(\mathbb{R}^k)$, there is a well-defined element

$$(A^{(i)} \circ_\alpha^\beta B^{(j)})(f) \in \mathcal{S}'(\mathbb{R}^k) \tag{3.5.21}$$

corresponding to

$$A^{(i)}_{(1,\ldots,i)} B^{(j)}_{(i+1,\ldots,i+\beta-1,\alpha,i+\beta,\ldots,k)}(f). \tag{3.5.22}$$

Let $g \in \mathcal{S}(\mathbb{R}^k)$. Now it follows from the assumption that $B^{(j)}$ has the good mapping property and Remark 3.3.4 that the bilinear map

$$(\tilde{f}, \tilde{g}) \mapsto \left\langle B^{(j)}_{(2,\ldots,\beta,1,\beta+1,\ldots,j)}(\tilde{f}(\underline{x}_{\alpha-1}, \cdot, \underline{x}_{\alpha+1;i}, \cdot)), (\cdot) \otimes \tilde{g}(\underline{x}'_i, \cdot) \right\rangle_{\mathcal{S}'(\mathbb{R}^j)-\mathcal{S}(\mathbb{R}^j)}, \tag{3.5.23}$$

which is a priori a bilinear continuous map

$$\mathcal{S}(\mathbb{R}^k) \times \mathcal{S}(\mathbb{R}^k) \to \mathcal{S}_{(\underline{x}_{\alpha-1}, \underline{x}_{\alpha+1;i}, \underline{x}'_i)}(\mathbb{R}^{\alpha-1} \times \mathbb{R}^{i-\alpha} \times \mathbb{R}^i; \mathcal{S}'_{x_\alpha}(\mathbb{R})), \tag{3.5.24}$$

is identifiable with a unique smooth map

$$\Phi_{B^{(j)},\alpha,\beta} : \mathcal{S}(\mathbb{R}^k) \times \mathcal{S}(\mathbb{R}^k) \to \mathcal{S}_{(\underline{x}_i; \underline{x}'_i)}(\mathbb{R}^{2i}). \tag{3.5.25}$$

Since we have the canonical isomorphism

$$\mathcal{L}(\mathcal{S}(\mathbb{R}^i), \mathcal{S}'(\mathbb{R}^i)) \cong \mathcal{S}'(\mathbb{R}^{2i}) \tag{3.5.26}$$

by the Schwartz kernel theorem, we therefore define the composition (3.5.21) by

$$\langle (A^{(i)} \circ_\alpha^\beta B^{(j)}) f, g \rangle_{\mathcal{S}'(\mathbb{R}^k) - \mathcal{S}(\mathbb{R}^k)} := \left\langle K_{A^{(i)}}, \Phi_{B^{(j)},\alpha,\beta}(f,g)^t \right\rangle_{\mathcal{S}'(\mathbb{R}^{2i}) - \mathcal{S}(\mathbb{R}^{2i})}, \tag{3.5.27}$$

where

$$\Phi_{B^{(j)},\alpha,\beta}(f,g)^t(\underline{x}_i; \underline{x}'_i) = \Phi_{B^{(j)},\alpha,\beta}(f,g)(\underline{x}'_i; \underline{x}_i), \qquad (\underline{x}_i, \underline{x}'_i) \in \mathbb{R}^{2i}.$$

Hence, taking (3.5.27) as the definition of (3.5.21) for $f \in \mathcal{S}(\mathbb{R}^k)$, we have defined an evidently linear map

$$A^{(i)} \circ_\alpha^\beta B^{(j)} : \mathcal{S}(\mathbb{R}^k) \to \mathcal{S}'(\mathbb{R}^k). \tag{3.5.28}$$

The continuity of this map follows from its definition as a composition of continuous maps. Bilinearity of $\circ_\alpha^\beta$ in $A^{(i)}$ and $B^{(j)}$ is obvious. Moreover, it is clear that if $B^{(j)}$ has the good mapping property, then $A^{(i)} \circ_\alpha^\beta B^{(j)}$ has the good mapping property. Lastly, the reader can check from the distributional Fubini-Tonelli theorem that our definition of $A^{(i)} \circ_\alpha^\beta B^{(j)}$ coincides with the composition (3.5.22) in the cse where $A^{(i)} \in \mathcal{L}(\mathcal{S}(\mathbb{R}^i), \mathcal{S}(\mathbb{R}^i))$ and $B^{(j)} \in \mathcal{L}(\mathcal{S}(\mathbb{R}^j), \mathcal{S}(\mathbb{R}^j))$ or $A^{(i)} \in \mathcal{L}(\mathcal{S}(\mathbb{R}^i), \mathcal{S}'(\mathbb{R}^i))$ and $B^{(j)} \in \mathcal{L}(\mathcal{S}'(\mathbb{R}^j), \mathcal{S}'(\mathbb{R}^j))$.

We now prove that the map

$$(\cdot) \circ_\alpha^\beta (\cdot) : \mathcal{L}(\mathcal{S}(\mathbb{R}^i), \mathcal{S}'(\mathbb{R}^i)) \times \mathcal{L}_{gmp}(\mathcal{S}(\mathbb{R}^j), \mathcal{S}'(\mathbb{R}^j)) \to \mathcal{L}_{gmp}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k)) \tag{3.5.29}$$

is continuous in the first entry, that is, for fixed $B^{(j)} \in \mathcal{L}_{gmp}(\mathcal{S}(\mathbb{R}^i), \mathcal{S}'(\mathbb{R}^i))$, the map

$$\mathcal{L}(\mathcal{S}(\mathbb{R}^i), \mathcal{S}'(\mathbb{R}^i)) \to \mathcal{L}_{gmp}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k)), \qquad A^{(i)} \mapsto A^{(i)} \circ_\alpha^\beta B^{(j)} \tag{3.5.30}$$

is continuous. By considerations of symmetry, it suffices to consider the case $(\alpha,\beta)=(1,1)$. To this end, it suffices to show that given a bounded subset $\mathfrak{R}^{(k)}\subset\mathcal{S}(\mathbb{R}^k)$, there exists a bounded subset $\mathfrak{R}^{(i)}\subset\mathcal{S}(\mathbb{R}^i)$ such that

$$\sup_{f^{(k)},g^{(k)}\in\mathfrak{R}^{(k)}}\left|\left\langle(A^{(i)}\circ_1^1 B^{(j)})f^{(k)}\middle|g^{(k)}\right\rangle\right|\lesssim\sup_{f^{(i)},g^{(i)}\in\mathfrak{R}^{(i)}}\left|\left\langle A^{(i)}f^{(i)}\middle|g^{(i)}\right\rangle\right|. \tag{3.5.31}$$

To see how to obtain the desired seminorm, first observe that

$$\begin{aligned}\left|\left\langle(A^{(i)}\circ_1^1 B^{(j)})f^{(k)}\middle|g^{(k)}\right\rangle\right| &= \left|\left\langle K_{A^{(i)}},\Phi_{B^{(j)},1,1}(f^{(k)},g^{(k)})^t\right\rangle_{\mathcal{S}'(\mathbb{R}^{2i})-\mathcal{S}(\mathbb{R}^{2i})}\right|\\ &=\left|\mathrm{Tr}_{1,\ldots,i}\big(A^{(i)}\Phi_{B^{(j)},1,1}(f^{(k)},g^{(k)})\big)\right|,\end{aligned} \tag{3.5.32}$$

where the ultimate equality follows from the definition of the generalized trace (recall Definition 3.2.1) and we commit an abuse of notation by using $\Phi_{B^{(j)},1,1}(f^{(k)},g^{(k)})$ to denote the operator in $\mathcal{L}(\mathcal{S}'(\mathbb{R}^i),\mathcal{S}(\mathbb{R}^i))$ defined by this integral kernel. Since $\mathfrak{R}^{(k)}$ is bounded, the image $\Phi_{B^{(j)},1,1}(\mathfrak{R}^{(k)}\times\mathfrak{R}^{(k)})$ is a bounded subset of $\mathcal{S}(\mathbb{R}^{2i})\cong\mathcal{L}(\mathcal{S}'(\mathbb{R}^i),\mathcal{S}(\mathbb{R}^i))$, and since $A^{(i)}$ is continuous, it follows that

$$\sup_{\gamma^{(i)}\in\Phi_{B^{(j)},1,1}(\mathfrak{R}^{(k)}\times\mathfrak{R}^{(k)})}\left|\mathrm{Tr}_{1,\ldots,i}\big(A^{(i)}\gamma^{(i)}\big)\right|<\infty. \tag{3.5.33}$$

Hence, there exists an element $\gamma_0^{(i)}\in\Phi_{B^{(j)},1,1}(\mathfrak{R}^{(k)}\times\mathfrak{R}^{(k)})$ such that

$$\left|\mathrm{Tr}_{1,\ldots,i}\left(A^{(i)}\gamma_0^{(i)}\right)\right|\geq\frac{1}{2}\sup_{\gamma^{(i)}\in\Phi_{B^{(j)},1,1}(\mathfrak{R}^{(k)}\times\mathfrak{R}^{(k)})}\left|\mathrm{Tr}_{1,\ldots,i}\big(A^{(i)}\gamma^{(i)}\big)\right|. \tag{3.5.34}$$

Since each element of $\mathcal{S}(\mathbb{R}^{2i})$ can be written as $\sum_{\ell=1}^{\infty}\lambda_\ell f_\ell^{(i)}\otimes g_\ell^{(i)}$, where $\sum_{\ell=1}^{\infty}|\lambda_\ell|\leq 1$, and $f_\ell^{(i)},g_\ell^{(i)}$ are sequences in $\mathcal{S}(\mathbb{R}^i)$ converging to zero, we see from the separate continuity of the generalized trace that

$$\begin{aligned}\left|\mathrm{Tr}_{1,\ldots,i}\Big(A^{(i)}\gamma_0^{(i)}\Big)\right| &\leq \sum_{\ell=1}^{\infty}|\lambda_\ell|\left|\mathrm{Tr}_{1,\ldots,i}\Big(A^{(i)}(f_{0,\ell}^{(i)}\otimes g_{0,\ell}^{(i)})\Big)\right| \\ &\leq \sup_{f^{(i)},g^{(i)}\in\{f_{0,\ell'}^{(i)},g_{0,\ell'}^{(i)}\}_{\ell'=1}^{\infty}}\left|\langle A^{(i)}f^{(i)},g^{(i)}\rangle_{\mathcal{S}'(\mathbb{R}^i)-\mathcal{S}(\mathbb{R}^i)}\right|.\end{aligned} \tag{3.5.35}$$

We claim that $\{f_{0,\ell}^{(i)},g_{0,\ell}^{(i)}\}_{\ell=1}^{\infty}$ is a bounded subset of $\mathcal{S}(\mathbb{R}^i)$, which then completes the proof. Indeed, this follows readily from the fact that $f_{0,\ell}^{(i)},g_{0,\ell}^{(i)}$ converge to zero. □

**Remark 3.5.3.** If we restrict the domain of the map $\circ_\alpha^\beta$ to the space

$$\mathcal{L}_{gmp,*}(\mathcal{S}(\mathbb{R}^i),\mathcal{S}'(\mathbb{R}^i))\times\mathcal{L}_{gmp,*}(\mathcal{S}(\mathbb{R}^j),\mathcal{S}'(\mathbb{R}^j))$$

consisting of distribution-valued operators satisfying the good mapping property such that their adjoints also satisfy the good mapping property, which we endow with the subspace topology, then it follows by duality that $\circ_\alpha^\beta$ is separately continuous on this space.

**Remark 3.5.4.** If $B^{(j)}\in\mathcal{L}_{gmp}(\mathcal{S}_s(\mathbb{R}^j),\mathcal{S}_s'(\mathbb{R}^j))$, then it follows from bosonic symmetry that for any $(\alpha,\beta)\in\mathbb{N}_{\leq i}\times\mathbb{N}_{\leq j}$,

$$A^{(i)}\circ_\alpha^\beta B^{(j)}=A^{(i)}\circ_\alpha^1 B^{(j)}. \tag{3.5.36}$$

**Remark 3.5.5.** If $A^{(i)}\in\mathcal{L}(\mathcal{S}_s(\mathbb{R}^i),\mathcal{S}'(\mathbb{R}^i))$ and $B^{(j)}\in\mathcal{L}_{gmp}(\mathcal{S}_s(\mathbb{R}^j),\mathcal{S}_s'(\mathbb{R}^j))$, then given two extensions $A_1^{(i)},A_2^{(i)}\in\mathcal{L}(\mathcal{S}(\mathbb{R}^i),\mathcal{S}'(\mathbb{R}^i))$ of $A^{(i)}$, we claim that

$$\sum_{\alpha=1}^{i}A_1^{(i)}\circ_\alpha^1 B^{(j)}=\sum_{\alpha=1}^{i}A_2^{(i)}\circ_\alpha^1 B^{(j)}\in\mathcal{L}(\mathcal{S}_s(\mathbb{R}^k),\mathcal{S}'(\mathbb{R}^k)). \tag{3.5.37}$$

Indeed, for $f\in\mathcal{S}_s(\mathbb{R}^k),g\in\mathcal{S}(\mathbb{R}^k)$, we have that

$$\sum_{\alpha=1}^{i}\langle g,(A_1^{(i)}\circ_\alpha^1 B^{(j)})f\rangle_{\mathcal{S}(\mathbb{R}^k)-\mathcal{S}'(\mathbb{R}^k)}=\sum_{\alpha=1}^{i}\Big\langle K_{A_1^{(i)}},\Phi_{B^{(j)},\alpha,1}(f,g)^t\Big\rangle_{\mathcal{S}'(\mathbb{R}^{2i})-\mathcal{S}(\mathbb{R}^{2i})}. \tag{3.5.38}$$

Since each $\Phi_{B^{(j)},\alpha,1}(f,g) \in \mathcal{S}(\mathbb{R}^{2i})$ and $f \in \mathcal{S}_s(\mathbb{R}^k)$, we see that

$$\sum_{\alpha=1}^{i} \Phi_{B^{(j)},\alpha,1}(f,g)(\pi(\underline{x}_i);\underline{x}_i') = \sum_{\alpha=1}^{i} \Phi_{B^{(j)},\alpha,1}(f,g)(\underline{x}_i;\underline{x}_i'), \qquad (\underline{x}_i,\underline{x}_i') \in \mathbb{R}^{2i}, \tag{3.5.39}$$

for any permutation $\pi \in \mathbb{S}_i$. Consequently, for fixed $\underline{x}_i' \in \mathbb{R}^i$, the function $\sum_{\alpha=1}^{i} \Phi_{B^{(j)},\alpha,1}(f,g)(\cdot,\underline{x}_i')$ belongs to $\mathcal{S}_s(\mathbb{R}^i)$ on which the two extensions $A_1^{(i)}$ and $A_2^{(i)}$ agree. It then follows from the Schwartz kernel theorem that

$$\left\langle K_{A_1^{(i)}}, \left(\sum_{\alpha=1}^{i} \Phi_{B^{(j)},\alpha,1}(f,g)\right)^t \right\rangle_{\mathcal{S}'(\mathbb{R}^{2i})-\mathcal{S}(\mathbb{R}^{2i})} = \left\langle K_{A_2^{(i)}}, \left(\sum_{\alpha=1}^{i} \Phi_{B^{(j)},\alpha,1}(f,g)\right)^t \right\rangle_{\mathcal{S}'(\mathbb{R}^{2i})-\mathcal{S}(\mathbb{R}^{2i})}, \tag{3.5.40}$$

and therefore

$$\sum_{\alpha=1}^{i} \langle g, (A_1^{(i)} \circ_\alpha^1 B^{(j)})f\rangle_{\mathcal{S}(\mathbb{R}^k)-\mathcal{S}'(\mathbb{R}^k)} = \sum_{\alpha=1}^{i} \langle g, (A_2^{(i)} \circ_\alpha^1 B^{(j)})f\rangle_{\mathcal{S}(\mathbb{R}^k)-\mathcal{S}'(\mathbb{R}^k)}, \tag{3.5.41}$$

which establishes our claim.

By Lemma 3.5.1,

$$A^{(\ell)} \circ_\alpha^\beta B^{(j)} \in \mathcal{L}_{gmp}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k)), \qquad \text{for } \ell + j - 1 = k. \tag{3.5.42}$$

Hence, by definition of the bracket $[\cdot,\cdot]_1$ and Remark 3.5.2,

$$\sum_{\substack{\ell,j\geq 1 \\ \ell+j-1=k}} \left[A^{(\ell)}, B^{(j)}\right]_1 \in \mathcal{L}_{gmp}(\mathcal{S}_s(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k)). \tag{3.5.43}$$

Thus it remains to show two properties: first that the symmetrization of an operator preserves the good mapping property, which will then establish that $C^{(k)} \in \mathcal{L}_{gmp}(\mathcal{S}_s(\mathbb{R}^k), \mathcal{S}_s'(\mathbb{R}^k))$, where $C^{(k)}$ is defined according to (3.5.9), and second that $C^{(k)}$ is skew-adjoint. We begin with the following lemma which establishes the desired property of the symmetrization operators.

**Lemma 3.5.6.** *If $A = (A^{(k)})_{k\in\mathbb{N}} \in \bigoplus_{k=1}^{\infty} \mathcal{L}_{gmp}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k))$, then*

$$\mathrm{Sym}(A) \in \bigoplus_{k=1}^{\infty} \mathcal{L}_{gmp}(\mathcal{S}_s(\mathbb{R}^k), \mathcal{S}_s'(\mathbb{R}^k)).$$

*Proof.* It suffices to show that for each $k \in \mathbb{N}$, if $A^{(k)} \in \mathcal{L}_{gmp}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k))$, then

$$\mathrm{Sym}_k(A^{(k)}) \in \mathcal{L}_{gmp}(\mathcal{S}_s(\mathbb{R}^k), \mathcal{S}_s'(\mathbb{R}^k)).$$

Let $\alpha \in \mathbb{N}_{\leq k}$. We need to show that the map

$$\begin{aligned} &\mathcal{S}_s(\mathbb{R}^k) \times \mathcal{S}_s(\mathbb{R}^k) \to \mathcal{S}(\mathbb{R}; \mathcal{S}'(\mathbb{R})) \\ &(f^{(k)}, g^{(k)}) \mapsto \left\langle \mathrm{Sym}_k(A^{(k)})(f^{(k)}), (\cdot) \otimes_\alpha g(\cdot, x_\alpha', \cdot) \right\rangle_{\mathcal{S}'(\mathbb{R}^k)-\mathcal{S}(\mathbb{R}^k)} \end{aligned} \tag{3.5.44}$$

may be identified with a continuous map $\mathcal{S}_s(\mathbb{R}^k) \times \mathcal{S}_s(\mathbb{R}^k) \to \mathcal{S}(\mathbb{R}^2)$. By definition of the $\mathrm{Sym}_k$ operator and bilinearity of the distributional pairing, we have that

$$\begin{aligned} &\left\langle \mathrm{Sym}_k(A^{(k)}) f^{(k)}, (\cdot) \otimes_\alpha g^{(k)}(\cdot, x_\alpha', \cdot) \right\rangle_{\mathcal{S}'(\mathbb{R}^k)-\mathcal{S}(\mathbb{R}^k)} \\ &= \frac{1}{k!} \sum_{\pi \in \mathbb{S}_k} \left\langle A^{(k)}_{(\pi(1),\ldots,\pi(k))} f^{(k)}, (\cdot) \otimes_\alpha g^{(k)}(\cdot, x_\alpha', \cdot) \right\rangle_{\mathcal{S}'(\mathbb{R}^k)-\mathcal{S}(\mathbb{R}^k)}. \end{aligned} \tag{3.5.45}$$

By definition of the notation $A^{(k)}_{(\pi(1),\ldots,\pi(k))} = \pi \circ A^{(k)}_{1,\ldots,k} \circ \pi^{-1}$, we have that

$$\begin{aligned} &\left\langle A^{(k)}_{(\pi(1),\ldots,\pi(k))} f^{(k)}, (\cdot) \otimes_\alpha g^{(k)}(\cdot, x_\alpha', \cdot) \right\rangle_{\mathcal{S}'(\mathbb{R}^k)-\mathcal{S}(\mathbb{R}^k)} \\ &= \left\langle A^{(k)}(f^{(k)} \circ \pi^{-1}) \circ \pi, (\cdot) \otimes_\alpha g^{(k)}(\cdot, x_\alpha', \cdot) \right\rangle_{\mathcal{S}'(\mathbb{R}^k)-\mathcal{S}(\mathbb{R}^k)} \\ &= \left\langle A^{(k)}(f^{(k)}) \circ \pi, (\cdot) \otimes_\alpha g^{(k)}(\cdot, x_\alpha', \cdot) \right\rangle_{\mathcal{S}'(\mathbb{R}^k)-\mathcal{S}(\mathbb{R}^k)}, \end{aligned} \tag{3.5.46}$$

where the ultimate equality follows from the assumption $f^{(k)} \in \mathcal{S}_s(\mathbb{R}^k)$. Let $\phi \in \mathcal{S}(\mathbb{R})$ be a test function. Then by definition of the permutation of a distribution,

$$\left\langle A^{(k)}(f^{(k)}) \circ \pi, \phi \otimes_\alpha g^{(k)}(\cdot, x_\alpha', \cdot) \right\rangle_{\mathcal{S}'(\mathbb{R}^k)-\mathcal{S}(\mathbb{R}^k)} = \left\langle A^{(k)} f^{(k)}, (\phi \otimes_\alpha g^{(k)}(\cdot, x_\alpha', \cdot)) \circ \pi^{-1} \right\rangle_{\mathcal{S}'(\mathbb{R}^k)-\mathcal{S}(\mathbb{R}^k)}. \tag{3.5.47}$$

Observing that

$$((\phi \otimes_\alpha g^{(k)}(\cdot, x'_\alpha, \cdot)) \circ \pi^{-1})(\underline{x}_k) = g^{(k)}(x_{\pi^{-1}(1)}, \ldots, x_{\pi^{-1}(\alpha-1)}, x'_\alpha, x_{\pi^{-1}(\alpha+1)}, \ldots, x_{\pi^{-1}(k)})\phi(x_{\pi^{-1}(\alpha)}), \quad \underline{x}_k \in \mathbb{R}^k, \tag{3.5.48}$$

upon setting $j := \pi^{-1}(\alpha)$ and using the bosonic symmetry of $g^{(k)}$, we obtain that

$$((\phi \otimes_\alpha g^{(k)}(\cdot, x'_\alpha, \cdot)) \circ \pi^{-1})(\underline{x}_k) = g^{(k)}(\underline{x}_{j-1}, x'_\alpha, \underline{x}_{j+1;k})\phi(x_j) = (\phi \otimes_j g^{(k)}(\cdot, x'_\alpha, \cdot))(\underline{x}_k). \tag{3.5.49}$$

Since $A^{(k)}$ has the good mapping property, we have that

$$\left\langle A^{(k)} f^{(k)}, \phi \otimes_j g^{(k)}(\cdot, x'_\alpha, \cdot) \right\rangle_{\mathcal{S}'(\mathbb{R}^k)-\mathcal{S}(\mathbb{R}^k)} = \left\langle \Phi_{A^{(k)},j}(f^{(k)}, g^{(k)})(\cdot, x'_\alpha), \phi \right\rangle_{\mathcal{S}'(\mathbb{R})-\mathcal{S}(\mathbb{R})}, \tag{3.5.50}$$

where $\Phi_{A^{(k)},j} : \mathcal{S}(\mathbb{R}^k) \times \mathcal{S}(\mathbb{R}^k) \to \mathcal{S}(\mathbb{R}^2)$ is a continuous bilinear map. Since $\mathcal{S}_s(\mathbb{R}^k)$ continuously embeds (trivially) in $\mathcal{S}(\mathbb{R}^k)$ and since $\alpha \in \mathbb{N}_{\leq k}$ was arbitrary, we conclude that (3.5.45) is identifiable with a finite sum of continuous bilinear maps $\mathcal{S}_s(\mathbb{R}^k) \times \mathcal{S}_s(\mathbb{R}^k) \to \mathcal{S}(\mathbb{R}^2)$, and the proof of the lemma is complete. □

Finally, to conclude our proof that the Lie bracket is well-defined, we only need to verify that $C^{(k)}$ defined according to (3.5.9) is skew-adjoint. This is a consequence of Remark 3.5.2, Remark 3.5.5, and the following lemma.

**Lemma 3.5.7.** *Let $i, j \in \mathbb{N}$, and define $k := i + j - 1$. Let $A^{(i)} \in \mathcal{L}_{gmp}(\mathcal{S}(\mathbb{R}^i), \mathcal{S}'(\mathbb{R}^i))$ and $B^{(j)} \in \mathcal{L}_{gmp}(\mathcal{S}(\mathbb{R}^j), \mathcal{S}'(\mathbb{R}^j))$ be skew-adjoint distribution-valued operators. Then for any $(\alpha, \beta) \in \mathbb{N}_{\leq i} \times \mathbb{N}_{\leq j}$,*

$$(A^{(i)} \circ_\alpha^\beta B^{(j)})^* = (B^{(j)} \circ_\beta^\alpha A^{(i)})_{(i+1,\ldots,i+\beta-1,\alpha,i+\beta,\ldots,k,1,\ldots,i)} \in \mathcal{L}_{gmp}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k)). \tag{3.5.51}$$

*Proof.* By considerations of symmetry, it suffices to consider the case where $(\alpha, \beta) = (1, 1)$. Recalling the definition of the adjoint of a distribution-valued operator, see Lemma 3.1.1, we need to show that

$$\begin{aligned}&\left\langle (B^{(j)} \circ_1^1 A^{(i)})_{(1,i+1,\ldots,k,2,\ldots,i)} g, \bar{f} \right\rangle_{\mathcal{S}'(\mathbb{R}^k)-\mathcal{S}(\mathbb{R}^k)} \\ &= \overline{\langle (A^{(i)} \circ_1^1 B^{(j)}) f, \overline{g} \rangle_{\mathcal{S}'(\mathbb{R}^k)-\mathcal{S}(\mathbb{R}^k)}},\end{aligned} \tag{3.5.52}$$

for any $f, g \in \mathcal{S}(\mathbb{R}^k)$. By Lemma 3.3.2,

$$A^{(i)}_{(1,\ldots,i)} \text{ and } B^{(j)}_{(1,i+1,\ldots,k)}$$

are both skew-adjoint elements of $\mathcal{L}_{gmp}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k))$. Now by density of linear combinations of pure tensors, linearity, and the continuity of the operators $A^{(i)}_{(1,\ldots,i)}$, $B^{(j)}_{(1,i+1,\ldots,k)}$, and $A^{(i)} \circ_1^1 B^{(j)}$, it suffices to consider the expression

$$\overline{\langle (A^{(i)} \circ_1^1 B^{(j)}) f, \overline{g} \rangle_{\mathcal{S}'(\mathbb{R}^k)-\mathcal{S}(\mathbb{R}^k)}} \tag{3.5.53}$$

in the case where $f, g \in \mathcal{S}(\mathbb{R}^k)$ are pure tensors of the form

$$f = \bigotimes_{a=1}^{k} f_a \text{ and } g = \bigotimes_{a=1}^{k} g_a, \tag{3.5.54}$$

respectively, where $f_1, \ldots, f_k, g_1, \ldots, g_k \in \mathcal{S}(\mathbb{R})$. Recalling the definition (3.5.27) for $A^{(i)} \circ_1^1 B^{(j)}$, we have that

$$\overline{\langle (A^{(i)} \circ_1^1 B^{(j)}) f, \bar{g} \rangle_{\mathcal{S}'(\mathbb{R}^k)-\mathcal{S}(\mathbb{R}^k)}} = \overline{\left\langle K_{A^{(i)}}, \Phi_{B^{(j)},1,1}(f, \bar{g})^t \right\rangle_{\mathcal{S}'(\mathbb{R}^{2i})-\mathcal{S}(\mathbb{R}^{2i})}}.$$

An examination of the $\Phi_{B^{(j)}}(f, \bar{g})$ together with the tensor product structure of $f$ and $g$

reveals that

$$\Phi_{B^{(j)},1,1}(f,\bar{g})(\underline{x}_i;\underline{x}_i') = \underbrace{(\bigotimes_{a=2}^{i} f_a)}_{=:f^{(i-1)}}(\underline{x}_{2;i}) \quad \underbrace{(\bigotimes_{a=1}^{i}\overline{g_a})}_{=:\overline{g^{(1)}}\otimes\overline{g^{(i-1)}}} \; (\underline{x}_i') \tag{3.5.55}$$
$$\times \left\langle B^{(j)}\left(f_1 \otimes \bigotimes_{a=i+1}^{k} f_a\right), (\cdot)\otimes \bigotimes_{a=i+1}^{k}\overline{g_a}\right\rangle_{\mathcal{S}'(\mathbb{R}^j)-\mathcal{S}(\mathbb{R}^j)} (x_1).$$

Since $B^{(j)}$ has the good mapping property, it follows that the element of $\mathcal{S}'_{x_1}(\mathbb{R})$ defined by the second factor in the right-hand side of (3.5.55) is in fact an element of $\mathcal{S}(\mathbb{R})$, which we denote by

$$\phi_{B^{(j)},1}\left(f_1 \otimes \bigotimes_{a=i+1}^{k} f_a, \bigotimes_{a=i+1}^{k}\overline{g_a}\right) =: \phi_{B^{(j)},1}(f^{(j)},\overline{g^{(j-1)}}). \tag{3.5.56}$$

Thus, using (3.5.56) and (3.5.55), we can write

$$\Phi_{B^{(j)},1,1}(f,\bar{g})(\underline{x}_i;\underline{x}_i') = \phi_{B^{(j)},1}(f^{(j)},\overline{g^{(j-1)}})(x_1)f^{(i-1)}(\underline{x}_{2;i})\overline{g^{(1)}}(x_1')\overline{g^{(i-1)}}(\underline{x}_{2;i}'), \qquad (\underline{x}_i,\underline{x}_i') \in \mathbb{R}^{2i}, \tag{3.5.57}$$

and

$$\begin{aligned} &\overline{\left\langle K_{A^{(i)}}, \Phi_{B^{(j)},1,1}(f,\bar{g})^t\right\rangle_{\mathcal{S}'(\mathbb{R}^{2i})-\mathcal{S}(\mathbb{R}^{2i})}} \\ &= \overline{\left\langle A^{(i)}\left(\phi_{B^{(j)},1}(f^{(j)},\overline{g^{(j-1)}})\otimes f^{(i-1)}\right), \overline{g^{(1)}\otimes g^{(i-1)}}\right\rangle_{\mathcal{S}'(\mathbb{R}^i)-\mathcal{S}(\mathbb{R}^i)}} \end{aligned} \tag{3.5.58}$$

by the Schwartz kernel theorem. Since $A^{(i)}$ is skew-adjoint, we have that this last expression equals

$$-\left\langle A^{(i)}\left(g^{(1)}\otimes g^{(i-1)}\right), \overline{\phi_{B^{(j)},1}(f^{(j)},\overline{g^{(j-1)}})\otimes f^{(i-1)}}\right\rangle_{\mathcal{S}'(\mathbb{R}^i)-\mathcal{S}(\mathbb{R}^i)}. \tag{3.5.59}$$

Now since $A^{(i)}$ also has the good mapping property by assumption, the element of $\mathcal{S}'_{x_1}(\mathbb{R})$ defined by

$$-\left\langle A^{(i)}\left(g^{(1)}\otimes g^{(i-1)}\right), (\cdot)\otimes\overline{f^{(i-1)}}\right\rangle_{\mathcal{S}'(\mathbb{R}^i)-\mathcal{S}(\mathbb{R}^i)} \tag{3.5.60}$$

is identifiable with a unique element of $\mathcal{S}_{x_1}(\mathbb{R})$, which we denote by

$$-\phi_{A^{(i)},1}(g^{(1)} \otimes g^{(i-1)}, \overline{f^{(i-1)}}). \tag{3.5.61}$$

Using (3.5.61), we see that

$$(3.5.59) = -\int_{\mathbb{R}} dx \phi_{A^{(i)},1}(g^{(1)} \otimes g^{(i-1)}, \overline{f^{(i-1)}})(x)\overline{\phi_{B^{(j)},1}(f^{(j)}, \overline{g^{(j-1)}})(x)}. \tag{3.5.62}$$

After unpacking the definition of the Schwartz function $\phi_{B^{(j)},1}(f^{(j)}, \overline{g^{(j-1)}})$ given in (3.5.55) and (3.5.56), it follows that

$$\begin{aligned}
(3.5.62) &= \overline{\left\langle B^{(j)} f^{(j)}, \overline{\phi_{A^{(i)},1}(g^{(1)} \otimes g^{(i-1)}, \overline{f^{(i-1)}}) \otimes g^{(j-1)}} \right\rangle_{\mathcal{S}'(\mathbb{R}^j)-\mathcal{S}(\mathbb{R}^j)}} \\
&= \left\langle B^{(j)} \left( \phi_{A^{(i)},1}(g^{(1)} \otimes g^{(i-1)}, \overline{f^{(i-1)}}) \otimes g^{(j-1)} \right), \overline{f^{(j)}} \right\rangle_{\mathcal{S}'(\mathbb{R}^j)-\mathcal{S}(\mathbb{R}^j)} \\
&= \left\langle K_{B^{(j)}}, \left( \left( \phi_{A^{(i)},1}(g^{(1)} \otimes g^{(i-1)}, \overline{f^{(i-1)}}) \otimes g^{(j-1)} \right) \otimes \overline{f^{(j)}} \right)^t \right\rangle_{\mathcal{S}'(\mathbb{R}^{2j})-\mathcal{S}(\mathbb{R}^{2j})},
\end{aligned} \tag{3.5.63}$$

where we use the skew-adjointness of $B^{(j)}$ to obtain the penultimate equality and the Schwartz kernel theorem to obtain the ultimate equality.

Our goal now is to show that

$$\begin{aligned}
&\left( \phi_{A^{(i)},1}(g^{(1)} \otimes g^{(i-1)}, \overline{f^{(i-1)}}) \otimes g^{(j-1)} \right) \otimes \overline{f^{(j)}}(\underline{x}_j; \underline{x}_j') \\
&= \Phi_{A^{(i)},1,1}(g \circ \pi, \bar{f} \circ \pi)(\underline{x}_j; \underline{x}_j')
\end{aligned} \tag{3.5.64}$$

where $\pi \in \mathbb{S}_k$ is the permutation

$$\pi(a) = \begin{cases} 1, & a = 1 \\ a + j - 1, & 2 \leq a \leq i \\ a - i + 1, & i + 1 \leq a \leq k. \end{cases} \tag{3.5.65}$$

With (3.5.64), we then have by definition of the composite distribution $B^{(j)} \circ_1^1 A^{(i)}$, see (3.5.27), and the notation

$$(B^{(j)} \circ_1^1 A^{(i)})_{(1,i+1,\ldots,k,2,\ldots,i)},$$

see Proposition 3.3.1, that

$$
\begin{aligned}
(3.5.63) &= \left\langle K_{B^{(j)}}, \Phi_{A^{(i)},1,1}(g\circ\pi, \bar{f}\circ\pi)^t\right\rangle_{\mathcal{S}'(\mathbb{R}^{2j})-\mathcal{S}(\mathbb{R}^{2j})} \\
&= \left\langle (B^{(j)} \circ_1^1 A^{(i)})(g\circ\pi), \bar{f}\circ\pi\right\rangle_{\mathcal{S}'(\mathbb{R}^{k})-\mathcal{S}(\mathbb{R}^{k})} \\
&= \left\langle (B^{(j)} \circ_1^1 A^{(i)})_{(1,i+1,\ldots,k,2,\ldots,i)} g, \bar{f}\right\rangle_{\mathcal{S}'(\mathbb{R}^{k})-\mathcal{S}(\mathbb{R}^{k})}, \qquad (3.5.66)
\end{aligned}
$$

which is exactly what we needed to show.

Turning to (3.5.64), observe that

$$
(g\circ\pi)(\underline{x}_k) = g(x_1, x_{j+1}, \ldots, x_k, x_2, \ldots, x_j) = g_1(x_1)(\bigotimes_{a=2}^{i} g_a)(\underline{x}_{j+1;k})(\bigotimes_{a=i+1}^{k} g_a)(\underline{x}_{2;j}), \quad (3.5.67)
$$

and similarly for $(\bar{f}\circ\pi)$. By the same analysis as in (3.5.55), it then follows that

$$
\begin{aligned}
\Phi_{A^{(i)},1,1}(g\circ\pi, \bar{f}\circ\pi)(\underline{x}_j; \underline{x}'_j) &= (\bigotimes_{a=i+1}^{k} g_a)(\underline{x}_{2;j})(\bigotimes_{a=i+1}^{k} \overline{f_a})(\underline{x}'_{2;j})\overline{f_1}(x'_1) \\
&\quad \times \left\langle A^{(i)}(\bigotimes_{a=1}^{i} g_a), (\cdot)\otimes \bigotimes_{a=2}^{i-1}\overline{f_a}\right\rangle_{\mathcal{S}'(\mathbb{R}^{i})-\mathcal{S}(\mathbb{R}^{i})} (x_1) \\
&= \phi_{A^{(i)},1}(g^{(1)}\otimes g^{(i-1)}, \overline{f^{(i-1)}})(x_1) g^{(j-1)}(\underline{x}_{2;j}) f^{(j)}(x'_j), \quad (3.5.68)
\end{aligned}
$$

as desired. $\square$

We now turn to the proof of Proposition 3.1.7.

*Proof of Proposition 3.1.7.* We first verify the Lie bracket properties (L1)-(L3) in Definition 3.3.14. Bilinearity and anti-symmetry are immediate from the linearity of the bosonic symmetrization Sym operator, see (3.3.43) above, and the bilinearity and anti-symmetry of the bracket $[\cdot,\cdot]_1$.

To verify the Jacobi identity

$$[A,[B,C]]^{(k)} + [C,[A,B]]^{(k)} + [B,[C,A]]^{(k)} = 0, \tag{3.5.69}$$

we use our convergence result Proposition 3.1.4 together with the fact that $[\cdot,\cdot]_{\mathfrak{G}_N}$ is a Lie bracket by Proposition 3.1.1. Let $A, B, C \in \mathfrak{G}_\infty$, where $A = (A^{(k)})_{k\in\mathbb{N}}, B = (B^{(k)})_{k\in\mathbb{N}}, C = (C^{(k)})_{k\in\mathbb{N}}$. Note that since $\mathfrak{G}_\infty$ is a direct sum, there exists an $N_0 \in \mathbb{N}$ such that $A^{(k)} = B^{(k)} = C^{(k)} = 0$ for $k \geq N_0$. Now by mollifying and truncating the Schwartz kernels of the $k$-particle components $A^{(k)}, B^{(k)}, C^{(k)}$, we obtain approximating sequences

$$A_{n_1} := (A_{n_1}^{(k)})_{k\in\mathbb{N}},\ B_{n_2} := (B_{n_2}^{(k)})_{k\in\mathbb{N}},\ C_{n_3} := (C_{n_3}^{(k)})_{k\in\mathbb{N}} \in \mathfrak{G}_\infty \cap \bigoplus_{k=1}^{\infty} \mathcal{L}(\mathcal{S}_s'(\mathbb{R}^k), \mathcal{S}_s(\mathbb{R}^k)) \tag{3.5.70}$$

such that for all $(n_1, n_2, n_3) \in \mathbb{N}^3$, $A_{n_1}^{(k)} = B_{n_2}^{(k)} = C_{n_3}^{(k)} = 0 \in \mathfrak{g}_{k,gmp}$ for $k \geq N_0$. In particular, $A_{n_1}, B_{n_2}, C_{n_3} \in \mathfrak{G}_M$ for any integer $M \geq N_0$. Now for such $M$, we know from the Jacobi identity for $[\cdot,\cdot]_{\mathfrak{G}_M}$ that

$$\left[A_{n_1}, [B_{n_2}, C_{n_3}]_{\mathfrak{G}_M}\right]_{\mathfrak{G}_M} + \left[C_{n_3}, [A_{n_1}, B_{n_2}]_{\mathfrak{G}_M}\right]_{\mathfrak{G}_M} + \left[B_{n_2}, [C_{n_3}, A_{n_1}]_{\mathfrak{G}_M}\right]_{\mathfrak{G}_M} = 0 \in \mathfrak{G}_M \subset \mathfrak{G}_\infty. \tag{3.5.71}$$

Consequently, for fixed $(n_1, n_2, n_3) \in \mathbb{N}^3$, we obtain from Proposition 3.1.4 that

$$\begin{aligned} 0 &= \lim_{M\to\infty} \Big( \left[A_{n_1}, [B_{n_2}, C_{n_3}]_{\mathfrak{G}_M}\right]_{\mathfrak{G}_M} + \left[C_{n_3}, [A_{n_1}, B_{n_2}]_{\mathfrak{G}_M}\right]_{\mathfrak{G}_M} + \left[B_{n_2}, [C_{n_3}, A_{n_1}]_{\mathfrak{G}_M}\right]_{\mathfrak{G}_M} \Big) \\ &= \left[A_{n_1}, [B_{n_2}, C_{n_3}]_{\mathfrak{G}_\infty}\right]_{\mathfrak{G}_\infty} + \left[C_{n_3}, [A_{n_1}, B_{n_2}]_{\mathfrak{G}_\infty}\right]_{\mathfrak{G}_\infty} + \left[B_{n_2}, [C_{n_3}, A_{n_1}]_{\mathfrak{G}_\infty}\right]_{\mathfrak{G}_\infty}. \end{aligned} \tag{3.5.72}$$

Next, using three applications of the separate continuity of the bracket $[\cdot,\cdot]_{\mathfrak{G}_\infty}$ established below, we have that

$$\big[A, [B,C]_{\mathfrak{G}_\infty}\big]_{\mathfrak{G}_\infty} = \lim_{n_1\to\infty}\lim_{n_2\to\infty}\lim_{n_3\to\infty} \big[A_{n_1}, [B_{n_2}, C_{n_3}]_{\mathfrak{G}_\infty}\big]_{\mathfrak{G}_\infty}, \tag{3.5.73}$$

$$\big[C, [A,B]_{\mathfrak{G}_\infty}\big]_{\mathfrak{G}_\infty} = \lim_{n_1\to\infty}\lim_{n_2\to\infty}\lim_{n_3\to\infty} \big[C_{n_3}, [A_{n_1}, B_{n_2}]_{\mathfrak{G}_\infty}\big]_{\mathfrak{G}_\infty}, \tag{3.5.74}$$

$$\big[B, [C,A]_{\mathfrak{G}_\infty}\big]_{\mathfrak{G}_\infty} = \lim_{n_1\to\infty}\lim_{n_2\to\infty}\lim_{n_3\to\infty} \big[B_{n_2}, [C_{n_3}, A_{n_1}]_{\mathfrak{G}_\infty}\big]_{\mathfrak{G}_\infty}. \tag{3.5.75}$$

Summarizing our computations, we have shown that

$$\begin{aligned} 0 &= \lim_{n_1\to\infty}\lim_{n_2\to\infty}\lim_{n_3\to\infty}\lim_{M\to\infty}\Big(\big[A_{n_1}, [B_{n_2}, C_{n_3}]_{\mathfrak{G}_M}\big]_{\mathfrak{G}_M} + \big[C_{n_3}, [A_{n_1}, B_{n_2}]_{\mathfrak{G}_M}\big]_{\mathfrak{G}_M} \\ &\qquad + \big[B_{n_2}, [C_{n_3}, A_{n_1}]_{\mathfrak{G}_M}\big]_{\mathfrak{G}_M}\Big) \\ &= \big[A, [B,C]_{\mathfrak{G}_\infty}\big]_{\mathfrak{G}_\infty} + \big[C, [A,B]_{\mathfrak{G}_\infty}\big]_{\mathfrak{G}_\infty} + \big[B, [C,A]_{\mathfrak{G}_\infty}\big]_{\mathfrak{G}_\infty}, \end{aligned} \tag{3.5.76}$$

which completes the proof of the Jacobi identity.

Finally, we check that the map $[\cdot,\cdot]_{\mathfrak{G}_\infty}$ is separately continuous. By linearity, it suffices to show that for each fixed $\ell, j \in \mathbb{N}$ and fixed $\alpha \in \mathbb{N}_{\leq \ell}$, the binary operation $\circ_\alpha^1$ is separately continuous as a map

$$\circ_\alpha^1 : \mathfrak{g}_{\ell,gmp} \times \mathfrak{g}_{j,gmp} \to \mathcal{L}_{gmp,*}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k)) \tag{3.5.77}$$

where $k := \ell + j - 1$ and where the space $\mathcal{L}_{gmp,*}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k))$ consists of distribution-valued operators satisfying the good mapping property such that their adjoints also satisfy the good mapping property, endowed with the subspace topology. This property follows from Remark 3.5.3 together with the fact that the adjoints of elements in $\mathfrak{g}_{\ell,gmp}$ and $\mathfrak{g}_{j,gmp}$ also satisfy the good mapping property by skew-adjointness. Thus, the proof of the proposition is complete. □

### 3.5.3 The Lie-Poisson Manifold $\mathfrak{G}_\infty^*$ of Density Matrix $\infty$-Hierarchies

In this subsection, we define the Poisson structure on $\mathfrak{G}_\infty^*$, which will be used in the sequel in order to establish Hamiltonian properties of the GP hierarchy. Since many of the proofs from Section 3.4.2 carry over with trivial modification, as they do not make use of the good mapping property, we focus instead in this section on the parts of the construction which require the good mapping property. To begin, we define the real topological vector space

$$\mathfrak{G}_\infty^* := \{\Gamma = (\gamma^{(k)})_{k\in\mathbb{N}} \in \prod_{k=1}^{\infty} \mathcal{L}(\mathcal{S}_s'(\mathbb{R}^k), \mathcal{S}_s(\mathbb{R}^k)) : \gamma^{(k)} = (\gamma^{(k)})^* \ \ \forall k \in \mathbb{N}\}, \tag{3.5.78}$$

endowed with the product topology.[19] Analogous to Lemma 3.4.14, it holds that $\mathfrak{G}_\infty^*$ is isomorphic to the dual of $(\mathfrak{G}_\infty)^*$.

**Lemma 3.5.8** (Dual of $\mathfrak{G}_\infty$)**.** *The topological dual of $\mathfrak{G}_\infty$, denoted by $(\mathfrak{G}_\infty)^*$ and endowed with the strong dual topology, is isomorphic to $\mathfrak{G}_\infty^*$.*

We now need to established the existence of a Poisson structure on $\mathfrak{G}_\infty^*$. We start by specifying a unital sub-algebra of $C^\infty(\mathfrak{G}_\infty^*; \mathbb{R})$.

**Definition 3.5.9.** Let $\mathcal{A}_\infty$ be the algebra with respect to point-wise product generated by functionals in

$$\{F \in C^\infty(\mathfrak{G}_\infty^*; \mathbb{R}) : F(\cdot) = i\,\mathrm{Tr}(A\cdot),\ A \in \mathfrak{G}_\infty\} \cup \{F \in C^\infty(\mathfrak{G}_\infty^*; \mathbb{R}) : F(\cdot) \equiv C \in \mathbb{R}\}. \tag{3.5.79}$$

[19]We remark that $\mathfrak{G}_\infty^*$ is the projective limit of the spaces $\{\mathfrak{G}_N^*\}_{N\in\mathbb{N}}$ directed with respect to reverse inclusion.

In other words, $\mathcal{A}_\infty$ is the algebra (under point-wise product) generated by constants and the image of $\mathfrak{G}_\infty$ under the canonical embedding into $(\mathfrak{G}_\infty^*)^*$. We note that our previous remarks Remark 3.4.16, Remark 3.4.17, Remark 3.4.18 carry over with $\mathcal{A}_{H,N}$ replaced by $\mathcal{A}_\infty$.

We now wish to define the Lie-Poisson bracket $\{\cdot,\cdot\}_{\mathfrak{G}_\infty^*}$ on $\mathcal{A}_\infty \times \mathcal{A}_\infty$ using the Lie bracket constructed in Section 3.5.2. In order to so, we first need an identification of continuous linear functionals as skew-adjoint operators, which follows from Lemma 3.4.19.

**Lemma 3.5.10** (Dual of $\mathfrak{G}_\infty^*$)**.** *The topological dual of $\mathfrak{G}_\infty^*$, denoted by $(\mathfrak{G}_\infty^*)^*$ and endowed with the strong dual topology, is isomorphic to*

$$\widetilde{\mathfrak{G}}_\infty := \{A \in \bigoplus_{k=1}^\infty \mathcal{L}(\mathcal{S}_s(\mathbb{R}^k), \mathcal{S}_s'(\mathbb{R}^k)) : (A^{(k)})^* = -A^{(k)}\}, \tag{3.5.80}$$

*equipped with the subspace topology induced by $\bigoplus_{k=1}^\infty \mathcal{L}(\mathcal{S}_s(\mathbb{R}^k), \mathcal{S}_s'(\mathbb{R}^k))$, via the canonical bilinear form*

$$i\,\mathrm{Tr}(A \cdot \Gamma) = i\sum_{k=1}^\infty \mathrm{Tr}_{1,\ldots,k}(A^{(k)}\gamma^{(k)}), \qquad \Gamma = (\gamma^{(k)})_{k\in\mathbb{N}} \in \mathfrak{G}_\infty^*. \tag{3.5.81}$$

**Remark 3.5.11.** The previous lemma implies that, given a smooth real-valued functional $F : \mathfrak{G}_\infty^* \to \mathbb{R}$ and a point $\Gamma \in \mathfrak{G}_\infty^*$, we may identify the continuous linear functional $dF[\Gamma]$, given by the Gâteaux derivative of $F$ at $\Gamma$, as a skew-adjoint element of $\bigoplus_{k=1}^\infty \mathcal{L}(\mathcal{S}_s(\mathbb{R}^k), \mathcal{S}_s'(\mathbb{R}^k))$. We will abuse notation by denoting this element by $dF[\Gamma] = (dF[\Gamma]^{(k)})_{k\in\mathbb{N}}$.

We are now prepared to introduce the Lie-Poisson bracket $\{\cdot,\cdot\}_{\mathfrak{G}_\infty^*}$ on $\mathcal{A}_\infty \times \mathcal{A}_\infty$.

**Definition 3.5.12.** For $F, G \in \mathcal{A}_\infty$, we define

$$\{F,G\}_{\mathfrak{G}_\infty^*}(\Gamma) := i\,\mathrm{Tr}\big([dF[\Gamma], dG[\Gamma]]_{\mathfrak{G}_\infty} \cdot \Gamma\big), \qquad \forall \Gamma \in \mathfrak{G}_\infty^*. \tag{3.5.82}$$

**Remark 3.5.13** (Existence of Casimirs)**.** The functional $F(\Gamma) \coloneqq \mathrm{Tr}_1(\gamma^{(1)})$ is a Casimir[20] for the Poisson bracket $\{\cdot,\cdot\}_{\mathfrak{G}_\infty^*}$. Consequently, the Poisson bracket $\{\cdot,\cdot\}_{\mathfrak{G}_\infty^*}$ is not canonically induced by a symplectic structure on $\mathfrak{G}_\infty^*$.

We now turn to our ultimate goal of this subsection, that is, proving the following:

**Proposition 3.1.8.** *$(\mathfrak{G}_\infty^*, \mathcal{A}_\infty, \{\cdot,\cdot\}_{\mathfrak{G}_\infty^*})$ is a weak Poisson manifold.*

Properties (P1) and (P2) in Definition 3.3.1 for weak Poisson manifolds are readily proved using the same arguments in the proofs of Lemma 3.4.22 and Lemma 3.4.23, respectively, together with the following technical result, which in turn follows from the same argument as in Lemma 3.4.21. We omit the details of the verification of these properties.

**Lemma 3.5.14.** *Suppose that $G_j \in \mathcal{A}_\infty$ is a trace functional $G_j(\Gamma) = i\,\mathrm{Tr}(dG_j[0]\cdot\Gamma)$ for $j = 1, 2$. Then for all $\Gamma \in \mathfrak{G}_\infty^*$, the Gâteaux derivative $d\{G_1, G_2\}_{\mathfrak{G}_\infty^*}[\Gamma]$ at the point $\Gamma$ may be identified with the element*

$$[dG_1[0], dG_2[0]]_{\mathfrak{G}_\infty} \in \mathfrak{G}_\infty \tag{3.5.83}$$

*via the canonical trace pairing. If $G_1$ is a trace functional and $G_2 = G_{2,1}G_{2,2}$ is the product of two trace functionals in $\mathcal{A}_\infty$, then $d\{G_1, G_2\}_{\mathfrak{G}_\infty^*}[\Gamma]$ may be identified with*

$$G_{2,1}(\Gamma)[dG_1[0], dG_{2,2}[0]]_{\mathfrak{G}_\infty} + G_{2,2}(\Gamma)[dG_1[0], dG_{2,1}[0]]_{\mathfrak{G}_\infty} \tag{3.5.84}$$

*for all $\Gamma \in \mathfrak{G}_\infty^*$ via the canonical trace pairing.*

[20] i.e. it Poisson commutes with every functional in $\mathcal{A}_\infty$.

Property (P3) is more delicate: to show that the Hamiltonian vector field is well-defined, we have to exploit the good mapping property. Analogous to the proof of Proposition 3.1.7, rather than prove directly the well-definedness of the Hamiltonian vector field, we can use our earlier investment of work in proving Lemma 3.4.24, which gives an explicit formula for the $N$-body vector field, together with our convergence result Proposition 3.1.4 and an approximation argument.

**Lemma 3.5.15.** $(\mathfrak{G}_\infty^*, \mathcal{A}_\infty, \{\cdot,\cdot\}_{\mathfrak{G}_\infty^*})$ *satisfies property (P3) in Definition 3.3.1. Furthermore, if $H \in \mathcal{A}_\infty$, then we have the following formula for the Hamiltonian vector field $X_H$:*

$$X_H(\Gamma)^{(\ell)} = \sum_{j=1}^{\infty} j \operatorname{Tr}_{\ell+1,\ldots,\ell+j-1}\left(\left[\sum_{\alpha=1}^{\ell} dH[\Gamma]^{(j)}_{(\alpha,\ell+1,\ldots,\ell+j-1)}, \gamma^{(\ell+j-1)}\right]\right). \tag{3.5.85}$$

*Proof.* Let $F, H \in \mathcal{A}_\infty$. In order to find a candidate Hamiltonian vector field, we compute $\{F,H\}_{\mathfrak{G}_\infty^*}$ using an approximation to reduce to the case where $F$ and $G$ belong to $\mathcal{A}_{H,N}$, for all $N$ sufficiently large, together with the $N$-hierarchy Hamiltonian vector field result Lemma 3.4.24 and our convergence result Proposition 3.1.4. Once we have found a candidate, we then verify that the vector field is a smooth map $\mathfrak{G}_\infty^* \rightarrow \mathfrak{G}_\infty^*$, which then completes the proof by the uniqueness guaranteed by Remark 3.3.2.

By definition of $\mathcal{A}_\infty$, the functionals $F$ and $H$ are finite linear combinations of finite products of trace functionals generated by elements in $\mathfrak{G}_\infty$:

$$F(\Gamma) = \sum_{a=1}^{M_F} C_{a,F} \prod_{b=1}^{M_{a,F}} i \operatorname{Tr}(A_{b,F} \cdot \Gamma), \qquad H(\Gamma) = \sum_{a=1}^{M_H} C_{a,H} \prod_{b=1}^{M_{a,H}} i \operatorname{Tr}(A_{b,H} \cdot \Gamma), \tag{3.5.86}$$

where $M_F, M_H, M_{a,F}, M_{a,H} \in \mathbb{N}$, $C_{a,F}, C_{a,H} \in \mathbb{R}$, and $A_{b,F} = (A_{b,F}^{(k)})_{k\in\mathbb{N}}, A_{b,H} = (A_{b,H}^{(k)})_{k\in\mathbb{N}} \in \mathfrak{G}_\infty$. Additionally, since $\mathfrak{G}_\infty$ is a direct sum, there exists an integer $N_0 \in \mathbb{N}$ such that for each $1 \leq a \leq M_F$ and $1 \leq b \leq M_{a,F}$,

$$A_{b,F}^{(k)} = 0 \in \mathfrak{g}_{k,gmp}, \qquad \forall 1 \leq k \leq N_0 \tag{3.5.87}$$

and similarly for $A_{b,H}^{(k)}$. So by mollifying and truncating the Schwartz kernels of each $A_{b,F}^{(k)}, A_{b,H}^{(k)}$, we obtain approximating sequences $A_{n,b,F} := (A_{n,b,F}^{(k)})_{k\in\mathbb{N}}$ and $A_{n,b,H} := (A_{n,b,H}^{(k)})_{k\in\mathbb{N}}$, such that

$$A_{n,b,F}, A_{n,b,H} \in \mathfrak{G}_\infty \cap \bigoplus_{k=1}^{\infty} \mathcal{L}(\mathcal{S}_s'(\mathbb{R}^k), \mathcal{S}_s(\mathbb{R}^k)), \tag{3.5.88}$$

$A_{n,b,F} \to A_{b,F}$, and $A_{n,b,H} \to A_{b,H}$ in $\mathfrak{G}_\infty$ as $n \to \infty$. In particular, each $A_{n,b,F}, A_{n,b,H} \in \mathfrak{G}_M$ for every integer $M \geq N_0$. Now using the approximants $A_{n,b,F}$ and $A_{n,b,H}$, we can define sequences $(F_n)_{n\in\mathbb{N}}$ and $(H_n)_{n\in\mathbb{N}}$ of functionals in $\mathcal{A}_\infty$ by

$$F_n(\Gamma) := \sum_{a=1}^{M_F} C_{a,F} \prod_{b=1}^{M_{a,F}} i\,\mathrm{Tr}(A_{n,b,F} \cdot \Gamma), \qquad H_n(\Gamma) := \sum_{a=1}^{M_H} C_{a,H} \prod_{b=1}^{M_{a,H}} i\,\mathrm{Tr}(A_{n,b,H} \cdot \Gamma), \tag{3.5.89}$$

such that $F_n(\Gamma) \to F(\Gamma)$ and $H_n(\Gamma) \to H(\Gamma)$ as $n \to \infty$ uniformly on bounded subsets of $\mathfrak{G}_\infty^*$. Lastly, note that by the Leibnitz rule for the Gâteaux derivative,

$$dF_n[\Gamma], dH_n[\Gamma] \in \mathfrak{G}_M, \qquad \forall M \geq N_0 \tag{3.5.90}$$

and $dF_n[\Gamma] \to dF[\Gamma]$ and $dH_n[\Gamma] \to dH[\Gamma]$ in $\bigoplus_{k=1}^{\infty} \mathcal{L}(\mathcal{S}_s(\mathbb{R}^k), \mathcal{S}_s'(\mathbb{R}^k))$, as $n \to \infty$, uniformly on bounded subsets of $\mathfrak{G}_\infty^*$.

Now by separate continuity of the Lie bracket $[\cdot,\cdot]_{\mathfrak{G}_\infty}$ and the separate continuity of the generalized trace (see Proposition 3.2.3), we obtain from the definition of $\{\cdot,\cdot\}_{\mathfrak{G}_\infty^*}$ that

$$\begin{aligned}\{F,H\}_{\mathfrak{G}_\infty^*}(\Gamma) &= i\,\mathrm{Tr}\big([dF[\Gamma], dH[\Gamma]]_{\mathfrak{G}_\infty} \cdot \Gamma\big) \\ &= i \lim_{n_1\to\infty} \lim_{n_2\to\infty} \mathrm{Tr}\big([dF_{n_1}[\Gamma], dH_{n_2}[\Gamma]]_{\mathfrak{G}_\infty} \cdot \Gamma\big) \\ &= \lim_{n_1\to\infty} \lim_{n_2\to\infty} \{F_{n_1}, H_{n_2}\}_{\mathfrak{G}_\infty^*}(\Gamma),\end{aligned} \tag{3.5.91}$$

for each $\Gamma \in \mathfrak{G}_\infty^*$. Since

$$dF_{n_1}[\Gamma]^{(k)} = dH_{n_2}[\Gamma]^{(k)} = 0 \in \mathfrak{g}_{k,gmp}, \qquad \forall k \geq N_0, \;\; (n_1, n_2) \in \mathbb{N}^2, \;\; \Gamma \in \mathfrak{G}_\infty^*, \tag{3.5.92}$$

it follows from an examination of the definition of $[dF_{n_1}[\Gamma], dH_{n_2}[\Gamma]]_{\mathfrak{G}_\infty}$ that

$$[dF_{n_1}[\Gamma], dH_{n_2}[\Gamma]]_{\mathfrak{G}_\infty}^{(k)} = 0 \in \mathfrak{g}_{k,gmp}, \qquad \forall k \geq 2N_0 + 1, \;\; (n_1, n_2) \in \mathbb{N}^2, \;\; \Gamma \in \mathfrak{G}_\infty^*. \tag{3.5.93}$$

Therefore, if $\Gamma = (\gamma^{(k)})_{k \in \mathbb{N}} \in \mathfrak{G}_\infty^*$, then letting $\Gamma_M := (\gamma^{(k)})_{k=1}^M$ be the projection onto an element of $\mathfrak{G}_M^*$, for $M \geq 2N_0 + 1$, we see that

$$\begin{aligned}\mathrm{Tr}\Big([dF_{n_1}[\Gamma], dH_{n_2}[\Gamma]]_{\mathfrak{G}_\infty} \cdot \Gamma\Big) &= \mathrm{Tr}\Big([dF_{n_1}[\Gamma], dH_{n_2}[\Gamma]]_{\mathfrak{G}_\infty} \cdot \Gamma_{2N_0+1}\Big) \\ &= \mathrm{Tr}\Big([dF_{n_1}[\Gamma_{2N_0+1}], dH_{n_2}[\Gamma_{2N_0+1}]]_{\mathfrak{G}_\infty} \cdot \Gamma_{2N_0+1}\Big).\end{aligned} \tag{3.5.94}$$

For each $(n_1, n_2) \in \mathbb{N}^2$, we have by Proposition 3.1.4 and the separate continuity of the generalized trace that

$$\mathrm{Tr}\Big([dF_{n_1}[\Gamma_{2N_0+1}], dH_{n_2}[\Gamma_{2N_0+1}]]_{\mathfrak{G}_\infty} \cdot \Gamma_{2N_0+1}\Big) = \lim_{M \to \infty} \mathrm{Tr}\Big([dF_{n_1}[\Gamma_{2N_0+1}], dH_{n_2}[\Gamma_{2N_0+1}]]_{\mathfrak{G}_M} \cdot \Gamma_{2N_0+1}\Big). \tag{3.5.95}$$

For $M \gg 2_{N_0+1}$, we have by Lemma 3.4.24 that

$$\begin{aligned}i\,\mathrm{Tr}\Big([dF_{n_1}[\Gamma_{2N_0+1}], dH_{n_2}[\Gamma_{2N_0+1}]]_{\mathfrak{G}_M} \cdot \Gamma_{2N_0+1}\Big) &= \{F_{n_1}, H_{n_2}\}_{\mathfrak{G}_M^*}(\Gamma_{2N_0+1}) \\ &= \sum_{\ell=1}^{N_0} i\,\mathrm{Tr}_{1,\ldots,\ell}\Big(dF_{n_1}[\Gamma_{2N_0+1}]^{(\ell)} X_{H_{n_2},\mathfrak{G}_M^*}(\Gamma_{2N_0+1})^{(\ell)}\Big),\end{aligned} \tag{3.5.96}$$

where

$$\begin{aligned}&X_{H_{n_2},\mathfrak{G}_M^*}(\Gamma_{2N_0+1})^{(\ell)} \\ &= \sum_{j=1}^{M} \sum_{r=r_0}^{\min\{\ell,j\}} C'_{\ell jkrM}\,\mathrm{Tr}_{\ell+1,\ldots,k}\left(\left[\sum_{\underline{\alpha}_r \in P_r^\ell} dH_{n_2}[\Gamma_{2N_0+1}]^{(j)}_{(\underline{\alpha}_r,\ell+1,\ldots,\min\{\ell+j-r,k\})}, \gamma_{2N_0+1}^{(k)}\right]\right)\end{aligned} \tag{3.5.97}$$

and where

$$k := \min\{\ell + j - 1, M\}, \quad r_0 := \max\{1, \min\{\ell, j\} - (M - \max\{\ell, j\})\}, \tag{3.5.98}$$

and

$$C'_{\ell jkrM} := \binom{j}{r} \frac{MC_{\ell,M}C_{j,M}}{C_{k,M}\prod_{m=1}^{r-1}(M-k+m)}. \tag{3.5.99}$$

Since $dF_{n_1}[\Gamma_{2N_0+1}]^{(\ell)} = 0 \in \mathfrak{g}_\ell$ and $dH_{n_2}[\Gamma_{2N_0+1}]^{(j)} = 0 \in \mathfrak{g}_j$, for $\ell, j \geq N_0$, we see upon substituting the right-hand side of (3.5.97) into (3.5.96) that, for any $M \geq 2N_0 + 1$, only pairs $(\ell, j)$ satisfying $\ell + j - 1 \leq M$ give a nonzero contribution to the resulting expression. Similarly, only pairs $(\ell, j)$ such that $r_0 = 1$ give a nonzero contribution to (3.5.96). Therefore, we may write

$$\begin{aligned} &X_{H_{n_2},\mathfrak{G}_M^*}(\Gamma_{2N_0+1})^{(\ell)} \\ &= \sum_{j=1}^{M}\sum_{r=1}^{\min\{\ell,j\}} C'_{\ell jkrM}\,\mathrm{Tr}_{\ell+1,\ldots,\ell+j-1}\left(\left[\sum_{\underline{\alpha}_r\in P_r^\ell} dH_{n_2}[\Gamma_{2N_0+1}]^{(j)}_{(\underline{\alpha}_r,\ell+1,\ldots,\ell+j-r)}, \gamma_{2N_0+1}^{(\ell+j-1)}\right]\right). \end{aligned} \tag{3.5.100}$$

By the analysis from the proof of Proposition 3.1.4, we have that

$$\lim_{M\to\infty} C'_{\ell jkrM} = \begin{cases} j, & r = 1 \\ 0, & 2 \leq r \leq \min\{\ell, j\} \end{cases}. \tag{3.5.101}$$

Since the summands in (3.5.100) are zero for $j \geq N_0$, it then follows that

$$X_{H_{n_2},\mathfrak{G}_M^*}(\Gamma_{2N_0+1})^{(\ell)} \xrightarrow[M\to\infty]{\mathfrak{g}_\ell^*} \underbrace{\sum_{j=1}^{\infty} j\,\mathrm{Tr}_{\ell+1,\ldots,\ell+j-1}\left(\left[\sum_{\alpha=1}^{\ell} dH_{n_2}[\Gamma_{2N_0+1}]^{(j)}_{(\alpha,\ell+1,\ldots,\ell+j-1)}, \gamma_{2N_0+1}^{(\ell+j-1)}\right]\right)}_{=:X_{H_{n_2},\mathfrak{G}_\infty^*}(\Gamma_{2N_0+1})^{(\ell)}}. \tag{3.5.102}$$

The preceding convergence result implies, by the separate continuity of the generalized trace, that for fixed $(n_1, n_2) \in \mathbb{N}^2$,

$$\begin{aligned}
&\lim_{M\to\infty} \sum_{\ell=1}^{N_0} i\,\mathrm{Tr}_{1,\ldots,\ell}\Big(dF_{n_1}[\Gamma_{2N_0+1}]^{(\ell)} X_{H_{n_2},\mathfrak{G}_M^*}(\Gamma_{2N_0+1})^{(\ell)}\Big) \\
&= \sum_{\ell=1}^{N_0} i\,\mathrm{Tr}_{1,\ldots,\ell}\Big(dF_{n_1}[\Gamma_{2N_0+1}]^{(\ell)} X_{H_{n_2},\mathfrak{G}_\infty^*}(\Gamma_{2N_0+1})^{(\ell)}\Big).
\end{aligned} \tag{3.5.103}$$

Recalling from (3.5.92) that $dH_{n_2}[\Gamma_{2N_0+1}]^{(j)} = dH_{n_2}[\Gamma]^{(j)}$, for all $j \in \mathbb{N}$, and

$$\gamma_{2N_0+1}^{(\ell+j-1)} = \gamma^{(\ell+j-1)}, \quad \text{for } \ell + j - 1 \leq 2N_0 + 1,$$

by definition of the projection $\Gamma_{2N_0+1}$, we obtain that

$$X_{H_{n_2},\mathfrak{G}_\infty^*}(\Gamma_{2N_0+1})^{(\ell)} = \underbrace{\sum_{j=1}^{\infty} j\,\mathrm{Tr}_{\ell+1,\ldots,\ell+j-1}\left(\left[\sum_{\alpha=1}^{\ell} dH_{n_2}[\Gamma]^{(j)}_{(\alpha,\ell+1,\ldots,\ell+j-1)}, \gamma^{(\ell+j-1)}\right]\right)}_{=:X_{H_{n_2}}(\Gamma)^{(\ell)}}, \tag{3.5.104}$$

for $\ell \in \mathbb{N}_{\leq N_0}$. Similarly, by (3.5.92), $dF_{n_1}[\Gamma_{2N_0+1}]^{(\ell)} = dF_{n_1}[\Gamma]^{(\ell)}$, and so we have that

$$\sum_{\ell=1}^{N_0} i\,\mathrm{Tr}_{1,\ldots,\ell}\Big(dF_{n_1}[\Gamma_{2N_0+1}]^{(\ell)} X_{H_{n_2},\mathfrak{G}_\infty^*}(\Gamma_{2N_0+1})^{(\ell)}\Big) = \sum_{\ell=1}^{N_0} i\,\mathrm{Tr}_{1,\ldots,\ell}\Big(dF_{n_1}[\Gamma]^{(\ell)} X_{H_{n_2}}(\Gamma)^{(\ell)}\Big). \tag{3.5.105}$$

We now proceed to the analysis of the iterative limits $n_2 \to \infty$ followed by $n_1 \to \infty$. Since

$$dH_{n_2}[\Gamma] \to dH[\Gamma]$$

in $\mathfrak{G}_\infty$, as $n_2 \to \infty$, it follows from Proposition 3.3.1 and the universal property of the tensor product that the $(\ell + j - 1)$-particle extensions

$$dH_{n_2}[\Gamma]^{(j)}_{(\alpha,\ell+1,\ldots,\ell+j-1)} \longrightarrow dH[\Gamma]^{(j)}_{(\alpha,\ell+1,\ldots,\ell+j-1)}, \tag{3.5.106}$$

in $\mathcal{L}_{gmp}(\mathcal{S}(\mathbb{R}^{\ell+j-1}), \mathcal{S}'(\mathbb{R}^{\ell+j-1}))$ as $M \to \infty$. for $\Gamma \in \mathfrak{G}_\infty^*$ fixed. The continuity of the commutator bracket, the good mapping property, and the separate continuity of the generalized trace imply that

$$X_{H_{n_2}}(\Gamma) \longrightarrow X_H(\Gamma). \tag{3.5.107}$$

in $\prod_{k=1}^\infty \mathcal{L}(\mathcal{S}_s'(\mathbb{R}^k), \mathcal{S}_s(\mathbb{R}^k))$ as $n_2 \to \infty$. Moreover, the continuity of the adjoint operation (see Lemma 3.1.1) and the self-adjointness of $X_{H_{n_2}}(\Gamma)$ imply that $X_H(\Gamma)$ is self-adjoint, hence an element of $\mathfrak{G}_\infty^*$. We note that writing $X_H(\Gamma)$ is a slight abuse of notation since we have not yet verified that $X_H$ satisfies all of the desired properties, but this limit, $X_H$, will be our candidate Hamiltonian vector field from the statement of the lemma.

For each $n_1 \in \mathbb{N}$ fixed, the separate continuity of the generalized trace and the fact that $dF_{n_1}[\Gamma]^{(\ell)} = 0$, for $\ell \geq N_0$, then implies

$$\lim_{n_2 \to \infty} i \operatorname{Tr}\left(dF_{n_1}[\Gamma] \cdot X_{H_{n_2}}(\Gamma)\right) = i \operatorname{Tr}(dF_{n_1}[\Gamma] \cdot X_H(\Gamma)). \tag{3.5.108}$$

Since $dF_{n_1}[\Gamma] \to dF[\Gamma]$ in $\mathfrak{G}_\infty$, as $n_1 \to \infty$, by construction of the approximations $F_{n_1}$, another application of the separate continuity of the generalized trace yields

$$\lim_{n_1 \to \infty} i \operatorname{Tr}(dF_{n_1}[\Gamma] \cdot X_H(\Gamma)) = i \operatorname{Tr}(dF[\Gamma] \cdot X_H(\Gamma)). \tag{3.5.109}$$

After a little bookkeeping, we have shown that for every $\Gamma \in \mathfrak{G}_\infty^*$,

$$\begin{aligned}
\{F, G\}_{\mathfrak{G}_\infty^*}(\Gamma) &= \lim_{n_1 \to \infty} \lim_{n_2 \to \infty} \lim_{M \to \infty} i \operatorname{Tr}\left([dF_{n_1}[\Gamma_{2N_0+1}], dH_{n_2}[\Gamma_{2N_0+1}]]_{\mathfrak{G}_M} \cdot \Gamma_{2N_0+1}\right) \\
&= \lim_{n_1 \to \infty} \lim_{n_2 \to \infty} \lim_{M \to \infty} i \operatorname{Tr}\left(dF[\Gamma_{2N_0+1}] \cdot X_{H_{n_2}, \mathfrak{G}_M}(\Gamma_{2N_0+1})\right) \\
&= \lim_{n_1 \to \infty} \lim_{n_2 \to \infty} i \operatorname{Tr}\left(dF_{n_1}[\Gamma] \cdot X_{H_{n_2}}(\Gamma)\right) \\
&= i \operatorname{Tr}(dF[\Gamma] \cdot X_H(\Gamma)).
\end{aligned} \tag{3.5.110}$$

We now verify that $X_H$ is a smooth map $\mathfrak{G}_\infty^* \to \mathfrak{G}_\infty^*$ in order to conclude by Remark 3.3.2. It remains only to check the smoothness property. If $H$ is a trace functional, then since $dH[\Gamma]^{(j)} = dH[0]^{(j)}$ satisfies the good mapping property, the desired conclusion is immediate. The general case then follows by the Leibnitz rule for the Gâteaux derivative, since constant functionals and trace functionals generate $\mathcal{A}_\infty$. $\square$

### 3.5.4 The Poisson Morphism $\iota : \mathcal{S}(\mathbb{R}) \to \mathfrak{G}_\infty^*$

We now turn to the proof of Theorem 3.1.12. We recall that we are considering the map

$$\iota : \mathcal{S}(\mathbb{R}) \to \mathfrak{G}_\infty^*, \qquad \iota(\phi) := \left( |\phi^{\otimes k}\rangle \, \langle \phi^{\otimes k}| \right)_{k\in\mathbb{N}}, \tag{3.5.111}$$

which sends a 1-particle wave function to a density matrix $\infty$-hierarchy. We recall the definition

$$\mathcal{A}_{\mathcal{S}} = \left\{ H \,:\, \nabla_s H \in C^\infty(\mathcal{S}(\mathbb{R}); \mathcal{S}(\mathbb{R})) \right\} \subset C^\infty(\mathcal{S}(\mathbb{R}); \mathbb{R}).$$

and we restate Theorem 3.1.12 here for the reader's convenience.

**Theorem 3.1.12.** *The map $\iota$ is a Poisson morphism of $(\mathcal{S}(\mathbb{R}^d), \mathcal{A}_{\mathcal{S}}, \{\cdot,\cdot\}_{L^2})$ into $(\mathfrak{G}_\infty^*, \mathcal{A}_\infty, \{\cdot,\cdot\}_{\mathfrak{G}_\infty^*})$, i.e. it is a smooth map such that*

$$\{F\circ\iota, G\circ\iota\}_{L^2}(\phi) = \{F, G\}_{\mathfrak{G}_\infty^*}(\iota(\phi)), \qquad \forall \phi \in \mathcal{S}(\mathbb{R}^d), \tag{3.1.39}$$

*for all functionals $F, G \in \mathcal{A}_\infty$.*

We recall that although we set $d = 1$ in the proof, it works in any dimension. To prove Theorem 3.1.12, we will need the following technical result which gives a formula for the Gâteaux derivative of $\iota$.

**Lemma 3.5.16** (Formula for $d\iota$)**.** *Let $\phi, \psi \in \mathcal{S}(\mathbb{R})$. Then for all $k \in \mathbb{N}$,*

$$d\iota[\phi](\psi)^{(k)} = \sum_{m=1}^{k} |\phi^{\otimes(m-1)} \otimes \psi \otimes \phi^{\otimes(k-m)}\rangle \, \langle \phi^{\otimes k}| + \sum_{m=1}^{k} |\phi^{\otimes k}\rangle \, \langle \phi^{\otimes m-1} \otimes \psi \otimes \phi^{\otimes(k-m)}| \, . \tag{3.5.112}$$

*Proof.* The desired formula follows readily from the product rule. □

**Remark 3.5.17.** We record here the observation that for $\phi \in \mathcal{S}(\mathbb{R})$ fixed, each sum in (3.5.112) has co-domain $\mathcal{L}(\mathcal{S}_s'(\mathbb{R}^k), \mathcal{S}_s(\mathbb{R}^k))$. We will use this observation throughout the proof of Theorem 3.1.12 below.

*Proof of Theorem 3.1.12.* Smoothness of $\iota$ follows readily from Lemma 3.5.16 and induction on $k$, therefore, it remains to check that

(i) $\iota^*\mathcal{A}_\infty \subset \mathcal{A}_{\mathcal{S}}$,

(ii) $\iota^*\{\cdot, \cdot\}_{\mathfrak{G}_\infty^*} = \{\iota^*\cdot, \iota^*\cdot\}_{\mathcal{S}(\mathbb{R})}$.

We prove assertion (i). Let $F \in \mathcal{A}_\infty$. We need to show that $f \coloneqq F \circ \iota \in \mathcal{A}_{\mathcal{S}}$, that is, we need to show the symplectic $L^2$ gradient of $f$ exists and is a smooth $\mathcal{S}(\mathbb{R})$-valued map. To this end, observe that by the chain rule, for any $\phi, \delta\phi \in \mathcal{S}(\mathbb{R})$, we have

$$\begin{aligned} df[\phi](\delta\phi) &= dF[\iota(\phi)](d\iota[\phi](\delta\phi)) \\ &= i \operatorname{Tr}(dF[\iota(\phi)] \cdot d\iota[\phi](\delta\phi)) \\ &= i \sum_{k=1}^{\infty} \operatorname{Tr}_{1,\ldots,k}\big(dF[\iota(\phi)]^{(k)} d\iota[\phi]^{(k)}(\delta\phi)\big), \end{aligned} \tag{3.5.113}$$

where the penultimate equality follows from the identification of $dF[\iota(\phi)]$ as an element of $\widetilde{\mathfrak{G}_\infty}$, the bi-dual of $\mathfrak{G}_\infty$, via the canonical trace pairing and the ultimate equality follows from the definition of the dot product. Now applying Lemma 3.5.16 and the bilinearity of the generalized trace, we see that

$$\begin{aligned}
\mathrm{Tr}_{1,\ldots,k}\big(dF[\iota(\phi)]^{(k)} d\iota[\phi]^{(k)}(\delta\phi)\big) &= \mathrm{Tr}_{1,\ldots,k}\left(dF[\iota(\phi)]^{(k)}\left(\sum_{m=1}^{k} \left|\phi^{\otimes(m-1)} \otimes \delta\phi \otimes \phi^{\otimes(k-m)}\right\rangle \left\langle \phi^{\otimes k}\right|\right)\right) \\
&\quad + \mathrm{Tr}_{1,\ldots,k}\left(dF[\iota(\phi)]^{(k)}\left(\sum_{m=1}^{k} \left|\phi^{\otimes k}\right\rangle \left\langle \phi^{\otimes(m-1)} \otimes \delta\phi \otimes \phi^{\otimes(k-m)}\right|\right)\right) \\
&= \left\langle \phi^{\otimes k} \middle| dF[\iota(\phi)]^{(k)}\left(\sum_{m=1}^{k} \phi^{\otimes(m-1)} \otimes \delta\phi \otimes \phi^{\otimes(k-m)}\right)\right\rangle \\
&\quad + \left\langle \sum_{m=1}^{k} \phi^{\otimes(m-1)} \otimes \delta\phi \otimes \phi^{\otimes(k-m)} \middle| dF[\iota(\phi)]^{(k)}\phi^{\otimes k}\right\rangle,
\end{aligned} \tag{3.5.114}$$

where the ultimate equality is just applying the definition of the generalized trace. Since $dF[\iota(\phi)]^{(k)}$ is skew-adjoint, we have that

$$\begin{aligned}
&\left\langle \phi^{\otimes k} \middle| dF[\iota(\phi)]^{(k)}\left(\sum_{m=1}^{k} \phi^{\otimes(m-1)} \otimes \delta\phi \otimes \phi^{\otimes(k-m)}\right)\right\rangle \\
&= -\left\langle dF[\iota(\phi)]^{(k)}\phi^{\otimes k} \middle| \sum_{m=1}^{k} \phi^{\otimes(m-1)} \otimes \delta\phi \otimes \phi^{\otimes(k-m)}\right\rangle.
\end{aligned} \tag{3.5.115}$$

Since $dF[\iota(\phi)]^{(k)}$ satisfies the good mapping property, the preceding expression can be written as $-\langle \psi_{F,k}|\delta\phi\rangle$, where $\psi_{F,k} \in \mathcal{S}(\mathbb{R})$ is the unique Schwartz function coinciding with the bosonic tempered distribution

$$\left\langle \sum_{\alpha=1}^{k} (\cdot) \otimes_\alpha \phi^{\otimes(k-1)} \middle| dF[\iota(\phi)]^{(k)}\phi^{\otimes k}\right\rangle, \tag{3.5.116}$$

and we recall the notation $(\cdot) \otimes_\alpha \phi^{\otimes(k-1)}$ introduced in (3.5.14). Similarly,

$$\left\langle \sum_{m=1}^{k} \phi^{\otimes(m-1)} \otimes \delta\phi \otimes \phi^{\otimes(k-m)} \middle| dF[\iota(\phi)]^{(k)}\phi^{\otimes k}\right\rangle = \langle \delta\phi|\psi_{F,k}\rangle\,. \tag{3.5.117}$$

Therefore, we have shown that

$$\begin{aligned}
&\left\langle \phi^{\otimes k} \middle| dF[\iota(\phi)]^{(k)} \left( \sum_{m=1}^{k} \phi^{\otimes (m-1)} \otimes \delta\phi \otimes \phi^{\otimes (k-m)} \right) \right\rangle \\
&\quad + \left\langle \sum_{m=1}^{k} \phi^{\otimes (m-1)} \otimes \delta\phi \otimes \phi^{\otimes (k-m)} \middle| dF[\iota(\phi)]^{(k)} \phi^{\otimes k} \right\rangle \\
&= 2i \operatorname{Im}\{\langle \delta\phi | \psi_{F,k} \rangle\} \\
&= i\omega_{L^2}(\delta\phi, \psi_{F,k}) \qquad (3.5.118)
\end{aligned}$$

and consequently by (3.5.113), (3.5.114), (3.5.118) and bilinearity

$$i \sum_{k=1}^{\infty} \operatorname{Tr}_{1,\ldots,k}\big( dF[\iota(\phi)]^{(k)} d\iota[\phi]^{(k)}(\delta\phi) \big) = -\sum_{k=1}^{\infty} \omega_{L^2}(\delta\phi, \psi_{F,k}) = \omega_{L^2}(\psi_F, \delta\phi), \qquad (3.5.119)$$

where we have defined $\psi_F \coloneqq \sum_{k=1}^{\infty} \psi_{F,k}$ and used the anti-symmetry of $\omega_{L^2}$ to obtain the ultimate equality. Note that moving the summation inside the second entry of $\omega_{L^2}$ is justified by the bilinearity of the symplectic form since $dF[\iota(\phi)]^{(k)} = 0$ for all but finitely many $k$, by assumption that $F \in \mathcal{A}_\infty$ and the generating structure of $\mathcal{A}_\infty$. Consequently, $\psi_{F,k} \equiv 0$ for all but finitely many $k$. We conclude that

$$df[\phi](\delta\phi) = \omega_{L^2}(\psi_F, \delta\phi), \qquad (3.5.120)$$

and hence, recalling the definition of the symplectic $L^2$ gradient in Remark 3.3.12, we have that

$$\boldsymbol{\nabla}_s f(\phi) = \psi_F \in \mathcal{S}(\mathbb{R}). \qquad (3.5.121)$$

Lastly, using the identity (3.5.121), we prove assertion (ii). By definition of the Hamiltonian vector field $X_G(\iota(\phi))$ in (P3) together with Lemma 3.5.15, which gives a formula for $X_G(\iota(\phi))$, we have that for $F, G \in \mathcal{A}_\infty$,

$$\begin{aligned}
&\{F, G\}_{\mathfrak{G}_\infty^*}(\iota(\phi)) \\
&= dF[\iota(\phi)](X_G(\iota(\phi))) \\
&= i\sum_{k=1}^{\infty} \mathrm{Tr}_{1,\ldots,k}\left(dF[\iota(\phi)]^{(k)} \sum_{j=1}^{\infty} j\, \mathrm{Tr}_{k+1,\ldots,k+j-1}\left(\left[\sum_{\alpha=1}^{k} dG[\iota(\phi)]^{(j)}_{(\alpha,k+1,\ldots,k+j-1)}, \iota(\phi)^{(k+j-1)}\right]\right)\right).
\end{aligned} \tag{3.5.122}$$

Observe that

$$dG[\iota(\phi)]^{(j)}_{(\alpha,k+1,\ldots,k+j-1)}\iota(\phi)^{(k+j-1)} = \left|\phi^{\otimes(k-1)} \otimes^{\alpha} dG[\iota(\phi)]^{(j)}(\phi^{\otimes j})\right\rangle \left\langle\phi^{\otimes(k+j-1)}\right|, \tag{3.5.123}$$

where $\phi^{\otimes(k-1)} \otimes^{\alpha} dG[\iota(\phi)]^{(j)}(\phi^{\otimes j})$ is the tempered distribution in $\mathcal{S}'(\mathbb{R}^{k+j-1})$ defined by

$$\begin{aligned}
&\left(\phi^{\otimes(k-1)} \otimes^{\alpha} dG[\iota(\phi)]^{(j)}(\phi^{\otimes j})\right)(\underline{x}_{k+j-1}) \\
&:= \phi^{\otimes(\alpha-1)}(\underline{x}_{\alpha-1})\phi^{\otimes(k-\alpha)}(\underline{x}_{\alpha+1;k})dG[\iota(\phi)]^{(j)}(x_\alpha, \underline{x}_{k+1;k+j-1}).
\end{aligned} \tag{3.5.124}$$

Since $dG[\iota(\phi)]^{(j)}$ has the good mapping property by assumption $G \in \mathcal{A}_\infty$, it follows from Remark 3.3.4 and the definition of the generalized partial trace that

$$\begin{aligned}
&\mathrm{Tr}_{k+1,\ldots,k+j-1}\left(dG[\iota(\phi)]^{(j)}_{(\alpha,k+1,\ldots,k+j-1)}\iota(\phi)^{(k+j-1)}\right) \\
&= \left|\phi^{\otimes(\alpha-1)} \otimes \psi_{G,j,\alpha} \otimes \phi^{\otimes(k-\alpha)}\right\rangle \left\langle\phi^{\otimes k}\right|,
\end{aligned} \tag{3.5.125}$$

where $\psi_{G,j,\alpha} \in \mathcal{S}(\mathbb{R})$ is the unique Schwartz function such that

$$\langle\delta\phi|\psi_{G,j,\alpha}\rangle = \left\langle\delta\phi \otimes_\alpha \phi^{\otimes(j-1)}\middle|dG[\iota(\phi)]^{(j)}(\phi^{\otimes j})\right\rangle, \qquad \forall\delta\phi \in \mathcal{S}(\mathbb{R}). \tag{3.5.126}$$

Moreover, since $dG[\iota(\phi)]^{(j)}(\phi^{\otimes j}) \in \mathcal{S}_s'(\mathbb{R}^j)$, it follows from Lemma 3.3.27 that

$$\left\langle\delta\phi \otimes_\alpha \phi^{\otimes(j-1)}\middle|dG[\iota(\phi)]^{(j)}(\phi^{\otimes j})\right\rangle = \left\langle\delta\phi \otimes_{\alpha'} \phi^{\otimes(j-1)}\middle|dG[\iota(\phi)]^{(j)}(\phi^{\otimes j})\right\rangle, \tag{3.5.127}$$

for any $1 \leq \alpha, \alpha' \leq j$, and therefore $\psi_{G,j,\alpha} = \psi_{G,j,\alpha'}$. Hence,

$$\begin{aligned}&\mathrm{Tr}_{k+1,\ldots,k+j-1}\Big(dG[\iota(\phi)]^{(j)}_{(\alpha,k+1,\ldots,k+j-1)}\iota(\phi)^{(k+j-1)}\Big)\\&= \frac{1}{j}\left|\phi^{\otimes(\alpha-1)} \otimes \psi_{G,j} \otimes \phi^{\otimes(k-\alpha)}\right\rangle \left\langle \phi^{\otimes k}\right|,\end{aligned} \tag{3.5.128}$$

where $\psi_{G,j}$ is defined the same as $\psi_{F,k}$ above, except with $(F,k)$ replaced by $(G,j)$. By completely analogous reasoning together with the skew-adjointness of $dG[\iota(\phi)]^{(j)}$, we also obtain that

$$\begin{aligned}&\mathrm{Tr}_{k+1,\ldots,k+j-1}\Big(\iota(\phi)^{(k+j-1)}dG[\iota(\phi)]^{(j)}_{(\alpha,k+1,\ldots,k+j-1)}\Big)\\&= -\frac{1}{j}\left|\phi^{\otimes k}\right\rangle \left\langle \phi^{\otimes(\alpha-1)} \otimes \psi_{G,j} \otimes \phi^{\otimes(k-\alpha)}\right|,\end{aligned} \tag{3.5.129}$$

Substituting the identities (3.5.128) and (3.5.129) into (3.5.122), we obtain the expression

$$\begin{aligned}&i\sum_{k=1}^{\infty}\mathrm{Tr}_{1,\ldots,k}\Bigg(dF[\iota(\phi)]^{(k)}\Bigg(\sum_{j=1}^{\infty}\sum_{\alpha=1}^{k}\left|\phi^{\otimes(\alpha-1)}\otimes\psi_{G,j}\otimes\phi^{\otimes(k-\alpha)}\right\rangle\left\langle\phi^{\otimes k}\right|\\&\qquad\qquad+\left|\phi^{\otimes k}\right\rangle\left\langle\phi^{\otimes(\alpha-1)}\otimes\psi_{G,j}\otimes\phi^{\otimes(k-\alpha)}\right|\Bigg)\Bigg)\\&= i\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\left\langle\phi^{\otimes k}\middle|dF[\iota(\phi)]^{(k)}\left(\sum_{\alpha=1}^{k}\phi^{\otimes(\alpha-1)}\otimes\psi_{G,j}\otimes\phi^{\otimes(k-\alpha)}\right)\right\rangle\\&\qquad\qquad+\left\langle\sum_{\alpha=1}^{k}\phi^{\otimes(\alpha-1)}\otimes\psi_{G,j}\otimes\phi^{\otimes(k-\alpha)}\middle|dF[\iota(\phi)]^{(k)}\phi^{\otimes k}\right\rangle\\&= -2\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\mathrm{Im}\left\{\left\langle\sum_{\alpha=1}^{k}\phi^{\otimes(\alpha-1)}\otimes\psi_{G,j}\otimes\phi^{\otimes(k-\alpha)}\middle|dF[\iota(\phi)]^{(k)}\phi^{\otimes k}\right\rangle\right\}\\&= -2\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\mathrm{Im}\{\langle\psi_{G,j}|\psi_{F,k}\rangle\},\end{aligned} \tag{3.5.130}$$

where the penultimate equality follows from the skew-adjointness of $dF[\iota(\phi)]^{(k)}$ and the ultimate equality follows from the definition of $\psi_{F,k}$. Since $\psi_{F,k} = \psi_{G,j} \equiv 0$ for all but finitely many $j, k$, we are justified in writing

$$-2\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\operatorname{Im}\{\langle\psi_{G,j}|\psi_{F,k}\rangle\} = -2\operatorname{Im}\{\langle\psi_G|\psi_F\rangle\}, \tag{3.5.131}$$

where $\psi_F$ is defined as above and $\psi_G := \sum_{j=1}^{\infty}\psi_{G,j}$ is defined completely analogously. Recalling (3.3.15) for the definition of $\omega_{L^2}$ and identity (3.5.121) for the symplectic gradient, we obtain that

$$-2\operatorname{Im}\{\langle\psi_G|\psi_F\rangle\} = \omega_{L^2}(\boldsymbol{\nabla}_s f(\phi), \boldsymbol{\nabla}_s g(\phi)). \tag{3.5.132}$$

After a little bookkeeping, we realize that we have shown that

$$\{F, G\}_{\mathfrak{G}_\infty^*}(\iota(\phi)) = \omega_{L^2}(\boldsymbol{\nabla}_s f(\phi), \boldsymbol{\nabla}_s g(\phi)). \tag{3.5.133}$$

Since the symplectic form $\omega_{L^2}$ canonically induces the Poisson bracket $\{\cdot,\cdot\}_{L^2}$ through

$$\{f, g\}_{L^2}(\phi) = \omega_{L^2}(\boldsymbol{\nabla}_s f(\phi), \boldsymbol{\nabla}_s g(\phi)), \tag{3.5.134}$$

the proof of assertion (ii) is complete. $\square$

## 3.6 GP Hamiltonian Flows

In this last section, we prove Theorem 3.1.3 and its limiting version Theorem 3.1.10.

### 3.6.1 BBGKY Hamiltonian Flow

For the reader's benefit, we recall that the BBGKY Hamiltonian $\mathcal{H}_{BBGKY,N}$ is the trace functional given by

$$\mathcal{H}_{BBGKY,N}(\Gamma_N) = \operatorname{Tr}(\mathbf{W}_{BBGKY,N}\cdot\Gamma_N), \tag{3.6.1}$$

where

$$\mathbf{W}_{BBGKY,N} = (-\Delta_x, \kappa V_N(X_1 - X_2), 0, \ldots), \tag{3.6.2}$$

with $\kappa$ and $V_N$ as in (3.1.3). We also recall here the statement of Theorem 3.1.3.

**Theorem 3.1.3.** *Let $I \subset \mathbb{R}$ be a compact interval. Then $\Gamma_N = (\gamma_N^{(k)})_{k=1}^N \in C^\infty(I; \mathfrak{G}_N^*)$ is a solution to the BBGKY hierarchy* (3.1.4) *if and only if*

$$\frac{d}{dt}\Gamma_N = X_{\mathcal{H}_{BBGKY,N}}(\Gamma_N), \tag{3.1.18}$$

*where $X_{\mathcal{H}_{BBGKY,N}}$ is the unique vector field defined by $\mathcal{H}_{BBGKY,N}$ (see Definition 3.3.1) with respect to the weak Poisson structure $(\mathfrak{G}_N^*, \mathcal{A}_{H,N}, \{\cdot,\cdot\}_{\mathfrak{G}_N^*})$.*

We now proceed to proving Theorem 3.1.3. Since by Lemma 3.4.24, we have the formula

$$\begin{aligned} &X_{\mathcal{H}_{BBGKY,N}}(\Gamma_N)^{(\ell)} \\ &= \sum_{j=1}^N \sum_{r=r_0}^{\min\{\ell,j\}} C'_{\ell jkrN} \operatorname{Tr}_{\ell+1,\ldots,k}\left(\left[\sum_{\underline{\alpha}_r \in P_r^\ell} d\mathcal{H}_{BBGKY,N}[\Gamma_N]^{(j)}_{(\underline{\alpha}_r,\ell+1,\ldots,\min\{\ell+j-r,k\})}, \gamma_N^{(k)}\right]\right), \end{aligned} \tag{3.6.3}$$

where

$$k \coloneqq \min\{\ell+j-1, N\}, \quad r_0 \coloneqq \max\{1, \min\{\ell,j\} - (N - \max\{\ell,j\})\}, \tag{3.6.4}$$

and

$$C'_{\ell jkrN} \coloneqq \frac{N C_{\ell,N} C_{j,N}}{C_{k,N}\prod_{m=1}^{r-1}(N-k+m)}\binom{j}{r},$$

our task reduces to simplifying the expression in the right-hand side of (3.6.3).

To this end, we first need a formula for the Gâteaux derivative $d\mathcal{H}_{BBGKY,N}$ of $\mathcal{H}_{BBGKY,N}$ and its identification with an observable $N$-hierarchy via the canonical trace pairing. Indeed, let $N \in \mathbb{N}$. Then for any $\Gamma_N = (\gamma_N^{(k)})_{k=1}^N \in \mathfrak{G}_N^*$, we have that

$$d\mathcal{H}_{BBGKY,N}[\Gamma_N](\delta\Gamma_N) = \mathrm{Tr}(\mathbf{W}_{BBGKY,N} \cdot \delta\Gamma_N), \qquad \forall \delta\Gamma_N \in \mathfrak{G}_N^*. \tag{3.6.5}$$

Therefore, $d\mathcal{H}_{BBGKY,N}[\Gamma_N] = d\mathcal{H}_{BBGKY,N}[0]$ is uniquely identifiable with the observable 2-hierarchy $-i\mathbf{W}_{BBGKY,N}$. As a consequence, we see that

$$d\mathcal{H}_{BBGKY,N}[\Gamma_N]^{(j)}_{(\underline{\alpha}_r,\ell+1,\ldots,\min\{\ell+j-r,k\})} = 0 \tag{3.6.6}$$

for $3 \leq j \leq N$. Therefore, by (3.6.3), we have

$$\begin{aligned} X_{\mathcal{H}_{BBGKY,N}}(\Gamma_N)^{(\ell)} = -iC'_{\ell 1 \ell 1 N} \sum_{\alpha=1}^{\ell} \Big[(-\Delta_{x_1})_{(\alpha)}, \gamma_N^{(\ell)}\Big] \\ - i\kappa \sum_{r=r_0}^{\min\{\ell,2\}} C'_{\ell 2 k r N} \sum_{\underline{\alpha}_r \in P_r^\ell} \mathrm{Tr}_{\ell+1,\ldots,k}\Big(\Big[(V_N(X_1 - X_2))_{(\underline{\alpha}_r,\ell+1,\ldots,\min\{\ell+2-r,k\})}, \gamma_N^{(k)}\Big]\Big) \\ =: \mathrm{Term}_{1,\ell} + \mathrm{Term}_{2,\ell}. \end{aligned} \tag{3.6.7}$$

We first consider $\mathrm{Term}_{1,\ell}$. Note that $(-\Delta_x)_{(\alpha)} = -\Delta_{x_\alpha}$. Now unpacking the definition of the normalizing constant $C'_{\ell 1 \ell 1 N}$, we find that

$$C'_{\ell 1 \ell 1 N} = \frac{NC_{\ell,N}C_{1,N}}{C_{\ell,N}} = NC_{1,N} = 1, \tag{3.6.8}$$

where the ultimate equality follows from the fact that $C_{1,N} = 1/|P_1^N| = 1/N$. Hence,

$$\mathrm{Term}_{1,\ell} = -i\sum_{\alpha=1}^{\ell} \Big[-\Delta_{x_\alpha}, \gamma_N^{(\ell)}\Big]. \tag{3.6.9}$$

We next consider $\mathrm{Term}_{2,\ell}$. We divide into cases based on the values of $\ell \in \{1,\ldots,N\}$.

- If $\ell = 1$, then

$$\mathrm{Term}_{2,1} = -i\kappa C'_{1221N}\,\mathrm{Tr}_2\Big(\Big[(V_N(X_1 - X_2)_{(1,2)}, \gamma_N^{(2)}\Big]\Big), \tag{3.6.10}$$

where we use that $k = 2$. Since $(V_N(X_1 - X_2))_{(1,2)} = V_N(X_1 - X_2)$, it follows that

$$\mathrm{Term}_{2,1} = -i\kappa C'_{1221N} \,\mathrm{Tr}_2\Big(\Big[V_N(X_1 - X_2), \gamma_N^{(2)}\Big]\Big). \tag{3.6.11}$$

Unpacking the definition of the constant $C'_{1221N}$, we see that

$$C'_{1221N} = \frac{NC_{1,N}C_{2,N}}{C_{2,N}}\binom{2}{1} = 2NC_{1,N} = 2, \tag{3.6.12}$$

hence,

$$\mathrm{Term}_{2,1} = -2i\kappa \,\mathrm{Tr}_2\Big(\Big[V_N(X_1 - X_2), \gamma_N^{(2)}\Big]\Big). \tag{3.6.13}$$

- If $2 \leq \ell \leq N - 1$, then

$$r_0 = \max\{\min\{\ell, 2\} - (N - \max\{\ell, 2\}), 1\} = \max\{2 - (N - \ell), 1\} = 1 \tag{3.6.14}$$

and therefore

$$\mathrm{Term}_{2,\ell} = -i\kappa \sum_{r=1}^{2} C'_{\ell 2(\ell+1) rN} \sum_{\underline{\alpha}_r \in P_r^\ell} \mathrm{Tr}_{\ell+1}\Big(\Big[V_N(X_1 - X_2)_{(\underline{\alpha}_r, \ell+1)}, \gamma_N^{(\ell+1)}\Big]\Big), \tag{3.6.15}$$

where we use that $k = \ell + 1$. If $r = 1$, then

$$\sum_{\underline{\alpha}_1 \in P_1^\ell} \mathrm{Tr}_{\ell+1}\Big(\Big[V_N(X_1 - X_2)_{(\underline{\alpha}_1, \ell+1)}, \gamma_N^{(\ell+1)}\Big]\Big) = \sum_{\alpha=1}^{\ell} \mathrm{Tr}_{\ell+1}\Big(\Big[V_N(X_\alpha - X_{\ell+1}), \gamma_N^{(\ell+1)}\Big]\Big), \tag{3.6.16}$$

and recalling (3.4.9), we have

$$C'_{\ell 2(\ell+1)1N} = \frac{NC_{\ell,N}C_{2,N}}{C_{\ell+1,N}}\binom{2}{1} = \frac{2(N-\ell)}{(N-1)}. \tag{3.6.17}$$

If $r = 2$, then $\min\{\ell + 2 - r, k\} = \ell$, which per our notation implies that

$$\sum_{\underline{\alpha}_r \in P_r^\ell} \mathrm{Tr}_{\ell+1}\Big(\Big[V_N(X_1 - X_2)_{(\underline{\alpha}_r, \ell+1)}, \gamma_N^{(\ell+1)}\Big]\Big) = \sum_{(\alpha_1,\alpha_2) \in P_2^\ell} \mathrm{Tr}_{\ell+1}\Big(\Big[(V_N(X_1 - X_2)_{(\alpha_1,\alpha_2)}, \gamma_N^{(\ell+1)}\Big]\Big). \tag{3.6.18}$$

Since $\alpha_1, \alpha_2 \in \mathbb{N}_{\leq \ell}$ and $V_N(X_1 - X_2)_{(\alpha_1,\alpha_2)} = V_N(X_{\alpha_1} - X_{\alpha_2})$, we have that

$$\mathrm{Tr}_{\ell+1}\Big(\Big[(V_N(X_1 - X_2)_{(\alpha_1,\alpha_2)}, \gamma_N^{(\ell+1)}\Big]\Big) = \Big[V_N(X_{\alpha_1} - X_{\alpha_2}), \gamma_N^{(\ell)}\Big]. \tag{3.6.19}$$

Now since $k = \ell + 1$, it follows from our computation in (3.6.17) that

$$C'_{\ell 2(\ell+1)2N} = \frac{NC_{\ell,N}C_{2,N}}{C_{\ell+1,N}(N-k+1)}\binom{2}{2} = \frac{1}{N-1}. \tag{3.6.20}$$

Since $V_N(X_{\alpha_1} - X_{\alpha_2}) = V_N(X_{\alpha_2} - X_{\alpha_1})$ by the evenness of the potential $V$, it follows that

$$\sum_{\underline{\alpha}_2 \in P_2^\ell} \Big[V_N(X_{\alpha_1} - X_{\alpha_2}), \gamma_N^{(\ell)}\Big] = \frac{2}{N-1} \sum_{1 \leq \alpha_1 < \alpha_2 \leq \ell} \Big[V_N(X_{\alpha_1} - X_{\alpha_2}), \gamma_N^{(\ell)}\Big]. \tag{3.6.21}$$

After a little bookkeeping, we obtain that

$$\begin{aligned} \mathrm{Term}_{2,\ell} = -i\kappa \frac{2(N-\ell)}{N-1} \sum_{\alpha=1}^{\ell} \mathrm{Tr}_{\ell+1}\Big(\Big[V_N(X_\alpha - X_{\ell+1}), \gamma_N^{(\ell+1)}\Big]\Big) \\ - i\kappa \frac{2}{N-1} \sum_{1 \leq \alpha_1 < \alpha_2 \leq \ell} \Big[V_N(X_{\alpha_1} - X_{\alpha_2}), \gamma_N^{(\ell)}\Big]. \end{aligned} \tag{3.6.22}$$

- Lastly, if $\ell = N$, then

$$r_0 = \max\{\min\{N, 2\} - (N - \max\{N, 2\}), 1\} = 2. \tag{3.6.23}$$

Moreover, $k = N$, so that

$$\mathrm{Term}_{2,N} = -i\kappa C'_{N2N2N} \sum_{\underline{\alpha}_2 \in P_2^N} \Big[(V_N(X_1 - X_2))_{(\underline{\alpha}_2)}, \gamma_N^{(N)}\Big]. \tag{3.6.24}$$

Since

$$C'_{N2N2N} = \frac{NC_{N,N}C_{2,N}}{C_{N,N}}\binom{2}{2} = \frac{1}{N-1}, \tag{3.6.25}$$

we can again use the evenness of the potential $V$ to conclude that

$$\mathrm{Term}_{2,N} = -\frac{2i\kappa}{N-1}\sum_{1\leq\alpha_1<\alpha_2\leq N}\left[V_N(X_{\alpha_1}-X_{\alpha_2}),\gamma_N^{(N)}\right]. \tag{3.6.26}$$

Putting our case analysis together, we obtain

$$X_{\mathcal{H}_{BBGKY,N}}(\Gamma_N)^{(1)} = -i\left[-\Delta_{x_1},\gamma_N^{(1)}\right] - 2i\kappa\,\mathrm{Tr}_2\left(\left[V_N(X_1-X_2),\gamma_N^{(2)}\right]\right), \tag{3.6.27}$$

while for $2\leq\ell\leq N-1$ we have

$$\begin{aligned} X_{\mathcal{H}_{BBGKY,N}}(\Gamma_N)^{(\ell)} = -i\sum_{\alpha=1}^{\ell}\left[-\Delta_{x_\alpha},\gamma_N^{(\ell)}\right] - \frac{2i\kappa}{N-1}\sum_{1\leq\alpha_1<\alpha_2\leq\ell}\left[V_N(X_{\alpha_1}-X_{\alpha_2}),\gamma_N^{(\ell)}\right] \\ - \frac{2i\kappa(N-\ell)}{N-1}\sum_{\alpha=1}^{\ell}\mathrm{Tr}_{\ell+1}\left(\left[V_N(X_\alpha-X_{\ell+1}),\gamma_N^{(\ell+1)}\right]\right), \end{aligned} \tag{3.6.28}$$

and finally

$$X_{\mathcal{H}_{BBGKY,N}}(\Gamma_N)^{(N)} = -i\sum_{\alpha=1}^{N}\left[-\Delta_{x_\alpha},\gamma_N^{(\ell)}\right] - \frac{2i\kappa}{N-1}\sum_{1\leq\alpha_1<\alpha_2\leq N}\left[V_N(X_{\alpha_1}-X_{\alpha_2}),\gamma_N^{(N)}\right], \tag{3.6.29}$$

which we see, upon comparison with (3.1.4), are precisely the equations for solutions to the BBGKY hierarchy, thus completing the proof.

### 3.6.2 GP Hamiltonian Flow

In this subsection, we prove Theorem 3.1.10. For the reader's benefit, we recall that the GP Hamiltonian $\mathcal{H}_{GP}$ is the trace functional given by

$$\mathcal{H}_{GP}(\Gamma) \coloneqq \mathrm{Tr}(\mathbf{W}_{GP}\cdot\Gamma),\ \ \Gamma\in\mathfrak{G}_\infty^*; \qquad \mathbf{W}_{GP} = (-\Delta_x, \kappa\delta(X_1-X_2), 0, \ldots). \tag{3.6.30}$$

We recall the statement of the theorem.

**Theorem 3.1.10** (Hamiltonian structure for GP)**.** *Let $I \subset \mathbb{R}$ be a compact interval. Then $\Gamma \in C^\infty(I; \mathfrak{G}_\infty^*)$ is a solution to the GP hierarchy* (3.1.5) *if and only if*

$$\left(\frac{d}{dt}\Gamma\right)(t) = X_{\mathcal{H}_{GP}}(\Gamma(t)), \qquad \forall t \in I, \tag{3.1.31}$$

*where $X_{\mathcal{H}_{GP}}$ is the unique Hamiltonian vector field defined by $\mathcal{H}_{GP}$ with respect to the weak Poisson structure $(\mathfrak{G}_\infty^*, \mathcal{A}_\infty, \{\cdot,\cdot\}_{\mathfrak{G}_\infty^*})$.*

The proof is similar to the proof that the BBGKY hierarchy is a Hamiltonian equation of motion, and Theorem 3.1.10 may be viewed as the $N \to \infty$ limit of Theorem 3.1.3. In Chapter 4, we will obtain Theorem 3.1.10 for the 1D cubic GP hierarchy as part of a more general theorem which connects the Hamiltonian structure of an infinte coupled system of linear equations, which we call the $n$-th GP hierarchy, to the Hamiltonian structure of the $n$-th equation of the nonlinear Schrödinger hierarchy, which is of fundamental interest in the study of the NLS as an integrable system (see, for instance, the survey of Palais [74]). The GP hierarchy under consideration here then corresponds to the $n = 3$ equation of the aforementioned family of equations.

We now proceed to proving Theorem 3.1.10. Recalling equation (3.1.5) for the GP hierarchy, we need to show that

$$X_{\mathcal{H}_{GP}}(\Gamma)^{(k)} = -i\big(\big[-\Delta_{\underline{x}_k}, \gamma^{(k)}\big] + 2\kappa B_{k+1}\gamma^{(k+1)}\big), \qquad k \in \mathbb{N}, \tag{3.6.31}$$

for any $\Gamma = (\gamma^{(k)}) \in \mathfrak{G}_\infty^*$, which we do by direct computation.

Let $\Gamma \in \mathfrak{G}_\infty^*$. By application of Lemma 3.5.15 to $\mathcal{H}_{GP}$ together with the identification

$$d\mathcal{H}_{GP}[\Gamma] = -i\mathbf{W}_{GP}, \tag{3.6.32}$$

which is immediate from the fact that $\mathcal{H}_{GP}$ is a trace functional, we know that

$$X_{\mathcal{H}_{GP}}(\Gamma)^{(k)} = \sum_{j=1}^{\infty} j \operatorname{Tr}_{k+1,\ldots,k+j-1}\left(\left[\sum_{\alpha=1}^{k} d\mathcal{H}_{GP}[\Gamma]_{(\alpha,k+1,\ldots,k+j-1)}^{(j)}, \gamma^{(k+j-1)}\right]\right). \tag{3.6.33}$$

Since $-i\mathbf{W}_{GP}^{(j)} = 0 \in \mathfrak{g}_{j,gmp}$, for $j \geq 3$, we see from (3.6.30) that the formula for $X_{\mathcal{H}_{GP}}(\Gamma)$ simplifies to

$$\begin{aligned} X_{\mathcal{H}_{GP}}(\Gamma)^{(k)} = &-i\sum_{\alpha=1}^{k}\big((-\Delta_{x_1})_{(\alpha)}\gamma^{(k)} - \gamma^{(k)}(-\Delta_{x_1})_{(\alpha)}\big) \\ &- i2\kappa\sum_{\alpha=1}^{k}\operatorname{Tr}_{k+1}\big(\delta(X_1-X_2)_{(\alpha,k+1)}\gamma^{(k+1)}\big) - \operatorname{Tr}_{k+1}\big(\gamma^{(k+1)}\delta(X_1-X_2)_{(\alpha,k+1)}\big), \end{aligned} \tag{3.6.34}$$

for $k \in \mathbb{N}$.

Since $(-\Delta_{x_1})_{(\alpha)} = -\Delta_{x_\alpha}$ and $\Delta_{\underline{x}_k} = \sum_{\alpha=1}^{k}\Delta_{x_\alpha}$ by definition, it follows that

$$-i\sum_{\alpha=1}^{k}\big((-\Delta_{x_1})_{(\alpha)}\gamma^{(k)} - \gamma^{(k)}(-\Delta_{x_1})_{(\alpha)}\big) = -i\big[-\Delta_{\underline{x}_k}, \gamma^{(k)}\big]. \tag{3.6.35}$$

Since $\delta(X_1 - X_2)_{(\alpha,k+1)} = \delta(X_\alpha - X_{k+1})$, it follows from Proposition 3.2.4 for the generalized partial trace that $\operatorname{Tr}_{k+1}(\delta(X_\alpha - X_{k+1})\gamma^{(k+1)})$ is the element of $\mathcal{L}(\mathcal{S}_s'(\mathbb{R}^k), \mathcal{S}(\mathbb{R}^k))$ with Schwartz kernel

$$\int_{\mathbb{R}} dx_{k+1}\delta(x_\alpha - x_{k+1})\gamma^{(k+1)}(\underline{x}_{k+1}; \underline{x}_k', x_{k+1}) = \gamma^{(k+1)}(\underline{x}_k, x_\alpha; \underline{x}_k', x_\alpha) = B_{\alpha;k+1}^{+}\gamma^{(k+1)}(\underline{x}_k; \underline{x}_k'). \tag{3.6.36}$$

Similarly, $\operatorname{Tr}_{k+1}(\gamma^{(k+1)}\delta(X_\alpha - X_{k+1}))$ is the operator with Schwartz kernel

$$\int_{\mathbb{R}} dx_{k+1}'\delta(x_\alpha' - x_{k+1})\gamma^{(k+1)}(\underline{x}_k, x_{k+1}'; \underline{x}_{k+1}') = \gamma^{(k+1)}(\underline{x}_k, x_\alpha'; \underline{x}_k', x_\alpha') = B_{\alpha;k+1}^{-}\gamma^{(k+1)}(\underline{x}_k; \underline{x}_k'). \tag{3.6.37}$$

Since $B_{k+1} = \sum_{\alpha=1}^{k} B^{+}_{\alpha;k+1} - B^{-}_{\alpha;k+1}$ by definition, we conclude that

$$\begin{aligned} &-2\kappa i \sum_{\alpha=1}^{k} \mathrm{Tr}_{k+1}\big(\delta(X_1 - X_2)_{(\alpha,k+1)}\gamma^{(k+1)}\big) - \mathrm{Tr}_{k+1}\big(\gamma^{(k+1)}\delta(X_1 - X_2)_{(\alpha,k+1)}\big) \\ &= -2\kappa i B_{k+1}\gamma^{(k+1)}. \end{aligned} \tag{3.6.38}$$

After a little bookkeeping, we see that we have shown (3.6.31), thus completing the proof of Theorem 3.1.10.

| Symbol | Definition |
|---|---|
| $(\underline{x}_k)$, $\underline{x}_k$ | $(x_1, \ldots, x_k)$ |
| $\underline{x}_{m_1;m_k}$ | $(x_{m_1}, \ldots, x_{m_k})$ |
| $\underline{x}_{i;i+k}$ | $(x_i, \ldots, x_{i+k})$ |
| $d\underline{x}_k$ | $dx_1 \cdots dx_k$ |
| $d\underline{x}_{i;i+k}$ | $dx_i \cdots dx_{i+k}$ |
| $\mathbb{N}_{\leq i}$ or $\mathbb{N}_{\geq i}$ | $\{n \in \mathbb{N} : n \leq i\}$ or $\{n \in \mathbb{N} : n \geq i\}$ |
| $\mathbb{S}_k$ | symmetric group on $k$ elements |
| $\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k)$ | Schwartz space on $\mathbb{R}^k$ and tempered distributions on $\mathbb{R}^k$ |
| $\mathcal{D}'(\mathbb{R}^k)$ | distributions on $\mathbb{R}^k$ |
| $\mathcal{S}_s(\mathbb{R}^k), \mathcal{S}_s'(\mathbb{R}^k)$ | symmetric Schwartz space, Definition 3.3.24, and symmetric tempered distributions |
| $\mathcal{L}(E; F)$ | continuous linear maps between locally convex spaces $E$ and $F$ |
| $\tilde{\mathcal{L}}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}(\mathbb{R}^k))$ | $\mathcal{L}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}(\mathbb{R}^k))$ equipped with the subspace topology induced by $\mathcal{L}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k))$ |
| $\tilde{\mathcal{L}}(\mathcal{S}_s(\mathbb{R}^k), \mathcal{S}_s(\mathbb{R}^k))$ | analogous to previous definition |
| $dF$ | the Gâteaux derivative of $F$, Definition 2.1.4 |
| $\boldsymbol{\nabla}$ or $\boldsymbol{\nabla}_s$ | the real or symplectic $L^2$ gradients, Definition 3.3.11 and Remark 3.3.12 |
| $A_{(\pi(1),\ldots,\pi(k))}$ | conjugation of an operator by a permutation, see (3.3.42) |
| $\mathrm{Sym}(f)$ | symmetrization operator for functions, Definition 3.3.23 |
| $\mathrm{Sym}(A)$ | symmetrization operator for operators, Definition 3.3.30 |
| $L_s^2(\mathbb{R}^k)$ | symmetric wave functions, Definition 3.3.29 |
| $B_{i;j}^{\pm}, B_{i;j}$ | contraction operators, Definition 3.3.34 |
| $\phi^{\otimes k}$ | $k$-fold tensor of $\phi$ with itself, (3.3.64) |
| $\omega_{L^2}$ | symplectic form on $L^2(\mathbb{R}^k)$, (3.3.15) |
| $\mathcal{A}_{\mathcal{S}}$ | see Proposition 3.3.13 and (3.3.20) |
| $\{\cdot,\cdot\}_{L^2}$ | Poisson bracket on $L^2(\mathbb{R}^k)$, (3.3.21) |
| $A_{(j_1,\ldots,j_k)}^{(k)}$ | $k$-particle extension, (3.4.5) |
| $\mathfrak{g}_k$ | locally convex space of $k$-body bosonic observables, (3.4.1) |
| $(\mathfrak{G}_N, [\cdot,\cdot]_{\mathfrak{G}_N})$ | Lie algebra of observable $N$-hierarchies, (3.4.49) |
| $\circ_r$ | r-fold contraction, (3.4.30) |
| $(\mathfrak{G}_N^*, \mathcal{A}_\infty, \{\cdot,\cdot\}_{\mathfrak{G}_N^*})$ | Lie-Poisson manifold of density matrix $N$-hierarchies, (3.4.64) |
| $\mathfrak{g}_{k,gmp}$ | locally convex space of $k$-body observables satisfying the good mapping property, (3.5.7) |
| $(\mathfrak{G}_\infty, [\cdot,\cdot]_{\mathfrak{G}_\infty})$ | Lie algebra of observable $\infty$-hierarchies, (3.5.8) and (3.5.9) |
| $\circ_\alpha^\beta$ | contraction operator, Lemma 3.5.1 |
| $(\mathfrak{G}_\infty^*, \mathcal{A}_\infty, \{\cdot,\cdot\}_{\mathfrak{G}_\infty^*})$ | Lie-Poisson manifold of density matrix $\infty$-hierarchies, (3.5.78), Definition 3.5.9 and (3.5.82) |
| $\mathrm{Tr}_{1,\ldots,N}$ | generalized trace, Definition 3.2.1 |
| $\mathrm{Tr}_{k+1,\ldots,N}$ | generalized partial trace, Proposition 3.2.4 |

Table 3.1: Notation

# Chapter 4

# Poisson Commuting Energies for a System of Infinitely Many Bosons[1]

## 4.1 Statement of Main Results and Blueprint of Proofs

We provide an outline and discussion of the main results of this chapter and their proofs. We begin by recalling in Section 4.1.1 several of the main geometric results from Chapter 3 which are needed in the current chapter.

### 4.1.1 Review of Chapter 3

As we saw in Chapter 3, a major soure of difficulty is the construction of an infinite-dimensional Lie algebra of observable $\infty$-hierarchies and its dual weak Lie-Poisson manifold of density matrix $\infty$-hierarchies, which together form the geometric foundation of the Hamiltonian formulation of the GP hierarchy. The analytic difficulties in this definition stem primarily from the fact that the GP Hamiltonian $\mathcal{H}_{GP} = \mathcal{H}_3$ is the trace functional associated to a *distribution-valued operator (DVO)*.[2] The natural Lie bracket for such operators requires composition of two operators in a given particle coordinate. Such a definition is not possible in general since the composition of two DVOs may be ill-defined. Overcoming these

[1]This chapter is based on an equal collaboration with D. Mendelson, A.R. Nahmod, N. Pavlović, and G. Staffilani.

[2]Not to be confused with operator-valued distributions in quantum field theory.

difficulties necessitated the identification of a property for DVOs which we termed the *good mapping property*, whose definition we recall here.

**Definition 4.1.1** (Good mapping property)**.** Let $\ell \in \mathbb{N}$. We say that an operator $A^{(\ell)} \in \mathcal{L}(\mathcal{S}(\mathbb{R}^\ell), \mathcal{S}'(\mathbb{R}^\ell))$ has the *good mapping property* if for any $\alpha \in \mathbb{N}_{\leq \ell}$, the continuous bilinear map

$$\mathcal{S}(\mathbb{R}^\ell) \times \mathcal{S}(\mathbb{R}^\ell) \to \mathcal{S}_{x'_\alpha}(\mathbb{R}; \mathcal{S}'_{x_\alpha}(\mathbb{R}))$$
$$(f^{(\ell)}, g^{(\ell)}) \mapsto \int_{\mathbb{R}^{\ell-1}} dx_1 \ldots dx_{\alpha-1} dx_{\alpha+1} \ldots dx_\ell A^{(\ell)}(f^{(\ell)})(x_1, \ldots, x_\ell) g^{(\ell)}(x_1, \ldots, x_{\alpha-1}, x'_\alpha, x_{\alpha+1}, \ldots, x_\ell),$$

may be identified with a continuous bilinear map $\mathcal{S}(\mathbb{R}^\ell) \times \mathcal{S}(\mathbb{R}^\ell) \to \mathcal{S}(\mathbb{R}^2)$.[3]

The good mapping property has the following important consequence: let $(\alpha, \beta) \in \mathbb{N}_{\leq \ell} \times \mathbb{N}_{\leq j}$, and let $A^{(\ell)} \in \mathcal{L}(\mathcal{S}(\mathbb{R}^\ell), \mathcal{S}'(\mathbb{R}^\ell))$ and $B^{(j)} \in \mathcal{L}(\mathcal{S}(\mathbb{R}^j), \mathcal{S}'(\mathbb{R}^j))$ have the good mapping property. If $k := \ell + j - 1$, then the bilinear map

$$\mathcal{S}(\mathbb{R}^k)^2 \to \mathcal{S}_{(\underline{x}_{\alpha-1}, \underline{x}_{\alpha+1;\ell}, \underline{x}'_\ell)}(\mathbb{R}^{\alpha-1} \times \mathbb{R}^{\ell-\alpha} \times \mathbb{R}^\ell; \mathcal{S}'_{x_\alpha}(\mathbb{R}))$$
$$(f^{(k)}, g^{(k)}) \mapsto \begin{cases} \left\langle B^{(j)}_{(1,\ldots,j)}(f^{(k)}(\underline{x}_{\alpha-1}, \cdot, \underline{x}_{\alpha+1;\ell}, \cdot)), (\cdot) \otimes g^{(k)}(\underline{x}'_\ell, \cdot) \right\rangle_{\mathcal{S}'(\mathbb{R}^j)-\mathcal{S}(\mathbb{R}^j)}, & \beta = 1 \\ \left\langle B^{(j)}_{(2,\ldots,\beta,1,\beta+1,\ldots,j)}(f^{(k)}(\underline{x}_{\alpha-1}, \cdot, \underline{x}_{\alpha+1;\ell}, \cdot)), (\cdot) \otimes g^{(k)}(\underline{x}'_\ell, \cdot) \right\rangle_{\mathcal{S}'(\mathbb{R}^j)-\mathcal{S}(\mathbb{R}^j)}, & \beta \neq 1 \end{cases} \tag{4.1.1}$$

may be identified with a unique smooth bilinear map

$$\Phi_{B^{(j)},\alpha,\beta} : \mathcal{S}(\mathbb{R}^k) \times \mathcal{S}(\mathbb{R}^k) \to \mathcal{S}_{(\underline{x}_\ell, \underline{x}'_\ell)}(\mathbb{R}^{2\ell}) \tag{4.1.2}$$

[3] Here and throughout this chapter, an integral involving a distribution should be understood as a distributional pairing unless specified otherwise.

via

$$\int_{\mathbb{R}} dx_\alpha \Phi_{B^{(j)},\alpha,\beta}(f^{(k)},g^{(k)})(\underline{x}_\ell;\underline{x}'_\ell)\phi(x_\alpha)$$
$$= \begin{cases} \left\langle B^{(j)}_{(1,\ldots,j)}(f^{(k)}(\underline{x}_{\alpha-1},\cdot,\underline{x}_{\alpha+1;\ell},\cdot)), \phi\otimes g^{(k)}(\underline{x}'_\ell,\cdot)\right\rangle_{\mathcal{S}'(\mathbb{R}^j)-\mathcal{S}(\mathbb{R}^j)}, & \beta=1 \\ \left\langle B^{(j)}_{(2,\ldots,\beta,1,\beta+1,\ldots,j)}(f^{(k)}(\underline{x}_{\alpha-1},\cdot,\underline{x}_{\alpha+1;\ell},\cdot)), \phi\otimes g^{(k)}(\underline{x}'_\ell,\cdot)\right\rangle_{\mathcal{S}'(\mathbb{R}^j)-\mathcal{S}(\mathbb{R}^j)}, & \beta\neq 1, \end{cases} \tag{4.1.3}$$

for any $\phi \in \mathcal{S}(\mathbb{R})$ and $(\underline{x}_{1;\alpha-1},\underline{x}_{\alpha+1;\ell},\underline{x}'_\ell) \in \mathbb{R}^{2\ell-1}$. Here, the subscript $(2,\ldots,\beta,1,\beta+1,\ldots,j)$ is to be interpreted in the sense of the subscript notation in (4.1.13) (see also Proposition 3.3.1).[4] Hence, by the Schwartz kernel theorem isomorphism

$$\mathcal{L}(\mathcal{S}(\mathbb{R}^k),\mathcal{S}'(\mathbb{R}^k)) \cong \mathcal{S}(\mathbb{R}^{2k}), \tag{4.1.4}$$

we can define the following composition as an element

$$(A^{(\ell)} \circ_\alpha^\beta B^{(j)}) \in \mathcal{L}(\mathcal{S}(\mathbb{R}^k),\mathcal{S}'(\mathbb{R}^k)) \tag{4.1.5}$$

by

$$\left\langle (A^{(\ell)} \circ_\alpha^\beta B^{(j)})f^{(k)},g^{(k)}\right\rangle_{\mathcal{S}'(\mathbb{R}^k)-\mathcal{S}(\mathbb{R}^k)} := \left\langle K_{A^{(\ell)}}, \Phi^t_{B^{(j)},\alpha,\beta}(f^{(k)},g^{(k)})\right\rangle_{\mathcal{S}'(\mathbb{R}^{2k})-\mathcal{S}(\mathbb{R}^{2k})}, \tag{4.1.6}$$

where $K_{A^{(\ell)}}$ denotes the Schwartz kernel of $A^{(\ell)}$ and $\Phi^t_{B^{(j)},\alpha,\beta}(f^{(k)},g^{(k)})$ denotes the transpose of $\Phi_{B^{(j)},\alpha,\beta}(f^{(k)},g^{(k)})$ defined by

$$\Phi^t_{B^{(j)},\alpha,\beta}(f^{(k)},g^{(k)})(\underline{x}_j;\underline{x}'_j) := \Phi_{B^{(j)},\alpha,\beta}(f^{(k)},g^{(k)})(\underline{x}'_j;\underline{x}_j), \qquad \forall (\underline{x}_j,\underline{x}'_j)\in\mathbb{R}^{2j}. \tag{4.1.7}$$

Note that $A^{(\ell)} \circ_\alpha^\beta B^{(j)}$ coincides with the composition

$$A^{(\ell)}_{(1,\ldots,\ell)} B^{(j)}_{(\ell+1,\ldots,\ell+\beta-1,\alpha,\ell+\beta,\ldots,k)} \tag{4.1.8}$$

[4] So as to avoid a cumbersome consideration of cases in the sequel, we will not distinguish between the $\beta=1$ and $\beta\neq 1$ cases going forward.

when the latter is defined. We let $\mathcal{L}_{gmp}(\mathcal{S}(\mathbb{R}^\ell), \mathcal{S}'(\mathbb{R}^\ell))$ denote the subset of $\mathcal{L}(\mathcal{S}(\mathbb{R}^\ell), \mathcal{S}'(\mathbb{R}^\ell))$ of elements with the good mapping property, and $\mathcal{L}_{gmp,*}(\mathcal{S}(\mathbb{R}^\ell), \mathcal{S}'(\mathbb{R}^\ell))$ denote the further subset of elements which are skew-adjoint (see Lemma 3.1.1 and Definition 3.1.3 for the definitions of adjoint and skew-adjoint for a DVO). We established in Lemma 3.5.1 and Remark 3.5.3 that the composition

$$(\cdot) \circ_\alpha^\beta (\cdot) : \mathcal{L}_{gmp,*}(\mathcal{S}(\mathbb{R}^\ell), \mathcal{S}'(\mathbb{R}^\ell)) \times \mathcal{L}_{gmp,*}(\mathcal{S}(\mathbb{R}^j), \mathcal{S}'(\mathbb{R}^j)) \to \mathcal{L}_{gmp,*}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k)) \tag{4.1.9}$$

is a separately continuous, bilinear map.

With the composition map $(\cdot) \circ_\alpha^\beta (\cdot)$ in hand, we proceed to reviewing the main geometric actors from Chapter 3. We recall that

$$\mathfrak{g}_{k,gmp} \coloneqq \{A^{(k)} \in \mathcal{L}_{gmp}(\mathcal{S}_s(\mathbb{R}^k), \mathcal{S}_s'(\mathbb{R}^k)) : (A^{(k)})^* = -A^{(k)}\}, \tag{4.1.10}$$

where $\mathcal{S}_s(\mathbb{R}^k)$ is the subspace of $\mathcal{S}(\mathbb{R}^k)$ consisting of functions invariant under permutation of coordinates (see Definition 3.3.24), and

$$\mathfrak{G}_\infty \coloneqq \bigoplus_{k=1}^\infty \mathfrak{g}_{k,gmp} \tag{4.1.11}$$

endowed with the locally convex topology. We equip $\mathfrak{G}_\infty$ with a Lie bracket given by

$$\begin{aligned} &[A,B]_{\mathfrak{G}_\infty} = C = (C^{(k)})_{k\in\mathbb{N}} \\ &C^{(k)} \coloneqq \mathrm{Sym}_k\left(\sum_{\ell,j\geq 1; \ell+j-1=k} \sum_{\alpha=1}^{\ell}\sum_{\beta=1}^{j} \left((A^{(\ell)} \circ_\alpha^\beta B^{(j)}) - (B^{(j)} \circ_\beta^\alpha A^{(\ell)})\right)\right),^{5} \end{aligned} \tag{4.1.12}$$

where $\mathrm{Sym}_k$ denotes the bosonic symmetrization operator given by

$$\mathrm{Sym}_k(A^{(k)}) \coloneqq \frac{1}{k!}\sum_{\pi\in\mathbb{S}_k} A^{(k)}_{(\pi(1),\ldots,\pi(k))}, \quad A^{(k)}_{(\pi(1),\ldots,\pi(k))} = \pi \circ A^{(k)} \circ \pi^{-1}. \tag{4.1.13}$$

[5] Strictly speaking, a priori it is not the operators $A^{(\ell)}$ and $B^{(j)}$ that appear in the right-hand side, but instead extensions $\tilde{A}^{(\ell)} \in \mathcal{L}_{gmp}(\mathcal{S}(\mathbb{R}^\ell), \mathcal{S}'(\mathbb{R}^\ell))$ and $\tilde{B}^{(j)} \in \mathcal{L}_{gmp}(\mathcal{S}(\mathbb{R}^j), \mathcal{S}'(\mathbb{R}^j))$. The right-hand side is independent of the choice of extension, as shown in Remark 3.5.5, and therefore we will not comment on this technical point in the sequel.

**Proposition 4.1.2** (Proposition 3.1.7)**.** $(\mathfrak{G}_\infty, [\cdot,\cdot]_{\mathfrak{G}_\infty})$ *is a Lie algebra.*

Next, we recall the definition of the weak Lie-Poisson manifold $(\mathfrak{G}_\infty^*, \mathcal{A}_\infty, \{\cdot,\cdot\}_{\mathfrak{G}_\infty^*})$, which is the phase space underlying the GP hierarchy. We define the real topological vector space

$$\mathfrak{g}_k^* := \left\{\gamma^{(k)} \in \mathcal{L}(\mathcal{S}_s'(\mathbb{R}^k), \mathcal{S}_s(\mathbb{R}^k)) : \gamma^{(k)} = (\gamma^{(k)})^*\right\} \tag{4.1.14}$$

and define the topological direct product

$$\mathfrak{G}_\infty^* := \prod_{k=1}^\infty \mathfrak{g}_k^*. \tag{4.1.15}$$

Attached to $\mathfrak{G}_\infty^*$ is the admissible algebra of functionals $\mathcal{A}_\infty$ defined to be the real algebra with respect to point-wise product generated by functionals in the set

$$\{F \in C^\infty(\mathfrak{G}_\infty^*; \mathbb{R}) : F(\cdot) = i\,\mathrm{Tr}(\mathbf{W}\cdot),\ \mathbf{W} \in \mathfrak{G}_\infty\} \cup \{F \in C^\infty(\mathfrak{G}_\infty^*; \mathbb{R}) : F(\cdot) \equiv C \in \mathbb{R}\}. \tag{4.1.16}$$

Most importantly, our choice of $\mathcal{A}_\infty$ contains the trace functionals associated to the observable $\infty$-hierarchies $\{-i\mathbf{W}_n\}_{n=1}^\infty$. We can then define the Poisson bracket of functionals $F, G \in \mathcal{A}_\infty$ by

$$\{F, G\}_{\mathfrak{G}_\infty^*}(\Gamma) = i\,\mathrm{Tr}\Big([dF[\Gamma], dG[\Gamma]]_{\mathfrak{G}_\infty} \cdot \Gamma\Big), \qquad \forall \Gamma \in \mathfrak{G}_\infty^*. \tag{4.1.17}$$

In the right-hand side of (4.1.17), we identify the Gâteaux derivatives $dF[\Gamma]$ and $dG[\Gamma]$, which are a priori continuous linear functionals, as elements of $\mathfrak{G}_\infty$. This identification is possible thanks to the definition of $\mathcal{A}_\infty$ and the next lemma, which characterizes the dual of $\mathfrak{G}_\infty^*$.

**Lemma 4.1.3** (Lemma 3.5.8)**.** *The topological dual of $\mathfrak{G}_\infty^*$, denoted by $(\mathfrak{G}_\infty^*)^*$ and endowed with the strong dual topology, is isomorphic to*

$$\widetilde{\mathfrak{G}}_\infty := \{A \in \bigoplus_{k=1}^{\infty} \mathcal{L}(\mathcal{S}_s(\mathbb{R}^k), \mathcal{S}_s'(\mathbb{R}^k)) : (A^{(k)})^* = -A^{(k)}\}, \tag{4.1.18}$$

*equipped with the subspace topology induced by $\bigoplus_{k=1}^{\infty} \mathcal{L}(\mathcal{S}_s(\mathbb{R}^k), \mathcal{S}_s'(\mathbb{R}^k))$, via the canonical bilinear form*

$$i\,\mathrm{Tr}(A\cdot\Gamma) = i\sum_{k=1}^{\infty} \mathrm{Tr}_{1,\ldots,k}(A^{(k)}\gamma^{(k)}), \qquad \forall \Gamma = (\gamma^{(k)})_{k\in\mathbb{N}} \in \mathfrak{G}_\infty^*, \;\; A = (A^{(k)})_{k\in\mathbb{N}} \in \widetilde{\mathfrak{G}}_\infty. \tag{4.1.19}$$

In Chapter 3, classical results on the existence of a Lie-Poisson manifold associated to a Lie algebra were unavailable to us due to functional analytic difficulties, such as the fact that $\mathfrak{G}_\infty \subsetneq \widetilde{\mathfrak{G}}_\infty$. Nevertheless, we verified directly that our choices for $\mathfrak{G}_\infty^*$, $\mathcal{A}_\infty$, and $\{\cdot,\cdot\}_{\mathfrak{G}_\infty^*}$ satisfy the weak Poisson axioms of Definition 3.3.1, thereby establishing the following result.

**Proposition 4.1.4** (Proposition 3.1.8 and Lemma 3.5.15)**.** $(\mathfrak{G}_\infty^*, \mathcal{A}_\infty, \{\cdot,\cdot\}_{\mathfrak{G}_\infty^*})$ *is a weak Poisson manifold. Furthermore, for any $F \in \mathcal{A}_\infty$, the Hamiltonian vector field $X_F$ is given by the formula*

$$X_F(\Gamma)^{(\ell)} = \sum_{j=1}^{\infty} j\,\mathrm{Tr}_{\ell+1,\ldots,\ell+j-1}\left(\left[\sum_{\alpha=1}^{\ell} dH[\Gamma]^{(j)}_{(\alpha,\ell+1,\ldots,\ell+j-1)}, \gamma^{(\ell+j-1)}\right]\right), \qquad \ell\in\mathbb{N}, \;\; \Gamma \in \mathfrak{G}_\infty^*, \tag{4.1.20}$$

*where the extension $dH[\Gamma]^{(j)}_{(\alpha,\ell+1,\ldots,\ell+j-1)}$ is defined via Proposition 3.3.1.*

### 4.1.2 Statement of Main Results

Having reviewed the results from Chapter 3 presently germane, we are now prepared to state the main results of the current work. We previously introduced the GP hierarchy in

(1.2.10), which we recall now. We say that a sequence of time-dependent kernels $(\gamma^{(k)})_{k\in\mathbb{N}}$ of $k$-particle density matrices is a solution to the GP hierarchy if

$$i\partial_t \gamma^{(k)} = -\left[\Delta_{\underline{x}_k}, \gamma^{(k)}\right] + 2\kappa B_{k+1}(\gamma^{(k+1)}), \qquad k \in \mathbb{N}, \tag{4.1.21}$$

with $\kappa \in \{\pm 1\}$, and

$$B_{k+1}(\gamma^{(k+1)}) = \sum_{j=1}^{k} \left(B_{j;k+1}^{+} - B_{j;k+1}^{-}\right)(\gamma^{(k+1)}), \tag{4.1.22}$$

where for every $(\underline{x}_k, \underline{x}_k') \in \mathbb{R}^{2k}$,

$$\begin{aligned} B_{j;k+1}^{+}(\gamma^{(k+1)})(t, \underline{x}_k; \underline{x}_k') &\coloneqq \gamma^{(k+1)}(t, \underline{x}_k, x_j; \underline{x}_k', x_j), \\ B_{j;k+1}^{-}(\gamma^{(k+1)})(t, \underline{x}_k; \underline{x}_k') &\coloneqq \gamma^{(k+1)}(t, \underline{x}_k, x_j'; \underline{x}_k', x_j'). \end{aligned} \tag{4.1.23}$$

When $\kappa = 1$, we say that the hierarchy is *defocusing* and for $\kappa = -1$, we say that the hierarchy is *focusing* (in analogy with the defocusing and focusing NLS, respectively).

To address Question 1.3.3, we must first establish the existence of an infinite sequence of observable $\infty$-hierarchies $\{-i\mathbf{W}_n\}_{n\in\mathbb{N}} \in \mathfrak{G}_\infty$ by a recursion argument inspired by that for the operators $w_n$ in (1.3.8). Due to analytic difficulties, once again stemming primarily from the need to consider the composition of DVOs, we proceed in three steps.

The first step consists of constructing an element

$$\widetilde{\mathbf{W}}_n \in \bigoplus_{k=1}^{\infty} \mathcal{L}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k))$$

by the recursive formula

$$\begin{aligned} &\widetilde{\mathbf{W}}_1 \coloneqq \mathbf{E}_1 = (Id_1, 0, \ldots) \\ &\widetilde{\mathbf{W}}_{n+1}^{(k)} \coloneqq (-i\partial_{x_1})\widetilde{\mathbf{W}}_n^{(k)} + \kappa \sum_{m=1}^{n-1} \sum_{\ell,j\geq 1;\ell+j=k} \delta(X_1 - X_{\ell+1})\left(\widetilde{\mathbf{W}}_m^{(\ell)} \otimes \widetilde{\mathbf{W}}_{n-m}^{(j)}\right), \qquad \forall k \in \mathbb{N}, \end{aligned} \tag{4.1.24}$$

Note the structural similarity between this recursion and the one for the operators $w_n$ stated in (1.3.8). While the DVO $\widetilde{\mathbf{W}}_m^{(\ell)} \otimes \widetilde{\mathbf{W}}_{n-m}^{(j)}$ is well-defined by the universal property of the tensor product, the composition

$$\delta(X_1 - X_{\ell+1})\Big(\widetilde{\mathbf{W}}_m^{(\ell)} \otimes \widetilde{\mathbf{W}}_{n-m}^{(j)}\Big) \tag{4.1.25}$$

is a priori purely formal, since evaluation on a Schwartz function leads to products of distributions, in particular products of $\delta$ functions and their higher-order derivatives. Thus, the challenge is to give meaning to this composition. The key property which allow us to make sense of the composition is that if we formally expand the recursion, we will only find products such as $\delta(x_1 - x_2)\delta(x_2 - x_3)$, which is well-defined as the Lebesgue measure on the hyperplane $\{\underline{x}_k \in \mathbb{R}^k : x_1 = x_2 = x_3\}$. To systematically handle the products of distributions, we use the wave front set and a useful criterion of Hörmander for the multiplication of distributions (see Proposition 4.0.14 and more generally, Appendix 4).

A priori, Hörmander's criterion only yields that the product of two tempered distributions is a distribution, not necessarily tempered, which is problematic since we work exclusively with tempered distributions. Moreover, we wish any definition of the composition (4.1.25) to satisfy the property

$$\begin{aligned}
&\Big\langle \delta(X_1 - X_{\ell+1})\Big(\widetilde{\mathbf{W}}_m^{(\ell)} \otimes \widetilde{\mathbf{W}}_{n-m}^{(j)}\Big)(f^{(\ell)} \otimes f^{(j)}), g^{(\ell)} \otimes g^{(j)} \Big\rangle_{\mathcal{S}'(\mathbb{R}^k)-\mathcal{S}(\mathbb{R}^k)} \\
&\quad = \int_{\mathbb{R}} dx\, \Phi_{\widetilde{\mathbf{W}}_m^{(\ell)}}(f^{(\ell)}, g^{(\ell)})(x,x) \Phi_{\widetilde{\mathbf{W}}_{n-m}^{(j)}}(f^{(j)}, g^{(j)})(x,x),
\end{aligned} \tag{4.1.26}$$

where

$$\Phi_{\widetilde{\mathbf{W}}_m^{(\ell)}} : \mathcal{S}(\mathbb{R}^\ell)^2 \to \mathcal{S}(\mathbb{R}^2), \quad \Phi_{\widetilde{\mathbf{W}}_{n-m}^{(j)}} : \mathcal{S}(\mathbb{R}^j)^2 \to \mathcal{S}(\mathbb{R}^2) \tag{4.1.27}$$

are the necessarily unique maps identifiable with

$$\begin{aligned}
&\mathcal{S}(\mathbb{R}^{\ell})^2 \to \mathcal{S}_{x'}(\mathbb{R};\mathcal{S}_x'(\mathbb{R})) \quad (f^{(\ell)}, g^{(\ell)}) \mapsto \left\langle \widetilde{\mathbf{W}}_m^{(\ell)} f^{(\ell)}, (\cdot)\otimes g^{(\ell)}(x',\cdot)\right\rangle_{\mathcal{S}'(\mathbb{R}^{\ell})-\mathcal{S}(\mathbb{R}^{\ell})}, \\
&\mathcal{S}(\mathbb{R}^{j})^2 \to \mathcal{S}_{x'}(\mathbb{R};\mathcal{S}_x'(\mathbb{R})) \quad (f^{(j)}, g^{(j)}) \mapsto \left\langle \widetilde{\mathbf{W}}_{n-m}^{(j)} f^{(j)}, (\cdot)\otimes g^{(j)}(x',\cdot)\right\rangle_{\mathcal{S}'(\mathbb{R}^{j})-\mathcal{S}(\mathbb{R}^{j})}
\end{aligned} \tag{4.1.28}$$

via

$$\begin{aligned}
\int_{\mathbb{R}} dx \Phi_{\widetilde{\mathbf{W}}_m^{(\ell)}}(f^{(\ell)}, g^{(\ell)})(x;x')\phi(x) &= \left\langle \widetilde{\mathbf{W}}_m^{(\ell)} f^{(\ell)}, \phi\otimes g^{(\ell)}(x',\cdot)\right\rangle_{\mathcal{S}'(\mathbb{R}^{\ell})-\mathcal{S}(\mathbb{R}^{\ell})}, \\
\int_{\mathbb{R}} dx \Phi_{\widetilde{\mathbf{W}}_{n-m}^{(j)}}(f^{(j)}, g^{(j)})(x;x')\phi(x) &= \left\langle \widetilde{\mathbf{W}}_{n-m}^{(j)} f^{(j)}, \phi\otimes g^{(j)}(x',\cdot)\right\rangle_{\mathcal{S}'(\mathbb{R}^{j})-\mathcal{S}(\mathbb{R}^{j})},
\end{aligned} \tag{4.1.29}$$

for any $\phi \in \mathcal{S}(\mathbb{R})$.

We ensure that this is achieved thanks once more to the good mapping property of Definition 4.1.1. Indeed, proceeding inductively and exploiting the recursion formula and the induction hypothesis that

$$\widetilde{\mathbf{W}}_1, \ldots, \widetilde{\mathbf{W}}_n \in \bigoplus_{k=1}^{\infty} \mathcal{L}_{gmp}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k))$$

together with some Fourier analysis, we show that the composition (4.1.25) is exactly what we think it should be, namely, the unique distribution in $\mathcal{D}'(\mathbb{R}^k)$ satisfying (4.1.26), which can then be shown to be tempered. Moreover, by further appealing to the good mapping property and the universal property of the tensor product, we can show that the composition (4.1.25) indeed belongs to $\mathcal{L}_{gmp}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k))$. The preceding discussion is summarized by the following proposition.

**Proposition 4.1.5.** *For each $n \in \mathbb{N}$, there exists an element*

$$\widetilde{\mathbf{W}}_n \in \bigoplus_{k=1}^{\infty} \mathcal{L}_{gmp}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k))$$

*defined according to the recursive formula (4.1.24), where the composition* (4.1.25) *is well-defined in the sense of Proposition 4.0.14.*

Since we are interested in the action of the elements $\widetilde{\mathbf{W}}_n$ on density matrices, which are self-adjoint, the second step in the construction is to make each $\widetilde{\mathbf{W}}_n$ self-adjoint in the sense of Definition 3.1.3. By the involution property of the adjoint operation (see Lemma 3.1.1), the DVO

$$\mathbf{W}_{n,sa} \coloneqq \frac{1}{2}\Big(\widetilde{\mathbf{W}}_n + \widetilde{\mathbf{W}}_n^*\Big) \tag{4.1.30}$$

is a self-adjoint element of $\mathcal{L}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k))$. Since we want to preserve the good mapping property throughout each step of the construction, the challenge is to show that $\widetilde{\mathbf{W}}_n^*$ also has the good mapping property. Naively taking the adjoint of the recursive formula (4.1.24), we should formally have that

$$\widetilde{\mathbf{W}}_{n+1}^{(k),*} \text{ ``}=\text{'' } \widetilde{\mathbf{W}}_n^{(k),*}(-i\partial_{x_1}) + \kappa \sum_{m=1}^{n-1} \sum_{\ell,j\geq 1; \ell+j=k} \Big(\widetilde{\mathbf{W}}_m^{(\ell),*} \otimes \widetilde{\mathbf{W}}_{n-m}^{(j),*}\Big)\delta(X_1 - X_{\ell+1}). \tag{4.1.31}$$

While the expression on the right-hand side is, a priori, meaningless,[6] by inducting on the statement that $\widetilde{\mathbf{W}}_1^*, \ldots, \widetilde{\mathbf{W}}_{n-1}^*$ having the good mapping property and exploiting duality, the recursion for $\widetilde{\mathbf{W}}_n$, and the good mapping property for $\widetilde{\mathbf{W}}_n$, we are able to prove that the $\widetilde{\mathbf{W}}_n^*$ have the good mapping property, as desired.

The third, final, and easiest step of the construction is to symmetrize the $\mathbf{W}_{n,sa}$, so that we obtain an $\infty$-hierarchy which belongs to $\mathfrak{G}_\infty$. The motivation is that we always restrict to permutation-invariant test functions, reflecting the bosonic nature of the underlying physics. To obtain a formula for $\mathbf{W}_n$ from $\mathbf{W}_{n,sa}$ is straightforward. We record this definition in the following proposition:

[6] Among other issues, we note that for $f^{(k)} \in \mathcal{S}(\mathbb{R}^k)$, the tempered distribution $\delta(x_1 - x_{\ell+1})f^{(k)}$ does not belong to the domain of $\widetilde{\mathbf{W}}_m^{(\ell),*} \otimes \widetilde{\mathbf{W}}_{n-m}^{(j),*}$.

**Proposition 4.1.6.** *For each $n \in \mathbb{N}$,*

$$-i\mathbf{W}_n := -i\,\mathrm{Sym}(\mathbf{W}_{n,sa}) = -\frac{i}{2}\left(\mathrm{Sym}\left(\widetilde{\mathbf{W}}_n\right) + \mathrm{Sym}\left(\widetilde{\mathbf{W}}_n^*\right)\right) \in \mathfrak{G}_\infty, \tag{4.1.32}$$

*where* Sym *is a bosonic symmetrization operator, the definition of which is given in Definition 3.3.30.*

Having constructed the $\infty$-hierarchies $\{-i\mathbf{W}_n\}_{n=1}^{\infty}$, we define trace functionals $\mathcal{H}_n \in \mathcal{A}_\infty$ by

$$\mathcal{H}_n(\Gamma) := \mathrm{Tr}(\mathbf{W}_n \cdot \Gamma), \qquad \Gamma \in \mathfrak{G}_\infty^*. \tag{4.1.33}$$

Since the functionals $I_n$ are generated by the operators $w_n$, much in the same manner as the trace functionals $\mathcal{H}_n$ are generated by the $\mathbf{W}_n$, our next task is to relate $\mathbf{W}_n$ to the one-particle nonlinear operators $w_n$ defined in (1.3.8). Doing so necessitates understanding the action of the $k$-particle components $\widetilde{\mathbf{W}}_n^{(k)}$ and $\widetilde{\mathbf{W}}_n^{(k),*}$ on pure tensors of the form

$$|\phi_1 \otimes \cdots \otimes \phi_k\rangle \, \langle \psi_1 \otimes \cdots \otimes \psi_k| \,, \qquad \phi_1, \ldots, \phi_k, \psi_1, \ldots, \psi_k \in \mathcal{S}(\mathbb{R}). \tag{4.1.34}$$

To make this connection precise for the arguments in Section 4.7, our strategy is to replace the nonlinear operator $w_n$ with a multilinear operator by generalizing the recursion (1.3.8). See Section 4.5.1 for more details. As most of the results in Section 4.5 are of a technical nature, and perhaps not so enlightening at this stage, we mention only the following result, which connects $\mathcal{H}_n$ to the functionals $I_n$ and can be obtained as an easy corollary of Proposition 4.6.2:

$$\mathcal{H}_n(\Gamma) = I_n(\phi), \qquad \forall \Gamma = (\,|\phi^{\otimes k}\rangle \, \langle \phi^{\otimes k}|)_{k \in \mathbb{N}}, \;\; \phi \in \mathcal{S}(\mathbb{R}). \tag{4.1.35}$$

Next, we turn to establishing the involution statement of Question 1.3.3, which we record in the following theorem:

**Theorem 4.1.7** (Involution theorem)**.** *Let $n, m \in \mathbb{N}$. Then*

$$\{\mathcal{H}_n, \mathcal{H}_m\}_{\mathfrak{G}_\infty^*} \equiv 0 \text{ on } \mathfrak{G}_\infty^*. \tag{4.1.36}$$

To prove Theorem 4.1.7, we proceed on both the one-particle and infinite-particle fronts. We prove that there is an equivalence between the involution of the functionals $\mathcal{H}_n$ and the involution of certain real-valued functionals $I_{b,n}$, defined in (4.1.40) below, on a weak Poisson manifold of mixed states. We find this equivalence, explicitly stated in Theorem 4.1.9 below, quite interesting its own right. We now provide some details of the proof of this equivalence.

On the one-particle front, we relax (1.3.7) to a system

$$\begin{cases} i\partial_t\phi_1 = -\Delta\phi_1 + 2\kappa\phi_1^2\phi_2, \\ i\partial_t\phi_2 = \Delta\phi_2 - 2\kappa\phi_2^2\phi_1 \end{cases}, \tag{4.1.37}$$

where $\phi_1, \phi_2 : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$. We study (4.1.37) as an integrable system on a *complex* weak Poisson manifold $(\mathcal{S}(\mathbb{R}^2), \mathcal{A}_{\mathcal{S},\mathbb{C}}, \{\cdot,\cdot\}_{L^2,\mathbb{C}})$, see Proposition 4.3.5 for the precise definition of this manifold, by revisiting in detail the treatment of the NLS (1.3.7) in [28]. Specifically, we show that there are functionals

$$\tilde{I}_n(\phi_1, \phi_2) \coloneqq \int_{\mathbb{R}} dx \phi_2(x) w_{n,(\phi_1,\phi_2)}(x), \qquad \forall (\phi_1, \phi_2) \in \mathcal{S}(\mathbb{R})^2, \ \ n \in \mathbb{N}, \tag{4.1.38}$$

where $w_{n,(\phi_1,\phi_2)}(x)$ satisfies a similar recursion formula to the $w_n$, see (1.2.12), such that $\tilde{I}_3$ is the Hamiltonian for NLS system (4.1.37), and such that the $\tilde{I}_n$ commute on $(\mathcal{S}(\mathbb{R}^2), \mathcal{A}_{\mathcal{S},\mathbb{C}}, \{\cdot,\cdot\}_{L^2,\mathbb{C}})$.

Since we are ultimately interested in *real*, not complex, weak Poisson manifolds, we pass to another weak Poisson manifold of *mixed states*, $(\mathcal{S}(\mathbb{R};\mathcal{V}), \mathcal{A}_{\mathcal{S},\mathcal{V}}, \{\cdot,\cdot\}_{L^2,\mathcal{V}})$, where the space $\mathcal{S}(\mathbb{R};\mathcal{V})$ consists of Schwartz functions $\gamma$ taking values in the space $\mathcal{V}$ of self-adjoint, off-diagonal $4\times 4$ complex matrices:

$$\gamma = \frac{1}{2}\mathrm{odiag}(\phi_1, \overline{\phi_2}, \phi_2, \overline{\phi_1}) = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & \phi_1 \\ 0 & 0 & \overline{\phi_2} & 0 \\ 0 & \phi_2 & 0 & 0 \\ \overline{\phi_1} & 0 & 0 & 0 \end{pmatrix}, \qquad \phi_1, \phi_2 \in \mathcal{S}(\mathbb{R}). \tag{4.1.39}$$

We refer to (4.3.17), (4.3.19), and Proposition 4.3.2 for the precise definition and properties of this weak Poisson manifold.

We use the $\tilde{I}_n$ to define real-valued functionals $I_{b,n} \in \mathcal{A}_{\mathcal{S},\mathcal{V}}$ on the manifold $(\mathcal{S}(\mathbb{R};\mathcal{V}), \mathcal{A}_{\mathcal{S},\mathcal{V}}, \{\cdot,\cdot\}_{L^2,\mathcal{V}})$ via the formula

$$I_{b,n}(\gamma) := \frac{1}{2}\Big(\tilde{I}_n(\phi_1, \overline{\phi_2}) + \tilde{I}_n(\phi_2, \overline{\phi_1})\Big), \tag{4.1.40}$$

and we show in Proposition 1.3.7 that the family $\{I_{b,n}\}_{n\in\mathbb{N}}$ is in mutual involution with respect to the Poisson bracket $\{\cdot,\cdot\}_{L^2,\mathcal{V}}$. As we do not feel the results described in this paragraph are the primary contribution of this work, but nevertheless believe they may be of independent interest to the community, we have placed them in Appendix 1 and not the main body of the chapter.

On the infinite-particle front, we first demonstrate that there is a Poisson morphism

$$\begin{aligned} &\iota_{\mathfrak{m}} : (\mathcal{S}(\mathbb{R};\mathcal{V}), \mathcal{A}_{\mathcal{S},\mathcal{V}}, \{\cdot,\cdot\}_{L^2,\mathcal{V}}) \to (\mathfrak{G}_\infty^*, \mathcal{A}_\infty, \{\cdot,\cdot\}_{\mathfrak{G}_\infty^*}) \\ &\iota_{\mathfrak{m}}(\gamma) := \frac{1}{2}\big(\,|\phi_1^{\otimes k}\rangle\,\langle\phi_2^{\otimes k}| + |\phi_2^{\otimes k}\rangle\,\langle\phi_1^{\otimes k}|\big)_{k\in\mathbb{N}}, \qquad \gamma = \frac{1}{2}\mathrm{odiag}(\phi_1, \overline{\phi_2}, \phi_2, \overline{\phi_1}). \end{aligned} \tag{4.1.41}$$

The subscript $\mathfrak{m}$ signifies that $\iota_{\mathfrak{m}}$ produces a mixed state element of $\mathfrak{G}_\infty^*$.

**Theorem 4.1.8.** *The map $\iota_{\mathfrak{m}}$ is a Poisson morphism of $(\mathcal{S}(\mathbb{R};\mathcal{V}), \mathcal{A}_{\mathcal{S},\mathcal{V}}, \{\cdot,\cdot\}_{L^2,\mathcal{V}})$ into $(\mathfrak{G}_\infty^*, \mathcal{A}_\infty, \{\cdot,\cdot\}_{\mathfrak{G}_\infty^*})$; i.e., it is a smooth map with the property that*

$$\iota_{\mathfrak{m}}^*\{\cdot,\cdot\}_{\mathfrak{G}_\infty^*} = \{\iota_{\mathfrak{m}}^*\cdot, \iota_{\mathfrak{m}}^*\cdot\}_{L^2,\mathcal{V}}, \tag{4.1.42}$$

*where $\iota_{\mathfrak{m}}^*$ denotes the pullback of $\iota_{\mathfrak{m}}$.*

Theorem 4.1.8 is a generalization of Theorem 3.1.12 from Chapter 3 and, in fact, recovers this previous theorem since Proposition 4.3.2 demonstrates that there is also a Poisson morphism

$$\iota_{\mathfrak{pm}} : (\mathcal{S}(\mathbb{R}), \mathcal{A}_{\mathcal{S}}, \{\cdot,\cdot\}_{L^2}) \to (\mathcal{S}(\mathbb{R};\mathcal{V}), \mathcal{A}_{\mathcal{S},\mathcal{V}}, \{\cdot,\cdot\}_{L^2,\mathcal{V}}), \quad \phi \mapsto \frac{1}{2}\mathrm{odiag}(\phi, \overline{\phi}, \phi, \overline{\phi}), \tag{4.1.43}$$

and the composition of Poisson morphisms is again a Poisson morphism.

The motivation for Theorem 4.1.8 is the following. Since

$$I_{b,n}(\gamma) = \mathcal{H}_n(\iota_{\mathfrak{m}}(\gamma)), \qquad \forall \gamma \in \mathcal{S}(\mathbb{R};\mathcal{V}) \tag{4.1.44}$$

by Proposition 4.6.2, and since $\{I_{b,n}, I_{b,m}\}_{L^2,\mathcal{V}} \equiv 0$ on $\mathcal{S}(\mathbb{R};\mathcal{V})$, for any $n, m \in \mathbb{N}$, by Proposition 1.3.7, Theorem 4.1.8 implies that

$$0 = \{\mathcal{H}_n, \mathcal{H}_m\}_{\mathfrak{G}_\infty^*}(\iota_{\mathfrak{m}}(\gamma)) = \frac{1}{2}\sum_{k=1}^{\infty} i\,\mathrm{Tr}_{1,\ldots,k}\Big([-i\mathbf{W}_n, -i\mathbf{W}_m]_{\mathfrak{G}_\infty}^{(k)}\big(|\phi_1^{\otimes k}\rangle\,\langle\phi_2^{\otimes k}| + |\phi_2^{\otimes k}\rangle\,\langle\phi_1^{\otimes k}|\big)\Big). \tag{4.1.45}$$

Note that only finitely many terms in the above summation are nonzero. Next, we use a scaling argument to show that (4.1.45) implies that each of the summands in the right-hand side of (4.1.45) are identically zero:

$$\frac{i}{2}\,\mathrm{Tr}_{1,\ldots,k}\Big([-i\mathbf{W}_n, -i\mathbf{W}_m]_{\mathfrak{G}_\infty}^{(k)}\big(|\phi_1^{\otimes k}\rangle\,\langle\phi_2^{\otimes k}| + |\phi_2^{\otimes k}\rangle\,\langle\phi_1^{\otimes k}|\big)\Big) = 0, \qquad \forall \phi_1, \phi_2 \in \mathcal{S}(\mathbb{R}),\ \ k \in \mathbb{N}. \tag{4.1.46}$$

The intuition is that if a polynomial is identically zero then all of its coefficients are zero. By unpacking the definition of the Poisson bracket $\{\mathcal{H}_n, \mathcal{H}_m\}_{\mathfrak{G}_\infty^*}$, (4.1.46) yields

$$\{\mathcal{H}_n, \mathcal{H}_m\}_{\mathfrak{G}_\infty^*}(\Gamma) = 0, \qquad \forall \Gamma = \frac{1}{2}\big(|\phi_{k,1}^{\otimes k}\rangle\,\langle\phi_{k,2}^{\otimes k}| + |\phi_{k,2}^{\otimes k}\rangle\,\langle\phi_{k,1}^{\otimes k}|\big)_{k\in\mathbb{N}}, \tag{4.1.47}$$

where $\phi_{k,1}, \phi_{k,2} \in \mathcal{S}(\mathbb{R})$ for every $k \in \mathbb{N}$. By then using an approximation argument from Appendix 5 involving symmetric-rank-1 approximations (see Corollary 5.0.24) together with the continuity of $\{\mathcal{H}_n, \mathcal{H}_m\}_{\mathfrak{G}^*_\infty}$, we obtain from (4.1.47) that Poisson commutativity of the $I_{b,n}$ implies the Poisson commutativity of $\mathcal{H}_n$. The reverse implication is a straightforward consequence of Theorem 4.1.8. Summarizing the preceding discussion, we have the following equivalence result:

**Theorem 4.1.9** (Poisson commutativity equivalence)**.** *For any* $n, m \in \mathbb{N}$*,*

$$\{I_{b,n}, I_{b,m}\}_{L^2,\mathcal{V}}(\gamma) = 0, \qquad \forall \gamma \in \mathcal{S}(\mathbb{R}; \mathcal{V}), \tag{4.1.48}$$

*if and only if*

$$\{\mathcal{H}_n, \mathcal{H}_m\}_{\mathfrak{G}^*_\infty}(\Gamma) = 0, \qquad \forall \Gamma \in \mathfrak{G}^*_\infty. \tag{4.1.49}$$

In light of Proposition 1.3.7, which asserts the validity of (4.1.48), we then obtain Theorem 4.1.7 from Theorem 4.1.9, thus answering Question 1.3.3.

Having resolved Question 1.3.3, we turn to answering Question 1.3.4. For each $n \in \mathbb{N}$, we define the *n-th GP hierarchy (nGP)* to be the Hamiltonian equation of motion generated by the functional $\mathcal{H}_n$ with respect to the Poisson structure on $\mathfrak{G}^*_\infty$:

$$\left(\frac{d}{dt}\Gamma\right) = X_{\mathcal{H}_n}(\Gamma), \tag{4.1.50}$$

where $X_{\mathcal{H}_n}$ is the unique Hamiltonian vector field defined by $\mathcal{H}_n$. See (P3) of Definition 3.3.1 for the definition of the Hamiltonian vector field. We generalize the fact that solutions to the NLS generate a special class of factorized solutions to the GP hierarchy by proving that the same correspondence is true for the (nNLS) and (nGP). Thus, we are led to our final main theorem, providing an affirmative answer to Question 1.3.4.

**Theorem 4.1.10** (Connection between (nGP) and (nNLS))**.** *Let $n \in \mathbb{N}$. Let $I \subset \mathbb{R}$ be a compact interval and let $\phi \in C^\infty(I; \mathcal{S}(\mathbb{R}))$ be a solution to the (nNLS) with lifespan $I$. If we define*

$$\Gamma \in C^\infty(I; \mathfrak{G}_\infty^*), \qquad \Gamma := \big( |\phi^{\otimes k}\rangle \, \langle \phi^{\otimes k}| \big)_{k\in\mathbb{N}}, \tag{4.1.51}$$

*then $\Gamma$ is a solution to the (nGP).*

**Remark 4.1.11.** In Chapter 3, we defined the *Gross-Pitaevskii Hamiltonian functional $\mathcal{H}_{GP}$* by

$$\mathcal{H}_{GP}(\Gamma) := \mathrm{Tr}_1\big(-\Delta_{x_1}\gamma^{(1)}\big) + \kappa \, \mathrm{Tr}_{1,2}\big(\delta(X_1 - X_2)\gamma^{(2)}\big), \qquad \forall \Gamma = (\gamma^{(k)})_{k\in\mathbb{N}} \in \mathfrak{G}_\infty^*. \tag{4.1.52}$$

In particular, $\mathcal{H}_{GP} = \mathcal{H}_3$, and in the one-dimensional case, we recover Theorem 3.1.10 from Chapter 3, which asserts that the GP hierarchy (4.1.21) is the Hamiltonian equation of motion on $(\mathfrak{G}_\infty^*, \mathcal{A}_\infty, \{\cdot,\cdot\}_{\mathfrak{G}_\infty^*})$ induced by $\mathcal{H}_{GP}$.

**Remark 4.1.12.** Theorem 4.1.10 does not assert that the factorized solution $( |\phi^{\otimes k}\rangle \, \langle \phi^{\otimes k}|)_{k\in\mathbb{N}}$ is the unique solution to the $n$-th GP hierarchy starting from factorized initial data, only that it is *a* particular solution. More generally, Theorem 4.1.10 makes no assertion about the uniqueness of solutions to the (nGP) in the class $C^\infty(I; \mathfrak{G}_\infty^*)$. While the (nNLS) are known to be globally well-posed in the Schwartz class by the work of Beals and Coifman [8] and Zhou [102], unconditional uniqueness of the $n$-th GP hierarchy in the class $C^\infty(I; \mathfrak{G}_\infty^*)$, for some compact interval $I$, is an open problem, the resolution of which we do not address in this work.

To prove Theorem 4.1.10, we need to show that the $n$-th GP Hamiltonian vector field

$X_{\mathcal{H}_n}$ can be written as

$$X_{\mathcal{H}_n}(\Gamma)^{(k)} = \sum_{\alpha=1}^{k} \big( |\phi^{\otimes(\alpha-1)} \otimes \nabla_s I_n(\phi) \otimes \phi^{(k-\alpha)}\rangle \, \langle \phi^{\otimes k}| + |\phi^{\otimes k}\rangle \, \langle \phi^{\otimes(\alpha-1)} \otimes \nabla_s I_n(\phi) \otimes \phi^{(k-\alpha)}| \big), \tag{4.1.53}$$

for $\Gamma$ as in the statement of Theorem 4.1.10. We remind the reader that $\nabla_s I_n$ denotes the symplectic gradient of $I_n$ with respect to the form $\omega_{L^2}$, see Definition 3.3.11. To establish the identity (4.1.53), we use a formula from Section 4.5.2 for $\nabla_s I_n$, which is in terms of the Gâteaux derivatives of the nonlinear operators $w_n$. Combining this formula with the computation of $X_{\mathcal{H}_n}(\Gamma)$ for factorized $\Gamma$ (see Lemma 4.7.2), which extensively uses the good mapping property of the generators of the $\mathcal{H}_n$ (i.e. $-i\mathbf{W}_n$), we obtain (4.1.53) and hence the desired conclusion.

### 4.1.3 Organization of the Chapter

We close Section 4.1 by commenting on the organization of the chapter. In Section 4.3, we introduce several extensions of the weak Poisson manifold $(\mathcal{S}(\mathbb{R}^k), \mathcal{A}_{\mathcal{S}}, \{\cdot,\cdot\}_{L^2})$ from Section 3.3.1. We have omitted a review of calculus in the locally convex setting, tensor products, Lie algebras, and general weak symplectic/Poisson manifolds, as a review of these subjects is contained in Section 3.3.

In Section 4.4, we construct our observable $\infty$-hierarchies $-i\mathbf{W}_n$, thereby proving Proposition 4.1.6. The section is divided into three subsections corresponding to each stage of the construction: the preliminary version, followed by the self-adjoint version, followed by the final bosonic, self-adjoint version.

Section 4.5 is devoted to analyzing the correspondence between the $w_n$ and the $\mathbf{W}_n$ and the consequences of this correspondence. Section 4.5.1 contains the "multilinearization" of the $w_n$. Section 4.5.2 contains the proof of a formula for the symplectic gradients of the $I_n$. Section 4.5.3 connects the multilinearizations of the $w_n$ from Section 4.5.1 with the partial traces of the $\mathbf{W}_n$.

In Section 4.6, we prove our involution result, Theorem 4.1.7, in addition to the main auxiliary results involved in the proof of this theorem, which might be of independent interest. This section is broken down into four subsections in order to make the presentation more modular. Section 4.6.1 contains the proof of the Poisson morphism result, Theorem 4.1.8. Section 4.6.2 connects the infinite-particle functionals $\mathcal{H}_n$ to the one-particle functions $I_{b,n}$ via the Poisson morphism of Theorem 4.1.8 and the correspondence results of Section 4.5.3. Section 4.6.3 contains the proofs of the Poisson commutativity equivalence result, Theorem 4.1.9, and the involution result, Theorem 4.1.7. Lastly, Section 4.6.4 contains the proof of Proposition 4.6.3, which asserts that there is at least one functional which does not Poisson commute with a given $\mathcal{H}_n$.

In the last section, Section 4.7, of the chapter, we prove our $n$-th GP/$n$-th NLS correspodence result, Theorem 4.1.10. Section 4.7.1 is devoted to the computation of the Hamiltonian vector fields of the $\mathcal{H}_n$ evaluated on factorized states, and Section 4.7.2 is devoted to the proof of Theorem 4.1.10. To close the section, we compute in Section 4.7.3 the fourth GP hierarchy, which corresponds to the complex mKdV equation.

## 4.2 Notation

### 4.2.1 Index of Notation

We include Table 4.1, located at the end of the chapter, as a guide for the frequently used symbols in this work. In this table, we either provide a definition of the notation or a reference for where the symbol is defined.

## 4.3 Preliminaries

We need several examples of weak Poisson/symplectic manifolds in this work. An example we discussed at length in Section 3.3.1 is the Schwartz space $\mathcal{S}(\mathbb{R}^k)$, as well as its bosonic counterpart $\mathcal{S}_s(\mathbb{R}^k)$. However, we shall also need several generalizations of these examples. We begin with some comments on variational derivatives.

**Remark 4.3.1** (Variational derivatives)**.** For functionals $F, G \in C^\infty(\mathcal{S}(\mathbb{R}^k);\mathbb{R})$ having a special form discussed below, there is a computationally more convenient way to express their symplectic gradients and Poisson bracket in terms of *variational derivatives.* Given a smooth functional $\tilde{F} : \mathcal{S}(\mathbb{R}^k)^2 \to \mathbb{C}$, we define the variational derivatives $\boldsymbol{\nabla}_1\tilde{F}$ and $\boldsymbol{\nabla}_{\bar{2}}\tilde{F}$ by the property[7]

$$d\tilde{F}[\phi_1, \overline{\phi_2}](\delta\phi_1, \delta\overline{\phi_2}) = \int_{\mathbb{R}^k} d\underline{x}_k \Big(\boldsymbol{\nabla}_1\tilde{F}(\phi_1, \overline{\phi_2})\delta\phi_1 + \boldsymbol{\nabla}_{\bar{2}}\tilde{F}(\phi_1, \overline{\phi_2})\delta\overline{\phi_2}\Big)(\underline{x}_k),\ \forall(\phi_1, \overline{\phi_2}), (\delta\phi_1, \delta\overline{\phi_2}) \in \mathcal{S}(\mathbb{R}^k)^2. \tag{4.3.1}$$

The reader can verify that the variational derivatives, if they exist, are unique.

Let $F, G \in C^\infty(\mathcal{S}(\mathbb{R}^k);\mathbb{R})$. Suppose that

$$F(\phi) = \tilde{F}(\phi, \overline{\phi}), \qquad \tilde{F} \in C^\infty(\mathcal{S}(\mathbb{R}^k)^2;\mathbb{C}), \tag{4.3.2}$$

where $\tilde{F}$ satisfies the conditions

$$\overline{\tilde{F}(\phi_1, \overline{\phi_2})} = \tilde{F}(\phi_2, \overline{\phi_1}), \qquad \boldsymbol{\nabla}_1\tilde{F},\ \boldsymbol{\nabla}_{\bar{2}}\tilde{F} \in C^\infty(\mathcal{S}(\mathbb{R}^k)^2;\mathcal{S}(\mathbb{R}^k)), \tag{4.3.3}$$

[7] Our notation for variational derivatives is nonstandard. In the calculus of variations literature, one typically finds $\frac{\delta f}{\delta\phi_1}$ and $\frac{\delta f}{\delta\overline{\phi_2}}$ instead of $\boldsymbol{\nabla}_1 f(\phi_1, \overline{\phi_2})$ and $\boldsymbol{\nabla}_{\bar{2}}(\phi_1, \overline{\phi_2})$, respectively. We prefer our notation as it emphasizes the nature of the variational derivatives as vector fields. The motivations for the seemingly odd use of the subscript $\bar{2}$, as opposed to just 2, will become clear later in this subsection.

and similarly for $G$ and $\tilde{G}$. Then we claim that $F, G \in A_{\mathcal{S}}$ and their Poisson bracket $\{F, G\}_{L^2}$ may be rewritten as

$$\{F, G\}_{L^2}(\phi) = -i \int_{\mathbb{R}} dx \Big( \boldsymbol{\nabla}_1 \tilde{F}(\phi, \overline{\phi}) \boldsymbol{\nabla}_{\bar{2}} \tilde{G}(\phi, \overline{\phi}) - \boldsymbol{\nabla}_{\bar{2}} \tilde{F}(\phi, \overline{\phi}) \boldsymbol{\nabla}_1 \tilde{G}(\phi, \overline{\phi}) \Big)(x). \tag{4.3.4}$$

Indeed, observe that

$$\begin{aligned} d\tilde{F}[\phi_1, \overline{\phi_2}](\delta\phi_1, \delta\overline{\phi_2}) &= \lim_{\varepsilon \to 0} \frac{\tilde{F}(\phi_1 + \varepsilon\delta\phi_1, \overline{\phi_2} + \varepsilon\delta\overline{\phi_2}) - \tilde{F}(\phi_1, \overline{\phi_2})}{\varepsilon} \\ &= \overline{\lim_{\varepsilon \to 0} \frac{\overline{\tilde{F}(\phi_2 + \varepsilon\overline{\delta\overline{\phi_2}}, \overline{\phi_1} + \varepsilon\overline{\delta\phi_1}) - \tilde{F}(\phi_2, \overline{\phi_1})}}{\varepsilon}} \\ &= \overline{d\tilde{F}[\phi_2, \overline{\phi_1}](\overline{\delta\overline{\phi_2}}, \overline{\delta\phi_1})} \\ &= \int_{\mathbb{R}^k} d\underline{x}_k \Big( \overline{\boldsymbol{\nabla}_1 \tilde{F}(\phi_2, \overline{\phi_1})} \delta\overline{\phi_2} + \overline{\boldsymbol{\nabla}_{\bar{2}} \tilde{F}(\phi_2, \overline{\phi_1})} \delta\phi_1 \Big)(\underline{x}_k), \end{aligned} \tag{4.3.5}$$

where the ultimate equality follows by definition of the variational derivatives. Since

$$d\tilde{F}[\phi_1, \overline{\phi_2}](\delta\phi_1, \delta\overline{\phi_2}) = \int_{\mathbb{R}^k} d\underline{x}_k \Big( \boldsymbol{\nabla}_1 \tilde{F}(\phi_1, \overline{\phi_2}) \delta\phi_1 + \boldsymbol{\nabla}_{\bar{2}} \tilde{F}(\phi_1, \overline{\phi_2}) \delta\overline{\phi_2} \Big)(\underline{x}_k), \tag{4.3.6}$$

we conclude by uniqueness of variational derivatives that

$$\boldsymbol{\nabla}_1 \tilde{F}(\phi_1, \overline{\phi_2}) = \overline{\boldsymbol{\nabla}_{\bar{2}} \tilde{F}(\phi_2, \overline{\phi_1})}, \qquad \boldsymbol{\nabla}_{\bar{2}} \tilde{F}(\phi_1, \overline{\phi_2}) = \overline{\boldsymbol{\nabla}_1 \tilde{F}(\phi_2, \overline{\phi_1})}. \tag{4.3.7}$$

Now recalling the definition of the symplectic gradient, we have that

$$\begin{aligned} \omega_{L^2}(\boldsymbol{\nabla}_s F(\phi), \psi) &= dF[\phi](\psi) \\ &= d\tilde{F}[\phi, \overline{\phi}](\psi, \overline{\psi}) \\ &= \int_{\mathbb{R}^k} d\underline{x}_k \Big( \boldsymbol{\nabla}_1 \tilde{F}(\phi, \overline{\phi}) \psi + \boldsymbol{\nabla}_{\bar{2}} \tilde{F}(\phi, \overline{\phi}) \overline{\psi} \Big)(\underline{x}_k) \\ &= 2\,\mathrm{Re}\Big\{ \int_{\mathbb{R}^k} d\underline{x}_k \boldsymbol{\nabla}_1 \tilde{F}(\phi, \overline{\phi})(\underline{x}_k) \psi(\underline{x}_k) \Big\}, \end{aligned} \tag{4.3.8}$$

where the ultimate equality follows from the relations (4.3.7). By uniqueness of the symplectic gradient, we conclude that

$$\nabla_s F(\phi) = -i\overline{\nabla_1 \tilde{F}(\phi, \overline{\phi})} = -i\nabla_{\bar{2}}\tilde{F}(\phi, \overline{\phi}) = \frac{1}{2}\Big(\overline{i\nabla_1 \tilde{F}(\phi, \overline{\phi})} - i\nabla_{\bar{2}}\tilde{F}(\phi, \overline{\phi})\Big). \tag{4.3.9}$$

Since the right-hand side of the preceding identity defines an element of $C^\infty(\mathcal{S}(\mathbb{R}^k); \mathcal{S}(\mathbb{R}^k))$, we obtain that $F \in \mathcal{A}_{\mathcal{S}}$. Now we can rewrite the Poisson bracket as

$$\begin{aligned}
\omega_{L^2}(\nabla_s F(\phi), \nabla_s G(\phi)) &= 2\,\mathrm{Im}\bigg\{\int_{\mathbb{R}^k} d\underline{x}_k \Big(i\nabla_1 \tilde{F}(\phi, \overline{\phi})\overline{i\nabla_1 \tilde{G}(\phi, \overline{\phi})}\Big)(\underline{x}_k)\bigg\} \\
&= -i\int_{\mathbb{R}^k} d\underline{x}_k \Big(\nabla_1 \tilde{F}(\phi, \overline{\phi})\overline{\nabla_1 \tilde{G}(\phi, \overline{\phi})} - \overline{\nabla_1 \tilde{F}(\phi, \overline{\phi})}\nabla_1 \tilde{G}(\phi, \overline{\phi})\Big)(\underline{x}_k) \\
&= -i\int_{\mathbb{R}^k} d\underline{x}_k \Big(\nabla_1 \tilde{F}(\phi, \overline{\phi})\nabla_{\bar{2}} \tilde{G}(\phi, \overline{\phi}) - \nabla_{\bar{2}} \tilde{F}(\phi, \overline{\phi})\nabla_1 \tilde{G}(\phi, \overline{\phi})\Big)(\underline{x}_k),
\end{aligned} \tag{4.3.10}$$

where the ultimate equality follows from the relations (4.3.7).

In the sequel, all of the functionals we consider will satisfy the requirements (4.3.3). Consequently, we will pass between the variational derivative formulation (4.3.4) and the symplectic gradient formulation of the Poisson bracket without comment.

To motivate our next extension of the weak Poisson manifold $(\mathcal{S}(\mathbb{R}^k), \mathcal{A}_{\mathcal{S}}, \{\cdot,\cdot\}_{L^2})$, we observe that we can identify a one-particle wave function $\phi$ with the pure state

$$|\phi\rangle\ \langle\phi|\,.$$

We can define a real topological vector space of pure states by considering the space of Schwartz functions taking values in the space of self-adjoint, off-diagonal $2 \times 2$ complex matrices:

$$\begin{pmatrix} 0 & \phi \\ \overline{\phi} & 0 \end{pmatrix}. \tag{4.3.11}$$

The natural generalization of a pure state is a mixed state,

$$\frac{1}{2}(\,|\phi_1\rangle\,\langle\phi_2| + |\phi_2\rangle\,\langle\phi_1|),$$

and we can define a real topological vector space of mixed states as follows: let $\mathcal{V}$ denote the real vector space of self-adjoint, off-diagonal $4 \times 4$ matrices of the form

$$\frac{1}{2}\mathrm{odiag}(a, \overline{b}, b, \overline{a}), \qquad a, b \in \mathbb{C}. \tag{4.3.12}$$

We let $\mathcal{S}(\mathbb{R}^k; \mathcal{V})$ denote the space of Schwartz functions taking values in the space $\mathcal{V}$. Elements of $\mathcal{S}(\mathbb{R}^k; \mathcal{V})$ have the form

$$\gamma(\underline{x}_k) = \frac{1}{2}\mathrm{odiag}(\phi_1(\underline{x}_k), \overline{\phi_2}(\underline{x}_k), \phi_2(\underline{x}_k), \overline{\phi_1}(\underline{x}_k)), \qquad \forall \underline{x}_k \in \mathbb{R}^k,\ \ \phi_1, \phi_2 \in \mathcal{S}(\mathbb{R}^k). \tag{4.3.13}$$

We can define a real pre-Hilbert inner product on $\mathcal{S}(\mathbb{R}^k; \mathcal{V})$ by

$$\langle \gamma_1 | \gamma_2 \rangle_{\mathrm{Re}, \mathcal{V}} := 2 \int_{\mathbb{R}^k} d\underline{x}_k \, \mathrm{tr}_{\mathbb{C}^2 \otimes \mathbb{C}^2}(\gamma_1(\underline{x}_k)\gamma_{2,swap}(\underline{x}_k)), \qquad \forall \gamma_1, \gamma_2 \in \mathcal{S}(\mathbb{R}^k; \mathcal{V}), \tag{4.3.14}$$

where $\mathrm{tr}_{\mathbb{C}^2 \otimes \mathbb{C}^2}$ denotes the $4 \times 4$ matrix trace and

$$\gamma_{2,swap} = \frac{1}{2}\mathrm{odiag}(\phi_2, \overline{\phi_1}, \phi_1, \overline{\phi_2}), \qquad \gamma_2 = \frac{1}{2}\mathrm{odiag}(\phi_1, \overline{\phi_2}, \phi_2, \overline{\phi_1}). \tag{4.3.15}$$

The matrix left-multiplication operator

$$J : \mathcal{S}(\mathbb{R}^k; \mathcal{V}) \to \mathcal{S}(\mathbb{R}^k; \mathcal{V}), \qquad J = \mathrm{diag}(i, -i, i, -i) \tag{4.3.16}$$

defines an almost complex structure. We can then define a symplectic form $\omega_{L^2, \mathcal{V}}$ by

$$\omega_{L^2, \mathcal{V}}(\gamma_1, \gamma_2) := \langle J\gamma_1 | \gamma_{2,swap} \rangle_{\mathrm{Re}, \mathcal{V}} \,. \tag{4.3.17}$$

Analogous to Proposition 3.3.13, we have that $(\mathcal{S}(\mathbb{R}^k;\mathcal{V}),\omega_{L^2,\mathcal{V}})$ is a weak symplectic manifold. Moreover, the obvious map

$$\iota_{\mathfrak{pm}} : \mathcal{S}(\mathbb{R}^k) \to \mathcal{S}(\mathbb{R}^k;\mathcal{V}), \qquad \phi \mapsto \frac{1}{2}\mathrm{odiag}(\phi,\overline{\phi},\phi,\overline{\phi}) \tag{4.3.18}$$

is a symplectomorphism. Additionally, if we denote the symplectic gradient with respect to the form $\omega_{L^2,\mathcal{V}}$ by $\boldsymbol{\nabla}_{s,\mathcal{V}}$, then one can show that if we define

$$\mathcal{A}_{\mathcal{S},\mathcal{V}} \coloneqq \{F \in C^\infty(\mathcal{S}(\mathbb{R}^k;\mathcal{V});\mathbb{R}) : \boldsymbol{\nabla}_{s,\mathcal{V}}F \in C^\infty(\mathcal{S}(\mathbb{R}^k;\mathcal{V}),\mathcal{S}(\mathbb{R}^k;\mathcal{V}))\}, \tag{4.3.19}$$

and let $\{\cdot,\cdot\}_{L^2,\mathcal{V}}$ be the Poisson bracket canonically induced by the form $\omega_{L^2,\mathcal{V}}$, then the triple

$$(\mathcal{S}(\mathbb{R}^k;\mathcal{V}),\mathcal{A}_{\mathcal{S},\mathcal{V}},\{\cdot,\cdot\}_{L^2,\mathcal{V}}) \tag{4.3.20}$$

is a weak Poisson manifold. We summarize the preceding discussion with the following proposition.

**Proposition 4.3.2.** $(\mathcal{S}(\mathbb{R}^k;\mathcal{V}),\omega_{L^2,\mathcal{V}})$ *is a weak symplectic manifold, and* $(\mathcal{S}(\mathbb{R}^k;\mathcal{V}),\mathcal{A}_{\mathcal{S},\mathcal{V}},\{\cdot,\cdot\}_{L^2,\mathcal{V}})$ *is a weak Poisson manifold, where*

$$\{F,G\}_{L^2,\mathcal{V}}(\gamma) \coloneqq \omega_{L^2,\mathcal{V}}(\boldsymbol{\nabla}_{s,\mathcal{V}}F(\gamma),\boldsymbol{\nabla}_{s,\mathcal{V}}G(\gamma)). \tag{4.3.21}$$

*Furthermore, the map* $\iota_{\mathfrak{pm}}$ *is a symplectomorphism; i.e., it is a smooth map such that*

$$\iota_{\mathfrak{pm}}^*\omega_{L^2,\mathcal{V}} = \omega_{L^2}, \tag{4.3.22}$$

*where* $\iota_{\mathfrak{pm}}^*$ *denotes the pullback of* $\iota_{\mathfrak{pm}}$*, so that*

$$\iota_{\mathfrak{pm}} : (\mathcal{S}(\mathbb{R}^k),\mathcal{A}_{\mathcal{S}},\{\cdot,\cdot\}_{L^2}) \to (\mathcal{S}(\mathbb{R}^k;\mathcal{V}),\mathcal{A}_{\mathcal{S},\mathcal{V}},\{\cdot,\cdot\}_{L^2,\mathcal{V}}) \tag{4.3.23}$$

*is a Poisson morphism.*

**Remark 4.3.3.** Remark 4.3.1 carries over to the setting of $\mathcal{S}(\mathbb{R}^k;\mathcal{V})$. More precisely, suppose $F \in C^\infty(\mathcal{S}(\mathbb{R}^k;\mathcal{V});\mathbb{R})$ is such that

$$F(\gamma) = \tilde{F}(\phi_1, \overline{\phi_2}, \phi_2, \overline{\phi_1}), \qquad \gamma = \frac{1}{2}\mathrm{odiag}(\phi_1, \overline{\phi_2}, \phi_2, \overline{\phi_1}) \in \mathcal{S}(\mathbb{R}^k;\mathcal{V}), \tag{4.3.24}$$

where $\tilde{F} \in C^\infty(\mathcal{S}(\mathbb{R}^k)^4;\mathbb{C})$, is such that

$$\boldsymbol{\nabla}_1\tilde{F}, \;\; \boldsymbol{\nabla}_{\bar{2}}\tilde{F}, \;\; \boldsymbol{\nabla}_2\tilde{F}, \;\; \boldsymbol{\nabla}_{\bar{1}}\tilde{F} \in C^\infty(\mathcal{S}(\mathbb{R}^k)^4;\mathcal{S}(\mathbb{R}^k)), \tag{4.3.25}$$

where the four variational derivatives are uniquely defined by

$$\begin{aligned}
&d\tilde{F}[\phi_1, \phi_{\bar{2}}, \phi_2, \phi_{\bar{1}}](\delta\phi_1, \delta\phi_{\bar{2}}, \delta\phi_2, \delta\phi_{\bar{1}}) \\
&= \int_{\mathbb{R}^k} d\underline{x}_k \Big(\Big(\boldsymbol{\nabla}_1\tilde{F}\delta\phi_1 + \boldsymbol{\nabla}_{\bar{2}}\tilde{F}\delta\phi_{\bar{2}} + \boldsymbol{\nabla}_2\tilde{F}\delta\phi_2 + \boldsymbol{\nabla}_{\bar{1}}\tilde{F}\delta\phi_{\bar{1}}\Big)(\phi_1, \phi_{\bar{2}}, \phi_2, \phi_{\bar{1}})\Big)(\underline{x}_k),
\end{aligned} \tag{4.3.26}$$

and $\tilde{F}$ has the involution property

$$\tilde{F}(\phi_1, \phi_{\bar{2}}, \phi_2, \phi_{\bar{1}}) = \overline{\tilde{F}(\overline{\phi_{\bar{1}}}, \overline{\phi_2}, \overline{\phi_{\bar{2}}}, \overline{\phi_1})}. \tag{4.3.27}$$

Then $F \in \mathcal{A}_{\mathcal{S},\mathcal{V}}$. Additionally, if $F, G$ are two such functionals, then their Poisson bracket may be rewritten as

$$\begin{aligned}
\{F, G\}_{L^2,\mathcal{V}}(\gamma) = &-i\int_{\mathbb{R}^k} d\underline{x}_k \Big(\boldsymbol{\nabla}_1\tilde{F}(\gamma)\boldsymbol{\nabla}_{\bar{2}}\tilde{G}(\gamma) - \boldsymbol{\nabla}_{\bar{2}}\tilde{F}(\gamma)\boldsymbol{\nabla}_1\tilde{G}(\gamma)\Big)(\underline{x}_k) \\
&- i\int_{\mathbb{R}^k} d\underline{x}_k \Big(\boldsymbol{\nabla}_2\tilde{F}(\gamma)\boldsymbol{\nabla}_{\bar{1}}\tilde{G}(\gamma) - \boldsymbol{\nabla}_{\bar{1}}\tilde{F}(\gamma)\boldsymbol{\nabla}_2\tilde{G}(\gamma)\Big)(\underline{x}_k),
\end{aligned} \tag{4.3.28}$$

where we identify $\gamma$ with the 4-tuple $(\phi_1, \overline{\phi_2}, \phi_2, \overline{\phi_1})$ for the sake of more compact notation.

In the sequel, all the functionals on $\mathcal{S}(\mathbb{R}^k;\mathcal{V})$ we consider satisfy the conditions of the remark. Consequently, we will pass between the variational derivative and symplectic gradient formulations for the Poisson bracket without comment.

Lastly, we make heavy use of a "complexified" version of the weak symplectic manifold $(\mathcal{S}(\mathbb{R}^k), \omega_{L^2})$. More precisely, consider the cartesian product $\mathcal{S}(\mathbb{R}^k)^2$ and define a complex-valued map

$$\omega_{L^2,\mathbb{C}}(\underline{f}_2, \underline{g}_2) \coloneqq \int_{\mathbb{R}^k} d\underline{x}_k \, \mathrm{tr}_{\mathbb{C}^2}(J_{\mathbb{C}} \underline{f}_2 \underline{g}_2)(\underline{x}_k), \tag{4.3.29}$$

where

$$\underline{f}_2 = \begin{pmatrix} 0 & f_1 \\ f_2 & 0 \end{pmatrix}, \; \underline{g}_2 = \begin{pmatrix} 0 & g_1 \\ g_2 & 0 \end{pmatrix} \in \mathcal{S}(\mathbb{R}^k)^2, \tag{4.3.30}$$

$\mathrm{tr}_{\mathbb{C}^2}$ denotes the $2 \times 2$ matrix trace, and $J_{\mathbb{C}}$ is the left-matrix multiplication operator $\mathrm{diag}(i, -i)$. Here, we identify a Schwartz function taking values in the space of off-diagonal $2 \times 2$ matrices with an element of $\mathcal{S}(\mathbb{R}^k)^2$ in the obvious manner.

**Remark 4.3.4.** Note that if $\underline{f}_2 = \mathrm{odiag}(f, \overline{f})$ and $\underline{g}_2 = \mathrm{odiag}(g, \overline{g})$, for $f, g \in \mathcal{S}(\mathbb{R}^k)$, then

$$\omega_{L^2,\mathbb{C}}(\underline{f}_2, \underline{g}_2) = i \int_{\mathbb{R}^k} d\underline{x}_k \big(f\overline{g} - \overline{f}g\big)(\underline{x}_k) = 2 \,\mathrm{Im}\bigg\{ \int_{\mathbb{R}^k} d\underline{x}_k \overline{f(\underline{x}_k)} g(\underline{x}_k) \bigg\} = \omega_{L^2}(f, g). \tag{4.3.31}$$

**Proposition 4.3.5.** *Define a subset* $A_{\mathcal{S},\mathbb{C}} \subset C^\infty(\mathcal{S}(\mathbb{R}^k)^2; \mathbb{C})$ *by*

$$\mathcal{A}_{\mathcal{S},\mathbb{C}} \coloneqq \big\{ H \in C^\infty(\mathcal{S}(\mathbb{R}^k); \mathbb{C}) \,:\, \boldsymbol{\nabla}_{s,\mathbb{C}} H \in C^\infty(\mathcal{S}(\mathbb{R})^2; \mathcal{S}(\mathbb{R})^2) \big\}, \tag{4.3.32}$$

*and define a bracket* $\{\cdot, \cdot\}_{L^2,\mathbb{C}}$ *by*

$$\{F, G\}_{L^2,\mathbb{C}} \coloneqq \omega_{L^2,\mathbb{C}}(\boldsymbol{\nabla}_{s,\mathbb{C}} F, \boldsymbol{\nabla}_{s,\mathbb{C}} G). \tag{4.3.33}$$

*Then* $(\mathcal{S}(\mathbb{R}^k)^2, \mathcal{A}_{\mathcal{S},\mathbb{C}}, \{\cdot, \cdot\}_{L^2,\mathbb{C}})$ *is a weak Poisson manifold.*

**Remark 4.3.6.** As before, if $F, G \in C^\infty(\mathcal{S}(\mathbb{R}^k)^2; \mathbb{C})$ satisfy the condition (4.3.3), then $F, G \in \mathcal{A}_{\mathcal{S},\mathbb{C}}$ and

$$\{F, G\}_{L^2,\mathbb{C}}(\phi_1, \overline{\phi_2}) = -i \int_{\mathbb{R}^k} d\underline{x}_k \big( \boldsymbol{\nabla}_1 F(\phi_1, \overline{\phi_2}) \boldsymbol{\nabla}_{\bar{2}} G(\phi_1, \overline{\phi_2}) - \boldsymbol{\nabla}_{\bar{2}} F(\phi_1, \overline{\phi_2}) \boldsymbol{\nabla}_1 G(\phi_1, \overline{\phi_2}) \big)(\underline{x}_k). \tag{4.3.34}$$

**Remark 4.3.7.** All the Schwartz space examples given in this subsection have their $2L$-periodic analogues, where $\mathcal{S}(\mathbb{R}^k)$ is replaced by $C^\infty(\mathbb{T}_L^k)$. We will need the periodic examples in Appendix 1.

## 4.4 The Construction: Defining the $\mathbf{W}_n$

We now define the operators $\mathbf{W}_n$ giving rise to the Hamiltonian functionals $\mathcal{H}_n$. As detailed in Section 4.1, in order to construct the operators $\mathbf{W}_n$, we proceed incrementally.

### 4.4.1 Step 1: Preliminary Definition of Operators

Let

$$\widetilde{\mathbf{W}}_1 = (\widetilde{\mathbf{W}}_1^{(k)})_{k\in\mathbb{N}} \in \bigoplus_{k=1}^{\infty} \mathcal{L}_{gmp}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k)), \qquad \widetilde{\mathbf{W}}_1 := \mathbf{E}_1, \tag{4.4.1}$$

where we recall that

$$\mathbf{E}_j = (\mathbf{E}_j^{(k)})_{k\in\mathbb{N}} \in \bigoplus_{k=1}^{\infty} \mathcal{L}_{gmp}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k)), \qquad \mathbf{E}_j^{(k)} := Id_k\, \delta_{jk}, \tag{4.4.2}$$

where $Id_k$ is the identity operator in $\mathcal{L}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k))$ and $\delta_{jk}$ is the Kronecker delta function. We regard $\mathbf{E}_j$ as the $j^{th}$ coordinate element of $\bigoplus_{k=1}^{\infty} \mathcal{L}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k))$. It is clear that these operators satisfy the good mapping property.

We would like to recursively define

$$\widetilde{\mathbf{W}}_{n+1} = (\widetilde{\mathbf{W}}_{n+1}^{(k)})_{k\in\mathbb{N}} \in \bigoplus_{k=1}^{\infty} \mathcal{L}_{gmp}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k)) \tag{4.4.3}$$

by the formula

$$\widetilde{\mathbf{W}}_{n+1}^{(k)} := -i\partial_{x_1} \widetilde{\mathbf{W}}_n^{(k)} + \kappa \sum_{m=1}^{n-1} \sum_{\ell,j\geq 1;\ell+j=k} \delta(X_1 - X_{\ell+1}) \Big(\widetilde{\mathbf{W}}_m^{(\ell)} \otimes \widetilde{\mathbf{W}}_{n-m}^{(j)}\Big), \qquad k \in \mathbb{N}, \tag{4.4.4}$$

where we regard the multiplier operator $-i\partial_{x_1}$ as a $k$-particle operator by tensoring with the identity in the $X_2, \ldots, X_k$ coordinates. Similarly, we regard the multiplication $\delta(X_1 - X_{\ell+1})$ as $k$-particle operator simply by tensoring with the identity in the $X_2, \ldots, X_\ell, X_{\ell+2}, \ldots, X_k$ coordinates.

Our aim is then two-fold. First, we need to make sense of the definition (4.4.4). At first glance, the right-hand side of (4.4.4) is purely formal, since for $n \geq 4$, the sum will contain products of $\delta$ functions. However, as we will prove in the next lemma, the operators in (4.4.4) are well-defined elements of $\mathcal{L}_{gmp}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k))$. Intuitively, this is because the products in (4.4.4) never contain delta functions with identical arguments, such as $\delta^2(X_1 - X_2)$. Subsequently, we will show that all but finitely many terms in the recursion are non-zero, which justifies our use of the direct sum notation. Thus, we are led to Proposition 4.1.5, the statement of which we recall below.

**Proposition 4.1.5.** *For each $n \in \mathbb{N}$, there exists an element*

$$\widetilde{\mathbf{W}}_n \in \bigoplus_{k=1}^{\infty} \mathcal{L}_{gmp}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k))$$

*defined according to the recursive formula (4.1.24), where the composition* (4.1.25) *is well-defined in the sense of Proposition 4.0.14.*

We begin the proof of Proposition 4.1.5 with establishing the recursion (4.4.4).

**Lemma 4.4.1** (Rigorous recursion)**.** *For every $k, n \in \mathbb{N}$, the distribution-valued operator $\widetilde{\mathbf{W}}_n^{(k)}$ is an element of $\mathcal{L}_{gmp}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k))$ and satisfies the following:*

*(R1) There exists a finite subset* $\mathsf{A}_n^{(k)} \subset \mathbb{N}_0^k$ *of multi-indices such that*

$$\widetilde{\mathbf{W}}_n^{(k)} f^{(k)} = \sum_{\underline{\alpha}_k \in \mathsf{A}_n^{(k)}} u_{\underline{\alpha}_k, n} \partial_{\underline{x}_k}^{\underline{\alpha}_k} f^{(k)}, \qquad \forall f^{(k)} \in \mathcal{S}(\mathbb{R}^k), \tag{4.4.5}$$

*where* $u_{\underline{\alpha}_k, n} \in \mathcal{S}'(\mathbb{R}^k)$.

*(R2) For every* $\underline{\alpha}_k \in \mathsf{A}_n^{(k)}$, *either*

**Case 1** $\mathrm{WF}(u_{\underline{\alpha}_k, n}) = \emptyset$, *or*

**Case 2** $\mathrm{WF}(u_{\underline{\alpha}_k, n}) \neq \emptyset$ *and satisfies the* non-vanishing pair property*:*

$$(\underline{x}_k, \underline{\xi}_k) \in \mathrm{WF}(u_{\underline{\alpha}_k, n}) \Longrightarrow \exists \ell, j \in \mathbb{N}_{\leq k} \ \textit{s.t.} \ \ell < j \ \textit{and both} \ \xi_\ell \neq 0 \ \textit{and} \ \xi_j \neq 0. \tag{4.4.6}$$

**Remark 4.4.2.** In other words, (R1) means that $\widetilde{\mathbf{W}}_n^{(k)}$ can be written as a linear combination of terms, where each term consists of a differential operator left-composed with a distributional multiplication operator. The motivation for the non-vanishing pair property is to exploit the fact that the products of delta functions in (4.4.4) do not have the same arguments.

*Proof of Lemma 4.4.1.* We prove the assertion by strong induction on $n \geq 1$. The base case, namely that the claims hold for $n = 1$, is clear. Next, let $n \geq 1$ and suppose that for every $k \in \mathbb{N}$, we have that

$$\widetilde{\mathbf{W}}_1^{(k)}, \ldots, \widetilde{\mathbf{W}}_n^{(k)} \in \mathcal{L}_{gmp}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k)) \tag{4.4.7}$$

are defined according to (4.4.1) and (4.4.4) and satisfy the properties (R1) and (R2). We will show that for any $k \in \mathbb{N}$, the observable $\widetilde{\mathbf{W}}_{n+1}^{(k)}$ is a well-defined element of $\mathcal{L}_{gmp}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k))$ and satisfies the properties (R1) and (R2). We organize our argument into several steps:

**Step I:** We first prove (R1). If $\mathsf{A}_n^{(k)} \subset \mathbb{N}_0^k$ is a finite subset of multi-indices such that

$$\widetilde{\mathbf{W}}_n^{(k)} f^{(k)} = \sum_{\underline{\alpha}_k \in \mathsf{A}_n^{(k)}} u_{\underline{\alpha}_k,n} \partial_{\underline{x}_k}^{\underline{\alpha}_k} f^{(k)}, \qquad \forall f^{(k)} \in \mathcal{S}(\mathbb{R}^k), \tag{4.4.8}$$

where $u_{\underline{\alpha}_k,n} \in \mathcal{S}'(\mathbb{R}^k)$, then by the product rule,

$$(-i\partial_{x_1})\widetilde{\mathbf{W}}_n^{(k)} f^{(k)} = \sum_{\underline{\alpha}_k \in \mathsf{A}_n^{(k)}} \Big( (-i\partial_{x_1} u_{\underline{\alpha}_k,n}) \partial_{\underline{x}_k}^{\underline{\alpha}_k} f^{(k)} - i u_{\underline{\alpha}_k,n} \partial_{x_1} \partial_{\underline{x}_k}^{\underline{\alpha}_k} f^{(k)} \Big), \qquad \forall f^{(k)} \in \mathcal{S}(\mathbb{R}^k). \tag{4.4.9}$$

Let $\mathsf{A}_m^{(\ell)}$ and $\mathsf{A}_{n-m}^{(j)}$ be finite subsets of $\mathbb{N}_0^\ell$ and $\mathbb{N}_0^j$, respectively, such that

$$\widetilde{\mathbf{W}}_m^{(\ell)} f^{(\ell)} = \sum_{\underline{\alpha}_\ell \in \mathsf{A}_m^{(\ell)}} u_{\underline{\alpha}_\ell,m} \partial_{\underline{x}_\ell}^{\underline{\alpha}_\ell} f^{(\ell)}, \qquad \forall f^{(\ell)} \in \mathcal{S}(\mathbb{R}^\ell) \tag{4.4.10}$$

$$\widetilde{\mathbf{W}}_{n-m}^{(j)} f^{(j)} = \sum_{\underline{\alpha}_j \in \mathsf{A}_{n-m}^{(j)}} u_{\underline{\alpha}_j,n-m} \partial_{\underline{x}_j}^{\underline{\alpha}_j} f^{(j)}, \qquad \forall f^{(j)} \in \mathcal{S}(\mathbb{R}^j), \tag{4.4.11}$$

where $u_{\underline{\alpha}_\ell,m} \in \mathcal{S}'(\mathbb{R}^\ell)$ and $u_{\underline{\alpha}_j,n-m} \in \mathcal{S}'(\mathbb{R}^j)$. Define the set

$$\mathsf{A}_{n,m}^{(k)} \coloneqq \mathsf{A}_m^{(\ell)} \times \mathsf{A}_{n-m}^{(j)} \subseteq \mathbb{N}_0^\ell \times \mathbb{N}_0^j \tag{4.4.12}$$

so that

$$\Big(\widetilde{\mathbf{W}}_m^{(\ell)} \otimes \widetilde{\mathbf{W}}_{n-m}^{(j)}\Big) f^{(k)} = \sum_{(\underline{\alpha}_\ell, \underline{\alpha}_j) \in \mathsf{A}_{n,m}^{(k)}} \Big( u_{\underline{\alpha}_\ell,m} \otimes u_{\underline{\alpha}_j,n-m} \Big) \Big( \partial_{\underline{x}_\ell}^{\underline{\alpha}_\ell} \otimes \partial_{\underline{x}_j}^{\underline{\alpha}_j} \Big) f^{(k)}, \qquad \forall f^{(k)} \in \mathcal{S}(\mathbb{R}^k). \tag{4.4.13}$$

Hence, to prove the claim, it suffices to show that

$$\delta(X_1 - X_{\ell+1}) \Big( \widetilde{\mathbf{W}}_m^{(\ell)} \otimes \widetilde{\mathbf{W}}_{n-m}^{(j)} \Big) \tag{4.4.14}$$

is well-defined in $\mathcal{L}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k))$, and that for all $f^{(k)} \in \mathcal{S}(\mathbb{R}^k)$, (4.4.14) admits the representation

$$\Big(\delta(X_1 - X_{\ell+1})\Big(\widetilde{\mathbf{W}}_m^{(\ell)} \otimes \widetilde{\mathbf{W}}_{n-m}^{(j)}\Big)\Big) f^{(k)} = \sum_{(\underline{\alpha}_\ell, \underline{\alpha}_j) \in \mathsf{A}_{n,m}^{(k)}} \delta(x_1 - x_{\ell+1}) \Big(u_{\underline{\alpha}_\ell, m} \otimes u_{\underline{\alpha}_j, n-m}\Big) \Big(\partial_{\underline{x}_\ell}^{\underline{\alpha}_\ell} \otimes \partial_{\underline{x}_j}^{\underline{\alpha}_j}\Big) f^{(k)}, \tag{4.4.15}$$

where $\delta(x_1 - x_{\ell+1})(u_{\underline{\alpha}_\ell, m} \otimes u_{\underline{\alpha}_j, n-m})$ is well-defined in $\mathcal{S}'(\mathbb{R}^k)$. We will do this in two steps:

- First, we will show that (4.4.14) admits the representation (4.4.15) for all $f^{(k)} \in \mathcal{S}(\mathbb{R}^k)$, and that $\delta(x_1 - x_{\ell+1})(u_{\underline{\alpha}_\ell, m} \otimes u_{\underline{\alpha}_j, n-m}) \in \mathcal{D}'(\mathbb{R}^k)$ in the Hörmander product sense of Proposition 4.0.14.
- Second, we will show that the products are, in fact, tempered distributions.

To show that the product of distributions

$$\delta(x_1 - x_{\ell+1})\Big(\widetilde{\mathbf{W}}_m^{(\ell)} \otimes \widetilde{\mathbf{W}}_{n-m}^{(j)}\Big)(f^{(k)}) \tag{4.4.16}$$

is well-defined in $\mathcal{D}'(\mathbb{R}^k)$ for every $f^{(k)} \in \mathcal{S}(\mathbb{R}^k)$, it suffices by Hörmander's criterion (Proposition 4.0.14) to show that

$$(\underline{x}_k, \underline{\xi}_k) \in \mathrm{WF}(\delta(x_1 - x_{\ell+1})) \Longrightarrow (\underline{x}_k, -\underline{\xi}_k) \notin \mathrm{WF}\Big(\Big(\widetilde{\mathbf{W}}_m^{(\ell)} \otimes \widetilde{\mathbf{W}}_{n-m}^{(j)}\Big) f^{(k)}\Big). \tag{4.4.17}$$

By Lemma 4.0.10, which computes the wave front set of $\delta(x_1 - x_{\ell+1})$, we need to show that if $\xi_1 \neq 0$, then

$$((x_1, \underline{x}_{2;\ell}, x_1, \underline{x}_{\ell+2;k}), (\xi_1, \underline{0}_{2;\ell}, -\xi_1, \underline{0}_{\ell+2;k})) \notin \mathrm{WF}\Big(\Big(\widetilde{\mathbf{W}}_m^{(\ell)} \otimes \widetilde{\mathbf{W}}_{n-m}^{(j)}\Big) f^{(k)}\Big). \tag{4.4.18}$$

Since for any $(\underline{\alpha}_\ell, \underline{\alpha}_j) \in \mathsf{A}_{n,m}^{(k)}$ and for any $g^{(k)} \in \mathcal{S}(\mathbb{R}^k)$, we have the inclusion

$$\mathrm{WF}\Big(\Big(u_{\underline{\alpha}_\ell,m} \otimes u_{\underline{\alpha}_j,n-m}\Big)g^{(k)}\Big) \subset \mathrm{WF}\Big(u_{\underline{\alpha}_\ell,m} \otimes u_{\underline{\alpha}_j,n-m}\Big), \tag{4.4.19}$$

by Proposition 4.0.9(f), it follows from Proposition 4.0.9(c) and (4.4.13) that

$$\mathrm{WF}\Big(\Big(\widetilde{\mathbf{W}}_m^{(\ell)} \otimes \widetilde{\mathbf{W}}_{n-m}^{(j)}\Big)f^{(k)}\Big) \subset \bigcup_{(\underline{\alpha}_\ell,\underline{\alpha}_j)\in \mathsf{A}_{n,m}^{(k)}} \mathrm{WF}\Big(u_{\underline{\alpha}_\ell,m} \otimes u_{\underline{\alpha}_j,n-m}\Big), \qquad \forall f^{(k)} \in \mathcal{S}(\mathbb{R}^k). \tag{4.4.20}$$

Now by Proposition 4.0.9(e), we have that

$$\begin{aligned}\mathrm{WF}\Big(u_{\underline{\alpha}_\ell,m} \otimes u_{\underline{\alpha}_j,n-m}\Big) \subset{} & \Big(\mathrm{WF}\big(u_{\underline{\alpha}_\ell,m}\big) \times \mathrm{WF}\Big(u_{\underline{\alpha}_j,n-m}\Big)\Big) \\ & \cup \big(\mathrm{supp}\big(u_{\underline{\alpha}_\ell,m}\big) \times \{\underline{0}_\ell\}\big) \times \mathrm{WF}\Big(u_{\underline{\alpha}_j,n-m}\Big) \\ & \cup \mathrm{WF}\big(u_{\underline{\alpha}_\ell,m}\big) \times \Big(\mathrm{supp}\Big(u_{\underline{\alpha}_j,n-m}\Big) \times \{\underline{0}_j\}\Big).\end{aligned} \tag{4.4.21}$$

Note that we abuse notation with the cartesian products on the right-hand side of the preceding inclusion in the following sense: we denote an element of $\mathrm{WF}(u_{\underline{\alpha}_\ell,m}) \times \mathrm{WF}(u_{\underline{\alpha}_j,n-m})$ by

$$(\underline{x}_\ell, \underline{x}_{\ell+1;k}, \underline{\xi}_\ell, \underline{\xi}_{\ell+1;k}), \tag{4.4.22}$$

where

$$(\underline{x}_\ell, \underline{\xi}_\ell) \in \mathrm{WF}(u_{\underline{\alpha}_\ell,m}), \quad (\underline{x}_{\ell+1;k}, \underline{\xi}_{\ell+1;k}) \in \mathrm{WF}(u_{\underline{\alpha}_j,n-m})$$

and similarly for elements of $(\mathrm{supp}(u_{\underline{\alpha}_\ell,m})\times\{\underline{0}_\ell\})\times\mathrm{WF}(u_{\underline{\alpha}_j,n-m})$ and $\mathrm{WF}(u_{\underline{\alpha}_\ell,m})\times(\mathrm{supp}(u_{\underline{\alpha}_j,n-m})\times\{\underline{0}_j\})$. We now consider three cases based on the values of the sets $\mathrm{WF}(u_{\underline{\alpha}_\ell,m})$ and $\mathrm{WF}(u_{\underline{\alpha}_j,n-m})$.

(i) Suppose that $\mathrm{WF}\big(u_{\underline{\alpha}_\ell,m}\big)$ and $\mathrm{WF}\Big(u_{\underline{\alpha}_j,n-m}\Big)$ are both empty. Then it follows readily from (4.4.21) that

$$\mathrm{WF}\Big(u_{\underline{\alpha}_\ell,m} \otimes u_{\underline{\alpha}_j,n-m}\Big) = \emptyset, \tag{4.4.23}$$

and so (4.4.18) is satisfied.

(ii) Without loss of generality, suppose that $\mathrm{WF}\Big(u_{\underline{\alpha}_j,n-m}\Big) = \emptyset$ and that $\mathrm{WF}\big(u_{\underline{\alpha}_\ell,m}\big) \neq \emptyset$ and satisfies the non-vanishing pair property. Then by (4.4.21), we have

$$\mathrm{WF}\Big(u_{\underline{\alpha}_\ell,m} \otimes u_{\underline{\alpha}_j,n-m}\Big) \subset \mathrm{WF}\big(u_{\underline{\alpha}_\ell,m}\big) \times \Big(\mathrm{supp}\Big(u_{\underline{\alpha}_j,n-m}\Big) \times \{\underline{0}_j\}\Big). \tag{4.4.24}$$

Observe that the set on the right-hand side does not contain an element of the form

$$((x_1, \underline{x}_{2;\ell}, x_1, \underline{x}_{\ell+2;k}), (\xi_1, \underline{0}_{2;\ell}, -\xi_1, \underline{0}_{\ell+2;k})), \qquad \xi_1 \neq 0. \tag{4.4.25}$$

since $\mathrm{WF}(u_{\underline{\alpha}_\ell,m})$ is nonempty and satisfies the non-vanishing pair property.

(iii) Suppose that both $\mathrm{WF}\big(u_{\underline{\alpha}_\ell,m}\big)$ and $\mathrm{WF}\Big(u_{\underline{\alpha}_j,n-m}\Big)$ are both nonempty and satisfy the non-vanishing pair property. Then if $(\underline{x}_k, \underline{\xi}_k) \in \mathrm{WF}\Big(u_{\underline{\alpha}_\ell,m} \otimes u_{\underline{\alpha}_j,n-m}\Big)$, one of three sub-cases must occur:

1. $\underline{\xi}_\ell = 0$ and there exists $l_1, l_2 \in \{\ell+1, \ldots, \ell+j\}$ such that $\xi_{l_1} \neq 0$ and $\xi_{l_2} \neq 0$.
2. $\underline{\xi}_{\ell+1;k} = 0$ and there exists $l_1, l_2 \in \{1, \ldots, \ell\}$ such that $\xi_{l_1} \neq 0$ and $\xi_{l_2} \neq 0$.
3. $\underline{\xi}_\ell \neq 0$, $\underline{\xi}_{\ell+1;k} \neq 0$, and there exist $l_1, l_2 \in \{1, \ldots, \ell\}$ and $l_3, l_4 \in \{\ell+1, \ldots, k\}$ such that $\xi_{l_1} \neq 0$, $\xi_{l_2} \neq 0$, $\xi_{l_3} \neq 0$ and $\xi_{l_4} \neq 0$.

Any of these three sub-cases guarantees (4.4.18).

To summarize, we have shown that

$$((x_1, \underline{x}_{2;\ell}, x_1, \underline{x}_{\ell+2;k}), (\xi_1, \underline{0}_{2;\ell}, -\xi_1, \underline{0}_{\ell+2;k})) \notin \bigcup_{(\underline{\alpha}_\ell,\underline{\alpha}_j) \in \mathsf{A}_{n,m}^{(k)}} \mathrm{WF}\Big(u_{\underline{\alpha}_\ell,m} \otimes u_{\underline{\alpha}_j,n-m}\Big), \tag{4.4.26}$$

and therefore

$$\delta(x_1 - x_2)\Big(\widetilde{\mathbf{W}}_m^{(\ell)} \otimes \widetilde{\mathbf{W}}_{n-m}^{(j)}\Big)(f^{(k)}) \tag{4.4.27}$$

is defined in $\mathcal{D}'(\mathbb{R}^k)$ according to Proposition 4.0.14, proving the first claim.

We now show that this Hörmander product is tempered:

$$\delta(x_1 - x_{\ell+1})\Big(\widetilde{\mathbf{W}}_m^{(\ell)} \otimes \widetilde{\mathbf{W}}_{n-m}^{(j)}\Big)(f^{(k)}) \in \mathcal{S}'(\mathbb{R}^k), \qquad \forall f^{(k)} \in \mathcal{S}'(\mathbb{R}^k). \tag{4.4.28}$$

Since by the inductive hypothesis, $\widetilde{\mathbf{W}}_m^{(\ell)}$ and $\widetilde{\mathbf{W}}_{n-m}^{(j)}$ satisfy the good mapping property of Definition 4.1.1 (and we refer to Appendix 3.3 for more details on the good mapping property), there exist unique continuous bilinear maps

$$\Phi_{\widetilde{\mathbf{W}}_m^{(\ell)},\alpha} : \mathcal{S}(\mathbb{R}^\ell)^2 \to \mathcal{S}_{(x_\alpha,x'_\alpha)}(\mathbb{R}^2), \;\; \Phi_{\widetilde{\mathbf{W}}_{n-m}^{(j)},\beta} : \mathcal{S}(\mathbb{R}^j)^2 \to \mathcal{S}_{(x_\beta,x'_\beta)}(\mathbb{R}^2), \qquad \alpha \in \mathbb{N}_{\leq \ell}, \;\; \beta \in \mathbb{N}_{\leq j} \tag{4.4.29}$$

identifiable with the maps

$$\begin{aligned}
&\mathcal{S}(\mathbb{R}^\ell)^2 \to \mathcal{S}_{x'_\alpha}(\mathbb{R};\mathcal{S}'_{x_\alpha}(\mathbb{R})), \quad (f^{(\ell)}, g^{(\ell)}) \mapsto \Big\langle \widetilde{\mathbf{W}}_m^{(\ell)} f^{(\ell)}, (\cdot) \otimes_\alpha g^{(\ell)}(\cdot, x'_\alpha, \cdot)\Big\rangle_{\mathcal{S}'(\mathbb{R}^\ell)-\mathcal{S}(\mathbb{R}^\ell)},\\
&\mathcal{S}(\mathbb{R}^j)^2 \to \mathcal{S}_{x'_\beta}(\mathbb{R};\mathcal{S}'_{x_\beta}(\mathbb{R})), \quad (f^{(j)}, g^{(j)}) \mapsto \Big\langle \widetilde{\mathbf{W}}_m^{(j)} f^{(j)}, (\cdot) \otimes_\beta g^{(j)}(\cdot, x'_\beta, \cdot)\Big\rangle_{\mathcal{S}'(\mathbb{R}^j)-\mathcal{S}(\mathbb{R}^j)},
\end{aligned} \tag{4.4.30}$$

via

$$\begin{aligned}
&\int_{\mathbb{R}} dx_\alpha \Phi_{\widetilde{\mathbf{W}}_m^{(\ell)},\alpha}(f^{(\ell)}, g^{(\ell)})(x_\alpha; x'_\alpha)\phi(x_\alpha) = \Big\langle \widetilde{\mathbf{W}}_n^{(\ell)} f^{(\ell)}, \phi \otimes_\alpha g^{(\ell)}(\cdot, x'_\alpha, \cdot)\Big\rangle_{\mathcal{S}'(\mathbb{R}^\ell)-\mathcal{S}(\mathbb{R}^\ell)},\\
&\int_{\mathbb{R}} dx_\beta \Phi_{\widetilde{\mathbf{W}}_{n-m}^{(j)},\beta}(f^{(j)}, g^{(j)})(x_\beta; x'_\beta)\phi(x_\beta) = \Big\langle \widetilde{\mathbf{W}}_{n-m}^{(j)} f^{(j)}, \phi \otimes_\beta g^{(j)}(\cdot, x'_\beta, \cdot)\Big\rangle_{\mathcal{S}'(\mathbb{R}^j)-\mathcal{S}(\mathbb{R}^j)},
\end{aligned} \tag{4.4.31}$$

for $\phi \in \mathcal{S}(\mathbb{R})$, respectively. Above, the notation $(\cdot) \otimes_\alpha g^{(\ell)}(\cdot, x'_\alpha, \cdot)$ and $(\cdot) \otimes_\beta g^{(j)}(\cdot, x'_\beta, \cdot)$ is defined by

$$\left.\begin{aligned}
&\big(\phi \otimes_\alpha g^{(\ell)}(\cdot, x'_\alpha, \cdot)\big)(\underline{y}_\alpha) \coloneqq \phi(y_\alpha) g^{(\ell)}(\underline{y}_{1;\alpha-1}, x'_\alpha, \underline{y}_{\alpha+1;\ell}), \qquad \forall \underline{y}_\ell \in \mathbb{R}^\ell\\
&\big(\phi \otimes_\beta g^{(j)}(\cdot, x'_\beta, \cdot)\big)(\underline{y}_\beta) \coloneqq \phi(y_\beta) g^{(j)}(\underline{y}_{1;\beta-1}, x'_\beta, \underline{y}_{\beta+1;j}), \qquad \forall \underline{y}_j \in \mathbb{R}^j
\end{aligned}\right., \qquad \forall \phi \in \mathcal{S}(\mathbb{R}). \tag{4.4.32}$$

Now given $f^{(k)}, g^{(k)} \in \mathcal{S}(\mathbb{R}^k)$, we see that

$$(\underline{x}_\ell, \underline{x}'_\ell) \mapsto \Phi_{\widetilde{\mathbf{W}}^{(j)}_{n-m},1}(f^{(k)}(\underline{x}_\ell, \cdot), g^{(k)}(\underline{x}'_\ell, \cdot)) \in \mathcal{S}_{(\underline{x}_\ell, \underline{x}'_\ell)}(\mathbb{R}^{2\ell}; \mathcal{S}_{(y_1,y'_1)}(\mathbb{R}^2)). \tag{4.4.33}$$

Thus, we can define a map $\Psi_{\widetilde{\mathbf{W}}^{(j)}_{n-m},1} : \mathcal{S}(\mathbb{R}^k)^2 \to \mathcal{S}(\mathbb{R}^{2(\ell+1)})$

$$\begin{aligned}
&\Psi_{\widetilde{\mathbf{W}}^{(j)}_{n-m},1}(f^{(k)}, g^{(k)})(\underline{x}_{\ell+1}; \underline{x}'_{\ell+1}) \\
&\coloneqq \Phi_{\widetilde{\mathbf{W}}^{(j)}_{n-m},1}(f^{(k)}(\underline{x}_\ell, \cdot), g^{(k)}(\underline{x}'_\ell, \cdot))(x_{\ell+1}; x'_{\ell+1}), \qquad \forall (\underline{x}_{\ell+1}, \underline{x}'_{\ell+1}) \in \mathbb{R}^{2(\ell+1)},
\end{aligned} \tag{4.4.34}$$

which is bilinear and continuous. Now since $\Phi_{\widetilde{\mathbf{W}}^{(\ell)}_{m},1} : \mathcal{S}(\mathbb{R}^\ell)^2 \to \mathcal{S}(\mathbb{R}^2)$ is bilinear and continuous, the universal property of the tensor product and the identification of $\mathcal{S}(\mathbb{R}^{2\ell}) \cong \mathcal{S}(\mathbb{R}^\ell)\hat{\otimes}\mathcal{S}(\mathbb{R}^\ell)$ implies that there exists a unique continuous linear map

$$\bar{\Phi}_{\widetilde{\mathbf{W}}^{(\ell)}_{m},1} : \mathcal{S}(\mathbb{R}^{2\ell}) \to \mathcal{S}(\mathbb{R}^2), \tag{4.4.35}$$

with the property that

$$\Phi_{\widetilde{\mathbf{W}}^{(\ell)}_{m},1}(f^{(\ell)}, g^{(\ell)}) = \bar{\Phi}_{\widetilde{\mathbf{W}}^{(\ell)}_{m},1}(f^{(\ell)} \otimes g^{(\ell)}), \qquad \forall f^{(\ell)}, g^{(\ell)} \in \mathcal{S}(\mathbb{R}^\ell). \tag{4.4.36}$$

Hence, the function

$$\bar{\Phi}_{\widetilde{\mathbf{W}}^{(\ell)}_{m,1},1}\Big(\Psi_{\widetilde{\mathbf{W}}^{(j)}_{n-m},1}(f^{(k)}, g^{(k)})(\cdot, x_{\ell+1}; \cdot, x'_{\ell+1})\Big)(x_1; x'_1), \qquad \forall (x_1, x_{\ell+1}, x'_1, x'_{\ell+1}) \in \mathbb{R}^4$$

defines an element of $\mathcal{S}(\mathbb{R}^4)$, and moreover,

$$\begin{aligned}
&\mathcal{S}(\mathbb{R}^k)^2 \to \mathcal{S}(\mathbb{R}^4), \\
&(f^{(k)}, g^{(k)}) \mapsto \bar{\Phi}_{\widetilde{\mathbf{W}}^{(\ell)}_{m,1},1}\Big(\Psi_{\widetilde{\mathbf{W}}^{(j)}_{n-m},1}(f^{(k)}, g^{(k)})(\cdot, x_{\ell+1}; \cdot, x'_{\ell+1})\Big)(x_1; x'_1), \qquad \forall (x_1, x_{\ell+1}, x'_1, x'_{\ell+1}) \in \mathbb{R}^4
\end{aligned} \tag{4.4.37}$$

is a continuous bilinear map. Thus, we may define a functional $u_{f^{(k)}}$ on $\mathcal{S}(\mathbb{R}^k)$ by

$$\begin{aligned}
&\langle u_{f^{(k)}}, g^{(k)} \rangle_{\mathcal{S}'(\mathbb{R}^k)-\mathcal{S}(\mathbb{R}^k)} \\
&\coloneqq \int_{\mathbb{R}^2} dx_1 dx_{\ell+1} \delta(x_1 - x_{\ell+1}) \bar{\Phi}_{\widetilde{\mathbf{W}}^{(\ell)}_{m},1}\Big(\Psi_{\widetilde{\mathbf{W}}^{(j)}_{n-m},1}(f^{(k)}, g^{(k)})(\cdot, x_{\ell+1}; \cdot, x_{\ell+1})\Big)(x_1; x_1), \qquad \forall g^{(k)} \in \mathcal{S}(\mathbb{R}^k).
\end{aligned} \tag{4.4.38}$$

This functional $u_{f^{(k)}}$ is evidently linear, and it follows from the continuity of $\bar{\Phi}_{\widetilde{\mathbf{W}}_m^{(\ell)},1}$ and $\Psi_{\widetilde{\mathbf{W}}_{n-m}^{(j)},1}$ that it is continuous $\mathcal{S}(\mathbb{R}^k) \to \mathbb{C}$, hence a tempered distribution. Furthermore, we claim that the map

$$\mathcal{S}(\mathbb{R}^k) \to \mathcal{S}'(\mathbb{R}^k), \qquad f^{(k)} \mapsto \langle u_{f^{(k)}}, \cdot \rangle_{\mathcal{S}'(\mathbb{R}^k)-\mathcal{S}(\mathbb{R}^k)} \tag{4.4.39}$$

satisfies the good mapping property. Indeed, replacing $f^{(k)}, g^{(k)}$ with $\pi f^{(k)}, \pi g^{(k)}$, for any $\pi \in \mathbb{S}_k$, it suffices to verify this assertion for the case $\alpha = 1$ in Definition 4.1.1. Additionally, it suffices by the universal property of the tensor product and the Schwartz kernel theorem isomorphism $\mathcal{S}(\mathbb{R}^k) \cong \mathcal{S}(\mathbb{R}^\ell) \hat{\otimes} \mathcal{S}(\mathbb{R}^j)$ to show that there is a (necessarily unique) continuous, multilinear map

$$\Phi_u : \left(\mathcal{S}(\mathbb{R}^\ell) \times \mathcal{S}(\mathbb{R}^j)\right)^2 \to \mathcal{S}(\mathbb{R}^2),$$

such that for $f^{(\ell)}, g^{(\ell)} \in \mathcal{S}(\mathbb{R}^\ell)$ and $f^{(j)}, g^{(j)} \in \mathcal{S}(\mathbb{R}^j)$,

$$\begin{aligned} &\int_{\mathbb{R}} dx \Phi_u(f^{(\ell)}, f^{(j)}, g^{(\ell)}, g^{(j)})(x;x')\phi(x) \\ &= \langle u_{f^{(\ell)} \otimes f^{(j)}}, \phi \otimes (g^{(\ell)} \otimes g^{(j)})(x', \cdot) \rangle_{\mathcal{S}'(\mathbb{R}^k)-\mathcal{S}(\mathbb{R}^k)}, \qquad \forall \phi \in \mathcal{S}(\mathbb{R}),\ x' \in \mathbb{R}. \end{aligned} \tag{4.4.40}$$

Now for any $\phi \in \mathcal{S}(\mathbb{R})$, the bilinearity of $\Phi_{\widetilde{\mathbf{W}}_{n-m}^{(j)},1}$ implies

$$\begin{aligned} &\Phi_{\widetilde{\mathbf{W}}_{n-m}^{(j)},1}\big((f^{(\ell)} \otimes f^{(j)})(\underline{x}_\ell, \cdot), (\phi \otimes (g^{(\ell)} \otimes g^{(j)})(x', \cdot))(\underline{x}'_\ell, \cdot)\big)(x_{\ell+1}; x'_{\ell+1}) \\ &= f^{(\ell)}(\underline{x}_\ell)\phi(x'_1)g^{(\ell)}(x', \underline{x}'_{2;\ell})\Phi_{\widetilde{\mathbf{W}}_{n-m}^{(j)},1}\big(f^{(j)}, g^{(j)}\big)(x_{\ell+1}; x'_{\ell+1}), \qquad \forall (\underline{x}_{\ell+1}, \underline{x}'_{\ell+1}, x') \in \mathbb{R}^{2\ell+3}. \end{aligned} \tag{4.4.41}$$

Hence,

$$\begin{aligned} &\Psi_{\widetilde{\mathbf{W}}_{n-m}^{(j)},1}\big(f^{(\ell)} \otimes f^{(j)}, \phi \otimes (g^{(\ell)} \otimes g^{(j)})(x', \cdot)\big)(\underline{x}_{\ell+1}; \underline{x}'_{\ell+1}) \\ &= f^{(\ell)}(\underline{x}_\ell)\phi(x'_1)g^{(\ell)}(x', \underline{x}'_{2;\ell})\Phi_{\widetilde{\mathbf{W}}_{n-m}^{(j)},1}(f^{(j)}, g^{(j)})(x_{\ell+1}; x'_{\ell+1}), \qquad \forall (\underline{x}_{\ell+1}, \underline{x}'_{\ell+1}) \in \mathbb{R}^{2(\ell+1)}. \end{aligned} \tag{4.4.42}$$

For $x' \in \mathbb{R}$ and $\phi \in \mathcal{S}(\mathbb{R})$, define the function $\tilde{g}^{(\ell)}_{x',\phi} \in \mathcal{S}(\mathbb{R}^\ell)$ by

$$\tilde{g}^{(\ell)}_{x',\phi}(\underline{x}'_\ell) := \phi(x'_1)g^{(\ell)}(x', \underline{x}'_{2;\ell}), \qquad \forall \underline{x}'_\ell \in \mathbb{R}^\ell, \tag{4.4.43}$$

so that we can write

$$\begin{aligned} &\Psi_{\widetilde{\mathbf{W}}^{(j)}_{n-m},1}\big(f^{(\ell)} \otimes f^{(j)}, \phi \otimes (g^{(\ell)} \otimes g^{(j)})(x', \cdot)\big)(\underline{x}_{\ell+1}; \underline{x}'_{\ell+1}) \\ &= (f^{(\ell)} \otimes \tilde{g}^{(\ell)}_{x',\phi})(\underline{x}_\ell; \underline{x}'_\ell)\Phi_{\widetilde{\mathbf{W}}^{(j)}_{n-m},1}(f^{(j)}, g^{(j)})(x_{\ell+1}; x'_{\ell+1}), \qquad \forall (\underline{x}_{\ell+1}, \underline{x}'_{\ell+1}) \in \mathbb{R}^{2(\ell+1)}. \end{aligned} \tag{4.4.44}$$

Therefore, using identity (4.4.44) and the linearity of the map $\bar{\Phi}_{\widetilde{\mathbf{W}}^{(\ell)}_m,1}$, we see that

$$\begin{aligned} &\bar{\Phi}_{\widetilde{\mathbf{W}}^{(\ell)}_m,1}\Big(\Psi_{\widetilde{\mathbf{W}}^{(j)}_{n-m},1}\big(f^{(\ell)} \otimes f^{(j)}, \phi \otimes (g^{(\ell)} \otimes g^{(j)})(x', \cdot)\big)(\cdot, x_{\ell+1}; \cdot, x'_{\ell+1})\Big)(x_1; x'_1) \\ &= \Phi_{\widetilde{\mathbf{W}}^{(j)}_{n-m},1}(f^{(j)}, g^{(j)})(x_{\ell+1}; x'_{\ell+1})\bar{\Phi}_{\widetilde{\mathbf{W}}^{(\ell)}_m,1}\Big(f^{(\ell)} \otimes \tilde{g}^{(\ell)}_{x',\phi}\Big)(x_1; x'_1) \\ &= \Phi_{\widetilde{\mathbf{W}}^{(j)}_{n-m},1}(f^{(j)}, g^{(j)})(x_{\ell+1}; x'_{\ell+1})\Phi_{\widetilde{\mathbf{W}}^{(\ell)}_m,1}(f^{(\ell)}, \tilde{g}^{(\ell)}_{x',\phi})(x_1; x'_1), \end{aligned} \tag{4.4.45}$$

where the ultimate equality follows from the property (4.4.36). Recalling the definition (4.4.38) for $u_{f^{(k)}}$, we obtain that

$$\begin{aligned} &\big\langle u_{f^{(\ell)} \otimes f^{(j)}}, \phi \otimes (g^{(\ell)} \otimes g^{(j)})(x', \cdot)\big\rangle_{\mathcal{S}'(\mathbb{R}^k)-\mathcal{S}(\mathbb{R}^k)} \\ &= \int_{\mathbb{R}^2} dx_1 dx_{\ell+1}\delta(x_1 - x_{\ell+1})\bar{\Phi}_{\widetilde{\mathbf{W}}^{(\ell)}_m,1}\Big(\Psi_{\widetilde{\mathbf{W}}^{(j)}_{n-m},1}\big(f^{(\ell)} \otimes f^{(j)}, \phi \otimes (g^{(\ell)} \otimes g^{(j)})(x', \cdot)\big)(\cdot, x_{\ell+1}; \cdot, x_{\ell+1})\Big)(x_1; x_1) \\ &= \int_{\mathbb{R}^2} dx_1 dx_{\ell+1}\delta(x_1 - x_{\ell+1})\Phi_{\widetilde{\mathbf{W}}^{(j)}_{n-m},1}(f^{(j)}, g^{(j)})(x_{\ell+1}; x_{\ell+1})\Phi_{\widetilde{\mathbf{W}}^{(\ell)}_m,1}(f^{(\ell)}, \tilde{g}^{(\ell)}_{x',\phi})(x_1; x_1) \\ &= \int_{\mathbb{R}} dx \Phi_{\widetilde{\mathbf{W}}^{(j)}_{n-m},1}(f^{(j)}, g^{(j)})(x; x)\Phi_{\widetilde{\mathbf{W}}^{(\ell)}_m,1}(f^{(\ell)}, \tilde{g}^{(\ell)}_{x',\phi})(x; x) \\ &= \Big\langle \widetilde{\mathbf{W}}^{(\ell)}_m f^{(\ell)}, \Phi_{\widetilde{\mathbf{W}}^{(j)}_{n-m},1}(f^{(j)}, g^{(j)})|_{y=y'}\tilde{g}^{(\ell)}_{x',\phi}\Big\rangle_{\mathcal{S}'(\mathbb{R}^\ell)-\mathcal{S}(\mathbb{R}^\ell)}, \end{aligned}$$

where $\Phi_{\widetilde{\mathbf{W}}^{(j)}_{n-m},1}(f^{(j)}, g^{(j)})|_{y=y'}$ denotes the restriction to the hyperplane $\{(y, y') : y = y'\} \subset \mathbb{R}^2$ and the ultimate equality follows from the definition of $\Phi_{\widetilde{\mathbf{W}}^{(\ell)}_m,1}$ in (4.4.29). Unpacking the definition of $\tilde{g}^{(\ell)}_{x',\phi}$ from (4.4.43) and applying the definition of $\Phi_{\widetilde{\mathbf{W}}^{(\ell)}_m,1}$ once more, we conclude that

$$\begin{aligned}
&\left\langle \widetilde{\mathbf{W}}^{(\ell)}_m f^{(\ell)}, \Phi_{\widetilde{\mathbf{W}}^{(j)}_{n-m},1}(f^{(j)}, g^{(j)})|_{y=y'} \tilde{g}^{(\ell)}_{x',\phi} \right\rangle_{\mathcal{S}'(\mathbb{R}^\ell)-\mathcal{S}(\mathbb{R}^\ell)} \\
&= \left\langle \widetilde{\mathbf{W}}^{(\ell)}_m f^{(\ell)}, (\phi \Phi_{\widetilde{\mathbf{W}}^{(j)}_{n-m},1}(f^{(j)}, g^{(j)})|_{y=y'}) \otimes g^{(\ell)}(x', \cdot) \right\rangle_{\mathcal{S}'(\mathbb{R}^\ell)-\mathcal{S}(\mathbb{R}^\ell)} \\
&= \int_{\mathbb{R}} dx \Phi_{\widetilde{\mathbf{W}}^{(\ell)}_m,1}(f^{(\ell)}, g^{(\ell)})(x; x') \phi(x) \Phi_{\widetilde{\mathbf{W}}^{(j)}_{n-m},1}(f^{(j)}, g^{(j)})(x; x). \qquad (4.4.46)
\end{aligned}$$

Therefore, the desired map $\Phi_u$ is given by

$$\Phi_u(f^{(\ell)}, f^{(j)}, g^{(\ell)}, g^{(j)})(x; x') \coloneqq \Phi_{\widetilde{\mathbf{W}}^{(\ell)}_m,1}(f^{(\ell)}, g^{(\ell)})(x; x') \Phi_{\widetilde{\mathbf{W}}^{(j)}_{n-m},1}(f^{(j)}, g^{(j)})(x; x), \qquad (4.4.47)$$

which is evidently multilinear and continuous $(\mathcal{S}(\mathbb{R}^\ell) \times \mathcal{S}(\mathbb{R}^j))^2 \to \mathcal{S}(\mathbb{R}^2)$ being the composition maps. Thus, the proof that $f^{(k)} \mapsto u_{f^{(k)}}$ has the good mapping property is complete.

Lastly, we claim that $u_{f^{(k)}}$ coincides with the Hörmander product

$$\delta(x_1 - x_{\ell+1}) \Big( \widetilde{\mathbf{W}}^{(\ell)}_m \otimes \widetilde{\mathbf{W}}^{(j)}_{n-m} \Big)(f^{(k)})$$

defined above via Proposition 4.0.14. To prove the claim, we rely on the uniqueness criterion for the product. We set

$$g^{(k)} \coloneqq g^{(1)} \otimes g^{(\ell-1)} \otimes \tilde{g}^{(1)} \otimes g^{(j-1)}, \quad \phi^{(k)} \coloneqq \phi^{(1)} \otimes \phi^{(\ell-1)} \otimes \tilde{\phi}^{(1)} \otimes \phi^{(j-1)} \qquad (4.4.48)$$

for $g^{(1)}, \tilde{g}^{(1)}, \phi^{(1)}, \tilde{\phi}^{(1)} \in \mathcal{S}(\mathbb{R})$, $g^{(\ell-1)}, \phi^{(\ell-1)} \in \mathcal{S}(\mathbb{R}^{i-1})$, and $g^{(j-1)}, \phi^{(j-1)} \in \mathcal{S}(\mathbb{R}^{j-1})$. By density of linear combinations of tensor products, it suffices to show that

$$\langle \mathcal{F}({g^{(k)}}^2 u_{f^{(k)}}), \phi^{(k)} \rangle_{\mathcal{S}'(\mathbb{R}^k)-\mathcal{S}(\mathbb{R}^k)} = \langle \mathcal{F}(g^{(k)} \delta(x_1 - x_{\ell+1})) * \mathcal{F}(g^{(k)} (\widetilde{\mathbf{W}}^{(\ell)}_m \otimes \widetilde{\mathbf{W}}^{(j)}_{n-m})(f^{(k)})), \phi^{(k)} \rangle_{\mathcal{S}'(\mathbb{R}^k)-\mathcal{S}(\mathbb{R}^k)}, \qquad (4.4.49)$$

since pointwise equality then follows from the localization lemma (see Chapter 2, §2 of [40]) together with the continuity of the Fourier transforms involved. This is then an exercise, the details of which we leave to the reader, relying on the good mapping property and the distributional Plancherel theorem.

**Step II:** The property (R2) is readily established by the arguments in the previous step and the fact that $\mathsf{A}_{n,m}^{(k)}$ defined in (4.4.12) has finite cardinality, it then follows from another application of Proposition 4.0.9(c) that either

$$\mathrm{WF}\Big(\widetilde{\mathbf{W}}_{n+1}^{(k)} f^{(k)}\Big) = \emptyset$$

or

$$\mathrm{WF}\Big(\widetilde{\mathbf{W}}_{n+1}^{(k)} f^{(k)}\Big) \neq \emptyset \text{ and satisfies the non-vanishing pair property.}$$

**Step III:** Next, we show that the map $f^{(k)} \mapsto \widetilde{\mathbf{W}}_{n+1}^{(k)} f^{(k)}$ satisfies the good mapping property for every $k \in \mathbb{N}$. Since differentiation is a continuous endomorphism of $\mathcal{S}'(\mathbb{R}^k)$, it is immediate from the induction hypothesis that

$$-i\partial_{x_1} \widetilde{\mathbf{W}}_n^{(k)} \in \mathcal{L}_{gmp}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k)). \tag{4.4.50}$$

Since $\mathcal{L}_{gmp}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k))$ is a vector space, it remains to show that

$$f^{(k)} \mapsto \delta(x_1 - x_{\ell+1})\Big(\widetilde{\mathbf{W}}_m^{(\ell)} \otimes \widetilde{\mathbf{W}}_{n-m}^{(j)}\Big)(f^{(k)}) \tag{4.4.51}$$

satisfies the good mapping property for every $\ell, j \in \mathbb{N}$ with $\ell + j = k$ and $m \in \mathbb{N}_{\leq n-1}$. But this follows from Step II, where we showed that $u_{f^{(k)}}$ defined in (4.4.38) coincides with the Hörmander product in the right-hand side of (4.4.51) and that the DVO $f^{(k)} \mapsto u_{f^{(k)}}$ defined in (4.4.38) has the good mapping property.

**Step IV:** Finally, we show that

$$\widetilde{\mathbf{W}}_n^{(k)} : \mathcal{S}(\mathbb{R}^k) \to \mathcal{S}'(\mathbb{R}^k)$$

is a continuous map. As argued before, it suffices to show that the map

$$(f^{(\ell)}, f^{(j)}) \mapsto \delta(x_1 - x_{\ell+1})\Big(\widetilde{\mathbf{W}}_m^{(\ell)} \otimes \widetilde{\mathbf{W}}_{n-m}^{(j)}\Big)(f^{(\ell)} \otimes f^{(j)}) \tag{4.4.52}$$

is a continuous bilinear map $\mathcal{S}(\mathbb{R}^\ell) \times \mathcal{S}(\mathbb{R}^j) \to \mathcal{S}'(\mathbb{R}^k)$. Bilinearity is obvious. For continuity, suppose that $(f_r^{(\ell)}, f_r^{(j)}) \to 0 \in \mathcal{S}(\mathbb{R}^\ell) \times \mathcal{S}(\mathbb{R}^j)$ as $r \to \infty$. We need to show that for any bounded subset $\mathfrak{R}$ of $\mathcal{S}(\mathbb{R}^k)$,

$$\lim_{r \to \infty} \sup_{g^{(k)} \in \mathfrak{R}} \left| \langle \delta(x_1 - x_{\ell+1})\Big(\widetilde{\mathbf{W}}_m^{(\ell)} \otimes \widetilde{\mathbf{W}}_{n-m}^{(j)}\Big)(f_r^{(\ell)} \otimes f_r^{(j)}), g^{(k)} \rangle_{\mathcal{S}'(\mathbb{R}^k)-\mathcal{S}(\mathbb{R}^k)} \right| = 0. \tag{4.4.53}$$

But this follows from our analysis proving the good mapping property of the map $f^{(k)} \mapsto u_{f^{(k)}}$ in Step II. $\square$

We now turn to showing that only finitely many components of $\widetilde{\mathbf{W}}_n$ are nonzero for a given $n \in \mathbb{N}$. This property justifies our use of the direct sum notation.

**Lemma 4.4.3.** *For all $n \in \mathbb{N}$, we have*

$$\widetilde{\mathbf{W}}_{2n}^{(k)} = 0 \in \mathcal{L}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k)) \qquad k \in \mathbb{N}_{\geq n+1}, \tag{4.4.54}$$

*and*

$$\widetilde{\mathbf{W}}_{2n+1}^{(k)} = 0 \in \mathcal{L}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k)), \qquad k \in \mathbb{N}_{\geq n+2}. \tag{4.4.55}$$

*Proof.* We prove the lemma by strong induction on $n$. We first establish the base case $n = 1$. It follows from the recursion (4.4.4) that

$$\widetilde{\mathbf{W}}_2 = -i\partial_{x_1}\mathbf{E}_1. \tag{4.4.56}$$

Since $\mathbf{E}_1^{(k)} = 0$ for $k \geq 2$, it follows that $\widetilde{\mathbf{W}}_2^{(k)} = 0$ for $k \geq 2$. To see that $\widetilde{\mathbf{W}}_3^{(k)} = 0$ for $k \geq 3$, observe that

$$(-i\partial_{x_1})\widetilde{\mathbf{W}}_2^{(k)} = 0 \in \mathcal{L}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k)), \tag{4.4.57}$$

since $\widetilde{\mathbf{W}}_2^{(k)} = 0$. If $k \geq 3$ and $\ell, j \in \mathbb{N}$ satisfy $\ell + j = k$, then $\max\{\ell, j\} \geq 2$. Since $\widetilde{\mathbf{W}}_1^{(m)} = 0$ for $m \geq 2$, we obtain that

$$\widetilde{\mathbf{W}}_1^{(\ell)} \otimes \widetilde{\mathbf{W}}_1^{(j)} = 0 \in \mathcal{L}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k)), \tag{4.4.58}$$

which implies that $\delta(X_1 - X_{\ell+1})\Big(\widetilde{\mathbf{W}}_1^{(\ell)} \otimes \widetilde{\mathbf{W}}_1^{(j)}\Big) = 0$.

We now proceed to the inductive step. Let $n \in \mathbb{N}_{\geq 2}$ and suppose that for all integers $m \in \mathbb{N}_{\leq n}$,

$$\widetilde{\mathbf{W}}_{2m}^{(k)} = 0 \in \mathcal{L}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k)), \qquad \forall k \in \mathbb{N}_{\geq m+1} \tag{4.4.59}$$

$$\widetilde{\mathbf{W}}_{2m+1}^{(k)} = 0 \in \mathcal{L}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k)), \qquad \forall k \in \mathbb{N}_{\geq m+2}. \tag{4.4.60}$$

We now need to show that these identities hold with $m = n+1$. We first handle the case of even indices. Specifically, we show that

$$\widetilde{\mathbf{W}}_{2(n+1)}^{(k)} = 0 \in \mathcal{L}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k)), \quad k \in \mathbb{N}_{\geq n+2}.$$

Observe that if $k \geq n+2$, then by the induction hypothesis, $\widetilde{\mathbf{W}}_{2(n+1)-1}^{(k)} = 0$ and therefore

$$-i\partial_{x_1}\widetilde{\mathbf{W}}_{2(n+1)-1}^{(k)} = 0 \in \mathcal{L}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k)). \tag{4.4.61}$$

We now consider the Hörmander product terms

$$\delta(X_1 - X_{\ell+1})\Big(\widetilde{\mathbf{W}}_m^{(\ell)} \otimes \widetilde{\mathbf{W}}_{2n+1-m}^{(j)}\Big), \qquad \ell + j = k \tag{4.4.62}$$

arising in the recursion relation (4.4.4) for $\widetilde{\mathbf{W}}_{2(n+1)}^{(k)}$. By symmetry, it suffices to consider the following case: if $m$ is odd (i.e. $m = 2r+1$ for some $r \in \mathbb{N}_0$) then $2n+1-m$ is even (i.e. $2n+1-m = 2r'$ for some $r' \in \mathbb{N}$), and we can write $n = r + r'$. By the induction hypothesis

$$\widetilde{\mathbf{W}}_m^{(\ell)} = 0, \quad \forall \ell \in \mathbb{N}_{\geq r+2} \tag{4.4.63}$$

$$\widetilde{\mathbf{W}}_{2n+1-m}^{(j)} = 0, \quad \forall j \in \mathbb{N}_{\geq r'+1}. \tag{4.4.64}$$

If $k \geq n+2 = r+r'+2$, then either $\ell \geq r+2$ or $j \geq r'+1$, since if both $\ell \leq r+1$ and $j \leq r'$, then

$$k = \ell + j \leq r + r' + 1. \tag{4.4.65}$$

Thus,

$$\delta(X_1 - X_{\ell+1})\Big(\widetilde{\mathbf{W}}_m^{(\ell)} \otimes \widetilde{\mathbf{W}}_{2n+1-m}^{(j)}\Big) = 0 \in \mathcal{L}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k)), \tag{4.4.66}$$

and so it follows from the recursion relation (4.4.4) that $\widetilde{\mathbf{W}}_{2(n+1)}^{(k)} = 0 \in \mathcal{L}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k))$ for $k \geq n+2$.

We next handle the case of odd indices, namely we show that

$$\widetilde{\mathbf{W}}_{2(n+1)+1}^{(k)} = 0 \in \mathcal{L}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k)), \qquad k \geq n+3. \tag{4.4.67}$$

As before, observe that if $k \geq n+3$, then

$$(-i\partial_{x_1})\widetilde{\mathbf{W}}_{2(n+1)}^{(k)} = 0 \in \mathcal{L}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k)) \tag{4.4.68}$$

by the result of the preceding paragraph. Now consider the Hörmander product terms

$$\delta(X_1 - X_{\ell+1})\Big(\widetilde{\mathbf{W}}_m^{(\ell)} \otimes \widetilde{\mathbf{W}}_{2n+2-m}^{(j)}\Big) \tag{4.4.69}$$

in the recursion relation (4.4.4) for $\widetilde{\mathbf{W}}_{2(n+1)+1}^{(k)}$. We consider two cases:

C1. Suppose $m$ is odd (i.e. $m = 2r+1$ for some $r \in \mathbb{N}_0$). Then $2n+2-m$ is odd (i.e. $2n+2-m = 2r'+1$ for some $r' \in \mathbb{N}_0$), and we can write $2(n+1)+1 = 2(r+r'+1)+1$. If $k \geq (r+r'+1)+2$, then either $\ell \geq r+2$ or $j \geq r'+2$, since if both $\ell \leq r+1$ and $j \leq r'+1$, we have that

$$k = \ell + j \leq (r+r'+1)+1. \tag{4.4.70}$$

Hence applying the induction hypothesis to obtain $\widetilde{\mathbf{W}}_m^{(\ell)} = 0$ or $\widetilde{\mathbf{W}}_{2n+2-m}^{(j)} = 0$, respectively, we conclude that

$$\delta(X_1 - X_{\ell+1})\Big(\widetilde{\mathbf{W}}_m^{(\ell)} \otimes \widetilde{\mathbf{W}}_{2n+2-m}^{(j)}\Big) = 0 \in \mathcal{L}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k)). \tag{4.4.71}$$

C2. Suppose $m$ is even (i.e. $m = 2r$ for some $r \in \mathbb{N}$). Then $2n+2-m$ is even (i.e. $2n+2-m = 2r'$ for some $r' \in \mathbb{N}$), and we can write $2n+2 = 2(r+r')$. Once again, if $k \geq r+r'+1$, then either $\ell \geq r+1$ or $j \geq r'+1$, since if $\ell \leq r$ and $j \leq r'$, then

$$k = \ell + j \leq r + r'. \tag{4.4.72}$$

Hence, we obtain again that

$$\delta(X_1 - X_{\ell+1})\Big(\widetilde{\mathbf{W}}_m^{(\ell)} \otimes \widetilde{\mathbf{W}}_{2n+2-m}^{(j)}\Big) = 0 \in \mathcal{L}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k)). \tag{4.4.73}$$

by the induction hypothesis.

In now follows from the recursion relation (4.4.4) that $\widetilde{\mathbf{W}}_{2(n+1)+1}^{(k)} = 0 \in \mathcal{L}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k))$ for $k \geq n+3$, completing the proof of the inductive step. $\square$

### 4.4.2 Step 2: Defining Self-Adjoint Operators

Our goal is now to define the self-adjoint elements $\mathbf{W}_{n,sa}$, proving the following:

**Proposition 4.4.4.** *For each $n \in \mathbb{N}$, there exists an element*

$$\mathbf{W}_{n,sa} \in \bigoplus_{k=1}^{\infty} \mathcal{L}_{gmp,*}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k)),$$

*given by*

$$\mathbf{W}_{n,sa} \coloneqq \frac{1}{2}\Big(\widetilde{\mathbf{W}}_n + \widetilde{\mathbf{W}}_n^*\Big). \tag{4.4.74}$$

**Remark 4.4.5.** Recall that

$$(\widetilde{\mathbf{W}}_n^*)^{(k)} \coloneqq \widetilde{\mathbf{W}}_n^{(k),*}.$$

is the adjoint operator defined in Lemma 3.1.1.

It follows readily from Lemma 3.1.1 that

$$\mathbf{W}_{n,sa} \in \bigoplus_{k=1}^{\infty} \mathcal{L}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k))$$

and is self-adjoint. Thus, in order to prove Proposition 4.4.4, we only need to verify each $\mathbf{W}_{n,sa}$ satisfies the good mapping property, for which it suffices by linearity and the fact that each $\widetilde{\mathbf{W}}_n^{(k)} \in \mathcal{L}_{gmp}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k))$ to prove that

$$\widetilde{\mathbf{W}}_n^{(k),*} \in \mathcal{L}_{gmp}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k)), \qquad \forall k \in \mathbb{N}. \tag{4.4.75}$$

Using the recursion (4.4.4), the linearity of the adjoint operation, and the fact that

$$\Big(-i\partial_{x_1}\widetilde{\mathbf{W}}_n^{(k)}\Big)^* = \widetilde{\mathbf{W}}_n^{(k),*}(-i\partial_{x_1}) \in \mathcal{L}_{gmp}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k)) \tag{4.4.76}$$

by Lemma 3.1.2, we just need to show that

$$\left(\delta(X_1 - X_{\ell+1})\left(\widetilde{\mathbf{W}}_m^{(\ell)} \otimes \widetilde{\mathbf{W}}_{n-m}^{(j)}\right)\right)^* \in \mathcal{L}_{gmp}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k)) \tag{4.4.77}$$

for any $m \in \mathbb{N}_{\leq n-1}$ and $\ell, j \in \mathbb{N}$ satisfying $\ell + j = k$. We prove this assertion by another induction argument.

**Lemma 4.4.6.** *Let $n \in \mathbb{N}_{\geq 2}$, and suppose that $\widetilde{\mathbf{W}}_1^*, \ldots, \widetilde{\mathbf{W}}_{n-1}^* \in \bigoplus_{k=1}^{\infty} \mathcal{L}_{gmp}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k))$. Then* (4.4.77) *holds.*

*Proof.* Let $k \in \mathbb{N}$. Given $f^{(k)} \in \mathcal{S}(\mathbb{R}^k)$, we define the tempered distribution $v_{f^{(k)}}$ by

$$g^{(k)} \mapsto \left\langle f^{(k)} \middle| \delta(X_1 - X_{\ell+1})\left(\widetilde{\mathbf{W}}_m^{(\ell)} \otimes \widetilde{\mathbf{W}}_{n-m}^{(j)}\right) g^{(k)} \right\rangle, \tag{4.4.78}$$

where the composition $\delta(X_1 - X_{\ell+1})(\widetilde{\mathbf{W}}_m^{(\ell)} \otimes \widetilde{\mathbf{W}}_{n-m}^{(j)})$ is well-defined by Lemma 4.4.1. It is easy to check that the map

$$\mathcal{S}(\mathbb{R}^k) \to \mathcal{S}'(\mathbb{R}^k), \qquad f^{(k)} \mapsto v_{f^{(k)}} \tag{4.4.79}$$

is a continuous linear map, so it remains for us to verify the good mapping property. As in the proof of Lemma 4.4.1, it suffices to show that for any $\alpha \in \mathbb{N}_{\leq k}$, the map

$$\begin{aligned} &(\mathcal{S}(\mathbb{R}^\ell) \times \mathcal{S}(\mathbb{R}^j))^2 \to \mathcal{S}_{x'_\alpha}(\mathbb{R}; \mathcal{S}'_{x_\alpha}(\mathbb{R})) \\ &(f^{(\ell)}, f^{(j)}, g^{(\ell)}, g^{(j)}) \mapsto \left\langle v_{f^{(\ell)} \otimes f^{(j)}} \middle| (\cdot) \otimes_\alpha (g^{(\ell)} \otimes g^{(j)})(\cdot, x'_\alpha, \cdot) \right\rangle, \qquad x'_\alpha \in \mathbb{R}. \end{aligned} \tag{4.4.80}$$

may be identified with a (necessarily unique) continuous map $(\mathcal{S}(\mathbb{R}^\ell) \times \mathcal{S}(\mathbb{R}^j))^2 \to \mathcal{S}(\mathbb{R}^2)$, which is antilinear in the $f^{(\ell)}, f^{(j)}$ variables and linear in the $g^{(\ell)}, g^{(j)}$ variables. The reader will recall that the notation $\otimes_\alpha$ is defined in (4.4.32). To simplify the presentation, we will assume $\alpha \leq \ell$. The case $\ell < \alpha \leq k$ follows mutatis mutandis. Moreover, by replacing

$f^{(\ell)}, g^{(\ell)}$ with $\pi f^{(\ell)}, \pi g^{(\ell)}$, for $\pi \in \mathbb{S}_\ell$, we may assume that $\alpha = 1$. For any $\phi \in \mathcal{S}(\mathbb{R})$, we have by the distributional Fubini-Tonelli theorem that,

$$\begin{aligned}
&\left\langle \left\langle v_{f^{(\ell)}\otimes f^{(j)}} \middle| (\cdot) \otimes (g^{(\ell)} \otimes g^{(j)})(x_1', \cdot) \right\rangle, \phi \right\rangle_{\mathcal{S}'(\mathbb{R})-\mathcal{S}(\mathbb{R})} \\
&= \left\langle v_{f^{(\ell)}\otimes f^{(j)}} \middle| \phi \otimes (g^{(\ell)} \otimes g^{(j)})(x_1', \cdot) \right\rangle \\
&= \left\langle f^{(\ell)} \otimes f^{(j)} \middle| \delta(x_1 - x_{\ell+1}) \left( \widetilde{\mathbf{W}}_m^{(\ell)} \otimes \widetilde{\mathbf{W}}_{n-m}^{(j)} \right) \left( \phi \otimes g^{(\ell)}(x_1', \cdot) \otimes g^{(j)} \right) \right\rangle \\
&= \left\langle \delta(x_1 - x_{\ell+1}) \left( \widetilde{\mathbf{W}}_m^{(\ell)} \otimes \widetilde{\mathbf{W}}_{n-m}^{(j)} \right) \left( \phi \otimes g^{(\ell)}(x_1', \cdot) \otimes g^{(j)} \right), \overline{f^{(\ell)} \otimes f^{(j)}} \right\rangle_{\mathcal{S}'(\mathbb{R}^k)-\mathcal{S}(\mathbb{R}^k)}. \qquad (4.4.81)
\end{aligned}$$

Using the identifications of (4.4.31) and the action of the DVO $\delta(X_1 - X_{\ell+1})(\widetilde{\mathbf{W}}_m^{(\ell)} \otimes \widetilde{\mathbf{W}}_{n-m}^{(j)})$ given by (4.4.38) in Step II of the proof of Lemma 4.4.1, we find that

$$\begin{aligned}
(4.4.81) &= \int_{\mathbb{R}} dx_1 \Phi_{\widetilde{\mathbf{W}}_{n-m}^{(j)},1}(g^{(j)}, \overline{f^{(j)}})(x_1; x_1) \Phi_{\widetilde{\mathbf{W}}_m^{(\ell)},1}(\phi \otimes g^{(\ell)}(x_1', \cdot), \overline{f^{(\ell)}})(x_1; x_1) \\
&= \left\langle f^{(\ell)} \overline{\Phi_{\mathbf{W}_{n-m}^{(j)},1}(g^{(j)}, \overline{f^{(j)}})|_{y=y'}} \middle| \widetilde{\mathbf{W}}_m^{(\ell)} \left( \phi \otimes g^{(\ell)}(x_1', \cdot) \right) \right\rangle \\
&= \left\langle \widetilde{\mathbf{W}}_m^{(\ell),*} \left( f^{(\ell)} \overline{\Phi_{\mathbf{W}_{n-m}^{(j)},1}(g^{(j)}, \overline{f^{(j)}})|_{y=y'}} \right) \middle| \phi \otimes g^{(\ell)}(x_1', \cdot) \right\rangle, \qquad (4.4.82)
\end{aligned}$$

where the ultimate equality follows from the definition of the adjoint of a DVO, see Lemma 3.1.1. As before, the notation $|_{y=y'}$ denotes restriction to the hyperplane $\{(y, y') : y = y'\} \subset \mathbb{R}^2$. By the induction hypothesis, $\widetilde{\mathbf{W}}_m^{(\ell),*}$ possesses the good mapping property. Therefore, for any $\alpha \in \mathbb{N}_{\leq \ell}$, we can uniquely identify the map

$$\mathcal{S}(\mathbb{R}^\ell)^2 \to \mathcal{S}_{x_\alpha'}(\mathbb{R}; \mathcal{S}_{x_\alpha}'(\mathbb{R})), \qquad (\tilde{f}^{(\ell)}, \tilde{g}^{(\ell)}) \mapsto \left\langle \widetilde{\mathbf{W}}_m^{(\ell),*} \tilde{f}^{(\ell)}, (\cdot) \otimes_\alpha \tilde{g}^{(\ell)}(\cdot, x_\alpha', \cdot) \right\rangle_{\mathcal{S}'(\mathbb{R}^\ell)-\mathcal{S}(\mathbb{R}^\ell)} \qquad (4.4.83)$$

with a continuous bilinear map

$$\Phi_{\widetilde{\mathbf{W}}_m^{(\ell),*},\alpha} : \mathcal{S}(\mathbb{R}^\ell)^2 \to \mathcal{S}_{(x_\alpha, x_\alpha')}(\mathbb{R}^2)$$

$$\int_{\mathbb{R}} dx_\alpha \Phi_{\widetilde{\mathbf{W}}_m^{(\ell),*},\alpha}(\tilde{f}^{(\ell)}, \tilde{g}^{(\ell)})(x_\alpha; x_\alpha') \phi(x_\alpha) = \left\langle \widetilde{\mathbf{W}}_m^{(\ell),*} \tilde{f}^{(\ell)}, \phi \otimes_\alpha \tilde{g}^{(\ell)}(\cdot, x_\alpha', \cdot) \right\rangle_{\mathcal{S}'(\mathbb{R}^\ell)-\mathcal{S}(\mathbb{R}^\ell)}, \qquad \phi \in \mathcal{S}(\mathbb{R}). \qquad (4.4.84)$$

Hence,

$$
\begin{aligned}
(4.4.82) &= \overline{\left\langle \widetilde{\mathbf{W}}_m^{(\ell),*}\Big(f^{(\ell)}\overline{\Phi_{\mathbf{W}_{n-m}^{(j)},1}(g^{(j)},\overline{f^{(j)}})|_{y=y'}}\Big), \overline{\phi \otimes g^{(\ell)}(x_1',\cdot)}\right\rangle_{\mathcal{S}'(\mathbb{R}^\ell)-\mathcal{S}(\mathbb{R}^\ell)}} \\
&= \overline{\int_{\mathbb{R}} dx_1 \Phi_{\widetilde{\mathbf{W}}_m^{(\ell),*},1}(f^{(\ell)}\overline{\Phi_{\mathbf{W}_{n-m}^{(j)},1}(g^{(j)},\overline{f^{(j)}})|_{y=y'}}, \overline{g^{(\ell)}})(x_1;x_1')\overline{\phi(x_1)}} \\
&= \int_{\mathbb{R}} dx_1 \overline{\Phi_{\widetilde{\mathbf{W}}_m^{(\ell),*},1}(f^{(\ell)}\overline{\Phi_{\mathbf{W}_{n-m}^{(j)},1}(g^{(j)},\overline{f^{(j)}})|_{y=y'}}, \overline{g^{(\ell)}})}(x_1;x_1')\phi(x_1).
\end{aligned} \tag{4.4.85}
$$

Defining the map

$$
(f^{(\ell)}, f^{(j)}, g^{(\ell)}, g^{(j)}) \mapsto \overline{\Phi_{\widetilde{\mathbf{W}}_m^{(\ell),*},1}(f^{(\ell)}\overline{\Phi_{\mathbf{W}_{n-m}^{(j)}}(g^{(j)},\overline{f^{(j)}})|_{y=y'}}, \overline{g^{(\ell)}})} \tag{4.4.86}
$$

yields the desired conclusion, being the composition of continuous maps, antilinear in the $f^{(\ell)}, f^{(j)}$ variables, and linear in the $g^{(\ell)}, g^{(j)}$ variables. □

Since the base case $\widetilde{\mathbf{W}}_1^{(k),*} \in \mathcal{L}_{gmp}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k))$ for every $k \in \mathbb{N}$ is trivial, the lemma and the remarks preceding it imply the Proposition 4.4.4.

### 4.4.3 Step 3: Bosonic Symmetrization

We now modify the definition of the operators $\mathbf{W}_{n,sa}$ from the previous subsection in order to obtain a bosonic operator which generates the same trace functional as $\mathbf{W}_{n,sa}$ when evaluated on elements of $\mathfrak{G}_\infty^*$. As an immediate consequence of Lemma 3.3.32, we obtain Proposition 4.1.6, completing the main objective of Section 4.4. We conclude this subsection by explicitly computing $\mathbf{W}_3$ and $\mathbf{W}_4$.

**Example 4.4.7** (Computation of $\mathbf{W}_3$)**.** From the recursion (4.4.4), we have that

$$\begin{aligned}\widetilde{\mathbf{W}}_3^{(k)} &= (-i\partial_{x_1})\widetilde{\mathbf{W}}_2^{(k)} + \kappa\sum_{\ell+j=k}\delta(X_1-X_{\ell+1})\Big(\widetilde{\mathbf{W}}_1^{(\ell)}\otimes\widetilde{\mathbf{W}}_1^{(j)}\Big)\\ &= \begin{cases}(-i\partial_{x_1})^2, & k=1\\ \kappa\delta(X_1-X_2)Id_2 = \kappa\delta(X_1-X_2), & k=2\\ 0_k, & k\geq 3.\end{cases}\end{aligned} \tag{4.4.87}$$

Since the components $\widetilde{\mathbf{W}}_3^{(k)}$ are already self-adjoint and bosonic, it follows that

$$\mathbf{W}_3 = \widetilde{\mathbf{W}}_3 = \big((-i\partial_{x_1})^2, \kappa\delta(X_1-X_2), 0_3, \ldots\big). \tag{4.4.88}$$

**Example 4.4.8** (Computation of $\mathbf{W}_4$)**.** Similarly, from the recursion (4.4.4), we have that

$$\widetilde{\mathbf{W}}_4^{(k)} = (-i\partial_{x_1})\widetilde{\mathbf{W}}_3^{(k)} + \kappa\sum_{m=1}^{2}\sum_{\ell+j=k}\delta(X_1-X_{\ell+1})\Big(\widetilde{\mathbf{W}}_m^{(\ell)}\otimes\widetilde{\mathbf{W}}_{3-m}^{(j)}\Big). \tag{4.4.89}$$

If $k=1$, then

$$\widetilde{\mathbf{W}}_4^{(1)} = (-i\partial_{x_1})\widetilde{\mathbf{W}}_3^{(1)} = (-i\partial_{x_1})^3 = \mathbf{W}_4^{(1)}, \tag{4.4.90}$$

since $(-i\partial_{x_1})^3$ is self-adjoint and bosonic. If $k=2$, then

$$\begin{aligned}\widetilde{\mathbf{W}}_4^{(2)} &= (-i\partial_{x_1})\widetilde{\mathbf{W}}_3^{(2)} + \kappa\delta(X_1-X_2)\Big(\widetilde{\mathbf{W}}_1^{(1)}\otimes\widetilde{\mathbf{W}}_2^{(1)}\Big) + \kappa\delta(X_1-X_2)\Big(\widetilde{\mathbf{W}}_2^{(1)}\otimes\widetilde{\mathbf{W}}_1^{(1)}\Big)\\ &= \kappa((-i\partial_{x_1})\delta(X_1-X_2) + \delta(X_1-X_2)(Id_1\otimes(-i\partial_x)) + \delta(X_1-X_2)((-i\partial_x)\otimes Id_1))\\ &= -i\kappa(\partial_{x_1}\delta(X_1-X_2) + \delta(X_1-X_2)(\partial_{x_1}+\partial_{x_2})).\end{aligned} \tag{4.4.91}$$

The term $-i\delta(X_1-X_2)(\partial_{x_1}+\partial_{x_2})$ is evidently bosonic, and it is self-adjoint since

$$[\partial_{x_1}+\partial_{x_2}, \delta(X_1-X_2)] = 0.$$

For the term $-i\partial_{x_1}\delta(X_1 - X_2)$, Lemma 3.1.2 implies that the adjoint is given by $-i\delta(X_1 - X_2)\partial_{x_1}$, and therefore

$$\begin{aligned}
&\frac{\kappa}{2}\operatorname{Sym}_2((-i\partial_{x_1})\delta(X_1 - X_2) + \delta(X_1 - X_2)(-i\partial_{x_1})) \\
&= \frac{\kappa}{4}((-i\partial_{x_1} - i\partial_{x_2})\delta(X_1 - X_2) + \delta(X_1 - X_2)(-i\partial_{x_1} - i\partial_{x_2})) \\
&= \frac{\kappa}{2}(-i\partial_{x_1} - i\partial_{x_2})\delta(X_1 - X_2),
\end{aligned} \tag{4.4.92}$$

where we use that $\delta$ is an even distribution and again that $[\partial_{x_1} + \partial_{x_2}, \delta(X_1 - X_2)] = 0$. We conclude that

$$\mathbf{W}_4^{(2)} = \frac{3\kappa}{2}(-i\partial_{x_1} - i\partial_{x_2})\delta(X_1 - X_2). \tag{4.4.93}$$

Finally, it is evident that $\mathbf{W}_4^{(k)} = 0_k$ for $k \geq 3$.

## 4.5 The Correspondence: $\mathbf{W}_n$ and $w_n$

### 4.5.1 Multilinear Forms $w_n$

In this subsection, we analyze the structure of the nonlinear operators $w_n$ as sums of restricted multilinear forms. For each $k \in \mathbb{N}$, we define a $(2k-1)$-$\mathbb{C}$-linear operator

$$w_n^{(k)} : \mathcal{S}(\mathbb{R})^k \times \mathcal{S}(\mathbb{R})^{k-1} \to \mathcal{S}(\mathbb{R}), \quad (\phi_1, \ldots, \phi_k; \psi_2, \ldots, \psi_k) \mapsto w_n^{(k)}[\phi_1, \ldots, \phi_k; \psi_2, \ldots, \psi_k], \tag{4.5.1}$$

recursively by

$$\begin{aligned}
&w_1^{(k)}[\phi_1,\dots,\phi_k;\psi_2,\dots,\psi_k] \coloneqq \phi_1\delta_{k1},\\
&w_{n+1}^{(k)}[\phi_1,\dots,\phi_k;\psi_2,\dots,\psi_k]\\
&= (-i\partial_x)w_n^{(k)}[\phi_1,\dots,\phi_k;\psi_2,\dots,\psi_k]\\
&\qquad + \kappa\sum_{m=1}^{n-1}\sum_{\ell,j\geq 1;\ell+j=k}\psi_{\ell+1}w_m^{(\ell)}[\phi_1,\dots,\phi_\ell;\psi_2,\dots,\psi_\ell]w_{n-m}^{(j)}[\phi_{\ell+1},\dots,\phi_k;\psi_{\ell+2},\dots,\psi_k],
\end{aligned} \tag{4.5.2}$$

where $\delta_{k1}$ denotes the usual Kronecker delta. The next lemma establishes several important structural properties of the $w_n$, including that $w_n^{(k)}$ is identically zero for all but finitely many $k\in\mathbb{N}$.

**Lemma 4.5.1** (Properties of $w_n^{(k)}$). *The following properties hold:*

- *For each odd $n\in\mathbb{N}$, $w_n^{(k)}\equiv 0$ for $k>\frac{n+1}{2}$ and for $k\leq\frac{n+1}{2}$ we have*

$$w_n^{(k)}[\phi_1,\dots,\phi_k;\psi_2,\dots,\psi_k] = \sum_{\substack{(\underline{\alpha}_k,\underline{\alpha}'_{k-1})\in\mathbb{N}_0^{2k-1}\\ |\underline{\alpha}_k|+|\underline{\alpha}'_{k-1}|=n-1-2(k-1)}} a_{n,(\underline{\alpha}_k,\underline{\alpha}'_{k-1})}\Big(\prod_{r=1}^k\partial_x^{\alpha_r}\phi_r\Big)\Big(\prod_{r=2}^k\partial_x^{\alpha'_r}\psi_r\Big), \tag{4.5.3}$$

  *where $a_{n,(\underline{\alpha}_k,\underline{\alpha}'_{k-1})}\in\mathbb{R}$.*

- *For each even $n\in\mathbb{N}$, $w_n^{(k)}\equiv 0$ for $k>\frac{n}{2}$ and for $k\leq\frac{n}{2}$ we have*

$$w_n^{(k)}[\phi_1,\dots,\phi_k;\psi_2,\dots,\psi_k] = i\sum_{\substack{(\underline{\alpha}_k,\underline{\alpha}'_{k-1})\in\mathbb{N}_0^{2k-1}\\ |\underline{\alpha}_k|+|\underline{\alpha}'_{k-1}|=n-1-2(k-1)}} a_{n,(\underline{\alpha}_k,\underline{\alpha}'_{k-1})}\Big(\prod_{r=1}^k\partial_x^{\alpha_r}\phi_r\Big)\Big(\prod_{r=2}^k\partial_x^{\alpha'_r}\psi_r\Big), \tag{4.5.4}$$

  *where $a_{n,(\underline{\alpha}_k,\underline{\alpha}'_{k-1})}\in\mathbb{R}$.*

*Proof.* We prove the lemma by strong induction on $n$. We begin with the base case $n = 1$. That (4.5.3) holds for $n = 1$ is tautological. For the induction step, suppose that there exists some $n \in \mathbb{N}$ such that either (4.5.3) or (4.5.4) holds for every odd or even $j \in \mathbb{N}_{\leq n}$, respectively. We consider two cases based on whether $n$ is even or odd.

Consider the even index case. We first show that $w_n^{(k)} \equiv 0$ for $k > \frac{n}{2}$. Since $n-1$ is odd, the induction hypothesis implies that

$$(-i\partial_x)w_{n-1}^{(k)} \equiv 0, \qquad k > \frac{n}{2}. \tag{4.5.5}$$

Now suppose that $\ell, j \in \mathbb{N}$ are such that $\ell + j = k$ and

$$w_m^{(\ell)} \otimes w_{n-1-m}^{(j)} \not\equiv 0, \tag{4.5.6}$$

where $1 \leq m \leq n-2$. By symmetry, it suffices to consider when $m$ is odd and $n-1-m$ is even. By the induction hypothesis,

$$w_m^{(\ell)} \equiv 0, \;\; \ell > \frac{m+1}{2} \qquad \text{and} \qquad w_{n-1-m}^{(j)} \equiv 0, \;\; j > \frac{n-1-m}{2}. \tag{4.5.7}$$

Consequently, we must have that

$$k = \ell + j \leq \frac{m+1}{2} + \frac{n-1-m}{2} = \frac{n}{2}. \tag{4.5.8}$$

It then follows from the recursion (4.5.2) that $w_n^{(k)} \equiv 0$ for $k > \frac{n}{2}$.

Next we establish the asserted expansion formula. By the induction hypothesis,

$$w_{n-1}^{(k)}[\phi_1, \ldots, \phi_k; \psi_2, \ldots, \psi_k] = \sum_{\substack{(\underline{\alpha}_k, \underline{\alpha}'_{k-1}) \in \mathbb{N}_0^{2k-1} \\ |\underline{\alpha}_k| + |\underline{\alpha}'_{k-1}| = n-2-2(k-1)}} a_{n-1,(\underline{\alpha}_k, \underline{\alpha}'_{k-1})} \Big(\prod_{r=1}^k \partial_x^{\alpha_r} \phi_r\Big)\Big(\prod_{r=2}^k \partial_x^{\alpha'_r} \psi_r\Big), \tag{4.5.9}$$

where the coefficients $a_{n-1,(\underline{\alpha}_k,\underline{\alpha}'_{k-1})}$ are real. Hence by the Leibnitz rule, we can define real coefficients $b_{n,(\underline{\alpha}_k,\underline{\alpha}_{k-1})}$ such that

$$-i\partial_x w^{(k)}_{n-1}[\phi_1,\ldots,\phi_k;\psi_2,\ldots,\psi_k] = i \sum_{\substack{(\underline{\alpha}_k,\underline{\alpha}'_{k-1})\in\mathbb{N}_0^{2k-1}\\ |\underline{\alpha}_k|+|\underline{\alpha}'_{k-1}|=n-1-2(k-1)}} b_{n,(\underline{\alpha}_k,\underline{\alpha}_{k-1})}(\prod_{r=1}^{k}\partial_x^{\alpha_r}\phi_r)(\prod_{r=2}^{k}\partial_x^{\alpha'_r}\psi_r). \tag{4.5.10}$$

Similarly, for $m\in\mathbb{N}_{\le n-2}$ and $\ell, j\in\mathbb{N}$, the induction hypothesis implies that

$$w^{(\ell)}_m[\phi_1,\ldots,\phi_\ell;\psi_2,\ldots,\psi_\ell] = \begin{cases} \displaystyle\sum_{\substack{(\underline{\alpha}_\ell,\underline{\alpha}'_{\ell-1})\in\mathbb{N}_0^{2\ell-1}\\ |\underline{\alpha}_\ell|+|\underline{\alpha}'_{\ell-1}|=m-1-2(\ell-1)}} a_{m,(\underline{\alpha}_\ell,\underline{\alpha}'_{\ell-1})}(\prod_{r=1}^{\ell}\partial_x^{\alpha_r}\phi_r)(\prod_{r=2}^{\ell}\partial_x^{\alpha'_r}\psi_r), & m \text{ odd},\\ i\displaystyle\sum_{\substack{(\underline{\alpha}_\ell,\underline{\alpha}'_{\ell-1})\in\mathbb{N}_0^{2\ell-1}\\ |\underline{\alpha}_\ell|+|\underline{\alpha}'_{\ell-1}|=m-1-2(\ell-1)}} a_{m,(\underline{\alpha}_\ell,\underline{\alpha}'_{\ell-1})}(\prod_{r=1}^{\ell}\partial_x^{\alpha_r}\phi_r)(\prod_{r=2}^{\ell}\partial_x^{\alpha'_r}\psi_r), & m \text{ even}\end{cases} \tag{4.5.11}$$

and

$$\begin{aligned} &w^{(j)}_{n-1-m}[\phi_{\ell+1,\ldots,\phi_k};\psi_{\ell+2},\ldots,\psi_k]\\ &= \begin{cases} i\displaystyle\sum_{\substack{(\underline{\alpha}_j,\underline{\alpha}'_{j-1})\in\mathbb{N}_0^{2j-1}\\ |\underline{\alpha}_j|+|\underline{\alpha}'_{j-1}|=n-2-m-2(j-1)}} a_{n-1-m,(\underline{\alpha}_j,\underline{\alpha}'_{j-1})}(\prod_{r=\ell+1}^{k}\partial_x^{\alpha_r}\phi_r)(\prod_{r=\ell+2}^{k}\partial_x^{\alpha'_r}\psi_r), & m \text{ odd}\\ \displaystyle\sum_{\substack{(\underline{\alpha}_j,\underline{\alpha}'_{j-1})\in\mathbb{N}_0^{2j-1}\\ |\underline{\alpha}_j|+|\underline{\alpha}'_{j-1}|=n-2-m-2(j-1)}} a_{n-1-m,(\underline{\alpha}_j,\underline{\alpha}'_{j-1})}(\prod_{r=\ell+1}^{k}\partial_x^{\alpha_r}\phi_r)(\prod_{r=\ell+2}^{k}\partial_x^{\alpha'_r}\psi_r), & m \text{ even}\end{cases}, \end{aligned} \tag{4.5.12}$$

where $a_{n-1-m,(\underline{\alpha}_\ell,\underline{\alpha}'_{\ell-1})}, a_{n-1-m,(\underline{\alpha}_j,\underline{\alpha}'_{j-1})}\in\mathbb{R}$. For $\ell+j=k$ and $(\underline{\alpha}_\ell,\underline{\alpha}'_{\ell-1}),(\underline{\alpha}_j,\underline{\alpha}'_{j-1})$ as in the summations above, the multi-index

$$(\underline{\alpha}_\ell,\underline{\alpha}_j,\underline{\alpha}'_{\ell-1},\underline{\alpha}'_{j-1})\in\mathbb{N}_0^{2k-2}$$

satisfies

$$|(\underline{\alpha}_\ell,\underline{\alpha}_j)|+|(\underline{\alpha}'_{\ell-1},\underline{\alpha}'_{j-1})| = m-1-2(\ell-1)+n-2-m-2(j-1) = n-1-2(k-1). \tag{4.5.13}$$

Consequently, we can define real coefficients $c_{n,(\underline{\alpha}_k,\underline{\alpha}'_{k-1})}$ such that

$$\sum_{m=1}^{n-1} \psi_{\ell+1} w_m^{(\ell)}[\phi_1,\ldots,\phi_\ell;\psi_2,\ldots,\psi_j] w_{n-1-m}^{(j)}[\phi_{\ell+1},\ldots,\phi_k;\psi_{\ell+2},\ldots,\psi_k]$$
$$= i \sum_{\substack{(\underline{\alpha}_k,\underline{\alpha}'_{k-1})\in\mathbb{N}_0^{2k-1} \\ |\underline{\alpha}_k|+|\underline{\alpha}'_{k-1}|=n-1-2(k-1)}} c_{n,(\underline{\alpha}_k,\underline{\alpha}'_{k-1})} (\prod_{r=1}^k \partial_x^{\alpha_r}\phi_r)(\prod_{r=2}^k \partial_x^{\alpha'_r}\psi_r). \tag{4.5.14}$$

Defining

$$a_{n,(\underline{\alpha}_k,\underline{\alpha}'_{k-1})} := b_{n,(\underline{\alpha}_k,\underline{\alpha}'_{k-1})} + c_{n,(\underline{\alpha}_k,\underline{\alpha}'_{k-1})}, \tag{4.5.15}$$

and summing (4.5.10) and (4.5.14) shows that (4.5.4) holds.

Next, consider the odd index case. To establish that $w_n^{(k)} \equiv 0$ for $k > \frac{n+1}{2}$, we have by our previous discussion in the even case, that

$$-i\partial_x w_{n-1}^{(k)} = 0, \qquad k > \frac{n-1}{2}. \tag{4.5.16}$$

Suppose that $\ell, j \in \mathbb{N}$ are such that $\ell + j = k$ and

$$w_m^{(\ell)} \otimes w_{n-1-m}^{(j)} \not\equiv 0, \tag{4.5.17}$$

where $1 \leq m \leq n-2$. If $m$ is odd, then $n-1-m$ is odd, and so by the induction hypothesis,

$$w_m^{(\ell)} \equiv 0, \;\; \ell > \frac{m+1}{2} \qquad \text{and} \qquad w_{n-1-m}^{(j)} \equiv 0, \;\; j > \frac{n-m}{2}. \tag{4.5.18}$$

Consequently, we must have that

$$k = \ell + j \leq \frac{m+1}{2} + \frac{n-m}{2} = \frac{n+1}{2}. \tag{4.5.19}$$

Similarly, if $m$ is even, then $n-1-m$ is even, and so by the induction hypothesis

$$w_m^{(\ell)} \equiv 0, \;\; \ell > \frac{m}{2} \qquad \text{and} \qquad w_{n-1-m}^{(j)} \equiv 0, \;\; j > \frac{n-m-1}{2}. \tag{4.5.20}$$

Consequently, we must have that

$$k = \ell + j \le \frac{m}{2} + \frac{n-m-1}{2} = \frac{n-1}{2}. \tag{4.5.21}$$

It now follows from the recursion (4.5.2) that $w_n^{(k)} \equiv 0$ for $k > \frac{n+1}{2}$. Repeating the proof mutatis mutandis from the $n$ even case, we see that $w_n^{(k)}$ has the representation (4.5.3). Thus, the proof of the induction step is complete. $\square$

We establish now some notation we will use here and in the sequel. For $k, n \in \mathbb{N}$, we define densities

$$P_n^{(k)}[\phi_1, \ldots, \phi_k; \psi_1, \ldots, \psi_k] \coloneqq \psi_1 w_n^{(k)}[\phi_1, \ldots, \phi_k; \psi_2, \ldots, \psi_k] \in \mathcal{S}(\mathbb{R}), \tag{4.5.22}$$

and we define

$$I_n^{(k)}[\phi_1, \ldots, \phi_k; \psi_1, \ldots, \psi_k] \coloneqq \int_{\mathbb{R}} dx P_n^{(k)}[\phi_1, \ldots, \phi_k; \psi_1, \ldots, \psi_k](x). \tag{4.5.23}$$

It is clear from Lemma 4.5.1, that $P_n^{(k)} : \mathcal{S}(\mathbb{R})^{2k} \to \mathcal{S}(\mathbb{R})$ is a $2k$-$\mathbb{C}$-linear, continuous map, and thus $I_n^{(k)} : \mathcal{S}(\mathbb{R})^{2k} \to \mathbb{C}$ is a $2k$-$\mathbb{C}$-linear, continuous map. For $k \in \mathbb{N}$, we recall the notation $\phi^{\times k}$ from (3.3.65) to denote the measurable function $\phi^{\times k} : \mathbb{R}^m \to \mathbb{C}^k$

$$\phi^{\times k}(\underline{x}_m) \coloneqq (\phi(\underline{x}_m), \ldots, \phi(\underline{x}_m)), \tag{4.5.24}$$

and similarly for $\psi^{\times k}$.

**Remark 4.5.2.** It is clear from the recursion (4.5.2) that

$$I_n(\phi) = \sum_{k=1}^{\infty} I_n^{(k)}[\phi^{\times k}; \overline{\phi}^{\times k}], \qquad \forall \phi \in \mathcal{S}(\mathbb{R}), \tag{4.5.25}$$

where $I_n$ is as defined in (1.3.9).

Remark 4.5.2 and the structure result Lemma 4.5.1 allow us to give a proof of the seemingly obvious fact that the functionals $I_n$ are not constant on $\mathcal{S}(\mathbb{R})$. We obtain this fact as a consequence of a more general lemma. Note that since $I_n(0) = 0$, the nonconstancy of $I_n$ is equivalent to $I_n \not\equiv 0$.

**Lemma 4.5.3.** *Let $n \in \mathbb{N}$, and let $\underline{c} = \{c_k\}_{k\in\mathbb{N}} \subset \mathbb{C}$ such that $c_1 \neq 0$. Define the map*

$$I_{n,\underline{c}} : \mathcal{S}(\mathbb{R}) \to \mathbb{C}, \qquad I_{n,\underline{c}}(\phi) := \sum_{k=1}^{\infty} c_k I_n^{(k)}[\phi^{\times k}; \overline{\phi}^{\times k}], \qquad \forall \phi \in \mathcal{S}(\mathbb{R}). \tag{4.5.26}$$

*Then $I_{n,\underline{c}} \not\equiv 0$.*

*Proof.* Assume the contrary. Then for any $\lambda \in \mathbb{C}$, we find from the $2k$-complex linearity of the functionals $I_n^{(k)}$ that

$$0 = I_{n,\underline{c}}(\lambda\phi) = \sum_{k=1}^{\infty} c_k I_n^{(k)}[(\lambda\phi)^{\times k}; \overline{(\lambda\phi)}^{\times k}] = \sum_{k=1}^{\infty} c_k |\lambda|^{2k} I_n^{(k)}[\phi^{\times k}; \overline{\phi}^{\times k}], \qquad \forall \phi \in \mathcal{S}(\mathbb{R}). \tag{4.5.27}$$

Now fix $\phi \in \mathcal{S}(\mathbb{R})$ and define a function

$$\rho_{\phi,\underline{c}} : \mathbb{C} \to \mathbb{C}, \qquad \rho_{\phi,\underline{c}}(\lambda) := \sum_{k=1}^{\infty} c_k |\lambda|^{2k} I_n^{(k)}[\phi^{\times k}; \overline{\phi}^{\times k}], \tag{4.5.28}$$

which is well-defined and smooth since $I_n^{(k)} \equiv 0$ for all but finitely many indices $k$. Now observe that

$$0 = (\partial_\lambda \partial_{\overline{\lambda}} \rho_{\phi,\underline{c}})(0) = c_1 I_n^{(1)}[\phi; \overline{\phi}] = c_1 \int_{\mathbb{R}} dx\, \overline{\phi}(x)(-i\partial_x)^{n-1}\phi(x). \tag{4.5.29}$$

Choosing $\phi \in \mathcal{S}(\mathbb{R})$ to be a function whose Fourier transform $\widehat{\phi}$ satisfies $0 \leq \widehat{\phi} \leq 1$,

$$\widehat{\phi}(\xi) = \begin{cases} 1, & 2 \leq \xi \leq 3 \\ 0, & \xi \leq 1,\ \xi \geq 4 \end{cases}, \tag{4.5.30}$$

we obtain a contradiction from Plancherel's theorem, since $c_1 \neq 0$ by assumption. $\square$

### 4.5.2 Variational Derivatives

In this subsection, we show that the functionals $I_n$ satisfy the conditions of Remark 4.3.1 and explicitly compute their symplectic gradients. To this end, we record here a recursive formula for the functions $w_{n,(\psi_1,\psi_2)}$, which generalizes the recursive formula (1.3.8) for $w_n$, given by

$$\begin{aligned} &w_{1,(\psi_1,\psi_2)}(x) = \psi_1(x) \\ &w_{n+1,(\psi_1,\psi_2)}(x) = -i\partial_x w_{n,(\psi_1,\psi_2)}(x) + \kappa\psi_2(x)\sum_{m=1}^{n-1} w_{m,(\psi_1,\psi_2)}(x) w_{n-m,(\psi_1,\psi_2)}(x)' \end{aligned} \tag{4.5.31}$$

and we refer to (1.2.19) for more details. We define $\tilde{I}_n : \mathcal{S}(\mathbb{R})^2 \to \mathbb{C}$ by

$$\tilde{I}_n(\psi_1,\psi_2) := \int_{\mathbb{R}} dx \psi_2(x) w_{n,(\psi_1,\psi_2)}(x), \qquad \forall (\psi_1,\psi_2) \in \mathcal{S}(\mathbb{R})^2. \tag{4.5.32}$$

**Remark 4.5.4.** By comparing the recursion (4.5.31) to the recursion (4.5.2), we see that

$$w_{n,(\psi_1,\psi_2)} = \sum_{k=1}^{\infty} w_n^{(k)}[\psi_1^{\times k}; \psi_2^{\times(k-1)}] \tag{4.5.33}$$

and consequently

$$\tilde{I}_n(\psi_1,\psi_2) = \sum_{k=1}^{\infty} I_n^{(k)}[\psi_1^{\times k}; \psi_2^{\times k}]. \tag{4.5.34}$$

We now use the multilinear $w_n^{(k)}$ introduced in the previous subsection in order to compute the variational derivatives, defined in (4.3.1), of the functions $\tilde{I}_n$. We first dispense with a technical lemma asserting the existence of a partial transpose for the $w_n^{(k)}$ in $C^\infty(\mathcal{S}(\mathbb{R})^{2k-1}; \mathcal{S}(\mathbb{R}))$. The proof follows from the structural formula of Lemma 4.5.1 and integration by parts; we leave the details to the reader.

**Lemma 4.5.5.** *Let $n, k \in \mathbb{N}$. Then for $1 \leq j \leq k$, there exists a unique partial transpose $w_{n,j}^{(k),t} \in C^\infty(\mathcal{S}(\mathbb{R})^{2k-1}; \mathcal{S}(\mathbb{R}))$, such that for every $\delta\phi \in \mathcal{S}(\mathbb{R})$ and $\phi_1, \ldots, \phi_k, \psi_2, \ldots \psi_k \in \mathcal{S}(\mathbb{R})$ we have*

$$\begin{aligned}&\int_{\mathbb{R}} dx \delta\phi(x) w_{n,j}^{(k),t}[\phi_1, \ldots, \phi_k; \psi_2, \ldots, \psi_k](x) \\ &= \int_{\mathbb{R}} dx \phi_j(x) w_n^{(k)}[\phi_1, \ldots, \phi_{j-1}, \delta\phi, \phi_{j+1}, \ldots, \phi_k; \psi_2, \ldots, \psi_k](x),\end{aligned} \tag{4.5.35}$$

*Similarly, for $2 \leq j \leq k$, there exists a unique partial transpose $w_{n,j'}^{(k),t} \in C^\infty(\mathcal{S}(\mathbb{R})^{2k-1}; \mathcal{S}(\mathbb{R}))$, such that for every $\delta\psi \in \mathcal{S}(\mathbb{R})$ and $\phi_1, \ldots, \phi_k, \psi_2, \ldots \psi_k \in \mathcal{S}(\mathbb{R})$ we have*

$$\begin{aligned}&\int_{\mathbb{R}} dx \delta\psi(x) w_{n,j'}^{(k),t}[\phi_1, \ldots, \phi_k; \psi_2, \ldots, \psi_k](x) \\ &= \int_{\mathbb{R}} dx \psi_j(x) w_n^{(k)}[\phi_1, \ldots, \phi_k; \psi_2, \ldots, \psi_{j-1}, \delta\psi, \psi_{j+1}, \ldots, \psi_k](x).\end{aligned} \tag{4.5.36}$$

*For convenience of notation, we define $w_{n,1'}^{(k),t} \in C^\infty(\mathcal{S}(\mathbb{R})^{2k-1}; \mathcal{S}(\mathbb{R}))$ by*

$$w_{n,1'}^{(k),t}[\phi_1, \ldots, \phi_k; \psi_2, \ldots, \psi_k] \coloneqq w_n^{(k)}[\phi_1, \ldots, \phi_k; \psi_2, \ldots, \psi_k]. \tag{4.5.37}$$

We may now proceed to establish formulae for the variational derivatives of the $\tilde{I}_n$.

**Lemma 4.5.6.** *For $n \in \mathbb{N}$, we have that*

$$\boldsymbol{\nabla}_1 \tilde{I}_n(\phi, \psi) = \sum_{k=1}^{\infty} \sum_{j=1}^{k} w_{n,j}^{(k),t}[\phi^{\times(j-1)}, \psi, \phi^{\times(k-j)}; \psi^{\times(k-1)}], \tag{4.5.38}$$

$$\boldsymbol{\nabla}_{\bar{2}} \tilde{I}_n(\phi, \psi) = \sum_{k=1}^{\infty} \sum_{j=1}^{k} w_{n,j'}^{(k),t}[\phi^{\times k}; \psi^{\times(k-1)}] \tag{4.5.39}$$

*for every* $(\phi,\psi)\in\mathcal{S}(\mathbb{R})^2$. *In particular,*

$$\begin{aligned}\nabla_s I_n(\phi) &= -i\sum_{k=1}^{\infty}\sum_{j=1}^{k}\overline{w_{n,j}^{(k),t}[\phi^{\times(j-1)},\overline{\phi},\phi^{\times(k-j)};\overline{\phi}^{\times(k-1)}]}\\ &= -i\sum_{k=1}^{\infty}\sum_{j=1}^{k}w_{n,j'}^{(k),t}[\phi^{\times k};\overline{\phi}^{\times(k-1)}]\\ &= -\frac{i}{2}\sum_{k=1}^{\infty}\sum_{j=1}^{k}\left(\overline{w_{n,j}^{(k),t}[\phi^{\times(j-1)},\overline{\phi},\phi^{\times(k-j)};\overline{\phi}^{\times(k-1)}]}+w_{n,j'}^{(k),t}[\phi^{\times k};\overline{\phi}^{\times(k-1)}]\right).\end{aligned} \tag{4.5.40}$$

*Proof.* Fix a point $(\phi,\psi)\in\mathcal{S}(\mathbb{R})^2$. Unpacking the definition of $\tilde{I}_n$ and using the chain rule for the Gâteaux derivative, we obtain that

$$\begin{aligned}d\tilde{I}_n[\phi,\psi](\delta\phi,\delta\psi) = \sum_{k=1}^{\infty}\sum_{j=1}^{k}\Bigg(&\int_{\mathbb{R}}dxP_n^{(k)}[\phi^{\times(j-1)},\delta\phi,\phi^{\times(k-j)};\psi^{\times k}](x)\\ &+\int_{\mathbb{R}}dxP_n^{(k)}[\phi^{\times k};\psi^{\times(j-1)},\delta\psi,\psi^{\times(k-j)}](x)\Bigg).\end{aligned} \tag{4.5.41}$$

Since

$$P_n^{(k)}[\phi^{\times(j-1)},\delta\phi,\phi^{\times(k-j)};\psi^{\times k}] = \psi w_n^{(k)}[\phi^{\times(j-1)},\delta\phi,\phi^{\times(k-j)};\psi^{\times(k-1)}] \tag{4.5.42}$$

and

$$P_n^{(k)}[\phi^{\times k};\psi^{\times(j-1)},\delta\psi,\psi^{\times(k-j)}] = \begin{cases}\delta\psi w_n^{(k)}[\phi^{\times k};\psi^{\times(k-1)}], & j=1\\ \psi w_n^{(k)}[\phi^{\times k};\psi^{\times(j-2)},\delta\psi,\psi^{\times(k-j)}], & 2\leq j\leq k\end{cases}, \tag{4.5.43}$$

upon application of Lemma 4.5.5, we see that

$$\begin{aligned}&\int_{\mathbb{R}}dxP_n^{(k)}[\phi^{\times(j-1)},\delta\phi,\phi^{\times(k-j)};\psi^{\times k}](x)\\ &=\int_{\mathbb{R}}dx\delta\phi(x)w_{n,j}^{(k),t}[\phi^{\times(j-1)},\psi,\phi^{\times(k-j)};\psi^{\times(k-1)}](x)\end{aligned} \tag{4.5.44}$$

and

$$\int_{\mathbb{R}}dxP_n^{(k)}[\phi^{\times k};\psi^{\times(j-1)},\delta\psi,\psi^{\times(k-j)}](x) = \int_{\mathbb{R}}dx\delta\psi(x)w_{n,j'}^{(k),t}[\phi^{\times k};\psi^{\times(k-1)}](x). \tag{4.5.45}$$

Substituting (4.5.44) and (4.5.45) into (4.5.41), we arrive at the identity

$$\begin{aligned} d\tilde{I}_n[\phi,\psi](\delta\phi,\delta\psi) = \sum_{k=1}^{\infty}\sum_{j=1}^{k}\Bigg(&\int_{\mathbb{R}} dx\delta\phi(x)w_{n,j}^{(k),t}[\phi^{\times(j-1)},\psi,\phi^{\times(k-j)};\psi^{\times(k-1)}](x) \\ &+\int_{\mathbb{R}} dx\delta\psi(x)w_{n,j'}^{(k),t}[\phi^{\times k};\psi^{\times(k-1)}](x)\Bigg). \end{aligned} \tag{4.5.46}$$

Using that there are only finitely many indices $k$ yielding a nonzero contribution by Lemma 4.5.1, we can move the summations inside the integral to conclude that

$$\begin{aligned} d\tilde{I}_n[\phi,\psi](\delta\phi,\delta\psi) = &\int_{\mathbb{R}} dx\delta\phi(x)\left(\sum_{k=1}^{\infty}\sum_{j=1}^{k} w_{n,j}^{(k),t}[\phi^{\times(j-1)},\psi,\phi^{\times(k-j)};\psi^{\times(k-1)}](x)\right) \\ &+\int_{\mathbb{R}} dx\delta\psi(x)\left(\sum_{k=1}^{\infty}\sum_{j=1}^{k} w_{n,j'}^{(k),t}[\phi^{\times k};\psi^{\times(k-1)}](x)\right), \end{aligned} \tag{4.5.47}$$

which yields the desired formula for the variational derivatives in light of (4.3.1).

To see the second assertion for the symplectic gradient $\mathbf{\nabla}_s I_n(\phi)$, we recall that from the fact that $I_n(\phi) = \tilde{I}_n(\phi,\overline{\phi})$, Remark 4.3.1, and (4.3.9) that we have the the identity

$$\mathbf{\nabla}_s I_n(\phi) = -i\overline{\mathbf{\nabla}_1\tilde{I}_n(\phi,\overline{\phi})} = -i\mathbf{\nabla}_{\bar{2}}\tilde{I}_n(\phi,\overline{\phi}).$$

Substituting the identities for $\mathbf{\nabla}_1\tilde{I}_n(\phi,\overline{\phi})$, $\mathbf{\nabla}_{\bar{2}}\tilde{I}_n(\phi,\overline{\phi})$ into the right-hand side of the previous equality completes the proof. $\square$

### 4.5.3 Partial Trace Connection of $\mathbf{W}_n$ to $w_n$

We next connect the linear DVOs $\widetilde{\mathbf{W}}_n^{(k)}$ constructed in Section 4.4 to the multilinear Schwartz-valued operators $w_n^{(k)}$ constructed in Section 4.5.1. We note that since the definition of the $\mathbf{W}_n$ is fairly straightforward given the definition of $\widetilde{\mathbf{W}}_n$, it will suffice to establish these connections for the latter operators.

It will be important to remember the following consequence of the fact that $\widetilde{\mathbf{W}}_n^{(k)}$ satisfies the good mapping property: the generalized partial trace

$$\mathrm{Tr}_{2,\ldots,k}\left(\widetilde{\mathbf{W}}_n^{(k)} \,|\bigotimes_{\ell=1}^{k}\phi_\ell\rangle \; \langle\bigotimes_{\ell=1}^{k}\psi_\ell|\right), \tag{4.5.48}$$

which is a priori the element of $\mathcal{L}(\mathcal{S}(\mathbb{R}), \mathcal{S}'(\mathbb{R}))$ given by the property

$$\begin{aligned} &\left\langle \mathrm{Tr}_{2,\ldots,k}\left(\widetilde{\mathbf{W}}_n^{(k)} \,|\bigotimes_{\ell=1}^{k}\phi_\ell\rangle \; \langle\bigotimes_{\ell=1}^{k}\psi_\ell|\right)\phi, \psi \right\rangle_{\mathcal{S}'(\mathbb{R})-\mathcal{S}(\mathbb{R})} \\ &= \left\langle \widetilde{\mathbf{W}}_n^{(k)} \bigotimes_{\ell=1}^{k}\phi_\ell, \psi \otimes \left\langle \bigotimes_{\ell=1}^{k}\overline{\psi_\ell}, \phi \right\rangle_{\mathcal{S}'_{x_1}(\mathbb{R})-\mathcal{S}_{x_1}(\mathbb{R})} \right\rangle_{\mathcal{S}'(\mathbb{R}^k)-\mathcal{S}(\mathbb{R}^k)} \\ &= \langle\psi_1|\phi\rangle \left\langle \widetilde{\mathbf{W}}_n^{(k)} \bigotimes_{\ell=1}^{k}\phi_\ell, \psi \otimes \bigotimes_{\ell=2}^{k}\overline{\psi_\ell} \right\rangle_{\mathcal{S}'(\mathbb{R}^k)-\mathcal{S}(\mathbb{R}^k)}, \end{aligned} \tag{4.5.49}$$

for every $\phi, \psi \in \mathcal{S}(\mathbb{R})$, is in fact uniquely identifiable with the element in $\mathcal{S}(\mathbb{R}^2)$ which we denote by

$$\Phi_{\widetilde{\mathbf{W}}_n^{(k)}}(\phi_1, \ldots, \phi_k; \overline{\psi_1}, \ldots, \overline{\psi_k})$$

via

$$\begin{aligned} &\left\langle \mathrm{Tr}_{2,\ldots,k}\left(\widetilde{\mathbf{W}}_n^{(k)} \,|\bigotimes_{\ell=1}^{k}\phi_\ell\rangle \; \langle\bigotimes_{\ell=1}^{k}\psi_\ell|\right) f, g \right\rangle_{\mathcal{S}'(\mathbb{R})-\mathcal{S}(\mathbb{R})} \\ &= \int_{\mathbb{R}^2} dx dx' \Phi_{\widetilde{\mathbf{W}}_n^{(k)}}(\phi_1, \ldots, \phi_k; \overline{\psi_1}, \ldots, \overline{\psi_k})(x; x') f(x') g(x). \end{aligned} \tag{4.5.50}$$

Moreover, the map

$$\mathcal{S}(\mathbb{R})^{2k} \to \mathcal{S}(\mathbb{R}^2), \qquad (\phi_1, \ldots, \phi_k, \psi_1, \ldots, \psi_k) \mapsto \Phi_{\widetilde{\mathbf{W}}_n^{(k)}}(\phi_1, \ldots, \phi_k; \overline{\psi_1}, \ldots, \overline{\psi_k}) \tag{4.5.51}$$

is continuous. The objective of the next lemma is to obtain a formula for $\Phi_{\widetilde{\mathbf{W}}_n^{(k)}}$ in terms of $w_n^{(k)}$.

**Lemma 4.5.7.** *Let $k, n \in \mathbb{N}$. Then the following properties hold:*

- *For any $\pi \in \mathbb{S}_k$ with $\pi(1) = 1$, we have that for all $(x, x') \in \mathbb{R}^2$,*

$$\begin{aligned}&\Phi_{\widetilde{\mathbf{W}}^{(k)}_{n,(\pi(1),\ldots,\pi(k))}}(\phi_1, \ldots, \phi_k; \overline{\psi_1}, \ldots, \overline{\psi_k})(x; x')\\ &\qquad = \overline{\psi_1(x')} w^{(k)}_n[\phi_{\pi(1)}, \ldots, \phi_{\pi(k)}; \overline{\psi_{\pi(2)}}, \ldots, \overline{\psi_{\pi(k)}}](x),\end{aligned} \tag{4.5.52}$$

*and*

$$\begin{aligned}&\Phi_{\widetilde{\mathbf{W}}^{(k),*}_{n,(\pi(1),\ldots,\pi(k))}}(\phi_1, \ldots, \phi_k; \overline{\psi_1}, \ldots, \overline{\psi_k})(x; x')\\ &\qquad = \overline{\psi_1(x') w^{(k),t}_{n,1}[\overline{\phi_1}, \psi_{\pi(2)}, \ldots, \psi_{\pi(k)}; \overline{\phi_{\pi(2)}}, \ldots, \overline{\phi_{\pi(k)}}](x)}.\end{aligned} \tag{4.5.53}$$

- *For any $\pi \in \mathbb{S}_k$ with $\pi(1) \neq 1$, we have that for all $(x, x') \in \mathbb{R}^2$,*

$$\begin{aligned}&\Phi_{\widetilde{\mathbf{W}}^{(k)}_{n,(\pi(1),\ldots,\pi(k))}}(\phi_1, \ldots, \phi_k; \overline{\psi_1}, \ldots, \overline{\psi_k})(x; x')\\ &= \overline{\psi_1(x')} w^{(k),t}_{n,\pi^{-1}(1)'}[\phi_{\pi(1)}, \ldots, \phi_{\pi(k)}; \overline{\psi_{\pi(2)}}, \ldots, \overline{\psi_{\pi(\pi^{-1}(1)-1)}}, \overline{\psi_{\pi(1)}}, \overline{\psi_{\pi(\pi^{-1}(1)+1)}}, \ldots, \overline{\psi_{\pi(k)}}](x),\end{aligned} \tag{4.5.54}$$

*and*

$$\begin{aligned}&\Phi_{\widetilde{\mathbf{W}}^{(k),*}_{n,(\pi(1),\ldots,\pi(k))}}(\phi_1, \ldots, \phi_k; \overline{\psi_1}, \ldots, \overline{\psi_k})(x; x')\\ &= \overline{\psi_1(x') w^{(k),t}_{n,\pi^{-1}(1)}[\psi_{\pi(1)}, \ldots, \psi_{\pi(\pi^{-1}(1)-1)}, \overline{\phi_{\pi(1)}}, \psi_{\pi(\pi^{-1}(1)+1)}, \ldots, \psi_{\pi(k)}; \overline{\phi_{\pi(2)}}, \ldots, \overline{\phi_{\pi(k)}}](x)},\end{aligned} \tag{4.5.55}$$

*Proof.* We will begin by establishing the first claim for the identity permutation, that is, for each $k \in \mathbb{N}$ and for any $\phi_1, \ldots, \phi_k, \psi_1, \ldots, \psi_k \in \mathcal{S}(\mathbb{R})$, we have that

$$\begin{aligned}&\Phi_{\widetilde{\mathbf{W}}^{(k)}_n}(\phi_1, \ldots, \phi_k; \overline{\psi_1}, \ldots, \overline{\psi_k})(x; x')\\ &= \overline{\psi_1(x')} w^{(k)}_n[\phi_1, \ldots, \phi_k; \overline{\psi_2}, \ldots, \overline{\psi_k}](x), \qquad \forall (x, x') \in \mathbb{R}^2.\end{aligned} \tag{4.5.56}$$

By Lemma 4.4.3, it suffices to prove (4.5.56) by induction on

$$\{(k, n) \in \mathbb{N}^2 : k \leq n\}. \tag{4.5.57}$$

We begin with the base case, $(k,n)=(1,1)$. Since $\widetilde{\mathbf{W}}_1^{(1)} = Id_1 \in \mathcal{L}(\mathcal{S}(\mathbb{R}), \mathcal{S}'(\mathbb{R}))$, we have the Schwartz kernel identity

$$\Big(\widetilde{\mathbf{W}}_1^{(1)} \left|\phi_1\right\rangle \left\langle\psi_1\right|\Big)(x_1;x_1') = \phi_1(x_1)\overline{\psi_1(x_1')} = \overline{\psi_1(x_1')}w_1^{(1)}[\phi_1](x_1), \qquad \forall (x_1,x_1') \in \mathbb{R}^2, \tag{4.5.58}$$

which proves (4.5.56) for the base case.

For the induction step, suppose that there is some $n \in \mathbb{N}$ such that for all integers $j \in \mathbb{N}_{\leq n}$ the following assertion holds: for all integers $k \in \mathbb{N}_{\leq j}$ and for all functions $\phi_1, \ldots, \phi_k, \psi_1, \ldots, \psi_k \in \mathcal{S}(\mathbb{R})$, we have that

$$\Phi_{\widetilde{\mathbf{W}}_j^{(k)}}(\phi_1,\ldots,\phi_k;\overline{\psi_1},\ldots,\overline{\psi_k})(x;x') = \overline{\psi_1}(x')w_j^{(k)}[\phi_1,\ldots,\phi_k;\overline{\psi}_2,\ldots,\overline{\psi}_k](x), \qquad \forall (x,x') \in \mathbb{R}^2. \tag{4.5.59}$$

We now prove (4.5.59) with $j = n+1$. By the recursion relation (4.4.4) and the bilinearity of the generalized trace, we have that

$$\begin{aligned}
&\mathrm{Tr}_{2,\ldots,k}\left(\widetilde{\mathbf{W}}_{n+1}^{(k)} \left|\bigotimes_{\ell=1}^k \phi_\ell\right\rangle \left\langle\bigotimes_{\ell=1}^k \psi_\ell\right|\right) \\
&= \mathrm{Tr}_{2,\ldots,k}\left((-i\partial_{x_1})\widetilde{\mathbf{W}}_n^{(k)} \left|\bigotimes_{r=1}^k \phi_r\right\rangle \left\langle\bigotimes_{r=1}^k \psi_r\right|\right) \\
&\qquad + \kappa \sum_{m=1}^{n-1}\sum_{\ell+j=k} \mathrm{Tr}_{2,\ldots,k}\left(\delta(X_1 - X_{\ell+1})\Big(\widetilde{\mathbf{W}}_m^{(\ell)} \otimes \widetilde{\mathbf{W}}_{n-m}^{(j)}\Big) \left|\bigotimes_{r=1}^k \phi_r\right\rangle \left\langle\bigotimes_{r=1}^k \psi_r\right|\right) \\
&=: \mathrm{Term}_{1,k} + \mathrm{Term}_{2,k}.
\end{aligned} \tag{4.5.60}$$

We first analyze $\mathrm{Term}_{1,k}$. Since $(-i\partial_{x_1})\widetilde{\mathbf{W}}_n^{(k)} \in \mathcal{L}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k))$, it follows from the definition of the generalized partial trace that

$$\mathrm{Term}_{1,k} = (-i\partial_x)\,\mathrm{Tr}_{2,\ldots,k}\left(\widetilde{\mathbf{W}}_n^{(k)} \left|\bigotimes_{r=1}^k \phi_r\right\rangle \left\langle\bigotimes_{r=1}^k \psi_r\right|\right). \tag{4.5.61}$$

It follows from the induction hypothesis that

$$\begin{aligned}&(-i\partial_x)\,\mathrm{Tr}_{2,\ldots,k}\left(\widetilde{\mathbf{W}}_n^{(k)}\,|\bigotimes_{r=1}^k \phi_r\rangle\,\langle\bigotimes_{r=1}^k \psi_r|\right)(x;x')\\&=\left((-i\partial_x)\Phi_{\widetilde{\mathbf{W}}_n^{(k)}}(\phi_1,\ldots,\phi_k;\overline{\psi_1},\ldots,\overline{\psi_k})\right)(x;x')\\&=\overline{\psi_1(x')}(-i\partial_x)w_n^{(k)}[\phi_1,\ldots,\phi_k;\overline{\psi_2},\ldots,\overline{\psi_k}](x)\end{aligned} \tag{4.5.62}$$

with equality in the sense of tempered distributions on $\mathbb{R}^2$. Substituting (4.5.62) into (4.5.61), we obtain that

$$\mathrm{Term}_{1,k}=\overline{\psi_1(x')}(-i\partial_x)w_n^{(k)}[\phi_1,\ldots,\phi_k;\overline{\psi_2},\ldots,\overline{\psi_k}](x). \tag{4.5.63}$$

We next analyze $\mathrm{Term}_{2,k}$. By the computed action of the Hörmander product $\delta(X_1-X_{\ell+1})\left(\widetilde{\mathbf{W}}_m^{(\ell)}\otimes\widetilde{\mathbf{W}}_{n-m}^{(j)}\right)$ given by (4.4.38) and the definition of $\Phi_{\widetilde{\mathbf{W}}_m^{(\ell)}}$ and $\Phi_{\widetilde{\mathbf{W}}_{n-m}^{(j)}}$ we have that

$$\begin{aligned}&\mathrm{Tr}_{2,\ldots,k}\left(\delta(X_1-X_{\ell+1})\left(\widetilde{\mathbf{W}}_m^{(\ell)}\otimes\widetilde{\mathbf{W}}_{n-m}^{(j)}\right)|\bigotimes_{r=1}^k \phi_r\rangle\,\langle\bigotimes_{r=1}^k \psi_r|\right)(x;x')\\&=\Phi_{\widetilde{\mathbf{W}}_m^{(\ell)}}(\phi_1,\ldots,\phi_\ell;\overline{\psi_1},\ldots,\overline{\psi_\ell})(x;x')\Phi_{\widetilde{\mathbf{W}}_{n-m}^{(j)}}(\phi_{\ell+1},\ldots,\phi_k;\overline{\psi_{\ell+1}},\ldots,\overline{\psi_k})(x;x)\end{aligned} \tag{4.5.64}$$

in the sense of tempered distributions. Using the induction hypothesis for $\widetilde{\mathbf{W}}_m^{(\ell)}$ and $\widetilde{\mathbf{W}}_{n-m}^{(j)}$, respectively, we also have that

$$\begin{aligned}&\Phi_{\widetilde{\mathbf{W}}_m^{(\ell)}}(\phi_1,\ldots,\phi_\ell;\overline{\psi_1},\ldots,\overline{\psi_\ell})(x;x')\\&=\overline{\psi_1}(x')w_m^{(\ell)}[\phi_1,\ldots,\phi_\ell;\overline{\psi_2},\ldots,\overline{\psi_\ell}](x),\qquad\forall(x,x')\in\mathbb{R}^2\end{aligned} \tag{4.5.65}$$

and

$$\begin{aligned}&\Phi_{\widetilde{\mathbf{W}}_{n-m}^{(j)}}(\phi_{\ell+1},\ldots,\phi_k;\overline{\psi_{\ell+1}},\ldots,\overline{\psi_k}](x;x')\\&=\overline{\psi_{\ell+1}}(x')w_{n-m}^{(j)}[\phi_{\ell+1},\ldots,\phi_k;\overline{\psi_{\ell+2}},\ldots,\overline{\psi_k}](x),\qquad\forall(x,x')\in\mathbb{R}^2.\end{aligned} \tag{4.5.66}$$

Substituting the two preceding expressions into (4.5.64), we find that

$$(4.5.64) = \overline{\psi_1(x')\psi_{\ell+1}(x)}w_m^{(\ell)}[\phi_1,\ldots,\phi_\ell;\overline{\psi_2},\ldots,\overline{\psi_\ell}](x)w_{n-m}^{(j)}[\phi_{\ell+1},\ldots,\phi_k;\overline{\psi_{\ell+2}},\ldots,\overline{\psi_k}](x). \tag{4.5.67}$$

Hence,

$$\begin{aligned}&\mathrm{Term}_{2,k}(x;x') \\ &= \kappa\sum_{m=1}^{n-1}\sum_{\ell+j=k}\overline{\psi_1(x')\psi_{\ell+1}(x)}w_m^{(\ell)}[\phi_1,\ldots,\phi_\ell;\overline{\psi_2},\ldots,\overline{\psi_\ell}](x)w_{n-m}^{(j)}[\phi_{\ell+1},\ldots,\phi_k;\overline{\psi_{\ell+2}},\ldots,\overline{\psi_k}](x).\end{aligned} \tag{4.5.68}$$

Combining our identities for $\mathrm{Term}_{1,k}$ and $\mathrm{Term}_{2,k}$, we obtain that

$$\begin{aligned}&(\mathrm{Term}_{1,k}+\mathrm{Term}_{2,k})(x;x') \\ &= \overline{\psi_1(x')}(-i\partial_x)w_n^{(k)}[\phi_1,\ldots,\phi_k;\overline{\psi_2},\ldots,\overline{\psi_k}](x) \\ &\quad+\kappa\sum_{m=1}^{n-1}\sum_{\ell+j=k}\overline{\psi_1(x')\psi_{\ell+1}(x)}w_m^{(\ell)}[\phi_1,\ldots,\phi_\ell;\overline{\psi_2},\ldots,\overline{\psi_\ell}](x)w_{n-m}^{(j)}[\phi_{\ell+1},\ldots,\phi_k;\overline{\psi_{\ell+2}},\ldots,\overline{\psi_k}](x),\end{aligned}$$

with equality in $\mathcal{S}'(\mathbb{R}^2)$. Now applying the recursive relation (4.5.2) for $w_{n+1}^{(k)}[\phi_1,\ldots,\phi_k;\overline{\psi_2},\ldots,\overline{\psi_k}]$, we find that

$$(\mathrm{Term}_{1,k}+\mathrm{Term}_{2,k})(x;x') = \overline{\psi_1(x')}w_{n+1}^{(k)}[\phi_1,\ldots,\phi_k;\overline{\psi_2},\ldots,\overline{\psi_k}](x), \tag{4.5.69}$$

which completes the proof of the induction step for showing (4.5.56).

We now use (4.5.56) to prove the adjoint assertion of the lemma. For $f,g\in\mathcal{S}(\mathbb{R})$, we have by definition of the generalized partial trace (see Proposition 3.2.4) that

$$\begin{aligned}&\left\langle \mathrm{Tr}_{2,\ldots,k}\left(\widetilde{\mathbf{W}}_n^{(k),*}\,|\bigotimes_{r=1}^k\phi_r\rangle\,\langle\bigotimes_{r=1}^k\psi_r|\right)f,g\right\rangle_{\mathcal{S}'(\mathbb{R})-\mathcal{S}(\mathbb{R})} \\ &= \langle\psi_1|f\rangle\left\langle\widetilde{\mathbf{W}}_n^{(k),*}\bigotimes_{r=1}^k\phi_r, g\otimes\bigotimes_{r=2}^k\overline{\psi_r}\right\rangle_{\mathcal{S}'(\mathbb{R}^k)-\mathcal{S}(\mathbb{R}^k)}.\end{aligned} \tag{4.5.70}$$

By Lemma 3.1.1,

$$\left\langle \widetilde{\mathbf{W}}_n^{(k),*} \bigotimes_{r=1}^k \phi_r, \overline{\overline{g} \otimes \bigotimes_{r=2}^k \psi_r} \right\rangle_{\mathcal{S}'(\mathbb{R}^k)-\mathcal{S}(\mathbb{R}^k)} = \overline{\left\langle \widetilde{\mathbf{W}}_n^{(k)} (\overline{g} \otimes \bigotimes_{r=2}^k \psi_r), \bigotimes_{r=1}^k \overline{\phi_r} \right\rangle_{\mathcal{S}'(\mathbb{R}^k)-\mathcal{S}(\mathbb{R}^k)}}. \tag{4.5.71}$$

We can rewrite

$$\begin{aligned}
&\langle \psi_1 | f \rangle \overline{\left\langle \widetilde{\mathbf{W}}_n^{(k)} (\overline{g} \otimes \bigotimes_{r=2}^k \psi_r), \bigotimes_{r=1}^k \overline{\phi_r} \right\rangle_{\mathcal{S}'(\mathbb{R}^k)-\mathcal{S}(\mathbb{R}^k)}} \\
&= \overline{\left\langle \mathrm{Tr}_{2,\ldots,k} \left( \widetilde{\mathbf{W}}_n^{(k)} \, |\overline{g} \otimes \bigotimes_{r=2}^k \psi_r \rangle \, \langle f \otimes \bigotimes_{r=2}^k \phi_r | \right) \psi_1, \overline{\phi_1} \right\rangle_{\mathcal{S}'(\mathbb{R})-\mathcal{S}(\mathbb{R})}}.
\end{aligned} \tag{4.5.72}$$

Now applying (4.5.56) to this expression, we obtain that the right-hand side of (4.5.72) equals

$$\begin{aligned}
&\overline{\int_{\mathbb{R}^2} dx dx' \Phi_{\widetilde{\mathbf{W}}_n^{(k)}} (\overline{g}, \psi_2, \ldots, \psi_k; \overline{f}, \overline{\phi_2}, \ldots, \overline{\phi_k})(x; x') \psi_1(x') \overline{\phi_1(x)}} \\
&= \overline{\int_{\mathbb{R}^2} dx dx' \overline{f}(x') w_n^{(k)} [\overline{g}, \psi_2, \ldots, \psi_k; \overline{\phi_2}, \ldots, \overline{\phi_k}](x) \psi_1(x') \overline{\phi_1}(x)} \\
&= \int_{\mathbb{R}^2} dx dx' f(x') \overline{w_n^{(k)} [\overline{g}, \psi_2, \ldots, \psi_k; \overline{\phi_2}, \ldots, \overline{\phi_k}]}(x) \overline{\psi_1}(x') \phi_1(x).
\end{aligned} \tag{4.5.73}$$

Next, using the Fubini-Tonelli theorem and applying Lemma 4.5.5 in the $x$-integration, we find that

$$\begin{aligned}
(4.5.73) &= \langle \psi_1 | f \rangle \overline{\int_{\mathbb{R}} dx w_{n,1}^{(k),t} [\overline{\phi_1}, \psi_2, \ldots, \psi_k; \overline{\phi_2}, \ldots, \overline{\phi_k}](x) \overline{g}(x)} \\
&= \langle \psi_1 | f \rangle \int_{\mathbb{R}} dx \overline{w_{n,1}^{(k),t} [\overline{\phi_1}, \psi_2, \ldots, \psi_k; \overline{\phi_2}, \ldots, \overline{\phi_k}]}(x) g(x).
\end{aligned} \tag{4.5.74}$$

Since $f, g \in \mathcal{S}(\mathbb{R})$ were arbitrary, going back to the left-hand side of (4.5.70) and using the uniqueness and properties of $\Phi_{\mathbf{W}_n^{(k),*}}$, we conclude the pointwise in $\mathbb{R}^2$ identity

$$\Phi_{\mathbf{W}_n^{(k),*}} (\phi_1, \ldots, \phi_k; \overline{\psi_1}, \ldots, \overline{\psi_k})(x; x') = \overline{\psi_1(x') w_{n,1}^{(k),t} [\overline{\phi_1}, \psi_2, \ldots, \psi_k; \overline{\phi_2}, \ldots, \overline{\phi_k}](x)}. \tag{4.5.75}$$

We next need to generalize (4.5.56) and (4.5.75) to arbitrary permutations $\pi \in \mathbb{S}_k$. By definition of the notation

$$\widetilde{\mathbf{W}}_{n,(\pi(1),\ldots,\pi(k))}^{(k)} := \pi \circ \widetilde{\mathbf{W}}_n^{(k)} \circ \pi^{-1},$$

we have that for any $\phi_1, \ldots, \phi_k \in \mathcal{S}(\mathbb{R})$,

$$\widetilde{\mathbf{W}}_{n,(\pi(1),\ldots,\pi(k))}^{(k)}(\bigotimes_{r=1}^{k} \phi_r) = \pi \circ \widetilde{\mathbf{W}}_n^{(k)}((\bigotimes_{r=1}^{k} \phi_r) \circ \pi^{-1}), \tag{4.5.76}$$

where the reader will recall from (3.3.28) and (3.3.29) how a permutation acts on vectors and functions, respectively. Setting $f^{(k)} := \bigotimes_{r=1}^{k} \phi_r$, we have by definition that

$$(f^{(k)} \circ \pi^{-1})(\underline{x}_k) = f^{(k)}(x_{\pi^{-1}(1)}, \ldots, x_{\pi^{-1}(k)}) = \prod_{r=1}^{k} \phi_r(x_{\pi^{-1}(r)}). \tag{4.5.77}$$

Making the change of variable $r' = \pi^{-1}(r)$, we see that

$$\prod_{r=1}^{k} \phi_r(x_{\pi^{-1}(r)}) = \prod_{r'=1}^{k} \phi_{\pi(r')}(x_{r'}) = (\bigotimes_{r=1}^{k} \phi_{\pi(r)})(\underline{x}_k). \tag{4.5.78}$$

Therefore,

$$\mathrm{Tr}_{2,\ldots,k}\left(\widetilde{\mathbf{W}}_{n,(\pi(1),\ldots,\pi(k))}^{(k)} \, |\otimes_{\ell=1}^{k} \phi_\ell\rangle \, \langle \otimes_{\ell=1}^{k} \psi_\ell|\right) = \mathrm{Tr}_{2,\ldots,k}\left(\left(\pi \circ \widetilde{\mathbf{W}}_n^{(k)}\right) |\bigotimes_{\ell=1}^{k} \phi_{\pi(\ell)}\rangle \, \langle \bigotimes_{\ell=1}^{k} \psi_\ell|\right) \tag{4.5.79}$$

as elements of $\mathcal{L}_{gmp}(\mathcal{S}(\mathbb{R}), \mathcal{S}'(\mathbb{R}))$. Next, it follows from the characterizing property of the generalized partial trace and the fact that we define a permutation to act on tempered distribution by duality that

$$\begin{aligned} &\left\langle \mathrm{Tr}_{2,\ldots,k}\left(\left(\pi \circ \widetilde{\mathbf{W}}_n^{(k)}\right) |\bigotimes_{\ell=1}^{k} \phi_{\pi(\ell)}\rangle \, \langle \bigotimes_{\ell=1}^{k} \psi_\ell|\right) f, g \right\rangle_{\mathcal{S}'(\mathbb{R})-\mathcal{S}(\mathbb{R})} \\ &= \langle \psi_1 | f \rangle \left\langle \widetilde{\mathbf{W}}_n^{(k)} \bigotimes_{\ell=1}^{k} \phi_{\pi(\ell)}, (g \otimes \bigotimes_{\ell=2}^{k} \overline{\psi_\ell}) \circ \pi^{-1} \right\rangle_{\mathcal{S}'(\mathbb{R}^k)-\mathcal{S}(\mathbb{R}^k)}. \end{aligned} \tag{4.5.80}$$

Repeating the computation which yielded (4.5.78), we find that

$$(g \otimes \bigotimes_{\ell=2}^{k} \overline{\psi_\ell}) \circ \pi^{-1} = (\bigotimes_{\ell=1}^{\pi^{-1}(1)-1} \overline{\psi_{\pi(\ell)}}) \otimes g \otimes (\bigotimes_{\ell=\pi^{-1}(1)+1}^{k} \overline{\psi_{\pi(\ell)}}), \tag{4.5.81}$$

where per our notation convention, the tensor product on the right-hand side is to be interpreted as $g \otimes \bigotimes_{\ell=2}^{k} \overline{\psi_{\pi(\ell)}}$ if $\pi(1) = 1$. Thus,

$$\begin{aligned}
(4.5.80) &= \langle \psi_1 | f \rangle \left\langle \widetilde{\mathbf{W}}_n^{(k)} \bigotimes_{\ell=1}^{k} \phi_{\pi(\ell)}, (\bigotimes_{\ell=1}^{\pi^{-1}(1)-1} \overline{\psi_{\pi(\ell)}}) \otimes g \otimes (\bigotimes_{\ell=\pi^{-1}(1)+1}^{k} \overline{\psi_{\pi(\ell)}}) \right\rangle_{\mathcal{S}'(\mathbb{R}^k)-\mathcal{S}(\mathbb{R}^k)} \\
&= \left\langle \mathrm{Tr}_{2,\ldots,k} \left( \widetilde{\mathbf{W}}_n^{(k)} | \bigotimes_{\ell=1}^{k} \phi_{\pi(\ell)} \rangle \, \langle \psi_1 \otimes (\bigotimes_{\ell=2}^{\pi^{-1}(1)-1} \psi_{\pi(\ell)}) \otimes \overline{g} \otimes (\bigotimes_{\ell=\pi^{-1}(1)+1}^{k} \psi_{\pi(\ell)} | \right) f, \overline{\psi_{\pi(1)}} \right\rangle_{\mathcal{S}'(\mathbb{R})-\mathcal{S}(\mathbb{R})}.
\end{aligned}$$

By definition of $\Phi_{\widetilde{\mathbf{W}}_n^{(k)}}$, this last expression equals

$$\int_{\mathbb{R}^2} dx dx' \Phi_{\widetilde{\mathbf{W}}_n^{(k)}}(\phi_{\pi(1)}, \ldots, \phi_{\pi(k)}; \overline{\psi_1}, \overline{\psi_{\pi(2)}}, \ldots, \overline{\psi_{\pi(\pi^{-1}(1)-1)}}, g, \overline{\psi_{\pi(\pi^{-1}(1)+1)}}, \ldots, \overline{\psi_{\pi(k)}})(x; x') f(x') \overline{\psi_{\pi(1)}(x)}.$$

Applying the result we have just established for the identity permutation, recorded in (4.5.56), and using the Fubini-Tonelli theorem and Lemma 4.5.5, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^2} dx dx' \overline{\psi_1(x')} w_n^{(k)}[\phi_{\pi(1)}, \ldots, \phi_{\pi(k)}; \overline{\psi_{\pi(2)}}, \ldots, \overline{\psi_{\pi(\pi^{-1}(1)-1)}}, g, \overline{\psi_{\pi(\pi^{-1}(1)+1)}}, \ldots, \overline{\psi_{\pi(k)}})(x) f(x') \overline{\psi_{\pi(1)}(x)} \\
&= \int_{\mathbb{R}^2} dx dx' w_{n,\pi^{-1}(1)'}^{(k),t}[\phi_{\pi(1)}, \ldots, \phi_{\pi(k)}; \overline{\psi_{\pi(2)}}, \ldots, \overline{\psi_{\pi(\pi^{-1}(1)-1)}}, \overline{\psi_{\pi(1)}}, \overline{\psi_{\pi(\pi^{-1}(1)+1)}}, \ldots, \overline{\psi_{\pi(k)}}](x) \\
&\qquad \times \overline{\psi_1(x')} g(x) f(x').
\end{aligned}$$

Since $f, g \in \mathcal{S}(\mathbb{R})$ were arbitrary, we conclude that

$$\begin{aligned}
&\Phi_{\mathbf{W}_{n,(\pi(1),\ldots,\pi(k))}^{(k)}}(\phi_1, \ldots, \phi_k; \overline{\psi_1}, \ldots, \overline{\psi_k})(x; x') \\
&= \overline{\psi_1(x')} w_{n,\pi^{-1}(1)'}^{(k),t}[\phi_{\pi(1)}, \ldots, \phi_{\pi(k)}; \overline{\psi_{\pi(2)}}, \ldots, \overline{\psi_{\pi(\pi^{-1}(1)-1)}}, \overline{\psi_{\pi(1)}}, \overline{\psi_{\pi(\pi^{-1}(1)+1)}}, \ldots, \overline{\psi_{\pi(k)}}](x), \qquad (x, x') \in \mathbb{R}^2.
\end{aligned} \tag{4.5.82}$$

For the assertions about the adjoint, consider the expression

$$\int_{\mathbb{R}^2} dxdx' \Phi_{\widetilde{\mathbf{W}}_n^{(k),*}}(\phi_{\pi(1)}, \ldots, \phi_{\pi(k)}; \overline{\psi_1}, \overline{\psi_{\pi(2)}}, \ldots, \overline{\psi_{\pi(\pi^{-1}(1)-1)}}, g, \overline{\psi_{\pi(\pi^{-1}(1)+1)}}, \ldots, \overline{\psi_{\pi(k)}})(x;x') f(x') \overline{\psi_{\pi(1)}(x)}. \tag{4.5.83}$$

By (4.5.75), we have

$$\begin{aligned}&\Phi_{\widetilde{\mathbf{W}}_n^{(k),*}}(\phi_{\pi(1)}, \ldots, \phi_{\pi(k)}; \overline{\psi_1}, \overline{\psi_{\pi(2)}}, \ldots, \overline{\psi_{\pi(\pi^{-1}(1)-1)}}, g, \overline{\psi_{\pi(\pi^{-1}(1)+1)}}, \ldots, \overline{\psi_{\pi(k)}})(x;x') \\ &= \overline{\psi_1(x')} \overline{w_{n,1}^{(k),t}[\overline{\phi_{\pi(1)}}, \psi_{\pi(2)}, \ldots, \psi_{\pi(\pi^{-1}(1)-1)}, \overline{g}, \psi_{\pi(\pi^{-1}(1)+1)}, \ldots, \psi_{\pi(k)}; \overline{\phi_{\pi(2)}}, \ldots, \overline{\phi_{\pi(k)}}](x)}.\end{aligned} \tag{4.5.84}$$

By the characterizing property of $w_{n,1}^{(k),t}$ from Lemma 4.5.5, followed by a second application of Lemma 4.5.5, we have that

$$\begin{aligned}&\int_{\mathbb{R}} dx \; \overline{\psi_{\pi(1)}(x) w_{n,1}^{(k),t}[\overline{\phi_{\pi(1)}}, \psi_{\pi(2)}, \ldots, \psi_{\pi(\pi^{-1}(1)-1)}, \overline{g}, \psi_{\pi(\pi^{-1}(1)+1)}, \ldots, \psi_{\pi(k)}; \overline{\phi_{\pi(2)}}, \ldots, \overline{\phi_{\pi(k)}}](x)} \\ &= \overline{\int_{\mathbb{R}} dx \; \overline{\phi_{\pi(1)}(x)} w_n^{(j)}[\psi_{\pi(1)}, \ldots, \psi_{\pi(\pi^{-1}(1)-1)}, \overline{g}, \psi_{\pi(\pi^{-1}(1)+1)}, \ldots, \psi_{\pi(k)}; \overline{\phi_{\pi(2)}}, \ldots, \overline{\phi_{\pi(k)}}](x)} \\ &= \overline{\int_{\mathbb{R}} dx \; \overline{g(x)} w_{n,\pi^{-1}(1)}^{(k),t}[\psi_{\pi(1)}, \ldots, \psi_{\pi(\pi^{-1}(1)-1)}, \overline{\phi_{\pi(1)}}, \psi_{\pi(\pi^{-1}(1)+1)}, \ldots, \psi_{\pi(k)}; \overline{\phi_{\pi(2)}}, \ldots, \overline{\phi_{\pi(k)}}](x)}.\end{aligned} \tag{4.5.85}$$

By substituting (4.5.84) into (4.5.83), then using Fubini-Tonelli theorem and the preceding identity, we conclude that

$$\begin{aligned}&\Phi_{\mathbf{W}_{n,(\pi(1),\ldots,\pi(k))}^{(k),*}}(\phi_1, \ldots, \phi_k; \overline{\psi_1}, \ldots, \overline{\psi_k})(x;x') \\ &= \overline{\psi_1(x') w_{n,\pi^{-1}(1)}^{(k),t}[\psi_{\pi(1)}, \ldots, \psi_{\pi(\pi^{-1}(1)-1)}, \overline{\phi_{\pi(1)}}, \psi_{\pi(\pi^{-1}(1)+1)}, \ldots, \psi_{\pi(k)}; \overline{\phi_{\pi(2)}}, \ldots, \overline{\phi_{\pi(k)}}](x)}\end{aligned} \tag{4.5.86}$$

point-wise in $\mathbb{R}^2$, which establishes the final claim and completes the proof. $\square$

By taking the (1-particle) trace of the DVOs

$$\mathrm{Tr}_{2,\ldots,k}\left(\widetilde{\mathbf{W}}_{n,(\pi(1),\ldots,\pi(k))}^{(k)} |\bigotimes_{\ell=1}^k \phi_\ell\rangle \langle \bigotimes_{\ell=1}^k \psi_\ell|\right), \quad \mathrm{Tr}_{2,\ldots,k}\left(\widetilde{\mathbf{W}}_{n,(\pi(1),\ldots,\pi(k))}^{(k),*} |\bigotimes_{\ell=1}^k \phi_\ell\rangle \langle \bigotimes_{\ell=1}^k \psi_\ell|\right)$$

and using the definition (4.5.23) of $I_n^{(k)}$, we obtain the following corollary of Lemma 4.5.7:

**Corollary 4.5.8.** *Let $k,n \in \mathbb{N}$. Then for any permutation $\pi \in \mathbb{S}_k$ and any functions $\phi_1,\ldots,\phi_k,\psi_1,\ldots,\psi_k \in \mathcal{S}(\mathbb{R})$, we have the identities*

$$\mathrm{Tr}_{1,\ldots,k}\left(\widetilde{\mathbf{W}}_{n,(\pi(1),\ldots,\pi(k))}^{(k)} \left|\otimes_{\ell=1}^k\phi_\ell\right\rangle \left\langle\otimes_{\ell=1}^k\psi_\ell\right|\right) = I_n^{(k)}[\phi_{\pi(1)},\ldots,\phi_{\pi(k)};\overline{\psi_{\pi(1)}},\ldots,\overline{\psi_{\pi(k)}}], \tag{4.5.87}$$

$$\mathrm{Tr}_{1,\ldots,k}\left(\widetilde{\mathbf{W}}_{n,(\pi(1),\ldots,\pi(k))}^{(k),*} \left|\otimes_{\ell=1}^k\phi_\ell\right\rangle \left\langle\otimes_{\ell=1}^k\psi_\ell\right|\right) = \overline{I_n^{(k)}[\psi_{\pi(1)},\ldots,\psi_{\pi(k)};\overline{\phi_{\pi(1)}},\ldots,\overline{\phi_{\pi(k)}}]}. \tag{4.5.88}$$

## 4.6 The Involution: $\mathcal{H}_n$ and $I_{b,n}$

In this section, we prove Theorem 4.1.7. We recall the definition of the trace functionals

$$\mathcal{H}_n(\Gamma) \coloneqq \mathrm{Tr}(\mathbf{W}_n \cdot \Gamma), \qquad \forall \Gamma \in \mathfrak{G}_\infty^*. \tag{4.6.1}$$

The statement of the theorem is then the following:

**Theorem 4.1.7** (Involution theorem)**.** *Let $n,m \in \mathbb{N}$. Then*

$$\{\mathcal{H}_n,\mathcal{H}_m\}_{\mathfrak{G}_\infty^*} \equiv 0 \text{ on } \mathfrak{G}_\infty^*. \tag{4.1.36}$$

As discussed in the introduction, we prove Theorem 4.1.7 by showing that the Poisson commutativity of the functionals $\mathcal{H}_n$ on the weak Poisson manifold $(\mathfrak{G}_\infty^*, \mathcal{A}_\infty, \{\cdot,\cdot\}_{\mathfrak{G}_\infty^*})$ is equivalent to the Poisson commutativity of the functionals $I_{b,n}$ on the weak Poisson manifold $(\mathcal{S}(\mathbb{R};\mathcal{V}), \mathcal{A}_{\mathcal{S},\mathcal{V}}, \{\cdot,\cdot\}_{L^2,\mathcal{V}})$. See (4.3.17), (4.3.19), and Proposition 4.3.2 for definition and properties of this manifold. Since the Poisson commutativity of the $I_{b,n}$ is established in Proposition 1.3.7, this equivalence will complete the proof of Theorem 4.1.7.

Establishing this equivalence relies on the detailed correspondence between the observable $\infty$-hierarchies $-i\mathbf{W}_n$ and the multilinear forms $w_n$ which we have obtained in

Section 4.5, the reduction to symmetric-rank-1 tensors described in Appendix 5, and the demonstration of a Poisson morphism

$$\iota_{\mathfrak{m}} : (\mathcal{S}(\mathbb{R};\mathcal{V}), \mathcal{A}_{\mathcal{S},\mathcal{V}}, \{\cdot,\cdot\}_{L^2,\mathcal{V}}) \to (\mathfrak{G}_\infty^*, \mathcal{A}_\infty, \{\cdot,\cdot\}_{\mathfrak{G}_\infty^*}).$$

We establish the existence of this Poisson morphism in the next subsection.

### 4.6.1 The Mixed State Poisson Morphism

Analogous to Theorem 3.1.12 from Chapter 3, which shows that there is a Poisson morphism between $(\mathcal{S}(\mathbb{R}), \mathcal{A}_{\mathcal{S}}, \{\cdot,\cdot\}_{L^2})$ and $(\mathfrak{G}_\infty^*, \mathcal{A}_\infty, \{\cdot,\cdot\}_{\mathfrak{G}_\infty^*})$ given by

$$\iota(\phi) := (\,|\phi^{\otimes k}\rangle\ \langle\phi^{\otimes k}|)_{k\in\mathbb{N}}, \qquad \forall \phi \in \mathcal{S}(\mathbb{R}) \tag{4.6.2}$$

Theorem 4.1.8 stated below demonstrates that we have a Poisson morphism $\iota_{\mathfrak{m}}$ between the weak Poisson manifolds $(\mathcal{S}(\mathbb{R};\mathcal{V}), \mathcal{A}_{\mathcal{S},\mathcal{V}}, \{\cdot,\cdot\}_{L^2,\mathcal{V}})$ and $(\mathfrak{G}_\infty^*, \mathcal{A}_\infty, \{\cdot,\cdot\}_{\mathfrak{G}_\infty^*})$ given by

$$\iota_{\mathfrak{m}}(\gamma) := \frac{1}{2}(\,|\phi_1^{\otimes k}\rangle\ \langle\phi_2^{\otimes k}| + |\phi_2^{\otimes k}\rangle\ \langle\phi_1^{\otimes k}|)_{k\in\mathbb{N}}, \qquad \forall \gamma = \frac{1}{2}\mathrm{odiag}(\phi_1, \overline{\phi_2}, \phi_2, \overline{\phi_1}) \in \mathcal{S}(\mathbb{R};\mathcal{V}). \tag{4.6.3}$$

**Theorem 4.1.8.** *The map $\iota_{\mathfrak{m}}$ is a Poisson morphism of $(\mathcal{S}(\mathbb{R};\mathcal{V}), \mathcal{A}_{\mathcal{S},\mathcal{V}}, \{\cdot,\cdot\}_{L^2,\mathcal{V}})$ into $(\mathfrak{G}_\infty^*, \mathcal{A}_\infty, \{\cdot,\cdot\}_{\mathfrak{G}_\infty^*})$; i.e., it is a smooth map with the property that*

$$\iota_{\mathfrak{m}}^*\{\cdot,\cdot\}_{\mathfrak{G}_\infty^*} = \{\iota_{\mathfrak{m}}^*\cdot, \iota_{\mathfrak{m}}^*\cdot\}_{L^2,\mathcal{V}}, \tag{4.1.42}$$

*where $\iota_{\mathfrak{m}}^*$ denotes the pullback of $\iota_{\mathfrak{m}}$.*

Before proceeding with the proof of Theorem 4.1.8, we first record the Gâteaux derivative of the map $\iota_{\mathfrak{m}}$, which is used in the proof of the theorem. The computation is an easy exercise relying on multilinearity which we leave to the reader.

**Lemma 4.6.1** (Derivative of $\iota_{\mathfrak{m}}$)**.** *The Gâteaux derivative of the map $\iota_{\mathfrak{m}}$ is given by*

$$d\iota_{\mathfrak{m}}[\gamma](\delta\gamma)^{(k)} = \frac{1}{2}\sum_{\alpha=1}^{k}\Big( |\phi_1^{\otimes(\alpha-1)} \otimes \delta\phi_1 \otimes \phi_1^{\otimes k-\alpha}\rangle \langle \phi_2^{\otimes k}| + |\phi_2^{\otimes k}\rangle \langle \phi_1^{\otimes(\alpha-1)} \otimes \delta\phi_1 \otimes \phi_1^{\otimes(k-\alpha)}| \\ + |\phi_1^{\otimes k}\rangle \langle \phi_2^{\otimes(\alpha-1)} \otimes \delta\phi_2 \otimes \phi_2^{\otimes(k-\alpha)}| + |\phi_2^{\otimes(\alpha-1)} \otimes \delta\phi_2 \otimes \phi_2^{\otimes(k-\alpha)}\rangle \langle \phi_1^{\otimes k}|\Big), \tag{4.6.4}$$

*for every $k \in \mathbb{N}$, where*

$$\gamma = \frac{1}{2}\mathrm{odiag}(\phi_1, \overline{\phi_2}, \phi_2, \overline{\phi_1}),\ \ \delta\gamma = \frac{1}{2}\mathrm{odiag}(\delta\phi_1, \overline{\delta\phi_2}, \delta\phi_2, \overline{\delta\phi_1}) \in \mathcal{S}(\mathbb{R};\mathcal{V}). \tag{4.6.5}$$

We now turn the proof of Theorem 4.1.8.

*Proof of Theorem 4.1.8.* The proof of this result proceeds similarly to the proof of Theorem 3.1.12 from Chapter 3. Smoothness of $\iota_{\mathfrak{m}}$ follows from its multilinear structure, therefore it remains to check that

(i) $\iota_{\mathfrak{m}}^{*}\mathcal{A}_\infty \subset \mathcal{A}_{\mathcal{S},\mathcal{V}}$,

(ii) $\iota_{\mathfrak{m}}^{*}\{\cdot,\cdot\}_{\mathfrak{G}_\infty^*} = \{\iota_{\mathfrak{m}}^{*}\cdot, \iota_{\mathfrak{m}}^{*}\cdot\}_{L^2,\mathcal{V}}$.

We first prove assertion (i). Let $F \in \mathcal{A}_\infty$, and set $f := F \circ \iota_{\mathfrak{m}}$. By the chain rule for the Gâteaux derivative, we have that

$$\begin{aligned}
df[\gamma](\delta\gamma) &= dF[\iota_{\mathfrak{m}}(\gamma)](d\iota_{\mathfrak{m}}[\gamma](\delta\gamma)) \\
&= i\sum_{k=1}^{\infty}\mathrm{Tr}_{1,\dots,k}\big(dF[\iota_{\mathfrak{m}}(\gamma)]^{(k)}d\iota_{\mathfrak{m}}[\gamma](\delta\gamma)^{(k)}\big) \\
&= \frac{i}{2}\sum_{k=1}^{\infty}\mathrm{Tr}_{1,\dots,k}\left(dF[\iota_{\mathfrak{m}}(\gamma)]^{(k)}\,|\sum_{\alpha=1}^{k}\phi_1^{\otimes(\alpha-1)}\otimes\delta\phi_1\otimes\phi_1^{\otimes(k-\alpha)}\rangle\ \langle\phi_2^{\otimes k}|\right) \\
&\qquad\qquad + \mathrm{Tr}_{1,\dots,k}\left(dF[\iota_{\mathfrak{m}}(\gamma)]^{(k)}\,|\phi_2^{\otimes k}\rangle\ \langle\sum_{\alpha=1}^{k}\phi_1^{\otimes(\alpha-1)}\otimes\delta\phi_1\otimes\phi_1^{\otimes(k-\alpha)}|\right) \\
&\qquad\qquad + \mathrm{Tr}_{1,\dots,k}\left(dF[\iota_{\mathfrak{m}}(\gamma)]^{(k)}\,|\sum_{\alpha=1}^{k}\phi_2^{\otimes(\alpha-1)}\otimes\delta\phi_2\otimes\phi_2^{\otimes(k-\alpha)}\rangle\ \langle\phi_1^{\otimes k}|\right) \\
&\qquad\qquad + \mathrm{Tr}_{1,\dots,k}\left(dF[\iota_{\mathfrak{m}}(\gamma)]^{(k)}\,|\phi_1^{\otimes k}\rangle\ \langle\sum_{\alpha=1}^{k}\phi_2^{\otimes(\alpha-1)}\otimes\delta\phi_2\otimes\phi_2^{\otimes(k-\alpha)}|\right),
\end{aligned} \tag{4.6.6}$$

where the ultimate equality follows from application of Lemma 4.6.1.

Next, observe that by Definition 3.2.1 for the generalized trace and Definition 4.1.1 for the good mapping property, we have that

$$\begin{aligned}
&\mathrm{Tr}_{1,\dots,k}\left(dF[\iota_{\mathfrak{m}}(\gamma)]^{(k)}\,|\phi_2^{\otimes k}\rangle\ \langle\sum_{\alpha=1}^{k}\phi_1^{\otimes(\alpha-1)}\otimes\delta\phi_1\otimes\phi_1^{\otimes(k-\alpha)}|\right) \\
&= \left\langle\sum_{\alpha=1}^{k}\phi_1^{\otimes(\alpha-1)}\otimes\delta\phi_1\otimes\phi_1^{\otimes(k-\alpha)}\middle| dF[\iota_{\mathfrak{m}}(\gamma)]^{(k)}\phi_2^{\otimes k}\right\rangle \\
&= \langle\delta\phi_1|\psi_{F,2,k}\rangle\,,
\end{aligned} \tag{4.6.7}$$

where $\psi_{F,2,k}\in\mathcal{S}(\mathbb{R})$ is the necessarily unique Schwartz function coinciding with the antilinear functional

$$\delta\phi_1 \mapsto \left\langle \left\langle \sum_{\alpha=1}^{k} (\cdot) \otimes_\alpha \phi_1^{\otimes(k-1)} \middle| dF[\iota_{\mathfrak{m}}(\gamma)]^{(k)} \phi_2^{\otimes k} \right\rangle \ , \delta\phi_1 \right\rangle_{\mathcal{S}'(\mathbb{R})-\mathcal{S}(\mathbb{R})}$$
$$:= \left\langle \sum_{\alpha=1}^{k} \phi_1^{\otimes(\alpha-1)} \otimes \delta\phi_1 \otimes \phi_1^{\otimes(k-\alpha)} \middle| dF[\iota_{\mathfrak{m}}(\gamma)]^{(k)} \phi_2^{\otimes k} \right\rangle \tag{4.6.8}$$

and where the reader will recall the definition of the notation $\otimes_\alpha$ from (4.4.32). By the same reasoning,

$$\mathrm{Tr}_{1,\ldots,k}\left( dF[\iota_{\mathfrak{m}}(\gamma)]^{(k)} \left|\phi_1^{\otimes k}\right\rangle \left\langle \sum_{\alpha=1}^{k} \phi_2^{\otimes(\alpha-1)} \otimes \delta\phi_2 \otimes \phi_2^{\otimes(k-\alpha)} \right| \right) = \langle \delta\phi_2 | \psi_{F,1,k} \rangle \,, \tag{4.6.9}$$

where $\psi_{F,1,k}$ is the necessarily unique Schwartz function coinciding with the antilinear functional

$$\left\langle \sum_{\alpha=1}^{k} (\cdot) \otimes_\alpha \phi_2^{\otimes(k-1)} \middle| dF[\iota_{\mathfrak{m}}(\gamma)]^{(k)} \phi_1^{\otimes k} \right\rangle . \tag{4.6.10}$$

Next, using that $dF[\iota_{\mathfrak{m}}(\gamma)]^{(k)}$ is skew-adjoint,

$$\begin{aligned}
&\mathrm{Tr}_{1,\ldots,k}\left( dF[\iota_{\mathfrak{m}}(\gamma)]^{(k)} \left|\sum_{\alpha=1}^{k} \phi_1^{\otimes(\alpha-1)} \otimes \delta\phi_1 \otimes \phi_1^{\otimes(k-\alpha)}\right\rangle \left\langle \phi_2^{\otimes k}\right| \right) \\
&= -\overline{\left\langle dF[\iota_{\mathfrak{m}}(\gamma)]\phi_2^{\otimes k} \middle| \sum_{\alpha=1}^{k} \phi_1^{\otimes(\alpha-1)} \otimes \delta\phi_1 \otimes \phi_1^{\otimes(k-\alpha)} \right\rangle} \\
&= -\overline{\left\langle \sum_{\alpha=1}^{k} \phi_1^{\otimes(\alpha-1)} \otimes \delta\phi_1 \otimes \phi_1^{\otimes(k-\alpha)} \middle| dF[\iota_{\mathfrak{m}}(\gamma)]\phi_2^{\otimes k} \right\rangle} \\
&= -\overline{\langle \delta\phi_1 | \psi_{F,2,k} \rangle} \\
&= -\langle \psi_{F,2,k} | \delta\phi_1 \rangle \,.
\end{aligned} \tag{4.6.11}$$

By the same reasoning,

$$\mathrm{Tr}_{1,\ldots,k}\left( dF[\iota_{\mathfrak{m}}(\gamma)]^{(k)} \left|\sum_{\alpha=1}^{k} \phi_2^{\otimes(\alpha-1)} \otimes \delta\phi_2 \otimes \phi_2^{\otimes(k-\alpha)}\right\rangle \left\langle \phi_1^{\otimes k}\right| \right) = -\langle \psi_{F,1,k} | \delta\phi_2 \rangle \,. \tag{4.6.12}$$

Substituting identities (4.6.7), (4.6.9), (4.6.11), and (4.6.12) into (4.6.6), we find that

$$\begin{aligned} df[\iota_{\mathfrak{m}}(\gamma)](\delta\gamma) &= \frac{i}{2}\sum_{k=1}^{\infty}(\langle\delta\phi_1|\psi_{F,2,k}\rangle + \langle\delta\phi_2|\psi_{F,1,k}\rangle - \langle\psi_{F,2,k}|\delta\phi_1\rangle - \langle\psi_{F,1,k}|\delta\phi_2\rangle) \\ &= \frac{i}{2}(\langle\delta\phi_1|\psi_{F,2}\rangle + \langle\delta\phi_2|\psi_{F,1}\rangle - \langle\psi_{F,2}|\delta\phi_1\rangle - \langle\psi_{F,1}|\delta\phi_2\rangle), \end{aligned} \tag{4.6.13}$$

where we have defined $\psi_{F,1} := \sum_{k=1}^{\infty}\psi_{F,1,k}$ and similarly for $\psi_{F,2}$. Note that these are well-defined Schwartz functions since $dF^{(k)}$ is zero for all but finitely many $k$ by assumption that $F \in \mathcal{A}_\infty$ (recall that $\mathcal{A}_\infty$ is generated by the set (4.1.16)). The preceding formula can be rewritten as

$$df[\iota_m(\gamma)](\delta\gamma) = \frac{1}{2}\operatorname{tr}_{\mathbb{C}^2\otimes\mathbb{C}^2}\big(J\mathrm{odiag}(\psi_{F,1}, \overline{\psi_{F,2}}, \psi_{F,2}, \overline{\psi_{F,1}})\mathrm{odiag}(\delta\phi_2, \overline{\delta\phi_1}, \delta\phi_1, \overline{\delta\phi_2})\big), \tag{4.6.14}$$

where $J = \mathrm{diag}(i, -i, i, -i)$. Recalling definition (4.3.17) for the symplectic form $\omega_{L^2,\mathcal{V}}$, we then see from (4.6.14) that the symplectic gradient of $f$ with respect to the form $\omega_{L^2,\mathcal{V}}$, which we denote by $\boldsymbol{\nabla}_{s,\mathcal{V}}f$, is given by

$$\boldsymbol{\nabla}_{s,\mathcal{V}}f(\gamma) = \frac{1}{2}\mathrm{odiag}(\psi_{F,1}, \overline{\psi_{F,2}}, \psi_{F,2}, \overline{\psi_{F,1}}). \tag{4.6.15}$$

That the map

$$\mathcal{S}(\mathbb{R};\mathcal{V}) \to \mathcal{S}(\mathbb{R};\mathcal{V}), \qquad \gamma \mapsto \boldsymbol{\nabla}_{s,\mathcal{V}}f(\gamma) \tag{4.6.16}$$

is smooth follows from the fact that $\gamma$ depends smoothly on $(\psi_{F,1}, \psi_{F,2})$, a consequence of the good mapping property. This completes our verification of assertion (i).

We now verify assertion (ii) using the formula (4.6.15). By definition of the Hamiltonian vector field in (P3) of Definition 3.3.1 together with Proposition 4.1.4, which gives a formula for the vector field $X_G(\iota_{\mathfrak{m}}(\gamma))$, we have that

$$\begin{aligned}
&\{F,G\}_{\mathfrak{G}_\infty^*}(\iota_{\mathfrak{m}}(\gamma)) \\
&= dF[\iota_{\mathfrak{m}}(\gamma)](X_G(\iota_{\mathfrak{m}}(\gamma)) \\
&= \sum_{k=1}^{\infty} i\,\mathrm{Tr}_{1,\ldots,k}\left(dF[\iota_{\mathfrak{m}}(\gamma)]^{(k)}\left(\sum_{j=1}^{\infty} j\,\mathrm{Tr}_{k+1,\ldots,k+j-1}\left(\left[\sum_{\alpha=1}^{k} dG[\iota_{\mathfrak{m}}(\gamma)]^{(j)}_{(\alpha,k+1,\ldots,k+j-1)}, \iota_{\mathfrak{m}}(\gamma)^{(k+j-1)}\right]\right)\right)\right)
\end{aligned}$$

By the bosonic symmetry of $dG[\iota_{\mathfrak{m}}(\gamma)]^{(j)}$,

$$\begin{aligned}
&\sum_{j=1}^{\infty} j\,\mathrm{Tr}_{k+1,\ldots,k+j-1}\left(\left[\sum_{\alpha=1}^{k} dG[\iota_{\mathfrak{m}}(\gamma)]^{(j)}_{(\alpha,k+1,\ldots,k+j-1)}, \iota_{\mathfrak{m}}(\gamma)^{(k+j-1)}\right]\right) \\
&\quad = \sum_{j=1}^{\infty} \mathrm{Tr}_{k+1,\ldots,k+j-1}\left(\left[\sum_{\alpha=1}^{k}\sum_{\beta=1}^{j} dG[\iota_{\mathfrak{m}}(\gamma)]^{(j)}_{(k+1,\ldots,k+\beta-1,\alpha,k+\beta,\ldots,\ldots,k+j-1)}, \iota_{\mathfrak{m}}(\gamma)^{(k+j-1)}\right]\right).
\end{aligned} \tag{4.6.17}$$

It is then a short computation using the Schwartz kernel theorem and the definition of $\iota_{\mathfrak{m}}$ that

$$\begin{aligned}
&\sum_{\beta=1}^{j} dG[\iota_{\mathfrak{m}}(\gamma)]^{(j)}_{(k+1,\ldots,k+\beta-1,\alpha,k+\beta,\ldots,k+j-1)}\iota_{\mathfrak{m}}(\gamma)^{(k+j-1)} \\
&= \frac{1}{2}\Big( |\phi_1^{\otimes(k-1)} \otimes^{\alpha} dG[\iota_{\mathfrak{m}}(\gamma)]^{(j)}(\phi_1^{\otimes j})\rangle\ \langle \phi_2^{\otimes(k+j-1)}| + |\phi_2^{\otimes(k-1)} \otimes^{\alpha} dG[\iota_{\mathfrak{m}}(\gamma)]^{(j)}(\phi_2^{\otimes j})\rangle\ \langle\phi_1^{\otimes(k+j-1)}|\Big),
\end{aligned} \tag{4.6.18}$$

where $\phi_1^{\otimes(k-1)} \otimes^{\alpha} dG[\iota_{\mathfrak{m}}(\gamma)]^{(j)}(\phi_1^{\otimes j})$ is the element of $\mathcal{S}'(\mathbb{R}^{k+j-1})$ defined by

$$\begin{aligned}
&\left(\phi_1^{\otimes(k-1)} \otimes^{\alpha} dG[\iota_{\mathfrak{m}}(\gamma)]^{(j)}(\phi_1^{\otimes j})\right)(\underline{x}_{k+j-1}) \\
&\coloneqq \phi_1^{\otimes(\alpha-1)}(\underline{x}_{\alpha-1})\phi_1^{\otimes(k-\alpha)}(\underline{x}_{\alpha+1;k})\left(\sum_{\beta=1}^{j} dG[\iota_{\mathfrak{m}}(\gamma)]^{(j)}(\phi_1^{\otimes j})(\underline{x}_{k+1;k+\beta-1}, x_\alpha, \underline{x}_{k+\beta;k+j-1})\right),
\end{aligned} \tag{4.6.19}$$

and similarly for $\phi_2^{\otimes(k-1)}\otimes^{\alpha} dG[\iota_{\mathfrak{m}}(\gamma)]^{(j)}(\phi_2^{\otimes j})$. Since $dG[\iota_{\mathfrak{m}}(\gamma)]$ has the good mapping property by assumption that $G \in \mathcal{A}_\infty$, Remark 3.3.4 and the definition of the generalized trace imply that for every $1 \leq \alpha \leq k$,

$$\begin{aligned}
&\mathrm{Tr}_{k+1,\ldots,k+j-1}\left(\sum_{\beta=1}^{j} dG[\iota_{\mathfrak{m}}(\gamma)]^{(j)}_{(k+1,\ldots,k+\beta-1,\alpha,k+\beta,\ldots,k+j-1)}\iota_{\mathfrak{m}}(\gamma)^{(k+j-1)}\right) \\
&= \frac{1}{2}\Big( \,|\phi_1^{\otimes(\alpha-1)} \otimes \psi_{G,1,j} \otimes \phi_1^{\otimes(k-\alpha)}\rangle \,\langle \phi_2^{\otimes k}| + \,|\phi_2^{\otimes(\alpha-1)} \otimes \psi_{G,2,j} \otimes \phi_2^{\otimes(k-\alpha)}\rangle \,\langle \phi_1^{\otimes k}|\Big),
\end{aligned} \tag{4.6.20}$$

where $\psi_{G,1,j}, \psi_{G,2,j} \in \mathcal{S}(\mathbb{R})$ are the necessarily unique Schwartz functions satisfying

$$\langle \phi | \psi_{G,1,j}\rangle = \left\langle \sum_{\beta=1}^{j} \phi \otimes_\beta \phi_2^{\otimes(j-1)} \middle| dG[\iota_{\mathfrak{m}}(\gamma)]^{(j)}\phi_1^{\otimes j}\right\rangle \tag{4.6.21}$$

$$\langle \phi | \psi_{G,2,j}\rangle = \left\langle \sum_{\beta=1}^{j} \phi \otimes_\beta \phi_1^{\otimes(j-1)} \middle| dG[\iota_{\mathfrak{m}}(\gamma)]\phi_2^{\otimes j}\right\rangle, \qquad \forall \phi \in \mathcal{S}(\mathbb{R}). \tag{4.6.22}$$

By repeating the same arguments and now using that the skew-adjointness of $dG[\iota_{\mathfrak{m}}(\gamma)]^{(j)}$, we also obtain that for every $1 \leq \alpha \leq k$,

$$\begin{aligned}
&\mathrm{Tr}_{k+1,\ldots,k+j-1}\left(\sum_{\beta=1}^{j} \iota_{\mathfrak{m}}(\gamma)^{(k+j-1)} dG[\iota_{\mathfrak{m}}(\gamma)]^{(j)}_{(\alpha,k+1,\ldots,k+j-1)}\right) \\
&= -\frac{1}{2}\Big( \,|\phi_1^{\otimes k}\rangle \,\langle \phi_2^{\otimes(\alpha-1)} \otimes \psi_{G,2,j} \otimes \phi_2^{\otimes(k-\alpha)}| + \,|\phi_2^{\otimes k}\rangle \,\langle \phi_1^{\otimes(\alpha-1)} \otimes \psi_{G,1,j} \otimes \phi_1^{\otimes(k-\alpha)}|\Big).
\end{aligned} \tag{4.6.23}$$

Substituting identities (4.6.20) and (4.6.23) into (4.6.17) above, we find that

$$\begin{aligned}
&\{F,G\}_{\mathfrak{G}_\infty^*}(\iota_{\mathfrak{m}}(\gamma))\\
&=\frac{i}{2}\sum_{k=1}^{\infty}\mathrm{Tr}_{1,\ldots,k}\Bigg(dF[\iota_{\mathfrak{m}}(\gamma)]^{(k)}\Bigg(\sum_{j=1}^{\infty}|\sum_{\alpha=1}^{k}\phi_1^{\otimes(\alpha-1)}\otimes\psi_{G,1,j}\otimes\phi_1^{\otimes(k-\alpha)}\rangle\;\langle\phi_2^{\otimes k}|\\
&\qquad\qquad\qquad\qquad\qquad\qquad+|\sum_{\alpha=1}^{k}\phi_2^{\otimes(\alpha-1)}\otimes\psi_{G,2,j}\otimes\phi_2^{\otimes(k-\alpha)}\rangle\;\langle\phi_1^{\otimes k}|\Bigg)\Bigg)\\
&\quad+\frac{i}{2}\sum_{k=1}^{\infty}\mathrm{Tr}_{1,\ldots,k}\Bigg(dF[\iota_{\mathfrak{m}}(\gamma)]^{(k)}\Bigg(\sum_{j=1}^{\infty}|\phi_2^{\otimes k}\rangle\;\langle\sum_{\alpha=1}^{k}\phi_1^{\otimes(\alpha-1)}\otimes\psi_{G,1,j}\otimes\phi_1^{\otimes(k-\alpha)}|\\
&\qquad\qquad\qquad\qquad\qquad\qquad+|\phi_1^{\otimes k}\rangle\;\langle\sum_{\alpha=1}^{k}\phi_2^{\otimes(\alpha-1)}\otimes\psi_{G,2,j}\otimes\phi_2^{\otimes(k-\alpha)}|\Bigg)\Bigg)\\
&=\frac{i}{2}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\left\langle\phi_2^{\otimes k}\middle|dF[\iota_{\mathfrak{m}}(\gamma)]^{(k)}\left(\sum_{\alpha=1}^{k}\phi_1^{\otimes(\alpha-1)}\otimes\psi_{G,1,j}\otimes\phi_1^{\otimes(k-\alpha)}\right)\right\rangle\\
&\qquad\qquad+\left\langle\phi_1^{\otimes k}\middle|dF[\iota_{\mathfrak{m}}(\gamma)]^{(k)}\left(\sum_{\alpha=1}^{k}\phi_2^{\otimes(\alpha-1)}\otimes\psi_{G,2,j}\otimes\phi_2^{\otimes(k-\alpha)}\right)\right\rangle\\
&\qquad\qquad+\left\langle\sum_{\alpha=1}^{k}\phi_1^{\otimes(\alpha-1)}\otimes\psi_{G,1,j}\otimes\phi_1^{\otimes(k-\alpha)}\middle|dF[\iota_{\mathfrak{m}}(\gamma)]^{(k)}\phi_2^{\otimes k}\right\rangle\\
&\qquad\qquad+\left\langle\sum_{\alpha=1}^{k}\phi_2^{\otimes(\alpha-1)}\otimes\psi_{G,2,j}\otimes\phi_2^{\otimes(k-\alpha)}\middle|dF[\iota_{\mathfrak{m}}(\gamma)]^{(k)}\phi_1^{\otimes k}\right\rangle,
\end{aligned}\tag{4.6.24}$$

where the ultimate equality is immediate from the definition of the generalized trace. Recalling the definitions of $\psi_{F,1,k}$ and $\psi_{F,2,k}$ in (4.6.7) and (4.6.9), respectively, we have that

$$\left\langle\sum_{\alpha=1}^{k}\phi_1^{\otimes(\alpha-1)}\otimes\psi_{G,1,j}\otimes\phi_1^{\otimes(k-\alpha)}\middle|dF[\iota_{\mathfrak{m}}(\gamma)]^{(k)}\phi_2^{\otimes k}\right\rangle=\langle\psi_{G,1,j}|\psi_{F,2,k}\rangle\,,\tag{4.6.25}$$

$$\left\langle\sum_{\alpha=1}^{k}\phi_2^{\otimes(\alpha-1)}\otimes\psi_{G,2,j}\otimes\phi_2^{\otimes(k-\alpha)}\middle|dF[\iota_{\mathfrak{m}}(\gamma)]^{(k)}\phi_1^{\otimes k}\right\rangle=\langle\psi_{G,2,j}|\psi_{F,1,k}\rangle\,.\tag{4.6.26}$$

Now using the skew-adjointness of $dF[\iota_{\mathfrak{m}}(\gamma)]^{(k)}$, we find that

$$\begin{aligned}
&\overline{\left\langle \phi_2^{\otimes k} \middle| dF[\iota_{\mathfrak{m}}(\gamma)]^{(k)} \left( \sum_{\alpha=1}^{k} \phi_1^{\otimes(\alpha-1)} \otimes \psi_{G,1,j} \otimes \phi_1^{\otimes(k-\alpha)} \right) \right\rangle} \\
&= -\left\langle \sum_{\alpha=1}^{k} \phi_1^{\otimes(\alpha-1)} \otimes \psi_{G,1,j} \otimes \phi_1^{\otimes(k-\alpha)} \middle| dF[\iota_{\mathfrak{m}}(\gamma)]^{(k)} \phi_2^{\otimes k} \right\rangle \\
&= -\left\langle \psi_{F,2,k} | \psi_{G,1,j} \right\rangle . \qquad (4.6.27)
\end{aligned}$$

Similarly,

$$\left\langle \phi_1^{\otimes k} \middle| dF[\iota_{\mathfrak{m}}(\gamma)]^{(k)} \left( \sum_{\alpha=1}^{k} \phi_2^{\otimes(\alpha-1)} \otimes \psi_{G,2,j} \otimes \phi_2^{\otimes(k-\alpha)} \right) \right\rangle = -\left\langle \psi_{F,1,k} | \psi_{G,2,j} \right\rangle . \qquad (4.6.28)$$

Hence,

$$\begin{aligned}
\{F, G\}_{\mathfrak{G}_\infty^*}(\iota_{\mathfrak{m}}(\gamma)) &= \frac{i}{2} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \langle \psi_{G,1,j} | \psi_{F,2,k} \rangle + \langle \psi_{G,2,j} | \psi_{F,1,k} \rangle - \langle \psi_{F,2,k} | \psi_{G,1,j} \rangle - \langle \psi_{F,1,k} | \psi_{G,2,j} \rangle \\
&= \frac{i}{2} (\langle \psi_{G,1} | \psi_{F,2} \rangle + \langle \psi_{G,2} | \psi_{F,1} \rangle - \langle \psi_{F,2} | \psi_{G,1} \rangle - \langle \psi_{F,1} | \psi_{G,2} \rangle), \qquad (4.6.29)
\end{aligned}$$

where we have defined $\psi_{F,\ell} := \sum_{k=1}^{\infty} \psi_{F,\ell,k}$, for $\ell \in \{1, 2\}$, and similarly for $\psi_{G,\ell}$. Note that these are well-defined elements of $\mathcal{S}(\mathbb{R})$ since $\psi_{F,\ell,k}, \psi_{G,\ell,j}$ are identically zero for all but finitely many $k, j$. By (4.6.15), we know that

$$\begin{aligned}
\nabla_{s,\mathcal{V}} f(\gamma) &= \frac{1}{2} \mathrm{odiag}(\psi_{F,1}, \overline{\psi_{F,2}}, \psi_{F,2}, \overline{\psi_{F,1}}), \qquad (4.6.30) \\
\nabla_{s,\mathcal{V}} g(\gamma) &= \frac{1}{2} \mathrm{odiag}(\psi_{G,1}, \overline{\psi_{G,2}}, \psi_{G,2}, \overline{\psi_{G,1}}). \qquad (4.6.31)
\end{aligned}$$

Hence by recalling the definition (4.3.17) for the symplectic form $\omega_{L^2,\mathcal{V}}$ and Proposition 4.3.2, then proceeding by direct computation, we find that

$$\begin{aligned}
&\{f,g\}_{L^2,\mathcal{V}}(\gamma)\\
&= \omega_{L^2,\mathcal{V}}(\boldsymbol{\nabla}_{s,\mathcal{V}} f(\gamma), \boldsymbol{\nabla}_{s,\mathcal{V}} g(\gamma))\\
&= \frac{1}{2}\int_{\mathbb{R}} dx\, \mathrm{tr}_{\mathbb{C}^2\otimes\mathbb{C}^2}\Big(\mathrm{diag}(i,-i,i,-i)\mathrm{odiag}(\psi_{F,1},\overline{\psi_{F,2}},\psi_{F,2},\overline{\psi_{F,1}})\mathrm{odiag}(\psi_{G,2},\overline{\psi_{G,1}},\psi_{G,1},\overline{\psi_{G,2}})\Big)(x)\\
&= (4.6.29).
\end{aligned} \tag{4.6.32}$$

Therefore, we have shown that

$$\{F,G\}_{\mathfrak{G}_\infty^*}(\iota_{\mathfrak{m}}(\gamma)) = \{f,g\}_{L^2,\mathcal{V}}(\gamma), \tag{4.6.33}$$

completing the proof. □

### 4.6.2 Relating the Functionals $\mathcal{H}_n$ and $I_{b,n}$

We now use the analysis of Section 4.5.3 to relate the functionals $\mathcal{H}_n$, defined in (4.1.33), on the infinite-particle phase space $\mathfrak{G}_\infty^*$ to the functionals $I_{b,n}$, defined in (4.1.40), on the one-particle mixed-state phase space $\mathcal{S}(\mathbb{R};\mathcal{V})$, defined in (4.1.39).

**Proposition 4.6.2.** *For every $n \in \mathbb{N}$, it holds that*

$$\mathcal{H}_n(\iota_{\mathfrak{m}}(\gamma)) = I_{b,n}(\gamma), \qquad \forall \gamma \in \mathcal{S}(\mathbb{R};\mathcal{V}). \tag{4.6.34}$$

*Proof.* Fix $n \in \mathbb{N}$ and let $\gamma = \frac{1}{2}\mathrm{odiag}(\phi_1,\overline{\phi_2},\phi_2,\overline{\phi_1})$, for $\phi_1,\phi_2 \in \mathcal{S}(\mathbb{R})$. Unpacking the definition (4.1.33) of $\mathcal{H}_n$, the definition (4.1.32) for $\mathbf{W}_n$, and the bilinearity of the generalized trace, we see that

$$\begin{aligned}
\mathcal{H}_n(\iota_{\mathfrak{m}}(\gamma)) = \frac{1}{4}\sum_{k=1}^\infty \frac{1}{k!}\sum_{\pi\in\mathbb{S}_k} &\mathrm{Tr}_{1,\ldots,k}\Big(\widetilde{\mathbf{W}}^{(k)}_{n,(\pi(1),\ldots,\pi(k))}\,|\phi_1^{\otimes k}\rangle\,\langle\phi_2^{\otimes k}|\Big) + \mathrm{Tr}_{1,\ldots,k}\Big(\widetilde{\mathbf{W}}^{(k)}_{n,(\pi(1),\ldots,\pi(k))}\,|\phi_2^{\otimes k}\rangle\,\langle\phi_1^{\otimes k}|\Big)\\
&+ \mathrm{Tr}_{1,\ldots,k}\Big(\widetilde{\mathbf{W}}^{(k),*}_{n,(\pi(1),\ldots,\pi(k))}\,|\phi_1^{\otimes k}\rangle\,\langle\phi_2^{\otimes k}|\Big) + \mathrm{Tr}_{1,\ldots,k}\Big(\widetilde{\mathbf{W}}^{(k),*}_{n,(\pi(1),\ldots,\pi(k))}\,|\phi_2^{\otimes k}\rangle\,\langle\phi_1^{\otimes k}|\Big).
\end{aligned} \tag{4.6.35}$$

By Corollary 4.5.8, we have the identities

$$\begin{aligned}
\mathrm{Tr}_{1,\ldots,k}\left(\widetilde{\mathbf{W}}^{(k)}_{n,(\pi(1),\ldots,\pi(k))}\left|\phi_1^{\otimes k}\right\rangle\left\langle\phi_2^{\otimes k}\right|\right) &= I_n^{(k)}(\phi_1^{\times k};\overline{\phi_2}^{\times k}),\\
\mathrm{Tr}_{1,\ldots,k}\left(\widetilde{\mathbf{W}}^{(k)}_{n,(\pi(1),\ldots,\pi(k))}\left|\phi_2^{\otimes k}\right\rangle\left\langle\phi_1^{\otimes k}\right|\right) &= I_n^{(k)}(\phi_2^{\times k};\overline{\phi_1}^{\times k}),\\
\mathrm{Tr}_{1,\ldots,k}\left(\widetilde{\mathbf{W}}^{(k),*}_{n,(\pi(1),\ldots,\pi(k))}\left|\phi_1^{\otimes k}\right\rangle\left\langle\phi_2^{\otimes k}\right|\right) &= \overline{I_n^{(k)}(\phi_2^{\times k};\overline{\phi_1}^{\times k})},\\
\mathrm{Tr}_{1,\ldots,k}\left(\widetilde{\mathbf{W}}^{(k),*}_{n,(\pi(1),\ldots,\pi(k))}\left|\phi_2^{\otimes k}\right\rangle\left\langle\phi_1^{\otimes k}\right|\right) &= \overline{I_n^{(k)}(\phi_1^{\times k};\overline{\phi_2}^{\times k})}.
\end{aligned} \tag{4.6.36}$$

for every $k \in \mathbb{N}$ and $\pi \in \mathbb{S}_k$. Consequently, by Remark 4.5.4,

$$\begin{aligned}
\mathcal{H}_n(\iota_{\mathfrak{m}}(\gamma)) &= \frac{1}{4}\sum_{k=1}^{\infty}\left(I_n^{(k)}(\phi_1^{\times k};\overline{\phi_2}^{\times k}) + I_n^{(k)}(\phi_2^{\times k};\overline{\phi_1}^{\times k}) + \overline{I_n^{(k)}(\phi_2^{\times k};\overline{\phi_1}^{\times k})} + \overline{I_n^{(k)}(\phi_1^{\times k};\overline{\phi_2}^{\times k})}\right)\\
&= \frac{1}{4}\left(\tilde{I}_n(\phi_1,\overline{\phi_2}) + \tilde{I}_n(\phi_2,\overline{\phi_1}) + \overline{\tilde{I}_n(\phi_1,\overline{\phi_2})} + \overline{\tilde{I}_n(\phi_2,\overline{\phi_1})}\right).
\end{aligned} \tag{4.6.37}$$

By (1.2.26), we know that the $\tilde{I}_n$ have the involution property

$$\tilde{I}_n(f,\overline{g}) = \overline{\tilde{I}_n(g,\overline{f})}, \qquad \forall f,g \in \mathcal{S}(\mathbb{R}). \tag{4.6.38}$$

So, we obtain by the definition of $I_{b,n}$ in (4.1.40) that

$$\mathcal{H}_n(\iota_{\mathfrak{m}}(\gamma)) = \frac{1}{2}\left(\tilde{I}_n(\phi_1,\overline{\phi_2}) + \tilde{I}_n(\phi_2,\overline{\phi_1})\right) = I_{b,n}(\gamma), \tag{4.6.39}$$

as required. □

### 4.6.3 Proof of Theorem 4.1.7 and Theorem 4.1.9

The goal of this subsection is to complete the proof of Theorem 4.1.7:

**Theorem 4.1.7** (Involution theorem)**.** *Let $n, m \in \mathbb{N}$. Then*

$$\{\mathcal{H}_n, \mathcal{H}_m\}_{\mathfrak{G}_\infty^*} \equiv 0 \ \textit{on}\ \mathfrak{G}_\infty^*. \tag{4.1.36}$$

As detailed in the introduction, we will establish Theorem 4.1.7 by proving Theorem 4.1.9, the statement of which we recall here.

**Theorem 4.1.9** (Poisson commutativity equivalence). *For any $n, m \in \mathbb{N}$,*

$$\{I_{b,n}, I_{b,m}\}_{L^2,\mathcal{V}}(\gamma) = 0, \qquad \forall \gamma \in \mathcal{S}(\mathbb{R};\mathcal{V}), \tag{4.1.48}$$

*if and only if*

$$\{\mathcal{H}_n, \mathcal{H}_m\}_{\mathfrak{G}_\infty^*}(\Gamma) = 0, \qquad \forall \Gamma \in \mathfrak{G}_\infty^*. \tag{4.1.49}$$

We refer to (4.1.40) for the definition of $I_{b,n}$. In light of Proposition 1.3.7 which establishes the validity of (4.1.48), Theorem 4.1.7 is then an immediate corollary of Theorem 4.1.9. Thus we focus on proving Theorem 4.1.9.

*Proof of Theorem 4.1.9.* The implication that

$$\{\mathcal{H}_n, \mathcal{H}_m\}_{\mathfrak{G}_\infty^*} \equiv 0 \Longrightarrow \{I_{b,n}, I_{b,m}\}_{L^2,\mathcal{V}} \equiv 0$$

is a consequence of Theorem 4.1.8 and Proposition 4.6.2. Indeed, the latter states that

$$\mathcal{H}_n(\iota_{\mathfrak{m}}(\gamma)) = I_{b,n}(\gamma),$$

and hence by Theorem 4.1.8, we have

$$\{I_{b,n}, I_{b,m}\}_{L^2,\mathcal{V}}(\gamma) = \{\mathcal{H}_n, \mathcal{H}_m\}_{\mathfrak{G}_\infty^*}(\iota_{\mathfrak{m}}(\gamma)) = 0.$$

To show the reverse implication, we first claim that it suffices to show that

$$\{\mathcal{H}_n, \mathcal{H}_m\}_{\mathfrak{G}_\infty^*}(\Gamma) = 0, \quad \forall \Gamma = (\gamma^{(k)})_{k\in\mathbb{N}},\ \gamma^{(k)} = \frac{1}{2}\big(|f_k^{\otimes k}\rangle\,\langle g_k^{\otimes k}| + |g_k^{\otimes k}\rangle\,\langle f_k^{\otimes k}|\big),\ f_k, g_k \in \mathcal{S}(\mathbb{R}). \tag{4.6.40}$$

Indeed, for any $k \in \mathbb{N}$, Corollary 5.0.24 gives that finite linear combinations of the form

$$\sum_{j=1}^{N_k} \frac{a_j}{2}\big( |f_j^{\otimes k}\rangle \langle g_j^{\otimes k}| + |g_j^{\otimes k}\rangle \langle f_j^{\otimes k}|\big), \quad a_j \in \mathbb{C},\ f_j, g_j \in \mathcal{S}(\mathbb{R}),\ N_k \in \mathbb{N} \tag{4.6.41}$$

are dense in $\mathfrak{g}_k^*$ (recall (4.1.14)). Since by definition $\mathfrak{G}_\infty^*$ is the topological direct product of the $\mathfrak{g}_k^*$ (recall (4.1.15)), elements $\Gamma = (\gamma^{(k)})_{k\in\mathbb{N}} \in \mathfrak{G}_\infty^*$ of the form

$$\gamma^{(k)} = \sum_{j=1}^{\infty} \frac{a_{jk}}{2}\big( |f_{jk}^{\otimes k}\rangle \langle g_{jk}^{\otimes k}| + |g_{jk}^{\otimes k}\rangle \langle f_{jk}^{\otimes k}|\big), \qquad k \in \mathbb{N}, \tag{4.6.42}$$

where $f_{jk}, g_{jk} \in \mathcal{S}(\mathbb{R})$ and $a_{jk} \in \mathbb{C}$ with $a_{jk} = 0$ for all but finitely many $j \in \mathbb{N}$, are dense in $\mathfrak{G}_\infty^*$. Now recalling the definition (4.1.17) for the Poisson bracket $\{\mathcal{H}_n, \mathcal{H}_m\}_{\mathfrak{G}_\infty^*}$ and using the bilinearity of the generalized trace, we need to show that for $\Gamma$ in the form of (4.6.42),

$$\begin{aligned} 0 &= \{\mathcal{H}_n, \mathcal{H}_m\}_{\mathfrak{G}_\infty^*}(\Gamma) \\ &= \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \frac{i a_{jk}}{2} \operatorname{Tr}_{1,\ldots,k}\Big( [-i\mathbf{W}_n, -i\mathbf{W}_m]_{\mathfrak{G}_\infty}^{(k)} \big( |f_{jk}^{\otimes k}\rangle \langle g_{jk}^{\otimes k}| + |g_{jk}^{\otimes k}\rangle \langle f_{jk}^{\otimes k}|\big)\Big) \\ &= \sum_{j=1}^{\infty} a_{jk} \{\mathcal{H}_n, \mathcal{H}_m\}_{\mathfrak{G}_\infty^*}(\Gamma_j), \end{aligned} \tag{4.6.43}$$

where

$$\Gamma_j = (\gamma_j^{(k)})_{k\in\mathbb{N}}, \qquad \gamma_j^{(k)} := \frac{1}{2}\big( |f_{jk}^{\otimes k}\rangle \langle g_{jk}^{\otimes k}| + |g_{jk}^{\otimes k}\rangle \langle f_{jk}^{\otimes k}|\big). \tag{4.6.44}$$

Note that because $[-i\mathbf{W}_n, -i\mathbf{W}_m]_{\mathfrak{G}_\infty}^{(k)}$ is zero for all but finitely many $k$, and for each fixed $k \in \mathbb{N}$, $a_{jk}$ is zero for all but finitely many $j$, it follows that there are only finitely many nonzero terms in the double series above, and consequently, there are no issues of convergence. (4.6.40) will imply that each summand in (4.6.43) is zero, so by continuity of $\{\mathcal{H}_n, \mathcal{H}_m\}_{\mathfrak{G}_\infty^*}$ and by density of elements of the form (4.6.42) in $\mathfrak{G}_\infty^*$, we arrive at the desired implication.

Thus, we proceed to show (4.6.40). Unpacking the definition of $\{\mathcal{H}_n, \mathcal{H}_m\}_{\mathfrak{G}^*_\infty}(\Gamma)$, we see that

$$\{\mathcal{H}_n, \mathcal{H}_m\}_{\mathfrak{G}^*_\infty}(\Gamma) = \frac{i}{2}\sum_{k=1}^{\infty} \mathrm{Tr}_{1,\ldots,k}\Big([-i\mathbf{W}_n, -i\mathbf{W}_m]^{(k)}_{\mathfrak{G}_\infty}\big(\,|f_k^{\otimes k}\rangle\,\langle g_k^{\otimes k}| + |g_k^{\otimes k}\rangle\,\langle f_k^{\otimes k}|\big)\Big) \quad (4.6.45)$$

For each $k \in \mathbb{N}$ and $\lambda \in \mathbb{C}$, consider the element $\gamma_{k,\lambda} \in \mathcal{S}(\mathbb{R}; \mathcal{V})$ defined by

$$\gamma_{k,\lambda} \coloneqq \frac{1}{2}\mathrm{odiag}(\lambda f_k, \overline{\lambda g_k}, \lambda g_k, \overline{\lambda f_k}) \quad (4.6.46)$$

Then by the assumption (4.1.48) and Theorem 4.1.8,

$$\begin{aligned}
0 = \{I_{b,n}, I_{b,m}\}_{L^2,\mathcal{V}}(\gamma_{k,\lambda}) &= \{\mathcal{H}_n, \mathcal{H}_m\}_{\mathfrak{G}^*_\infty}(\iota_{\mathfrak{m}}(\gamma_{k,\lambda})) \\
&= \sum_{j=1}^{\infty} i\,\mathrm{Tr}_{1,\ldots,j}\Big([-i\mathbf{W}_n, -i\mathbf{W}_m]^{(j)}_{\mathfrak{G}_\infty}\,\iota_{\mathfrak{m}}(\gamma_{k,\lambda})^{(j)}\Big) \\
&= \frac{i}{2}\sum_{j=1}^{\infty} |\lambda|^{2j}\,\mathrm{Tr}_{1,\ldots,j}\Big([-i\mathbf{W}_n, -i\mathbf{W}_m]^{(j)}_{\mathfrak{G}_\infty}\big(\,|f_k^{\otimes j}\rangle\,\langle g_k^{\otimes j}| + |g_k^{\otimes j}\rangle\,\langle f_k^{\otimes j}|\big)\Big) \\
&=: \frac{i}{2}\rho_k(\lambda). \qquad (4.6.47)
\end{aligned}$$

$\rho_k$ is well-defined on $\mathbb{C}$, since there are only finitely many indices $j$ for which the summand is nonzero. Since for any $r \in \mathbb{N}$,

$$0 = ((\partial_\lambda \partial_{\overline{\lambda}})^r \rho_k)(0) = r!\,\mathrm{Tr}_{1,\ldots,r}\Big([-i\mathbf{W}_n, -i\mathbf{W}_m]^{(r)}_{\mathfrak{G}_\infty}\big(\,|f_k^{\otimes r}\rangle\,\langle g_k^{\otimes r}| + |g_k^{\otimes r}\rangle\,\langle f_k^{\otimes r}|\big)\Big), \quad (4.6.48)$$

it follows that

$$\mathrm{Tr}_{1,\ldots,k}\Big([-i\mathbf{W}_n, -i\mathbf{W}_m]^{(k)}_{\mathfrak{G}_\infty}\big(\,|f_k^{\otimes k}\rangle\,\langle g_k^{\otimes k}| + |g_k^{\otimes k}\rangle\,\langle f_k^{\otimes k}|\big)\Big) = 0. \quad (4.6.49)$$

Therefore, each summand in the right-hand side of (4.6.45) vanishes, yielding (4.6.40). Thus, the proof of Theorem 4.1.8 is complete. □

### 4.6.4 Nontriviality

In this subsection, we prove that the statement of Theorem 4.1.7 is nontrivial in the sense that the functionals $\mathcal{H}_n$ do not Poisson commute with every element of $\mathcal{A}_\infty$. The proof of this fact proceeds by a reduction to proving a one-particle result.

**Proposition 4.6.3.** *For every $n \in \mathbb{N}$, there exists a functional $F \in \mathcal{A}_\infty$ and an element $\Gamma \in \mathfrak{G}_\infty^*$ such that*

$$\{F, \mathcal{H}_n\}_{\mathfrak{G}_\infty^*}(\Gamma) \neq 0. \tag{4.6.50}$$

*Proof.* We proceed by contradiction and suppose that for every $F \in \mathcal{A}_\infty$, it holds that $\{F, \mathcal{H}_n\}_{\mathfrak{G}_\infty^*} \equiv 0$ on $\mathfrak{G}_\infty^*$. So by the Definition 3.3.1(P3) for the Hamiltonian vector field, we have that

$$0 = \{F, \mathcal{H}_n\}_{\mathfrak{G}_\infty^*}(\Gamma) = dF[\Gamma](X_{\mathcal{H}_n}(\Gamma)). \tag{4.6.51}$$

By duality, it follows that $X_{\mathcal{H}_n} \equiv 0$ on $\mathfrak{G}_\infty^*$. In particular, for any pure state $\Gamma = \iota(\phi)$, where $\iota$ is as in (4.6.2) and $\phi \in \mathcal{S}(\mathbb{R})$, we have by Theorem 4.1.10 (to be proved in the next section) that

$$X_{\mathcal{H}_n}(\iota(\phi))^{(1)} = |\phi\rangle \langle \boldsymbol{\nabla}_s I_n(\phi)| + |\boldsymbol{\nabla}_s I_n(\phi)\rangle \langle\phi| = 0 \in \mathfrak{g}_1^*. \tag{4.6.52}$$

Taking the 1-particle trace of the right-hand side and using the characterization of the symplectic gradient (see Definition 3.3.11), we obtain that

$$0 = dI_n[\phi](\phi) = \sum_{k=1}^{\infty} 2k I_n^{(k)}[\phi^{\times k}; \overline{\phi}^{\times k}], \tag{4.6.53}$$

where the ultimate equality follows by direct computation. However, (4.6.53) is a contradiction by Lemma 4.5.3, and therefore the proof is complete. $\square$

## 4.7 The Equations of Motion: $n$GP and $n$NLS

In this last section, we prove Theorem 4.1.10. Before recalling the statement of this theorem, we first recall that for each $n \in \mathbb{N}$, the Hamiltonian functionals $\mathcal{H}_n$ are given by the formula

$$\mathcal{H}_n(\Gamma) \coloneqq \mathrm{Tr}(\mathbf{W}_n \cdot \Gamma), \qquad \forall \Gamma \in \mathfrak{G}_\infty^* \tag{4.7.1}$$

and the Hamiltonian equation of motion defined by the functional $\mathcal{H}_n$ on $\mathfrak{G}_\infty^*$, which we have called the *$n$-th GP-hierarchy (nGP)*, is given by

$$\frac{d}{dt}\Gamma = X_{\mathcal{H}_n}(\Gamma), \tag{4.7.2}$$

where $X_{\mathcal{H}_n}$ is the Hamiltonian vector field associated to $\mathcal{H}_n$.

**Theorem 4.1.10** (Connection between (nGP) and (nNLS))**.** *Let $n \in \mathbb{N}$. Let $I \subset \mathbb{R}$ be a compact interval and let $\phi \in C^\infty(I; \mathcal{S}(\mathbb{R}))$ be a solution to the (nNLS) with lifespan $I$. If we define*

$$\Gamma \in C^\infty(I; \mathfrak{G}_\infty^*), \qquad \Gamma \coloneqq \left( |\phi^{\otimes k}\rangle \, \langle \phi^{\otimes k}| \right)_{k \in \mathbb{N}}, \tag{4.1.51}$$

*then $\Gamma$ is a solution to the (nGP).*

Theorem 4.1.10 asserts that (nGP) admits a special class of factorized solutions of the form

$$\Gamma = (\gamma^{(k)})_{k \in \mathbb{N}}, \qquad \gamma^{(k)} \coloneqq |\phi^{\otimes k}\rangle \, \langle \phi^{\otimes k}| \,, \qquad \phi \in C^\infty(I; \mathcal{S}(\mathbb{R})), \tag{4.7.3}$$

where $\phi$ solves the $n$-th nonlinear Schrödinger equation (nNLS):

$$\left(\frac{d}{dt}\phi\right)(t) = \boldsymbol{\nabla}_s I_n(\phi(t)), \qquad \forall t \in I, \tag{4.7.4}$$

and where $\boldsymbol{\nabla}_s$ is the symplectic gradient with respect to the $L^2$ standard symplectic structure (recall Definition 3.3.11 and Remark 3.3.12). We note that existence and uniqueness for the (nNLS) equation in the class $C^\infty(I;\mathcal{S}(\mathbb{R}))$ follows from the inverse scattering results of [8, 102, 103].

### 4.7.1 nGP Hamiltonian Vector Fields

We first relate the formula given by Proposition 4.1.4 for the Hamiltonian vector field $X_{\mathcal{H}_n}$ to the nonlinear operators $w_n$. This connection underpins the proof of Theorem 4.1.10. For $n \in \mathbb{N}$, Proposition 4.1.4 gives

$$X_{\mathcal{H}_n}(\Gamma)^{(\ell)} = \sum_{j=1}^{\infty} j \operatorname{Tr}_{\ell+1,\ldots,\ell+j-1}\left(\left[\sum_{\alpha=1}^{\ell}(-i\mathbf{W}_n)^{(j)}_{(\alpha,\ell+1,\ldots,\ell+j-1)}, \gamma^{(\ell+j-1)}\right]\right), \qquad \ell \in \mathbb{N}, \ \Gamma \in \mathfrak{G}^*_\infty. \tag{4.7.5}$$

The main lemma is a formula for

$$\operatorname{Tr}_{\ell+1,\ldots,\ell+j-1}\left(\left[\sum_{\alpha=1}^{\ell}(-i\mathbf{W}_n)^{(j)}_{(\alpha,\ell+1,\ldots,\ell+j-1)}, \gamma^{(\ell+j-1)}\right]\right)$$

in the special case where $\gamma^{(\ell+j-1)}$ is a mixed state, i.e.

$$\gamma^{(\ell+j-1)} = \frac{1}{2}\left(\left|f^{\otimes(\ell+j-1)}\right\rangle \left\langle g^{\otimes(\ell+j-1)}\right| + \left|g^{\otimes(\ell+j-1)}\right\rangle \left\langle f^{\otimes(\ell+j-1)}\right|\right), \qquad f,g \in \mathcal{S}(\mathbb{R}). \tag{4.7.6}$$

**Lemma 4.7.1.** *Let $\ell, j \in \mathbb{N}$. Suppose that $\gamma^{(\ell+j-1)}$ is of the form (4.7.6). Then for any*

*$\alpha \in \mathbb{N}_{\leq \ell}$ and $\beta \in \mathbb{N}_{\leq j}$, it holds that*

$$\begin{aligned}
&\mathrm{Tr}_{\ell+1,\ldots,\ell+j-1}\Big((\mathbf{W}_{n,sa})^{(j)}_{(\ell+1,\ldots,\ell+\beta-1,\alpha,\ell+\beta,\ldots,\ell+j-1)}\gamma^{(\ell+j-1)}\Big)(\underline{x}_\ell;\underline{x}'_\ell) \\
&= \frac{1}{4}f^{\otimes(\ell-1)}(\underline{x}_{\alpha-1},\underline{x}_{\alpha+1;\ell})\overline{g^{\otimes \ell}(\underline{x}'_\ell)} \\
&\qquad \times \left(w^{(j),t}_{n,\beta'}[f^{\times j};\overline{g}^{\times(j-1)}](x_\alpha) + \overline{w^{(j),t}_{n,\beta}[g^{\times(\beta-1)},\overline{f},g^{(j-\beta)};\overline{f}^{\times(j-1)}](x_\alpha)}\right), \\
&\quad + \frac{1}{4}g^{\otimes(\ell-1)}(\underline{x}_{\alpha-1},\underline{x}_{\alpha+1;\ell})\overline{f^{\otimes \ell}(\underline{x}'_\ell)} \\
&\qquad \times \left(w^{(j),t}_{n,\beta'}[g^{\times j};\overline{f}^{\times(j-1)}](x_\alpha) + \overline{w^{(j),t}_{n,\beta}[f^{\times(\beta-1)},\overline{g},f^{\times(j-\beta)};\overline{g}^{\times(j-1)}](x_\alpha)}\right)
\end{aligned} \tag{4.7.7}$$

*and*

$$\begin{aligned}
&\mathrm{Tr}_{\ell+1,\ldots,\ell+j-1}\Big(\gamma^{(\ell+j-1)}(\mathbf{W}_{n,sa})^{(j)}_{(\ell+1,\ldots,\ell+\beta-1,\alpha,\ell+\beta,\ldots,\ell+j-1)}\Big)(\underline{x}_\ell;\underline{x}'_\ell) \\
&= \frac{1}{4}g^{\otimes \ell}(\underline{x}_\ell)\overline{f^{\otimes(\ell-1)}(\underline{x}'_{\alpha-1},\underline{x}'_{\alpha+1;\ell})} \\
&\qquad \times \left(\overline{w^{(j),t}_{n,\beta'}[f^{\times j};\overline{g}^{\times(j-1)}](x'_\alpha)} + w^{(j),t}_{n,\beta}[g^{\times(\beta-1)},\overline{f},g^{\times(j-\beta)};\overline{f}^{\times(j-1)}](x'_\alpha)\right). \\
&\quad + \frac{1}{4}f^{\otimes \ell}(\underline{x}_\ell)\overline{g^{\otimes(\ell-1)}(\underline{x}'_{\alpha-1},\underline{x}'_{\alpha+1;\ell})} \\
&\qquad \times \left(\overline{w^{(j),t}_{n,\beta'}[g^{\times j};\overline{f}^{\times(j-1)}](x'_\alpha)} + w^{(j),t}_{n,\beta}[f^{\times(\beta-1)},\overline{g},f^{\times(j-\beta)};\overline{g}^{\times(j-1)}](x'_\alpha)\right)
\end{aligned} \tag{4.7.8}$$

*In all cases, equality holds in the sense of tempered distributions.*

*Proof.* By considerations of symmetry, it suffices to consider the case $\alpha = \ell$. Then by Proposition 3.3.1 for the $(\ell + j - 1)$-particle extension, Proposition 3.2.4 for the generalized partial trace, and the definition (4.4.74) for $\mathbf{W}_{n,sa}$, we find that

$$\begin{aligned}
&\mathrm{Tr}_{\ell+1,\ldots,\ell+j-1}\Big(\mathbf{W}^{(j)}_{n,sa,(\ell+1,\ldots,\ell+\beta-1,\ell,\ell+\beta,\ldots,\ell+j-1)}\gamma^{(\ell+j-1)}\Big) \\
&= \frac{1}{4}\,\mathrm{Tr}_{\ell+1,\ldots,\ell+j-1}\Big(\widetilde{\mathbf{W}}^{(j)}_{n,(\ell+1,\ldots,\ell+\beta-1,\ell,\ell+\beta,\ldots,\ell+j-1)}\,|f^{\otimes(\ell+j-1)}\rangle\,\langle g^{\otimes(\ell+j-1)}|\Big) \\
&\quad + \frac{1}{4}\,\mathrm{Tr}_{\ell+1,\ldots,\ell+j-1}\Big(\widetilde{\mathbf{W}}^{(j),*}_{n,(\ell+1,\ldots,\ell+\beta-1,\ell,\ell+\beta,\ldots,\ell+j-1)}\,|f^{\otimes(\ell+j-1)}\rangle\,\langle g^{\otimes(\ell+j-1)}|\Big) \\
&\quad + \frac{1}{4}\,\mathrm{Tr}_{\ell+1,\ldots,\ell+j-1}\Big(\widetilde{\mathbf{W}}^{(j)}_{n,(\ell+1,\ldots,\ell+\beta-1,\ell,\ell+\beta,\ldots,\ell+j-1)}\,|g^{\otimes(\ell+j-1)}\rangle\,\langle f^{\otimes(\ell+j-1)}|\Big) \\
&\quad + \frac{1}{4}\,\mathrm{Tr}_{\ell+1,\ldots,\ell+j-1}\Big(\widetilde{\mathbf{W}}^{(j),*}_{n,(\ell+1,\ldots,\ell+\beta-1,\ell,\ell+\beta,\ldots,\ell+j-1)}\,|g^{\otimes(\ell+j-1)}\rangle\,\langle f^{\otimes(\ell+j-1)}|\Big) \qquad (4.7.9) \\
&= \frac{1}{4}\,|f^{\otimes(\ell-1)}\rangle\,\langle g^{\otimes(\ell-1)}| \otimes \Big(\mathrm{Tr}_{2,\ldots,j}\Big(\widetilde{\mathbf{W}}^{(j)}_{n,(2,\ldots,\beta,1,\beta+1,\ldots,j)}\,|f^{\otimes j}\rangle\,\langle g^{\otimes j}|\Big) \\
&\qquad\qquad + \mathrm{Tr}_{2,\ldots,j}\Big(\widetilde{\mathbf{W}}^{(j),*}_{n,(2,\ldots,\beta,1,\beta+1,\ldots,j)}\,|f^{\otimes j}\rangle\,\langle g^{\otimes j}|\Big)\Big) \\
&\quad + \frac{1}{4}\,|g^{\otimes(\ell-1)}\rangle\,\langle f^{\otimes(\ell-1)}| \otimes \Big(\mathrm{Tr}_{2,\ldots,j}\Big(\widetilde{\mathbf{W}}^{(j)}_{n,(2,\ldots,\beta,1,\beta+1,\ldots,j)}\,|g^{\otimes j}\rangle\,\langle f^{\otimes j}|\Big) \\
&\qquad\qquad + \mathrm{Tr}_{2,\ldots,j}\Big(\widetilde{\mathbf{W}}^{(j),*}_{n,(2,\ldots,\beta,1,\beta+1,\ldots,j)}\,|g^{\otimes j}\rangle\,\langle f^{\otimes j}|\Big)\Big),
\end{aligned}$$

where the ultimate equality follows from the tensor product structure. We introduce the permutation $\pi \in \mathbb{S}_j$ defined by

$$\pi(a) := \begin{cases} a+1, & 1 \leq a \leq \beta-1 \\ 1, & a = \beta \\ a, & \beta+1 \leq a \leq j \end{cases}, \qquad (4.7.10)$$

so that we can then write

$$\widetilde{\mathbf{W}}^{(j)}_{n,(2,\ldots,\beta,1,\beta+1,\ldots,j)} = \widetilde{\mathbf{W}}^{(j)}_{n,(\pi(1),\ldots,\pi(j))} \qquad (4.7.11)$$

and similarly for the adjoint. Using the notation $\Phi_{\widetilde{\mathbf{W}}^{(j)}_{n,(\pi(1),\ldots,\pi(j))}}$ introduced in (4.5.50), and similarly for the adjoint, we have that

$$\begin{aligned}
&\mathrm{Tr}_{2,\ldots,j}\Big(\widetilde{\mathbf{W}}^{(j)}_{n,(\pi(1),\ldots,\pi(j))}\,|f^{\otimes j}\rangle\,\langle g^{\otimes j}|\Big)(x;x') + \mathrm{Tr}_{2,\ldots,j}\Big(\widetilde{\mathbf{W}}^{(j),*}_{n,(\pi(1),\ldots,\pi(j))}\,|f^{\otimes j}\rangle\,\langle g^{\otimes j}|\Big)(x;x') \\
&= \Phi_{\widetilde{\mathbf{W}}^{(j)}_{n,(\pi(1),\ldots,\pi(j))}}(f,\ldots,f;\overline{g},\ldots,\overline{g})(x;x') + \Phi_{\widetilde{\mathbf{W}}^{(j),*}_{n,(\pi(1),\ldots,\pi(j))}}(f,\ldots,f;\overline{g},\ldots,\overline{g})(x;x')
\end{aligned} \qquad (4.7.12)$$

and

$$\mathrm{Tr}_{2,\ldots,j}\Big(\widetilde{\mathbf{W}}^{(j)}_{n,(\pi(1),\ldots,\pi(j))}\left|g^{\otimes j}\right\rangle\left\langle f^{\otimes j}\right|\Big)(x;x') + \mathrm{Tr}_{2,\ldots,j}\Big(\widetilde{\mathbf{W}}^{(j),*}_{n,(\pi(1),\ldots,\pi(j))}\left|g^{\otimes j}\right\rangle\left\langle f^{\otimes j}\right|\Big)(x;x')$$
$$= \Phi_{\widetilde{\mathbf{W}}^{(j)}_{n,(\pi(1),\ldots,\pi(j))}}(g,\ldots,g;\overline{f},\ldots,\overline{f})(x;x') + \Phi_{\widetilde{\mathbf{W}}^{(j),*}_{n,(\pi(1),\ldots,\pi(j))}}(g,\ldots,g;\overline{f},\ldots,\overline{f})(x;x') \tag{4.7.13}$$

in the sense of tempered distributions on $\mathbb{R}^2$. Next, applying Lemma 4.5.7, we obtain that for $\pi(1)=1$ it holds that

$$(4.7.12) = \overline{g(x')}\Big(w_n^{(j)}[f^{\times j};\overline{g}^{\times(j-1)}](x) + \overline{w_{n,1}^{(j),t}[\overline{f},g^{\times(j-1)};\overline{f}^{\times(j-1)}](x)}\Big), \tag{4.7.14}$$

and

$$(4.7.13) = \overline{f(x')}\Big(w_n^{(j)}[g^{\times j};\overline{f}^{\times(j-1)}](x) + \overline{w_{n,1}^{(j),t}[\overline{g},f^{\times(j-1)};\overline{g}^{\times(j-1)}](x)}\Big), \tag{4.7.15}$$

while if $\pi(1)\neq 1$, we have

$$(4.7.12) = \overline{g(x')}\Big(w^{(j),t}_{n,\pi^{-1}(1)'}[f^{\times j};\overline{g}^{\times(j-1)}](x) + \overline{w^{(j),t}_{n,\pi^{-1}(1)}[g^{\times(\pi^{-1}(1)-1)},\overline{f},g^{\times(j-\pi^{-1}(1))};\overline{f}^{\times(j-1)}](x)}\Big), \tag{4.7.16}$$

and

$$(4.7.13) = \overline{f(x')}\Big(w^{(j),t}_{n,\pi^{-1}(1)'}[g^{\times j};\overline{f}^{\times(j-1)}](x) + \overline{w^{(j),t}_{n,\pi^{-1}(1)}[f^{(\pi^{-1}(1)-1)},\overline{g},f^{\times(j-\pi^{-1}(1))};\overline{g}^{\times(j-1)}](x)}\Big). \tag{4.7.17}$$

Since $\pi^{-1}(1)=\beta$ by definition of the permutation $\pi$, we obtain (4.7.7) after a little book-keeping.

To obtain (4.7.8) from (4.7.7), observe that the self-adjointness of $\mathbf{W}^{(j)}_{n,sa}$ and $\gamma^{(\ell+j-1)}$ implies the Schwartz kernel identity

$$\begin{aligned}&\overline{\mathrm{Tr}_{\ell+1,\ldots,\ell+j-1}\Big(\mathbf{W}^{(j)}_{n,sa,(\ell+1,\ldots,\ell+\beta-1,\alpha,\ell+\beta,\ldots,\ell+j-1)}\gamma^{(\ell+j-1)}\Big)(\underline{x}'_\ell;\underline{x}_\ell)}\\ &= \mathrm{Tr}_{\ell+1,\ldots,\ell+j-1}\Big(\gamma^{(\ell+j-1)}\mathbf{W}^{(j)}_{n,sa,(\ell+1,\ldots,\ell+\beta-1,\alpha,\ell+\beta,\ldots,\ell+j-1)}\Big)(\underline{x}_\ell;\underline{x}'_\ell).\end{aligned} \tag{4.7.18}$$

Substituting (4.7.7) into the left-hand side of the preceding identity yields the desired conclusion. □

We conclude this subsection by recording the required formula of the Hamiltonian vector field $X_{\mathcal{H}_n}$ which follows from the previous lemma and some algebraic manipulations.

**Lemma 4.7.2.** *Suppose that* $\Gamma = (|\phi^{\otimes k}\rangle\langle\phi^{\otimes k}|)_{k\in\mathbb{N}}$, *for some* $\phi \in \mathcal{S}(\mathbb{R})$. *Then for any* $n \in \mathbb{N}$, *we have the Schwartz kernel identity*

$$\begin{aligned}
&X_{\mathcal{H}_n}(\Gamma)^{(\ell)}(\underline{x}_\ell;\underline{x}'_\ell) \\
&= -\frac{i}{2}\sum_{j=1}^{\infty}\sum_{\alpha=1}^{\ell} |\phi^{\otimes(\ell-1)}\rangle\langle\phi^{\otimes(\ell-1)}|(\underline{x}_{\alpha-1},\underline{x}_{\alpha+1;\ell};\underline{x}'_{\alpha-1},\underline{x}'_{\alpha+1;\ell}) \\
&\qquad\times\Bigg(\overline{\phi(x'_\alpha)}\sum_{\beta=1}^{j}\Big(w_{n,\beta'}^{(j),t}[\phi^{\times j};\overline{\phi}^{\times(j-1)}] + \overline{w_{n,\beta}^{(j),t}[\phi^{\times(\beta-1)},\overline{\phi},\phi^{(j-\beta)};\overline{\phi}^{\times(j-1)}]}\Big)(x_\alpha) \\
&\qquad\qquad -\phi(x_\alpha)\sum_{\beta=1}^{j}\Big(\overline{w_{n,\beta'}^{(j),t}[\phi^{\times j};\overline{\phi}^{\times(j-1)}]} + w_{n,\beta}^{(j),t}[\phi^{\times(\beta-1)},\overline{\phi},\phi^{(j-\beta)};\overline{\phi}^{\times(j-1)}]\Big)(x'_\alpha)\Bigg)
\end{aligned} \tag{4.7.19}$$

*for every* $\ell \in \mathbb{N}$.

*Proof.* We use the formula (4.7.5) and recalling definition (4.1.32) for $\mathbf{W}_n$, we obtain that

$$\begin{aligned}
&X_{\mathcal{H}_n}(\Gamma)^{(\ell)}(\underline{x}_\ell;\underline{x}'_\ell) \\
&= -i\sum_{j=1}^{\infty}\frac{1}{(j-1)!}\sum_{\pi\in\mathbb{S}_j}\mathrm{Tr}_{\ell+1,\ldots,\ell+-1}\left(\left[\sum_{\alpha=1}^{\ell}\mathbf{W}^{(j)}_{n,sa,(\pi(\alpha),\pi(\ell+1),\ldots,\pi(\ell+\beta-1))},\gamma^{(\ell+j-1)}\right]\right),
\end{aligned} \tag{4.7.20}$$

where here, $\mathbb{S}_j$ denotes the symmetric group on the set $\{\alpha,\ell+1,\ldots,\ell+j-1\}$. We can decompose $\mathbb{S}_j$ by

$$\mathbb{S}_j = \bigcup_{r\in\{\alpha,\ell+1,\ldots,\ell+j-1\}}\{\pi\in\mathbb{S}_j : \pi^{-1}(\alpha) = r\} =: \mathbb{S}_{j,r}. \tag{4.7.21}$$

Note that each set in the partition has cardinality $(j-1)!$. It is a straightforward computation using the bosonic symmetry of $\gamma^{(\ell+j-1)}$ that

$$\begin{aligned}&\mathrm{Tr}_{\ell+1,\ldots,\ell+j-1}\Big(\Big[\mathbf{W}^{(j)}_{n,sa,(\pi(\alpha),\pi(\ell+1),\ldots,\pi(\ell+j-1))},\gamma^{(\ell+j-1)}\Big]\Big)\\ &=\begin{cases}\mathrm{Tr}_{\ell+1,\ldots,\ell+j-1}\Big(\Big[\mathbf{W}^{(j)}_{n,sa,(\alpha,\ell+1,\ldots,\ell+j-1)},\gamma^{(\ell+j-1)}\Big]\Big), & r=\alpha\\ \mathrm{Tr}_{\ell+1,\ldots,\ell+j-1}\Big(\Big[\mathbf{W}^{(j)}_{n,sa,(\ell+1,\ldots,r,\alpha,r+1,\ldots,\ell+j-1)},\gamma^{(\ell+j-1)}\Big]\Big), & r\in\{\ell+1,\ldots,\ell+j-1\}\end{cases}.\end{aligned} \tag{4.7.22}$$

Using these observations and applying Lemma 4.7.1 to (4.7.20), we obtain the Schwartz kernel identity

$$\begin{aligned}(4.7.20)&=-i\sum_{j=1}^{\infty}\sum_{\alpha=1}^{\ell}\sum_{\beta=1}^{j}\mathrm{Tr}_{\ell+1,\ldots,\ell+j-1}\Big(\Big[\mathbf{W}^{(j)}_{n,sa,(\ell+1,\ldots,\ell+\beta-1,\alpha,\ell+\beta,\ldots,\ell+j-1)},\gamma^{(\ell+j-1)}\Big]\Big)(\underline{x}_\ell;\underline{x}'_\ell)\\ &=-\frac{i}{2}\sum_{j=1}^{\infty}\sum_{\alpha=1}^{\ell}|\phi^{\otimes(\ell-1)}\rangle\,\langle\phi^{\otimes(\ell-1)}|\,(\underline{x}_{\alpha-1},\underline{x}_{\alpha+1;\ell};\underline{x}'_{\alpha-1},\underline{x}'_{\alpha+1;\ell})\\ &\qquad\times\Bigg(\overline{\phi(x'_\alpha)}\sum_{\beta=1}^{j}\Big(w^{(j),t}_{n,\beta'}[\phi^{\times j};\overline{\phi}^{\times(j-1)}]+\overline{w^{(j),t}_{n,\beta}[\phi^{\times(\beta-1)},\overline{\phi},\phi^{\times(j-\beta)};\overline{\phi}^{\times(j-1)}]}\Big)(x_\alpha)\\ &\qquad\quad-\phi(x_\alpha)\sum_{\beta=1}^{j}\Big(\overline{w^{(j),t}_{n,\beta'}[\phi^{\times j};\overline{\phi}^{\times(j-1)}]}+w^{(j),t}_{n,\beta}[\phi^{\times(\beta-1)},\overline{\phi},\phi^{\times(j-\beta)};\overline{\phi}^{\times(j-1)}]\Big)(x'_\alpha)\Bigg)\end{aligned} \tag{4.7.23}$$

This yields the desired formula. □

### 4.7.2 Proof of Theorem 4.1.10

In this subsection, we prove Theorem 4.1.10.

*Proof of Theorem 4.1.10.* Fix $n\in\mathbb{N}$. We would like to establish that $\Gamma=(\,|\phi^{\otimes k}\rangle\,\langle\phi^{\otimes k}|)_{k\in\mathbb{N}}$, where $\phi\in C^\infty(I;\mathcal{S}(\mathbb{R}))$, satisfies

$$\frac{d}{dt}\Gamma=X_{\mathcal{H}_n}(\Gamma), \tag{4.7.24}$$

i.e. $\Gamma$ is a solution to the $n$-th GP hierarchy, if

$$\frac{d}{dt}\phi = \boldsymbol{\nabla}_s I_n(\phi), \tag{4.7.25}$$

i.e. $\phi$ is a solution to the $n$-th NLS. By the Leibnitz rule,

$$\left(\frac{d}{dt}\Gamma\right)^{(\ell)} = \sum_{\alpha=1}^{\ell} \left|\phi^{\otimes(\alpha-1)} \otimes \frac{d}{dt}\phi \otimes \phi^{\otimes(\ell-\alpha)}\right\rangle \left\langle \phi^{\otimes \ell}\right| + \left|\phi^{\otimes\ell}\right\rangle \left\langle \phi^{\otimes(\alpha-1)} \otimes \frac{d}{dt}\phi \otimes \phi^{\otimes(\ell-\alpha)}\right|. \tag{4.7.26}$$

Substituting equation (4.7.25) into the right-hand side of the preceding equality, we obtain that

$$\left(\frac{d}{dt}\Gamma\right)^{(\ell)} = \sum_{\alpha=1}^{\ell} \left|\phi^{\otimes(\alpha-1)} \otimes \boldsymbol{\nabla}_s I_n(\phi) \otimes \phi^{\otimes(\ell-\alpha)}\right\rangle \left\langle \phi^{\otimes \ell}\right| + \left|\phi^{\otimes\ell}\right\rangle \left\langle \phi^{\otimes(\alpha-1)} \otimes \boldsymbol{\nabla}_s I_n(\phi) \otimes \phi^{\otimes(\ell-\alpha)}\right|. \tag{4.7.27}$$

Now the reader will recall that $\boldsymbol{\nabla}_s I_n$ is the symplectic gradient with respect to the form $\omega_{L^2}$ and by (4.5.40) is given by the formula

$$\boldsymbol{\nabla}_s I_n(\phi) = -\frac{i}{2}\sum_{j=1}^{\infty}\left\{\sum_{\beta=1}^{j}\left(\overline{w_{n,\beta}^{(j),t}[\phi^{\times(\beta-1)}, \overline{\phi}, \phi^{\times(j-\beta)}; \overline{\phi}^{\times(j-1)}]} + w_{n,\beta'}^{(j),t}[\phi^{\times j}; \overline{\phi}^{\times(j-1)}]\right)\right\}. \tag{4.7.28}$$

Substituting identity (4.7.28) into the right-hand side of (4.7.27) and comparing the resulting expression with the formula (4.7.19) given by Lemma 4.7.2 yields the desired conclusion. □

### 4.7.3 An Example: the Fourth GP Hierarchy

We conclude this section with an example computation of one the $n$-th GP hierarchies. Specifically, we explicitly compute the equation of motion for the fourth GP hierarchy, which is the next one after the usual GP hierarchy (the third one in our terminology). In light of our Theorem 4.1.10, the fourth GP hierarchy corresponds to the complex mKdV equation

$$\partial_t\phi = \partial_x^3\phi - 6\kappa|\phi|^2\partial_x\phi, \qquad \kappa \in \{\pm 1\}. \tag{4.7.29}$$

**Example 4.7.3** (Fourth GP hierarchy)**.** We first recall from Example 4.4.8 that the

$$\mathbf{W}_4 = \Big((-i\partial_{x_1})^3, -\frac{3\kappa i}{2}(\partial_{x_1}+\partial_{x_2})\delta(X_1-X_2), 0, \ldots\Big). \tag{4.7.30}$$

Substituting (4.7.30) into the right-hand side of the (4.1.50), using Lemma 3.5.10 and the fact that $dH[\Gamma]^{(j)} = -i\mathbf{W}_n^{(j)}$ once again, the fourth GP equation, written in operator form, simplifies to

$$\begin{aligned}
\partial_t\gamma^{(\ell)} &= \sum_{\alpha=1}^{\ell}\sum_{j=1}^{2}\sum_{\beta=1}^{j}\mathrm{Tr}_{\ell,\ldots,\ell+j-1}\Big((-i\mathbf{W}_4^{(j)})_{(\ell+1,\ldots,\ell+\beta-1,\alpha,\ell+\beta,\ldots,\ell+j-1)}\gamma^{(\ell+j-1)}\Big)\\
&\qquad\qquad - \mathrm{Tr}_{\ell+1,\ldots,\ell+j-1}\Big(\gamma^{(\ell+j-1)}(-i\mathbf{W}_4^{(j)})_{(\ell+1,\ldots,\ell+\beta-1,\alpha,\ell+\beta,\ldots,\ell+j-1)}\Big)\\
&= -i\sum_{\alpha=1}^{\ell}\Bigg(\mathbf{W}_{4,(\alpha)}^{(1)}\gamma^{(\ell)} + \gamma^{(\ell)}\mathbf{W}_{4,(\alpha)}^{(1)} + \mathrm{Tr}_{\ell+1}\Big(\mathbf{W}_{4,(\alpha,\ell+1)}^{(2)}\gamma^{(\ell+1)} - \gamma^{(\ell+1)}\mathbf{W}_{4,(\alpha,\ell+1)}^{(2)}\Big)\\
&\qquad\qquad + \mathrm{Tr}_{\ell+1}\Big(\mathbf{W}_{4,(\ell+1,\alpha)}^{(2)}\gamma^{(\ell+1)} - \gamma^{(\ell+1)}\mathbf{W}_{4,(\ell+1,\alpha)}^{(2)}\Big)\Bigg),
\end{aligned}$$

where we recall that the subscript notation is used to specify the variables on which the $\mathbf{W}_n^{(j)}$ operators act. By direct computation, this expression simplifies to yield

$$\partial_t\gamma^{(\ell+1)} = \sum_{\alpha=1}^{\ell}(\partial_{x_\alpha}^3 + \partial_{x'_\alpha}^3)\gamma^{(\ell)} - 6\kappa\big(B^+_{\alpha;\ell+1}(\partial_{x_\alpha}\gamma^{(\ell+1)}) + B^-_{\alpha;\ell+1}(\partial_{x'_\alpha}\gamma^{(\ell+1)})\big), \tag{4.7.31}$$

which is the fourth GP hierarchy, and which can readily be seen to yield (4.7.29) for factorized solutions.

| Symbol | Definition |
|---|---|
| $\underline{x}_k$ or $\underline{x}_{i;i+k}$ | $(x_1,\ldots,x_k)$ or $(x_i,\ldots,x_{i+k})$ |
| $d\underline{x}_k$ or $d\underline{x}_{i;i+k}$ | $dx_1\cdots dx_k$ or $dx_i\cdots dx_{i+k}$ |
| $\mathbb{N}$ or $\mathbb{N}_0$ | natural numbers or natural numbers inclusive of zero |
| $\mathbb{N}_{\leq i}$ or $\mathbb{N}_{\geq i}$ | $\{n\in\mathbb{N} : n\leq i\}$ or $\{n\in\mathbb{N} : n\geq i\}$ |
| $\mathbb{S}_k$ | symmetric group on $k$ elements |
| $C_c^\infty(\mathbb{R}^k)$ or $\mathcal{D}(\mathbb{R}^k)$ | smooth, compactly supported functions on $\mathbb{R}^k$ |
| $\mathcal{S}(\mathbb{R}^k)$ or $\mathcal{S}_s(\mathbb{R}^k)$ | Schwartz space or bosonic Schwartz space on $\mathbb{R}^k$: Definition 3.3.24 |
| $\mathcal{S}(\mathbb{R}^k;\mathcal{V})$ | Schwartz functions on $\mathbb{R}^k$ with values in $\mathcal{V}$: (4.1.39), (4.3.12) |
| $\mathcal{S}'(\mathbb{R}^k)$ or $\mathcal{S}_s'(\mathbb{R}^k)$ | tempered distributions or bosonic tempered distributions on $\mathbb{R}^k$ |
| $\mathcal{D}'(\mathbb{R}^k)$ | distributions on $\mathbb{R}^k$ |
| $\mathcal{L}(E,F)$ | continuous linear maps between locally convex spaces $E$ and $F$ |
| $dF$ | the Gâteaux derivative of $F$: Definition 2.1.4 |
| $\boldsymbol{\nabla}$ or $\boldsymbol{\nabla}_s$, $\boldsymbol{\nabla}_{s,\mathcal{V}}$, $\boldsymbol{\nabla}_{s,\mathbb{C}}$ | the real or symplectic $L^2$ gradients: Definition 3.3.11 and Remark 3.3.12, Proposition 4.3.2, Proposition 4.3.5 |
| $\boldsymbol{\nabla}_1, \boldsymbol{\nabla}_{\bar{1}}, \boldsymbol{\nabla}_2, \boldsymbol{\nabla}_{\bar{2}}$ | variational derivatives: (4.3.1), (4.3.26) |
| $A^{(k)}_{(\pi(1),\ldots,\pi(k))}$ | conjugation of an operator by a permutation: see (3.3.42) |
| $\mathrm{Sym}_k(f)$ | symmetrization operator for functions: Definition 3.3.23 |
| $\mathrm{Sym}_k(A^{(k)})$, $\mathrm{Sym}(A)$ | symmetrization operator for operators: Definition 3.3.30 |
| $B^{\pm}_{i;j}, B_{i;j}$ | contraction operators: (4.1.22) (4.1.23) |
| $\phi^{\otimes k}$ or $\phi^{\times k}$ | $k$-fold tensor or cartesian product of $\phi$ with itself: (3.3.64) or (3.3.65) |
| $\omega_{L^2}$, $\omega_{L^2,\mathcal{V}}$, $\omega_{L^2,\mathcal{V}}$ | $L^2$ symplectic forms: (3.3.15), (4.3.17), (4.3.29) |
| $\mathcal{A}_{\mathcal{S}}$, $\mathcal{A}_{\mathcal{S},\mathcal{V}}$, $\mathcal{A}_{\mathcal{S},\mathbb{C}}$ | see (3.3.20), (4.3.19), (4.3.32) |
| $\{\cdot,\cdot\}_{L^2}, \{\cdot,\cdot\}_{L^2,\mathcal{V}}, \{\cdot,\cdot\}_{L^2,\mathbb{C}}$ | $L^2$ Poisson brackets: (3.3.21), (4.3.21), (4.3.33) |
| $(\mathfrak{G}_\infty, [\cdot,\cdot]_{\mathfrak{G}_\infty})$ | Lie algebra of observable $\infty$-hierarchies: see discussion around Proposition 4.1.2 |
| $(\mathfrak{G}_\infty^*, \mathcal{A}_\infty, \{\cdot,\cdot\}_{\mathfrak{G}_\infty^*})$ | Lie-Poisson manifold of density matrix $\infty$-hierarchies: (4.1.17) and discussion around Proposition 4.1.4 |
| $w_n$, $w_{n,(\psi_1,\psi_2)}$ | recursive functions: (1.3.8), (4.5.31) |
| $w_n^{(k)}$; $w_{n,j}^{(k),t}$, $w_{n,j'}^{(k),t}$ | $k$-particle component of $w_n$: (4.5.2); partial transposes of $w_n^{(k)}$: Lemma 4.5.5 |
| $I_n$, $\tilde{I}_n$, $I_{b,n}$ | involutive functionals: (1.3.9), (4.1.38), (4.1.40) |
| $\widetilde{\mathbf{W}}_n$ | the unsymmetrized operators: (4.1.24) |
| $\mathbf{W}_{n,sa}$ | the self-adjoint operators: (4.4.74) |
| $\mathbf{W}_n$ | the bosonic, self-adjoint operators: (4.1.32) |
| $\mathcal{H}_n$ | the $n$-th Hamiltonian functional: (4.1.33) |
| $\mathrm{Tr}_{1,\ldots,N}$ | generalized trace: Definition 3.2.1 |
| $\mathrm{Tr}_{k+1,\ldots,N}$ | generalized partial trace: Proposition 3.2.4 |
| $WF(u)$ | wave front set of a distribution $u$: Definition 4.0.6 |

Table 4.1: Notation

# Appendix

# Appendix 1

# The 1D NLS as an Integrable System

In this appendix, we sketch the proof that the 1-particle functionals $I_n$ are involution with respect to the Poisson bracket $\{\cdot,\cdot\}_{L^2}$. We generalize the presentation to allow for the case where the two Schwartz functions $\psi, \bar{\psi}$ are independent, since this is the actual 1-particle result that we use in Section 4.6. Hence, rather than considering the scalar NLS equation (1.3.7), we consider the system

$$\begin{cases} (i\partial_t + \Delta)\psi_1 = 2\kappa\psi_1^2\psi_2 \\ (i\partial_t - \Delta)\psi_2 = -2\kappa\psi_2^2\psi_1 \end{cases}, \qquad \kappa \in \{\pm 1\}. \tag{1.0.1}$$

Our presentation will proceed at a high level, following the exposition in [28, Chapter I and Chapter III]; however, the reader may consult Chapter I, §7 and Chapter III, §4 of the aforementioned reference to fill in any omitted analytic details. We also consider the $L$-periodic case rather than entire real line. The extension to the latter case follows from truncation and periodization to fundamental domain $[-L, L]$, application of the periodic result, and then passage to the limit $L \to \infty$.

## 1.1 Transition and Monodromy Matrices

We start by fixing some notation. For $L > 0$, we let $\mathbb{T}_L$ denote the domain $[-L, L]$ with periodic boundary conditions and $C^\infty(\mathbb{T}_L)$ the space of smooth functions on $\mathbb{T}_L$. Equivalently, $C^\infty(\mathbb{T}_L)$ is the space of smooth functions on the real line whose derivatives of all order are $2L$-periodic. Given a $(\mathbb{C}^2 \otimes \mathbb{C}^2)$-valued functional $M_{(\psi_1,\psi_2)}$ on $C^\infty(\mathbb{T}_L)$, we define

$$M^\dagger_{(\psi_1,\psi_2)} \coloneqq \overline{M_{(\overline{\psi_2},\overline{\psi_1})}} \tag{1.1.1}$$

where the complex conjugate of the matrix is taken entry-wise. Evidently, the $\dagger$ operation is involutive.

The system (1.0.1) is a compatibility condition for the overdetermined system of equations

$$\begin{cases} \partial_x F_{(\psi_1,\psi_2)}(t,x,\lambda) = U_{(\psi_1,\psi_2)}(t,x,\lambda) F_{(\psi_1,\psi_2)}(t,x,\lambda), \\ \partial_t F_{(\psi_1,\psi_2)}(t,x,\lambda) = V_{(\psi_1,\psi_2)}(t,x,\lambda) F_{(\psi_1,\psi_2)}(t,x,\lambda) \end{cases}, \tag{1.1.2}$$

where $F_{(\psi_1,\psi_2)}$ is a spacetime $\mathbb{C}^2$-valued column vector and $U_{(\psi_1,\psi_2)}$ and $V_{(\psi_1,\psi_2)}$ are $\lambda$-dependent $2\times 2$ matrices given by

$$U_{(\psi_1,\psi_2)}(\lambda) \coloneqq U_{0,(\psi_1,\psi_2)} + \lambda U_1, \qquad U_{0,(\psi_1,\psi_2)} \coloneqq \sqrt{\kappa}\begin{pmatrix} 0 & \psi_2 \\ \psi_1 & 0 \end{pmatrix}, \;\; U_1 \coloneqq \frac{1}{2i}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1.1.3}$$

and

$$\begin{aligned} &V_{(\psi_1,\psi_2)}(\lambda) \coloneqq V_{0,(\psi_1,\psi_2)} + \lambda V_{1,(\psi_1,\psi_2)} + \lambda^2 V_2, \\ &V_{0,(\psi_1,\psi_2)} \coloneqq i\sqrt{\kappa}\begin{pmatrix} \sqrt{\kappa}\psi_1\psi_2 & -\partial_x\psi_2 \\ \partial_x\psi_1 & -\sqrt{\kappa}\psi_1\psi_2 \end{pmatrix}, \;\; V_{1,(\psi_1,\psi_2)} \coloneqq -U_{0,(\psi_1,\psi_2)}, \;\; V_2 \coloneqq -U_1. \end{aligned} \tag{1.1.4}$$

In the preceding and following material, $\lambda$ plays the role of an auxiliary spectral parameter. It will be convenient going forward to introduce notation for the $2\times 2$ Pauli matrices:

$$\sigma_1 \coloneqq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \;\; \sigma_2 \coloneqq \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \;\; \sigma_3 \coloneqq \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \;\; \sigma_+ \coloneqq \frac{\sigma_1 + i\sigma_2}{2}, \;\; \sigma_- \coloneqq \frac{\sigma_1 - i\sigma_2}{2}. \tag{1.1.5}$$

Written using $U$ and $V$, the compatibility condition for the system (1.1.2) is then

$$\partial_t U_{(\psi_1,\psi_2)} - \partial_x V_{(\psi_1,\psi_2)} + \left[U_{(\psi_1,\psi_2)}, V_{(\psi_1,\psi_2)}\right] = 0 \tag{1.1.6}$$

point-wise in $\lambda$. In the sequel, we will omit the subscript $(\psi_1, \psi_2)$, which shows that the matrices are really matrix-valued functionals evaluated at a specific point, except when invoking the dependence is necessary. We hope that this omission will not result in any confusion on the reader's part.

There is a geometric interpretation to (1.1.6) in terms of local connection coefficients in the vector bundle $\mathbb{R}^2 \times \mathbb{C}^2$. Equation (1.1.6) then says that the $(U, V)$-connection has zero curvature. For this reason, (1.1.6) is often called the *zero curvature representation* in the literature. We will not emphasize this geometric aspect in the appendix, as it does not play a role for us.

Now fix a time $t_0$ and consider the *auxiliary linear problem*

$$\partial_x F = U(t_0, x, \lambda) F. \tag{1.1.7}$$

The object of interest associated to (1.1.7) is the *monodromy matrix*, which is the matrix of parallel transport along the contour $t = t_0$, $-L \leq x \leq L$ positively oriented:

$$T_L(\lambda, t_0) := \overset{\curvearrowleft}{\exp} \left( \int_{-L}^{L} dx U(x, t_0, \lambda) \right), \tag{1.1.8}$$

where $\overset{\curvearrowleft}{\exp}$ denotes the path-ordered exponential.[1] By using the superposition principle for parallel transport and the fact that parallel transport along a closed curve is trivial, one can show that the monodromy matrices are conjugate for different values of $t$. Consequently, the

[1]For $A \in L^\infty(\mathbb{T}_L; \mathbb{C}^n \otimes \mathbb{C}^n)$, the *path-ordered exponential* of $A$ is defined by

$$\overset{\curvearrowleft}{\exp} \left( \int_{-L}^{x} dz A(z) \right) := \sum_{n=0}^{\infty} \int_{-L}^{x} dx_n \int_{-L}^{x_n} dx_{n-1} \cdots \int_{-L}^{x_2} dx_1 A(x_n) \cdots A(x_1). \tag{1.1.9}$$

trace of the monodromy matrix is constant in time:

$$\operatorname{tr}_{\mathbb{C}^2} T_L(\lambda, t_2) = \operatorname{tr}_{\mathbb{C}^2} T_L(\lambda, t_1), \qquad \forall t_1, t_2 \in \mathbb{R}, \tag{1.1.10}$$

where $\operatorname{tr}_{\mathbb{C}^2}$ denotes the $2 \times 2$ matrix trace. Furthermore, one can show that the choice of fundamental domain $[-L, L]$ in the definition (1.1.8) is immaterial to computing the trace. We conclude that

$$\tilde{F}_L(\lambda) \coloneqq \operatorname{tr}_{\mathbb{C}^2} T_L(\lambda) \tag{1.1.11}$$

is a *generating function* for the conservation laws of (1.0.1).

More generally, we have the *transition matrix*, which is the matrix of parallel transport from $y$ to $x$ along the $x$-axis:

$$T(x, y, \lambda) \coloneqq \overset{\curvearrowleft}{\exp} \left( \int_y^x dz U(z, \lambda) \right). \tag{1.1.12}$$

The monodromy matrix is then the special case of the transition matrix obtained by setting $(x, y) = (L, -L)$. From the definition (1.1.12), it is immediate that the transition matrix satisfies the Cauchy problem

$$\begin{cases} \partial_x T(x, y, \lambda) = U(x, \lambda) T(x, y, \lambda) \\ T(x, y, \lambda)|_{x=y} = I_{\mathbb{C}^2} \end{cases}, \tag{1.1.13}$$

where $I_{\mathbb{C}^2}$ is the identity matrix on $\mathbb{C}^2$. $T(x, y, \lambda)$ is a smooth function of $(x, y)$ and is also analytic in $\lambda$ due to the analyticity of $U(x, \lambda)$ and the initial datum. By using that $\int_y^x = -\int_x^y$ in (1.1.12), we see that $T(x, y, \lambda)$ also satisfies the ODE

$$\partial_y T(x, y, \lambda) = -T(x, y, \lambda) U(y, \lambda). \tag{1.1.14}$$

Additionally, the transition matrix has several elementary properties, which we record with the following lemma.

**Lemma 1.1.1.** *The following properties hold:*

*(i)* $T(x,z,\lambda)T(z,y,\lambda) = T(x,y,\lambda)$,

*(ii)* $T(x,y,\lambda) = T^{-1}(y,x,\lambda)$,

*(iii)* $\det_{\mathbb{C}^2} T(x,y,\lambda) = 1$.

*Proof.* Properties (i) and (ii) are straightforward, and we leave them to the reader. For property (iii), the reader will recall Jacobi's formula that for any $n \times n$ matrix $A(t)$,

$$\frac{d}{dt}\det_{\mathbb{C}^n}(A(t)) = \operatorname{tr}_{\mathbb{C}^n}\left(\operatorname{adj}(A(t))\frac{dA(t)}{dt}\right), \tag{1.1.15}$$

where $\operatorname{adj}(A(t))$ is the adjugate of $A(t)$ (i.e. the transpose of the cofactor matrix of $A(t)$). Fixing $y, \lambda$ and applying Jacobi's formula to $T(x,y,\lambda)$ with independent variable $x$ instead of $t$ and also using the equation (1.1.13), we find that $\det_{\mathbb{C}^2}(T(x,y,\lambda))$ is a solution to the Cauchy problem

$$\begin{cases} \partial_x \det_{\mathbb{C}^2}(T(x,y,\lambda)) & = \operatorname{tr}_{\mathbb{C}^2}(\operatorname{adj}(T(x,y,\lambda))U(x,\lambda)T(x,y,\lambda)), \\ \det_{\mathbb{C}^2}(T(x,y,\lambda))|_{x=y} & = 1 \end{cases} \tag{1.1.16}$$

Since

$$\operatorname{adj}(T(x,y,\lambda)) = \begin{pmatrix} T^{22}(x,y,\lambda) & -T^{12}(x,y,\lambda) \\ -T^{21}(x,y,\lambda) & T^{11}(x,y,\lambda) \end{pmatrix}, \tag{1.1.17}$$

it follows by direct computation that

$$T(x,y,\lambda)\operatorname{adj}(T(x,y,\lambda)) = \det_{\mathbb{C}^2}(T(x,y,\lambda)) Id_{\mathbb{C}^2}. \tag{1.1.18}$$

So by the cyclicity and linearity of trace, $\det_{\mathbb{C}^2}(T(x,y,\lambda))$ is the unique constant solution to the Cauchy problem

$$\begin{cases} \partial_x \det_{\mathbb{C}^2}(T(x,y,\lambda))) & = \det_{\mathbb{C}^2}(T(x,y,\lambda)) \operatorname{tr}_{\mathbb{C}^2}(U(x,y,\lambda)I_{\mathbb{C}^2}) = 0 \\ \det_{\mathbb{C}^2}(T(x,y,\lambda))|_{x=y} & = 1 \end{cases}, \tag{1.1.19}$$

where we use that $U(x,y,\lambda)$ is trace-less. Thus, the proof of (iii) is complete. $\square$

It is evident from its definition (1.1.3) that

$$U^{\dagger}_{(\psi_1,\psi_2)}(x,\lambda) = \sigma U_{(\psi_1,\psi_2)}(x,\bar{\lambda})\sigma, \tag{1.1.20}$$

where

$$\sigma = \begin{cases} \sigma_1, & \kappa = 1 \\ \sigma_2, & \kappa = -1 \end{cases}, \tag{1.1.21}$$

where $\kappa$ is the defocusing/focusing parameter in (1.0.1) and $\sigma_1, \sigma_2$ are the Pauli matrices in (1.1.5). The transition matrix also satisfies an important involution relation leading to the special structure of the matrix $T(x,y,\lambda)$, which we isolate in the next lemma.

**Lemma 1.1.2.** *$T(x,y,\lambda)$ has the involution property*

$$\sigma T_{(\psi_1,\psi_2)}(x,y,\bar{\lambda})\sigma = T^{\dagger}_{(\psi_1,\psi_2)}(x,y,\lambda). \tag{1.1.22}$$

*Consequently, we can write the monodromy matrix $T_{L,\psi_1,\psi_2}(\lambda)$ as*

$$T_{L,(\psi_1,\psi_2)}(\lambda) = \begin{pmatrix} a_{L,(\psi_1,\psi_2)}(\lambda) & \operatorname{sgn}(\kappa) b^{\dagger}_{L,(\psi_1,\psi_2)}(\bar{\lambda}) \\ b_{L,(\psi_1,\psi_2)}(\lambda) & a^{\dagger}_{L,(\psi_1,\psi_2)}(\bar{\lambda}) \end{pmatrix}, \tag{1.1.23}$$

*where $a^{\dagger}_{L,(\psi_1,\psi_2)}(\lambda) := \overline{a_{L,(\overline{\psi_2},\overline{\psi_1})}(\lambda)}$ and analogously for $b^{\dagger}_{L,(\psi_1,\psi_2)}$.*

*Proof.* Since the Cauchy problem (1.1.13) has a unique solution and $\sigma^2 = I_{\mathbb{C}^2}$, it suffices to show that the matrix

$$\tilde{T}_{(\psi_1,\psi_2)}(x,y,\lambda) := \sigma T^{\dagger}_{(\psi_1,\psi_2)}(x,y,\bar{\lambda})\sigma \tag{1.1.24}$$

is a solution of (1.1.13).

It is evident from $T_{(\psi_1,\psi_2)}(x,y,\lambda)|_{x=y} = I_{\mathbb{C}^2}$ and $\sigma^2 = I_{\mathbb{C}^2}$ that the initial condition holds. Now using that $\partial_x$ commutes with left- (and right-) multiplication by a constant matrix and complex conjugation, we find that

$$\begin{aligned}\partial_x \tilde{T}_{(\psi_1,\psi_2)}(x,y,\lambda) &= \sigma\overline{\partial_x T_{(\overline{\psi_2},\overline{\psi_1})}(x,y,\bar{\lambda})}\sigma \\ &= \sigma\overline{U_{(\overline{\psi_2},\overline{\psi_1})}(x,\overline{\lambda})T_{(\overline{\psi_2},\overline{\psi_1})}(x,y,\bar{\lambda})}\sigma \\ &= \sigma U^{\dagger}_{(\psi_1,\psi_2)}(x,\overline{\lambda})T^{\dagger}_{(\psi_1,\psi_2)}(x,y,\bar{\lambda})\sigma,\end{aligned} \tag{1.1.25}$$

where the penultimate equality follows from application of (1.1.13) with $(\psi_1,\psi_2)$ replaced by $(\overline{\psi_2},\overline{\psi_1})$ and the ultimate equality follows from the definition of the dagger superscript. Since $\sigma^2 = I_{\mathbb{C}^2}$, we can use the associativity of matrix multiplication together with the identity (1.1.20) to write

$$\begin{aligned}\sigma U^{\dagger}_{(\psi_1,\psi_2)}(x,\overline{\lambda})T^{\dagger}_{(\psi_1,\psi_2)}(x,y,\overline{\lambda})\sigma &= \left(\sigma U^{\dagger}_{(\psi_1,\psi_2)}(x,\overline{\lambda})\sigma\right)\left(\sigma T^{\dagger}_{(\psi_1,\psi_2)}(x,y,\overline{\lambda})\sigma\right) \\ &= U_{(\psi_1,\psi_2)}(x,\lambda)\tilde{T}_{(\psi_1,\psi_2)}(x,y,\lambda),\end{aligned} \tag{1.1.26}$$

which is exactly what we needed to show.

We now show the second assertion concerning the structure of the monodromy matrix. We only present the details in the case $\kappa = 1$ and leave the $\kappa = -1$ case as an exercise for the reader. Writing

$$T_{(\psi_1,\psi_2)}(x,y,\lambda) = \begin{pmatrix} T^{11}_{(\psi_1,\psi_2)}(x,y,\lambda) & T^{12}_{(\psi_1,\psi_2)}(x,y,\lambda) \\ T^{21}_{(\psi_1,\psi_2)}(x,y,\lambda) & T^{22}_{(\psi_1,\psi_2)}(x,y,\lambda) \end{pmatrix}, \tag{1.1.27}$$

we see from direct computation that

$$\begin{aligned}\sigma T_{(\psi_1,\psi_2)}(x,y,\bar{\lambda})\sigma &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} T^{12}_{(\psi_1,\psi_2)}(x,y,\bar{\lambda}) & T^{11}_{(\psi_1,\psi_2)}(x,y,\bar{\lambda}) \\ T^{22}_{(\psi_1,\psi_2)}(x,y,\bar{\lambda}) & T^{21}_{(\psi_1,\psi_2)}(x,y,\bar{\lambda}) \end{pmatrix} \\ &= \begin{pmatrix} T^{22}_{(\psi_1,\psi_2)}(x,y,\bar{\lambda}) & T^{21}_{(\psi_1,\psi_2)}(x,y,\bar{\lambda}) \\ T^{12}_{(\psi_1,\psi_2)}(x,y,\bar{\lambda}) & T^{11}_{(\psi_1,\psi_2)}(x,y,\bar{\lambda}) \end{pmatrix}.\end{aligned} \tag{1.1.28}$$

Now by the involution property (1.1.22) and the definition of $T^{\dagger}_{(\psi_1,\psi_2)}$, we see that

$$\begin{aligned}\begin{pmatrix}\overline{T^{11}_{(\overline{\psi_2},\overline{\psi_1})}(x,y,\lambda)} & \overline{T^{12}_{(\overline{\psi_2},\overline{\psi_1})}(x,y,\lambda)} \\ \overline{T^{21}_{(\overline{\psi_2},\overline{\psi_1})}(x,y,\lambda)} & \overline{T^{22}_{(\overline{\psi_2},\overline{\psi_1})}(x,y,\lambda)}\end{pmatrix} &= T^{\dagger}_{(\psi_1,\psi_2)}(x,y,\lambda) \\ &= \begin{pmatrix}T^{22}_{(\psi_1,\psi_2)}(x,y,\bar{\lambda}) & T^{21}_{(\psi_1,\psi_2)}(x,y,\bar{\lambda}) \\ T^{12}_{(\psi_1,\psi_2)}(x,y,\bar{\lambda}) & T^{11}_{(\psi_1,\psi_2)}(x,y,\bar{\lambda})\end{pmatrix}.\end{aligned} \tag{1.1.29}$$

Evaluating this identity at $(x,y)=(L,-L)$ and defining

$$a_{L,(\psi_1,\psi_2)}(\lambda) \coloneqq T^{11}_{L,(\psi_1,\psi_2)}(\lambda), \quad b_{L,(\psi_1,\psi_2)}(\lambda) \coloneqq T^{21}_{L,(\psi_1,\psi_2)}(\lambda), \tag{1.1.30}$$

we obtain the desired conclusion. □

**Remark 1.1.3.** Since the transition matrix is an entire function of $\lambda$, it follows that the functions $a_{L,(\psi_1,\psi_2)}, a^{\dagger}_{L,(\psi_1,\psi_2)}, b_{L,(\psi_1,\psi_2)}, b^{\dagger}_{L,(\psi_1,\psi_2)}$ are entire functions as well. In fact, they are of exponential type $L$. Moreover, the unimodularity property (iii) for the transition matrix implies the normalization condition

$$a_{L,(\psi_1,\psi_2)}(\lambda)a^{\dagger}_{L,(\psi_1,\psi_2)}(\lambda) - \operatorname{sgn}(\kappa)b_{L,(\lambda_1,\lambda_2)}(\lambda)b^{\dagger}_{L,(\psi_1,\psi_2)}(\lambda) = 1, \qquad \lambda\in\mathbb{R}. \tag{1.1.31}$$

We close this subsection with an alternative way to see that the trace of the monodromy matrix, which we called $\tilde{F}_L(\lambda)$ in (1.1.11), is conserved in time. By differentiating both sides of equation (1.1.13) with respect to time and performing some algebraic manipulation, one finds that

$$\partial_t T(t,x,y,\lambda) = V(t,x,\lambda)T(t,x,y,\lambda) - T(t,x,y,\lambda)V(t,y,\lambda) \tag{1.1.32}$$

Since $V$ is $2L$-periodic and therefore $V(t,L,\lambda)=V(t,-L,\lambda)$, it follows that the monodromy matrix satisfies the von Neumann equation

$$\partial_t T_L(t,\lambda) = [V(t,L,\lambda), T_L(t,\lambda)]. \tag{1.1.33}$$

Since differentiation commutes with the trace and the trace of a commutator is zero, it follows that

$$\partial_t \operatorname{tr}_{\mathbb{C}^2}(T_L(t,\lambda)) = 0. \tag{1.1.34}$$

## 1.2 Integrals of Motion

We now use an asymptotic expansion for the generating functional $\tilde{F}_L(\lambda)$ (recall (1.1.11)) to identify conserved quantities for the system (1.0.1). We start by finding a gauge transformation that reduces the transition matrix to diagonal form $\exp Z(x,y,\lambda)$:

$$T(x,y,\lambda) = (I_{\mathbb{C}^2} + W(x,\lambda)) \exp(Z(x,y,\lambda))(I_{\mathbb{C}^2} + W(y,\lambda))^{-1}, \tag{1.2.1}$$

where $W$ and $Z$ are off-diagonal and diagonal matrices, respectively. We will see that $W$ and $Z$ have the large real $\lambda$ asymptotic expansions

$$W(x,\lambda) \sim \sum_{n=1}^{\infty} \frac{W_n(x)}{\lambda^n}, \;\; Z(x,y,\lambda) \sim \frac{(x-y)\lambda\sigma_3}{2i} + \sum_{n=1}^{\infty} \frac{Z_n(x,y,\lambda)}{\lambda^n}, \tag{1.2.2}$$

where the reader will recall the Pauli matrix $\sigma_3$ from (1.1.5). Here and throughout the appendix, the asymptotic should be interpreted as follows: for any $k \in \mathbb{N}$,

$$\begin{aligned} o(|\lambda|^{-k}) = &\sup_{-L \le x \le L} \|W(x,\lambda) - \sum_{n=1}^{k} \frac{W_n(x)}{\lambda^n}\| \\ &+ \sup_{-L \le x,y, \le L} \|Z(x,y,\lambda) - \frac{(x-y)\lambda\sigma_3}{2i} - \sum_{n=1}^{k} \frac{Z_n(x,y,\lambda)}{\lambda^n}\| \end{aligned} \tag{1.2.3}$$

as $|\lambda| \to \infty$ on the real line, where $\|\cdot\|$ denotes any matrix norm.

Proceeding formally to identify the relevant equations, we substitute (1.2.1) into the transition matrix differential equation (1.1.13) and use the Leibnitz rule to obtain that

$$\begin{aligned} & U(x,\lambda)(I_{\mathbb{C}^2}+W(x,\lambda))\exp(Z(x,y,\lambda))(I_{\mathbb{C}^2}+W(y,\lambda))^{-1} \\ &= \partial_x W(x,\lambda)\exp(Z(x,y,\lambda))(I_{\mathbb{C}^2}+W(y,\lambda))^{-1} \\ &\quad + (I_{\mathbb{C}^2}+W(x,\lambda))\partial_x Z(x,y,\lambda)\exp(Z(x,y,\lambda))(I_{\mathbb{C}^2}+W(y,\lambda))^{-1}, \end{aligned} \tag{1.2.4}$$

which can be manipulated to yield

$$U(x,\lambda)(I_{\mathbb{C}^2}+W(x,\lambda)) = \partial_x W(x,\lambda) + (I_{\mathbb{C}^2}+W(x,\lambda))\partial_x Z(x,y,\lambda). \tag{1.2.5}$$

Recalling from (1.1.3) that $U(x,\lambda) = U_0(x)+\lambda U_1$, where $U_0$ is off-diagonal and $U_1$ is diagonal, and decomposing both sides of (1.2.5) into off-diagonal and diagonal parts, we find that $W$ and $Z$ satisfy the coupled system of equations

$$\begin{cases} \partial_x W + W\partial_x Z = U_0 + \lambda U_1 W \\ \partial_x Z = U_0 W + \lambda U_1 \end{cases}. \tag{1.2.6}$$

Substituting the second equation into the first one and using that $U_1$ anticommutes with $W$, we find that $W$ satisfies the matrix Riccati equation

$$\partial_x W + i\lambda\sigma_3 W + WU_0W - U_0 = 0. \tag{1.2.7}$$

One can rewrite (1.2.7) as an integral equation and use the fixed-point method to show that (1.2.7) has a smooth solution on $\mathbb{T}_L$ for sufficiently large $\lambda$ depending on the data $(\|\phi\|_{L^1(\mathbb{T}_L)}, \|\phi\|_{L^\infty(\mathbb{T}_L)}, L)$, with the asymptotic expansion (1.2.2). We can then solve for $Z$ subject to the initial condition $Z(x,y,\lambda)|_{x=y} = 0_{\mathbb{C}^2}$ by

$$Z(x,y,\lambda) = \frac{\lambda(x-y)}{2i}\sigma_3 + \int_y^x dz\; U_0(z)W(z,\lambda). \tag{1.2.8}$$

In particular, the asymptotic expansion of $Z$ is then determined by the asymptotic expansion for $W$. $W$ and $Z$ satisfy (1.2.1) since both the left-hand side and right-hand side of the equation (1.2.1) are solutions to the same Cauchy problem, which has a unique solution.

Next, substituting the expansion $\sum_{n=1}^{\infty} \frac{W_n(x)}{\lambda^n}$ into equation (1.2.7), we find that the coefficients $W_n(x)$ satisfy the recursion relation

$$\begin{aligned} W_1(x) &= -i\sigma_3 U_0(x) = i\sqrt{\kappa}\begin{pmatrix} 0 & -\psi_2(x) \\ \psi_1(x) & 0 \end{pmatrix}, \\ W_{n+1}(x) &= i\sigma_3\left(\partial_x W_n(x) + \sum_{k=1}^{n-1} W_k(x)U_0(x)W_{n-k}(x)\right). \end{aligned} \tag{1.2.9}$$

Evidently, the matrices $W_n(x)$ are $2L$-periodic and are polynomials of the derivatives of $U_0(x)$. By equation (1.2.7) for $W$ and the continuity method together with the equation (1.2.8) for $Z$, one can show that the asymptotic (1.2.2) holds. In the next lemma, we record an important involution property of the $W_n$. As before with $U$, we include the subscripts $(\psi_1, \psi_2)$ in the sequel to denote the underlying dependence.

**Lemma 1.2.1.** *For every $n \in \mathbb{N}$, it holds that $W_n$ is off-diagonal and*

$$W^{\dagger}_{n,(\psi_1,\psi_2)}(x) = \sigma W_{n,(\psi_1,\psi_2)}(x)\sigma, \tag{1.2.10}$$

*where $\sigma$ is as in* (1.1.21). *Additionally, $W_{n,(\psi_1,\psi_2)}(x)$ has the form*

$$i\sqrt{\kappa}\begin{pmatrix} 0 & -w^{\dagger}_{n,(\psi_1,\psi_2)}(x) \\ w_{n,(\psi_1,\psi_2)}(x) & 0 \end{pmatrix}, \tag{1.2.11}$$

*where the functions $w_{n,(\psi_1,\psi_2)}(x)$ satisfy the recursion relation*

$$\begin{aligned} w_{1,(\psi_1,\psi_2)}(x) &= \psi_1(x), \\ w_{n+1,(\psi_1,\psi_2)}(x) &= -i\partial_x w_n(x) + \kappa\psi_2(x)\sum_{k=1}^{n-1} w_{k,(\psi_1,\psi_2)}(x)w_{n-k,(\psi_1,\psi_2)}(x). \end{aligned} \tag{1.2.12}$$

*Proof.* We prove the lemma by strong induction on $n$ using the recursion formula (1.2.9). The base case $n = 1$ follows from

$$U^{\dagger}_{0,(\psi_1,\psi_2)}(x) = \sigma U_{0,(\psi_1,\psi_2)}(x)\sigma \tag{1.2.13}$$

and the fact that $\sigma$ anti-commutes with $\sigma_3$.

For the induction step, suppose that for some $n \in \mathbb{N}$, the involution relation holds for all $k \in \mathbb{N}_{\leq n-1}$. Multiplying (1.2.9) by $\sigma$ on the left and right and using that $\sigma^2 = I_{\mathbb{C}^2}$, we find that

$$\begin{aligned}
&\sigma W_{n+1,(\psi_1,\psi_2)}(x)\sigma \\
&= i\sigma\sigma_3\left(\partial_x W_{n,(\psi_1,\psi_2)}(x) + \sum_{k=1}^{n-1} W_{k,(\psi_1,\psi_2)}(x)U_{0,(\psi_1,\psi_2)}(x)W_{n-k,(\psi_1,\psi_2)}(x)\right)\sigma \\
&= -i\sigma_3\left(\partial_x(\sigma W_{n,(\psi_1,\psi_2)}(x)\sigma) + \sum_{k=1}^{n-1}(\sigma W_{k,(\psi_1,\psi_2)}(x)\sigma)(\sigma U_{0,(\psi_1,\psi_2)}(x)\sigma)(\sigma W_{n-k,(\psi_1,\psi_2)}(x)\sigma)\right) \\
&= -i\sigma_3\left(\partial_x W^{\dagger}_{n,(\psi_1,\psi_2)}(x) + \sum_{k=1}^{n} W^{\dagger}_{k,(\psi_1,\psi_2)}(x)U^{\dagger}_{0,(\psi_1,\psi_2)}(x)W^{\dagger}_{n-k,(\psi_1,\psi_2)}(x)\right), \qquad (1.2.14)
\end{aligned}$$

where we again use (1.2.13) and the anti-commutativity of $\sigma$ and $\sigma_3$ to obtain the penultimate equality and the induction hypothesis to obtain the ultimate equality. Since $(i\sigma_3)^{\dagger} = -i\sigma_3$ and the $\dagger$ operation is a homomorphism of algebras which commutes with differentiation, (1.2.10) is proved. Since $W_{1,(\psi_1,\psi_2)}, \ldots, W_{n,(\psi_1,\psi_2)}$ are off-diagonal, it it follows from some basic algebra and the diagonality and off-diagonality of $\sigma_3$ and $U_0$, respectively, that $W_{n+1,(\psi_1,\psi_2)}$ is off-diagonal. Thus, the proof of the induction step is complete.

Now since $W_{n,(\psi_1,\psi_2)}$ is off-diagonal, it takes the form

$$W_{n,(\psi_1,\psi_2)} = \begin{pmatrix} 0 & w^{12}_{n,(\psi_1,\psi_2)} \\ w^{21}_{n,(\psi_1,\psi_2)} & 0 \end{pmatrix}, \qquad w^{12}_{n,(\psi_1,\psi_2)}, w^{21}_{n,(\psi_1,\psi_2)} \in C^{\infty}(\mathbb{T}_L), \qquad (1.2.15)$$

which by direct computation implies that

$$\sigma W_{n,(\psi_1,\psi_2)}\sigma = \begin{pmatrix} 0 & \mathrm{sgn}(\kappa) w^{21}_{n,(\psi_1,\psi_2)} \\ \mathrm{sgn}(\kappa) w^{12}_{n,(\psi_1,\psi_2)} & 0 \end{pmatrix}, \qquad (1.2.16)$$

Now the involution relation (1.2.10) implies the equality

$$\begin{pmatrix} 0 & \operatorname{sgn}(\kappa) w^{21}_{n,(\psi_1,\psi_2)} \\ \operatorname{sgn}(\kappa) w^{12}_{n,(\psi_1,\psi_2)} & 0 \end{pmatrix} = W^{\dagger}_{n,(\psi_1,\psi_2)} = \begin{pmatrix} 0 & w^{12,\dagger}_{n,(\psi_1,\psi_2)} \\ w^{21,\dagger}_{n,(\psi_1,\psi_2)} & 0 \end{pmatrix}. \tag{1.2.17}$$

Therefore, defining $w_{n,(\psi_1,\psi_2)} := w^{21}_{n,(\psi_1,\psi_2)}/(i\sqrt{\kappa})$, we can write $W_{n,(\psi_1,\psi_2)}$ in the form

$$W_{n,(\psi_1,\psi_2)} = i\sqrt{\kappa} \begin{pmatrix} 0 & -w^{\dagger}_{n,(\psi_1,\psi_2)}(x) \\ w_{n,(\psi_1,\psi_2)}(x) & 0 \end{pmatrix}, \tag{1.2.18}$$

where by (1.2.9), the functions $w_{n,(\psi_1,\psi_2)}(x)$ satisfy the recursion relation

$$\begin{aligned} w_{1,(\psi_1,\psi_2)}(x) &= \psi_1(x), \\ w_{n+1,(\psi_1,\psi_2)}(x) &= -i\partial_x w_{n,(\psi_1,\psi_2)}(x) + \kappa\psi_2(x) \sum_{k=1}^{n-1} w_{k,(\psi_1,\psi_2)}(x) w_{n-k,(\psi_1,\psi_2)}(x). \end{aligned} \tag{1.2.19}$$

Thus, the proof of the lemma is complete. $\square$

By using the equation (1.2.7), one can also show that $W_{(\psi_1,\psi_2)}(x,\lambda)$ satisfies the same involutive property as $W_n$. So we can write

$$W_{(\psi_1,\psi_2)}(x,\lambda) = i\sqrt{\kappa}\Big( w_{(\psi_1,\psi_2)}(x,\lambda)\sigma_- - w^{\dagger}_{(\psi_1,\psi_2)}(x,\bar{\lambda})\sigma_+ \Big), \tag{1.2.20}$$

where $\sigma_{\pm}$ are defined in (1.1.5) and where $w_{(\psi_1,\psi_2)}$ has the large real lambda asymptotic expansion

$$w_{(\psi_1,\psi_2)}(x,\lambda) \sim \sum_{n=1}^{\infty} \frac{w_{n,(\psi_1,\psi_2)}(x)}{\lambda^n}. \tag{1.2.21}$$

Using equation (1.2.8) for $Z_{(\psi_1,\psi_2)}(x,y,\lambda)$ and evaluating $(x,y)=(L,-L)$, we find that

$$\begin{aligned}
Z_{L,(\psi_1,\psi_2)}(\lambda) &\coloneqq Z_{(\psi_1,\psi_2)}(L,-L,\lambda) \\
&= \frac{\lambda L}{i}\sigma_3 + \int_{-L}^{L} dz U_{(\psi_1,\psi_2)}(z) W_{(\psi_1,\psi_2)}(z,\lambda) \\
&= \begin{pmatrix} -i\lambda L & 0 \\ 0 & i\lambda L \end{pmatrix} + \int_{-L}^{L} dz \begin{pmatrix} 0 & \sqrt{\kappa}\psi_2(z) \\ \sqrt{\kappa}\psi_1(z) & 0 \end{pmatrix} \begin{pmatrix} 0 & -i\sqrt{\kappa} w^{\dagger}_{(\psi_1,\psi_2)}(z,\lambda) \\ i\sqrt{\kappa} w_{(\psi_1,\psi_2)}(z,\lambda) & 0 \end{pmatrix} \\
&= \begin{pmatrix} -i\lambda L + i\kappa \int_{-L}^{L} dz \psi_2(z) w_{(\psi_1,\psi_2)}(z,\lambda) & 0 \\ 0 & i\lambda L - i\kappa \int_{-L}^{L} dz \psi_1(z) w^{\dagger}_{(\psi_1,\psi_2)}(z,\lambda) \end{pmatrix}
\end{aligned} \tag{1.2.22}$$

Evaluating both sides of equation (1.2.1) at $(x,y)=(L,-L)$, we find that the monodromy matrix $T_L(\lambda)$ has the representation

$$T_{L,(\psi_1,\overline{\psi_2})}(\lambda) = \Big(I_{\mathbb{C}^2} + W_{(\psi_1,\overline{\psi_2})}(L,\lambda)\Big) \exp\Big(Z_{L,(\psi_1,\overline{\psi_2})}(\lambda)\Big) \Big(I_{\mathbb{C}^2} + W_{(\psi_1,\overline{\psi_2})}(-L,\lambda)\Big)^{-1}. \tag{1.2.23}$$

We now turn to finding a formula for the generating function $\tilde{F}_L(\lambda)$ (recall (1.1.11)) in terms of the functions $w$ and $w^{\dagger}$. We first have an important involution property for the entries of $Z_L(\lambda)$.

**Lemma 1.2.2.** *For every $(\psi_1,\overline{\psi_2}) \in \mathcal{S}(\mathbb{R})^2$ and $\lambda \in \mathbb{R}$ sufficiently large so that $w_{(\psi_1,\overline{\psi_2})}(\cdot,\lambda)$ exists, it holds that*

$$\int_{-L}^{L} dx \overline{\psi_2}(x) w_{(\psi_1,\overline{\psi_2})}(x,\lambda) = \int_{-L}^{L} dx \psi_1(x) w^{\dagger}_{(\psi_1,\overline{\psi_2})}(x,\lambda) = \overline{\int_{-L}^{L} dx \overline{\psi_1}(x) w_{(\psi_2,\overline{\psi_1})}(x,\lambda)}. \tag{1.2.24}$$

*In particular, if for every $n \in \mathbb{N}$, we define*

$$\tilde{I}_n(\psi_1,\overline{\psi_2}) \coloneqq \int_{-L}^{L} dx \overline{\psi_2}(x) w_{n,(\psi_1,\overline{\psi_2})}(x), \qquad \forall (\psi_1,\overline{\psi_2}) \in \mathcal{S}(\mathbb{R})^2, \tag{1.2.25}$$

*then*

$$\tilde{I}_n(\psi_1, \overline{\psi_2}) = \overline{\tilde{I}_n(\psi_2, \overline{\psi_1})}. \tag{1.2.26}$$

*Proof.* Since $\det_{\mathbb{C}^2}(T_{L,(\psi_1,\overline{\psi_2})}(\lambda)) = 1$ by the unimodularity property Lemma 1.1.1(iii) and

$$\Big(I_{\mathbb{C}^2} + W_{(\psi_1,\overline{\psi_2})}(L,\lambda)\Big)^{-1} = I_{\mathbb{C}^2} + W_{(\psi_1,\overline{\psi_2})}(-L,\lambda) \tag{1.2.27}$$

by the $2L$-periodicity of $W(\cdot,\lambda)$, it follows from the multiplicative property of determinant that

$$1 = \det_{\mathbb{C}^2}(T_{L,(\psi_1,\overline{\psi_2})}(\lambda)) = \det_{\mathbb{C}^2}\Big(\exp Z_{L,(\psi_1,\overline{\psi_2})}(\lambda)\Big). \tag{1.2.28}$$

Now for any matrix $A \in \mathbb{C}^n \otimes \mathbb{C}^n$, Jacobi's formula implies the trace identity

$$\det_{\mathbb{C}^n}(e^A) = \exp(\operatorname{tr}_{\mathbb{C}^n} A). \tag{1.2.29}$$

Hence,

$$1 = \exp\Big(\operatorname{tr}_{\mathbb{C}^2} Z_{L,(\psi_1,\overline{\psi_2})}(\lambda)\Big) = 1 \Longrightarrow \operatorname{tr}_{\mathbb{C}^2} Z_{L,(\psi_1,\overline{\psi_2})}(\lambda) = 0. \tag{1.2.30}$$

So by identity (1.2.22), we obtain that

$$\int_{-L}^{L} dx\overline{\psi_2}(x) w_{(\psi_1,\overline{\psi_2})}(x,\lambda) = \int_{-L}^{L} dx\psi_1(x) w^{\dagger}_{(\psi_1,\overline{\psi_2})}(x,\lambda) = \overline{\int_{-L}^{L} dx\overline{\psi_1}(x) w_{(\psi_2,\overline{\psi_1})}(x,\lambda)}. \tag{1.2.31}$$

where the ultimate equality follows by definition of the $\dagger$ superscript. Substituting the asymptotic expansions (1.2.21) for $w_{(\psi_1,\overline{\psi_2})}(x,\lambda)$ and $w_{(\psi_2,\overline{\psi_1})}(x,\lambda)$ into the left-hand and right-hand sides of the preceding equation, respectively, and using the definition (1.2.25) for $\tilde{I}_n(\psi_1,\overline{\psi_2})$ and $\tilde{I}_n(\psi_2,\overline{\psi_1})$, the second assertion follows as well. $\square$

**Lemma 1.2.3.** *For every $(\psi_1, \overline{\psi_2}) \in \mathcal{S}(\mathbb{R})^2$ and $\lambda \in \mathbb{R}$ sufficiently large as in Lemma 1.2.2, it holds that*

$$\tilde{F}_L(\psi_1, \overline{\psi_2}; \lambda) = 2\cos\Big(-\lambda L + \kappa \int_{-L}^{L} dx\overline{\psi_2}(x) w_{(\psi_1,\overline{\psi_2})}(x,\lambda)\Big), \tag{1.2.32}$$

*where $\tilde{F}_L$ is defined in (1.1.11).*

*Proof.* Since the trace is invariant under unitary transformation and $W_{(\psi_1,\overline{\psi_2})}$ is $2L$-periodic, we have that

$$\tilde{F}_L(\psi_1,\overline{\psi_2};\lambda) = \mathrm{tr}_{\mathbb{C}^2}\, T_{L,(\psi_1,\overline{\psi_2})}(\lambda) = \mathrm{tr}_{\mathbb{C}^2} \exp\Big(Z_{L,(\psi_1,\overline{\psi_2})}(\lambda)\Big), \tag{1.2.33}$$

so we have reduced to considering the right-hand side expression.

Using that $Z_{L,(\psi_1,\overline{\psi_2})}(\lambda)$ is diagonal and applying formula (1.2.22) and Lemma 1.2.2, we find that

$$Z_{L,(\psi_1,\overline{\psi_2})}(\lambda) = \begin{pmatrix} -i\lambda L + i\kappa \int_{-L}^{L} dx\overline{\psi_2}(x) w_{(\psi_1,\overline{\psi_2})}(x,\lambda) & 0 \\ 0 & i\lambda L - i\kappa \int_{-L}^{L} dx\overline{\psi_2}(x) w_{(\psi_1,\overline{\psi_2})}(x,\lambda)) \end{pmatrix}, \tag{1.2.34}$$

it follows that the exponential of $Z_{L,(\psi_1,\overline{\psi_2})}(\lambda)$ is the diagonal matrix with the entries given by the exponential of the entries of $Z_L(\lambda)$. Using the elementary trigonometric identity

$$e^{iz} + e^{-iz} = 2\cos(z), \qquad z \in \mathbb{C}, \tag{1.2.35}$$

we then obtain that

$$\mathrm{tr}_{\mathbb{C}^2} \exp\Big(Z_{L,(\psi_1,\overline{\psi_2})}(\lambda)\Big) = 2\cos\Big(-\lambda L + \kappa \int_{-L}^{L} dx\overline{\psi_2}(x) w_{(\psi_1,\overline{\psi_2})}(x,\lambda)\Big), \tag{1.2.36}$$

which completes the proof of the lemma. $\square$

**Remark 1.2.4.** By Lemma 1.1.2, we have the involution relation

$$\mathrm{tr}_{\mathbb{C}^2}\, T_{L,(\psi_1,\overline{\psi_2})}(\lambda) = \mathrm{tr}_{\mathbb{C}^2}\Big(\sigma T^{\dagger}_{L,(\psi_1,\overline{\psi_2})}(\bar{\lambda})\sigma\Big) = \mathrm{tr}_{\mathbb{C}^2}\, T^{\dagger}_{L,(\psi_1,\overline{\psi_2})}(\bar{\lambda}) = \overline{\mathrm{tr}_{\mathbb{C}^2}\Big(T_{L,(\psi_2,\overline{\psi_1})}(\bar{\lambda})\Big)}, \tag{1.2.37}$$

where we use the cyclicity of trace and $\sigma^2 = I_{\mathbb{C}^2}$ to obtain the penultimate equality. Consequently, we have that

$$\tilde{F}_L(\psi_1, \overline{\psi_2}; \lambda) = \overline{\tilde{F}_L(\psi_2, \overline{\psi_1}; \bar{\lambda})}. \tag{1.2.38}$$

Consequently, if we take twice the real part of $\tilde{F}_L(\psi_1, \overline{\psi_2}; \lambda)$,

$$F_{L,\mathrm{Re}}(\psi_1, \overline{\psi_2}; \lambda) \coloneqq 2\,\mathrm{Re}\Big\{\tilde{F}_L(\psi_1, \overline{\psi_2}; \lambda)\Big\}, \qquad \forall (\psi_1, \overline{\psi_2}, \lambda) \in C^\infty(\mathbb{T}_L)^2 \times \mathbb{C}, \tag{1.2.39}$$

then we obtain from (1.2.32) that

$$\begin{aligned} F_{L,\mathrm{Re}}(\psi_1, \overline{\psi_2}; \lambda) = 2\cos\Big(-\lambda L + \kappa \int_{-L}^{L} dx \overline{\psi_2}(x) w_{(\psi_1,\overline{\psi_2})}(x,\lambda)\Big) \\ + 2\cos\Big(-\overline{\lambda} L + \kappa \int_{-L}^{L} dx \overline{\psi_1}(x) w_{(\psi_2,\overline{\psi_1})}(x,\bar{\lambda})\Big). \end{aligned} \tag{1.2.40}$$

Similarly, if we take twice the imaginary part of $\tilde{F}_L(\psi_1, \overline{\psi_2}; \lambda)$,

$$F_{L,\mathrm{Im}}(\psi_1, \overline{\psi_2}; \lambda) \coloneqq 2\,\mathrm{Im}\Big\{\tilde{F}_L(\psi_1, \overline{\psi_2})\Big\}, \tag{1.2.41}$$

then we have that

$$\begin{aligned} F_{L,\mathrm{Im}}(\psi_1, \overline{\psi_2}; \lambda) = -i\Bigg(2\cos\Big(-\lambda L + \kappa \int_{-L}^{L} dx \overline{\psi_2}(x) w_{(\psi_1,\overline{\psi_2})}(x,\lambda)\Big) \\ -2\cos\Big(-\overline{\lambda} L + \kappa \int_{-L}^{L} dx \overline{\psi_1}(x) w_{(\psi_2,\overline{\psi_1})}(x,\overline{\lambda})\Big)\Bigg). \end{aligned} \tag{1.2.42}$$

Moreover, we can regard $F_{L,\mathrm{Re}}(\cdot,\cdot;\lambda)$ and $F_{L,\mathrm{Im}}(\cdot,\cdot;\lambda)$, respectively, as restrictions of the complex functionals of four variables to the subspace $\psi_{\bar{1}} = \overline{\psi_1}, \psi_{\bar{2}} = \overline{\psi_2}$. More precisely, for fixed $\lambda \in \mathbb{C}$, define complex-valued functionals on $C^\infty(\mathbb{T}_L)^4$ by

$$\begin{aligned} \tilde{F}_{L,\mathrm{Re}}(\psi_1, \psi_{\bar{2}}, \psi_2, \psi_{\bar{1}}; \lambda) &\coloneqq \tilde{F}_L(\psi_1, \psi_{\bar{2}}; \lambda) + \tilde{F}_L(\psi_2, \psi_{\bar{1}}; \overline{\lambda}), \\ \tilde{F}_{L,\mathrm{Im}}(\psi_1, \psi_{\bar{2}}, \psi_2, \psi_{\bar{1}}; \lambda) &\coloneqq -i\Big(\tilde{F}_L(\psi_1, \psi_{\bar{2}}; \lambda) - \tilde{F}_L(\psi_2, \psi_{\bar{1}}; \overline{\lambda})\Big), \end{aligned} \tag{1.2.43}$$

so that

$$\begin{aligned} F_{L,\mathrm{Re}}(\psi_1, \psi_{\bar{2}}; \lambda) &= \tilde{F}_{L,\mathrm{Re}}(\psi_1, \overline{\psi_2}, \psi_2, \overline{\psi_1}; \lambda) \\ F_{L,\mathrm{Im}}(\psi_1, \psi_{\bar{2}}; \lambda) &= \tilde{F}_{L,\mathrm{Im}}(\psi_1, \overline{\psi_2}, \psi_2, \overline{\psi_1}; \lambda). \end{aligned} \tag{1.2.44}$$

Consequently, $F_{L,\mathrm{Re}}(\lambda)$ and $F_{L,\mathrm{Im}}(\lambda)$ extend with an abuse of notation to well-defined smooth functionals on the space $C^\infty(\mathbb{T}_L; \mathcal{V})$ (recall the space of matrices $\mathcal{V}$ in (4.3.12)) given by

$$\begin{cases} F_{L,\mathrm{Re}}(\gamma; \lambda) \coloneqq F_{L,\mathrm{Re}}(\phi_1, \overline{\phi_2}; \lambda), \\ F_{L,\mathrm{Im}}(\gamma; \lambda) \coloneqq F_{L,\mathrm{Im}}(\phi_1, \overline{\phi_2}; \lambda) \end{cases}, \qquad \forall \gamma = \frac{1}{2}\mathrm{odiag}(\phi_1, \overline{\phi_2}, \phi_2, \overline{\phi_1}), \tag{1.2.45}$$

which belong to the admissible algebra $\mathcal{A}_{\mathcal{S},\mathcal{V}}$, provided that $\tilde{F}_L \in \mathcal{A}_{\mathcal{S},\mathbb{C}}$, a result we postpone until the next subsection. By the same reasoning, the functionals

$$\begin{aligned} I_{b,n}(\gamma) &\coloneqq \frac{1}{2}\Big(\tilde{I}_n(\phi_1, \overline{\phi_2}) + \tilde{I}_n(\phi_2, \overline{\phi_1})\Big) \\ &= \frac{1}{2}\int_{-L}^{L} dx \Big(\overline{\phi_2}(x) w_{n,(\phi_1,\overline{\phi_2})}(x) + \overline{\phi_1}(x) w_{n,(\phi_2,\overline{\phi_1})}(x)\Big), \quad \forall \gamma = \frac{1}{2}\mathrm{odiag}(\phi_1, \overline{\phi_2}, \phi_2, \overline{\phi_1}), \end{aligned} \tag{1.2.46}$$

where the subscript $b$ is to denote the dependence on two inputs, extend to smooth functionals on $C^\infty(\mathbb{T}_L; \mathcal{V})$ which belong to $\mathcal{A}_{\mathcal{S},\mathcal{V}}$. This latter admissibility can be verified using the results of Section 4.5.2. Note that by Lemma 1.2.2, the functionals $I_{b,n}$ are real-valued.

## 1.3 Poisson Commutativity

In this last subsection of the appendix, we show that the functionals $I_{b,n}$ defined in (1.2.46) are in involution with respect to the Poisson bracket $\{\cdot,\cdot\}_{L^2,\mathcal{V}}$ defined in Proposition 4.3.2. We obtain this result by first showing that the generating functionals $\tilde{F}_L(\lambda), \tilde{F}_L(\mu)$, for $\lambda, \mu \in \mathbb{C}$, are in involution with respect to the Poisson bracket $\{\cdot,\cdot\}_{L^2,\mathbb{C}}$. The reader will recall that the $\tilde{F}_L$ was defined in (1.1.11) above.

Given two complex-valued functionals $F, G$ on $C^\infty(\mathbb{T}_L)^2$ satisfying the conditions of Remark 4.3.6, we recall their Poisson bracket is defined by

$$\{F, G\}_{L^2,\mathbb{C}}(\psi_1, \psi_2) = -i \int_{-L}^{L} dx(\boldsymbol{\nabla}_1 F(\psi_1, \psi_2)\boldsymbol{\nabla}_{\bar{2}} G(\psi_1, \psi_2) - \boldsymbol{\nabla}_{\bar{2}} F(\psi_1, \psi_2)\boldsymbol{\nabla}_1 G(\psi_1, \psi_2))(x), \tag{1.3.1}$$

where $\boldsymbol{\nabla}_1$ and $\boldsymbol{\nabla}_{\bar{2}}$ denote the variational derivatives defined in (4.3.1). Now let $A$ and $B$ be two complex-matrix-valued functionals on $C^\infty(\mathbb{T}_L)^2$. We introduce the notation

$$\{A \overset{\otimes}{,} B\}_{L^2,\mathbb{C}}(\psi_1, \psi_2) := -i \int_{-L}^{L} dx(\boldsymbol{\nabla}_1 A(\psi_1, \psi_2) \otimes \boldsymbol{\nabla}_{\bar{2}} B(\psi_1, \psi_2) - \boldsymbol{\nabla}_{\bar{2}} A(\psi_1, \psi_2) \otimes \boldsymbol{\nabla}_1 B(\psi_1, \psi_2))(x), \tag{1.3.2}$$

where our identification of the tensor product is the $4 \times 4$ matrix

$$(A \otimes B)_{jk,mn} = A_{jm} B_{kn}, \qquad j, m, k, n \in \{1, 2\}, \tag{1.3.3}$$

so that

$$\{A \overset{\otimes}{,} B\}_{L^2,\mathbb{C}_{jk,mn}} = \{A_{jm}, B_{kn}\}_{L^2,\mathbb{C}}. \tag{1.3.4}$$

**Remark 1.3.1.** An observation important for our identities in the sequel is that the notation $\{\overset{\otimes}{,}\}$ admits an obvious extension to general $n \times n$ matrices.

The reader may check that the above tensor Poisson bracket notation has the following properties:

**Skew-symmetry**

$$\{A \overset{\otimes}{,} B\}_{L^2,\mathbb{C}} = -P\{B \overset{\otimes}{,} A\}_{L^2,\mathbb{C}} P, \tag{1.3.5}$$

where $P$ is the permutation matrix in $\mathbb{C}^2 \otimes \mathbb{C}^2$ defined by $P(\xi \otimes \eta) = \eta \otimes \xi$, for $\xi, \eta \in \mathbb{C}^2$.

**Leibnitz rule**

$$\{A \overset{\otimes}{,} BC\}_{L^2,\mathbb{C}} = \{A \overset{\otimes}{,} B\}_{L^2,\mathbb{C}}(I_{\mathbb{C}^2} \otimes C) + (I_{\mathbb{C}^2} \otimes B)\{A \overset{\otimes}{,} C\}_{L^2,\mathbb{C}}, \tag{1.3.6}$$

**Jacobi identity**

$$\begin{aligned} 0 &= \{A \overset{\otimes}{,} \{B \overset{\otimes}{,} C\}_{L^2,\mathbb{C}}\}_{L^2,\mathbb{C}} + P_{13}P_{23}\{C \overset{\otimes}{,} \{A \overset{\otimes}{,} B\}_{L^2,\mathbb{C}}\}_{L^2,\mathbb{C}} P_{23}P_{13} \\ &\quad + P_{13}P_{12}\{B \overset{\otimes}{,} \{C \overset{\otimes}{,} A\}_{L^2,\mathbb{C}}\}_{L^2,\mathbb{C}} P_{12}P_{13}, \end{aligned} \tag{1.3.7}$$

where $P_{ij}$ is the permutation matrix in $(\mathbb{C}^2)^{\otimes 3}$ which swaps the $i^{th}$ and $j^{th}$ element of a tensor $\xi_1 \otimes \xi_2 \otimes \xi_3$, for $i, j \in \{1, 2, 3\}$.

**Remark 1.3.2.** The reader can also check that $P$ is idempotent (i.e. $P^2 = I_{\mathbb{C}^2}$) and $P(A \otimes B) = (B \otimes A)P$, for any $2 \times 2$ matrices $A, B$.

With the above notation in hand, we proceed to compute Poisson brackets. Let us consider $U_{(\psi_1,\psi_2)}(z, \lambda)$ from (1.1.3) as a functional of $(\psi_1, \psi_2)$, for fixed $(z, \lambda)$. For the reader's benefit, we recall that

$$U_{(\psi_1,\psi_2)}(x, \lambda) = \frac{\lambda}{2i}\sigma_3 + U_0(x) = \frac{\lambda}{2i}\sigma_3 + \sqrt{\kappa}(\psi_2(x)\sigma_+ + \psi_1(x)\sigma_-), \tag{1.3.8}$$

where $U_0(x)$ is defined in (1.1.3). The first objective is to prove the following lemma which gives the so-called *fundamental Poisson brackets.*

**Lemma 1.3.3** (Fundamental Poisson brackets)**.** *For any $(\lambda, \mu) \in \mathbb{C}^2$, we have the distributional (on $\mathbb{T}_L^2$) identity*

$$\{U(x, \lambda) \overset{\otimes}{,} U(y, \mu)\}_{L^2,\mathbb{C}} = -[r(\lambda - \mu), U(x, \lambda) \otimes I_{\mathbb{C}^2} + I_{\mathbb{C}^2} \otimes U(y, \mu)]\delta(x - y), \tag{1.3.9}$$

*where $r(\lambda - \mu) \coloneqq -\frac{\kappa}{(\lambda-\mu)}P$.*[2]

[2]This matrix $r$ is called an *r-matrix* in the integrable systems literature and is a central object in the study of such systems.

*Proof.* We recall the (classical) canonical commutation relations

$$\{\psi_1(x), \psi_1(y)\}_{L^2,\mathbb{C}} = \{\psi_2(x), \psi_2(y)\}_{L^2,\mathbb{C}} = 0, \quad \{\psi_1(x), \psi_2(y)\}_{L^2,\mathbb{C}} = -i\delta(x-y), \tag{1.3.10}$$

which should be interpreted in the sense of tempered distributions on $\mathbb{T}_L^2$. It then follows from (1.3.8) that

$$(\boldsymbol{\nabla}_1 U(x,\lambda))(\psi_1,\psi_2) = \sqrt{\kappa}\sigma_- \delta_x, \quad (\boldsymbol{\nabla}_{\bar{2}} U(x,\lambda))(\psi_1,\psi_2) = \sqrt{\kappa}\sigma_+ \delta_x, \tag{1.3.11}$$

where $\delta_x$ is the Dirac mass centered at the point $x$. Hence,

$$\begin{aligned}
&\{U(x,\lambda) \overset{\otimes}{,} U(y,\mu)\}_{L^2,\mathbb{C}}(\psi_1,\psi_2) \\
&= -i\int_{-L}^{L} dz((\boldsymbol{\nabla}_1 U(x,\lambda))(\psi_1,\psi_2)(\boldsymbol{\nabla}_{\bar{2}} U(y,\mu))(\psi_1,\psi_2) - (\boldsymbol{\nabla}_{\bar{2}} U(x,\lambda))(\psi_1,\psi_2)(\boldsymbol{\nabla}_1 U(y,\mu))(\psi_1,\psi_2))(z) \\
&= -i\kappa \int_{-L}^{L} dz \delta(z-x)\delta(z-y)(\sigma_- \otimes \sigma_+ - \sigma_+ \otimes \sigma_-) \\
&= -i\kappa\delta(x-y)(\sigma_- \otimes \sigma_+ - \sigma_+ \otimes \sigma_-).
\end{aligned}$$

One can check from the commutation relations for the Pauli matrices defined in (1.1.5) that

$$\sigma_- \otimes \sigma_+ - \sigma_+ \otimes \sigma_- = \frac{1}{2}[P, \sigma_3 \otimes I_{\mathbb{C}^2}] = -\frac{1}{2}[P, I_{\mathbb{C}^2} \otimes \sigma_3]. \tag{1.3.12}$$

Therefore,

$$\begin{aligned}
i\kappa(\sigma_- \otimes \sigma_+ - \sigma_+ \otimes \sigma_-) &= \frac{i\kappa\lambda}{\lambda-\mu}(\sigma_- \otimes \sigma_+ - \sigma_+ \otimes \sigma_-) - \frac{i\kappa\mu}{\lambda-\mu}(\sigma_- \otimes \sigma_+ - \sigma_+ \otimes \sigma_-) \\
&= -\frac{\kappa}{\lambda-\mu}\left(\frac{\lambda}{2i}[P, \sigma_3 \otimes I_{\mathbb{C}^2}] + \frac{\mu}{2i}[P, I_{\mathbb{C}^2} \otimes \sigma_3]\right).
\end{aligned} \tag{1.3.13}$$

Now recalling the definition of $U(x,\lambda)$ in (1.3.8) and that $P$ commutes with the tensor $U_0(x) \otimes I_{\mathbb{C}^2} + I_{\mathbb{C}^2} \otimes U_0(x)$ by the symmetry of the latter, we obtain the desired conclusion. $\square$

The importance of the fundamental Poisson brackets is that they yield a formula for the Poisson brackets between the entries of the transition matrices $T(x, y, \lambda)$ and $T(x, y, \mu)$, regarded as matrix-valued functionals, as the next lemma shows.

**Lemma 1.3.4.** *For fixed $-L < y < x < L$ and $(\lambda, \mu) \in \mathbb{C}^2$, regard $T(x, y, \lambda)$ as the $\mathbb{C}^2 \otimes \mathbb{C}^2$-matrix valued functional $C^\infty(\mathbb{T}_L)^2$ defined by $(\psi_1, \psi_2) \mapsto T_{(\psi_1,\psi_2)}(x, y, \lambda)$ and similarly for $T(x, y, \mu)$. Then it holds that*

$$\{T(x, y, \lambda) \overset{\otimes}{,} T(x, y, \mu)\}_{L^2,\mathbb{C}} = -[r(\lambda - \mu), T(x, y, \lambda) \otimes T(x, y, \mu)]. \tag{1.3.14}$$

*Proof.* We use the differential equations (1.1.13) and (1.1.14) for the transition matrix in order to prove the lemma. Since the $(a, b)$ entry of the matrix-valued functional $T(x, y, \lambda)$ depends on $(\psi_1, \psi_2)$ through the entries of the matrix-valued functional $U(z, \lambda)$ it follows from the definition of the Poisson bracket $\{\cdot, \cdot\}_{L^2,\mathbb{C}}$ reviewed in (1.3.1) and the chain rule that

$$\begin{aligned}
&\big\{T^{ab}(x, y, \lambda), T^{cd}(x, y, \mu)\big\}_{L^2,\mathbb{C}}(\psi_1, \psi_2) \\
&= \int_y^x \int_y^x dz dz' (\boldsymbol{\nabla}_{U^{jk}(\lambda)} T^{ab}(x, y, \lambda)(\psi_1, \psi_2))(z) \big\{U^{jk}(z, \lambda), U^{\ell m}(z', \mu)\big\}_{L^2,\mathbb{C}}(\psi_1, \psi_2) \\
&\qquad \times (\boldsymbol{\nabla}_{U^{\ell m}(\mu)} T^{cd}(x, y, \mu)(\psi_1, \psi_2))(z'),
\end{aligned} \tag{1.3.15}$$

where $\boldsymbol{\nabla}_{U^{jk}(\lambda)} T^{ab}(x, y, \lambda)$ and $\boldsymbol{\nabla}_{U^{\ell m}(\mu)} T^{cd}(x, y, \mu)$ are the variational derivatives uniquely defined by (a priori in the sense of distributions)

$$\begin{aligned}
dT^{ab}(x, y, \lambda)[\psi_1, \psi_2](\delta U^{jk}(\lambda)) &= \int_{-L}^{L} dz (\boldsymbol{\nabla}_{U^{jk}(\lambda)} T^{ab}(x, y, \lambda)(\psi_1, \psi_2))(z) \delta U^{jk}(z, \lambda), \\
dT^{cd}(x, y, \mu)[\psi_1, \psi_2](\delta U^{\ell m}(\mu)) &= \int_{-L}^{L} dz (\boldsymbol{\nabla}_{U^{\ell m}(\mu)} T^{cd}(x, y, \mu)(\psi_1, \psi_2))(z') \delta U^{\ell m}(z', \mu).
\end{aligned} \tag{1.3.16}$$

In (1.3.15), we use the convention of Einstein summation, so the summation over repeated indices is implicit.

We now seek a formula for $\boldsymbol{\nabla}_{U^{jk}(\lambda)}T^{ab}(x,y,\lambda)$ and $\boldsymbol{\nabla}_{U^{\ell m}(\mu)}T^{cd}(x,y,\mu)$. To find such a formula, we take the Gâteaux derivative of both sides of (1.1.13) at the point $U(\cdot,\lambda)$ in the direction $\delta U(\cdot,\lambda)$ to obtain the equation

$$\begin{cases}\partial_x dT(x,y,\lambda)[U(\cdot,\lambda)](\delta U(\cdot,\lambda)) = U(x,\lambda)dT(x,y,\lambda)[U(\cdot,\lambda)](\delta U(\cdot,\lambda)) + \delta U(x,\lambda)T(x,y,\lambda), \\ dT(x,y,\lambda)[U(\cdot,\lambda)](\delta U(\cdot,\lambda))|_{x=y} = I_{\mathbb{C}^2}. \end{cases} \tag{1.3.17}$$

The reader can check by direct computation that the solution to this equation is given by

$$dT(x,y,\lambda)[U(\cdot,\lambda)](\delta U(\cdot,\lambda)) = \int_y^x dz T(x,y,\lambda)\delta U(z,\lambda)T(z,y,\lambda). \tag{1.3.18}$$

Examining identity (1.3.18) entry-wise, we have that

$$\begin{aligned} dT^{ab}(x,y,\lambda)[U(\cdot,\lambda)](\delta U(\cdot,\lambda)) &= \int_y^x dz T^{aj}(x,y,\lambda)\delta U^{jk}(z,\lambda)T^{kb}(z,y,\lambda), \\ dT^{cd}(x,y,\mu)[U(\cdot,\lambda)](\delta U(\cdot,\lambda)) &= \int_y^x dz T^{c\ell}(x,y,\mu)\delta U^{\ell m}(z',\mu)T^{md}(z',y,\mu), \end{aligned} \tag{1.3.19}$$

which upon comparison with (1.3.16) yields the identity

$$\begin{aligned} (\boldsymbol{\nabla}_{U^{jk}(\lambda)}T^{ab}(x,y,\lambda)(\psi_1,\psi_2))(z) &= \begin{cases} T^{aj}_{(\psi_1,\psi_2)}(x,y,\lambda)T^{kb}_{(\psi_1,\psi_2)}(z,y,\lambda), & -L<y<z<x<L \\ 0, & \text{otherwise} \end{cases}, \\ (\boldsymbol{\nabla}_{U^{\ell m}(\lambda)}T^{cd}(x,y,\mu)(\psi_1,\psi_2))(z') &= \begin{cases} T^{c\ell}_{(\psi_1,\psi_2)}(x,y,\mu)T^{md}_{(\psi_1,\psi_2)}(z',y,\mu), & -L<y<z'<x<L \\ 0, & \text{otherwise} \end{cases}. \end{aligned} \tag{1.3.20}$$

Substituting the identity (1.3.20) into (1.3.15), we find that

$$\begin{aligned} &\{T(x,y,\lambda)\overset{\otimes}{,}T(x,y,\mu)\}_{L^2,\mathbb{C}}(\psi_1,\psi_2) \\ &= \int_y^x\int_y^x dzdz'\big(T_{(\psi_1,\psi_2)}(x,z,\lambda)\otimes T_{(\psi_1,\psi_2)}(x,z',\mu)\big)\{U(z,\lambda)\overset{\otimes}{,}U(z',\mu)\}_{L^2,\mathbb{C}}(\psi_1,\psi_2) \\ &\qquad\qquad \times \big(T_{(\psi_1,\psi_2)}(z,y,\lambda)\overset{\otimes}{,}T_{(\psi_1,\psi_2)}(z',y,\mu)\big). \end{aligned} \tag{1.3.21}$$

Using the formula given by Lemma 1.3.3, we obtain that the right-hand equals

$$-\int_y^x dz\big(T_{(\psi_1,\psi_2)}(x,z,\lambda)\otimes T_{(\psi_1,\psi_2)}(x,z,\mu)\big)[r(\lambda-\mu),U(z,\lambda)\otimes I_{\mathbb{C}^2}+I_{\mathbb{C}^2}\otimes U(z,\mu)] \\ \times\big(T_{(\psi_1,\psi_2)}(z,y,\lambda)\otimes T_{(\psi_1,\psi_2)}(z,y,\mu)\big). \tag{1.3.22}$$

We now claim that the integrand is the partial derivative with respect to $z$ of

$$\big(T_{(\psi_1,\psi_2)}(x,z,\lambda)\otimes T_{(\psi_1,\psi_2)}(x,z,\mu)\big)r(\lambda-\mu)\big(T_{(\psi_1,\psi_2)}(z,y,\lambda)\otimes T_{(\psi_1,\psi_2)}(z,y,\mu)\big), \tag{1.3.23}$$

which then completes the proof. Indeed, the reader may verify this is the case by direct computation using the Leibnitz rule and the equations (1.1.13) and (1.1.14) for the transition matrix. So upon application of the fundamental theorem of calculus and using the initial condition $T(x,y,\lambda)|_{x=y}=I_{\mathbb{C}^2}$, we obtain the desired conclusion. □

We next check that the functional $\tilde{F}_L(\lambda)$ defined in (1.1.11), is admissible (i.e. it belongs to $\mathcal{A}_{\mathcal{S},\mathbb{C}}$ defined in (4.3.32)). This admissibility will then imply that $F_{L,\mathrm{Re}}(\lambda)$ and $F_{L,\mathrm{Im}}(\lambda)$ defined in (1.2.39) and (1.2.41), respectively, belong to $\mathcal{A}_{\mathcal{S},\mathcal{V}}$ defined in (4.3.19). First, observe that by taking the direction

$$\delta U(z,\lambda)=\sqrt{\kappa}(\delta\psi_2(z)\sigma_+ + \delta\psi_1(z)\sigma_-) \tag{1.3.24}$$

in (1.3.18), we find that

$$\begin{aligned}(\boldsymbol{\nabla}_1 T(x,y,\lambda)(\psi_1,\psi_2))(z)&=\sqrt{\kappa}T_{(\psi_1,\psi_2)}(x,z,\lambda)\sigma_-T_{(\psi_1,\psi_2)}(z,y,\lambda),\\ (\boldsymbol{\nabla}_{\bar{2}} T(x,y,\lambda)(\psi_1,\psi_2))(z)&=\sqrt{\kappa}T_{(\psi_1,\psi_2)}(x,z,\lambda)\sigma_+T_{(\psi_1,\psi_2)}(z,y,\lambda),\end{aligned} \tag{1.3.25}$$

for $z\in[y,x]$, and zero for $z\in(-L,L)\setminus(y,x)$. Letting $x\to L^+$ and $y\to L^-$, we find that

$$\begin{aligned}(\boldsymbol{\nabla}_1 T_L(\lambda)(\psi_1,\psi_2))(z)&=\sqrt{\kappa}T_{(\psi_1,\psi_2)}(L,z,\lambda)\sigma_-T_{(\psi_1,\psi_2)}(z,-L,\lambda),\\ (\boldsymbol{\nabla}_{\bar{2}} T_L(\lambda)(\psi_1,\psi_2))(z)&=\sqrt{\kappa}T_{(\psi_1,\psi_2)}(L,z,\lambda)\sigma_+T_{(\psi_1,\psi_2)}(z,-L,\lambda).\end{aligned} \tag{1.3.26}$$

Note that $\boldsymbol{\nabla}_1 T_L(\lambda)(\psi_1,\psi_2), \boldsymbol{\nabla}_{\bar{2}} T_L(\lambda)(\psi_1,\psi_2)$ are smooth in $(-L,L)$ but discontinuous at the boundary, and consequently do no belong to $C^\infty(\mathbb{T}_L)$ (i.e. $T_L(\lambda)$ is not an admissible functional). However, if we take the $2\times 2$ matrix trace of both sides of the preceding identities and use that the variational derivative commutes with the trace together with the cyclicity of trace, we obtain that the resulting expressions extend smoothly periodically to the entire real line. We summarize the preceding discussion with the following lemma.

**Lemma 1.3.5.** *For any $\lambda \in \mathbb{C}$, $\tilde{F}_L \in \mathcal{A}_{\mathcal{S},\mathbb{C}}$. Consequently, $F_{L,\mathrm{Re}}(\lambda), F_{L,\mathrm{Im}}(\lambda) \in \mathcal{A}_{\mathcal{S},\mathcal{V}}$.*

We now show that traces $\tilde{F}_L(\lambda), \tilde{F}_L(\mu)$, for fixed $\mu, \lambda \in \mathbb{C}$, are in involution with respect to the Poisson bracket $\{\cdot,\cdot\}_{L^2,\mathbb{C}}$. They key ingredient of this result is the identity of Lemma 1.3.4 for the Poisson brackets between the entries of the transition matrices.

**Lemma 1.3.6.** *For any $\lambda, \mu \in \mathbb{C}$, we have that*

$$\left\{\tilde{F}_L(\lambda), \tilde{F}_L(\mu)\right\}_{L^2,\mathbb{C}} \equiv 0. \tag{1.3.27}$$

*Proof.* Applying Lemma 1.3.4, we have that

$$\begin{aligned}&\left[r(\lambda-\mu), T_{(\psi_1,\psi_2)}(x,y,\lambda)\otimes T_{(\psi_1,\psi_2)}(x,y,\mu)\right]\\ &= \int_{-L}^{L} dz(\boldsymbol{\nabla}_1 T(x,y,\lambda)\otimes \boldsymbol{\nabla}_{\bar{2}} T(x,y,\mu) - \boldsymbol{\nabla}_{\bar{2}} T(x,y,\lambda)\otimes \boldsymbol{\nabla}_1 T(x,y,\mu))(\phi_1,\phi_2)(z).\end{aligned} \tag{1.3.28}$$

Taking the $4\times 4$ matrix trace $\mathrm{tr}_{\mathbb{C}^2\otimes\mathbb{C}^2}$ of both sides and using that the trace of a commutator is zero together with the algebraic identity

$$\mathrm{tr}_{\mathbb{C}^2\otimes\mathbb{C}^2}(A\otimes B) = \mathrm{tr}_{\mathbb{C}^2}(A)\,\mathrm{tr}_{\mathbb{C}^2}(B), \tag{1.3.29}$$

for any $2\times 2$ matrices $A, B$, we obtain that

$$\begin{aligned}0 = -\int_{-L}^{L} dz\,&(\boldsymbol{\nabla}_1(\mathrm{tr}_{\mathbb{C}^2}(T(x,y,\lambda))\boldsymbol{\nabla}_{\bar{2}}\,\mathrm{tr}_{\mathbb{C}^2}(T(x,y,\mu)))(\phi_1,\phi_2)(z)\\ &-(\boldsymbol{\nabla}_{\bar{2}}\,\mathrm{tr}_{\mathbb{C}^2}(T(x,y,\lambda))\boldsymbol{\nabla}_1\,\mathrm{tr}_{\mathbb{C}^2}(T(x,y,\mu)))(\phi_1,\phi_2)(z))\,,\end{aligned} \tag{1.3.30}$$

where we also use that the trace commutes with the variational derivative. Now using the continuity in $(x,y)$ of the integrand, we can let $x \to L^-$ and $y \to -L^+$ and use that $\mathrm{tr}_{\mathbb{C}^2}(T_L(\lambda)) = \tilde{F}_L(\lambda)$ by definition (1.1.11) and $\mathrm{tr}_{\mathbb{C}^2}(T_L(\mu)) = \tilde{F}_L(\mu)$ to obtain the desired conclusion. $\square$

Now we show that the functionals $I_{b,n}$ defined in (1.2.46) are mutually involutive with respect to the Poisson structure on $C^\infty(\mathbb{T}_L;\mathcal{V})$. We begin by defining the generating functional

$$\tilde{p}_L(\phi_1,\overline{\phi_2};\lambda) := \arccos\left(\frac{1}{2}\tilde{F}_L(\phi_1,\overline{\phi_2};\lambda)\right), \qquad \forall(\phi_1,\overline{\phi_2},\lambda) \in C^\infty(\mathbb{T}_L)^2 \times \mathbb{C}, \tag{1.3.31}$$

where we take the principal branch of the function arccos. We first want to show that

$$\{\tilde{p}_L(\lambda),\tilde{p}_L(\mu)\}_{L^2,\mathbb{C}}(\phi_1,\overline{\phi_2}) = 0, \qquad \forall(\phi_1,\overline{\phi_2}) \in C^\infty(\mathbb{T}_L)^2, \tag{1.3.32}$$

for $\lambda,\mu \in \mathbb{R}$ with sufficiently large modulus, which requires us to compute the variational derivatives of $\tilde{p}_L(\lambda),\tilde{p}_L(\mu)$.

Recall from (1.2.32) that

$$\frac{1}{2}\tilde{F}_L(\phi_1,\overline{\phi_2};\lambda) = \cos\left(-\lambda L + \kappa\int_{-L}^{L} dx\overline{\phi_2}(x)w_{(\phi_1,\overline{\phi_2})}(x,\lambda)\right). \tag{1.3.33}$$

We want to show that we can choose $\lambda$ so that the cos in the right-hand side of the preceding equation is at positive distance from $\pm 1$ for all $(\phi_1,\overline{\phi_2})$ in a closed ball of $C^\infty(\mathbb{T}_L)$. To this end, we know from Appendix 1.2 that given $(\phi_1,\overline{\phi_2}) \in C^\infty(\mathbb{T}_L)^2$, we can choose

$$\lambda = \lambda(\|\phi_1\|_{L^1(\mathbb{T}_L)},\|\phi_1\|_{L^\infty(\mathbb{T}_L)},\|\phi_2\|_{L^1(\mathbb{T}_L)},\|\phi_2\|_{L^\infty(\mathbb{T}_L)},L) \in \mathbb{R}$$

with sufficiently large modulus so that there exists $w_{(\phi_1,\overline{\phi_2})}(\lambda)$ in (1.2.20) with the asymptotic expansion (1.2.21). Consequently, for any $k \in \mathbb{N}$, we have that

$$\begin{aligned}
\|w_{(\phi_1,\overline{\phi_2})}(\lambda)\|_{L^\infty(\mathbb{T}_L)} &\leq \left\| w_{(\phi_1,\overline{\phi_2})}(\lambda) - \sum_{n=1}^{k} \frac{w_{k,(\phi_1,\overline{\phi_2})}}{\lambda^n} \right\|_{L^\infty(\mathbb{T}_L)} + \sum_{n=1}^{k} \frac{\|w_{k,(\phi_1,\overline{\phi_2})}\|_{L^\infty(\mathbb{T}_L)}}{\lambda^n} \\
&= o(|\lambda|^k) + \sum_{n=1}^{k} \frac{\|w_{k,(\phi_1,\overline{\phi_2})}\|_{L^\infty(\mathbb{T}_L)}}{\lambda^n},
\end{aligned} \tag{1.3.34}$$

where the implicit constant in $o(|\lambda|^k)$ depends only the data $\|\partial_x^{n-1}\phi_j\|_{L^\infty(\mathbb{T}_L)}$ for $n \in \mathbb{N}_{\leq k+1}$ and $j \in \{1,2\}$. By the analysis of Section 4.5.1,

$$\|w_{k,(\phi_1,\overline{\phi_2})}\|_{L^\infty(\mathbb{T}_L)} \lesssim_k \sum_{n=0}^{k} \big( \|\partial_x^n \phi_1\|_{L^\infty(\mathbb{T}_L)} + \|\partial_x^n \phi_2\|_{L^\infty(\mathbb{T}_L)} \big). \tag{1.3.35}$$

Hence,

$$\begin{aligned}
\left| \int_{-L}^{L} dx \overline{\phi_2}(x) w_{(\phi_1,\overline{\phi_2})}(x,\lambda) \right| &\leq 2L \|\phi_2\|_{L^\infty(\mathbb{T}_L)} \|w_{(\phi_1,\overline{\phi_2})}(\lambda)\|_{L^\infty(\mathbb{T}_L)} \\
&\lesssim \frac{2L}{\lambda} \sum_{n=0}^{1} \big( \|\partial_x^n \phi_1\|_{L^\infty(\mathbb{T}_L)} + \|\partial_x^n \phi_2\|_{L^\infty(\mathbb{T}_L)} \big).
\end{aligned} \tag{1.3.36}$$

Thus, given $\varepsilon > 0$, we can choose $\lambda \in \mathbb{R}$ with sufficiently large modulus depending the data

$$(\varepsilon, L, \|\partial_x^n \phi_j\|_{L^\infty(\mathbb{T}_L)}), \qquad \forall (n,j) \in \{0,1\} \times \{1,2\},$$

so that

$$\left| \int_{-L}^{L} dx \overline{\phi_2}(x) w_{(\phi_1,\overline{\phi_2})}(x,\lambda) \right| < \varepsilon. \tag{1.3.37}$$

Also choosing $\lambda$ so that $\min_{k \in \mathbb{Z}}\{|\lambda L - k\pi|\} > 2\varepsilon$, we conclude that given $R > 0$,

$$\min_{k \in \mathbb{Z}} \left\{ \left| k\pi - \lambda L + \kappa \int_{-L}^{L} dx \overline{\phi_2}(x) w_{(\phi_1,\overline{\phi_2})}(x,\lambda) \right| \right\} \geq \delta > 0 \tag{1.3.38}$$

for all $\phi_1, \overline{\phi_2} \in C^\infty(\mathbb{T}_L)$ with $\|\partial_x^n \phi_1\|_{L^\infty(\mathbb{T}_L)}, \|\partial_x^n \phi_2\|_{L^\infty(\mathbb{T}_L)} \leq R$, for $n \in \{0, 1\}$. For such choice of $\lambda$, we have that

$$\tilde{p}_L(\phi_1, \overline{\phi_2}; \lambda) = -\lambda L + \kappa \int_{-L}^{L} dx \overline{\phi_2}(x) w_{(\phi_1, \overline{\phi_2})}(x, \lambda), \qquad \phi_1, \overline{\phi_2} \in C^\infty(\mathbb{T}_L), \tag{1.3.39}$$

for all $\phi_1, \overline{\phi_2} \in C^\infty(\mathbb{T}_L)$ with $\max\{\|\partial_x^n \phi_1\|_{L^\infty(\mathbb{T}_L)}, \|\phi_2\|_{L^\infty(\mathbb{T}_L)}\} \leq R,\ n \in \{0, 1\}$. Moreover, for such $\phi_1, \overline{\phi_2}$, we can use the chain rule without concern over the singularity of $\arccos(z)$ at $z = \pm 1$ to compute the variational derivatives $\tilde{p}_L$, finding

$$\begin{aligned}
(\boldsymbol{\nabla}_1 \tilde{p}_L(\lambda))(\phi_1, \overline{\phi_2}) &= \frac{1}{2}\left(1 - \left(\frac{\tilde{F}_L(\phi_1, \overline{\phi_2}; \lambda)}{2}\right)^2\right)^{-1/2} (\boldsymbol{\nabla}_1 \tilde{F}(\lambda))(\phi_1, \overline{\phi_2}), \\
(\boldsymbol{\nabla}_{\bar{2}} \tilde{p}_L(\lambda))(\phi_1, \overline{\phi_2}) &= \frac{1}{2}\left(1 - \left(\frac{\tilde{F}_L(\phi_1, \overline{\phi_2}; \lambda)}{2}\right)^2\right)^{-1/2} (\boldsymbol{\nabla}_{\bar{2}} \tilde{F}(\lambda))(\phi_1, \overline{\phi_2}),
\end{aligned} \tag{1.3.40}$$

where by Lemma 1.3.5, the variational derivatives of $\tilde{F}_L(\lambda)$ are elements of $C^\infty(C^\infty(\mathbb{T}_L)^2; C^\infty(\mathbb{T}_L))$. Recalling the definition (4.3.33) for the Poisson bracket $\{\cdot, \cdot\}_{L^2, \mathbb{C}}$, we then find that for appropriate $\lambda, \mu \in \mathbb{R}$,

$$\begin{aligned}
&\{\tilde{p}_L(\lambda), \tilde{p}_L(\mu)\}_{L^2, \mathbb{C}}(\phi_1, \overline{\phi_2}) \\
&= -\frac{i}{4}\left(1 - \left(\frac{\tilde{F}_L(\phi_1, \overline{\phi_2}; \lambda)}{2}\right)^2\right)^{-1/2} \left(1 - \left(\frac{\tilde{F}_L(\phi_1, \overline{\phi_2}; \mu)}{2}\right)^2\right)^{-1/2} \\
&\quad \times \int_{-L}^{L} dx \Big((\boldsymbol{\nabla}_1 \tilde{F}_L(\lambda))(\phi_1, \overline{\phi_2})(\boldsymbol{\nabla}_{\bar{2}} \tilde{F}_L(\mu))(\phi_1, \overline{\phi_2}) - (\boldsymbol{\nabla}_{\bar{2}} \tilde{F}_L(\lambda))(\phi_1, \overline{\phi_2})(\boldsymbol{\nabla}_1 \tilde{F}_L(\mu))(\phi_1, \overline{\phi_2})\Big)(x) \\
&= \frac{1}{4}\left(1 - \left(\frac{\tilde{F}_L(\phi_1, \overline{\phi_2}; \lambda)}{2}\right)^2\right)^{-1/2} \left(1 - \left(\frac{\tilde{F}_L(\phi_1, \overline{\phi_2}; \mu)}{2}\right)^2\right)^{-1/2} \Big\{\tilde{F}_L(\lambda), \tilde{F}_L(\mu)\Big\}_{L^2, \mathbb{C}}(\phi_1, \overline{\phi_2}) \\
&= 0,
\end{aligned}$$

where the ultimate equality follows from an application of Lemma 1.3.6.

We now use (1.3.32) to prove the mutual involution of the functionals $I_{b,n}$.

**Proposition 1.3.7.** *For any $n, m \in \mathbb{N}$, it holds that*

$$\{I_{b,n}, I_{b,m}\}_{L^2,\mathcal{V}} \equiv 0 \text{ on } C^\infty(\mathbb{T}_L;\mathcal{V}). \tag{1.3.41}$$

*Proof.* Fix $n, m \in \mathbb{N}$, and let $\gamma = \frac{1}{2}\mathrm{odiag}(\phi_1, \overline{\phi_2}, \phi_2, \overline{\phi_1}) \in C^\infty(\mathbb{T}_L;\mathcal{V})$. Let us first introduce some notation that will simplify the computations in the sequel. Define and

$$p_L(\gamma;\lambda) \coloneqq \tilde{p}_L(\phi_1, \overline{\phi_2};\lambda) + \tilde{p}_L(\phi_2, \overline{\phi_1};\lambda), \qquad \forall(\gamma,\lambda) \in C^\infty(\mathbb{T}_L;\mathcal{V}) \times \mathbb{C}, \tag{1.3.42}$$

where we recall that $\tilde{p}_L$ is defined in (1.3.31). Note that it is tautological that $p_L$ is the restriction of a complex-valued functional on $C^\infty(\mathbb{T}_L)^4$, which by an abuse of notation we write as

$$p_L(\phi_1, \phi_{\bar{2}}, \phi_2, \phi_{\bar{1}};\lambda) = \tilde{p}_L(\phi_1, \phi_{\bar{2}};\lambda) + \tilde{p}_L(\phi_2, \phi_{\bar{1}};\lambda), \qquad \phi_1, \phi_{\bar{1}}, \phi_2, \phi_{\bar{2}} \in C^\infty(\mathbb{T}_L). \tag{1.3.43}$$

Now for $\gamma \in C^\infty(\mathbb{T}_L;\mathcal{V})$, we have by the variational derivative formulation of the Poisson bracket $\{p_L(\lambda), p_L(\mu)\}_{L^2,\mathcal{V}}$ (recall (4.3.28)) and (1.3.43) that

$$\begin{aligned}
&\{p_L(\lambda), p_L(\mu)\}_{L^2,\mathcal{V}}(\gamma) \\
&= -i\int_{-L}^{L} dz((\boldsymbol{\nabla}_1 p_L(\lambda))(\boldsymbol{\nabla}_{\bar{2}} p_L(\mu)) - (\boldsymbol{\nabla}_{\bar{2}} p_L(\lambda))(\boldsymbol{\nabla}_1 p_L(\mu)))(\phi_1, \overline{\phi_2}, \phi_2, \overline{\phi_1})(z) \\
&\quad - i\int_{-L}^{L} dz((\boldsymbol{\nabla}_2 p_L(\lambda))(\boldsymbol{\nabla}_{\bar{1}} p_L(\mu)) - (\boldsymbol{\nabla}_{\bar{1}} p_L(\lambda))(\boldsymbol{\nabla}_2 p_L(\mu)))(\phi_1, \overline{\phi_2}, \phi_2, \overline{\phi_1})(z) \\
&= -i\int_{-L}^{L} dz((\boldsymbol{\nabla}_1 \tilde{p}_L(\lambda)(\boldsymbol{\nabla}_{\bar{2}} \tilde{p}_L(\mu)) - (\boldsymbol{\nabla}_{\bar{2}} \tilde{p}_L(\lambda))(\boldsymbol{\nabla}_1 \tilde{p}_L(\mu)))(\phi_1, \overline{\phi_2})(z) \\
&\quad - i\int_{-L}^{L} dz((\boldsymbol{\nabla}_1 \tilde{p}_L(\lambda)(\boldsymbol{\nabla}_{\bar{2}} \tilde{p}_L(\mu)) - (\boldsymbol{\nabla}_{\bar{2}} \tilde{p}_L(\lambda))(\boldsymbol{\nabla}_1 \tilde{p}_L(\mu)))(\phi_2, \overline{\phi_1})(z).
\end{aligned} \tag{1.3.44}$$

Recalling Remark 4.3.6 for the variational derivative formulation of the Poisson bracket $\{\cdot,\cdot\}_{L^2,\mathbb{C}}$, we can rewrite the right-hand side of the preceding equality to obtain that

$$\{p_L(\lambda), p_L(\mu)\}_{L^2,\mathcal{V}}(\gamma) = \{\tilde{p}_L(\lambda), \tilde{p}_L(\mu)\}_{L^2,\mathbb{C}}(\phi_1, \overline{\phi_2}) + \{\tilde{p}_L(\lambda), \tilde{p}_L(\mu)\}_{L^2,\mathbb{C}}(\phi_2, \overline{\phi_1}). \quad (1.3.45)$$

Given $R > 0$, for all $\gamma \in C^\infty(\mathbb{T}_L; \mathcal{V})$ with $\|\partial_x^n \gamma\|_{L^\infty(\mathbb{T}_L)} \leq R$, for $n \in \{0, 1\}$, we can choose $\lambda, \mu \in \mathbb{R}$ arbitrarily large to apply (1.3.32), yielding that both terms in the right-hand side of the preceding equality are zero. Hence,

$$\{p_L(\lambda), p_L(\mu)\}_{L^2,\mathcal{V}}(\gamma) = 0. \quad (1.3.46)$$

Now by the formula (1.3.39) for $\tilde{p}_L(\lambda)$ and the large real $\lambda$ asymptotic expansion (1.2.21) for $w_{(\phi_1,\overline{\phi_2})}(\lambda)$, we see that

$$\tilde{p}_L(\phi_1, \overline{\phi_2}; \lambda) \sim -\lambda L + \kappa \sum_{k=1}^\infty \frac{\int_{-L}^L dx \overline{\phi_2}(x) w_{k,(\phi_1,\overline{\phi_2})}(x)}{\lambda^k} = -\lambda L + \kappa \sum_{k=1}^\infty \frac{\tilde{I}_k(\phi_1, \overline{\phi_2})}{\lambda^k}, \quad (1.3.47)$$

where the ultimate equality follows from the definition (1.2.25) for $\tilde{I}_k$. Taking the variational derivatives of both sides of the preceding identity, we find that

$$\boldsymbol{\nabla}_1 \tilde{p}_L(\phi_1, \overline{\phi_2}; \lambda) \sim \kappa \sum_{k=1}^\infty \frac{\boldsymbol{\nabla}_1 \tilde{I}_k(\phi_1, \overline{\phi_2})}{\lambda^k}, \qquad \boldsymbol{\nabla}_{\bar{2}} \tilde{p}_L(\phi_1, \overline{\phi_2}; \lambda) \sim \kappa \sum_{k=1}^\infty \frac{\boldsymbol{\nabla}_{\bar{2}} \tilde{I}_k(\phi_1, \overline{\phi_2})}{\lambda^k}. \quad (1.3.48)$$

Substituting the asymptotic expansions (1.3.48) into (1.3.44), we see that

$$\begin{aligned}
0 &= \{p_L(\lambda), p_L(\mu)\}_{L^2,\mathcal{V}}(\gamma) \\
&\sim -i\kappa^2 \sum_{k,j=1}^\infty \frac{1}{\lambda^k \mu^j} \int_{-L}^L dz \left( \boldsymbol{\nabla}_1 \tilde{I}_k(\phi_1, \overline{\phi_2}) \boldsymbol{\nabla}_{\bar{2}} \tilde{I}_j(\phi_1, \overline{\phi_2}) - \boldsymbol{\nabla}_{\bar{2}} \tilde{I}_k(\phi_1, \overline{\phi_2}) \boldsymbol{\nabla}_1 \tilde{I}_j(\phi_1, \overline{\phi_2}) \right)(z) \\
&\quad - i\kappa^2 \sum_{k,j=1}^\infty \frac{1}{\lambda^k \mu^j} \int_{-L}^L dz \left( \boldsymbol{\nabla}_1 \tilde{I}_k(\phi_2, \overline{\phi_1}) \boldsymbol{\nabla}_{\bar{2}} \tilde{I}_j(\phi_2, \overline{\phi_1}) - \boldsymbol{\nabla}_{\bar{2}} \tilde{I}_k(\phi_2, \overline{\phi_1}) \boldsymbol{\nabla}_1 \tilde{I}_j(\phi_2, \overline{\phi_1}) \right)(z) \\
&= \sum_{k,j=1}^\infty \frac{4\{I_{b,k}, I_{b,j}\}_{L^2,\mathcal{V}}(\gamma)}{\lambda^k \mu^j},
\end{aligned} \quad (1.3.49)$$

where the ultimate equality follows from Remark 4.3.3 and the definition (1.2.46) of the functionals $I_{b,n}$. By the uniqueness of coefficients of asymptotic expansions, we conclude that $\{I_{b,k}, I_{b,j}\}_{L^2,\mathcal{V}} \equiv 0$ on $C^\infty(\mathbb{T}_L;\mathcal{V})$, completing the proof of the proposition. $\square$

# Appendix 2

# Locally Convex Spaces

## 2.1 Calculus on Locally Convex Spaces

The following material is intended as a crash course on calculus in the setting of locally convex topological vector spaces. Since we are in general not dealing with Banach spaces or Banach manifolds, the usual notion of the Fréchet derivative is not suitable for our purposes. Indeed, the prototypical example we ask the reader to keep in mind is the Schwartz space $\mathcal{S}(\mathbb{R})$.

One main issue posed by this more general setting is that there are several inequivalent notions of the derivative for maps between locally convex spaces. Here, we use the definition which is typically called the Gâteaux derivative, which has the property that $C^1$ maps are continuous,[1] and hence enables us to regard the derivative of a smooth real-valued functional $f$ at a point $x \in X$, which we denote by $df[x]$, as an element of the topological dual $X^*$.

The following material can be found in lecture notes by Milnor [65]. Many of the definitions we record are standard, but we include them for completeness. The proofs are omitted, but can be found in [38].

[1] For a notion of smoothness which allows for maps to be smooth but not continuous, we refer the reader to the monograph [48].

**Definition 2.1.1** (Topological vector space)**.** A real or complex *topological vector space (tvs)* $X$ is a vector space over a field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ with a topology $\tau$ which is Hausdorff and such that the operations of addition

$$+ : X \times X \to X, \qquad (x, y) \mapsto x + y \tag{2.1.1}$$

and scalar multiplication

$$\cdot : \mathbb{K} \times X \to X, \qquad (\lambda, x) \mapsto \lambda x \tag{2.1.2}$$

are continuous (the domains are equipped with the product topology).

**Definition 2.1.2** (Locally convex space)**.** A tvs $X$ is said to be *locally convex* if every neighborhood $U \ni 0$ contains a neighborhood $U' \ni 0$ which is convex.

A particularly nice consequence of local convexity is the following Hahn-Banach type result.

**Proposition 2.1.3** (Hahn-Banach)**.** *If $X$ is locally convex, then given two distinct vectors $x, y \in X$, there exists a continuous $\mathbb{K}$-linear map $\ell : X \to \mathbb{K}$ with $\ell(x) \neq \ell(y)$.*

**Definition 2.1.4** (Gâteaux derivative)**.** Let $X$ and $Y$ be locally convex $\mathbb{R}$-tvs, let $X_0 \subset X$ and $Y_0 \subset Y$ be open sets, and let $f : X_0 \to Y_0$ be a continuous map. Given a point $x \in X_0$ and a direction $v \in X$, we define the *directional derivative* or *Gâteaux derivative* of $f$ at $x$ in the direction $v$ to be the vector

$$f'(x; v) =: f'_x(v) := \lim_{t \to 0} \frac{f(x + tv) - f(x)}{t}, \tag{2.1.3}$$

if this limit exists. We call the map $f'_x : X \to Y$ the *derivative of $f$ at the point $x$*. We use the notation $df[x](v) := f'(x; v)$.

**Definition 2.1.5** ($C^1$ Gâteaux map)**.** Let $X_0, Y_0$, and $f$ be as above. The map $f : X_0 \to Y_0$ is $C^1$ if $f'(x; v)$ exists for all $x \in X_0, v \in X$ and is continuous as a map

$$f' : X_0 \times X \to Y, \tag{2.1.4}$$

where the domain is equipped with the product topology.

The Gâteaux derivative $f'_x$ of a map $f$ between two locally convex spaces may fail to be linear in the direction $v$. However, $C^1$ smoothness is enough to ensure linearity in the direction variable. We always work with $C^\infty$ functionals (see Definition 2.1.7), so the requisite $C^1$ smoothness is not problematic for our purposes.

**Proposition 2.1.6** (Linearity of derivative)**.** *If $f$ is $C^1$, then for all $x_0$ fixed, the map*

$$X \to Y, \qquad v \mapsto f'(x_0; v) \tag{2.1.5}$$

*is linear.*

Having defined the derivative and $C^1$ regularity, we can inductively define higher-order derivatives and regularity.

**Definition 2.1.7** (Higher derivatives)**.** The map $f : X_0 \to Y_0$ is $C^2$ *Gâteaux* if $f$ is a $C^1$ Gâteaux map and for each $v_1 \in X$ fixed, the map

$$X_0 \to Y, \qquad x \mapsto f'(x; v_1) \tag{2.1.6}$$

is $C^1$ with Gâteaux derivative

$$\lim_{t \to 0} \frac{f'(x + tv_2; v_1) - f'(x; v_1)}{t} \tag{2.1.7}$$

depending continuously on $(x; v_1, v_2) \in X_0 \times X \times X$ equipped with the product topology. If this limit exists, we call it the *second Gâteaux derivative* of $f$ at $x$ in the directions $v_1, v_2$ and denote it by $f''(x; v_1, v_2)$. We inductively define $C^r$ *maps* $X_0 \to Y_0$. If a map is $C^r$ for every $r \in \mathbb{N}$, then we say that $f$ is a $C^\infty$ *map* or alternatively, *smooth map*.

**Proposition 2.1.8** (Symmetry and $r$-linearity of $f^{(r)}_{x_0}$)**.** *If for $r \in \mathbb{N}$, the map $f$ is $C^r$, then for each fixed $x_0 \in X_0$, the map*

$$\underbrace{X \times \cdots \times X}_{r} \to Y, \qquad (v_1, \ldots, v_r) \mapsto f^{(r)}(x_0; v_1, \ldots, v_r) \tag{2.1.8}$$

*is $r$-linear and symmetric, i.e. for any permutation $\pi \in \mathbb{S}_r$,*

$$f^{(r)}(x_0; v_{\pi(1)}, \ldots, v_{\pi(r)}) = f^{(r)}(x_0; v_1, \ldots, v_r). \tag{2.1.9}$$

**Proposition 2.1.9** (Composition)**.** *If $f : X_0 \to Y_0$ and $g : Y_0 \to Z_0$ are $C^r$ maps, then $g \circ f : X_0 \to Z_0$ is $C^r$ and the derivative of $(g \circ f)$ at the point $x \in X_0$ is the map $g'_{f(x)} \circ f'_x : X \to Z$.*

## 2.2 Smooth Locally Convex Manifolds

In this subsection, we use the calculus reviewed in the preceding subsection to introduce the basics of smooth manifolds modeled on locally convex topological vector spaces, which is needed for the construction of the Lie-Poisson manifold structure in Section 3.5. Much of the theory parallels the finite-dimensional setting, where the model space $\mathbb{R}^d$ is now replaced by an arbitrary, possibly infinite-dimensional locally convex tvs. Consequently, many of the definitions below will be familiar to the reader with a minimal knowledge of differential topology, but we record them for completeness. As in the last subsection, we closely follow [65] in our presentation.

**Definition 2.2.1** (Smooth manifold)**.** A *smooth manifold* modeled on a locally convex space $V$ consists of a regular, Hausdorff topological space $M$ together with a collection of homeomorphisms $\varphi_\alpha : V_\alpha \to M_\alpha$ satisfying the following properties:

(M1) $V_\alpha \subset V$ is open.

(M2) $M_\alpha \subset M$ is open and $\bigcup_\alpha M_\alpha = M$.

(M3) $\varphi_\beta^{-1} \circ \varphi_\alpha : \varphi_\alpha^{-1}(M_\alpha \cap M_\beta) \to \varphi_\beta^{-1}(M_\alpha \cap M_\beta)$ is a smooth map between open subsets of $V$. We refer to the maps $\varphi_\alpha$ as *local coordinate systems* on $M$ and the maps $\varphi_\alpha^{-1}$ as *coordinate charts*.

**Remark 2.2.2.** We will sometimes say that the manifold $M$ is a *Fréchet manifold* if the locally convex model space $V$ is a Fréchet space.

Using the smooth structure together with the calculus from the last subsection, we can define the notion of a smooth map between manifolds.

**Definition 2.2.3** (Smooth map)**.** If $M_1$ and $M_2$ are smooth manifolds modeled on locally convex spaces $V_1$ and $V_2$, respectively, then a continuous function $f : M_1 \to M_2$ is *smooth* if the composition

$$\varphi_{\beta,2}^{-1} \circ f \circ \varphi_{\alpha,1} : \varphi_{\alpha,1}^{-1}\big(M_{1,\alpha} \cap f^{-1}(M_{2,\beta})\big) \to V_{2,\beta} \tag{2.2.1}$$

is smooth whenever $f(M_{1,\alpha}) \cap M_{2,\beta} \neq \emptyset$. We say that $f$ is a *diffeomorphism* if it is bijective and both $f$ and $f^{-1}$ are smooth.

**Definition 2.2.4** (Submanifold)**.** A subset $N$ of a smooth locally convex manifold $M$ is a *submanifold* if for each $m \in N$, there exists a chart $(M_\alpha, \varphi_\alpha^{-1})$ about the point $m$, such that

$\varphi_\alpha^{-1}(M_\alpha \cap N) = \varphi_\alpha^{-1}(M_\alpha) \cap W$, where $W$ is a closed subspace of the space $V$ on which $M$ is modeled.

**Remark 2.2.5.** The submanifold $N$ is smooth locally convex manifold modeled on $W$. Indeed, the reader may check that the maps $\varphi_\alpha|_{V_\alpha \cap W} : V_\alpha \cap W \to M_\alpha \cap N$ are homeomorphisms which satisfy properties (M1) - (M3).

In this work, we use the kinematic definition of tangent vectors (i.e. equivalence classes of smooth curves), as opposed to the operational definition (i.e. derivations). While these two definitions are equivalent in the finite-dimensional setting, they are in general inequivalent in the infinite-dimensional setting.

**Definition 2.2.6** (Tangent space)**.** Let $\varphi_\alpha : V_\alpha \to M_\alpha$ be a local coordinate system on $M$ with $x_0 \in M_\alpha$. Let $p_1, p_2 : I \to M$ be smooth maps on an open interval $I \subset \mathbb{R}$ with $p_i(0) = x_0$ for $i = 1, 2$. We say that $p_1 \sim p_2$ if and only if

$$\frac{d}{dt}\big(\varphi_\alpha^{-1} \circ p_1\big)|_{t=0} = \frac{d}{dt}\big(\varphi_\alpha^{-1} \circ p_2\big)|_{t=0}. \tag{2.2.2}$$

The reader may verify that $\sim$ defines an equivalence relation on smooth curves $p : I \to M$ with $p(0) = x_0$. The set of all such equivalence classes is called the *tangent space at* $x_0$, denoted by $T_{x_0}M$.

**Definition 2.2.7** (Tangent bundle)**.** We define the *tangent bundle* $TM$ as a set by

$$\coprod_{x \in M} T_x M.$$

We define a smooth locally convex structure on $TM$ modeled on $V \times V$ by the local coordinate systems

$$\psi_\alpha : V_\alpha \times V \to TM_\alpha \subset TM, \tag{2.2.3}$$

where $\psi_\alpha(u,v)$ is defined to be the equivalence class containing the smooth curve $t \mapsto \varphi_\alpha(u+tv)$ through the point $\varphi_\alpha(u) \in M$. The reader may verify that $\psi_\alpha$ maps $\{u\} \times V$ isomorphically onto the tangent space $T_{\varphi_\alpha(u)}M$.

**Definition 2.2.8** (Derivative)**.** Let $M_1$ and $M_2$ be smooth locally convex manifolds. A smooth map $f : M_1 \to M_2$ induces a continuous map

$$f'_x : T_x M_1 \to T_{f(x)} M_2, \qquad [p_1] \mapsto [f \circ p_1] \tag{2.2.4}$$

called the *derivative of* $f$ *at* $x$. Together, the maps $f'_x$ induce a smooth map

$$f_* : TM_1 \to TM_2, \qquad (x,v) \mapsto (f(x), f'_x(v)) \tag{2.2.5}$$

which maps $T_x M_1$ linearly into $T_{f(x)} M_2$.

**Definition 2.2.9** (Smooth vector field)**.** A *smooth vector field* on $M$ is a smooth map $X : M \to TM$ such that $X(x) \in T_x M$. We denote the vector space of smooth vector fields on $M$ by $\mathfrak{X}(M)$.

# Appendix 3

# Distribution-Valued Operators

We review and develop some properties of distribution-valued operators (DVOs), that is, elements of $\mathcal{L}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k))$, which are used extensively in this work. Most of these properties are a special case of a more general theory involving topological tensor products of locally convex spaces for which we refer the reader to [86, 41, 97] for further reading.

## 3.1 Adjoint

In this section, we record some properties of the adjoint of a DVO as well as some properties of the map taking a DVO to its adjoint. The proofs follow more or less readily from the definition and standard arguments, and are left to the reader.

**Lemma 3.1.1** (Adjoint map). *Let $k \in \mathbb{N}$, and let $A^{(k)} \in \mathcal{L}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k))$. Then there is a unique map $(A^{(k)})^* \in \mathcal{L}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k))$ such that*

$$\left\langle (A^{(k)})^* g^{(k)}, \overline{f^{(k)}} \right\rangle_{\mathcal{S}'(\mathbb{R}^k)-\mathcal{S}(\mathbb{R}^k)} = \overline{\left\langle A^{(k)} f^{(k)}, \overline{g^{(k)}} \right\rangle_{\mathcal{S}'(\mathbb{R}^k)-\mathcal{S}(\mathbb{R}^k)}}, \qquad \forall f^{(k)}, g^{(k)} \in \mathcal{S}(\mathbb{R}^k). \quad (3.1.1)$$

*Furthermore, the adjoint map*

$$* : \mathcal{L}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k)) \to \mathcal{L}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k)), \qquad A^{(k)} \mapsto (A^{(k)})^* \quad (3.1.2)$$

*is a continuous involution.*

*Additionally, for* $B^{(k)} \in \mathcal{L}(\mathcal{S}'(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k))$*, there exists a unique linear map in* $(B^{(k)})^* \in \mathcal{L}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}(\mathbb{R}^k))$ *such that*

$$\left\langle u^{(k)}, \overline{(B^{(k)})^* g^{(k)}} \right\rangle_{\mathcal{S}'(\mathbb{R}^k)-\mathcal{S}(\mathbb{R}^k)} = \left\langle B^{(k)} u^{(k)}, \overline{g^{(k)}} \right\rangle_{\mathcal{S}'(\mathbb{R}^k)-\mathcal{S}(\mathbb{R}^k)}, \quad \forall (g^{(k)}, u^{(k)}) \in \mathcal{S}(\mathbb{R}^k) \times \mathcal{S}'(\mathbb{R}^k). \tag{3.1.3}$$

*Moreover, the adjoint map*

$$* : \mathcal{L}(\mathcal{S}'(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k)) \to \mathcal{L}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}(\mathbb{R}^k)) \tag{3.1.4}$$

*is a continuous involution.*

The next lemma is useful for computing the adjoint of the composition of maps. We omit the proof, which is standard.

**Lemma 3.1.2.** *Let* $A^{(k)} \in \mathcal{L}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k))$ *and* $B^{(k)} \in \mathcal{L}(\mathcal{S}'(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k))$*. Then*

$$\left(B^{(k)} A^{(k)}\right)^* = (A^{(k)})^* (B^{(k)})^*. \tag{3.1.5}$$

**Definition 3.1.3** (Self- and skew-adjoint)**.** Given $k \in \mathbb{N}$, we say that an operator $A^{(k)} \in \mathcal{L}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k))$ is self-adjoint if $(A^{(k)})^* = A^{(k)}$. Similarly, we say that $A^{(k)} \in \mathcal{L}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k))$ is skew-adjoint if $(A^{(k)})^* = -A^{(k)}$.

**Remark 3.1.4.** Note that if $A^{(k)} \in \mathcal{L}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k))$ is an operator mapping $\mathcal{S}(\mathbb{R}^k) \to L^2(\mathbb{R}^k)$, then our definition of self-adjoint does *not* coincide with the usual Hilbert space definition for densely defined operators, but instead with the definition of a symmetric operator.

## 3.2 Trace and Partial Trace

In this section, we generalize the trace of an operator on a separable Hilbert space to the DVO setting. First, we record some remarks to motivate our definition. Since the operator $|f\rangle\langle g|$, where $f, g \in L^2(\mathbb{R}^N)$, has trace equal to $\langle f|g\rangle$, we might try to generalize the notion of trace to pure tensors of the form $f \otimes u$, where $u \in \mathcal{S}'(\mathbb{R}^N)$ and $f \in \mathcal{S}(\mathbb{R}^N)$, by defining

$$\mathrm{Tr}_{1,\ldots,N}(f \otimes u) = \langle u, f\rangle_{\mathcal{S}'(\mathbb{R}^N)-\mathcal{S}(\mathbb{R}^N)} \tag{3.2.1}$$

and hope to extend this definition to $\mathcal{S}(\mathbb{R}^N)\hat{\otimes}\mathcal{S}'(\mathbb{R}^N)$ through linearity, continuity, and density. However, the evaluation map

$$\mathcal{S}(\mathbb{R}^N) \times \mathcal{S}'(\mathbb{R}^N) \to \mathbb{C}, \qquad (f, u) \mapsto \langle u, f\rangle_{\mathcal{S}'(\mathbb{R}^N)-\mathcal{S}(\mathbb{R}^N)}, \tag{3.2.2}$$

is not continuous, but only separately continuous, preventing us from appealing to the universal property of the tensor product to guarantee the existence of a *unique* generalized trace

$$\mathrm{Tr}_{1,\ldots,N} : \mathcal{S}(\mathbb{R}^N)\hat{\otimes}\mathcal{S}'(\mathbb{R}^N) \to \mathbb{C} \tag{3.2.3}$$

satisfying (3.2.1).

Nonetheless, by viewing the trace as a *bilinear* map and using the canonical isomorphisms

$$\mathcal{L}(\mathcal{S}(\mathbb{R}^N), \mathcal{S}'(\mathbb{R}^N)) \cong \mathcal{S}'(\mathbb{R}^{2N}) \text{ and } \mathcal{L}(\mathcal{S}'(\mathbb{R}^N), \mathcal{S}(\mathbb{R}^N)) \cong \mathcal{S}(\mathbb{R}^{2N}), \tag{3.2.4}$$

we can uniquely define the generalized trace of the right-composition of an operator in $\mathcal{L}(\mathcal{S}(\mathbb{R}^N), \mathcal{S}'(\mathbb{R}^N))$ with an operator in $\mathcal{L}(\mathcal{S}'(\mathbb{R}^N), \mathcal{S}(\mathbb{R}^N))$ through the pairing of their Schwartz kernels. More precisely,

$$\mathrm{Tr}_{1,\ldots,N}(A^{(N)}\gamma^{(N)}) = \langle A^{(N)}, (\gamma^{(N)})^t\rangle_{\mathcal{S}'(\mathbb{R}^{2N})-\mathcal{S}(\mathbb{R}^{2N})} \tag{3.2.5}$$

is, with an abuse of notation, the distributional pairing of the Schwartz kernel of $A^{(N)}$, which belongs to $\mathcal{S}'(\mathbb{R}^{2N})$, with the Schwartz kernel of the transpose of $\gamma^{(N)}$,[1], which belongs to $\mathcal{S}(\mathbb{R}^{2N})$. Equivalently, for each fixed $A^{(N)} \in \mathcal{L}(\mathcal{S}(\mathbb{R}^N), \mathcal{S}'(\mathbb{R}^N))$, the Schwartz kernel theorem implies the existence of a unique linear map $\mathcal{L}(\mathcal{S}'(\mathbb{R}^N), \mathcal{S}(\mathbb{R}^N)) \to \mathbb{C}$, such that

$$\mathrm{Tr}_{1,\ldots,N}\big(A^{(N)}(f \otimes g)\big) = \langle A^{(N)} f, g\rangle_{\mathcal{S}'(\mathbb{R}^N)-\mathcal{S}(\mathbb{R}^N)} \tag{3.2.6}$$

for all $f, g \in \mathcal{S}(\mathbb{R}^N)$.

**Definition 3.2.1** (Generalized trace)**.** We define

$$\begin{aligned}
&\mathrm{Tr}_{1,\ldots,N} : \mathcal{L}(\mathcal{S}(\mathbb{R}^N), \mathcal{S}'(\mathbb{R}^N)) \times \mathcal{L}(\mathcal{S}'(\mathbb{R}^N), \mathcal{S}(\mathbb{R}^N)) \to \mathbb{C} \\
&\mathrm{Tr}_{1,\ldots,N}\big(A^{(N)}\gamma^{(N)}\big) \coloneqq \langle A^{(N)}, (\gamma^{(N)})^t\rangle_{\mathcal{S}'(\mathbb{R}^{2N})-\mathcal{S}(\mathbb{R}^{2N})}.
\end{aligned} \tag{3.2.7}$$

**Remark 3.2.2.** The reader can check that if $A^{(N)} \in \mathcal{L}(\mathcal{S}(\mathbb{R}^N), \mathcal{S}'(\mathbb{R}^N))$ and $\gamma^{(N)} \in \mathcal{L}(\mathcal{S}'(\mathbb{R}^N), \mathcal{S}(\mathbb{R}^N))$ are such that $A^{(N)}\gamma^{(N)}$ is a trace-class operator $\rho^{(N)}$, then our definition of the generalized trace of $A^{(N)}\gamma^{(N)}$ coincides with the usual definition of the trace of $\rho^{(N)}$ as an operator on the Hilbert space $L^2(\mathbb{R}^N)$.

We now establish some properties of the generalized trace which are reminiscent of properties of the usual trace encountered in functional analysis.

**Proposition 3.2.3** (Properties of generalized trace)**.** *Let* $A^{(N)} \in \mathcal{L}(\mathcal{S}(\mathbb{R}^N), \mathcal{S}'(\mathbb{R}^N))$*, and let* $\gamma^{(N)} \in \mathcal{L}(\mathcal{S}'(\mathbb{R}^N), \mathcal{S}(\mathbb{R}^N))$*. The following properties hold:*

*(i)* $\mathrm{Tr}_{1,\ldots,N}$ *is separately continuous.*

[1] $(\gamma^{(N)})^t$ is the operator $f \mapsto \int_{\mathbb{R}^N} d\underline{x}'_N \gamma(\underline{x}'_N; \underline{x}_N) f(\underline{x}'_N)$.

*(ii) We have the following identity:*

$$\mathrm{Tr}_{1,\ldots,N}\big((A^{(N)})^*\gamma^{(N)}\big) = \overline{\mathrm{Tr}_{1,\ldots,N}(A^{(N)}(\gamma^{(N)})^*)}. \tag{3.2.8}$$

*(iii) If $B^{(N)} \in \mathcal{L}(\mathcal{S}'(\mathbb{R}^N), \mathcal{S}'(\mathbb{R}^N))$, then $\mathrm{Tr}_{1,\ldots,N}$ satisfies the cyclicity property*

$$\mathrm{Tr}_{1,\ldots,N}\big(\big(B^{(N)}A^{(N)}\big)\gamma^{(N)}\big) = \mathrm{Tr}_{1,\ldots,N}\big(A^{(N)}\big(\gamma^{(N)}B^{(N)}\big)\big). \tag{3.2.9}$$

*Proof.* Assertion (i) follows from the separate continuity of the distributional pairing $\langle\cdot,\cdot\rangle_{\mathcal{S}'(\mathbb{R}^{2N})-\mathcal{S}(\mathbb{R}^{2N})}$.

To prove assertion (ii), it suffices by density of finite linear combinations of pure tensors together with bilinearity and separate continuity of the generalized trace to consider the case where $\gamma^{(N)} = f^{(N)} \otimes g^{(N)}$, for $f^{(N)}, g^{(N)} \in \mathcal{S}(\mathbb{R}^N)$. By definition of the generalized trace,

$$\mathrm{Tr}_{1,\ldots,N}\big((A^{(N)})^*(f^{(N)} \otimes g^{(N)})\big) = \big\langle (A^{(N)})^* f^{(N)}, g^{(N)} \big\rangle_{\mathcal{S}'(\mathbb{R}^N)-\mathcal{S}(\mathbb{R}^N)}, \tag{3.2.10}$$

and by definition of the adjoint in Lemma 3.1.1,

$$\big\langle (A^{(N)})^* f^{(N)}, g^{(N)} \big\rangle_{\mathcal{S}'(\mathbb{R}^N)-\mathcal{S}(\mathbb{R}^N)} = \overline{\Big\langle A^{(N)}\overline{g^{(N)}}, \overline{f^{(N)}} \Big\rangle}_{\mathcal{S}'(\mathbb{R}^N)-\mathcal{S}(\mathbb{R}^N)}. \tag{3.2.11}$$

Since $(\gamma^{(N)})^* = \overline{g^{(N)}} \otimes \overline{f^{(N)}}$, the desired conclusion then follows from another application of the definition of the generalized trace.

To prove assertion (iii), we note that since

$$B^{(N)}A^{(N)} \in \mathcal{L}(\mathcal{S}(\mathbb{R}^N), \mathcal{S}'(\mathbb{R}^N)), \qquad \gamma^{(N)}B^{(N)} \in \mathcal{L}(\mathcal{S}'(\mathbb{R}^N), \mathcal{S}(\mathbb{R}^N)), \tag{3.2.12}$$

all expressions are well-defined. As before, it suffices to consider the case where $\gamma^{(N)} = f^{(N)} \otimes g^{(N)}$, for $f^{(N)}, g^{(N)} \in \mathcal{S}(\mathbb{R}^N)$. The proof then follows readily using the involution property of the adjoint and the definition of generalized trace. $\square$

We now extend the partial trace map to our setting using our bilinear perspective.

**Proposition 3.2.4** (Generalized partial trace)**.** *Let $N \in \mathbb{N}$ and let $k \in \{0, \ldots, N-1\}$. Then there exists a unique bilinear, separately continuous map*

$$\mathrm{Tr}_{k+1,\ldots,N} : \mathcal{L}(\mathcal{S}(\mathbb{R}^N), \mathcal{S}'(\mathbb{R}^N)) \times \mathcal{L}(\mathcal{S}'(\mathbb{R}^N), \mathcal{S}(\mathbb{R}^N)) \to \mathcal{L}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k)), \tag{3.2.13}$$

*which satisfies*

$$\mathrm{Tr}_{k+1,\ldots,N}\big(A^{(N)}(f^{(N)} \otimes g^{(N)})\big) = \int_{\mathbb{R}^{N-k}} d\underline{x}_{k+1;N} (A^{(N)} f^{(N)})(\underline{x}_k, \underline{x}_{k+1;N}) g^{(N)}(\underline{x}_k', \underline{x}_{k+1;N}). \tag{3.2.14}$$

*for all $A^{(N)} \in \mathcal{L}(\mathcal{S}(\mathbb{R}^N), \mathcal{S}'(\mathbb{R}^N))$, and $f^{(N)}, g^{(N)} \in \mathcal{S}(\mathbb{R}^N)$. That is,*

$$\begin{aligned} &\big\langle \mathrm{Tr}_{k+1,\ldots,N}\big(A^{(N)}(f^{(N)} \otimes g^{(N)})\big)\phi^{(k)}, \psi^{(k)} \big\rangle_{\mathcal{S}'(\mathbb{R}^k)-\mathcal{S}(\mathbb{R}^k)} \\ &= \Big\langle A^{(N)} f^{(N)}, \psi^{(k)} \otimes \langle g^{(N)}, \phi^{(k)} \rangle_{\mathcal{S}'_{\underline{x}_k}(\mathbb{R}^k)-\mathcal{S}_{\underline{x}_k}(\mathbb{R}^k)} \Big\rangle_{\mathcal{S}'(\mathbb{R}^N)-\mathcal{S}(\mathbb{R}^N)}, \end{aligned} \tag{3.2.15}$$

*for all $\phi^{(k)}, \psi^{(k)} \in \mathcal{S}(\mathbb{R}^k)$.*

**Remark 3.2.5.** Our notation $\mathrm{Tr}_{k+1,\ldots,N}$ implies a partial trace over the variables with indices belonging to the index set $\{i : k+1 \le i \le N\}$. To alleviate some notational complications, we will use the convention that if the index set of the partial trace is empty, we do not take a partial trace.

*Proof.* We first show uniqueness. Fix $N \in \mathbb{N}$ and $k \in \{0, \ldots, N-1\}$. Fix $A^{(N)} \in \mathcal{L}(\mathcal{S}(\mathbb{R}^N), \mathcal{S}'(\mathbb{R}^N))$. Suppose that there are two maps $\mathrm{Tr}_{k+1,\ldots,N}$ and $\widehat{\mathrm{Tr}}_{k+1,\ldots,N}$ satisfying (3.2.14). Since every element $\gamma^{(N)} \in \mathcal{L}(\mathcal{S}'(\mathbb{R}^N), \mathcal{S}(\mathbb{R}^N))$ is of the form

$$\gamma^{(N)} = \sum_{j=1}^{\infty} \lambda_j f_j^{(k)} \otimes f_j^{(N-k)} \otimes g_j^{(k)} \otimes g_j^{(N-k)}, \tag{3.2.16}$$

where $\{\lambda_j\}_{j\in\mathbb{N}} \in \ell^1$ and $f_j^{(k)}, g_j^{(k)}$ and $f_j^{(N-k)}, g_j^{(N-k)}$ are sequences converging to zero in $\mathcal{S}(\mathbb{R}^k)$ and $\mathcal{S}(\mathbb{R}^{N-k})$, respectively. Since the partial sums converge in $\mathcal{L}(\mathcal{S}'(\mathbb{R}^N), \mathcal{S}(\mathbb{R}^N))$, we have by separate continuity that

$$\begin{aligned}
\mathrm{Tr}_{k+1,\ldots,N}\big(A^{(N)}\gamma^{(N)}\big) &= \sum_{j=1}^{\infty} \lambda_j \,\mathrm{Tr}_{k+1,\ldots,N}\Big(A^{(N)}\Big(f_j^{(k)} \otimes f_j^{(N-k)} \otimes g_j^{(k)} \otimes g_j^{(N-k)}\Big)\Big) \\
&= \sum_{j=1}^{\infty} \lambda_j \widehat{\mathrm{Tr}}_{k+1,\ldots,N}\Big(A^{(N)}\Big(f_j^{(k)} \otimes f_j^{(N-k)} \otimes g_j^{(k)} \otimes g_j^{(N-k)}\Big)\Big) \\
&= \widehat{\mathrm{Tr}}_{k+1,\ldots,N}\big(A^{(N)}\gamma^{(N)}\big),
\end{aligned} \tag{3.2.17}$$

which completes the proof of uniqueness.

We now prove existence. Let $N, k$ and $A^{(N)}$ be fixed as above. For $f^{(k)}, g^{(k)} \in \mathcal{S}(\mathbb{R}^k)$ and $\gamma^{(N)} \in \mathcal{L}(\mathcal{S}'(\mathbb{R}^N), \mathcal{S}(\mathbb{R}^N))$, we define the integral kernel

$$K_{f^{(k)},g^{(k)},\gamma^{(N)}}(\underline{x}_N; \underline{x}_N') \coloneqq g^{(k)}(\underline{x}_k') \int_{\mathbb{R}^k} d\underline{y}_k \gamma^{(N)}(\underline{x}_N; \underline{y}_k, \underline{x}_{k+1;N}') f^{(k)}(\underline{y}_k), \qquad (\underline{x}_N, \underline{x}_N') \in \mathbb{R}^{2N}. \tag{3.2.18}$$

It is evident that $K_{f^{(k)},g^{(k)},\gamma^{(N)}} \in \mathcal{S}(\mathbb{R}^{2N})$. Moreover, it is straightforward to check that the trilinear map

$$\mathcal{S}(\mathbb{R}^k) \times \mathcal{S}(\mathbb{R}^k) \times \mathcal{S}(\mathbb{R}^{2N}) \to \mathcal{S}(\mathbb{R}^{2N}), \qquad (f^{(k)}, g^{(k)}, \gamma^{(N)}) \mapsto K_{f^{(k)},g^{(k)},\gamma^{(N)}} \tag{3.2.19}$$

is continuous, where we abuse notation by using $\gamma^{(N)}$ to denote the Schwartz kernel as well as the operator. Therefore by the Schwartz kernel theorem and the fact that $A^{(N)} \in \mathcal{L}(\mathcal{S}(\mathbb{R}^N), \mathcal{S}'(\mathbb{R}^N))$ by assumption, for fixed $f^{(k)} \in \mathcal{S}(\mathbb{R}^k)$, the map

$$\mathcal{S}(\mathbb{R}^k) \to \mathbb{C}, \qquad g^{(k)} \mapsto \Big\langle K_{A^{(N)}}, K^t_{f^{(k)},g^{(k)},\gamma^{(N)}} \Big\rangle_{\mathcal{S}'(\mathbb{R}^{2N})-\mathcal{S}(\mathbb{R}^{2N})} \tag{3.2.20}$$

defines an element of $\mathcal{S}'(\mathbb{R}^k)$ and the map

$$\mathcal{S}(\mathbb{R}^k) \to \mathcal{S}'(\mathbb{R}^k), \qquad f^{(k)} \mapsto \Big\langle K_{A^{(N)}}, K^t_{f^{(k)},\cdot,\gamma^{(N)}} \Big\rangle_{\mathcal{S}'(\mathbb{R}^{2N})-\mathcal{S}(\mathbb{R}^{2N})} \tag{3.2.21}$$

is continuous. We therefore define $\mathrm{Tr}_{k+1,\ldots,N}(A^{(N)}\gamma^{(N)})$ to be the element of $\mathcal{L}(\mathcal{S}(\mathbb{R}^k),\mathcal{S}'(\mathbb{R}^k))$ given by

$$\left\langle \mathrm{Tr}_{k+1,\ldots,N}(A^{(N)}\gamma^{(N)})f^{(k)},g^{(k)}\right\rangle_{\mathcal{S}'(\mathbb{R}^k)-\mathcal{S}(\mathbb{R}^k)} \coloneqq \left\langle K_{A^{(N)}},K^t_{f^{(k)},g^{(k)},\gamma^{(N)}}\right\rangle_{\mathcal{S}'(\mathbb{R}^{2N})-\mathcal{S}(\mathbb{R}^{2N})}, \quad (3.2.22)$$

which is evidently bilinear in $(A^{(N)},\gamma^{(N)})$.

It remains for us to prove separate continuity. Implicit in our work in the preceding paragraph is continuity in the second entry for fixed $A^{(N)}$. Continuity in the first entry for fixed $\gamma^{(N)} \in \mathcal{L}(\mathcal{S}'(\mathbb{R}^N),\mathcal{S}(\mathbb{R}^N))$ then follows by duality. □

## 3.3 Contractions and The "Good Mapping Property"

Given $A^{(i)} \in \mathcal{L}(\mathcal{S}(\mathbb{R}^i),\mathcal{S}'(\mathbb{R}^i))$, an integer $k \geq i$, and a cardinality-$i$ subset $\{\ell_1,\ldots,\ell_i\} \subset \mathbb{N}_{\leq k}$, we want to define to an operator acting only on the variables associated to $\{\ell_1,\ldots,\ell_i\}$. We have the following result.

**Proposition 3.3.1** ($k$-particle extensions)**.** *There exists a unique* $A^{(i)}_{(\ell_1,\ldots,\ell_i)} \in \mathcal{L}(\mathcal{S}(\mathbb{R}^k),\mathcal{S}'(\mathbb{R}^k))$*, which satisfies*

$$A^{(i)}_{(\ell_1,\ldots,\ell_i)}(f_1\otimes\cdots\otimes f_k)(\underline{x}_k) = A^{(i)}(f_{\ell_1}\otimes\cdots\otimes f_{\ell_i})(x_{\ell_1},\ldots,x_{\ell_i})\cdot\left(\prod_{\ell\in\mathbb{N}_{\leq k}\setminus\{\ell_1,\ldots,\ell_i\}} f_\ell(x_\ell)\right) \quad (3.3.1)$$

*in the sense of tempered distributions.*

*Proof.* We first consider the case $(\ell_1,\ldots,\ell_i) = (1,\ldots,i)$. By the universal property of the tensor product, there exists a unique continuous linear map

$$A^{(i)}_{(1,\ldots,i)} \coloneqq A^{(i)}\otimes Id_{k-i} : \mathcal{S}(\mathbb{R}^i)\hat{\otimes}\mathcal{S}(\mathbb{R}^{k-i}) \to \mathcal{S}'(\mathbb{R}^i)\hat{\otimes}\mathcal{S}'(\mathbb{R}^{k-i}), \quad (3.3.2)$$

satisfying

$$A^{(i)}_{(1,\ldots,i)}(f^{(i)} \otimes g^{(k-i)})(\underline{x}_k) = A^{(i)}(f^{(i)})(\underline{x}_i)g^{(k-i)}(\underline{x}_{k-i}), \qquad \forall f \in \mathcal{S}(\mathbb{R}^i), g \in \mathcal{S}(\mathbb{R}^{k-i}). \quad (3.3.3)$$

For the general cases where $(\ell_1, \ldots, \ell_i) \neq (1, \ldots, i)$, we set

$$A^{(i)}_{(\ell_1,\ldots,\ell_i)} \coloneqq \pi^{-1} \circ A^{(i)}_{(1,\ldots,i)} \circ \pi, \qquad (3.3.4)$$

where $\pi \in \mathbb{S}_k$ is any permutation such that $\pi(\ell_j) = j$ for $j \in \mathbb{N}_{\leq i}$ and we let $\pi$ act on measurable functions by (3.3.29) and on distributions by duality. Let $(\ell_1^*, \ldots, \ell_{k-i}^*)$ denote the increasing ordering of the elements of the set $\mathbb{N}_{\leq k} \setminus \{\ell_1, \ldots, \ell_i\}$. Then for test functions $f_1, \ldots, f_k, g_1, \ldots, g_k \in \mathcal{S}(\mathbb{R})$, we have

$$\begin{aligned}
&\left\langle (\pi^{-1} \circ A^{(i)}_{(1,\ldots,i)} \circ \pi)(\bigotimes_{\ell=1}^k f_\ell), \bigotimes_{\ell=1}^k g_\ell \right\rangle_{\mathcal{S}'(\mathbb{R}^i)-\mathcal{S}(\mathbb{R}^i)} \\
&= \left\langle A^{(i)}(\bigotimes_{j=1}^i f_{\ell_j}) \otimes \bigotimes_{j=1}^{k-i} f_{\ell_j^*}, (\bigotimes_{j=1}^k g_j) \circ \pi \right\rangle_{\mathcal{S}'(\mathbb{R}^k)-\mathcal{S}(\mathbb{R}^k)} \\
&= \left\langle A^{(i)}(\bigotimes_{j=1}^i f_{\ell_j}), \bigotimes_{j=1}^i g_{\ell_j} \right\rangle_{\mathcal{S}'(\mathbb{R}^i)-\mathcal{S}(\mathbb{R}^i)} \cdot \left\langle \bigotimes_{j=1}^{k-i} f_{\ell_j^*}, \bigotimes_{j=1}^{k-i} g_{\ell_j^*} \right\rangle_{\mathcal{S}'(\mathbb{R}^{k-i})-\mathcal{S}(\mathbb{R}^{k-i})} \\
&= \left\langle A^{(i)}(\bigotimes_{j=1}^i f_{\ell_j}), \bigotimes_{j=1}^i g_{\ell_j} \right\rangle_{\mathcal{S}'(\mathbb{R}^i)-\mathcal{S}(\mathbb{R}^i)} \cdot \prod_{j \in \mathbb{N}_{\leq k} \setminus \{\ell_1,\ldots,\ell_i\}} \langle f_j, g_j \rangle_{\mathcal{S}'(\mathbb{R})-\mathcal{S}(\mathbb{R})}, \qquad (3.3.5)
\end{aligned}$$

where the penultimate equality follows from the definition of the tensor product of two distributions. By the density of finite linear combinations of pure tensors in $\mathcal{S}(\mathbb{R}^k)$, it follows from the preceding equality that our definition (3.3.42) is independent of the choice of permutation $\pi \in \mathbb{S}_k$ satisfying $\pi(\ell_j) = j$ for every $j \in \mathbb{N}_{\leq i}$. $\square$

An important property of the above $k$-particle extension is that it preserves self- and skew-adjointness.

**Lemma 3.3.2.** *Let $i \in \mathbb{N}$, let $k \in \mathbb{N}_{\geq i}$, and let $A^{(i)} \in \mathcal{L}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^i))$ be self-adjoint (resp skew-adjoint). Then for any cardinality-$i$ subset $\{\ell_1, \ldots, \ell_i\} \subset \mathbb{N}_{\leq k}$, we have that $A^{(i)}_{(\ell_1,\ldots,\ell_i)}$ is self-adjoint (resp. skew-adjoint).*

*Proof.* Replacing $A^{(i)}$ by $iA^{(i)}$, it suffices to consider the self-adjoint case. By considerations of symmetry, it suffices to consider the case $(\ell_1, \ldots, \ell_i) = (1, \ldots, i)$. The desired conclusion then follows from the fact that

$$\begin{aligned}\left\langle A^{(i)}_{(1,\ldots,i)}(f^{(i)} \otimes f^{(k-i)}) \middle| g^{(i)} \otimes g^{(k-i)} \right\rangle &= \left\langle Af^{(i)} \middle| g^{(i)} \right\rangle \left\langle f^{(k-i)} \middle| g^{(k-i)} \right\rangle \\ &= \left\langle f^{(i)} \middle| A^{(i)} g^{(i)} \right\rangle \left\langle f^{(k-i)} \middle| g^{(k-i)} \right\rangle \\ &= \left\langle f^{(i)} \otimes f^{(k-i)} \middle| A^{(i)}_{(1,\ldots,i)}(g^{(i)} \otimes g^{(k-i)}) \right\rangle, \end{aligned} \tag{3.3.6}$$

for all $(f^{(i)}, f^{(k-i)}, g^{(i)}, g^{(k-i)}) \in (\mathcal{S}(\mathbb{R}^i) \times \mathcal{S}(\mathbb{R}^{k-i}))^2$, linearity, and density of linear combinations of such pure tensors in $\mathcal{S}(\mathbb{R}^k)$. $\square$

Now let $i, j \in \mathbb{N}$, let $k := i + j - 1$, and let $(\alpha, \beta) \in \mathbb{N}_{\leq i} \times \mathbb{N}_{\leq j}$. To construct a Lie bracket in Section 3.5.2, we need to give meaning to the composition

$$A^{(i)}_{(1,\ldots,i)} B^{(j)}_{(i+1,\ldots,i+\beta-1,\alpha,i+\beta,\ldots,k)} \tag{3.3.7}$$

as an operator in $\mathcal{L}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k))$, when $A^{(i)} \in \mathcal{L}(\mathcal{S}(\mathbb{R}^i), \mathcal{S}'(\mathbb{R}^i))$ and $B^{(j)} \in \mathcal{L}(\mathcal{S}(\mathbb{R}^j), \mathcal{S}'(\mathbb{R}^j))$.

**Remark 3.3.3.** Without further conditions on $A^{(i)}$ or $B^{(j)}$, the composition (3.3.7) may not be well-defined. Indeed, consider the operator $A \in \mathcal{L}(\mathcal{S}(\mathbb{R}^2), \mathcal{S}'(\mathbb{R}^2))$ defined by

$$Af := \delta_0 f, \qquad \forall f \in \mathcal{S}(\mathbb{R}^2), \tag{3.3.8}$$

where $\delta_0$ denotes the Dirac mass about the origin in $\mathbb{R}^2$. Then for $f, g \in \mathcal{S}(\mathbb{R})$,

$$\int_{\mathbb{R}} dx_2 (Af^{\otimes 2})(x_1, x_2) g^{\otimes 2}(x_1', x_2) = f(0)g(0)f(x_1)g(x_1')\delta_0(x_1) \in \mathcal{S}'(\mathbb{R}) \otimes \mathcal{S}(\mathbb{R}). \tag{3.3.9}$$

It is easy to show that $f\delta_0 \in \mathcal{S}'(\mathbb{R})$ does not coincide with a Schwartz function.

This issue leads us to a property we call the *good mapping property.* The intuition for the good mapping property is the basic fact from distribution theory that the convolution of a distribution of compact support with a Schwartz function is again a Schwartz function. We recall the definition of the good mapping property here.

**Remark 3.3.4.** By tensoring with identity, we see that if $A^{(i)}$ has the good mapping property, then $A^{(i)}_{(\ell_1,\ldots,\ell_i)}$ has the good mapping property, where $i$ is replaced by $k$ and $\alpha \in \mathbb{N}_{\leq k}$.

## 3.4 The Subspace $\mathcal{L}_{gmp}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k))$

In this section, we expand more on $\mathcal{L}_{gmp}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k))$ as a topological vector subspace of $\mathcal{L}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k))$ and more on the identification of its topological dual.

**Lemma 3.4.1.** $\mathcal{L}_{gmp}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k))$ *is a dense subspace of* $\mathcal{L}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k))$*.*

*Proof.* We first show density, beginning by recalling that $\mathcal{L}_{gmp}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k))$ is endowed with the subspace topology induced by $\mathcal{L}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k))$. Let $A^{(k)} \in \mathcal{L}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k))$, and let $K_{A^{(k)}} \in \mathcal{S}'(\mathbb{R}^{2k})$ denote the Schwartz kernel of $A^{(k)}$. Since $\mathcal{S}(\mathbb{R}^{2k})$ is dense in $\mathcal{S}'(\mathbb{R}^{2k})$, given any bounded subset $\mathfrak{R} \subset \mathcal{S}(\mathbb{R}^{2k})$ and $\varepsilon > 0$, there exists $K_{\mathfrak{R},\varepsilon} \in \mathcal{S}(\mathbb{R}^{2k})$ such that

$$\sup_{\tilde{K} \in \mathfrak{R}} \left| \langle K_{A^{(k)}} - K_{\mathfrak{R},\varepsilon}, \tilde{K} \rangle_{\mathcal{S}'(\mathbb{R}^{2k}) - \mathcal{S}(\mathbb{R}^{2k})} \right| < \varepsilon. \tag{3.4.1}$$

Since the integral operator defined by the kernel $K_{\mathfrak{R},\varepsilon}$ is a continuous endomorphism of $\mathcal{S}(\mathbb{R}^k)$, it belongs to $\mathcal{L}_{gmp}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k))$. Since any bounded subset $\mathfrak{S} \subset \mathcal{S}(\mathbb{R}^k)$ induces a bounded subset $\mathfrak{R} \subset \mathcal{S}(\mathbb{R}^{2k})$ by

$$\mathfrak{R} \coloneqq \mathfrak{S} \otimes \overline{\mathfrak{S}} \coloneqq \{f \otimes \bar{g} : f, g \in \mathfrak{S}\}, \tag{3.4.2}$$

we conclude that given any $\varepsilon > 0$ and bounded subset $\mathfrak{S} \subset \mathcal{S}(\mathbb{R}^k)$, there exists an element $A^{(k)}_{\mathfrak{S},\varepsilon} \in \mathcal{L}(\mathcal{S}'(\mathbb{R}^k), \mathcal{S}(\mathbb{R}^k))$ such that

$$\sup_{f,g \in \mathfrak{S}} \left| \left\langle (A^{(k)} - A^{(k)}_{\mathfrak{S},\varepsilon}) f \middle| g \right\rangle \right| < \varepsilon. \tag{3.4.3}$$

Since the preceding seminorms generate the topology for $\mathcal{L}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k))$, the proof of density is complete. $\square$

Using the preceding lemma, we can show that the strong dual of the subspace $\mathcal{L}_{gmp}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k))$ is isomorphic to the space of linear operators with Schwartz-class kernels.

**Lemma 3.4.2.** *The space $\mathcal{L}_{gmp}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k))^*$ endowed with the strong dual topology is isomorphic to $\mathcal{L}(\mathcal{S}'(\mathbb{R}^k), \mathcal{S}(\mathbb{R}^k))$.*

*Proof.* Since the canonical embedding $\iota : \mathcal{L}_{gmp}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k)) \to \mathcal{L}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k))$ is tautologically continuous, the adjoint map

$$\iota^* : \mathcal{L}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k))^* \to \mathcal{L}_{gmp}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k))^* \tag{3.4.4}$$

is continuous. Now since $\mathcal{L}_{gmp}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k))$ is dense in $\mathcal{L}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k))$, any linear functional

$$\ell \in \mathcal{L}_{gmp}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k))^* \tag{3.4.5}$$

extends to a unique element $\tilde{\ell} \in \mathcal{L}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k))^*$ by the Hahn-Banach theorem. Hence, $\iota^*$ is a continuous bijection. Since the domain of the canonical isomorphism

$$\Phi : \mathcal{L}(\mathcal{S}'(\mathbb{R}^k), \mathcal{S}(\mathbb{R}^k)) \to \mathcal{L}(\mathcal{S}(\mathbb{R}^k), \mathcal{S}'(\mathbb{R}^k))^* \tag{3.4.6}$$

is a Fréchet space, it follows from the open mapping theorem that $\iota^* \circ \Phi$ is an isomorphism. $\square$

# Appendix 4

# Products of Distributions

In this appendix, we review some basic facts from microlocal analysis about the wave front set of a distribution and its application to proving the well-definedness of the product of two distributions, as used in Section 4.4.1. We mostly follow the exposition in Chapter VIII of [40], but refer the reader to Chapter IX, §10 of [79] for a more pedestrian treatment.

**Definition 4.0.3** (Singular support)**.** Let $u \in \mathcal{D}'(\mathbb{R}^k)$. We say that $x \in \mathbb{R}^k$ is a *regular point* of $u$ if and only if there exists an open neighborhood $U \ni x$ and a function $f : U \to \mathbb{C}$ which is $C^\infty$ on $U$, such that

$$\langle u, \phi \rangle_{\mathcal{D}'(\mathbb{R}^k)-\mathcal{D}(\mathbb{R}^k)} = \int_{\mathbb{R}^k} f(x)\phi(x)dx, \qquad \forall \phi \in C_c^\infty(\mathbb{R}^k) \text{ with } \operatorname{supp}(\phi) \subset U. \tag{4.0.1}$$

We call the set

$$\mathbb{R}^k \setminus \{x \in \mathbb{R}^k : \; x \text{ is a regular point for } u\} \tag{4.0.2}$$

the *singular support* of $u$, denoted by $\operatorname{sing\,supp}(u)$.

**Remark 4.0.4.** It is evident that $\operatorname{sing\,supp}(u) \subset \operatorname{supp}(u)$. Since the set of regular points is open (any other point in the neighborhood $U$ above also belongs to the singular support), it follows that $\operatorname{sing\,supp}(u)$ is a closed subset of $\operatorname{supp}(u)$.

The singular support is useful for establishing the well-definedness of a product of distributions $uv$ via localization, as the next proposition shows.

**Proposition 4.0.5.** *Let $u, v \in \mathcal{D}'(\mathbb{R}^k)$, and suppose that* $\operatorname{sing\,supp}(u) \cap \operatorname{sing\,supp}(v) = \emptyset$. *Then there is a unique $w \in \mathcal{D}'(\mathbb{R}^k)$ such that the following holds:*

*(i) If $x \notin \operatorname{sing\,supp}(v)$ and $v = f$ in a neighborhood of $x$, where $f \in C^\infty(\mathbb{R}^k)$, then $w = fu$ in a neighborhood of $x$.*

*(ii) If $x \notin \operatorname{sing\,supp}(u)$ and $u = g$ in a neighborhood of $x$, where $g \in C^\infty(\mathbb{R}^k)$, then $w = gv$ in a neighborhood of $x$.*

*Proof.* See Theorem IX.42 in [79]. $\square$

Next, we introduce the wave front set of a distribution. While the singular support captures the location of the singularities of a distribution, the wave front set also contains information about the directions of the high frequencies that cause these singularities.

**Definition 4.0.6** (Wave front set)**.** Let $u \in \mathcal{D}'(\mathbb{R}^k)$. We say that a point $(\underline{x}_k, \underline{\xi}_k) \in \mathbb{R}^k \times (\mathbb{R}^k \setminus \{0\})$ is a *regular directed point* for $u$ if and only if there exist radii $\varepsilon_x, \varepsilon_\xi > 0$ and a function $\phi \in C_c^\infty(\mathbb{R}^k)$ which is identically one on the open ball $B(\underline{x}_k, \varepsilon_x)$, such that

$$\left|\widehat{\phi u}(\lambda \underline{\eta}_k)\right| \lesssim_N (1 + |\lambda|)^{-N}, \qquad \forall (\underline{\eta}_k, \lambda) \in B(\underline{\xi}_k, \varepsilon_\xi) \times [0, \infty),\ \ \forall N \in \mathbb{N}_0. \tag{4.0.3}$$

We define the *wave front set* of $u$ to be the complement in $\mathbb{R}^k \times (\mathbb{R}^k \setminus \{0\})$ of the set of regular directed points:

$$\mathrm{WF}(u) := \left(\mathbb{R}^k \times (\mathbb{R}^k \setminus \{0\})\right) \setminus \{(\underline{x}_k, \underline{\xi}_k) \in \mathbb{R}^k \times (\mathbb{R}^k \setminus \{0\}) : (\underline{x}_k, \underline{\xi}_k) \text{ is a regular directed point for } u\}. \tag{4.0.4}$$

**Remark 4.0.7.** In [40], Hörmander uses a definition of the wave front set of a distribution $u$, which is seemingly different from our Definition 4.0.6. More precisely, for any $\underline{x}_k \in \mathbb{R}^k$ and $\phi \in C_c^\infty(\mathbb{R}^k)$, such that $\phi(\underline{x}_k) \neq 0$, he defines the set $\Sigma(\phi u)$ consisting of all $\underline{\xi}_k \in \mathbb{R}^k \setminus \{0\}$ having no conic neighborhood $U$ such that

$$|\widehat{\phi u}(\underline{\xi}_k)| \lesssim_N \Big(1 + |\underline{\xi}_k|\Big)^{-N}, \qquad \forall \underline{\xi}_k \in U, \ \forall N \in \mathbb{N}. \tag{4.0.5}$$

He then defines the set $\Sigma_x(u)$ by

$$\Sigma_{\underline{x}_k}(u) \coloneqq \bigcap_{\phi} \Sigma(\phi u), \ \ \phi \in C_c^\infty(\mathbb{R}^k) \text{ s.t. } \phi(\underline{x}_k) \neq 0. \tag{4.0.6}$$

Hörmander's definition of the wave front set of $u$, which we denote by $\widetilde{\mathrm{WF}}(u)$, is then given by

$$\widetilde{\mathrm{WF}}(u) \coloneqq \{(\underline{x}_k, \underline{\xi}_k) \in \mathbb{R}^k \times (\mathbb{R}^k \setminus \{0\}) : \underline{\xi}_k \in \Sigma_{\underline{x}_k}(u)\}. \tag{4.0.7}$$

It follows from Lemma 4.0.8 below that $\widetilde{\mathrm{WF}}(u) = \mathrm{WF}(u)$ (i.e. the two definitions are equivalent).

We record some properties of the wave front set.

**Lemma 4.0.8.** *If $u \in \mathcal{D}'(\mathbb{R}^k)$ and $g \in C_c^\infty(\mathbb{R}^k)$, then $\mathrm{WF}(gu) \subset \mathrm{WF}(u)$. Similarly, if $u \in \mathcal{S}'(\mathbb{R}^k)$ and $g \in \mathcal{S}(\mathbb{R}^k)$, then $\mathrm{WF}(gu) \subset \mathrm{WF}(u)$.*

**Proposition 4.0.9.** *Let $u \in \mathcal{D}'(\mathbb{R}^k)$.*

*(a) $\mathrm{WF}(u)$ is a closed subset of $\mathbb{R}^k \times (\mathbb{R}^k \setminus \{0\})$.*

*(b) For each $\underline{x}_k \in \mathbb{R}^k$, the set*

$$\mathrm{WF}_{\underline{x}_k}(u) \coloneqq \{\underline{\xi}_k \in \mathbb{R}^k \setminus \{0\} : (\underline{x}_k, \underline{\xi}_k) \in \mathrm{WF}(u)\} \tag{4.0.8}$$

*is a cone.*

*(c) If* $v \in \mathcal{D}'(\mathbb{R}^k)$*, then*

$$\mathrm{WF}(u+v) \subset \mathrm{WF}(u) \cup \mathrm{WF}(v). \tag{4.0.9}$$

*(d)* $\operatorname{sing\,supp}(u) = \{\underline{x}_k \in \mathbb{R}^k : \mathrm{WF}_{\underline{x}_k}(u) \neq \emptyset\}$.

*(e) If* $v \in \mathcal{D}'(\mathbb{R}^j)$*, then*

$$\mathrm{WF}(u \otimes v) \subset (\mathrm{WF}(u) \times \mathrm{WF}(v)) \cup ((\mathrm{supp}(u) \times \{0\}) \times \mathrm{WF}(v)) \cup (\mathrm{WF}(u) \times (\mathrm{supp}(v) \times \{0\})). \tag{4.0.10}$$

*(f) If* $u \in \mathcal{S}'(\mathbb{R}^i), v \in \mathcal{S}'(\mathbb{R}^j)$ *and* $w \in \mathcal{S}(\mathbb{R}^{i+j})$ *then*

$$\mathrm{WF}((u \otimes v)w) \subset \mathrm{WF}(u \otimes v).$$

*Proof.* Properties (a) - (c) are quick consequences of the definition of the wave front set. For (d), see Theorem IX.44 in [79]. For property (e), see Theorem 8.2.9 in [40]. Property (f) follows from Lemma 4.0.8. □

In our proof of Lemma 4.4.1, we will need the following result.

**Lemma 4.0.10** (Wave front set of $\delta(x_i - x_j)$)**.** *Let* $k \in \mathbb{N}$*, and let* $i < j \in \mathbb{N}_{\leq k}$*. Then*

$$\mathrm{WF}(\delta(x_i - x_j)) = \{(\underline{x}_k, \underline{\xi}_k) \in \mathbb{R}^k \times (\mathbb{R}^k \setminus \{0\}) : x_i = x_j,\ \ \xi_i + \xi_j = 0,\ \ \textit{and}\ \ \xi_\ell = 0\ \ \forall l \in \mathbb{N}_{\leq k} \setminus \{i,j\}\}.$$

*Proof.* By symmetry, it suffices to consider the case $(i,j) = (1,2)$. Since $\delta(x_1 - x_2)$ has singular support in the hyperplane $\{x_1 = x_2\} \subset \mathbb{R}^k$, it follows from Proposition 4.0.9(d) that $(\underline{x}_k, \underline{\xi}_k) \in \mathrm{WF}(\delta(x_1 - x_2))$ implies that $x_1 = x_2$.

Now suppose that $(\underline{x}_k, \underline{\xi}_k) \in \mathbb{R}^k \times (\mathbb{R}^k \setminus \{0\})$ and $\xi_1 + \xi_2 \neq 0$. We claim that such a point is a regular directed point for $\delta(x_1 - x_2)$ (i.e. it does not belong to the wave front set). Indeed, let $\varphi \in C_c^\infty(\mathbb{R}^k)$ be such that $\varphi(\underline{x}_k) \neq 0$. Then

$$\mathcal{F}(\delta(x_1 - x_2)\varphi)(\underline{\xi}_k') = \int_{\mathbb{R}^{k-1}} d\underline{y}_{2;k} \varphi(y_2, \underline{y}_{2;k}) e^{-i(\xi_1' + \xi_2')y_2 + \underline{\xi}_{3;k}' \cdot \underline{y}_{3;k}}, \qquad \forall \underline{\xi}_k' \in \mathbb{R}^k. \tag{4.0.11}$$

Since $\varphi$ is Schwartz class, repeated integration by parts in $\underline{y}_{2;k}$ yields

$$\left|\mathcal{F}(\delta(x_1 - x_2)\varphi)(\underline{\xi}_k')\right| \lesssim_N \left(1 + |\xi_1' + \xi_2'| + |\underline{\xi}_{3;k}'|\right)^{-N}, \qquad \forall N \in \mathbb{N}_0. \tag{4.0.12}$$

We consider two cases based on the values of $\xi_1$ and $\xi_2$.

I. If $\operatorname{sgn}(\xi_2) = \operatorname{sgn}(\xi_1)$, then

$$|\xi_1 + \xi_2| \geq \max\{|\xi_1|, |\xi_2|\}, \tag{4.0.13}$$

which implies that

$$\left(1 + |\xi_1 + \xi_2| + |\underline{\xi}_{3;k}|\right)^{-N} \lesssim_N \left(1 + |\underline{\xi}_k|\right)^{-N}. \tag{4.0.14}$$

Hence, if $\varepsilon > 0$ is sufficiently small so that $\operatorname{sgn}(\xi_1') = \operatorname{sgn}(\xi_2')$ for all $\underline{\xi}_k' \in B(\underline{\xi}_k, \varepsilon)$, then

$$\left|\mathcal{F}(\delta(x_1 - x_2)\varphi)(\lambda\underline{\xi}_k')\right| \lesssim_N \left(1 + \lambda|\underline{\xi}_k|\right)^{-N}, \qquad \forall \underline{\xi}_k' \in B(\underline{\xi}_k, \varepsilon), \;\; \lambda \in [0, \infty). \tag{4.0.15}$$

II. If $\operatorname{sgn}(\xi_2) = -\operatorname{sgn}(\xi_1)$, then without loss of generality suppose that $|\xi_1| > |\xi_2|$. Then for $\varepsilon > 0$ sufficiently small, we have that there exists $\theta \in (0, 1)$ such that

$$\frac{|\xi_2'|}{|\xi_1'|} \geq \theta, \qquad \forall \underline{\xi}_k' \in B(\underline{\xi}_k, \varepsilon). \tag{4.0.16}$$

So by the reverse triangle inequality,

$$\left(1 + \lambda|\xi_1' + \xi_2'| + \lambda|\underline{\xi}_{3;k}'|\right)^{-N} \lesssim_{\theta,N} \left(1 + \lambda|\underline{\xi}_k|\right)^{-N}, \qquad \forall \underline{\xi}_k' \in B(\underline{\xi}_k, \varepsilon), \;\; \lambda \in [0, \infty). \tag{4.0.17}$$

Now suppose that $(\underline{x}_k, \underline{\xi}_k) \in \mathbb{R}^k \times (\mathbb{R}^k \setminus \{0\})$, $\xi_1 + \xi_2 = 0$, and $\underline{\xi}_{3;k} \neq 0 \in \mathbb{R}^{k-2}$. We claim that such a point is a regular directed point. We consider two cases based on the magnitude of $|\xi_2|$ relative to $|\underline{\xi}_{3;k}|$.

I. If $|\xi_1| \leq |\underline{\xi}_{3;k}|$, then for $\varepsilon > 0$ sufficiently small,

$$\Big(1 + \lambda|\xi_1' + \xi_2'| + \lambda|\underline{\xi}_{3;k}'|\Big)^{-N} \lesssim_N \Big(1 + \lambda|\underline{\xi}_k|\Big)^{-N}, \qquad \forall \underline{\xi}_k' \in B(\underline{\xi}_k, \varepsilon), \ \ \lambda \in [0, \infty). \tag{4.0.18}$$

II. If $|\xi_1| > |\underline{\xi}_{3;k}|$, then for $\varepsilon > 0$ sufficiently small, there exists $\theta \in (0,1)$ such that

$$\frac{|\underline{\xi}_{3;k}'|}{|\xi_1'|} \geq \theta, \qquad \forall \underline{\xi}_k' \in B(\underline{\xi}_k, \varepsilon). \tag{4.0.19}$$

Hence,

$$|\underline{\xi}_{3;k}'| \geq \frac{|\underline{\xi}_{3;k}'|}{2} + \frac{\theta}{4}\Big(|\underline{\xi}_1'| + |\underline{\xi}_2'|\Big), \tag{4.0.20}$$

which implies that

$$\Big(1 + \lambda|\underline{\xi}_{3;k}'|\Big)^{-N} \lesssim_{\theta,N} \Big(1 + \lambda|\underline{\xi}_k|\Big)^{-N}, \qquad \forall \underline{\xi}_k' \in B(\underline{\xi}_k, \varepsilon), \ \ \lambda \in [0, \infty). \tag{4.0.21}$$

Thus, we have shown that

$$\mathrm{WF}(\delta(x_1 - x_2)) \subset \{(\underline{x}_k, \underline{\xi}_k) \in \mathbb{R}^k \times (\mathbb{R}^k \setminus \{0\}) : x_1 = x_2, \ \ \xi_1 + \xi_2 = 0, \ \text{ and } \ \underline{\xi}_{3;k} = 0\}. \tag{4.0.22}$$

For the reverse inclusion, we claim that $(\underline{x}_k, (-\xi_2, \xi_2, \underline{0}_{3;k})) \in \mathbb{R}^k \times (\mathbb{R}^k \setminus \{0\})$ is not a regular directed point for $\delta(x_1 - x_2)$. Indeed, this claim follows from observing that for a bump function $\varphi \in C_c^\infty(\mathbb{R}^k)$ about $\underline{x}_k$, we have that for all $\lambda \in [0, \infty)$,

$$\big|\mathcal{F}(\delta(x_1 - x_2)\varphi)(-\lambda\xi_2, \lambda\xi_2, \underline{0}_{3;k})\big| = \int_{\mathbb{R}^{k-1}} d\underline{x}_{2;k} \varphi(x_2, \underline{x}_{2;k}). \tag{4.0.23}$$

$\square$

We now seek to systematically give meaning to the product of distributions and, in particular, preserve the property that the Fourier transform maps products to convolution. We accomplish this task with a useful criterion due to Hörmander–one which we make heavy use of in Section 4.4–for how to "canonically" define the product of two distributions. Before stating Hörmander's result, we need a few technical preliminaries.

For a closed cone $\Gamma \subset \mathbb{R}^k \times (\mathbb{R}^k \setminus \{0\})$, define the set

$$\mathcal{D}'_\Gamma(\mathbb{R}^k) := \{u \in \mathcal{D}'(\mathbb{R}^k) : \mathrm{WF}(u) \subset \Gamma\}. \tag{4.0.24}$$

**Lemma 4.0.11.** *$u \in \mathcal{D}'(\mathbb{R}^k)$ belongs to $\mathcal{D}'_\Gamma(\mathbb{R}^k)$ if and only if for every $\phi \in C_c^\infty(\mathbb{R}^k)$ and every closed cone $V \subset \mathbb{R}^k$ satisfying*

$$\Gamma \cap (\mathrm{supp}(\phi) \times V) = \emptyset, \tag{4.0.25}$$

*we have that*

$$\sup_{\underline{\xi}_k \in V} |\underline{\xi}_k|^N |\widehat{(\phi u)}(\underline{\xi}_k)| < \infty, \qquad \forall N \in \mathbb{N}. \tag{4.0.26}$$

*Proof.* See Lemma 8.2.1 in [40]. □

It is clear that $\mathcal{D}'_\Gamma(\mathbb{R}^k)$ is a subspace of $\mathcal{D}'(\mathbb{R}^k)$. We say that a sequence $\{u_j\}_{j=1}^\infty$ in $\mathcal{D}'_\Gamma(\mathbb{R}^k)$ and $u \in \mathcal{D}'_\Gamma(\mathbb{R}^k)$, we say that $u_j \to u$ in $\mathcal{D}'_\Gamma(\mathbb{R}^k)$ as $j \to \infty$ if $u_j \to u$ in the weak-* topology on $\mathcal{D}'(\mathbb{R}^k)$ and for every $N \in \mathbb{N}$,

$$\sup_{\underline{\xi}_k \in V} |\underline{\xi}_k|^N |\widehat{(\phi u)}(\underline{\xi}_k) - \widehat{(\phi u_j)}(\underline{\xi}_k)| \to 0, \tag{4.0.27}$$

as $j \to \infty$, for every $\phi \in C_c^\infty(\mathbb{R}^k)$ and closed cone $V \subset \mathbb{R}^k$ such that (4.0.25) holds.

The next lemma shows that $C_c^\infty(\mathbb{R}^k)$ is sequentially dense in the space $\mathcal{D}'_\Gamma(\mathbb{R}^k)$.

**Lemma 4.0.12.** *For every $u \in \mathcal{D}'_\Gamma(\mathbb{R}^k)$, there exists a sequence $u_j \in C_c^\infty(\mathbb{R}^k)$ such that $u_j \to u$ in $\mathcal{D}'_\Gamma(\mathbb{R}^k)$.*

*Proof.* See Theorem 8.2.3 in [40]. $\square$

**Lemma 4.0.13.** *Let $m, n \in \mathbb{N}$ and let $f : \mathbb{R}^m \to \mathbb{R}^n$ be a $C^\infty$ map. Define the set of normals of the map $f$ by*

$$N_f := \{(f(\underline{x}_m), \underline{\eta}_n) \in \mathbb{R}^n \times \mathbb{R}^n : f'(\underline{x}_m)^T \underline{\eta}_n = 0\}, \tag{4.0.28}$$

*where $f'(\underline{x}_m)^T$ denotes the transpose of the matrix $f'(\underline{x}_m)$. Then the pullback distribution $f^*u$ can be defined in one and only one way for all $u \in \mathcal{D}'(\mathbb{R}^n)$ with*

$$N_f \cap \mathrm{WF}(u) = \emptyset \tag{4.0.29}$$

*so that $f^*u = u \circ f$, when $u \in C^\infty(\mathbb{R}^n)$ and for any closed conic subset $\Gamma \subset \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$ satisfying $\Gamma \cap N_f = \emptyset$, we have a continuous map $f^* : \mathcal{D}'_\Gamma(\mathbb{R}^n) \to \mathcal{D}'_{f^*\Gamma}(\mathbb{R}^m)$, where*

$$f^*\Gamma := \{(\underline{x}_m, f'(\underline{x}_m)^T \underline{\eta}_n) : (f(\underline{x}_m), \underline{\eta}_n) \in \Gamma\}. \tag{4.0.30}$$

*In particular, for every $u \in \mathcal{D}'(\mathbb{R}^n)$ satisfying (4.0.29), we have that*

$$\mathrm{WF}(f^*u) \subset f^* \, \mathrm{WF}(u). \tag{4.0.31}$$

*Proof.* See Theorem 8.2.4 in [40]. $\square$

We are now prepared to state Hörmander's criterion for the existence of the product of two distributions.

**Proposition 4.0.14** (Hörmander's criterion)**.** *Let $u_1, u_2 \in \mathcal{D}'(\mathbb{R}^k)$, and suppose that*

$$\mathrm{WF}(u_1) \oplus \mathrm{WF}(u_2) := \{(\underline{x}_k, \underline{\xi}_k) \in \mathbb{R}^k \times (\mathbb{R}^k \backslash \{0\}) : \underline{\xi}_k = \underline{\xi}_{1,k} + \underline{\xi}_{2,k},\ \ (\underline{x}_k, \underline{\xi}_{j,k}) \in \mathrm{WF}(u_j)\ \ \text{for } j = 1, 2\} \tag{4.0.32}$$

*does not contain an element of the form $(\underline{x}_k, 0)$. Then the product $u_1 u_2$ can be defined as the pullback of the tensor product $u_1 \otimes u_2$ by the diagonal map $d : \mathbb{R}^k \to \mathbb{R}^{2k}$. Moreover,*

$$\mathrm{WF}(u_1 u_2) \subset \mathrm{WF}(u_1) \cup \mathrm{WF}(u_2) \cup (\mathrm{WF}(u_1) \oplus \mathrm{WF}(u_2)). \tag{4.0.33}$$

*We refer to this definition of the product $u_1 u_2$ as the* Hörmander product*.*

*Proof.* See Theorem 8.2.10 in [40]. □

Sometimes it is easy to make an ansatz for an explicit formula for the product of two distributions, for example $\delta(x_1 - x_2)\delta(x_2 - x_3)$. The next lemma is useful for verifying that the ansatz indeed coincides with the product distribution defined by Proposition 4.0.14.

**Lemma 4.0.15.** *Let $u, v \in \mathcal{D}'(\mathbb{R}^k)$. Then there exists at most one distribution $w \in \mathcal{D}'(\mathbb{R}^k)$ such that for every $\underline{x}_k \in \mathbb{R}^k$, there exists $\phi \in C_c^\infty(\mathbb{R}^k)$ which is $\equiv 1$ on $B(\underline{x}_k, \varepsilon)$, for some $\varepsilon > 0$, and such that for every $\underline{\xi}_k \in \mathbb{R}^k$,*

$$\mathcal{F}(\phi u) \cdot \mathcal{F}(\phi v)(\underline{\xi}_k - \cdot) \in L^1(\mathbb{R}^k), \tag{4.0.34}$$

*the map*

$$\mathbb{R}^k \to \mathbb{C}, \qquad \underline{\xi}_k \mapsto (\mathcal{F}(\phi u) * \mathcal{F}(\phi v))(\underline{\xi}_k) \tag{4.0.35}$$

*is polynomially bounded, and*

$$\mathcal{F}(\phi^2 w)(\underline{\xi}_k) = (2\pi)^{-k/2} \int_{\mathbb{R}^k} d\underline{\eta}_k \mathcal{F}(\phi u)(\underline{\eta}_k) \mathcal{F}(\phi v)(\underline{\xi}_k - \underline{\eta}_k). \tag{4.0.36}$$

*Proof.* We first claim that for any $\psi \in C_c^\infty(\mathbb{R}^k)$,

$$\mathcal{F}(\psi\phi^2 w)(\underline{\xi}_k) = (2\pi)^{-k/2}(\mathcal{F}(\psi\phi u_1) * \mathcal{F}(\phi u_2))(\underline{\xi}_k) = (2\pi)^{-k/2}(\mathcal{F}(\phi u_1) * \mathcal{F}(\psi\phi u_2))(\underline{\xi}_k), \tag{4.0.37}$$

for all $\underline{\xi}_k \in \mathbb{R}^k$ where the integrals defining the convolutions converge absolutely for $\underline{\xi}_k$ fixed. Indeed, since $\widehat{\psi}$ is Schwartz and $\mathcal{F}(\phi^2 w)$ is analytic,

$$\begin{aligned}\mathcal{F}(\psi\phi^2 w)(\underline{\xi}_k) &= (2\pi)^{-k/2}\int_{\mathbb{R}^k} d\underline{\eta}_k \mathcal{F}(\psi)(\underline{\xi}_k - \underline{\eta}_k)\mathcal{F}(\phi^2 w)(\underline{\eta}_k)\\ &= (2\pi)^{-k/2}\int_{\mathbb{R}^k} d\underline{\eta}_k \mathcal{F}(\psi)(\underline{\xi}_k - \underline{\eta}_k)\left(\int_{\mathbb{R}^k} d\underline{\eta}'_k \mathcal{F}(\phi u_1)(\underline{\eta}_k - \underline{\eta}'_k)\mathcal{F}(\phi u_2)(\underline{\eta}'_k)\right),\end{aligned} \tag{4.0.38}$$

where the integrals are absolutely convergent. Hence, by the Fubini-Tonelli theorem,

$$\begin{aligned}&\int_{\mathbb{R}^k} d\underline{\eta}_k \mathcal{F}(\psi)(\underline{\xi}_k - \underline{\eta}_k)\left(\int_{\mathbb{R}^k} d\underline{\eta}'_k \mathcal{F}(\phi u_1)(\underline{\eta}_k - \underline{\eta}'_k)\mathcal{F}(\phi u_2)(\underline{\eta}'_k)\right)\\ &= \int_{\mathbb{R}^k} d\underline{\eta}'_k \mathcal{F}(\phi u_2)(\underline{\eta}'_k)\left(\int_{\mathbb{R}^k} d\eta_k \mathcal{F}(\psi)(\underline{\xi}_k - \underline{\eta}_k)\mathcal{F}(\phi u_1)(\underline{\eta}_k - \underline{\eta}'_k)\right).\end{aligned} \tag{4.0.39}$$

By the translation invariance of the Lebesgue measure,

$$\begin{aligned}\int_{\mathbb{R}^k} d\underline{\eta}_k \mathcal{F}(\psi)(\underline{\xi}_k - \underline{\eta}_k)\mathcal{F}(\phi u_1)(\underline{\eta}_k - \underline{\eta}'_k) &= \int_{\mathbb{R}^k} d\underline{\eta}_k \mathcal{F}(\psi)(\underline{\xi}_k - \underline{\eta}'_k - \underline{\eta}_k)\mathcal{F}(\phi u_1)(\underline{\eta}_k)\\ &= (\mathcal{F}(\psi) * \mathcal{F}(\phi u_1))(\underline{\xi}_k - \underline{\eta}'_k)\\ &= (2\pi)^{k/2}\mathcal{F}(\psi\phi u_1)(\underline{\xi}_k - \underline{\eta}'_k),\end{aligned} \tag{4.0.40}$$

where the ultimate equality follows from Fourier inversion. Therefore,

$$(2\pi)^{-k/2}\int_{\mathbb{R}^k} d\underline{\eta}'_k \mathcal{F}(\phi u_2)(\underline{\eta}'_k)\left(\int_{\mathbb{R}^k} d\eta_k \mathcal{F}(\psi)(\underline{\xi}_k - \underline{\eta}_k)\mathcal{F}(\phi u_1)(\underline{\eta}_k - \underline{\eta}'_k)\right) = (\mathcal{F}(\psi\phi u_1) * \mathcal{F}(\phi u_2))(\underline{\xi}_k). \tag{4.0.41}$$

By symmetry, we have also shown that

$$\mathcal{F}(\psi\phi^2 w)(\underline{\xi}_k) = (\mathcal{F}(\phi u_1) * \mathcal{F}(\psi\phi u_2))(\underline{\xi}_k). \tag{4.0.42}$$

Now suppose that $w_1, w_2 \in \mathcal{D}'(\mathbb{R}^k)$ are two distributions such that there exist $\phi_1, \phi_2 \in C_c^\infty(\mathbb{R}^k)$ so that

$$\mathcal{F}(\phi_1^2 w_1) = (\mathcal{F}(\phi_1 u_1) * \mathcal{F}(\phi_1 u_2)) \tag{4.0.43}$$

$$\mathcal{F}(\phi_2^2 w_2) = (\mathcal{F}(\phi_2 u_1) * \mathcal{F}(\phi_2 u_2)), \tag{4.0.44}$$

where the integrals defining the convolutions are absolutely convergent for fixed $\underline{\xi}_k$ and there exists $N_1, N_2 \in \mathbb{N}_0$ so that

$$\sup_{\underline{\xi}_k \in \mathbb{R}^k} \langle \underline{\xi}_k \rangle^{-N_1} \int_{\mathbb{R}^k} d\underline{\eta}_k \left| \mathcal{F}(\phi_1 u_1)(\underline{\xi}_k - \underline{\eta}_k) \mathcal{F}(\phi_1 u_2)(\underline{\eta}_k) \right| < \infty \tag{4.0.45}$$

$$\sup_{\underline{\xi}_k \in \mathbb{R}^k} \langle \underline{\xi}_k \rangle^{-N_2} \int_{\mathbb{R}^k} d\underline{\eta}_k \left| \mathcal{F}(\phi_2 u_1)(\underline{\xi}_k - \underline{\eta}_k) \mathcal{F}(\phi_2 u_2)(\underline{\eta}_k) \right| < \infty. \tag{4.0.46}$$

Then by (4.0.37),

$$\begin{aligned} \mathcal{F}(\phi_1^2 \phi_2^2 w_1) = (2\pi)^{-k/2} \mathcal{F}(\phi_2) * \mathcal{F}(\phi_2 \phi_1^2 w_1) &= (2\pi)^{-k/2} \mathcal{F}(\phi_2) * (\mathcal{F}(\phi_1 u_1) * \mathcal{F}(\phi_1 \phi_2 u_2)) \\ &= (2\pi)^{-k/2} \mathcal{F}(\phi_2 \phi_1 u_1) * \mathcal{F}(\phi_2 \phi_1 u_2), \end{aligned} \tag{4.0.47}$$

where the ultimate equality is justified since $\mathcal{F}(\phi_2)$ is a Schwartz function and the fact that there exists some $N \in \mathbb{N}$ so that

$$\sup_{\underline{\xi}_k \in \mathbb{R}^k} \langle \underline{\xi}_k \rangle^{-N} \int_{\mathbb{R}^k} d\underline{\eta}_k \left| \mathcal{F}(\phi_1 u_1)(\underline{\xi}_k - \underline{\eta}_k) \mathcal{F}(\phi_1 \phi_2 u_2)(\underline{\eta}_k) \right| < \infty, \tag{4.0.48}$$

which is a consequence of (4.0.45). Similarly,

$$\mathcal{F}(\phi_1^2 \phi_2 w_2) = (2\pi)^{-k/2} \mathcal{F}(\phi_1 \phi_2 u_1) * \mathcal{F}(\phi_1 \phi_2 u_2), \tag{4.0.49}$$

which shows that $\mathcal{F}(\phi_1^2 \phi_2^2 w_1) = \mathcal{F}(\phi_1^2 \phi_2^2 w_2)$. By a localization argument (see, for instance, Theorem 2.2.1 in [40]), it follows that $w_1 = w_2$ in $\mathcal{D}'(\mathbb{R}^k)$, completing the proof of the lemma. □

Lastly, we record some basic properties of the product of two distributions, when it exists.

**Proposition 4.0.16** (Properties of product)**.** *The following properties hold:*

*(a) If $f \in \mathcal{D}(\mathbb{R}^k)$ and $u \in \mathcal{D}'(\mathbb{R}^k)$, then the usual definition of the $fu$ coincides with Proposition 4.0.14.*

*(b) If $u, v, w \in \mathcal{D}'(\mathbb{R}^k)$ and the products $uv$, $(uv)w$, $vw$, and $u(vw)$ all exists, then $u(vw) = (uv)w$. Furthermore, if $uv$ exists, then $vu$ also exists and $uv = vu$.*

*(c) If $u, v \in \mathcal{D}'(\mathbb{R}^k)$ have disjoint singular supports, then $uv$ exists and is given by the product distribution guaranteed by Proposition 4.0.5.*

*(d) If $u, v \in \mathcal{D}'(\mathbb{R}^k)$ and $uv$ exists, then $\operatorname{supp}(uv) \subset \operatorname{supp}(u) \cap \operatorname{supp}(v)$.*

*Proof.* See Theorem IX.43 in [79]. $\square$

# Appendix 5

# Multilinear Algebra

In this appendix, we review some useful facts from multilinear algebra about symmetric tensors, which we make use of to prove Theorem 4.1.7. Throughout this appendix, $V$ denotes a finite-dimensional complex vector space unless specified otherwise. For concreteness, the reader can just take $V = \mathbb{C}^d$, where $d$ is the dimension of $V$. For more details and the omitted proofs, we refer the reader to [37] and [17], in particular the latter for a concise, pedestrian exposition.

Let $n \in \mathbb{N}$, and let $V^{\times n} \to V^{\otimes n}$ be an algebraic $n$-fold tensor product[1] for $V$. Now given any $n$-linear map $T : V^{\times n} \to W$, where $W$ is another complex finite-dimensional vector space, the universal property of the tensor product asserts that there exists a unique linear map $\bar{T} : V^{\otimes n} \to W$, such that the following diagram commutes

$$\begin{tikzcd} V^{\times n} \arrow[r] \arrow[dr, "T"'] & V^{\otimes n} \arrow[d, dashed, "\bar{T}"] \\ & W \end{tikzcd} . \tag{5.0.1}$$

In particular, given any permutation $\pi \in \mathbb{S}_n$, there is a unique map $\bar{\pi} : V^{\otimes n} \to V^{\otimes n}$ with the property that

$$\bar{\pi}\big(v_1 \otimes \cdots \otimes v_n\big) = v_{\pi(1)} \otimes \cdots \otimes v_{\pi(n)}, \qquad \forall v_1, \ldots, v_n \in V. \tag{5.0.2}$$

[1] The reader will recall that the tensor product is only defined up to unique isomorphism.

Using these maps $\bar{\pi}$, we can define the symmetrization operator $\mathrm{Sym}_n$ on $V^{\otimes n}$ by

$$\mathrm{Sym}_n(u) \coloneqq \frac{1}{n!}\sum_{\pi\in\mathbb{S}_n}\bar{\pi}(u), \qquad \forall u \in \ V^{\otimes n} \tag{5.0.3}$$

and define what it means for a tensor to be symmetric.

**Definition 5.0.17** (Symmetric tensor)**.** We say that $u \in V^{\otimes n}$ is *symmetric* if $\mathrm{Sym}_n(u) = u$. Equivalently, $\bar{\pi}(u) = u$ for every $\pi \in \mathbb{S}_n$. We let $\mathrm{Sym}_n(V^{\otimes n})$, alternatively $\bigotimes_s^n V$ or $V^{\otimes_s^n}$, denote the subspace of $V^{\otimes n}$ consisting of symmetric tensors.

**Remark 5.0.18.** If $\{e_1, \ldots, e_d\}$ is a basis for $V$, then $\{e_{j_1}\otimes\cdots\otimes e_{j_n}\}_{j_1,\ldots,j_n=1}^d$ is a basis for $V^{\otimes n}$, so that $\dim(V^{\otimes n}) = d^n$. Similarly, $\{\mathrm{Sym}_n(e_{j_1}\otimes\cdots\otimes e_{j_n})\}_{1\le j_1\le\cdots\le j_n\le d}$ is a basis for $V^{\otimes_s^n}$, so that $\dim(V^{\otimes_s^n}) = \binom{d+n-1}{n}$.

We now claim that any element of $V^{\otimes_s^n}$ is uniquely identifiable with an element of $\mathbb{C}[x_1,\ldots,x_d]_n$, the space of homogeneous polynomials of degree $n$ in $d$ variables. Indeed, fix a basis $\{e_1,\ldots,e_d\}$ for $V$, so that $\{\mathrm{Sym}_n(e_{j_1}\otimes\cdots\otimes e_{j_n})\}_{1\le j_1\le\cdots\le j_n\le d}$ is a basis for $V^{\otimes_s^n}$. By mapping

$$\mathrm{Sym}_n(e_{j_1}\otimes\cdots\otimes e_{j_n}) \mapsto x_1^{\alpha_1}\cdots x_d^{\alpha_d} =: \underline{x}_d^{\underline{\alpha}_d}, \tag{5.0.4}$$

where $\underline{\alpha}_d$ is the multi-index defined by

$$\alpha_j \coloneqq \sum_{i=1}^n \delta_j(j_i), \qquad \forall j \in \mathbb{N}_{\le d}, \tag{5.0.5}$$

where $\delta_j$ is the discrete Dirac mass centered at $j$, one obtains a linear map from $V^{\otimes_s^n} \to \mathbb{C}[x_1,\ldots,x_d]_n$. One can show this map is, in fact, an isomorphism. Consequently, if

$$u = \sum_{1\le j_1\le\cdots\le j_n\le d} u_{j_1\cdots j_n}\,\mathrm{Sym}_n(e_{j_1}\otimes\cdots\otimes e_{j_n}) \tag{5.0.6}$$

is an element of $V^{\otimes_s^n}$, then $u$ is uniquely identifiable with the element $F \in \mathbb{C}[x_1, \ldots, x_d]_n$ given by

$$F(\underline{x}_d) = \sum_{1 \le j_1 \le \cdots \le j_n \le d} u_{j_1 \cdots j_n} \underline{x}_d^{\underline{\alpha}_d(\underline{j}_n)}, \tag{5.0.7}$$

where we write $\underline{\alpha}_d(\underline{j}_n)$ to emphasize that $\underline{\alpha}_d$ is intended as a function of $\underline{j}_n$ according to the rule (5.0.5).

There is a useful bilinear form on $\mathbb{C}[x_1, \ldots, x_d]_n$ defined as follows: if $F, G \in \mathbb{C}[x_1, \ldots, x_d]_n$ are respectively given by

$$F(\underline{x}_d) = \sum_{|\underline{\alpha}_d| = n} \binom{n}{\alpha_1, \ldots, \alpha_d} a_{\underline{\alpha}_d} \underline{x}_d^{\underline{\alpha}_d}, \qquad G(\underline{x}_d) = \sum_{|\underline{\alpha}_d| = n} \binom{n}{\alpha_1, \ldots, \alpha_d} b_{\underline{\alpha}_d} \underline{x}_d^{\underline{\alpha}_d}, \tag{5.0.8}$$

then we define

$$\langle F, G \rangle := \sum_{|\underline{\alpha}_d| = n} \binom{n}{\alpha_1, \ldots, \alpha_d} a_{\underline{\alpha}_d} b_{\underline{\alpha}_d}. \tag{5.0.9}$$

The form $\langle \cdot, \cdot \rangle$, which is evidently symmetric, has the important property of nondegeneracy, as the next lemma shows.

**Lemma 5.0.19** (Nondegeneracy)**.** *The symmetric bilinear form $\langle \cdot, \cdot \rangle : \mathbb{C}[x_1, \ldots, x_d]_n \times \mathbb{C}[x_1, \ldots, x_d]_n \to \mathbb{C}$ is nondegenerate: if $\langle F, G \rangle = 0$ for all $G \in \mathbb{C}[x_1, \ldots, x_d]_n$, then $F \equiv 0$.*

When $G$ is of the form $G(\underline{x}_d) = (\beta_1 x_1 + \cdots + \beta_d x_d)^n$ (i.e. an $n^{th}$ power of a linear form), then the next lemma provides an explicit formula for $\langle F, G \rangle$.

**Lemma 5.0.20.** *If $G(\underline{x}_d) = (\beta_1 x_1 + \cdots + \beta_d x_d)^n$, where $\underline{\beta}_d \in \mathbb{C}^d$, then for every $F \in \mathbb{C}[x_1, \ldots, x_d]_n$, we have that*

$$\langle F, G \rangle = F(\underline{\beta}_d). \tag{5.0.10}$$

We now use Lemma 5.0.20 to prove the existence of a special decomposition for elements of $V^{\otimes_s^n}$. We have included a proof as it is a nice argument.

**Lemma 5.0.21** (Symmetric rank-1 decomposition)**.** *For any $u \in V^{\otimes_s^n}$, there exists an integer $N \in \mathbb{N}$, coefficients $\{a_j\}_{j=1}^N \subset \mathbb{C}$, and elements $\{v_j\}_{j=1}^N \subset V$, such that*

$$u = \sum_{j=1}^{N} a_j v_j^{\otimes n}. \tag{5.0.11}$$

*Proof.* Let $W \subset V^{\otimes_s^n}$ denote the set of elements which admit a decomposition of the form (5.0.11). Evidently, $W$ is a subspace of $V^{\otimes_s^n}$. Fix a basis $\{e_1, \ldots, e_d\}$ for $V$. If $v = \beta_1 e_1 + \cdots + \beta_d e_d$, then one can check that under the isomorphism given by (5.0.7), $v^{\otimes n}$ is uniquely identifiable with the polynomial

$$(\beta_1 x_1 + \cdots + \beta_d x_d)^n,$$

i.e. an $n^{th}$ power of a linear form. Consequently, $W$ is isomorphic to the span of $n^{th}$ powers of linear forms in $\mathbb{C}[x_1, \ldots, x_d]_n$.

Assume for the sake of a contradiction that $W$ is a proper subspace, so that the orthogonal complement $W^\perp$ with respect to the form $\langle \cdot, \cdot \rangle$ is nontrivial. Then it follows from the embedding of $W \subset \mathbb{C}[x_1, \ldots, x_d]_n$ that there exists a nonzero polynomial $F \in \mathbb{C}[x_1, \ldots, x_d]_n$, such that $\langle F, G \rangle = 0$ for every $G \in W$. Lemma 5.0.20 then implies that $F(\underline{\beta}_d) = 0$ for every $\underline{\beta}_d \in \mathbb{C}^d$, which contradicts that $F$ is a nonzero polynomial. $\square$

**Remark 5.0.22.** Since Lemma 5.0.21 asserts that a decomposition of the form (5.0.11) always exists, one can define the *symmetric rank* of an element $u \in V^{\otimes_s^n}$ by the minimal integer $N$. Evidently, a tensor of the form $v^{\otimes n}$ has symmetric rank 1. Although we will not need the notion of symmetric rank in this work, we will refer to the decomposition (5.0.11) as a symmetric-rank-1 decomposition.

As an application of the symmetric-rank-1 tensor decomposition, we now show an approximation result for bosonic Schwartz functions (i.e. elements of $\mathcal{S}_s(\mathbb{R}^d)$).

**Lemma 5.0.23.** *Let $f \in \mathcal{S}_s(\mathbb{R}^d)$. Then given $\varepsilon > 0$ and a Schwartz seminorm $\mathcal{N}$, there exist $N \in \mathbb{N}$, elements $\{f_i\}_{i=1}^N \subset \mathcal{S}(\mathbb{R})$, and coefficients $\{a_i\}_{i=1}^N \subset \mathbb{C}$, such that*

$$\mathcal{N}\left(f - \sum_{i=1}^{N} a_i f_i^{\otimes d}\right) \leq \varepsilon. \tag{5.0.12}$$

*In other words, finite linear combinations of symmetric-rank-1 tensor products are dense in $\mathcal{S}_s(\mathbb{R}^d)$.*

*Proof.* Fix $f \in \mathcal{S}_s(\mathbb{R}^d)$, $\varepsilon > 0$, and seminorm $\mathcal{N}$. Since $\mathcal{S}_s(\mathbb{R}^d) \cong \hat{\bigotimes}_s^d \mathcal{S}(\mathbb{R})$, there exists an integer $M \in \mathbb{N}$, elements $\{g_{ij}\}_{\substack{1\leq i\leq d \\ 1\leq j\leq M}} \subset \mathcal{S}(\mathbb{R})$, and coefficients $\{a_j\}_{1\leq j\leq M} \subset \mathbb{C}$, such that

$$\mathcal{N}\left(f - \sum_{j=1}^{M} a_j \operatorname{Sym}_d\left(\bigotimes_{i=1}^{d} g_{ij}\right)\right) \leq \varepsilon. \tag{5.0.13}$$

Define the complex vector space

$$V \coloneqq \operatorname{span}_{\mathbb{C}}\{g_{ij} : 1 \leq i \leq d,\ 1 \leq j \leq M\}, \tag{5.0.14}$$

which is evidently finite-dimensional. For each $j \in \mathbb{N}_{\leq M}$, consider the symmetric tensor

$$\operatorname{Sym}_d\left(\bigotimes_{i=1}^{d} g_{ij}\right) \in V^{\otimes_s^d}. \tag{5.0.15}$$

By Lemma 5.0.21, there exists an integer $N_j \in \mathbb{N}$, elements $\{f_{j\ell}\}_{\ell=1}^{N_j} \subset V$, coefficients $\{a_{j\ell}\}_{\ell=1}^{N_j} \subset \mathbb{C}$, such that

$$\mathrm{Sym}_d\left(\bigotimes_{i=1}^{d} g_{ij}\right) = \sum_{\ell=1}^{N_j} a_{j\ell} f_{j\ell}^{\otimes_d}. \tag{5.0.16}$$

Consequently,

$$\sum_{j=1}^{M} a_j \,\mathrm{Sym}_d\left(\bigotimes_{i=1}^{d} g_{ij}\right) = \sum_{j=1}^{M}\sum_{\ell=1}^{N_j} a_j a_{j\ell} f_{j\ell}^{\otimes_d}, \tag{5.0.17}$$

so upon substitution of this identity into (5.0.13), we obtain the desired conclusion. □

As a corollary of Lemma 5.0.23, we obtain the following decomposition for elements in $\mathcal{L}(\mathcal{S}_s'(\mathbb{R}^d), \mathcal{S}_s(\mathbb{R}^d))$.

**Corollary 5.0.24.** *Let $\gamma^{(d)} \in \mathcal{L}(\mathcal{S}_s'(\mathbb{R}^d), \mathcal{S}_s(\mathbb{R}^d))$. Then given $\varepsilon > 0$ and a Schwartz seminorm $\mathcal{N}$, there exists $N \in \mathbb{N}$, elements $\{f_i, g_i\}_{i=1}^{N} \subset \mathcal{S}(\mathbb{R})$, and coefficients $\{a_i\}_{i=1}^{N} \subset \mathbb{C}$, such that*

$$\mathcal{N}\left(\gamma^{(d)} - \sum_{i=1}^{N} a_i f_i^{\otimes d} \otimes g_i^{\otimes d}\right) \leq \varepsilon. \tag{5.0.18}$$

*Proof.* Fix $\gamma^{(d)} \in \mathcal{L}(\mathcal{S}_s'(\mathbb{R}^d), \mathcal{S}_s(\mathbb{R}^d))$, $\varepsilon > 0$, and seminorm $\mathcal{N}$. Since

$$\mathcal{L}(\mathcal{S}_s'(\mathbb{R}^d), \mathcal{S}_s(\mathbb{R}^d)) \cong \mathcal{S}_s(\mathbb{R}^d)\hat{\otimes}\mathcal{S}_s(\mathbb{R}^d),$$

there exists an integer $N$, elements $\{\tilde{f}_i, \tilde{g}_i\}_{i=1}^{N} \subset \mathcal{S}_s(\mathbb{R}^d)$, and coefficients $\{a_i\}_{i=1}^{N} \subset \mathbb{C}$, such that

$$\mathcal{N}\left(\gamma^{(d)} - \sum_{i=1}^{N} a_i \tilde{f}_i \otimes \tilde{g}_i\right) \leq \varepsilon. \tag{5.0.19}$$

For each $i \in \mathbb{N}_{\leq N}$, Lemma 5.0.23 implies that there exist integers $N_{i,f}, N_{i,g} \in \mathbb{N}$, elements $\{f_{ij}\}_{j=1}^{N_{i,f}}, \{g_{ij}\}_{j=1}^{N_{i,g}} \subset \mathcal{S}(\mathbb{R})$, and coefficeints $\{a_{ij,f}\}_{j=1}^{N_{i,f}}, \{a_{ij,g}\}_{j=1}^{N_{i,g}} \subset \mathbb{C}$, such that

$$\tilde{f}_i = \sum_{j=1}^{N_{i,f}} a_{ij,f} f_{ij}^{\otimes d}, \qquad \tilde{g}_i = \sum_{j=1}^{N_{i,g}} a_{ij,g} g_{ij}^{\otimes d}. \tag{5.0.20}$$

By setting coefficients equal to zero, we may assume without loss of generality that $N_{i,f} = N_{i,g} = M \in \mathbb{N}$, for every $i \in \mathbb{N}_{\leq N}$. So by the bilinearity of tensor product, we obtain the decomposition

$$\sum_{i=1}^{N} a_i \tilde{f}_i \otimes \tilde{g}_i = \sum_{i=1}^{N} \sum_{j,j'=1}^{M} a_i a_{ij,f} a_{ij',g} f_{ij}^{\otimes d} \otimes g_{ij'}^{\otimes d}. \tag{5.0.21}$$

Substitution of this identity into (5.0.19) and relabeling/re-indexing of the summation yields the desired conclusion. □

# Bibliography


[1] Mark J Ablowitz, David J Kaup, Alan C Newell, and Harvey Segur. The inverse scattering transform-Fourier analysis for nonlinear problems. *Studies in Applied Mathematics*, 53(4):249–315, 1974.

[2] Ralph Abraham and Jerrold E. Marsden. *Foundations of mechanics.* Benjamin/Cummings Publishing Co., Inc., Advanced Book Program, Reading, Mass., 1978. Second edition, revised and enlarged, With the assistance of Tudor Raţiu and Richard Cushman.

[3] Riccardo Adami, Claude Bardos, François Golse, and Alessandro Teta. Towards a rigorous derivation of the cubic NLSE in dimension one. *Asymptotic Analysis*, 40(2):93–108, 2004.

[4] Riccardo Adami, François Golse, and Alessandro Teta. Rigorous derivation of the cubic NLS in dimension one. *Journal of Statistical Physics*, 127(6):1193–1220, 2007.

[5] Sergio Albeverio, Friedrich Gesztesy, Raphael Hø egh Krohn, and Helge Holden. *Solvable models in quantum mechanics.* Texts and Monographs in Physics. Springer-Verlag, New York, 1988.

[6] Z. Ammari and S. Breteaux. Propagation of chaos for many-boson systems in one dimension with a point pair-interaction. *Asymptot. Anal.*, 76(3-4):123–170, 2012.

[7] Richard Beals and R. R. Coifman. Scattering and inverse scattering for first-order systems. II. *Inverse Problems*, 3(4):577–593, 1987.

[8] Richard Beals and Ronald R Coifman. Scattering and inverse scattering for first order systems. *Communications on Pure and Applied Mathematics*, 37(1):39–90, 1984.

[9] Richard Beals and Ronald R Coifman. Inverse scattering and evolution equations. *Communications on pure and applied mathematics*, 38(1):29–42, 1985.

[10] Richard Beals and Ronald R Coifman. Linear spectral problems, non-linear equations and the $\delta$-method. *Inverse problems*, 5(2):87, 1989.

[11] H. Bethe. Zur theorie der metalle. *Zeitschrift für Physik*, 71(3):205–226, Mar 1931.

[12] F. Calogero. Why are certain nonlinear PDEs both widely applicable and integrable? In *What is integrability?*, Springer Ser. Nonlinear Dynam., pages 1–62. Springer, Berlin, 1991.

[13] Thierry Cazenave. *Semilinear Schrödinger equations*, volume 10 of *Courant Lecture Notes in Mathematics*. New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2003.

[14] Thomas Chen, Christian Hainzl, Nataša Pavlović, and Robert Seiringer. Unconditional uniqueness for the cubic Gross-Pitaevskii hierarchy via quantum de Finetti. *Comm. Pure Appl. Math.*, 68(10):1845–1884, 2015.

[15] Thomas Chen and Nataša Pavlović. The quintic NLS as the mean field limit of a boson gas with three-body interactions. *J. Funct. Anal.*, 260(4):959–997, 2011.

[16] Paul R. Chernoff and Jerrold E. Marsden. *Properties of infinite dimensional Hamiltonian systems.* Lecture Notes in Mathematics, Vol. 425. Springer-Verlag, Berlin-New York, 1974.

[17] Pierre Comon, Gene Golub, Lek-Heng Lim, and Bernard Mourrain. Symmetric tensors and symmetric tensor rank. *SIAM J. Matrix Anal. Appl.*, 30(3):1254–1279, 2008.

[18] B. Davies. Higher conservation laws for the quantum nonlinear Schrödinger equation. *Phys. A*, 167(2):433–456, 1990.

[19] P Deift. Fifty Years of KdV: An Integrable System . *arXiv:1902.10267.*

[20] Percy Deift and Xin Zhou. Long-time asymptotics for solutions of the NLS equation with initial data in a weighted Sobolev space. volume 56, pages 1029–1077. 2003. Dedicated to the memory of Jürgen K. Moser.

[21] S. Dettmer, D. Hellweg, P. Ryytty, J. J. Arlt, W. Ertmer, K. Sengstock, D. S. Petrov, G. V. Shlyapnikov, H. Kreutzmann, L. Santos, and M. Lewenstein. Observation of Phase Fluctuations in Elongated Bose-Einstein Condensates. *Phys. Rev. Lett.*, 87(16):160406, Oct 2001.

[22] D.S. Petrov, D.M. Gangardt, and G.V. Shlyapnikov. Low-dimensional trapped gases. *J. Phys. IV France*, 116:5–44, 2004.

[23] V. Dunjko, V. Lorent, and M. Olshanii. Bosons in Cigar-Shaped Traps: Thomas-Fermi Regime, Tonks-Girardeau Regime, and In Between. *Phys. Rev. Lett.*, 86(24):5413–5416, Jun 2001.

[24] László Erdös, Benjamin Schlein, and Horng-Tzer Yau. Derivation of the Gross-Pitaevskii hierarchy for the dynamics of Bose-Einstein condensate. *Comm. Pure Appl. Math.*, 59(12):1659–1741, 2006.

[25] László Erdös, Benjamin Schlein, and Horng-Tzer Yau. Derivation of the cubic non-linear Schrödinger equation from quantum dynamics of many-body systems. *Inven. Math.*, 167(3):515–614, 2007.

[26] László Erdös, Benjamin Schlein, and Horng-Tzer Yau. Derivation of the Gross-Pitaevskii equation for the dynamics of Bose-Einstein condensate. *Ann. Math.*, 172(1):291–370, 2010.

[27] J. Esteve, J. B. Trebbia, T. Schumm, A. Aspect, C. I. Westbrook, and I. Bouchoule. Observations of Density Fluctuations in an Elongated Bose Gas: Ideal Gas and Quasicondensate Regimes. *Phys. Rev. Lett.*, 96(13):130403, Apr 2006.

[28] Ludwig D Faddeev and Leon A Takhtajan. *Hamiltonian methods in the theory of solitons*. Classics in Mathematics. Springer, Berlin, english edition, 2007.

[29] J Fröhlich, A Knowles, and A Pizzo. Atomism and quantization. *J. Phys. A*, 40(12):3033–3045, 2007.

[30] Jürg Fröhlich, Antti Knowles, Benjamin Schlein, and Vedran Sohinger. Gibbs measures of nonlinear Schrödinger equations as limits of many-body quantum states in dimensions $d \leqslant 3$. *Comm. Math. Phys.*, 356(3):883–980, 2017.

[31] Jürg Fröhlich, Antti Knowles, and Simon Schwarz. On the mean-field limit of bosons with Coulomb two-body interaction. *Comm. Math. Phys.*, 288(3):1023–1059, 2009.

[32] Jürg Fröhlich, Tai-Peng Tsai, and Horng-Tzer Yau. On a classical limit of quantum theory and the non-linear Hartree equation. *Geom. Funct. Anal.*, (Special Volume, Part I):57–78, 2000.

[33] Jürg Fröhlich, Tai-Peng Tsai, and Horng-Tzer Yau. On a classical limit of quantum theory and the non-linear Hartree equation. Number Special Volume, Part I, pages 57–78. 2000. GAFA 2000 (Tel Aviv, 1999).

[34] Clifford S. Gardner, John M. Greene, Martin D. Kruskal, and Robert M. Miura. Korteweg-deVries equation and generalization. VI. Methods for exact solution. *Comm. Pure Appl. Math.*, 27:97–133, 1974.

[35] Michel Gaudin. *The Bethe wavefunction.* Cambridge University Press, New York, 2014. Translated from the 1983 French original by Jean-Sébastien Caux.

[36] J. Ginibre and G. Velo. The classical field limit of scattering theory for nonrelativistic many-boson systems. I. *Comm. Math. Phys.*, 66(1):37–76, 1979.

[37] Werner Greub. *Multilinear algebra.* Springer-Verlag, New York-Heidelberg, second edition, 1978. Universitext.

[38] Richard S Hamilton. The inverse function theorem of Nash and Moser. *Bull. Am. Math. Soc.*, 7(1):65–222, 1982.

[39] Klaus Hepp. The classical limit for quantum mechanical correlation functions. *Comm. Math. Phys.*, 35:265–277, 1974.

[40] Lars Hörmander. *The Analysis of Linear Partial Differential Operators: Distribution Theory and Fourier Analysis*. Springer-Verlag, 1983.

[41] J. Horváth. *Topological vector spaces and distributions*. Number v. 1 in Addison-Wesley series in mathematics. Addison-Wesley Pub. Co., 1966.

[42] R. L. Hudson and G. R. Moody. Locally normal symmetric states and an analogue of de Finetti's theorem. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 33(4):343–351, Dec 1976.

[43] A. D. Jackson and G. M. Kavoulakis. Lieb Mode in a Quasi-One-Dimensional Bose-Einstein Condensate of Atoms. *Phys. Rev. Lett.*, 89(7):070403, Jul 2002.

[44] Tosio Kato. *Perturbation theory for linear operators*. Classics in Mathematics. Springer-Verlag, Berlin, 1995. Reprint of the 1980 edition.

[45] Rowan Killip, Monica Vişan, and Xiaoyi Zhang. Low regularity conservation laws for integrable PDE. *Geom. Funct. Anal.*, 28(4):1062–1090, 2018.

[46] Antti Knowles and Peter Pickl. Mean-field dynamics: singular potentials and rate of convergence. *Comm. Math. Phys.*, 298(1):101–138, 2010.

[47] Herbert Koch and Daniel Tataru. Conserved energies for the cubic nonlinear Schrödinger equation in one dimension. *Duke Math. J.*, 167(17):3207–3313, 2018.

[48] Andreas Kriegl and Peter W. Michor. *The convenient setting of global analysis*, volume 53 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1997.

[49] Oscar E. Lanford, III. Time evolution of large classical systems. pages 1–111. Lecture Notes in Phys., Vol. 38, 1975.

[50] Oscar E. Lanford, III. On a derivation of the Boltzmann equation. pages 117–137. Astérisque, No. 40, 1976.

[51] Peter D. Lax. Integrals of nonlinear equations of evolution and solitary waves. *Comm. Pure Appl. Math.*, 21:467–490, 1968.

[52] Mathieu Lewin, Phan Thành Nam, and Nicolas Rougerie. Derivation of nonlinear Gibbs measures from many-body quantum mechanics. *J. Éc. polytech. Math.*, 2:65–115, 2015.

[53] Elliott H. Lieb. Exact analysis of an interacting Bose gas. II. The excitation spectrum. *Phys. Rev. (2)*, 130:1616–1624, 1963.

[54] Elliott H. Lieb and Werner Liniger. Exact analysis of an interacting Bose gas. I. The general solution and the ground state. *Phys. Rev. (2)*, 130:1605–1616, 1963.

[55] Elliott H. Lieb, Robert Seiringer, and Jakob Yngvason. One-Dimensional Bosons in Three-Dimensional Traps. *Phys. Rev. Lett.*, 91(15):150401, Oct 2003.

[56] Elliott H. Lieb, Robert Seiringer, and Jakob Yngvason. One-dimensional behavior of dilute, trapped Bose gases. *Comm. Math. Phys.*, 244(2):347–393, 2004.

[57] J. E. Marsden, A. Weinstein, T. Ratiu, R. Schmid, and R. G. Spencer. Hamiltonian systems with symmetry, coadjoint orbits and plasma physics. In *Proceedings of the*

*IUTAM-ISIMM symposium on modern developments in analytical mechanics, Vol. I (Torino, 1982)*, volume 117, pages 289–340, 1983.

[58] Jerrold Marsden and Alan Weinstein. Coadjoint orbits, vortices, and Clebsch variables for incompressible fluids. volume 7, pages 305–323. 1983. Order in chaos (Los Alamos, N.M., 1982).

[59] Jerrold E. Marsden, Philip J. Morrison, and Alan Weinstein. The Hamiltonian structure of the BBGKY hierarchy equations. In *Fluids and plasmas: geometry and dynamics (Boulder, Colo., 1983)*, volume 28 of *Contemp. Math.*, pages 115–124. Amer. Math. Soc., Providence, RI, 1984.

[60] Jerrold E Marsden and Tudor S Ratiu. *Introduction to mechanics and symmetry: a basic exposition of classical mechanical systems*, volume 17. Springer Science & Business Media, 2013.

[61] Jerrold E. Marsden and Alan Weinstein. The Hamiltonian structure of the Maxwell-Vlasov equations. *Phys. D*, 4(3):394–406, 1981/82.

[62] Dana Mendelson, Andrea R Nahmod, Nataša Pavlović, Matthew Rosenzweig, and Gigliola Staffilani. Poisson commuting energies for a system of infinitely many bosons. *arXiv preprint arXiv:1910.06959*, 2019.

[63] Dana Mendelson, Andrea R. Nahmod, Nataša Pavlović, Matthew Rosenzweig, and Gigliola Staffilani. A rigorous derivation of the Hamiltonian structure for the nonlinear Schrödinger equation. *Adv. Math.*, 365:107054, 2020.

[64] Dana Mendelson, Andrea R. Nahmod, Nataša Pavlović, and Gigliola Staffilani. An infinite sequence of conserved quantities for the cubic Gross-Pitaevskii hierarchy on $\mathbb{R}$. *Trans. Amer. Math. Soc.*, 371(7):5179–5202, 2019.

[65] John Milnor. Remarks on infinite-dimensional Lie groups. In *Relativ. groups Topol. 2*. 1984.

[66] Philip J. Morrison. The Maxwell-Vlasov equations as a continuous Hamiltonian system. *Phys. Lett. A*, 80(5-6):383–386, 1980.

[67] Philip J Morrison. Poisson brackets for fluids and plasmas. In *AIP Conference proceedings*, volume 88, pages 13–46. AIP, 1982.

[68] Philip J. Morrison and John M. Greene. Noncanonical Hamiltonian density formulation of hydrodynamics and ideal magnetohydrodynamics. *Phys. Rev. Lett.*, 45(10):790–794, 1980.

[69] K.-H. Neeb, H. Sahlmann, and T. Thiemann. Weak poisson structures on infinite dimensional manifolds and hamiltonian actions. In Vladimir Dobrev, editor, *Lie Theory and Its Applications in Physics*, pages 105–135, Tokyo, 2014. Springer Japan.

[70] M. Olshanii. Atomic Scattering in the Presence of an External Confinement and a Gas of Impenetrable Bosons. *Phys. Rev. Lett.*, 81(5):938–941, Aug 1998.

[71] Maxim Olshanii and Vanja Dunjko. Short-Distance Correlation Properties of the Lieb-Liniger System and Momentum Distributions of Trapped One-Dimensional Atomic Gases. *Phys. Rev. Lett.*, 91(9):090401, Aug 2003.

[72] Hideki Omori. *Infinite-dimensional Lie groups*, volume 158 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1997. Translated from the 1979 Japanese original and revised by the author.

[73] B. G. Pachpatte. *Inequalities for differential and integral equations*, volume 197 of *Mathematics in Science and Engineering*. Academic Press, Inc., San Diego, CA, 1998.

[74] Richard Palais. The symmetries of solitons. *Bulletin of the American Mathematical Society*, 34(4):339–403, 1997.

[75] D. S. Petrov, G. V. Shlyapnikov, and J. T. M. Walraven. Regimes of Quantum Degeneracy in Trapped 1D Gases. *Phys. Rev. Lett.*, 85(18):3745–3749, Oct 2000.

[76] Peter Pickl. Derivation of the time dependent Gross-Pitaevskii equation without positivity condition on the interaction. *J. Stat. Phys.*, 140(1):76–89, 2010.

[77] Peter Pickl. A simple derivation of mean field limits for quantum systems. *Lett. Math. Phys.*, 97(2):151–164, 2011.

[78] Peter Pickl. Derivation of the time dependent Gross-Pitaevskii equation with external fields. *Rev. Math. Phys.*, 27(1):1550003, 45, 2015.

[79] Michael Reed and Barry Simon. *Methods of modern mathematical physics. II. Fourier analysis, self-adjointness*. Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1975.

[80] Michael Reed and Barry Simon. *Methods of modern mathematical physics. I.* Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, second edition, 1980. Functional analysis.

[81] S. Richard, F. Gerbier, J. H. Thywissen, M. Hugbart, P. Bouyer, and A. Aspect. Momentum Spectroscopy of 1D Phase Fluctuations in Bose-Einstein Condensates. *Phys. Rev. Lett.*, 91(1):010405, Jul 2003.

[82] Igor Rodnianski and Benjamin Schlein. Quantum Fluctuations and Rate of Convergence Towards Mean Field Dynamics. *Commun. Math. Phys.*, 291(1):31–61, 2009.

[83] Matthew Rosenzweig. The mean-field limit of the lieb-liniger model. *arXiv preprint arXiv:1912.07585*, 2019.

[84] Nicolas Rougerie. De Finetti Theorems, Mean-Field Limits and Bose-Einstein Condensation. Lectures notes from a course at the LMU, Munich. Translated and slightly expanded version of my cours Peccot, hal-01060125v4, arXiv:1409.1182., June 2015.

[85] Benjamin Schlein. Derivation of effective evolution equations from microscopic quantum dynamics. In *Evol. equations*, volume 17 of *Clay Math. Proc.*, pages 511–572. Amer. Math. Soc., Providence, RI, 2013.

[86] Laurent Schwartz. *Théorie des distributions*. Publications de l'Institut de Mathématique de l'Université de Strasbourg, No. IX-X. Nouvelle édition, entiérement corrigée, refondue et augmentée. Hermann, Paris, 1966.

[87] Robert Seiringer and Jun Yin. The Lieb-Liniger model as a limit of dilute bosons in three dimensions. *Comm. Math. Phys.*, 284(2):459–479, 2008.

[88] Richard G. Spencer. The Hamiltonian structure of multispecies fluid electrodynamics. In *Mathematical methods in hydrodynamics and integrability in dynamical systems (La*

*Jolla, Calif., 1981)*, volume 88 of *AIP Conf. Proc.*, pages 121–126. Amer. Inst. Phys., New York, 1982.

[89] Richard G. Spencer and Allan N. Kaufman. Hamiltonian structure of two-fluid plasma dynamics. *Phys. Rev. A (3)*, 25(4):2437–2439, 1982.

[90] Herbert Spohn. Kinetic equations from Hamiltonian dynamics: Markovian limits. *Rev. Mod. Phys.*, 52(3):569–615, 1980.

[91] Erling Störmer. Symmetric states of infinite tensor products of $C^*$ Algebras. *J. Funct. Anal.*, 3:48–68, 1969.

[92] Terence Tao. *Nonlinear dispersive equations*, volume 106 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2006. Local and global analysis.

[93] Chuu-Lian Terng. Soliton equations and differential geometry. *J. Differential Geom*, 45(2):407–445, 1997.

[94] Chuu-Lian Terng and Karen Uhlenbeck. Poisson actions and scattering theory for integrable systems. *Surveys in Differential Geometry*, 4(1):315–402, 1998.

[95] H. B. Thacker. Polynomial conservation laws in (1 + 1)-dimensional classical and quantum field theory. *Phys. Rev. D*, 17:1031–1040, Feb 1978.

[96] B. Laburthe Tolra, K. M. O'Hara, J. H. Huckans, W. D. Phillips, S. L. Rolston, and J. V. Porto. Observation of Reduced Three-Body Recombination in a Correlated 1D Degenerate Bose Gas. *Phys. Rev. Lett.*, 92(19):190401, May 2004.

[97] François Trèves. *Topological vector spaces, distributions and kernels.* Academic Press, New York-London, 1967.

[98] Zhihui Xie. Derivation of a nonlinear Schrödinger equation with a general power-type nonlinearity in $d = 1, 2$. *Differential Integral Equations*, 28(5-6):455–504, 2015.

[99] V E Zaharov and A B Šabat. Integration of the nonlinear equations of mathematical physics by the method of the inverse scattering problem. II. *Funktsional. Anal. i Prilozhen.*, 13(3):13–22, 1979.

[100] V. E. Zakharov, editor. *What is integrability?* Springer Series in Nonlinear Dynamics. Springer-Verlag, Berlin, 1991.

[101] V E Zakharov and A B Shabat. Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media. *Ž. Èksper. Teoret. Fiz.*, 61(1):118–134, 1971.

[102] Xin Zhou. Direct and inverse scattering transforms with arbitrary spectral singularities. *Comm. Pure Appl. Math.*, 42(7):895–938, 1989.

[103] Xin Zhou. $L^2$-Sobolev space bijectivity of the scattering and inverse scattering transforms. *Comm. Pure Appl. Math.*, 51(7):697–731, 1998.