The Dissertation Committee for Kyudong Choi
certifies that this is the approved version of the following dissertation:

Estimates on higher derivatives for the Navier-Stokes equations and Hölder continuity for integro-differential equations

Committee:

Alexis Vasseur, Supervisor

Luis Caffarelli

Alessio Figalli

Irene Gamba

Philip Morrison

Natasa Pavlovic

Estimates on higher derivatives for the Navier-Stokes equations and Hölder continuity for integro-differential equations

by

Kyudong Choi, B.S.C.S.; B.S.Math.

DISSERTATION

Presented to the Faculty of the Graduate School of

The University of Texas at Austin

in Partial Fulfillment

of the Requirements

for the Degree of

DOCTOR OF PHILOSOPHY

THE UNIVERSITY OF TEXAS AT AUSTIN

August 2012

Dedicated to my wife Yunkyung.

Acknowledgments

First and foremost, I would like to thank my advisor, Alexis Vasseur. His keen insight and enthusiasm about Mathematics always stayed me inspired and moved me forward. He supported and advised me in various ways, both professionally and personally.

Additionally, I would like to thank my dissertation committee members for their advice and guidance. The department of Mathematics at UT has offered me a great environment to study. I wish to thank all professors, classmates, and staffs.

I would like to thank Chan Woo Yang who guided me in my early steps to Mathematics. Also, I would like to thank my family who are constantly supporting me a lot.

He named it Ebenezer, saying, “Thus far has the Lord helped us.” 1 Samuel 7:12.

Estimates on higher derivatives for the Navier-Stokes equations and Hölder continuity for integro-differential equations

Kyudong Choi, Ph.D.
The University of Texas at Austin, 2012

Supervisor: Alexis Vasseur

This thesis is divided into two independent parts. The first part concerns the 3D Navier-Stokes equations. The second part deals with regularity issues for a family of integro-differential equations. In the first part of this thesis, we consider weak solutions of the 3D Navier-Stokes equations with L^2 initial data. We prove that $\nabla^\alpha u$ is locally integrable in space-time for any real α such that $1 < \alpha < 3$. Up to now, only the second derivative $\nabla^2 u$ was known to be locally integrable by standard parabolic regularization. We also present sharp estimates of those quantities in local weak-$L^{4/(\alpha+1)}$. These estimates depend only on the L^2 norm of the initial data and on the domain of integration. Moreover, they are valid even for $\alpha \geq 3$ as long as u is smooth. The proof uses a standard approximation of Navier-Stokes from Leray and blow-up techniques. The local study is based on De Giorgi techniques with a new pressure decomposition. To handle the non-locality of fractional Laplacians, Hardy space and Maximal functions are introduced.

In the second part of this thesis, we consider non-local integro-differential equations under certain natural assumptions on the kernel, and obtain persistence of Hölder continuity for their solutions. In other words, we prove that a solution stays in C^β for all time if its initial data lies in C^β. Also, we prove a C^β-regularization effect from $L^\infty \cap L^1$ initial data. It provides an alternative proof to the result of Caffarelli, Chan and Vasseur [10], which was based on De Giorgi techniques. This result has an application for a fully non-linear problem, which is used in the field of image processing. In addition, we show Hölder regularity for solutions of drift diffusion equations with supercritical fractional diffusion under the assumption $b \in L^\infty C^{1-\alpha}$ on the divergent-free drift velocity. The proof is in the spirit of Kiselev and Nazarov [48] where they established Hölder continuity of the critical surface quasi-geostrophic (SQG) equation by observing the evolution of a dual class of test functions.

Table of Contents

Chapter 1

Introduction

Before starting this thesis, we point out that this dissertation is based on the author's two works [21] and [20]. In particular, all the material of Chapters 3 and 4 is taken directly from these two papers.

1.1 The Navier-Stokes equations

1.1.1 Background

Any derivative signs ($\nabla, \Delta, (-\Delta)^{\alpha/2}, D, \partial$ and etc) denote derivatives in the space variable $x \in \mathbb{R}^3$ unless the time variable $t \in \mathbb{R}$ is clearly specified. We study the 3D Navier-Stokes equations

$$\begin{aligned} &\partial_t u + (u\cdot\nabla)u + \nabla P - \Delta u = 0 \quad \text{and} \\ &\qquad\qquad \operatorname{div} u = 0, \qquad t \in (0,\infty), \quad x \in \mathbb{R}^3 \end{aligned} \tag{1.1}$$

with L^2 initial data

$$u_0 \in L^2(\mathbb{R}^3), \quad \operatorname{div} u_0 = 0. \tag{1.2}$$

The problem of global regularity of weak solutions for the 3D Navier-Stokes equations has a long history. Leray [57] 1930s and Hopf [45] 1950s proved the existence of a global-time weak solution for any given L^2 initial data. Such Leray-Hopf weak solutions u lie in $L^\infty(0,\infty; L^2(\mathbb{R}^3))$ and ∇u do in $L^2(0,\infty; L^2(\mathbb{R}^3))$, and they satisfy the equations in the sense of distributions in $(0,\infty)\times\mathbb{R}^3$: for $\phi\in C_c^\infty((0,\infty)\times\mathbb{R}^3)$ with $\operatorname{div}\phi=0$,

$$\int_0^\infty\int_{\mathbb{R}^3}\Big(-u\cdot\partial_t\phi+\partial_j\phi_i\partial_j u_i-(\partial_j\phi_i)u_iu_j\Big)dxdt=0.$$

They have the energy inequality:

$$\|u(t)\|^2_{L^2(\mathbb{R}^3)}+2\|\nabla u\|^2_{L^2(0,t;L^2(\mathbb{R}^3))}\le\|u_0\|^2_{L^2(\mathbb{R}^3)}\qquad\text{for a.e. } t<\infty.$$

The initial value is achieved by strong L^2-convergence:

$$\lim_{t\to 0+}\|u(t)-u_0\|_{L^2(\mathbb{R}^3)}=0.$$

Until now, regularity and uniqueness of such weak solutions are generally open.

Instead, many criteria which ensure regularity of weak solutions have been developed. For instance, the Ladyženskaja-Prodi-Serrin Criteria ([54],[62] and [69]), says that if a Leray-Hopf weak solution u lies in $L^p((0,T);L^q(\mathbb{R}^3))$ for some p and q satisfying $\frac{2}{p}+\frac{3}{q}=1$ and $p<\infty$, then it is regular on $(0,T)\times\mathbb{R}^3$. Recently, the limit case $p=\infty$ was established in the paper of Escauriaza, Serëgin and Šverák [35]. Similar criteria exist with various conditions on derivatives of velocity, vorticity, or pressure. (see Beale, Kato and Majda [3], Beirão da Veiga [4] and Berselli and Galdi [7]) Also, many other conditions exist (e.g. see Cheskidov and Shvydkoy [19], Chan [17], and Bjorland and Vasseur [8]).

On the other hand, many efforts have been given to measure the size of the possible singular set where singularities may occur. This approach has been initiated by Scheffer [65]. Then, Caffarelli, Kohn and Nirenberg [9] improved the result and showed that possible singular sets have zero Hausdorff measure of one dimension for a certain sub-class of Leray-Hopf weak solutions, which is called a suitable weak solutions satisfying the following additional inequality

$$\partial_t \frac{|u|^2}{2} + \operatorname{div}(u\frac{|u|^2}{2}) + \operatorname{div}(uP) + |\nabla u|^2 - \Delta\frac{|u|^2}{2} \le 0 \tag{1.3}$$

in the sense of distributions. Different proofs have been proposed later by several authors (e.g. see Lin [58], Vasseur [79] and Wolf [80]). Similar criteria for interior points with other quantities can be found in many places (e.g. see Struwe [76], Gustafson, Kang and Tsai [44], Serëgin [68] and Chae, Kang and Lee [16]). Also, Robinson and Sadowski [63] and Kukavica [52] studied box-counting dimensions of singular sets.

In this thesis, we consider space-time $L^p_{(t,x)} = L^p_t L^p_x$ estimates of higher derivatives for weak solutions assuming only L^2 initial data. The estimate $\nabla u \in L^2((0,\infty)\times\mathbb{R}^3)$ is obvious thanks to the energy inequality. A simple interpolation gives $u \in L^{10/3}$. For the second derivatives of weak solutions, a rough estimate $\nabla^2 u \in L^{5/4}$ can be obtained by considering $(u\cdot\nabla)u$ as a source term from the standard parabolic regularization theory (see Ladyženskaja, Solonnikov and Ural′ceva [55]). With different ideas, Constantin [23] showed $L^{\frac{4}{3}-\epsilon}$ for any small $\epsilon > 0$ in periodic setting, and later Lions [59] improved it up to weak-$L^{\frac{4}{3}}$ (or $L^{\frac{4}{3},\infty}$) by assuming ∇u_0 is lying in the space of all bounded measures in $\mathbb{R}^3$. They used natural structure of the equation with some interpolation technique. On the other hand, Foiaş, Guillopé and Temam [37] and Duff [33] obtained other kinds of estimates for higher derivatives of weak solutions while Giga and Sawada [40] and Dong and Du [31] covered mild solutions. For asymptotic behavior, we refer to Schonbek and Wiegner [66].

Recently in Vasseur [78], it has been shown that, for any small $\epsilon > 0$, any integer $d \geq 1$ and any smooth solution u on $(0, T)$, there exist uniform bounds on $\nabla^d u$ in $L_{loc}^{\frac{4}{d+1}-\epsilon}$, which depend only on the L^2 norm of the initial data once ϵ, d and the domain of integration are fixed. It can be considered as a natural extension of the result of [23] for higher derivatives. However, the method is very different. In [78], the proof uses the Galilean invariance of the equation and some regularity criterion of [79], which re-proves the famous result of [9] by using a parabolic version of the De Giorgi method [30]. For applications of this method, we refer to Caffarelli and Vasseur [14]. Note that that the techniques gives full regularity to the critical Surface Quasi-Geostrophic equation in Caffarelli and Vasseur [13]. The exponent $p = \frac{4}{d+1}$ appears in a nonlinear way from the following invariance of the Navier-Stokes scaling $u_\lambda(t,x) = \lambda u(\lambda^2 t, \lambda x)$:

$$\|\nabla^d u_\lambda\|_{L^p}^p = \lambda^{-1}\|\nabla^d u\|_{L^p}^p. \tag{1.4}$$

1.1.2 The main result

Our main result improves the previous result of [78] in the sense of the following three directions. First, we achieve the limit case weak-$L^{\frac{4}{d+1}}$ (or $L^{\frac{4}{d+1},\infty}$) as the paper [59] did for second derivatives. Second, we make similar bounds for fractional derivatives as well as classical derivatives. Last, we consider not only smooth solutions but also global-time weak solutions. These three improvements will give us that $\nabla^{3-\epsilon}u$, which is almost third derivatives of weak solutions, is locally integrable on $(0,\infty)\times\mathbb{R}^3$. Our precise result is the following:

Theorem 1.1.1. *There exist universal constants $C_{d,\alpha}$ which depend only on integer $d\geq 1$ and real $\alpha\in[0,2)$ with the following two properties (I) and (II):*

(I) Suppose that we have a smooth solution u of (1.1) on $(0,T)\times\mathbb{R}^3$ for some $0<T\leq\infty$ with some initial data (1.2). Then it satisfies

$$\|(-\Delta)^{\frac{\alpha}{2}}\nabla^d u\|_{L^{p,\infty}(t_0,T;L^{p,\infty}(K))}\leq C_{d,\alpha}\Big(\|u_0\|^2_{L^2(\mathbb{R}^3)}+\frac{|K|}{t_0}\Big)^{\frac{1}{p}} \tag{1.5}$$

for any $t_0\in(0,T)$, any integer $d\geq 1$, any $\alpha\in[0,2)$ and any bounded open subset K of $\mathbb{R}^3$, where $p=\frac{4}{d+\alpha+1}$ and $|\cdot|=$ the Lebesgue measure in $\mathbb{R}^3$.

(II) For any initial data (1.2), we can construct a suitable weak solution u of (1.1) on $(0,\infty)\times\mathbb{R}^3$ such that $(-\Delta)^{\frac{\alpha}{2}}\nabla^d u$ is locally integrable in $(0,\infty)\times\mathbb{R}^3$ for $d=1,2$ and for $\alpha\in[0,2)$ with $(d+\alpha)<3$. Moreover, the estimate (1.5) holds with $T=\infty$ under the same setting of the above part (I) as long as $(d+\alpha)<3$.

There are several remarks for this theorem.

Remark 1.1.1. For any suitable weak solution u, we can define $(-\Delta)^{\alpha/2}\nabla^d u$ in the sense of distributions $\mathcal{D}'$ for any integer $d\geq 0$ and for any real $\alpha\in[0,2)$:

$$<(-\Delta)^{\alpha/2}\nabla^d u;\psi>_{\mathcal{D}',\mathcal{D}}=(-1)^d\int_{(0,\infty)\times\mathbb{R}^3}u\cdot(-\Delta)^{\alpha/2}\nabla^d\psi\,dxdt \tag{1.6}$$

for any test function $\psi\in\mathcal{D}=C_c^\infty((0,\infty)\times\mathbb{R}^3)$ where $(-\Delta)^{\alpha/2}$ in the right hand side is the traditional fractional Laplacian in $\mathbb{R}^3$ defined by the Fourier transform.

Note that $(-\Delta)^{\alpha/2}\nabla^d\psi$ lies in $L_t^\infty L_x^2$. Thus, this definition from (1.6) makes sense due to $u \in L_t^\infty L_x^2$. For the case $\alpha = 0$, we define $(-\Delta)^0$ as the identity map. For more general extensions of this fractional Laplacian operator, we recommend Silvestre [71].

Remark 1.1.2. Since we impose only (1.2) to u_0, the estimate (1.5) is a (quantitative) regularization result to higher derivatives. Also, in the proof, we will see that $\|u_0\|^2_{L^2(\mathbb{R}^3)}$ in (1.5) can be relaxed to $\|\nabla u\|^2_{L^2((0,T)\times\mathbb{R}^3)}$. Thus it says that any (higher) derivatives can be controlled by having only L^2 estimate on the dissipation of energy.

Remark 1.1.3. The result of the part (I) for $\alpha = 0$ extends the result of the paper [78] since for any $0 < q < p < \infty$ and any bounded subset $\Omega \subset \mathbb{R}^n$, we have

$$\|f\|_{L^q(\Omega)} \leq C \cdot \|f\|_{L^{p,\infty}(\Omega)}$$

where C depends only on p, q, dimension n, and Lebesgue measure of Ω (e.g. see Grafakos [43]).

Remark 1.1.4. The "smoothness" assumption in the part (I) is about differentiability. For example, the result of the part (I) for $d \geq 1$ and $\alpha = 0$ holds as long as u is d-times differentiable. In addition, constants in (1.5) are independent of any possible blow-time T.

Remark 1.1.5. $p = 4/(d + \alpha + 1)$ is a very interesting relation as mentioned before. Due to this p, the estimate (1.5) is a non-linear estimate while many other *a priori* estimates are linear. Estimates for $(d + \alpha)$ very close to 3, show that almost third derivatives of weak solutions are locally integrable. It would be very interesting to extend those results to values of $d < 1$. The case $d = \alpha = 0$, for instance, would imply that this weak solution u lies in $L^{4,\infty}$ which is beyond the best known estimate $u \in L^{10/3}$ from L^2 initial data. This kind of extension, however, seems out of reach as now.

1.1.3 The key ideas

1.1.3.1 $\nabla^d u \in L^{4/(d+1),\infty}$ for integer $d \geq 1$, for smooth u

Prior to presenting the main idea to get $\nabla^d u \in L^{4/(d+1),\infty}$, we want to mention that the famous paper [9] contains two different kinds of local regularity criteria. The first one is quantitative, and it says that if $\|u\|_{L^3(Q(1))}$ and $\|P\|_{L^{3/2}(Q(1))}$ is small where $Q(r)$ is the parabolic cylinder $(-r^2, 0) \times B(r)$, then u is bounded by some universal constant in $Q(1/2)$. The second one says that u is locally bounded near the origin if $\limsup_{r\to 0} r^{-1}\|\nabla u\|^2_{L^2(Q(r))}$ is small. So it is qualitative in the sense that the conclusion does not say that u is bounded by a universal constant, but that $\sup |u|$ for some local neighborhood is not infinite. It also required local estimates on an infinite number of zooms (via the $\limsup$).

To explain the main idea and the scaling of the result, suppose that one could prove the following quantitative statement which requires only the smallness of $\|\nabla u\|_{L^2}$:

$$\text{if} \quad \|\nabla u\|_{L^2(Q(1))} \text{ is small }, \text{ then } |\nabla^d u| \leq C_d \text{ in } Q_{(1/2)} \text{ for } d \geq 1.$$

Then we could easily prove $\nabla^d u \in$ weak-$L^{4/(d+1)}_{loc}$ by using the contrapositive statement of the above one and the standard scaling together with Chebyshev's inequality. Indeed, applying the contrapositive statement on $u_\lambda(s,y) = \lambda u(t+\lambda^2 s, x+\lambda y)$ (and scaling back the result on u), we would get that for all (t,x) such that $|\nabla^d u(t,x)| \geq \frac{C_d}{\lambda^{d+1}}$, we have $\frac{1}{\lambda^5}\int_{Q_{(t,x)}(\lambda)} |\nabla u|^2\, ds\, dy \geq \lambda^{-4}$. Denoting $F_\lambda(t,x) = \frac{1}{\lambda^5}\int_{Q_{(t,x)}(\lambda)} |\nabla u|^2\, ds\, dy$, by Chebyshev, we get

$$\mathcal{L}(|\nabla^d u(t,x)| \geq \frac{C_d}{\lambda^{d+1}}) \leq \lambda^4 \int F_\lambda\, dx\, dt \leq \lambda^4 \int |\nabla u|^2\, dx\, dt.$$

This would give the result. Note that the dependence of $p(=\frac{4}{d+1})$ with respect to d is unusual due to the nonlinear estimate obtained through the Chebyshev's inequality.

Unfortunately, the quantitative statement from above cannot be proven. This is due, in particular, to the long range effect of the pressure. Energy outside of the fixed region $Q(1)$ can have an effect (via the pressure) on the higher derivatives of u in $Q(1/2)$. A different quantitative local regularity criterion has been proposed in [79], which showed that for any $p > 1$, there exists ϵ_p such that

$$\text{if} \quad \|u\|_{L^\infty_t L^2_x(Q(1))} + \|\nabla u\|_{L^2_{t,x}(Q(1))} + \|P\|_{L^p_t L^1_x(Q(1))} \leq \epsilon_p, \text{ then } |u| \leq 1 \text{ in } Q(\frac{1}{2}). \tag{1.7}$$

Recently, this criterion was used in [78] in order to obtain higher derivative estimates $\nabla^d u \in L_{loc}^{\frac{4}{d+1}-\epsilon}$ for any small $\epsilon > 0$. The main proposition in [78] says that if both $\||\nabla u|^2 + |\nabla^2 P|\|_{L^p(Q(1))}$ and some other quantity about pressure (the Maximal function of $\nabla^{2-\delta} P$) are small, then u is bounded by 1 at the origin once u has a mean zero property in space. We can observe that $\|\nabla u\|^2_{L^2(Q(1))}$ and $\|\nabla^2 P\|_{L^1(Q(1))}$ have the same best scaling factor $\frac{1}{\lambda}$ (see (1.4)) among all well-known quantities which we can obtain from L^2 initial data. However, the other quantity about pressure has a slightly worse scaling factor than that of $\|\nabla u\|_{L^2}^2$. This is why the the limit case $L^{\frac{4}{d+1},\infty}$ could not be proved in [78].

To cover the limit case $L^{\frac{4}{d+1},\infty}$, we prove an equivalent estimate of (1.7) for $p = 1$. This is achieved by introducing a new pressure decomposition (see Lemma 3.2.3). Roughly speaking, we will decompose the pressure term into three parts for each decreasing balls B_k: a part depending only on u locally, a bounded part depending on u non-locally, and a part depending on both u and P, which does not depend on k. We will see that the last piece can be absorbed into the velocity component. Then, the following will be shown in Section 3.4 by using the Galilean invariance and a blow-up technique with the standard scaling :

Let (u, P) be a smooth solution of the Navier-Stokes. We define $F := \Big(|\nabla u|^2 + |\nabla^2 P|\Big)$ whose $L^1_{t,x}$ norm can be controlled by the L^2 norm of the initial data u_0 (see Lemma 3.4.1)). Then for each point (t, x) and for any ϵ such that $0 < \epsilon^2 \leq t$, there exists an incompressible flow $X(\cdot)$ with the following property: if $\frac{1}{\epsilon} \iint_{Q_{(t,x)}(\epsilon)} F(s, y + X(s))\, dyds \leq \delta$, then $|\nabla^d u(t, x)| \leq C_d/\epsilon^{(d+1)}$ for $d \geq 1$.

The purpose of introducing the flow $X(\cdot)$ is to get a mean zero velocity. It enables us to avoid using any estimate of the velocity u itself whose scaling is weaker than that of its derivative ∇u. This kind of universal property comes from the the local parabolic regularization effect of the viscosity term. However, if we do not control the main drift in a fixed region $Q(1)$, it could impair this process. The flow can be very fast in a fixed region $Q(1)$ so the fluid pass through the region before having time to be locally regularized via the viscosity. Also, note that F has the right scaling factor. As a consequence, thanks to the incompressibility of the flow $X(\cdot)$, we can prove $\nabla^d u \in L_{loc}^{\frac{4}{d+1},\infty}$ for classical derivatives ($\alpha = 0$ case) of smooth solutions of the Navier-Stokes equations.

Remark 1.1.6. The above argument can also be applied to solutions of some approximation scheme of the Navier-Stokes by adding one more assumption about the smallness of $\|\mathcal{M}(|\nabla u|)\|_{L^2}$ (see Proposition 3.1.1). We use the Leray regularization of the Navier-Stokes equation. This makes the drift velocity depend on the velocity via a convolution. It is then, not any more local. We can still control it locally, via any zoom, thanks to the maximal function.

1.1.3.2 $\nabla^\alpha u \in L^{4/(\alpha+1),\infty}$ for real $\alpha \geq 1$

The result for fractional derivatives ($0 < \alpha < 2$ case) is not obvious at all because there is no proper interpolation theorem for $L_{loc}^{p,\infty}$ spaces. For example, due to the non-locality of the fractional Laplacian operator, the fact $\nabla^2 u \in L_{loc}^{\frac{4}{3},\infty}$ with $\nabla^3 u \in L_{loc}^{1,\infty}$ does not imply the case of fractional derivatives even if we assume that u is smooth. Moreover, even though we assume that $\nabla^2 u \in L^{\frac{4}{3}}(\mathbb{R}^3)$ and $\nabla^3 u \in L^1(\mathbb{R}^3)$ which we can NOT prove here, the standard interpolation theorem still requires $\nabla^3 u \in L^q(\mathbb{R}^3)$ for some $q > 1$ (we refer to Bergh and Löfström [6]).

To overcome the difficulty, we will use the Maximal function of ∇u which captures some behavior of u in long-range distance. We will add some quantities depending on the maximal functions of ∇u to F (e.g. see the assumption of Proposition 3.1.2). This process should be done carefully because we want to add only functions whose scalings are correct. Unfortunately, the second derivatives of the pressure, which lie in the Hardy space $\mathcal{H} \subset L^1(\mathbb{R}^3)$ from Coifman, Lions, Meyer and Semmes [22], do not have an integrable Maximal function since the Maximal operator is not bounded on L^1. In order to handle non-local effects of the pressure, we will use some property of Hardy space, which says that some integrable functions play a similar role of the Maximal function (see (2.2)). This is the origin of the last term inside of the integral in (3.12) in Proposition 3.1.2.

1.1.3.3 $\nabla^\alpha u_n \in L^{4/(\alpha+1),\infty}$ for the approximation $\{u_n\}_{n=1}^\infty$ of Leray

Finally, the result (II) of Theorem 1.1.1 for a weak solution comes from a specific approximation of Navier-Stokes equations that Leray [57] used in order to construct a global time weak solution : $\partial_t u_n + ((u_n * \phi_{(1/n)}) \cdot \nabla) u_n + \nabla P_n - \Delta u_n = 0$ and $\operatorname{div} u_n = 0$ where ϕ is a fixed mollifier in $\mathbb{R}^3$, and $\phi_{(1/n)}$ is defined by $\phi_{(1/n)}(\cdot) = n^3 \phi(n\ \cdot)$. The main advantage of adopting this approximation is that it has strong existence theory of global-time smooth solutions u_n for each n, and it is well-known that there exists a suitable weak solution u as a weak limit.

However, to prove (1.5) uniformly for the approximation is nontrivial because our proof is based on local study of De Giorgi-type argument while the approximation is not scaling-invariant with the standard Navier-Stokes scaling in the sense that $u * \phi_{(1/n)}$ becomes $v * \phi_{1/(n\epsilon)}$ (or see Remark 3.1.5). In other words, after the scaling, the convective velocity of the approximation scheme depends on the original velocity more non-locally than before. For example, once we fix n and let ϵ go to zero, then we need information of the velocity v in almost whole space to control the convective velocity $v * \phi_{1/(n\epsilon)}$ in $Q(1)$.

It will be solved by separating its proof into two lemmas 3.2.4 and 3.2.5. In the first lemma, the convective velocity is controlled by the maximal function of $|\nabla u|$, which is not strong enough if the parameter $r := \frac{1}{n\epsilon}$ is small. In the second lemma with small r, we use the fact that the convective velocity is not too different from the velocity itself for the first few steps. Then we can combine those two lemmas to get a uniform De Giorgi type estimate $U_k \leq C^k U_{k-1}^{\beta}$ for some $\beta > 1$ (see Subsection 3.2.4).

In addition, the free parameter $r \in [0, \infty)$ has to be handled carefully also in the final bootstrapping arguments to get locally the control of higher derivatives (see Subsection 3.3.3).

1.1.3.4 $\nabla^{\alpha} u \in L^{4/(\alpha+1),\infty}$ for $1 < \alpha < 3$ for a weak solution u

The above argument will give us certain bounds for u_n in the form of (1.5) with $T = \infty$, for any integer $d \geq 1$ and for any $\alpha \in [0, 2)$, which is uniform in n. Therefore, for the case $(d + \alpha) < 3$, thanks to $p = 4/(d + \alpha + 1) > 1$, we can know that $(-\Delta)^{\frac{\alpha}{2}}\nabla^d u$ exists as a locally integrable function from weak-compactness of L^p for $p > 1$.

1.2 Integro-differential equations

This introduction section for integro-differential equations is a summary of Section 4.1. Thus the reader may go to Section 4.1 directly if he or she wants to see the full settings, results, and references.

1.2.1 Background and settings

Let $N \geq 1$ be any dimension. We say that a function lies in $C^\infty(\overline{\mathbb{R}^d})$ if all derivatives of every order are continuously bounded. We consider the following evolution equation

$$\partial_t w(t,x) \underbrace{- \int_{\mathbb{R}^N} [w(t,y) - w(t,x)] K(t,x,y) dy}_{(*)} = 0 \tag{1.8}$$

where K satisfies special conditions. To simplify, we consider the case $0 < \alpha < 1$ first in this section chapter while the case $1 \leq \alpha < 2$ requires a certain additional condition, which will be discussed in Subsection 1.2.3.2 (or refer to Definition 4.1.3). We denote T_t^K as the integral operator corresponding to the above $(*)$. Thus, the equation above is equivalent to $(\partial_t w)(t,x) + T_t^K(w(t,\cdot))(x) = 0$. Let $0 < \zeta \leq \infty$ and $1 \leq \Lambda < \infty$. We impose the following conditions to the kernel K:

$$\odot \text{ Symmetry in } x, y\text{: } K(t,x,y) = K(t,y,x), \tag{1.9}$$

$$\odot \text{ Bounds: } \Lambda^{-1} \cdot \mathbf{1}_{|x-y|\leq\zeta} \leq K(t,x,y)|x-y|^{N+\alpha} \leq \Lambda. \tag{1.10}$$

For convenience, we define the associated function k by $k(t,x,z) = K(t,x,x+z)|z|^{N+\alpha}$. Then the above two conditions are equivalent to $k(t,x,y-x) = k(t,y,x-y)$ and $\Lambda^{-1} \cdot \mathbf{1}_{|z|\leq\zeta} \leq k(t,x,z) \leq \Lambda$, respectively.

Under theses setting for K, Caffarelli, Chan, and Vasseur [10] proved a C^β-regularization effect from L^2 initial data by using De Giorgi techniques. A similar result for the stationary case was obtained by Kassmann in [47].

Related to the above singular integral operator, there have been many interests recently, not only from the field of analysis, but also from the field of probability (e.g. Caffarelli and Silvestre [12], Bass and Levin [2], and references therein). The most well-known example is the fractional heat equation. Indeed, if $K := c_\alpha/|x-y|^{N+\alpha}$ (i.e. $k := c_\alpha$), then the equation (1.8) becomes the fractional heat equation $w_t - \Delta^{\alpha/2}w = 0$ (or $T_t^k := -\Delta^{\alpha/2}$). Some regularity results related to this equation can be found in Caffarelli and Figalli [11].

1.2.2 The main result with applications

Our main goal is to prove the result of [10] with different techniques. In particular, we prove persistence of Hölder continuity in $L^\infty(0,\infty; C^\beta(\mathbb{R}^N))$, by observing the evolution of a dual class of test functions. Also, we prove a C^β-regularization effect from $L^\infty \cap L^1$ initial data. In [48], Kiselev and Nazarov proposed this duality approach for the 2D-SQG equation. In Subsection 1.2.3.1, we will explain how the duality approach works under our settings.

Let $w_0 \in C^\infty(\overline{\mathbb{R}^N}) \cap L^1(\mathbb{R}^N)$ be a given function and let $0 < T \leq \infty$. In addition, we assume

$$k(\cdot,\cdot,\cdot) \in C^\infty_{t,x,z}\Big([0,T] \times \overline{\mathbb{R}^N} \times \overline{\mathbb{R}^N}\Big). \tag{1.11}$$

Suppose that $w \in L^\infty(0,T;(L^1 \cap L^\infty)(\mathbb{R}^N))$ is a smooth solution of (1.8) on $[0,T) \times \mathbb{R}^N$ for the initial data w_0. Our main result is the following:

Theorem 1.2.1 (or refer to Theorem 4.1.1)**.** *There exist $\beta > 0$ and $C > 0$, and we have the following estimates for any $t \in (0,T)$:*

$$\|w(t,\cdot)\|_{C^\beta(\mathbb{R}^N)} \leq C \cdot \|w_0\|_{C^\beta(\mathbb{R}^N)} \text{ and} \tag{1.12}$$

$$\|w(t,\cdot)\|_{C^\beta(\mathbb{R}^N)} \leq C \cdot \max\{1, \frac{1}{t^{\beta/\alpha}}\} \cdot \|w_0\|_{L^\infty(\mathbb{R}^N)} \tag{1.13}$$

where the above constants $\beta > 0$ and $C > 0$ depend only on the parameter set $\{\alpha, \zeta, \Lambda\}$.

The idea for the proof will be given in Subsection 1.2.3.1.

In addition, we introduce the following non-linear evolution problem:

$$\partial_t \theta(t,x) - \int_{\mathbb{R}^N} \phi'(\theta(t,y) - \theta(t,x)) G(y-x) dy = 0. \tag{1.14}$$

This non-linear problem can be found in the field of image processing (e.g. see Gilboa and Osher [41]).

We impose the following conditions to the equation (1.14): Let $\phi : \mathbb{R} \to [0,\infty)$ be an even function of class C^2 satisfying

$$\phi(0) = 0 \quad \text{and} \quad \sqrt{\Lambda^{-1}} \le \phi''(x) \le \sqrt{\Lambda}, \quad x \in \mathbb{R}.$$

We assume that the kernel $G : \mathbb{R}^N/\{0\} \to [0,\infty)$ satisfies $G(-x) = G(x)$ and

$$\sqrt{\Lambda^{-1}} \cdot \mathbf{1}_{|x|\le\zeta} \le G(x) \cdot |x|^{N+\alpha} \le \sqrt{\Lambda} \text{ for } x \in \mathbb{R}^N/\{0\}. \tag{1.15}$$

Following the approach of [10], we present the following important consequence of Theorem 1.2.1.

Theorem 1.2.2 (or refer to Theorem 4.1.2)**.** *Let* $\theta_0 \in (W^{1,1} \cap W^{1,\infty})(\mathbb{R}^N)$ *be a given function. Then there exist* $\beta > 0$ *and* $C > 0$*, and there exists a global-time weak solution* θ *of the equation* (1.14) *with the following estimates for a.e.* $t \in (0,\infty)$*:*

$$\|\nabla\theta(t,\cdot)\|_{C^\beta(\mathbb{R}^N)} \le C \cdot \|\nabla\theta_0\|_{C^\beta(\mathbb{R}^N)} \quad \textit{if} \quad \nabla\theta_0 \in C^\beta(\mathbb{R}^N) \qquad \textit{and}$$

$$\|\nabla\theta(t,\cdot)\|_{C^\beta(\mathbb{R}^N)} \le C \cdot \max\{1, \frac{1}{t^{\beta/\alpha}}\} \cdot \|\nabla\theta_0\|_{L^\infty(\mathbb{R}^N)}$$

where the above two constants $\beta > 0$ *and* $C > 0$ *depend only on the parameter set* $\{\alpha, \zeta, \Lambda\}$*.*

Once we differentiate the equation (1.14), the proof follows directly. Indeed, after taking differentiation on (1.14), it is easy to verify that $\nabla\theta$ is a solution of (1.8) with the kernel $K(t,x,y) := \phi''(\theta(t,y) - \theta(t,x))G(y-x)$. In addition, this kernel satisfies all the assumptions of Theorem 1.2.1.

As the last application, we introduce the following drift diffusion equation with fractional diffusion

$$\partial_t w + (b \cdot \nabla) w + (-\Delta)^{\alpha/2} w = 0 \tag{1.16}$$

under some additional assumptions on the drift velocity b. One of Mathematical issues is to find a minimal condition on b to ensure certain regularity of a solution w. From Caffarelli and Vasseur [13], we have the C^∞-regularity for $\alpha = 1$ and for $b \in L_t^\infty BMO_x$ with div $b = 0$ (also, see [48] for the same setting, Kiselev, Nazarov and Volberg [49] for the 2D-SQG equation).

Our result is about supercritical diffusion ($\alpha < 1$). We follow the framework of [48] in order to show a similar result for $\alpha \in (0, 1)$ and for $b \in L_t^\infty C_x^{1-\alpha}$ with div $b = 0$. We prove persistence of Hölder continuity even though it has been known that $b \in L_t^\infty C_x^{1-\alpha}$ gives a solution for the 2D-SQG equation Hölder regularity for any $t > 0$ in Constantin and Wu [27] (also, see Silvestre [72] without div $b = 0$).

Theorem 1.2.3 (or refer to Theorem 4.1.3)**.** *Let $\alpha \in (0, 1)$ and $B \in [0, \infty)$. Then, there exist C and $\beta > 0$ with the following property:*

Let $T > 0$. Suppose $b \in C^\infty(\overline{\mathbb{R}^N} \times [0, T])$ and $b(t) \in C^{1-\alpha}$ with $[b(t)]_{C^{1-\alpha}} \le B$ and div $b(t, \cdot) = 0$ for all $t \in [0, T]$. Assume $w \in L^\infty(0, T; (L^1 \cap L^\infty)(\mathbb{R}^N))$ is a smooth solution of (1.16) with a smooth initial data $w_0 \in (L^1 \cap L^\infty)(\mathbb{R}^N)$. Then, this solution w satisfies the estimate (1.12) if $w_0 \in C^\beta$ and the estimate (1.13) for any $t \in (0, T)$.

Note that this *a priori* C^β-regularity does not imply that a weak solution is classical. It is interesting that we obtain higher regularity once we impose slightly stronger assumptions on b (or w). For example, we refer to the papers of Constantin and Wu [26](with $w \in L_t^\infty C_x^{1-\alpha+\epsilon}$ for the 2D-SQG), Silvestre [73](with $b \in L_t^\infty C_x^{1-\alpha+\epsilon}$ without div $b = 0$), Dong and Pavlović [32] (with $b \in C_t C_x^{1-\alpha}$ for the 2D-SQG).

1.2.3 The key ideas

1.2.3.1 The linear problem: Theorem 1.2.1

First, we recall the characterization of the Hölder space (C^β space) via Littlewood-Paley projections Δ_j, which is defined by $\Delta_j(w) = w * \Psi_{2^{-j}}$ where $\Psi_t(x) = t^{-N}\Psi(x/t)$ and $\hat{\Psi}(\xi) = \eta(\xi) - \eta(2\xi)$ with $\eta \in C_0^\infty, 0 \leq \eta(\xi) \leq 1, \eta = 1$ for $|\xi| \leq 1$ and $\eta = 0$ for $|\xi| \geq 2$ (see Stein [75]). Let w be a bounded function in $\mathbb{R}^N$. Then, the characterization says that

$$\sup_{j=1,2,3,\cdots} 2^{\beta j}\|\Delta_j(w)\|_{L^\infty(\mathbb{R}^N)} < \infty \text{ if and only if } w \in C^\beta(\mathbb{R}^N).$$

Motivated by this criterion, we define the following class $\mathcal{U}_r$, which was introduced first in [48] (refer to Definition 4.2.1) : $\varphi(\cdot)$ on $\mathbb{R}^N$ lies in $\mathcal{U}_r$ for some $r \in (0,\infty)$ if φ satisfies the following four conditions:

$$\int_{\mathbb{R}^N} \varphi(x)dx = 0 \qquad \text{the mean zero-condition,}$$

$$\int_{\mathbb{R}^N} |\varphi(x)||x-x_0|^\gamma dx \leq r^\gamma \quad \text{for some } x_0 \in \mathbb{R}^N \quad \text{the concentration-condition,}$$

$$\|\varphi\|_{L^\infty(\mathbb{R}^N)} \leq \frac{A}{r^N} \qquad \text{the } L^\infty\text{-condition, and}$$

$$\|\varphi\|_{L^1(\mathbb{R}^N)} \leq 1 \qquad \text{the } L^1\text{-condition}$$

where $A \geq 1$ will be chosen in the proof. In addition, we say that φ lies in $a\mathfrak{U}_r$ for some $a > 0$ when $(1/a)\varphi \in \mathfrak{U}_r$. We call x_0 a center of φ.

It is easy to see $\Psi_{2^{-j}} \in a\mathfrak{U}_{2^{-j}}$ for some constant a. As a result, we can prove the following characterization of C^β space via $\mathfrak{U}_r$ (refer to Lemma 4.2.4): a bounded function w lies in C^β if and only if

$$\sup_{\varphi \in \mathfrak{U}_r, 0<r\leq 1} r^{-\beta}\Big|\int_{\mathbb{R}^3} w(x)\varphi(x)dx\Big| < \infty.$$

Therefore, in order to show a solution $w(t) \in C^\beta$, it is enough to show

$$\sup_{\varphi \in \mathfrak{U}_r, 0<r\leq 1} r^{-\beta}\Big|\int_{\mathbb{R}^3} w(t,x)\varphi(x)dx\Big| < \infty.$$

Remark 1.2.1. In short, we test a function w against the family $\{r^{-\beta}\varphi \mid \varphi \in \mathfrak{U}_r, 0 < r \leq 1\}$ to estimate the C^β-norm of w. This argument tells us that the family plays a similar role of the subclass of the dual space of C^β, whose elements have norms less than 1. These elements of the family are called molecules, which form the local Hardy space h^p for $p < 1$ where $\beta = n(p^{-1} - 1)$. For references, we recommend Taibleson and Weiss [77], Goldberg [42](or see Stein [75] for a modern text).

For simple presentation, we suppose for a while that the integral operator T_t^K does not depend on the time variable t. If w and φ are solutions of (1.8) on a time interval $[0, t]$ with some initial data w_0 and φ_0, then we can verify the time derivative $\frac{d}{ds}$ of $\int_{\mathbb{R}^3} w(s,x)\varphi(t-s,x)dx$ vanishes thanks to the dual structure of the problem (1.8). More precisely, it is due to the symmetry condition (1.9) (refer to Lemma 4.2.5). This observation implies that

$$\int_{\mathbb{R}^3} w(t,x)\varphi_0(x)dx = \int_{\mathbb{R}^3} w_0(x)\varphi(t,x)dx.$$

Thus, to get $w(t) \in C^\beta$, it suffices to show

$$\sup_{\varphi_0 \in \mathcal{U}_r, 0<r\le 1} r^{-\beta} \Big| \int_{\mathbb{R}^3} w_0(x)\varphi(t,x)dx \Big| < \infty.$$

In other words, we need to control the evolution of $\varphi(t)$ properly to get $w(t) \in C^\beta$.

Suppose we could prove $\varphi(s) \in \left(\frac{r}{z(r,s)}\right)^\beta \mathcal{U}_{z(r,s)}$ where $z(\cdot,\cdot) \le 1$ is a function for $r \in (0,1]$ and $s \in (0,\infty)$ (refer to Proposition 4.3.1). Then we have

$$r^{-\beta} \Big| \int_{\mathbb{R}^3} w_0(x)\varphi(t,x)dx \Big| \le r^{-\beta} \Big(\frac{r}{z}\Big)^\beta z^\beta \cdot C \cdot \|w_0\|_{C^\beta} = C \cdot \|w_0\|_{C^\beta}.$$

Thus, once we find $z(\cdot,\cdot)$ such that $\varphi(s) \in \left(\frac{r}{z(r,s)}\right)^\beta \mathcal{U}_{z(r,s)}$, then the proof is done.

First we prove a short-time evolution result (refer to Proposition 4.3.1). In order to obtain it, we need to manage the competition (refer to Remark 4.3.4) between the L^p condition and the concentration condition. The former condition, which can be proved from the lower bound $\Lambda^{-1} \cdot \mathbf{1}_{|z|\le\zeta}$ of the kernel K, has a regularization effect (see Lemma 4.3.4, Lemma 4.3.5) as a diffusion term in usual PDEs does. However, the latter condition comes from the upper bound Λ of the kernel, and this upper bound plays a similar role as a source term in usual PDEs (see Lemma 4.3.3).

Since Proposition 4.3.1 tells us about the evolution $\varphi(s)$ only for a short time, it should be verified that we can repeat the short-time evolution as many times as we want in order to reach any fixed time (refer to Remark 4.4.1).

1.2.3.2 The additional assumption for the case $\alpha \geq 1$

Until now, we have considered only for the case $\alpha < 1$ of the equation (1.8). In this section, we explain why we need an extra condition for the case $\alpha \geq 1$.

First of all, we note that, in the proof of Proposition 4.3.1, we need to use a pointwise estimate of $T_t^K(f)$ several times for a certain class of functions. Suppose $f \in C_c^\infty(\mathbb{R}^3)$. Then we have

$$\begin{aligned} T_t^K(f)(x) &= \int (f(x) - f(y))K(t,x,y)dy = \int \frac{f(x) - f(x+z)}{|z|^{N+\alpha}} k(t,x,z)dz \\ &= \int_{|z|<1} \frac{f(x) - f(x+z)}{z^{N+\alpha}} k(t,x,z)dz + \int_{|z|\geq 1} \frac{f(x) - f(x+z)}{z^{N+\alpha}} k(t,x,z)dz. \end{aligned}$$

From the Taylor expansion with the error estimate $|R_f(x, r\sigma)| \leq Cr^2$, we have

$$\begin{aligned} |T_t^K(f)(x)| &\leq \Big| \int_0^1 \int_{S^{N-1}} \frac{(\nabla f)(x) \cdot (r\sigma) + R_f(x, r\sigma)}{r^{1+\alpha}} k(t,x,r\sigma)d\sigma dr \Big| + C \\ &\leq \Big| \int_0^1 \int_{S^{N-1}} \frac{(\nabla f)(x) \cdot (r\sigma)}{r^{1+\alpha}} k(t,x,r\sigma)d\sigma dr \Big| + C \\ &\leq C \underbrace{\int_0^1 r^{-\alpha} \Big| \int_{S^{N-1}} \sigma k(t,x,r\sigma)d\sigma \Big| dr}_{(*)} + C \end{aligned}$$

where σ is a surface element on the unit sphere $S^{N-1} \subset \mathbb{R}^N$.

If $\alpha \in [1,2)$, then the above integral $(*)$ does not make sense without a certain cancellation property of the kernel K (or k). For this reason, we ask the kernel K to satisfy not only (1.9) and (1.10) but also the following condition (see Definition 4.1.3): there exists a constant ν such that $(\alpha - 1) < \nu$, and it satisfies

$$\odot \text{ Cancellation: } \Big| \int_{S^{N-1}} k(t,x,s\sigma)\sigma d\sigma \Big| \leq C \cdot s^{\nu} \text{ for } s > 0. \tag{1.17}$$

For the fractional heat equation (i.e. $k := c_\alpha$), then the kernel satisfies the above condition (1.17) since the integral is always zero. For another example, suppose that the kernel has the form not of $K(t,x,y)$ but of $K(t,x-y)$ (for more general cases, we refer to Silvestre [70]). Then the natural symmetry we would impose to the kernel is $K(t,x-y) = K(t,y-x)$, (i.e. $k(t,z) = k(t,-z)$). Then, the integral in (1.17) vanishes thanks to the cancellation from the symmetry (see Remark 4.1.5).

1.2.3.3 The drift-diffusion problem: Theorem 1.2.3

First, we observe that the dual problem of (1.16) is

$$\partial_t \varphi + (b^{\bar{T}} \cdot \nabla)\varphi + (-\Delta)^{\alpha/2}\varphi = 0 \tag{1.18}$$

where $b^{\bar{T}}(t,x) := -b(\bar{T} - t, x)$ due to div $b(s,\cdot) = 0$ in the sense of the conclusion of Lemma 4.2.5. In other words, if w is a solution of (1.16) and if φ is a solution of (1.18), then we have

$$\int_{\mathbb{R}^3} w(t,x)\varphi_0(x)dx = \int_{\mathbb{R}^3} w_0(x)\varphi(t,x)dx.$$

Thus our goal is to prove the same conclusion of Proposition 4.3.1 for the problem (1.18). Suppose φ is a smooth solution of $\partial_t \varphi + (b \cdot \nabla)\varphi + (-\Delta)^{\alpha/2}\varphi = 0$ with the initial data φ_0 with a center x_0. Then we define the curve $X(\cdot)$ by solving

$$\dot{X}(s) = (b(s,\cdot))_{X(s),(A^{-1/N}r)} \quad \text{with} \quad X(0) = x_0$$

where $(f)_{x,l}$ is defined by $\frac{1}{|B_l|}\int_{B_l(x)} f(y)dy$ for any L^1_{loc} function f. This definition of $X(\cdot)$ will help us to minimize the effect coming from the drift term $(b\cdot\nabla)\varphi$ in the concentration condition (see Lemma 4.5.2).

We will use the following scaling properties of C^β space for $0<\beta<1$: For $f\in C^\beta$, for $l\in(0,\infty)$, for $p\in[1,\infty)$ and for any integer $k\geq 1$, we have

$$\Big(\frac{1}{|B_l|}\int_{B_l(x)} |f(y)-(f)_{x,l}|^p dy\Big)^{1/p} \leq 2^\beta\cdot[f]_{C^\beta}\cdot l^\beta \quad \text{and}$$

$$|(f)_{x,l}-(f)_{x,2^k\cdot l}| \leq 2^{N+\beta}\cdot\Big(\sum_{j=1}^{k}(2^\beta)^j\Big)\cdot[f]_{C^\beta}\cdot l^\beta.$$

Note that the BMO space has these properties corresponding to $\beta=0$, which was one of the key ideas of [48].

We will see that the concentration condition is the only obstacle thanks to the divergent free condition of b. The main difficulty is to prove the following estimate:

$$\int |\varphi|\cdot|b-(b(s,\cdot))_{X(s),(A^{-1/N}r)}|\cdot|x-X(s)|^{\gamma-1}dx \leq C\cdot B\cdot A^{\frac{\alpha-\gamma}{N}}r^{\gamma-\alpha}$$

We recall that $\sup\limits_{0<t<T}\|b(t)\|_{C^\beta}\leq B$. Once we split this integral into the two parts $\int_{B_{(A^{-1/N}r)}(X(s))}$ and $\int_{\mathbb{R}^N-B_{(A^{-1/N}r)}(X(s))}$, the estimate follows directly by Hölder's inequality thanks to the above scaling properties of C^β.

Chapter 2

Preliminaries from Analysis

All the materials in this chapter can be found in usual advanced textbooks of Real Analysis. Among them, we recommend Stein [75] and Grafakos [43].

Notations for general purpose

We define $B(r)$ the ball in $\mathbb{R}^3$ centered at the origin with radius r, $Q(r) = (-r^2, 0) \times B(r)$, the cylinder in $\mathbb{R} \times \mathbb{R}^3$ and $B(x;r)$ the ball in $\mathbb{R}^3$ centered at x with radius r.

Remark 2.0.2. In Section 3.2, we will introduce more notations for balls B_k and cylinders Q_k (see (3.13)), which decrease as k increases. When using a De Giorgi type argument, these notations are natural, and they will be used only in Subsection 3.2.1, 3.2.2 and 3.2.3.

L^p, weak-L^p and Sobolev spaces $W^{n,p}$

Let K be a open subset K of $\mathbb{R}^n$. For $0 < p < \infty$, we denote $L^p(K)$ the usual space with with (quasi) norm $\|f\|_{L^p(K)} = (\int_K |f|^p dx)^{(1/p)}$. Also, for $0 < p < \infty$, the weak-$L^p(K)$ space (or $L^{p,\infty}(K)$) is defined by

$$L^{p,\infty}(K) = \{f \text{ measurable in } K \subset \mathbb{R}^d \quad : \sup_{\alpha>0} \Big(\alpha^p \cdot |\{|f| > \alpha\} \cap K|\Big) < \infty\}$$

with (quasi) norm $\|f\|_{L^{p,\infty}(K)} = \sup_{\alpha>0}\Big(\alpha \cdot |\{|f| > \alpha\} \cap K|^{1/p}\Big)$. From the Chebyshev's inequality, we have $\|f\|_{L^{p,\infty}(K)} \le \|f\|_{L^p(K)}$ for any $0 < p < \infty$. Also, for $0 < q < p < \infty$, $L^{p,\infty}(K) \subset L^q(K)$ once K is bounded (refer to Remark 1.1.3 in the beginning).

For any integer $n \ge 0$ and for any $p \in [1, \infty]$, we denote $W^{n,p}(\mathbb{R}^3)$ and $W^{n,p}(B(r))$ as the standard Sobolev spaces for the whole space $\mathbb{R}^3$ and for a ball $B(r)$ in $\mathbb{R}^3$, respectively.

The Maximal function $\mathcal{M}$ and the Riesz transform $\mathcal{R}_j$

The Maximal function $\mathcal{M}$ in $\mathbb{R}^d$ is defined by the following standard way:

$$\mathcal{M}(f)(x) = \sup_{r>0} \frac{1}{|B(r)|}\int_{B(r)} |f(x+y)|dy.$$

Also, we can express this Maximal operator as a supremum of convolutions: $\mathcal{M}(f) = C\sup_{\delta>0}\Big(\chi_\delta * |f|\Big)$ where $\chi = \mathbf{1}_{\{|x|<1\}}$ is the characteristic function of the unit ball, and $\chi_\delta(\cdot) = (1/\delta^3)\chi(\cdot/\delta)$. Note that $\mathcal{M}$ is bounded from $L^p(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d)$ for $p \in (1, \infty]$ and from $L^1(\mathbb{R}^d)$ to $L^{1,\infty}(\mathbb{R}^d)$. we denote $\mathcal{M}$ and $\mathcal{M}^{(t)}$ as the Maximal functions in $\mathbb{R}^3$ and in $\mathbb{R}^1$, respectively.

For $1 \leq j \leq 3$, the Riesz Transform $\mathcal{R}_j$ in $\mathbb{R}^3$ is defined by the Fourier transform:

$$\widehat{\mathcal{R}_j(f)}(x) = i\frac{x_j}{|x|}\hat{f}(x).$$

The operator $\mathcal{R}_j$ is bounded in L^p for $1 < p < \infty$.

The Hardy space $\mathcal{H}$

The Hardy space $\mathcal{H}$ in $\mathbb{R}^3$ is defined by

$$\mathcal{H}(\mathbb{R}^3) = \{f \in L^1(\mathbb{R}^3) \quad : \quad \sup_{\delta > 0} |\mathcal{P}_\delta * f| \in L^1(\mathbb{R}^3)\}$$

where $\mathcal{P} = C(1+|x|^2)^{-2}$ is the Poisson kernel and $\mathcal{P}_\delta$ is defined by $\mathcal{P}_\delta(\cdot) = \delta^{-3}\mathcal{P}(\cdot/\delta)$. A norm of $\mathcal{H}$ is defined by L^1 norm of $\sup_{\delta>0} |\mathcal{P}_\delta * f|$. Thus $\mathcal{H}$ is a subspace of $L^1(\mathbb{R}^3)$ and $\|f\|_{L^1(\mathbb{R}^3)} \leq \|f\|_{\mathcal{H}(\mathbb{R}^3)}$ for any $f \in \mathcal{H}$. Moreover, the Riesz Transform is bounded from $\mathcal{H}$ to $\mathcal{H}$.

One of important applications of the Hardy space is the compensated compactness (see Coifman, Lions, Meyer and Semmes [22]). Especially, it says that if $E, B \in L^2(\mathbb{R}^3)$ and $curl E = \operatorname{div} B = 0$ in distribution, then $E \cdot B \in \mathcal{H}(\mathbb{R}^3)$ and we have

$$\|E \cdot B\|_{\mathcal{H}(\mathbb{R}^3)} \leq C \cdot \|E\|_{L^2(\mathbb{R}^3)} \cdot \|B\|_{L^2(\mathbb{R}^3)}$$

for some universal constant C. In order to obtain some regularity of second derivative of pressure, we can combine compensated compactness with boundedness of the

Riesz transform in $\mathcal{H}(\mathbb{R}^3)$. For example, if u is a weak solution of the Navier-Stokes (1.1), then a corresponding pressure P satisfies

$$\|\nabla^2 P\|_{L^1(0,\infty;\mathcal{H}(\mathbb{R}^3))} \leq C \cdot \|\nabla u\|^2_{L^2(0,\infty;L^2(\mathbb{R}^3))} \tag{2.1}$$

(see Lions [59] or the lemma 7 in [78]).

Note that if we replace the Poisson kernel $\mathcal{P}$ with any function $\mathcal{G} \in C^\infty(\mathbb{R}^3)$ with compact support, then we have a constant C depending only on $\mathcal{G}$ such that

$$\|\sup_{\delta>0} |\mathcal{G}_\delta * f|\|_{L^1(\mathbb{R}^3)} \leq C\|\sup_{\delta>0} |\mathcal{P}_\delta * f|\|_{L^1(\mathbb{R}^3)} = C\|f\|_{\mathcal{H}(\mathbb{R}^3)} \tag{2.2}$$

where $\mathcal{G}_\delta(\cdot) = \mathcal{G}(\cdot/\delta)/\delta^3$. (see Fefferman and Stein [36] or see Stein [75], Grafakos [43] for modern texts). Even though the Maximal function $\sup_{\delta>0}\Big(\chi_\delta * |f|\Big)$ of any non-trivial Hardy space function f is not integrable, there exist integrable functions $\Big(\sup_{\delta>0}\Big|\mathcal{G}_\delta * f\Big|\Big)$, which can capture locally some non-local feature of the function f in a similar way Maximal functions do. However, (2.2) is slightly weaker than the Maximal function, since it controls only mean values of nonlocal quantities (not the absolute value).

The definition of the fractional Laplacian $(-\Delta)^{\alpha/2}$

For $-3 < \alpha \leq 2$ and for $f \in \mathcal{S}(\mathbb{R}^3)$ (the Schwartz space), $(-\Delta)^{\frac{\alpha}{2}} f$ is defined by the Fourier transform:

$$\widehat{(-\Delta)^{\frac{\alpha}{2}} f}(\xi) = |\xi|^\alpha \hat{f}(\xi). \tag{2.3}$$

Note that $(-\Delta)^0 = Id$. Especially, for $\alpha \in (0,2)$, the fractional Laplacian can also be defined by the singular integral for any $f \in \mathcal{S}$:

$$(-\Delta)^{\frac{\alpha}{2}} f(x) = C_\alpha \cdot P.V. \int_{\mathbb{R}^3} \frac{f(x) - f(y)}{|x-y|^{3+\alpha}} dy. \tag{2.4}$$

Chapter 3

Estimates on fractional higher derivatives of weak solutions for the Navier-Stokes equations

The chapter is dedicated to prove Theorem 1.1.1, and it is based on [21]. It is organized as follows. In the next section, the main propositions 3.1.1 and 3.1.2 will be introduced. Then we prove those propositions 3.1.1 and 3.1.2 in Section 3.2 and 3.3, respectively. Finally we will explain how Proposition 3.1.2 implies the part (II) of Theorem 1.1.1 for $\alpha = 0$ and for $0 < \alpha < 2$ in Subsection 3.4.2 and 3.4.3 respectively while the part (I) will be covered in Subsection 3.4.4. After that, the appendix contains some missing proofs of technical lemmas.

In this chapter, any derivatives, convolutions and Maximal functions are with respect to the space variable $x \in \mathbb{R}^3$ unless the time variable is specified.

3.1 Two approximations and the main local study propositions

We fix $\phi \in C^\infty(\mathbb{R}^3)$ satisfying:

$$\int_{\mathbb{R}^3} \phi(x)dx = 1, \quad supp(\phi) \subset B(1), \quad 0 \le \phi \le 1$$

$$\phi(x) = 1 \text{ for } |x| \le \frac{1}{2} \quad \text{and} \quad \phi \text{ is radial.}$$

For any real number $r > 0$, we define the functions $\phi_r \in C^\infty(\mathbb{R}^3)$ by $\phi_r(x) = \frac{1}{r^3}\phi(\frac{x}{r})$. When $r = 0$, we define $\phi_r = \phi_0 = \delta_0$ as the Dirac-delta function. From the Young's inequality for convolutions, we can observe

$$\|f * \phi_r\|_{L^p(B(a))} \le \|f\|_{L^p(B(a+r))} \tag{3.1}$$

due to $supp(\phi_r) \subset B(r)$ for any $p \in [1, \infty]$, for any $f \in L^p_{loc}$ and for any $a, r > 0$.

We introduce two approximations to the Navier-Stokes. The first one (Problem I-n) is the approximation, which Leray [57] used, while the second one (Problem II-r) will be used in our local study after we apply some certain scaling to (Problem I-n).

Definition of (Problem I-n): the first approximation to Navier-Stokes

Definition 3.1.1. Let $n \ge 1$ be either an integer or the infinity ∞, and let $0 < T \le \infty$. Suppose that u_0 satisfy (1.2). We say that $(u, P) \in [C^\infty\big((0,T) \times \mathbb{R}^3\big)]^2$ is a solution of (Problem I-n) on $(0, T)$ for the data u_0 if it satisfies

$$\begin{aligned} \partial_t u + ((u * \phi_{\frac{1}{n}}) \cdot \nabla)u + \nabla P - \Delta u = 0 \\ \operatorname{div} u = 0 \quad t \in (0,T), \quad x \in \mathbb{R}^3 \end{aligned} \tag{3.2}$$

and

$$u(t) \to u_0 * \phi_{\frac{1}{n}} \text{ in } L^2\text{-sense as } t \to 0. \tag{3.3}$$

Remark 3.1.1. When $n = \infty$, (3.2) is the Navier-Stokes on $(0, T) \times \mathbb{R}^3$ with the initial value u_0.

Remark 3.1.2. If n is not the infinity but a positive integer, then for any given u_0 of (1.2), we have existence and uniqueness theory of (Problem I-n) on $(0,\infty)$ with the energy equality

$$\|u(t)\|^2_{L^2(\mathbb{R}^3)} + 2\|\nabla u\|^2_{L^2(0,t;L^2(\mathbb{R}^3))} = \|u_0 * \phi_{\frac{1}{n}}\|^2_{L^2(\mathbb{R}^3)}. \tag{3.4}$$

for any $t < \infty$. It is well-known that we can extract a sub-sequence which converges to a suitable weak solution u of (1.1) and (1.3) with the initial data u_0 of (1.2) (see [57], or see Lions [59], Lemarié-Rieusset [56] for modern texts).

Remark 3.1.3. As mentioned in the introduction section, we can observe that, for $n < \infty$, this notion (Problem I-n) is not invariant under the standard Navier-Stokes scaling $u(t,x) \to \epsilon u(\epsilon^2 t, \epsilon x)$ due to the convective velocity $(u * \phi_{1/n})$.

Definition of (Problem II-r): the second approximation to Navier-Stokes

Definition 3.1.2. Let $0 \le r < \infty$ be real. We say that $(u,P) \in [C^\infty\big((-4,0)\times\mathbb{R}^3\big)]^2$ is a solution of (Problem II-r) if it satisfies

$$\begin{aligned} &\partial_t u + (w\cdot\nabla)u + \nabla P - \Delta u = 0 \\ &\qquad\qquad \operatorname{div} u = 0, \quad t\in(-4,0), \quad x\in\mathbb{R}^3 \end{aligned} \tag{3.5}$$

where w is the difference of two functions:

$$w(t,x) = w'(t,x) - w''(t), \quad t\in(-4,0), x\in\mathbb{R}^3 \tag{3.6}$$

which are defined by u in the following way:

$$w'(t,x) = (u*\phi_r)(t,x) \quad \text{and} \quad w''(t) = \int_{\mathbb{R}^3} \phi(y)(u*\phi_r)(t,y)dy.$$

Remark 3.1.4. This notion of (Problem II-r) gives us the mean zero property for the convective velocity w: $\int_{\mathbb{R}^3}\phi(x)w(t,x)dx = 0$ on $(-4,0)$. Also this w is divergent free from the definition. Moreover, by multiplying u to (3.5), we have

$$\partial_t\frac{|u|^2}{2} + \text{div}(w\frac{|u|^2}{2}) + \text{div}(uP) + |\nabla u|^2 - \Delta\frac{|u|^2}{2} = 0 \tag{3.7}$$

in classical sense because our definition assumes $(u,P)\in C^\infty$.

Remark 3.1.5. We will introduce some specially designed ϵ-scaling which will be a bridge between (Problem I-n) and (Problem II-r) (it can be found in (3.85) in Section 3.4). This scaling is based on the Galilean invariance in order to obtain the mean zero property for the velocity u: $\int_{\mathbb{R}^3}\phi(x)u(t,x)dx = 0$ on $(-4,0)$. Moreover, this ϵ-scaling applied to solutions of (Problem I-n), provides a solution to (Problem II-$\frac{1}{n\epsilon}$) (it can be found (3.86)). We need a local result which is independent on both ϵ and n. In other words, we have to consider the free parameter $r := \frac{1}{n\epsilon}\in[0,\infty)$.

Remark 3.1.6. When $r = 0$, the equation (3.5) is the Navier-Stokes on $(-4,0)\times\mathbb{R}^3$ once we assume the mean zero property for u.

Now we present two main local-study propositions which require the notion of (Problem II-r). These are kinds of partial regularity theorems for solutions of (Problem II-r). The main difficulty to prove them is that both $\bar{\eta} > 0$ and $\bar{\delta} > 0$ should be independent of any r in $[0,\infty)$. We will prove this independence very carefully, which is the heart of Section 3.2 and 3.3.

The first local study proposition for (Problem II-r)

The following result is a quantitative version of partial regularity theorems which extends that of Vasseur [79] up to $p = 1$. The proof will be based on the De Giorgi iteration with a new pressure decomposition (see Lemma 3.2.3).

Proposition 3.1.1. *There exists $\bar{\delta} > 0$ with the following property:*

If u is a solution of (Problem II-r) for some $0 \leq r < \infty$ verifying both

$$\|u\|_{L^\infty(-2,0;L^2(B(\frac{5}{4})))} + \|P\|_{L^1(-2,0;L^1(B(1)))} + \|\nabla u\|_{L^2(-2,0;L^2(B(\frac{5}{4})))} \leq \bar{\delta}$$

$$\textit{and} \qquad \|\mathcal{M}(|\nabla u|)\|_{L^2(-2,0;L^2(B(2)))} \leq \bar{\delta},$$

then we have

$$|u(t,x)| \leq 1 \ \textit{on} \ [-\frac{3}{2}, 0] \times B(\frac{1}{2}).$$

Remark 3.1.7. For the case $r = 0$, we do not need the smallness condition on $\|\mathcal{M}(|\nabla u|)\|_{L^2}$. Indeed, if $r = 0$, then, in Lemma 3.2.5, we have $k_r = k_0 = \infty$ without using the smallness of $\|\mathcal{M}(|\nabla u|)\|_{L^2}$. Then we can use Lemma 3.2.6 directly in order to get the above conclusion.

Remark 3.1.8. For the case $r > 0$, the smallness condition on $\|\nabla u\|_{L^2}$ is not necessary because we have $|\nabla u(x)| \leq M(|\nabla u|)(x)$ for a.e. x.

The above proposition will be proved in Section 3.2. The two terms $\|u\|_{L_t^\infty L_x^2}$ and $\|P\|_{L_t^1 L_x^1}$ do not have the correct scaling through the ε-zoom. The next proposition 3.1.2 deals only quantities which have the correct scaling. However, since we cannot control the mean value of u with such quantities, it will assume the mean-zero property on u. As mentioned in the introduction, we cannot expect a local parabolic regularization effect if the drift is too big.

Notations associated to the fractional derivatives $(-\Delta)^{\alpha/2}$

Before stating the second local study proposition, we introduce the following two definitions of ζ and h^α, which will be used only in the proof concerning fractional derivatives. The main purpose of these two functions is to overcome the weak point that $\mathcal{M}(\nabla^2 P)$ is not integrable due to $\nabla^2 P \in L^1$. We define ζ by $\zeta(x) = \phi(\frac{x}{2}) - \phi(x)$. Then we have

$$\begin{aligned} &\zeta \in C^\infty(\mathbb{R}^3), \quad supp(\zeta) \subset B(2), \quad \zeta(x) = 0 \text{ for } |x| \le \frac{1}{2} \\ &\text{and } \sum_{j=k}^{\infty} \zeta(\frac{x}{2^j}) = 1 \text{ for } |x| \ge 2^k \text{ for any integer } k. \end{aligned} \tag{3.8}$$

In addition, we define the function h^α for $\alpha > 0$ by $h^\alpha(x) = \zeta(x)/|x|^{3+\alpha}$. Also we define $(h^\alpha)_\delta$ and $(\nabla^d h^\alpha)_\delta$ by $(h^\alpha)_\delta(x) = \delta^{-3} h^\alpha(x/\delta)$ and $(\nabla^d h^\alpha)_\delta(x) = \delta^{-3}(\nabla^d h^\alpha)(x/\delta)$ for $\delta > 0$ and for positive integer d, respectively. Then they satisfy

$$\begin{aligned} &(h^\alpha)_\delta \in C^\infty(\mathbb{R}^3), \quad supp((h^\alpha)_\delta) \subset B(2\delta) - B(\delta/2), \\ &\text{and } \frac{1}{|x|^{3+\alpha}} \cdot \zeta(\frac{x}{2^j}) = \frac{1}{(2^j)^\alpha} \cdot (h^\alpha)_{2^j}(x) \text{ for any integer } j. \end{aligned} \tag{3.9}$$

The second local study proposition for (Problem II-r)

Proposition 3.1.2. *There exists $\bar{\eta} > 0$, and there exists a family of constants $C_{d,\alpha}$ with the following property:*

If u is a solution of (Problem II-r) for some $0 \le r < \infty$ verifying both

$$\int_{\mathbb{R}^3} \phi(x)u(t,x)dx = 0 \quad \text{for } t \in (-4,0) \text{ and} \tag{3.10}$$

$$\int_{-4}^{0}\int_{B(2)} \Big(|\nabla u|^2(t,x) + |\nabla^2 P|(t,x) + |\mathcal{M}(|\nabla u|)|^2(t,x)\Big)dxdt \le \bar{\eta}, \tag{3.11}$$

then $|\nabla^d u| \le C_{d,0}$ on $Q(\frac{1}{3}) = (-(\frac{1}{3})^2, 0) \times B(\frac{1}{3})$ for every integer $d \ge 0$.

Moreover if we assume further

$$\begin{aligned}\int_{-4}^{0}\int_{B(2)} \Big(&|\mathcal{M}(\mathcal{M}(|\nabla u|))|^2 + |\mathcal{M}(|\mathcal{M}(|\nabla u|)|^q)|^{2/q} \\ +&|\mathcal{M}(|\nabla u|^q)|^{2/q} + \sum_{m=d}^{d+4}\sup_{\delta>0}(|(\nabla^{m-1}h^\alpha)_\delta * \nabla^2 P|)\Big)dxdt \le \bar{\eta}\end{aligned} \tag{3.12}$$

for some integer $d \ge 1$ and for some real $\alpha \in (0,2)$ where $q = 12/(\alpha+6)$, then $|(-\Delta)^{\frac{\alpha}{2}}\nabla^d u| \le C_{d,\alpha}$ on $Q(\frac{1}{6})$ for such (d,α).

Remark 3.1.9. The functions h^α and $(\nabla^{m-1}h^\alpha)_\delta$ are defined in (3.9).

Remark 3.1.10. For the case $r = 0$, the smallness condition on $\|\mathcal{M}(|\nabla u|)\|_{L^2}$ in (3.11) is not necessary, while, for the case $r > 0$, we do not need the smallness condition on $\|\nabla u\|_{L^2}$ (refer to Remark 3.1.7, 3.1.8).

The proof will be given in Section 3.3 which will use the conclusion of the previous proposition 3.1.1. Moreover we will use an induction argument together with the integral representation of the fractional Laplacian in order to get estimates for the fractional case. The Maximal function term of (3.11) is introduced to estimate non-local part of the velocity u while the Maximal of Maximal function terms of (3.12) are to estimate non-local part of the drift velocity w, which depends on u non-locally. On the other hand, because $\nabla^2 P$ has only L^1 integrability, we can not have L^1 Maximal function of $\nabla^2 P$. Instead, we use some integrable functions, which is the last term of (3.12). This term plays the role which captures non-local information of pressure (see (2.2)). These will be stated clearly in Section 3.3 and 3.4.

3.2 Proof of the first local study proposition 3.1.1

This section is devoted to the proof of Proposition 3.1.1 which is a partial regularity theorem for (Problem II-r). Remember that we are looking for $\bar{\delta}$ which should be independent of $r \in [0, \infty)$.

In the first subsection 3.2.1, we present some lemmas related to the convective velocity w and a new pressure decomposition. Then, we prove two lemmas 3.2.4

and 3.2.5 in Subsection 3.2.2 and 3.2.3, which give us controls for large r and small r, respectively. Finally, the proof of Proposition 3.1.1 is given in the last subsection 3.2.4 where we combine those two lemmas.

3.2.1 A control on the convective velocity w and a new pressure decomposition

The following lemma says that the maximum of a convolution of any functions with ϕ_r can be controlled by just one point value of the Maximal function with some factor of $\frac{1}{r}$.

Lemma 3.2.1. *Let f be an integrable function in $\mathbb{R}^3$. Then for any integer $d \geq 0$, there exists $C = C(d)$ such that*

$$\|\nabla^d (f * \phi_r)\|_{L^\infty(B(2))} \leq \frac{C}{r^d} \cdot (1 + \frac{4}{r})^3 \cdot \inf_{x \in B(2)} \mathcal{M}f(x)$$

for any $0 < r < \infty$.

Proof. Let $z, x \in B(2)$. Then, we compute

$$\begin{aligned}
&|\nabla^d (f * \phi_r)(z)| = |(f * \nabla^d \phi_r)(z)| = |\int_{B(z,r)} f(y) \nabla^d \phi_r(z-y) dy| \\
&\leq \|\nabla^d \phi_r\|_{L^\infty} \int_{B(z,r)} |f(y)| dy = \frac{\|\nabla^d \phi\|_{L^\infty}}{r^{d+3}} \int_{B(z,r)} |f(y)| dy \\
&\leq \frac{\|\nabla^d \phi\|_{L^\infty}}{r^{d+3}} \frac{(r+4)^3}{(r+4)^3} \int_{B(x,r+4)} |f(y)| dy \leq \frac{C}{r^d} \cdot (1 + \frac{4}{r})^3 \cdot \mathcal{M}f(x).
\end{aligned}$$

We used $B(z,r) \subset B(x, r+4)$. Then we take sup in z and inf in x. Recall that $\phi(\cdot)$ is the fixed function in this chapter.

□

The following corollary is just an application of the previous lemma to solutions of (Problem II-r).

Corollary 3.2.2. *Let u be a solution of (Problem II-r) for $0 < r < \infty$. Then for any integer $d \geq 0$, there exists $C = C(d)$ such that*

$$\|w\|_{L^2(-4,0;L^\infty(B(2)))} \leq C \cdot (1 + \frac{4}{r})^3 \cdot \|\mathcal{M}(|\nabla u|)\|_{L^2(Q(2))}$$

and, for $d \geq 1$,

$$\|\nabla^d w\|_{L^2(-4,0;L^\infty(B(2)))} \leq \frac{C}{r^{d-1}} \cdot (1 + \frac{4}{r})^3 \cdot \|\mathcal{M}(|\nabla u|)\|_{L^2(Q(2))}.$$

Proof. Recall $\int_{\mathbb{R}^3} w(t,y)\phi(y)dy = 0$ and $supp(\phi) \subset B(1)$. Thus, for $z \in B(2)$, we compute

$$\begin{aligned}
|w(t,z)| =& \Big| \int_{\mathbb{R}^3} w(t,z)\phi(y)dy - \int_{\mathbb{R}^3} w(t,y)\phi(y)dy \Big| \\
\leq& \|\nabla w(t,\cdot)\|_{L^\infty(B(2))} \int_{\mathbb{R}^3} |z-y|\phi(y)dy \\
\leq& C\|(\nabla u) * \phi_r(t,\cdot)\|_{L^\infty(B(2))} \cdot \int_{\mathbb{R}^3} \phi(y)dy \\
\leq& C \cdot (1 + \frac{4}{r})^3 \cdot \inf_{x \in B(2)} \mathcal{M}(|\nabla u|)(t,x).
\end{aligned}$$

For the last inequality, we used Lemma 3.2.1 to ∇u. For $d \geq 1$, use $\nabla^d w = \nabla^{d-1}\Big[(\nabla u) * \phi_r\Big]$.

□

To use a De Giorgi type argument, we introduce the following notations, which will be used only in this section.

For real $k \geq 0$, we define

$$\begin{aligned} B_k &= \text{ the ball in } \mathbb{R}^3 \text{ centered at the origin with radius } \frac{1}{2}(1+\frac{1}{2^{3k}}), \\ T_k &= -\frac{1}{2}(3+\frac{1}{2^k}), \\ Q_k &= [T_k, 0] \times B_k \quad \text{ and} \\ s_k &= \text{ distance between } B_{k-1}^C \text{ and } B_{k-\frac{5}{6}} \\ &= 2^{-3k}\Big((\sqrt{2}-1)2\sqrt{2}\Big). \end{aligned} \tag{3.13}$$

Also we define $s_\infty = 0$. Note that $0 < s_1 < \frac{1}{4}$, and the sequence $\{s_k\}_{k=1}^{\infty}$ is strictly decreasing to zero as k goes to ∞.

For each integer $k \geq 0$, we define $\psi_k \in C^\infty(\mathbb{R}^3)$ satisfying:

$$\begin{aligned} &\psi_k = 1 \quad \text{ in } B_{k-\frac{2}{3}}, \quad \psi_k = 0 \quad \text{ in } B_{k-\frac{5}{6}}^C \\ &0 \leq \psi_k(x) \leq 1, \quad |\nabla \psi_k(x)| \leq C2^{3k} \text{ and } |\nabla^2 \psi_k(x)| \leq C2^{6k} \text{ for } x \in \mathbb{R}^3. \end{aligned} \tag{3.14}$$

This ψ_k plays a role of a cut-off function for B_k.

To prove Proposition 3.1.1, We need the following important lemma related to a new pressure decomposition. Here we decompose our pressure term into three parts: a non-local part depending on k, a local part depending on k, and a non-local part, which does not depend on k. The last part will be absorbed into the velocity component later.

Lemma 3.2.3. *There exists a constant $\Lambda_1 > 0$ with the following property: Suppose $A_{ij} \in L^1(B_0)$ $1 \leq i,j \leq 3$ and $P \in L^1(B_0)$ with $-\Delta P = \sum_{ij} \partial_i \partial_j A_{ij}$ in B_0.*

Then, there exist a function P_3 with $P_3|_{B_{\frac{2}{3}}} \in L^\infty$ such that, for any $k \geq 1$, we can decompose P by

$$P = P_{1,k} + P_{2,k} + P_3 \qquad \text{in } B_{\frac{1}{3}}, \tag{3.15}$$

and they satisfy

$$\|\nabla P_{1,k}\|_{L^\infty(B_{k-\frac{1}{3}}))} + \|P_{1,k}\|_{L^\infty(B_{k-\frac{1}{3}}))} \leq \Lambda_1 2^{12k} \sum_{ij} \|A_{ij}\|_{L^1(B_{\frac{1}{6}})}, \tag{3.16}$$

$$-\Delta P_{2,k} = \sum_{ij} \partial_i \partial_j (\psi_k A_{ij}) \qquad \text{in } \mathbb{R}^3 \quad \text{and} \tag{3.17}$$

$$\|\nabla P_3\|_{L^\infty(B_{\frac{2}{3}})} \leq \Lambda_1 (\|P\|_{L^1(B_{\frac{1}{6}})} + \sum_{ij} \|A_{ij}\|_{L^1(B_{\frac{1}{6}})}). \tag{3.18}$$

Note that Λ_1 is an independent constant.

Proof. The product rule and the hypothesis imply

$$\begin{aligned} -\Delta(\psi_1 P) &= -\psi_1 \Delta P - 2\operatorname{div}((\nabla\psi_1)P) + P\Delta\psi_1 \\ &= \psi_1 \sum_{ij} \partial_i \partial_j A_{ij} - 2\operatorname{div}((\nabla\psi_1)P) + P\Delta\psi_1 \\ &= -\Delta P_{1,k} - \Delta P_{2,k} - \Delta P_3 \end{aligned}$$

where $P_{1,k}$, $P_{2,k}$ and P_3 are defined by

$$\begin{aligned} -\Delta P_{1,k} &= \sum_{ij} \partial_i \partial_j ((\psi_1 - \psi_k) A_{ij}) \\ -\Delta P_{2,k} &= \sum_{ij} \partial_i \partial_j (\psi_k A_{ij}) \\ -\Delta P_3 &= -\sum_{ij} \partial_j [(\partial_i \psi_1)(A_{ij})] - \sum_{ij} \partial_i [(\partial_j \psi_1)(A_{ij})] \\ &\quad + \sum_{ij} (\partial_i \partial_j \psi_1)(A_{ij}) - 2\operatorname{div}((\nabla\psi_1)P) + P\Delta\psi_1. \end{aligned}$$

Here, $P_{1,k}$ and P_3 are defined by the representation formula $(-\Delta)^{-1}(f) = \frac{1}{4\pi}(\frac{1}{|x|} * f)$ while $P_{2,k}$ by the Riesz transforms.

Since $\psi_1 = 1$ on $B_{\frac{1}{3}}$, we have $\Delta P = \Delta(\psi_1 P)$ on $B_{\frac{1}{3}}$. Thus (3.15) holds.

By definition of $P_{2,k}$, (3.17) holds.

For (3.16) and (3.18), it follows the proof of the lemma 3 of [79] directly. For completeness, we present the proof. Note that $(\psi_1 - \psi_k)$ is supported in $(B_{\frac{1}{6}} - B_{k-\frac{2}{3}})$ and $\nabla\psi_1$ is supported in $(B_{\frac{1}{6}} - B_{\frac{1}{3}})$. Thus for $x \in B_{k-\frac{1}{3}}$, we get

$$\begin{aligned}
|P_{1,k}(x)| &= \left| \frac{1}{4\pi} \int_{(B_{\frac{1}{6}} - B_{k-\frac{2}{3}})} \frac{1}{|x-y|} \sum_{ij} (\partial_i \partial_j ((\psi_1 - \psi_k) A_{ij}))(y) dy \right| \\
&\leq \sup_{y \in B^C_{k-\frac{2}{3}}} (|\nabla_y^2 \frac{1}{|x-y|}|) \cdot \sum_{ij} \int_{B_{\frac{1}{6}}} |A_{ij}(x)| dy \\
&\leq C \cdot \sup_{y \in B^C_{k-\frac{2}{3}}} (\frac{1}{|x-y|^3}) \cdot \sum_{ij} \|A_{ij}\|_{L^1(B_{\frac{1}{6}})} \leq C_1 \cdot 2^{9k} \cdot \sum_{ij} \|A_{ij}\|_{L^1(B_{\frac{1}{6}})}.
\end{aligned}$$

We used integration by parts with the facts $|x-y| \geq 2^{-3k}$ and $|(\psi_1 - \psi_k)| \leq 1$.

In the same way, for $x \in B_{k-\frac{1}{3}}$, we compute

$$|\nabla P_{1,k}(x)| \leq C_2 \cdot 2^{12k} \cdot \sum_{ij} \|A_{ij}\|_{L^1(B_{\frac{1}{6}})}.$$

For $x \in B_{\frac{2}{3}}$, we get

$$\begin{aligned}|\nabla P_3(x)| =&\Big| \frac{1}{4\pi}\int_{(B_{\frac{1}{6}}-B_{\frac{1}{3}})}(\nabla_y \frac{1}{|x-y|})\Big[-\sum_{ij}\partial_j[(\partial_i\psi_1)(A_{ij})]-\sum_{ij}\partial_i[(\partial_j\psi_1)(A_{ij})]\\ &+\sum_{ij}(\partial_i\partial_j\psi_1)(A_{ij}))-2\,\mathrm{div}((\nabla\psi_1)P)+P\Delta\psi_1\Big]dy\Big|\\ \leq& C_3\Big(\sum_{ij}\|A_{ij}\|_{L^1(B_{\frac{1}{6}})}+\|P\|_{L^1(B_{\frac{1}{6}})}\Big).\end{aligned}$$

These prove (3.16) and (3.18), and we keep the constant $\Lambda_1 = max(C_1, C_2, C_3)$ for the future use.

□

Before presenting the De Giorgi arguments for large r and small r, we need more notations. In the following two main lemmas 3.2.4 and 3.2.5, P_3 will be constructed from solutions (u, P) for (Problem II-r) by using the previous lemma 3.2.3 and it will be clearly shown that ∇P_3 has the $L^1_t L^\infty_x$ bound. Thus we can define, for $t \in [-2, 0]$ and for $k \geq 0$,

$$E_k(t) = \frac{1}{2}(1-2^{-k}) + \int_{-1}^{t}\|\nabla P_3(s,\cdot)\|_{L^\infty(B_{\frac{2}{3}})}ds. \tag{3.19}$$

Note that E_k depends on t. We also define

$$\begin{aligned}v_k &= (|u|-E_k)_+,\\ d_k &= \sqrt{\frac{E_k\mathbf{1}_{\{|u|\geq E_k\}}}{|u|}|\nabla|u||^2+\frac{v_k}{|u|}|\nabla u|^2} \quad \text{and}\\ U_k &= \sup_{t\in[T_k,0]}\Big(\int_{B_k}|v_k|^2dx\Big)+\int\int_{Q_k}|d_k|^2dxdt\\ &= \|v_k\|^2_{L^\infty(T_k,0;L^2(B_k))}+\|d_k\|^2_{L^2(Q_k)}.\end{aligned}$$

It will be shown that P_3 can be absorbed into v_k, which is the key idea of the proof of Proposition 3.1.1.

3.2.2 De Giorgi argument to get a control for large r

The following lemma says that we can obtain a certain uniform non-linear estimate in the form of $W_k \leq C^k \cdot W_{k-1}^{\beta}$ when r is large. Then an elementary lemma can give us the conclusion (we will see Lemma 3.2.6 later). However, for small r, the factor $(1/r^3)$ blows up as r goes to zero. The case of small r will be treated in Lemma 3.2.5.

Lemma 3.2.4. *There exist constants $\delta_1 > 0$ and $\bar{C}_1 > 1$ such that if u is a solution of (Problem II-r) for some $0 < r < \infty$ verifying*

$$\|u\|_{L^{\infty}(-2,0;L^2(B(\frac{5}{4})))} + \|P\|_{L^1(-2,0;L^1(B(1)))} + \|\mathcal{M}(|\nabla u|)\|_{L^2(-2,0;L^2(B(2)))} \leq \delta_1,$$

then we have

$$U_k \leq \begin{cases} (\bar{C}_1)^k U_{k-1}^{\frac{7}{6}}, & \text{for any } k \geq 1 \quad \text{if } r \geq s_1 \\ \frac{1}{r^3} \cdot (\bar{C}_1)^k U_{k-1}^{\frac{7}{6}}, & \text{for any } k \geq 1 \quad \text{if } r < s_1. \end{cases}$$

Remark 3.2.1. Since $|f(x)| \leq M(f)(x)$ almost everywhere, the above assumption implies $\|\nabla u\|_{L^2(-2,0;L^2(B(2)))} \leq \delta_1$.

Remark 3.2.2. The parameter s_1 is the fixed constant defined in (3.13) such that $0 < s_1 < 1/4$, and $(\delta_1, \bar{C}_1)$ is independent of any $0 < r < \infty$. It will be clear that the exponent $7/6$ is not optimal and we can make it close to $(4/3)$ arbitrarily. However, any exponent bigger than 1 is enough for our study.

Proof. We assume $\delta_1 < 1$. First we claim that there exists a constant $\Lambda_2 \geq 1$ such that

$$\| |w| \cdot |u| \|_{L^2(-2,0;L^{3/2}(B_{\frac{1}{6}}))} \leq \Lambda_2 \cdot \delta_1 \quad \text{for any } 0 < r < \infty. \tag{3.20}$$

In order to prove the above claim (3.20), we separate it into **(I)-large** r **case** ($r \geq s_1$) and **(II)-small** r **case** ($r < s_1$):

(I)-large r **case.** From the corollary 3.2.2 if $r \geq s_1$, then we get

$$\begin{aligned} \|w\|_{L^2(-4,0;L^\infty(B(2)))} &\leq C \cdot (1 + \frac{4}{s_1})^3 \cdot \|\mathcal{M}(|\nabla u|)\|_{L^2(Q(2))} \\ &\leq C \|\mathcal{M}(|\nabla u|)\|_{L^2(Q(2))} \leq C\delta_1. \end{aligned} \tag{3.21}$$

Likewise, we obtain

$$\|\nabla w\|_{L^2(-4,0;L^\infty(B(2)))} \leq C\delta_1. \tag{3.22}$$

With Hölder's inequality and $B_{\frac{1}{6}} \subset B_0 = B(1) \subset B(\frac{5}{4}) \subset B(2)$, we get

$$\begin{aligned} \| |w| \cdot |u| \|_{L^2(-2,0;L^{3/2}(B_{\frac{1}{6}}))} &\leq C \|u\|_{L^\infty(-2,0;L^2(B(\frac{5}{4}))} \cdot \|w\|_{L^2(-4,0;L^\infty(B(2)))} \\ &\leq C \cdot {\delta_1}^2 \leq C_1 \cdot \delta_1. \end{aligned}$$

so we obtained (3.20) for $r \geq s_1$.

(II)-small r **case.** On the other hand, if $r < s_1$, then we get

$$\begin{aligned} \|w\|_{L^2(-4,0;L^\infty(B(2)))} &\leq C \cdot (1 + \frac{4}{r})^3 \cdot \|\mathcal{M}(|\nabla u|)\|_{L^2(Q(2))} \\ &\leq C \frac{1}{r^3} \|\mathcal{M}(|\nabla u|)\|_{L^2(Q(2))} \leq C \frac{1}{r^3} \delta_1 \end{aligned} \tag{3.23}$$

and

$$\|\nabla w\|_{L^2(-4,0;L^\infty(B(2)))} \leq C \frac{1}{r^3} \delta_1. \tag{3.24}$$

However, it is not enough to prove (3.20) because the factor $\frac{1}{r^3}$ blows up as r goes to zero. Instead, we use the fact that w and u are similar for small r in the following sense:

$$\|u\|_{L^4(-2,0;L^3(B_0))} \le C\Big(\|u\|_{L^\infty(-2,0;L^2(B_0))} + \|\nabla u\|_{L^2(-2,0;L^2(B_0))}\Big) \le C\delta_1$$

and

$$\|w'\|_{L^4(-2,0;L^3(B_{\frac{1}{6}}))} = \|u * \phi_r\|_{L^4(-2,0;L^3(B_{\frac{1}{6}}))} \le \|u\|_{L^4(-2,0;L^3(B_0))} \le C\delta_1$$

because $u * \phi_r$ in $B_{\frac{1}{6}}$ depends only on u in B_0 (recall that $r \le s_1$ and s_1 is the distance B_0^C and $B_{\frac{1}{6}}$ and refer to (3.1)). For w'', we compute

$$\begin{aligned}
\|w''\|_{L^\infty(-2,0;L^\infty(B(2)))} &= \|w''\|_{L^\infty_t((-2,0))} \\
&= \|\int_{\mathbb{R}^3} \phi(y)(u * \phi_r)(y)dy\|_{L^\infty_t((-2,0))} \\
&\le C\|\|u * \phi_r\|_{L^1_x(B(1))}\|_{L^\infty_t((-2,0))} \\
&\le C\|\|u\|_{L^1_x(B(\frac{5}{4}))}\|_{L^\infty_t((-2,0))} \\
&\le C\|u\|_{L^\infty(-2,0;L^2(B(\frac{5}{4})))} \\
&\le C\delta_1
\end{aligned} \tag{3.25}$$

because w'' is a constant in x, and $u * \phi_r$ in $B(1)$ depends only on u in $B(1+s_1)$ which is a subset of $B(\frac{5}{4})$. As a result, we have

$$\begin{aligned}
\||w| \cdot |u|\|_{L^2(-2,0;L^{3/2}(B_{\frac{1}{6}}))} &\le C\|u\|_{L^4(-2,0;L^3(B(1))} \cdot \|w\|_{L^4(-2,0;L^3(B(\frac{1}{6})))} \\
&\le C\delta_1 \cdot \||w'| + |w''|\|_{L^4(-2,0;L^3(B(\frac{1}{6})))} \\
&\le C \cdot {\delta_1}^2 \le C_2 \cdot \delta_1
\end{aligned} \tag{3.26}$$

so that we obtained (3.20) for $r \le s_1$.

Hence, by taking

$$\Lambda_2 = \max(C_1, C_2, 1), \tag{3.27}$$

we have (3.20), and Λ_2 is independent of $0 < r < \infty$ as long as $\delta_1 < 1$. From now on, we assume $\delta_1 < 1$ sufficiently small to satisfy $10 \cdot \Lambda_1 \cdot \Lambda_2 \cdot \delta_1 \leq 1/2$ (recall that Λ_1 comes from Lemma 3.2.3).

Thanks to Lemma 3.2.3 and (3.20), by putting $A_{ij} = w_i u_j$, we can decompose P by

$$P = P_{1,k} + P_{2,k} + P_3 \quad \text{ in } [-2,0] \times B_{\frac{1}{3}}$$

for each $k \geq 1$ with the following properties: for any $k \geq 1$,

$$\begin{aligned} \||\nabla P_{1,k}| + |P_{1,k}|\|_{L^2(-2,0;L^\infty(B_{k-\frac{1}{3}}))} &\leq \Lambda_1 2^{12k} \sum_{ij} \|w_i u_j\|_{L^2(-2,0;L^1(B_{\frac{1}{6}}))} \\ &\leq 9 \cdot \Lambda_1 \cdot \Lambda_2 \cdot \delta_1 \cdot 2^{12k} \leq 2^{12k}, \end{aligned} \tag{3.28}$$

$$-\Delta P_{2,k} = \sum_{ij} \partial_i \partial_j (\psi_k w_i u_j) \qquad \text{in } [-2,0] \times \mathbb{R}^3 \quad \text{ and } \tag{3.29}$$

$$\begin{aligned} \|\nabla P_3\|_{L^1(-2,0;L^\infty(B_{\frac{2}{3}}))} &\leq \Lambda_1 \Big(\|P\|_{L^1(-2,0;L^1(B(1))} + \sum_{ij} \|w_i u_j\|_{L^2(-2,0;L^1(B(1)))} \Big) \\ &\leq \Lambda_1(\delta_1 + 9 \cdot \Lambda_2 \cdot \delta_1) \leq 10 \cdot \Lambda_1 \cdot \Lambda_2 \cdot \delta_1 \leq \frac{1}{2}. \end{aligned} \tag{3.30}$$

Note that (3.30) enables E_k to be well-defined and it satisfies $0 \leq E_k \leq 1$ (see the definition of E_k in (3.19)).

In the following remarks 3.2.3–3.2.5, we gather some easy results, which were obtained in [79], without a proof. They can be found in the lemmas 4, 6 and the remark of the lemma 4 of [79]. Note that any constants C in the following remarks do not depend on k.

Remark 3.2.3. For any $k \geq 0$, the function u can be decomposed by $u = u\frac{v_k}{|u|} + u(1 - \frac{v_k}{|u|})$. Also we have

$$\begin{aligned} &\Big|u(1-\frac{v_k}{|u|})\Big| \leq 1, \quad \frac{v_k}{|u|}|\nabla u| \leq d_k, \quad \mathbf{1}_{|u|\geq E_k}|\nabla|u|| \leq d_k, \\ &|\nabla v_k| \leq d_k \quad \text{ and } \quad \big|\nabla \frac{uv_k}{|u|}\big| \leq 3d_k. \end{aligned} \tag{3.31}$$

Remark 3.2.4. For any $k \geq 1$ and for any $q \geq 1$, we have

$$\|\mathbf{1}_{v_k>0}\|_{L^q(Q_{k-1})} \leq C2^{\frac{10k}{3q}} U_{k-1}^{\frac{5}{3q}} \quad \text{ and } \quad \|\mathbf{1}_{v_k>0}\|_{L^\infty(T_{k-1},0;L^q(Q_{k-1}))} \leq C2^{\frac{2k}{q}} U_{k-1}^{\frac{1}{q}}.$$

Remark 3.2.5. For any $k \geq 1$, we have $\|v_{k-1}\|_{L^{\frac{10}{3}}(Q_{k-1})} \leq CU_{k-1}^{\frac{1}{2}}$.

By using the above remarks 3.2.3–3.2.5, we have, for any $1 \leq p \leq \frac{10}{3}$,

$$\begin{aligned} \|v_k\|_{L^p(Q_{k-1})} &= \|v_k \mathbf{1}_{v_k>0}\|_{L^p(Q_{k-1})} \\ &\leq \|v_k\|_{L^{\frac{10}{3}}(Q_{k-1})} \cdot \|\mathbf{1}_{v_k>0}\|_{L^{1/(\frac{1}{p}-\frac{3}{10})}(Q_{k-1})} \\ &\leq \|v_{k-1}\|_{L^{\frac{10}{3}}(Q_{k-1})} \cdot C2^{\frac{10k}{3}\cdot(\frac{1}{p}-\frac{3}{10})} U_{k-1}^{\frac{5}{3}\cdot(\frac{1}{p}-\frac{3}{10})} \\ &\leq C2^{\frac{7k}{3}} U_{k-1}^{\frac{5}{3p}}. \end{aligned} \tag{3.32}$$

Likewise, we get, for any $1 \leq p \leq 2$,

$$\|v_k\|_{L^\infty(T_{k-1},0;L^p(B_{k-1}))} \leq C2^k U_{k-1}^{\frac{1}{p}} \tag{3.33}$$

and

$$\|d_k\|_{L^p(Q_{k-1})} \leq C2^{\frac{5k}{3}} U_{k-1}^{\frac{5}{3p}-\frac{1}{3}}. \tag{3.34}$$

Second, we claim that for every $k \geq 1$, the function v_k verifies

$$\begin{aligned}\partial_t \frac{v_k^2}{2} + \operatorname{div}(w\frac{v_k^2}{2}) + d_k^2 - \Delta \frac{v_k^2}{2} \\ + \operatorname{div}(u(P_{1,k} + P_{2,k})) + (\frac{v_k}{|u|} - 1)u \cdot \nabla(P_{1,k} + P_{2,k}) \leq 0\end{aligned} \tag{3.35}$$

in $(-2, 0) \times B_{\frac{2}{3}}$.

Remark 3.2.6. The above inequality (3.35) does not contain P_3. We will see that this fact comes from the definition of $E_k(t)$ in (3.19).

Indeed, observe that $\frac{v_k^2}{2} = \frac{|u|^2}{2} + \frac{v_k^2 - |u|^2}{2}$ and note that E_k does not depend on the space variable but on the time variable. Thus we compute, for time derivatives,

$$\begin{aligned}\partial_t(\frac{v_k^2 - |u|^2}{2}) &= v_k \partial_t v_k - u\partial_t u = v_k \partial_t |u| - v_k \partial_t E_k - u\partial_t u \\ &= u(\frac{v_k}{|u|} - 1)\partial_t u - v_k \partial_t E_k = u(\frac{v_k}{|u|} - 1)\partial_t u - v_k \|\nabla P_3(t, \cdot)\|_{L^\infty(B_{\frac{2}{3}})}\end{aligned}$$

while, for any space derivatives ∂_α, we get

$$\partial_\alpha(\frac{v_k^2 - |u|^2}{2}) = u(\frac{v_k}{|u|} - 1)\partial_\alpha u.$$

Then we follow the same way as the lemma 5 of [79] did: First, we multiply (3.5) by $u(\frac{v_k}{|u|} - 1)$, and then we sum the result and (3.7). We omit the detail. As a result, we have

$$
\begin{aligned}
0 \geq \quad & \partial_t \frac{v_k^2}{2} + \mathrm{div}(w\frac{v_k^2}{2}) + d_k^2 - \Delta\frac{v_k^2}{2} + v_k\|\nabla P_3(t,\cdot)\|_{L^\infty(B_{\frac{2}{3}})} \\
& + \mathrm{div}(uP) + (\frac{v_k}{|u|} - 1)u \cdot \nabla P \\
= \quad & \partial_t \frac{v_k^2}{2} + \mathrm{div}(w\frac{v_k^2}{2}) + d_k^2 - \Delta\frac{v_k^2}{2} + \Big(v_k\|\nabla P_3(t,\cdot)\|_{L^\infty(B_{\frac{2}{3}})} + \frac{v_k}{|u|}u\cdot\nabla P_3\Big) \\
& + \mathrm{div}(u(P_{1,k} + P_{2,k})) + (\frac{v_k}{|u|} - 1)u\cdot\nabla(P_{1,k} + P_{2,k}).
\end{aligned}
$$

For the last equality, we used the fact $P = P_{1,k} + P_{2,k} + P_3$ in $B_{\frac{1}{3}}$ and

$$
\mathrm{div}(uP_3) + (\frac{v_k}{|u|} - 1)u\cdot\nabla P_3 = \frac{v_k}{|u|}u\cdot\nabla P_3. \tag{3.36}
$$

Thus we proved the claim (3.35) due to

$$
v_k\|\nabla P_3(t,\cdot)\|_{L^\infty(B_{\frac{2}{3}})} + \frac{v_k}{|u|}u\cdot\nabla P_3 \geq 0 \quad \text{ on } (-2,0)\times B_{\frac{2}{3}}.
$$

For any integer k, we introduce a cut-off function $\eta_k(x) \in C^\infty(\mathbb{R}^3)$ satisfying

$$
\begin{aligned}
&\eta_k = 1 \quad \text{in } B_k \quad , \quad \eta_k = 0 \quad \text{in } B^C_{k-\frac{1}{3}} \quad , \quad 0 \leq \eta_k \leq 1, \\
&|\nabla\eta_k| \leq C2^{3k} \quad \text{and} \quad |\nabla^2\eta_k| \leq C2^{6k}, \quad \text{for any } x \in \mathbb{R}^3.
\end{aligned}
$$

We multiply (3.35) by η_k and integrate on $[\sigma, t]\times\mathbb{R}^3$ for $T_{k-1} \leq \sigma \leq T_k \leq t \leq 0$ to get:

$$
\begin{aligned}
& \int_{\mathbb{R}^3} \eta_k(x)\frac{|v_k(t,x)|^2}{2}dx + \int_\sigma^t\int_{\mathbb{R}^3}\eta_k(x)d_k^2(s,x)dxds \\
& \leq \int_{\mathbb{R}^3}\eta_k(x)\frac{|v_k(\sigma,x)|^2}{2}dx \\
& + \int_\sigma^t\int_{\mathbb{R}^3}(\nabla\eta_k)(x)w(s,x)\frac{|v_k(s,x)|^2}{2}dxds + \int_\sigma^t\int_{\mathbb{R}^3}(\Delta\eta_k)(x)\frac{|v_k(s,x)|^2}{2}dxds \\
& - \int_\sigma^t\int_{\mathbb{R}^3}\eta_k(x)\Big(\mathrm{div}(u(P_{1,k}+P_{2,k})) + (\frac{v_k}{|u|} - 1)u\cdot\nabla(P_{1,k}+P_{2,k})\Big)(s,x)dxds.
\end{aligned}
$$

We integrate on $\sigma \in [T_{k-1}, T_k]$ and divide by $-(T_{k-1} - T_k) = 2^{-(k+1)}$ to get:

$$
\begin{aligned}
&\sup_{t\in[T_k,1]} \Big(\int_{\mathbb{R}^3} \eta_k(x) \frac{|v_k(t,x)|^2}{2} dx + \int_{T_k}^{t} \int_{\mathbb{R}^3} \eta_k(x) d_k^2(s,x) dx ds \Big) \\
&\le 2^{k+1} \cdot \int_{T_{k-1}}^{T_k} \int_{\mathbb{R}^3} \eta_k(x) \frac{|v_k(\sigma,x)|^2}{2} dx \\
&+ \int_{T_{k-1}}^{0} \Big| \int_{\mathbb{R}^3} \nabla \eta_k(x) w(s,x) \frac{|v_k(s,x)|^2}{2} dx \Big| ds + \int_{T_{k-1}}^{0} \Big| \int_{\mathbb{R}^3} \Delta \eta_k(x) \frac{|v_k(s,x)|^2}{2} dx \Big| ds \\
&+ \int_{T_{k-1}}^{0} \Big| \int_{\mathbb{R}^3} \eta_k(x) \Big(\operatorname{div}(u(P_{1,k} + P_{2,k})) + (\frac{v_k}{|u|} - 1) u \cdot \nabla (P_{1,k} + P_{2,k}) \Big)(s,x) dx \Big| ds.
\end{aligned}
$$

From $\eta_k = 1$ on B_k, we obtain

$$
\begin{aligned}
U_k &\le \sup_{t\in[T_k,1]} \Big(\int_{\mathbb{R}^3} \eta_k(x) \frac{|v_k(t,x)|^2}{2} dx \Big) + \int_{T_k}^{0} \int_{\mathbb{R}^3} \eta_k(x) d_k^2(s,x) dx ds \\
&\le 2 \cdot \sup_{t\in[T_k,1]} \Big(\int_{\mathbb{R}^3} \eta_k(x) \frac{|v_k(t,x)|^2}{2} dx + \int_{T_k}^{t} \int_{\mathbb{R}^3} \eta_k(x) d_k^2(s,x) dx ds \Big).
\end{aligned}
$$

Thus we have

$$
U_k \le (I) + (II) + (III) + (IV) \tag{3.37}
$$

where

$$
\begin{aligned}
(I) &= C 2^{6k} \int_{Q_{k-1}} |v_k(s,x)|^2 dx ds, \\
(II) &= \int_{Q_{k-1}} |\nabla \eta_k(x)| \cdot |w(s,x)| \cdot |v_k(s,x)|^2 dx ds, \\
(III) &= 2 \int_{T_{k-1}}^{0} \Big| \int_{\mathbb{R}^3} \eta_k(x) \Big(\operatorname{div}(u P_{1,k}) + (\frac{v_k}{|u|} - 1) u \cdot \nabla P_{1,k} \Big)(s,x) dx \Big| ds \quad \text{and} \\
(IV) &= 2 \int_{T_{k-1}}^{0} \Big| \int_{\mathbb{R}^3} \eta_k(x) \Big(\operatorname{div}(u P_{2,k}) + (\frac{v_k}{|u|} - 1) u \cdot \nabla P_{2,k} \Big)(s,x) dx \Big| ds.
\end{aligned} \tag{3.38}
$$

For (I), by using (3.32), we get, for any $0 < r < \infty$,

$$
(I) = C 2^{6k} \|v_k\|_{L^2(Q_{k-1})}^2 \le C 2^{10k} U_{k-1}^{\frac{5}{3}}. \tag{3.39}
$$

For (II) with $r \geq s_1$, by using (3.21) and (3.33), we compute

$$\begin{aligned}(II) &\leq C2^{3k}\|w\|_{L^2(-4,0;L^\infty(B(2)))} \cdot \||v_k|^2\|_{L^2(T_{k-1},0;L^1(B_{k-1}))} \\ &\leq C2^{3k}\delta_1\|v_k\|_{L^\infty(T_{k-1},0;L^{\frac{6}{5}}(B_{k-1}))} \cdot \|v_k\|_{L^2(T_{k-1},0;L^6(B_{k-1}))} \\ &\leq C2^{4k}\delta_1 U_{k-1}^{\frac{5}{6}} \cdot \Big(\|v_{k-1}\|_{L^\infty(T_{k-1},0;L^2(B_{k-1}))} + \|\nabla v_{k-1}\|_{L^2(T_{k-1},0;L^2(B_{k-1}))}\Big) \\ &\leq C2^{4k} \cdot \delta_1 \cdot U_{k-1}^{\frac{5}{6}} \cdot U_{k-1}^{\frac{1}{2}} \leq C2^{4k} \cdot \delta_1 \cdot U_{k-1}^{\frac{4}{3}} \leq C2^{4k} \cdot U_{k-1}^{\frac{4}{3}}.\end{aligned} \tag{3.40}$$

For $r < s_1$, we follow the above steps using (3.23) instead of (3.21), then we get

$$(II) \leq C\frac{1}{r^3}2^{4k} \cdot U_{k-1}^{\frac{4}{3}}. \tag{3.41}$$

For (III) (non-local pressure term), thanks to the smoothness of all functions, we observe that

$$\mathrm{div}(uP_{1,k}) + (\frac{v_k}{|u|} - 1)u \cdot \nabla P_{1,k} = \frac{v_k}{|u|}u \cdot \nabla P_{1,k}.$$

Thus, by using (3.28) and (3.32), we compute, for any $0 < r < \infty$,

$$\begin{aligned}(III) &\leq C \cdot \|\frac{v_k}{|u|}u \cdot \nabla P_{1,k}\|_{L^1(Q_{k-1})} \leq C\||v_k| \cdot |\nabla P_{1,k}|\|_{L^1(Q_{k-1})} \\ &\leq \|v_k\|_{L^2(T_{k-1},0;L^1(B_{k-1})))} \cdot \|\nabla P_{1,k}\|_{L^2(T_{k-1},0;L^\infty(B_{k-1}))} \\ &\leq \|\mathbf{1}_{v_k>0}\|_{L^2(T_{k-1},0;L^2(B_{k-1})))}\|v_k\|_{L^\infty(T_{k-1},0;L^2(B_{k-1})))} \cdot 2^{12k} \\ &\leq C2^{\frac{43k}{3}}U_{k-1}^{\frac{5}{6}}U_{k-1}^{\frac{1}{2}} \leq C2^{\frac{43k}{3}}U_{k-1}^{\frac{4}{3}}.\end{aligned} \tag{3.42}$$

For (IV) (local pressure term), as we did for (III), observe

$$\mathrm{div}(uP_{2,k}) + (\frac{v_k}{|u|} - 1)u \cdot \nabla P_{2,k} = \frac{v_k}{|u|}u \cdot \nabla P_{2,k}.$$

By the definition of $P_{2,k}$, we have

$$\begin{aligned}-\Delta P_{2,k} &= \sum_{ij} \partial_i\partial_j(\psi_k w_i u_j) = \sum_{ij} \partial_i((\partial_j\psi_k)w_i u_j + \psi_k(\partial_j w_i)u_j) \\ &= \sum_{ij} \partial_i\Big((\partial_j\psi_k)w_i u_j(1-\frac{v_k}{|u|}) + (\partial_j\psi_k)w_i u_j\frac{v_k}{|u|} \\ &\quad + \psi_k(\partial_j w_i)u_j(1-\frac{v_k}{|u|}) + \psi_k(\partial_j w_i)u_j\frac{v_k}{|u|}\Big)\end{aligned}$$

and

$$\begin{aligned}-\Delta(\nabla P_{2,k}) = \sum_{ij} \partial_i\nabla\Big(&(\partial_j\psi_k)w_i u_j(1-\frac{v_k}{|u|}) + (\partial_j\psi_k)w_i u_j\frac{v_k}{|u|} \\ &+ \psi_k(\partial_j w_i)u_j(1-\frac{v_k}{|u|}) + \psi_k(\partial_j w_i)u_j\frac{v_k}{|u|}\Big).\end{aligned}$$

Thus we can decompose $\nabla P_{2,k}$ by the Riesz transform into

$$\nabla P_{2,k} = G_{1,k} + G_{2,k} + G_{3,k} + G_{4,k}$$

where

$$\begin{aligned}G_{1,k} &= \sum_{ij}(\partial_i\nabla)(-\Delta)^{-1}\Big((\partial_j\psi_k)w_i u_j(1-\frac{v_k}{|u|})\Big), \\ G_{2,k} &= \sum_{ij}(\partial_i\nabla)(-\Delta)^{-1}\Big((\partial_j\psi_k)w_i u_j\frac{v_k}{|u|}\Big), \\ G_{3,k} &= \sum_{ij}(\partial_i\nabla)(-\Delta)^{-1}\Big(\psi_k(\partial_j w_i)u_j(1-\frac{v_k}{|u|})\Big) \quad \text{and} \\ G_{4,k} &= \sum_{ij}(\partial_i\nabla)(-\Delta)^{-1}\Big(\psi_k(\partial_j w_i)u_j\frac{v_k}{|u|}\Big).\end{aligned}$$

From L^p-boundedness of the Riesz transform with the fact $supp(\psi_k) \subset B_{k-(5/6)} \subset B_{k-1}$, we have

$$\|G_{2,k}\|_{L^2(T_{k-1},0;L^2(\mathbb{R}^3))} \leq C2^{3k}\|w\|_{L^2(T_{k-1},0;L^\infty(B_{k-1}))} \cdot \|v_k\|_{L^\infty(T_{k-1},0;L^2(B_{k-1}))},$$

$$\|G_{4,k}\|_{L^2(T_{k-1},0;L^2(\mathbb{R}^3))} \leq C \cdot \|\nabla w\|_{L^2(T_{k-1},0;L^\infty(B_{k-1}))} \cdot \|v_k\|_{L^\infty(T_{k-1},0;L^2(B_{k-1}))}.$$

Similarly, we have, for any $1 < p < \infty$,

$$\|G_{1,k}\|_{L^2(T_{k-1},0;L^p(\mathbb{R}^3))} \le C_p \cdot 2^{3k} \|w\|_{L^2(T_{k-1},0;L^\infty(B_{k-1}))} \quad \text{and}$$

$$\|G_{3,k}\|_{L^2(T_{k-1},0;L^p(\mathbb{R}^3))} \le C_p \cdot \|\nabla w\|_{L^2(T_{k-1},0;L^\infty(B_{k-1}))}.$$

Therefore, by using (3.22) and (3.24), we get

$$\||G_{2,k}| + |G_{4,k}|\|_{L^2(T_{k-1},0;L^2(\mathbb{R}^3))} \le \begin{cases} C \cdot 2^{3k} \cdot U_{k-1}^{\frac{1}{2}}, & \text{if } r \ge s_1 \\ C \cdot 2^{3k} \cdot \frac{1}{r^3} \cdot U_{k-1}^{\frac{1}{2}}, & \text{if } r < s_1 \end{cases}$$

and, for any $1 < p < \infty$,

$$\||G_{1,k}| + |G_{3,k}|\|_{L^2(T_{k-1},0;L^p(\mathbb{R}^3))} \le \begin{cases} C_p \cdot 2^{3k}, & \text{if } r \ge s_1 \\ C_p \cdot 2^{3k} \cdot \frac{1}{r^3}, & \text{if } r < s_1. \end{cases}$$

Thus, by using the above estimates and (3.32), for $r \ge s_1$ and $p > 5$, we compute

$$\begin{aligned}
(IV) &\le C \cdot \|\frac{v_k}{|u|} u \cdot \nabla P_{2,k}\|_{L^1(Q_{k-1})} \le C \||v_k| \cdot |\nabla P_{2,k}|\|_{L^1(Q_{k-1})} \\
&\le C \||v_k| \cdot (|G_{1,k}| + |G_{3,k}|)\|_{L^1(Q_{k-1})} + C \||v_k| \cdot (|G_{2,k}| + |G_{4,k}|)\|_{L^1(Q_{k-1})} \\
&\le \|v_k\|_{L^2(T_{k-1},0;L^{\frac{p}{p-1}}(B_{k-1})))} \cdot \||G_{1,k}| + |G_{3,k}|\|_{L^2(T_{k-1},0;L^p(B_{k-1}))} \\
&\quad + \|v_k\|_{L^2(T_{k-1},0;L^2(B_{k-1})))} \cdot \||G_{2,k}| + |G_{4,k}|\|_{L^2(T_{k-1},0;L^2(B_{k-1}))} \\
&\le C \cdot C_p \cdot 2^{\frac{16k}{3}} U_{k-1}^{\frac{4p-5}{3p}}.
\end{aligned}$$

By the same way, for $r < s_1$ and $p > 5$, we obtain

$$(IV) \le C \cdot C_p \cdot \frac{1}{r^3} 2^{\frac{16k}{3}} U_{k-1}^{\frac{4p-5}{3p}}.$$

Thus, by taking $p = 10$, we obtain

$$(IV) \le \begin{cases} C \cdot 2^{\frac{16k}{3}} U_{k-1}^{\frac{7}{6}}, & \text{if } r \ge s_1 \\ C \cdot \frac{1}{r^3} 2^{\frac{16k}{3}} U_{k-1}^{\frac{7}{6}}, & \text{if } r < s_1. \end{cases} \tag{3.43}$$

Finally, combining (3.39), (3.40), (3.41), (3.42) and (3.43) gives us

$$(I) + (II) + (III) + (IV) \le \begin{cases} C^k \cdot U_{k-1}^{\frac{7}{6}}, & \text{if } r \ge s_1 \\ \frac{1}{r^3} \cdot C^k \cdot U_{k-1}^{\frac{7}{6}}, & \text{if } r < s_1. \end{cases}$$

□

3.2.3 De Giorgi argument to get a control for small r

The next result handles the case of small r including the case $r = 0$.

Recall the definition of s_k in (3.13) first. It is the distance between B_{k-1}^C and $B_{k-\frac{5}{6}}$, and s_k is strictly decreasing to zero as $k \to \infty$. For any $0 < r < s_1$ we define k_r as the integer such that $s_{k_r+1} < r \leq s_{k_r}$. Note that $k_r \geq 1$, and it is increasing to ∞ as r goes to zero. For the case $r = 0$, we simply define $k_r = k_0 = \infty$.

Lemma 3.2.5. *There exist constants $\delta_2 > 0$ and $\bar{C}_2 > 1$ such that if u is a solution of (Problem II-r) for some $0 \leq r < s_1$ verifying*

$$\|u\|_{L^\infty(-2,0;L^2(B(\frac{5}{4}))} + \|P\|_{L^1(-2,0;L^1(B(1))} + \|\nabla u\|_{L^2(-2,0;L^2(B(\frac{5}{4}))} \leq \delta_2$$

then we have

$$U_k \leq (\bar{C}_2)^k U_{k-1}^{\frac{7}{6}} \quad \textit{for any integer } k \textit{ such that } 1 \leq k \leq k_r.$$

Remark 3.2.7. Note that δ_2 and $\bar{C}_2$ are independent of any $r \in [0, s_1)$, and the exponent 7/6 is not optimal.

Remark 3.2.8. This lemma says that even though r is very small, we can prove the above uniform estimate for the first few steps $k \leq k_r$. Moreover, the number k_r of these steps is increasing to the infinity with a certain rate as r goes to zero. In Subsection 3.2.4, we will see that this rate is enough to obtain a uniform estimate for any small r once we combine the two lemmas 3.2.4 and 3.2.5.

Proof. In this proof, we can borrow any estimates in the proof of the previous lemma 3.2.4 except those which depend on r and blow up as r goes to zero (recall that every estimate depending on r in the previous lemma 3.2.4 was obtained when and only when we used the smallness condition of L^2 norm of $\mathcal{M}(|\nabla u|)$, which we do not assume in Lemma 3.2.5).

Let $0 \leq r < s_1$. We take any integer k such that $1 \leq k \leq k_r$. As we chose δ_1 in the previous lemma 3.2.4, we assume $\delta_2 > 0$ first so small that

$$\delta_2 < 1, \quad 10\Lambda_1\Lambda_2\delta_2 \leq \frac{1}{2}.$$

We begin this proof by decomposing w' by

$$w' = u * \phi_r = \Big(u(1 - \frac{v_k}{|u|})\Big) * \phi_r + \Big(u\frac{v_k}{|u|}\Big) * \phi_r = w'^{,1} + w'^{,2}.$$

Thus the convective velocity w has a new decomposition: $w = w' - w'' = (w'^{,1} + w'^{,2}) - w'' = (w'^{,1} - w'') + w'^{,2}$. We will verify that $w'^{,1} - w''$ is bounded and $w'^{,2}$ can be controlled locally. First, for $w'^{,1}$,

$$|w'^{,1}(t,x)| = \Big|\Big(\Big(u(1 - \frac{v_k}{|u|})\Big) * \phi_r\Big)(t,x)\Big| \leq \|u(1 - \frac{v_k}{|u|})(t,\cdot)\|_{L^\infty(\mathbb{R}^3)} \leq 1 \tag{3.44}$$

for any $-4 \leq t$ and any $x \in \mathbb{R}^3$. From (3.25), we still have

$$\|w''\|_{L^\infty(-2,0;L^\infty(B(2)))} \leq C\bar{\delta} \leq C. \tag{3.45}$$

Combining above two results,

$$\||w'^{,1}| + |w''|\|_{L^\infty(-2,0;L^\infty(B(2)))} \leq C. \tag{3.46}$$

For $w^{\prime,2}$, we observe that any L^p norm of $w^{\prime,2} = \left(u\frac{v_k}{|u|}\right) * \phi_r$ in $B_{k-\frac{5}{6}}$ is less than or equal to that of v_k in B_{k-1} because $r \leq s_{k_r} \leq s_k$ and s_k is the distance between B_{k-1}^C and $B_{k-\frac{5}{6}}$ (see (3.1)). Thus we have, for any $1 \leq p \leq \infty$,

$$\begin{aligned} \|w^{\prime,2}\|_{L^p(T_{k-1},0;L^p(B_{k-\frac{5}{6}}))} &= \|\left(u\frac{v_k}{|u|}\right) * \phi_r\|_{L^p(T_{k-1},0;L^p(B_{k-\frac{5}{6}}))} \\ &= \||v_k| * \phi_r\|_{L^p(T_{k-1},0;L^p(B_{k-\frac{5}{6}}))} \leq \|v_k\|_{L^p(Q_{k-1})}. \end{aligned} \tag{3.47}$$

So, by using (3.32), we have

$$\|w^{\prime,2}\|_{L^p(T_{k-1},0;L^p(B_{k-\frac{5}{6}}))} \leq C2^{\frac{7k}{3}} U_{k-1}^{\frac{5}{3p}}, \qquad \text{for any } 1 \leq p \leq \frac{10}{3}. \tag{3.48}$$

Remark 3.2.9. The above computations show that, for any small r, the convective velocity w can be decomposed into one bounded part $(w^{\prime,1} - w^{\prime\prime})$ and the other part $w^{\prime,2}$, which has a certain contribution to the power of U_{k-1}.

Recall that the transport term estimate (3.20) is valid for $0 < r \leq s_1$ without having the smallness condition of $\|\mathcal{M}(|\nabla u|)\|_{L^2}$ (see (3.26)). Moreover, the argument in (3.26) says that (3.20) holds even for the case $r = 0$. As a result, for any $r \in [0, s_1)$, we have the same pressure estimates (3.28), (3.29) and (3.30). Thus we can follow the proof of the previous lemma 3.2.4 up to (3.37) without any single modification. It remains to control (I)–(IV).

For (I), the estimate (3.39) holds because (3.39) is independent of r.

For (II), by using (3.46) and (3.48) with the fact $supp(\eta_k) \subset B_{k-\frac{1}{3}} \subset B_{k-\frac{5}{6}}$, we have

$$
\begin{aligned}
(II) &= \| |\nabla \eta_k| \cdot |w| \cdot |v_k|^2 \|_{L^1(Q_{k-1})} \\
&\le C 2^{3k} \Big(\| (|w^{\prime,1}| + |w''|) \cdot |v_k|^2 \|_{L^1(Q_{k-1})} + \| |w^{\prime,2}| \cdot |v_k|^2 \|_{L^1(T_{k-1},0;L^1(B_{k-\frac{5}{6}}))} \Big) \\
&\le C 2^{3k} \|v_k\|^2_{L^2(Q_{k-1})} + C 2^{3k} \|w^{\prime,2}\|_{L^{\frac{10}{3}}(T_{k-1},0;L^{\frac{10}{3}}(B_{k-\frac{5}{6}}))} \cdot \| |v_k|^2 \|_{L^{\frac{10}{7}}(Q_{k-\frac{5}{6}})} \\
&\le C 2^{\frac{23k}{3}} U_{k-1}^{\frac{5}{3}} + C 2^{10k} U_{k-1}^{\frac{5}{3}} \le C 2^{10k} U_{k-1}^{\frac{5}{3}}.
\end{aligned} \tag{3.49}
$$

For (III)(non-local pressure term), we have (3.42) since (3.42) is independent of r.

For (IV)(local pressure term), by definition of $P_{2,k}$ and decomposition of w,

$$
\begin{aligned}
-\Delta P_{2,k} &= \sum_{ij} \partial_i \partial_j \Big(\psi_k w_i u_j (1 - \frac{v_k}{|u|}) + \psi_k w_i u_j \frac{v_k}{|u|} \Big) \\
&= \sum_{ij} \partial_i \partial_j \Big(\psi_k (w_i^{\prime,1} - w_i'') u_j (1 - \frac{v_k}{|u|}) + \psi_k w_i^{\prime,2} u_j (1 - \frac{v_k}{|u|}) \\
&\qquad + \psi_k (w_i^{\prime,1} - w_i'') u_j \frac{v_k}{|u|} + \psi_k w_i^{\prime,2} u_j \frac{v_k}{|u|} \Big).
\end{aligned}
$$

Thus we can decompose $P_{2,k}$ by

$$
P_{2,k} = P_{2,k,1} + P_{2,k,2} + P_{2,k,3} + P_{2,k,4}
$$

where

$$
\begin{aligned}
P_{2,k,1} &= \sum_{ij} (\partial_i \partial_j)(-\Delta)^{-1} \Big(\psi_k (w_i^{\prime,1} - w_i'') u_j (1 - \frac{v_k}{|u|}) \Big), \\
P_{2,k,2} &= \sum_{ij} (\partial_i \partial_j)(-\Delta)^{-1} \Big(\psi_k w_i^{\prime,2} u_j (1 - \frac{v_k}{|u|}) \Big), \\
P_{2,k,3} &= \sum_{ij} (\partial_i \partial_j)(-\Delta)^{-1} \Big(\psi_k (w_i^{\prime,1} - w_i'') u_j \frac{v_k}{|u|} \Big) \quad \text{and} \\
P_{2,k,4} &= \sum_{ij} (\partial_i \partial_j)(-\Delta)^{-1} \Big(\psi_k w_i^{\prime,2} u_j \frac{v_k}{|u|} \Big).
\end{aligned}
$$

By using $\left|u\left(1-\frac{v_k}{|u|}\right)\right| \leq 1$ and the fact that ψ_k is supported in $B_{k-\frac{5}{6}}$, we have

$$\|P_{2,k,1}\|_{L^p(T_{k-1},0;L^p(\mathbb{R}^3))} \leq C_p \quad \text{for } 1 < p < \infty, \text{ and} \tag{3.50}$$

$$\begin{aligned} \|P_{2,k,2}\|_{L^p(T_{k-1},0;L^p(\mathbb{R}^3))} &\leq C_p \||\psi_k| \cdot |w'^{,2}|\|_{L^p(T_{k-1},0;L^p(\mathbb{R}^3)))} \\ &\leq CC_p 2^{\frac{7k}{3}} U_{k-1}^{\frac{5}{3p}} \quad \text{for } 1 \leq p \leq \frac{10}{3} \end{aligned} \tag{3.51}$$

thanks to (3.46) and (3.48). Observe that for $i = 1, 2$,

$$\operatorname{div}\left(uG_i\right) + \left(\frac{v_k}{|u|} - 1\right) u \cdot \nabla G_i = \operatorname{div}\left(v_k \frac{u}{|u|} G_i\right) - G_i \operatorname{div}\left(\frac{uv_k}{|u|}\right). \tag{3.52}$$

For $P_{2,k,1}$, by using (3.31), (3.32), (3.34), (3.52) and (3.50) with $p = 10$, we have

$$\begin{aligned} &\int_{T_{k-1}}^0 \Big| \int_{\mathbb{R}^3} \eta_k(x)\Big(\operatorname{div}(uP_{2,k,1}) + (\frac{v_k}{|u|} - 1)u \cdot \nabla P_{2,k,1}\Big)(s,x)dx\Big| ds \\ &\leq C^{3k} \|v_k \cdot |P_{2,k,1}|\|_{L^1(Q_{k-1})} + 3\|d_k \cdot |P_{2,k,1}|\|_{L^1(Q_{k-1})} \\ &\leq C^{3k} \|v_k\|_{L^{\frac{10}{9}}(Q_{k-1})} \cdot \|P_{2,k,1}\|_{L^{10}(Q_{k-1})} + 3\|d_k\|_{L^{\frac{10}{9}}(Q_{k-1})} \cdot \|P_{2,k,1}\|_{L^{10}(Q_{k-1})} \\ &\leq C 2^{\frac{16k}{3}} U_{k-1}^{\frac{3}{2}} + C 2^{\frac{5k}{3}} U_{k-1}^{\frac{7}{6}} \leq C 2^{\frac{16k}{3}} U_{k-1}^{\frac{7}{6}}. \end{aligned} \tag{3.53}$$

Likewise, for $P_{2,k,2}$, by using (3.51) instead of (3.50), we have

$$\begin{aligned} &\int_{T_{k-1}}^0 \Big| \int_{\mathbb{R}^3} \eta_k(x)\Big(\operatorname{div}(uP_{2,k,2}) + (\frac{v_k}{|u|} - 1)u \cdot \nabla P_{2,k,2}\Big)(s,x)dx\Big| ds \\ &\leq C 2^{\frac{23k}{3}} U_{k-1}^{\frac{5}{3}} + C 2^{4k} U_{k-1}^{\frac{4}{3}} \leq C 2^{\frac{23k}{3}} U_{k-1}^{\frac{4}{3}}. \end{aligned} \tag{3.54}$$

From definitions of $P_{2,k,3}$ and $P_{2,k,4}$ with $\operatorname{div}(w) = 0$, we have

$$\begin{aligned} -\Delta\nabla(P_{2,k,3} + P_{2,k,4}) &= \sum_{ij} \partial_i \partial_j \nabla\Big(\psi_k w_i u_j \frac{v_k}{|u|}\Big) \\ &= \sum_{ij} \nabla \partial_j \Big((\partial_i \psi_k) w_i u_j \frac{v_k}{|u|} + \psi_k w_i \partial_i (u_j \frac{v_k}{|u|})\Big). \end{aligned}$$

Then we use the fact $w = (w'^{,1} - w'') + w'^{,2}$ so that we can decompose

$$\nabla(P_{2,k,3} + P_{2,k,4}) = H_{1,k} + H_{2,k} + H_{3,k} + H_{4,k}$$

where

$$H_{1,k}=\sum_{ij}(\nabla\partial_j)(-\Delta)^{-1}\Big((\partial_i\psi_k)(w_i^{\prime,1}-w_i^{\prime\prime})u_j\frac{v_k}{|u|}\Big),$$
$$H_{2,k}=\sum_{ij}(\nabla\partial_j)(-\Delta)^{-1}\Big((\partial_i\psi_k)w_i^{\prime,2}u_j\frac{v_k}{|u|}\Big),$$
$$H_{3,k}=\sum_{ij}(\nabla\partial_j)(-\Delta)^{-1}\Big(\psi_k(w_i^{\prime,1}-w_i^{\prime\prime})\partial_i(u_j\frac{v_k}{|u|})\Big)\quad\text{and}$$
$$H_{4,k}=\sum_{ij}(\nabla\partial_j)(-\Delta)^{-1}\Big(\psi_k w_i^{\prime,2}\partial_i(u_j\frac{v_k}{|u|})\Big).$$

By using $|u|\le 1+v_k$, we have

$$\begin{aligned}
&\int_{T_{k-1}}^0\Big|\int_{\mathbb{R}^3}\eta_k(x)\Big(\operatorname{div}(u(P_{2,k,3}+P_{2,k,4}))+(\frac{v_k}{|u|}-1)u\cdot\nabla(P_{2,k,3}+P_{2,k,4})\Big)dx\Big|ds\\
&\le C^{3k}\int_{Q_{k-1}}(1+v_k)\cdot|(P_{2,k,3}+P_{2,k,4})(s,x)|+|\nabla(P_{2,k,3}+P_{2,k,4})|dxds\\
&\le C^{3k}\Big(\|P_{2,k,3}\|_{L^1(Q_{k-1})}+\|v_k\cdot|P_{2,k,3}|\|_{L^1(Q_{k-1})}\\
&\qquad+\|P_{2,k,4}\|_{L^1(Q_{k-1})}+\|v_k\cdot|P_{2,k,4}|\|_{L^1(Q_{k-1})}\\
&\qquad+\|H_{1,k}\|_{L^1(Q_{k-1})}+\|H_{2,k}\|_{L^1(Q_{k-1})}+\|H_{3,k}\|_{L^1(Q_{k-1})}+\|H_{4,k}\|_{L^1(Q_{k-1})}\Big).
\end{aligned}\tag{3.55}$$

From (3.32) and (3.46) with the Riesz transform, we obtain

$$\|P_{2,k,3}\|_{L^1(Q_{k-1})}\le C\|P_{2,k,3}\|_{L^{\frac{10}{9}}(T_{k-1},0;L^{\frac{10}{9}}(\mathbb{R}^3))}\le C\|v_k\|_{L^{\frac{10}{9}}(Q_{k-1})}\le C2^{\frac{7k}{3}}U_{k-1}^{\frac{3}{2}},\tag{3.56}$$

$$\|H_{1,k}\|_{L^1(Q_{k-1})}\le C2^{\frac{16k}{3}}U_{k-1}^{\frac{3}{2}},\ \text{and}\tag{3.57}$$

$$\begin{aligned}
\|v_k\cdot|P_{2,k,3}|\|_{L^1(Q_{k-1})}&\le\|v_k\|_{L^2(Q_{k-1})}\|P_{2,k,3}\|_{L^2(Q_{k-1})}\\
&\le C2^{\frac{7k}{3}}U_{k-1}^{\frac{5}{6}}\cdot C2^{\frac{7k}{3}}U_{k-1}^{\frac{5}{6}}\le C2^{\frac{14k}{3}}U_{k-1}^{\frac{5}{3}}.
\end{aligned}\tag{3.58}$$

Using (3.32), (3.48), (3.31) and (3.34), we have

$$\|P_{2,k,4}\|_{L^1(Q_{k-1})}\le C2^{\frac{14k}{3}}U_{k-1}^{\frac{3}{2}},\tag{3.59}$$

$$\|H_{2,k}\|_{L^1(Q_{k-1})} \le C2^{\frac{23k}{3}} U_{k-1}^{\frac{3}{2}}, \tag{3.60}$$

$$\|v_k \cdot |P_{2,k,4}|\|_{L^1(Q_{k-1})} \le C2^{\frac{21k}{3}} U_{k-1}^{\frac{5}{3}}, \tag{3.61}$$

$$\|H_{3,k}\|_{L^1(Q_{k-1})} \le C2^{\frac{5k}{3}} U_{k-1}^{\frac{7}{6}}, \text{ and} \tag{3.62}$$

$$\|H_{4,k}\|_{L^1(Q_{k-1})} \le C2^{4k} U_{k-1}^{\frac{7}{6}}. \tag{3.63}$$

Combining (3.53), (3.54) and (3.55) together with (3.56), $\cdots$, (3.63), we obtain

$$(IV) \le C2^{\frac{23k}{3}} U_{k-1}^{\frac{7}{6}}. \tag{3.64}$$

Finally we combine (3.49) and (3.64) together with (3.39) and (3.42) in the previous lemma 3.2.4 in order to finish the proof of Lemma 3.2.5. □

3.2.4 Combining the two De Giorgi arguments

First we present the following lemma. Then the proof of Proposition 3.1.1 will follow. The following lemma says that certain non-linear estimates give the zero limit if the initial term is sufficiently small. This fact is one of key arguments of De Giorgi method.

Lemma 3.2.6. *Let $C > 1$ and $\beta > 1$. Then there exists a constant C_0^* such that for every sequence verifying both $0 \le W_0 < C_0^*$ and*

$$0 \le W_k \le C^k \cdot W_{k-1}^{\beta} \quad \textit{for any } k \ge 1,$$

we have $\lim_{k\to\infty} W_k = 0$.

Proof. It is quite standard, or see the lemma 1 in [79]. □

Finally we are ready to prove Proposition 3.1.1.

Proof of Proposition 3.1.1. Suppose that u is a solution of (Problem II-r) for some $0 \leq r < \infty$ verifying

$$\|u\|_{L^\infty(-2,0;L^2(B(\frac{5}{4})))} + \|P\|_{L^1(-2,0;L^1(B(1)))} + \|\nabla u\|_{L^2(-2,0;L^2(B(\frac{5}{4})))} \leq \delta$$

and $\|\mathcal{M}(|\nabla u|)\|_{L^2(-4,0;L^2(B(2)))} \leq \delta$

where δ will be chosen within the proof.

From Lemma 3.2.4 and 3.2.5, by assuming $\delta \leq \min(\delta_1, \delta_2)$, we have

$$U_k \leq \begin{cases} (\bar{C}_1)^k U_{k-1}^{\frac{7}{6}}, & \text{for any } k \geq 1 \quad \text{if } r \geq s_1. \\ \frac{1}{r^3} \cdot (\bar{C}_1)^k U_{k-1}^{\frac{7}{6}}, & \text{for any } k \geq 1 \quad \text{if } 0 < r < s_1. \\ (\bar{C}_2)^k U_{k-1}^{\frac{7}{6}} & \text{for } k = 1, 2, \cdots, k_r \quad \text{if } 0 \leq r < s_1. \end{cases} \tag{3.65}$$

Note that $k_r = \infty$ if $r = 0$. Thus we can combine the case $r = 0$ with the case $r \geq s_1$ into one estimate:

$$U_k \leq (\bar{C}_3)^k U_{k-1}^{\frac{7}{6}} \quad \text{for any } k \geq 1 \quad \text{if either } r \geq s_1 \text{ or } r = 0.$$

where we define $\bar{C}_3 = \max(\bar{C}_1, \bar{C}_2)$.

We consider now the case $0 < r < s_1$. Recall that $s_k = D \cdot 2^{-3k}$ where $D = \left((\sqrt{2} - 1)2\sqrt{2}\right) > 1$ and $s_{k_r+1} < r \leq s_{k_r}$ for any $r \in (0, s_1)$. It gives us $r \geq D \cdot 2^{-3(k_r+1)}$. Thus if $k \geq k_r$ and if $0 < r < s_1$, then the second line in (3.65) becomes

$$\begin{aligned} U_k &\leq \frac{1}{r^3} \cdot (\bar{C}_1)^k U_{k-1}^{\frac{7}{6}} \leq \frac{2^{9(k_r+1)}}{D^3} \cdot (\bar{C}_1)^k U_{k-1}^{\frac{7}{6}} \\ &\leq 2^{9(k+1)} \cdot (\bar{C}_1)^k U_{k-1}^{\frac{7}{6}} \leq (2^{18} \cdot \bar{C}_1)^k U_{k-1}^{\frac{7}{6}}. \end{aligned} \tag{3.66}$$

So we have for any $r \in (0, s_1)$,

$$U_k \leq \begin{cases} (2^{18} \cdot \bar{C}_1)^k U_{k-1}^{\frac{7}{6}}, & \text{for any } k \geq k_r. \\ (\bar{C}_2)^k U_{k-1}^{\frac{7}{6}} & \text{for } k = 1, 2, \cdots, k_r. \end{cases}$$

Define $\bar{C} = \max(2^{18} \cdot \bar{C}_1, \bar{C}_2, \bar{C}_3) = \max(2^{18} \cdot \bar{C}_1, \bar{C}_2)$. Then we can combine all three cases $r = 0$, $0 < r < s_1$, and $s_1 \leq r < \infty$ into one uniform estimate:

$$U_k \leq (\bar{C})^k U_{k-1}^{\frac{7}{6}} \quad \text{for any } k \geq 1 \quad \text{and for any } 0 \leq r < \infty.$$

Finally, by using the recursive lemma 3.2.6, we obtain C_0^* such that $U_k \to 0$ as $k \to 0$ whenever $U_0 < C_0^*$. This condition $U_0 < C_0^*$ is achievable once we assume δ so small that $\delta \leq \sqrt{\frac{C_0^*}{2}}$ because

$$U_0 \leq \Big(\|u\|_{L^\infty(-2,0;L^2(B(\frac{5}{4})))} + \|P\|_{L^1(-2,0;L^1(B(1)))} + \|\nabla u\|_{L^2(-2,0;L^2(B(\frac{5}{4})))} \Big)^2.$$

Thus we fix $\delta = \min(\sqrt{\frac{C_0^*}{2}}, \delta_1, \delta_2)$ which does not depend on any $r \in [0, \infty)$. Observe that for any $k \geq 1$,

$$\sup_{-\frac{3}{2} \leq t \leq 0} \int_{B(\frac{1}{2})} (|u(t,x)| - 1)_+^2 dx \leq U_k$$

from $E_k \leq 1$ and $(-\frac{3}{2}, 0) \times B(\frac{1}{2}) \subset Q_k$. Due to the fact $U_k \to 0$, the conclusion of Proposition 3.1.1 follows. □

3.3 Proof of the second local study proposition 3.1.2

First we present technical lemmas, whose proofs will be given in the appendix. In Subsection 3.3.2, it will be explained how to apply the previous local study proposition 3.1.1 in order to get a L^∞-bound of the velocity u . Then, Subsection 3.3.3 and 3.3.4 will give us L^∞-bounds for classical derivatives $\nabla^d u$ and for fractional derivatives $(-\Delta)^{\alpha/2} \nabla^d u$, respectively.

3.3.1 Some lemmas

The following lemma present estimates about higher derivatives of pressure which we can find a similar lemma in Vasseur [78]. However they are different in the sense that here we require $(n-1)$th order norm of v_1 to control nth derivatives of pressure (see (3.67)) while, in [78], nth order is required. This fact follows the divergence structure and it will be useful for the bootstrap argument in Subsection 3.3.3 for large r (we will see (3.73)).

Lemma 3.3.1. *Suppose that we have $v_1, v_2 \in (C^\infty(B(1)))^3$ with $\operatorname{div} v_1 = \operatorname{div} v_2 = 0$ and $P \in C^\infty(B(1))$ which satisfy*

$$-\Delta P = \operatorname{div}\operatorname{div}(v_2 \otimes v_1)$$

on $B(1) \subset \mathbb{R}^3$.

Then, for any $n \geq 2$, $0 < b < a < 1$ and $1 < p < \infty$, we have the two following estimates:

$$\|\nabla^n P\|_{L^p(B(b))} \leq C_{(a,b,n,p)}\Big(\|v_2\|_{W^{n-1,p_2}(B(a))} \cdot \|v_1\|_{W^{n-1,p_1}(B(a))} + \|P\|_{L^1(B(a))}\Big) \tag{3.67}$$

where $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, and

$$\|\nabla^n P\|_{L^\infty(B(b))} \leq C_{(a,b,n)}\Big(\|v_2\|_{W^{n,\infty}(B(a))} \cdot \|v_1\|_{W^{n,\infty}(B(a))} + \|P\|_{L^1(B(a))}\Big) \tag{3.68}$$

Note that such constants are independent of any v_1, v_2 and P. Also, ∞ is allowed for p_1 and p_2. e.g. if $p_1 = \infty$, then $p_2 = p$.

Proof. See the appendix. □

The following is a local result by using a parabolic regularization. It will be used in Subsection 3.3.3 to prove (3.71) and (3.73).

Lemma 3.3.2. *Suppose that we have smooth solution (v_1, v_2, P) on $Q(1) = (-1, 0)\times B(1)$ of*

$$\partial_t(v_1) + \mathrm{div}(v_2 \otimes v_1) + \nabla P - \Delta v_1 = 0$$

$$\mathrm{div}(v_1) = 0 \text{ and } \mathrm{div}(v_2) = 0.$$

Then, for any $n \geq 1$, $0 < b < a < 1$, $1 < p_1 < \infty$ and $1 < p_2 < \infty$, we have

$$\begin{aligned} \|\nabla^n v_1\|_{L^{p_1}(-(b)^2,0;L^{p_2}(B(b)))} \leq C_{(a,b,n,p_1,p_2)}\Big(&\|v_2 \otimes v_1\|_{L^{p_1}(-a^2,0;W^{n-1,p_2}(B(a)))} \\ &+ \|v_1\|_{L^{p_1}(-a^2,0;W^{n-1,p_2}(B(a)))} + \|P\|_{L^1(-a^2,0;L^1(B(a)))}\Big) \end{aligned} \tag{3.69}$$

where $v_2 \otimes v_1$ is the matrix whose (i,j) component is the product of j-th component $v_{2,j}$ of v_2 and i-th one $v_{1,i}$ of v_1 and $\Big(\mathrm{div}(v_2 \otimes v_1)\Big)_i = \sum_j \partial_j(v_{2,j}v_{1,i})$.

Note that such constants are independent of any v_1, v_2 and P.

Proof of Lemma 3.3.2 is omitted because it is based on the standard parabolic regularization result (e.g. Solonnikov [74]) and precise argument is essentially contained in [78] except that here we consider

$$(v_1)_t + \mathrm{div}(v_2 \otimes v_1) + \nabla P - \Delta v_1 = 0$$

while [78] covered

$$(u)_t + \mathrm{div}(u \otimes u) + \nabla P - \Delta u = 0.$$

The following lemma will be used in Subsection 3.3.3 , especially when we prove (3.73) for large r.

Lemma 3.3.3. *Suppose that we have smooth solution* (v_1, v_2, P) *on* $Q(1) = (-1, 0)\times B(1)$ *of*

$$\partial_t(v_1) + (v_2 \cdot \nabla)(v_1) + \nabla P - \Delta v_1 = 0$$

$$\mathrm{div}(v_1) = 0 \text{ and } \mathrm{div}(v_2) = 0.$$

Then, for any $n \geq 0$ *and* $0 < b < a < 1$*, we have*

$$\|\nabla^n v_1\|_{L^\infty(-(b)^2,0;L^1(B(b)))} \leq$$
$$C_{(a,b,n)}\Big[\Big(\|v_2\|_{L^2(-a^2,0;W^{n,\infty}(B(a)))} + 1\Big) \cdot \|v_1\|_{L^2(-a^2,0;W^{n,1}(B(a)))}$$
$$+ \|\nabla^{n+1} P\|_{L^1(-a^2,0;L^1(B(a)))}\Big]$$

and, for any $p \geq 1$,

$$\|\nabla^n v_1\|^{p+\frac{1}{2}}_{L^\infty(-(b)^2,0;L^{p+\frac{1}{2}}(B(b)))} \leq$$
$$C_{(a,b,n,p)}\Big[\Big(\|v_2\|_{L^2(-a^2,0;W^{n,\infty}(B(a)))} + 1\Big) \cdot \|v_1\|_{L^2(-(a)^2,0;W^{n,2p}(B(a)))}$$
$$+ \|\nabla^{n+1} P\|_{L^1(-(a)^2,0;L^{2p}(B(a)))}\Big] \cdot \|v_1\|^{p-\frac{1}{2}}_{L^\infty(-(a)^2,0;W^{n,p}(B(a)))}.$$

Note that such constants are independent of any v_1, v_2 *and* P.

Proof. See the appendix. □

The following nonlocal Sobolev-type lemma will be useful when we handle fractional derivatives by Maximal functions. We will see in Subsection 3.3.4 that the power $(1 + \frac{3}{p})$ of M on the right hand side of the following estimate is very important to obtain the estimate (3.79).

Lemma 3.3.4. *Let $M_0 > 0$ and $1 \le p < \infty$. Then there exist $C = C(M_0, p)$ with the following property:*

For any $M \ge M_0$ and for any $f \in C^1(\mathbb{R}^3)$ such that $\int_{\mathbb{R}^3} \phi(x) f(x) dx = 0$, we have

$$\|f\|_{L^p(B(M))} \le CM^{1+\frac{3}{p}} \cdot \Big(\|\mathcal{M}(|\nabla f|^p)\|^{1/p}_{L^1(B(1))} + \|\nabla f\|_{L^1(B(2))} \Big).$$

Proof. See the appendix. □

With the above lemmas, we are ready to prove Proposition 3.1.2.

Proof of Proposition 3.1.2. We divide this proof into three stages.

Stage 1 in Subsection 3.3.2: First, we will obtain a $L^\infty_t L^2_x$-local bound for u by using the mean-zero property of u and w. Then, a L^∞-local bound of u follows thanks to the first local study proposition 3.1.1.

Stage 2 in Subsection 3.3.3: We will get a L^∞-local bound for $\nabla^d u$ for $d \ge 1$ by using an induction argument with a boot-strapping. This is not obvious especially when r is large because w depends a non-local part of u while our knowledge about the L^∞-bound of u from the stage 1 is only local.

Stage 3 in Subsection 3.3.4: We will achieve a L^∞-local bound for $(-\Delta)^{\alpha/2}\nabla^d u$ for $d \ge 1$ with $0 < \alpha < 2$ from the integral representation of the fractional Laplacian. The non-locality of this fractional operator will lead us to adopting more complicated conditions (see (3.12)).

3.3.2 Stage 1: to obtain L^∞-local bound for u.

First we suppose that u satisfies all conditions of Proposition 3.1.2 without (3.12) (the condition (3.12) will be assumed only at the stage 3). Our goal is to find a sufficiently small $\bar{\eta} > 0$, which should be independent of $r \in [0, \infty)$.

Assume $\bar{\eta} \leq 1$ and define $\bar{r}_0 = \frac{1}{4}$ for this subsection. From (3.10), we get

$$\|u\|_{L^2(-4,0;L^6(B(2)))} \leq C\|\nabla u\|_{L^2(-4,0;L^2(B(2)))} \leq C \cdot \bar{\eta}.$$

From the corollary 3.2.2, if $r \geq \bar{r}_0$, then

$$\|w\|_{L^2(-4,0;L^\infty(B(2)))} \leq C \cdot \bar{\eta}.$$

On the other hand, if $0 \leq r < \bar{r}_0$, then

$$\|w'\|_{L^2(-4,0;L^6(B(\frac{7}{4})))} \leq C\|u\|_{L^2(-4,0;L^6(B(2)))} \leq C\bar{\eta}$$

because ϕ_r is supported in $B(r) \subset B(\bar{r})$, and $w = u * \phi_r$ (see (3.1)). For w'',

$$\begin{aligned} \|w''\|_{L^2(-4,0;L^\infty(B(2)))} &\leq \|\|u * \phi_r\|_{L^1(B(1))}\|_{L^2((-4,0))} \leq \|\|u\|_{L^1(B(2))}\|_{L^2((-4,0))} \\ &\leq C\|u\|_{L^2(-4,0;L^6(B(2)))} \leq C\bar{\eta}. \end{aligned}$$

Thus $\|w\|_{L^2(-4,0;L^6(B(\frac{7}{4})))} \leq C\bar{\eta}$ if $r < \bar{r}_0$ from $w = w' + w''$.

In sum, for any $0 \le r < \infty$,

$$\|w\|_{L^2(-4,0;L^6(B(\frac{7}{4})))} \le C\bar{\eta}. \tag{3.70}$$

Since the equation (3.5) depends only on ∇P, without loss of generality, we may assume $\int_{\mathbb{R}^3} \phi(x)P(t,x) = 0$ for $t \in (-4,0)$. Then with the mean zero property (3.10) of u, we have

$$\| \int_{\mathbb{R}^3} \phi(x)\nabla P(\cdot,x)dx\|_{L^1(-4,0)} \le C\bar{\eta}^{\frac{1}{2}}$$

after integrating in x.

From Sobolev's inequality, we have

$$\|\nabla P\|_{L^1(-4,0;L^{\frac{3}{2}}(B(\frac{7}{4}))} \le C\bar{\eta}^{\frac{1}{2}}$$
$$\text{and} \quad \|P\|_{L^1(-4,0;L^3(B(\frac{7}{4}))} \le C\bar{\eta}^{\frac{1}{2}}.$$

Then we follows step 1 and step 2 of the proof of the proposition 10 in [78], we can obtain

$$\|u\|_{L^\infty(-3,0;L^{\frac{3}{2}}(B(\frac{6}{4})))} \le C\bar{\eta}^{\frac{1}{3}}.$$

and then

$$\|u\|_{L^\infty(-2,0;L^2(B(\frac{5}{4})))} \le C\bar{\eta}^{\frac{1}{4}}$$

for $0 \le r < \infty$. Details are omitted.

Finally, by taking $0 < \bar{\eta} < 1$ such that $C\bar{\eta}^{\frac{1}{4}} \le \bar{\delta}$, we have all assumptions of Proposition 3.1.1. As a result, we have $|u(t,x)| \le 1$ on $[-\frac{3}{2},0] \times B(\frac{1}{2})$.

3.3.3 Stage 2: to obtain L^∞ local bound for $\nabla^d u$.

In this subsection, we cover only classical derivatives, i.e. the case $\alpha = 0$. For any integer $d \geq 1$, our goal is to find $C_{d,0}$ such that $|((-\Delta)^{\frac{0}{2}}\nabla^d)u(t,x)| = |\nabla^d u(t,x)| \leq C_{d,0}$ on $(-(\frac{1}{3})^2, 0) \times (B(\frac{1}{3}))$.

We define strictly decreasing sequences of balls and parabolic cylinders from $(-(\frac{1}{2})^2, 0) \times B(\frac{1}{2})$ to $(-(\frac{1}{3})^2, 0) \times (B(\frac{1}{3}))$ by

$$\bar{B}_n = B(\frac{1}{3} + \frac{1}{6}\cdot 2^{-n}) = B(l_n)$$
$$\bar{Q}_n = (-(\frac{1}{3} + \frac{1}{6}\cdot 2^{-n})^2, 0) \times \bar{B}_n = (-(l_n)^2, 0) \times \bar{B}_n$$

where $l_n = \frac{1}{3} + \frac{1}{6}\cdot 2^{-n}$. These notations will be used only in this subsection.

First, in order to cover the small r case, we claim the following:

There exist two positive sequences $\{\bar{r}_n\}_{n=0}^\infty$ and $\{C_{n,small}\}_{n=0}^\infty$ such that for any integer $n \geq 0$ and for any $r \in [0, \bar{r}_n)$,

$$\|\nabla^n u\|_{L^\infty(\bar{Q}_{11n})} \leq C_{n,small}. \tag{3.71}$$

Indeed, from the previous subsection 3.3.2 (the stage 1), the estimate (3.71) holds for $n = 0$ by taking $\bar{r}_0 = 1$ and $C_{0,small} = 1$. We define $\bar{r}_n =$ distance between B_{11n} and $(B_{11n-1})^C$ for $n \geq 1$. Then $\{\bar{r}_n\}_{n=0}^\infty$ is decreasing to zero as n goes to ∞. Moreover, we can control w by u as long as $0 \leq r < \bar{r}_n$: for any $n \geq 1$,

$$\begin{aligned} &\|w\|_{L^{p_1}(-(l_m)^2,0;L^{p_2}(\bar{B}_m))} \leq \Big(\|u\|_{L^{p_1}(-(l_{m-1})^2,0;L^{p_2}(\bar{B}_{m-1}))} + C\Big) \quad \text{and} \\ &\|\nabla^k w\|_{L^{p_1}(-(l_m)^2,0;L^{p_2}(\bar{B}_m))} \leq \|\nabla^k u\|_{L^{p_1}(-(l_{m-1})^2,0;L^{p_2}(\bar{B}_{m-1}))} \end{aligned} \tag{3.72}$$

for any integer m such that $m \leq 11 \cdot n$, for any $k \geq 1$ and for any $p_1 \in [1, \infty]$ and $p_2 \in [1, \infty]$ (see (3.1)).

We will use an induction with a boot-strapping. First we fix $d \geq 1$ and suppose that (3.71) is true up to $n = (d-1)$. It implies for any $r \in [0, \bar{r}_{d-1})$

$$\|u\|_{L^\infty(-l_s^2, 0; W^{d-1,\infty}(\bar{B}_s))} \leq C$$

where $s = 11(d-1)$. We want to show that (3.71) is also true for the case $n = d$.

From (3.72), $\|w\|_{L^\infty(-l_{s+1}^2, 0; W^{d-1,\infty}(\bar{B}_{s+1}))} \leq C$ and, From Lemma 3.3.2 with $v_2 = w$ and $v_1 = u$, $\|u\|_{L^{16}(-l_{s+2}^2, 0; W^{d,32}(\bar{B}_{s+2}))} \leq C$. Then, we use (3.72) and Lemma 3.3.2 in turn:

$\rightarrow w \in L^{16}(-l_{s+3}^2, 0; W^{d,32}(\bar{B}_{s+3})) \rightarrow u \in L^8(-l_{s+4}^2, 0; W^{d+1,16}(\bar{B}_{s+4}))$

$\rightarrow w \in L^8 W^{d+1,16} \rightarrow ... \rightarrow u \in L^2 W^{d+3,4}$

Then, from Sobolev's inequality,

$$\rightarrow u \in L^2 W^{d+2,\infty} \rightarrow w \in L^2(-l_{s+9}^2, 0; W^{d+2,\infty}(\bar{B}_{s+9})).$$

This estimate gives us

$$\Delta(\nabla^d u), \operatorname{div}(\nabla^d (w \otimes u)) \text{ and } \nabla(\nabla^d P) \in L^1(-l_{s+10}^2, 0; L^\infty(\bar{B}_{s+10}))$$

where we used (3.68) for the pressure term. Thus

$$\partial_t(\nabla^d u) \in L^1(-l_{s+10}^2, 0; L^\infty(\bar{B}_{s+10})).$$

Finally, we obtain that, for any $r \in [0, \bar{r}_d)$

$$\|\nabla^d u\|_{L^\infty(-l_{s+11}^2,0;L^\infty(\bar{B}_{s+11}))} \le C.$$

where C depends only on d. By this induction argument, we showed the above claim (3.71).

Now we introduce the second claim:

There exist a sequences $\{C_{n,large}\}_{n=0}^\infty$ such that for any integer $n \ge 0$ and for any $r \ge \bar{r}_n$,

$$\|\nabla^n u\|_{L^\infty(\bar{Q}_{21\cdot n})} \le C_{n,large} \tag{3.73}$$

where $\bar{r}_n$ comes from the previous claim (3.71).

Before proving the above second claim (3.73), we need the following two observations **(I),(II)** from Lemma 3.3.2 and 3.3.1:

(I). From the corollary 3.2.2 for any $n \ge 0$, if $r \ge \bar{r}_n$, then

$$\|w\|_{L^2(-4,0;W^{n,\infty}(B(2)))} \le C_n.$$

We use (3.69) in Lemma 3.3.2 with $v_1 = u$ and $v_2 = w$. Then it becomes

$$\|\nabla^n u\|_{L^{p_1}(-(l_m)^2,0;L^{p_2}(\bar{B}_m))} \le C_{(m,n,p_2)}\Big(\|u\|_{L^{\frac{2p_1}{2-p_1}}(-(l_{m-1})^2,0;W^{n-1,p_2}(\bar{B}_{m-1}))} + 1\Big) \tag{3.74}$$

for $n \geq 1$, $m \geq 1$, $1 < p_1 \leq 2$ and $1 < p_2 < \infty$ (for the case $p_1 = 2$, we may interpret $\frac{2p_1}{2-p_1} = \infty$).

(II). Moreover, (3.67) in Lemma 3.3.1 becomes

$$\|\nabla^n P\|_{L^1(-(l_m)^2,0;L^p(\bar{B}_m))} \leq C_{(m,n,p)}\Big(\|u\|_{L^2(-(l_{m-1})^2,0;W^{n-1,p}(\bar{B}_{m-1}))} + 1\Big) \qquad (3.75)$$

for $n \geq 2$ and $1 < p < \infty$.

Now we are ready to prove the second claim (3.73) by an induction with a boot-strapping. From the previous subsection 3.3.2 (the stage 1), (3.73) holds for $n = 0$ with $C_{0,large} = 1$. Fix $d \geq 1$ and suppose that we have (3.73) up to $n = (d-1)$. It implies for any $r \geq \bar{r}_{d-1}$

$$\|u\|_{L^\infty(-l_s^2,0;W^{d-1,\infty}(\bar{B}_s))} \leq C_{d-1,large}$$

where $s = 21(d-1)$. We want to show (3.73) for $n = d$.

By using (3.74) with $n = d, p_1 = 2$ and $p_2 = 11$, we have

$$\|u\|_{L^2(-l_{s+1}^2,0;W^{d,11}(\bar{B}_{s+1}))} \leq C$$

and, from (3.75) with $n = d+1, m = 0$ and $p = 11$, we get

$$\|\nabla^{d+1} P\|_{L^1(-l_{s+2}^2,0;L^{11}(\bar{B}_{s+2}))} \leq C.$$

Combining the above two results with Lemma 3.3.3 for $v_1 = u$ and $v_2 = w$, we can increase the integrability in space of u by 0.5 up to 6:

$$\|u\|_{L^\infty(-l_{s+3}^2,0;W^{d,1}(\bar{B}_{s+3}))} \leq C,$$

$$\|u\|_{L^\infty(-l_{s+4}^2,0;W^{d,1.5}(\bar{B}_{s+4}))} \leq C,$$

$$\cdots, \quad \text{and}$$

$$\|u\|_{L^\infty(-l_{s+13}^2,0;W^{d,6}(\bar{B}_{s+13}))} \leq C.$$

By using (3.74) and (3.75) again, we have

$$\|u\|_{L^2(-l_{s+14}^2,0;W^{d+1,6}(\bar{B}_{s+14}))} \leq C \quad \text{and}$$

$$\|\nabla^{d+2} P\|_{L^1(-l_{s+15}^2,0;L^6(\bar{B}_{s+15}))} \leq C.$$

Combining the above two results with Lemma 3.3.3 again , we have

$$\|u\|_{L^\infty(-l_{s+16}^2,0;W^{d+1,1}(\bar{B}_{s+16}))} \leq C,$$

$$\cdots, \quad \text{and}$$

$$\|u\|_{L^\infty(-l_{s+21}^2,0;W^{d+1,3.5}(\bar{B}_{s+21}))} \leq C.$$

Finally, from Sobolev's inequality,

$$\|\nabla^d u\|_{L^\infty(-l_{s+21}^2,0;L^\infty(\bar{B}_{s+21}))} \leq C$$

where C depends only on d as long as $r \geq \bar{r}_d$. From this induction, we proved the second claim (3.73).

Define for any $n \geq 0$, $C_{n,0} = \max(C_{n,small}, C_{n,large})$ where $C_{n,small}$ and $C_{n,large}$ come from (3.71) and (3.73) respectively. Then we have:

$$\|\nabla^n u\|_{L^\infty(Q(\frac{1}{3}))} \leq C_{n,0} \tag{3.76}$$

for any $n \geq 0$ and for any $0 \leq r < \infty$ due to $Q(\frac{1}{3}) \subset \bar{Q}_n$. It ends this stage 2.

3.3.4 Stage 3: to obtain L^∞ local bound for $(-\Delta)^{\alpha/2}\nabla^d u$.

From now on, we assume further that (u, P) satisfies (3.12) as well as all the other conditions of Proposition 3.1.2. In the following proof, we will not divide the proof into a small r part and a large r part.

Fix an integer $d \geq 1$ and a real α with $0 < \alpha < 2$. i.e. any constant which will appear may depend d and α. However, it will be clear that all constants are independent of any $r \in [0, \infty)$ and of any solution (u, P).

First, we claim:

There exists a constant $C = C(d, \alpha)$ such that

$$|(-\Delta)^{\frac{\alpha}{2}}\nabla^d u(t,x)| \leq C(d,\alpha) + \Big|\int_{|y|\geq(1/6)} \frac{\nabla^d u(t,x-y)}{|y|^{3+\alpha}}dy\Big| \tag{3.77}$$

for $|x| \leq (1/6)$ and for $-(1/3)^2 \leq t \leq 0$.

To prove (3.77), we first recall the Taylor expansion of any C^2 function f at x: $f(y) - f(x) = (\nabla f)(x)\cdot(y-x) + R(x,y)$, and we have an error estimate $|R| \leq C|x-y|^2\cdot\|\nabla^2 f\|_{L^\infty(B(x;|x-y|))}$. Note that if we integrate the first order term $(\nabla f)(x)\cdot(y-x)$ in y on any sphere with the center x, the integral vanishes thanks to the symmetry.

As a result, if we take (t,x) such that $|x| \leq (1/6)$ and $-(1/3)^2 \leq t \leq 0$, then we have

$$
\begin{aligned}
|(-\Delta)^{\frac{\alpha}{2}} \nabla^d u(t,x)| &= \Big| P.V. \int_{\mathbb{R}^3} \frac{\nabla^d u(t,x) - \nabla^d u(t,y)}{|x-y|^{3+\alpha}} dy \Big| \\
&\leq \sup_{z \in B((1/3))} (|\nabla^{d+2} u(t,z)|) \cdot \int_{|x-y|<(1/6)} \frac{1}{|x-y|^{3+\alpha-2}} dy \\
&\quad + \sup_{z \in B((1/3))} (|\nabla^d u(t,z)|) \cdot \int_{|x-y|\geq(1/6)} \frac{1}{|x-y|^{3+\alpha}} dy \\
&\quad + \Big| \int_{|x-y|\geq(1/6)} \frac{\nabla^d u(t,y)}{|x-y|^{3+\alpha}} dy \Big| \\
&\leq C(d,\alpha) + \Big| \int_{|y|\geq(1/6)} \frac{\nabla^d u(t,x-y)}{|y|^{3+\alpha}} dy \Big|
\end{aligned}
$$

where we used the result (3.76) of the previous subsection 3.3.3 (the stage 2) together with the Taylor expansion of $\nabla^d u(t,\cdot)$ at x in order to reduce certain amount of singularity at the origin $x = y$. We proved the first claim (3.77).

Second, we claim:

There exists $C = C(d,\alpha)$ such that

$$
\Big| \int_{|y|\geq(1/6)} \frac{\nabla^d u(t,x-y)}{|y|^{3+\alpha}} dy \Big| \leq C(d,\alpha) + \sum_{j=k}^{\infty} (\frac{1}{2^\alpha})^j \cdot |((h^\alpha)_{2^j} * \nabla^d u)(t,x)| \tag{3.78}
$$

for $|x| \leq (1/6)$ and for $-(1/3)^2 \leq t \leq 0$ where k is the integer such that $2^k \leq (1/6) < 2^{k+1}$.(i.e. from now on, we fix $k = -3$). Recall that h^α is defined in (3.9).

To prove the above second claim (3.78): (recall (3.8) and (3.9))

$$\begin{aligned}
&\Big|\int_{|y|\geq(1/6)}\frac{\nabla^d u(t,x-y)}{|y|^{3+\alpha}}dy\Big| = \Big|\int_{|y|\geq(1/6)}\sum_{j=k}^{\infty}\zeta(\frac{y}{2^j})\frac{\nabla^d u(t,x-y)}{|y|^{3+\alpha}}dy\Big| \\
&= \Big|\int_{|y|\geq(1/6)}\sum_{j=k}^{\infty}\frac{1}{(2^j)^\alpha}\cdot(h^\alpha)_{2^j}(y)\nabla^d u(t,x-y)dy\Big| \\
&\qquad \leq \sum_{j=k}^{k+1}\frac{1}{(2^j)^\alpha}\cdot\Big|\int_{|y|\geq(1/6)}(h^\alpha)_{2^j}(y)\nabla^d u(t,x-y)dy\Big| \\
&\qquad\quad + \sum_{j=k+2}^{\infty}\frac{1}{(2^j)^\alpha}\cdot\Big|\int_{|y|\geq(1/6)}(h^\alpha)_{2^j}(y)\nabla^d u(t,x-y)dy\Big| \\
&\qquad = (I)+(II).
\end{aligned}$$

For (I), we have

$$\begin{aligned}
(I) &\leq \sum_{j=k}^{k+1}\frac{1}{(2^j)^\alpha}\cdot\Big(\Big|\int_{\mathbb{R}^3}(h^\alpha)_{2^j}(y)\nabla^d u(t,x-y)dy\Big| \\
&\qquad\qquad + \int_{|y|\leq(1/6)}|(h^\alpha)_{2^j}(y)|\cdot|\nabla^d u(t,x-y)|dy\Big) \\
&\leq \sum_{j=k}^{k+1}\frac{1}{(2^j)^\alpha}\Big(|((h^\alpha)_{2^j}*\nabla^d u)(t,x)| + C\cdot\sup_{z\in B(1/3)}|\nabla^d u(t,z)|\Big) \\
&= \sum_{j=k}^{k+1}(\frac{1}{2^\alpha})^j\cdot|((h^\alpha)_{2^j}*\nabla^d u)(t,x)| + C(d,\alpha).
\end{aligned}$$

For (II), by using $supp(h^\alpha_{2^j})\subset(B(2^{j-1}))^C\subset(B(1/6))^C$ for any $j\geq k+2$,

$$\begin{aligned}
(II) &= \sum_{j=k+2}^{\infty}\frac{1}{(2^j)^\alpha}\cdot\Big|\int_{\mathbb{R}^3}(h^\alpha)_{2^j}(y)\nabla^d u(t,x-y)dy\Big| \\
&= \sum_{j=k+2}^{\infty}(\frac{1}{2^\alpha})^j\cdot|((h^\alpha)_{2^j}*\nabla^d u)(t,x)|.
\end{aligned}$$

We showed the second claim (3.78).

Third, we claim:

There exists $C = C(d,\alpha)$ such that

$$\|(h^\alpha)_M * \nabla^d u\|_{L^\infty(-(1/6)^2,0;L^1(B(1/6)))} \le C(d,\alpha)\cdot M^{1-d} \tag{3.79}$$

for any $M \ge 2^k$ (recall $k = -3$).

To prove the above third claim (3.79), we take the convolution with $\nabla^d[(h^\alpha)_M]$ to the equation (3.5). Then we have

$$\begin{aligned}(\nabla^d[(h^\alpha)_M] * u)_t + (\nabla^d[(h^\alpha)_M] * \Big((w\cdot\nabla)u\Big))\\ + (\nabla^d[(h^\alpha)_M] * \nabla P) - (\nabla^d[(h^\alpha)_M] * \Delta u) = 0\end{aligned}$$

so that we get

$$\begin{aligned}(\nabla^{d-1}[(h^\alpha)_M] * \nabla u)_t + (\nabla^d[(h^\alpha)_M] * \Big((w\cdot\nabla)u\Big))\\ + (\nabla^{d-1}[(h^\alpha)_M] * \nabla^2 P) - \Delta(\nabla^{d-1}[(h^\alpha)_M] * \nabla u) = 0.\end{aligned}$$

Define a cut-off $\Phi(t,x)$ by

$$0 \le \Phi(x) \le 1 \quad , \quad supp(\Phi) \subset (-4,0)\times B(2)$$

$$\Phi(t,x) = 1 \text{ for } (t,x) \in (-(1/6)^2, 0) \times B((1/6)).$$

We multiply $\Phi(t,x)\frac{(\nabla^{d-1}[(h^\alpha)_M]*\nabla u)(t,x)}{|(\nabla^{d-1}[(h^\alpha)_M]*\nabla u)(t,x)|}$, then integrate in x:

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^3}\Phi(t,x)|(\nabla^{d-1}[(h^\alpha)_M] * \nabla u)(t,x)|dx\\ &\le \int_{\mathbb{R}^3}(|\partial_t\Phi(t,x)| + |\Delta\Phi(t,x)|)|(\nabla^{d-1}[(h^\alpha)_M] * \nabla u)(t,x)|dx\\ &\quad + \int_{\mathbb{R}^3}|\Phi(t,x)||(\nabla^{d-1}[(h^\alpha)_M] * \nabla^2 P)|dx\\ &\quad + \int_{\mathbb{R}^3}|\Phi(t,x)||\nabla^d[(h^\alpha)_M] * \Big((w\cdot\nabla)u\Big)|dx.\end{aligned}$$

Then the integration on $[-4,t]$ for any $t \in [-(1/6),0]$ gives us

$$\begin{aligned}
&\|(h^\alpha)_M * \nabla^d u\|_{L^\infty(-(1/6)^2,0;L^1(B(1/6)))}\\
&= \|\nabla^{d-1}[(h^\alpha)_M] * \nabla u\|_{L^\infty(-(1/6)^2,0;L^1(B(1/6)))}\\
&\leq C\Big(\|\nabla^{d-1}[(h^\alpha)_M] * \nabla u\|_{L^1(-4,0;L^1(B(2)))}\\
&+ \|\nabla^{d-1}[(h^\alpha)_M] * \nabla^2 P\|_{L^1(-4,0;L^1(B(2)))}\\
&+ \|\nabla^{d}[(h^\alpha)_M] * \Big((w\cdot\nabla)u\Big)\|_{L^1(-4,0;L^1(B(2)))}\Big)\\
&= (I) + (II) + (III).
\end{aligned}$$

For (I), we use simple observations $\nabla^m[(f)_\delta] = \delta^{-m}\cdot(\nabla^m f)_\delta$ and $|(f)_\delta * \nabla u|(x) \leq C_f \cdot \mathcal{M}(|\nabla u|)(x)$ for any $f \in C_0^\infty(\mathbb{R}^3)$. Thus we get

$$\begin{aligned}
|(\nabla^{d-1}[(h^\alpha)_M] * \nabla u)(t,x)| &= M^{-(d-1)}\cdot|((\nabla^{d-1}h^\alpha)_M * \nabla u)(t,x)|\\
&\leq C\cdot M^{-(d-1)}\cdot \mathcal{M}(|\nabla u|)(t,x)
\end{aligned}$$

for any $0 < M < \infty$. It implies, for any $0 < M < \infty$,

$$\begin{aligned}
(I) &= \|(\nabla^{d-1}[(h^\alpha)_M] * \nabla u)\|_{L^1(-4,0;L^1(B(2)))}\\
&\leq C\cdot M^{-(d-1)}\cdot\|\mathcal{M}(|\nabla u|)\|_{L^1(-4,0;L^1(B(2)))}\\
&\leq C\cdot M^{-(d-1)}\cdot\|\mathcal{M}(|\nabla u|)\|_{L^2(-4,0;L^2(B(2)))} \leq C\cdot M^{1-d}.
\end{aligned}$$

For (II), we use our global information about the pressure in (3.12) thanks to the property of the Hardy space (2.2):

$$\begin{aligned}
(II) &= \|\nabla^{d-1}[(h^\alpha)_M] * \nabla^2 P\|_{L^1(-4,0;L^1(B(2)))}\\
&= M^{-(d-1)}\cdot\|(\nabla^{d-1}h^\alpha)_M * \nabla^2 P\|_{L^1(-4,0;L^1(B(2)))}\\
&\leq M^{-(d-1)}\cdot\|\sup_{\delta>0}(|(\nabla^{d-1}h^\alpha)_\delta * \nabla^2 P|)\|_{L^1(-4,0;L^1(B(2)))}\\
&\leq C\cdot M^{1-d}
\end{aligned} \tag{3.80}$$

for any $0 < M < \infty$.

For (III), we use the following useful facts **(1, $\cdots$, 5)**:

1. From $supp((h^\alpha)_M) \subset B(2M)$, we compute

$$
\begin{aligned}
&\|\nabla^d[(h^\alpha)_M] * \Big((w\cdot\nabla)u\Big)(t,\cdot)\|_{L^1(B(2))}\\
&\le \int_{B(2)}\int_{\mathbb{R}^3}\Big|\Big((w\cdot\nabla)u\Big)(t,y)\cdot(\nabla^d[(h^\alpha)_M])(x-y)\Big|dydx\\
&\le \int_{B(2M+2)}\Big|\Big((w\cdot\nabla)u\Big)(t,y)\Big|\cdot\Big[\int_{B(2)}|(\nabla^d[(h^\alpha)_M])(x-y)|dx\Big]dy\\
&\le C\|\nabla^d[(h^\alpha)_M]\|_{L^\infty(\mathbb{R}^3)}\cdot\|\Big((w\cdot\nabla)u\Big)(t,\cdot)\|_{L^1(B(2M+2))}\\
&\le C\cdot\frac{1}{M^{3+d}}\cdot\|\Big((w\cdot\nabla)u\Big)(t,\cdot)\|_{L^1(B(2M+2))}\\
&\le C\cdot\frac{1}{M^{3+d}}\cdot\|w(t,\cdot)\|_{L^{q'}(B(2M+2))}\cdot\|\nabla u(t,\cdot)\|_{L^q(B(2M+2))}
\end{aligned}
$$

where $q = 12/(\alpha+6)$ and $1/q + 1/q' = 1$. Note that $12/8 < q < 2$ due to $0 < \alpha < 2$.

2. For any $M \ge 2^k$, we have

$$
\begin{aligned}
\|w(t,\cdot)\|_{L^q(B(2M+2))} &\le CM^{1+\frac{3}{q}}\cdot\Big(\|\mathcal{M}(|\nabla w|^q)(t,\cdot)\|^{1/q}_{L^1(B(1))} + \|\nabla w(t,\cdot)\|_{L^1(B(2))}\Big)\\
&\le CM^{1+\frac{3}{q}}\cdot\Big(\|\mathcal{M}(|\mathcal{M}(|\nabla u|)|^q)(t,\cdot)\|^{1/q}_{L^1(B(1))} + \|\mathcal{M}(|\nabla u|)(t,\cdot)\|_{L^1(B(2))}\Big).
\end{aligned}
$$

where, for the first inequality, we used Lemma 3.3.4 and, for the second one, we used the fact $|\nabla w(t,x)| = |(\nabla u * \phi_r)(t,x)| \le C\cdot\mathcal{M}(|\nabla u|)(t,x)$. Note that C is independent of $r \in [0,\infty)$ thanks to the definitions of the convolution and the Maximal function.

So, for any $M \geq 2^k$, from (3.12), we get

$$\begin{aligned}
&\|w\|_{L^2(-4,0;L^q(B(2M+2)))} \\
&\leq CM^{1+\frac{3}{q}}\Big(\|\|\mathcal{M}(|\mathcal{M}(|\nabla u|)|^q)\|^{1/q}_{L^1_x(B(1))}\|_{L^2_t(-4,0)} \\
&\qquad + \|\|\mathcal{M}(|\nabla u|)\|_{L^1_x(B(2))}\|_{L^2_t(-4,0)}\Big) \\
&\leq CM^{1+\frac{3}{q}}\Big(\|\mathcal{M}(|\mathcal{M}(|\nabla u|)|^q)\|^{1/q}_{L^{2/q}(-4,0;L^1(B(2)))} \\
&\qquad + \|\|\mathcal{M}(|\nabla u|)\|_{L^2(-4,0;L^1(B(2)))}\Big) \\
&\leq CM^{1+\frac{3}{q}}.
\end{aligned}$$

Before stating the third fact, we need the following two observations:

From the standard Sobolev-Poincaré inequality on balls (e.g. see Saloff-Coste [64]), we have, for any $0 < M < \infty$ and for any f whose derivatives are in $L^q_{loc}(\mathbb{R}^3)$,

$$\|f - \bar{f}\|_{L^{3q/(3-q)}(B(M))} \leq C \cdot \|\nabla f\|_{L^q(B(M))} \tag{3.81}$$

where $\bar{f} = \int_B f dx/|B|$ is the mean value on B. Note that C is independent of M.

On the other hand, once we fix $M_0 > 0$, then there exists $C = C(M_0)$ with the following property:

For any p with $1 \leq p < \infty$, for any $M \geq M_0$ and for any $f \in L^p_{loc}(\mathbb{R}^3)$, we have

$$\|f\|_{L^p(B(M))} \leq CM^{\frac{3}{p}} \cdot \|\mathcal{M}(|f|^p)\|^{1/p}_{L^1(B(2))}. \tag{3.82}$$

Indeed, to prove (3.82), it is enough to show that

$$\|g\|_{L^1(B(M))} \le CM^3 \cdot \|\mathcal{M}(g)\|_{L^1(B(2))}.$$

For any $z \in B(2)$, we get

$$\begin{aligned}\int_{B(M)} |g(x)|dx &= \frac{(M+2)^3}{(M+2)^3} \cdot \int_{B(M+2)} |g(z+x)|dx \\ &\le (M+2)^3 \mathcal{M}(g)(z) \le C_{M_0} M^3 \mathcal{M}(g)(z)\end{aligned}$$

Then we take integral on $z \in B(2)$.

Now we state the third fact.

3. $\|w(t,\cdot)\|_{L^{3q/(3-q)}(B(2M+2))}$

$$\begin{aligned}&\le C \cdot \|\nabla w(t,\cdot)\|_{L^q(B(2M+2))} + \|\bar{w}(t,\cdot)\|_{L^{3q/(3-q)}(B(2M+2))} \\ &\le C \cdot M^{3/q} \cdot \|\mathcal{M}(|\nabla w|^q)(t,\cdot)\|^{1/q}_{L^1(B(2))} \\ &\quad + CM^{-3}\|w(t,\cdot)\|_{L^1(B(2M+2))} \cdot CM^{3\cdot\frac{3-q}{3q}} \\ &\le C \cdot M^{3/q} \cdot \|\mathcal{M}(|\mathcal{M}(|\nabla u|)|^q)(t,\cdot)\|^{1/q}_{L^1(B(2))} \\ &\quad + CM^{\frac{3}{q}-4}\|w(t,\cdot)\|_{L^1(B(2M+2))} \\ &\le C \cdot M^{3/q} \cdot \|\mathcal{M}(|\mathcal{M}(|\nabla u|)|^q)(t,\cdot)\|^{1/q}_{L^1(B(2))} \\ &\quad + CM^{\frac{3}{q}-4}CM^{1+\frac{3}{1}} \cdot \Big(\|\mathcal{M}(|\nabla w|^1)(t,\cdot)\|^{1/1}_{L^1(B(1))} + \|\nabla w(t,\cdot)\|_{L^1(B(2))}\Big) \\ &\le C \cdot M^{3/q} \cdot \|\mathcal{M}(|\mathcal{M}(|\nabla u|)|^q)(t,\cdot)\|^{1/q}_{L^1(B(2))} \\ &\quad + CM^{\frac{3}{q}}\Big(\|\mathcal{M}(|\mathcal{M}(|\nabla u|)|)(t,\cdot)\|_{L^1(B(1))} + \|\mathcal{M}(|\nabla u|)(t,\cdot)\|_{L^1(B(2))}\Big)\end{aligned}$$

where we used (3.81) for the first inequality while we used (3.82) and definition of mean value for the second one. For fourth and fifth ones, we used $|\nabla w(t,x)| \le$

$C|\mathcal{M}(|\nabla u|)(t,x)|$ and Lemma 3.3.4. So, by taking L^2-norm on time $[-4,0]$ with (3.12), we get

$$\|w\|_{L^2(-4,0;L^{\frac{3q}{3-q}}(B(2M+2)))} \leq CM^{\frac{3}{q}}$$

for any $M \geq 2^k$.

4. $\|w(t,\cdot)\|_{L^{q'}(B(2M+2))} \leq \|w(t,\cdot)\|^{\theta}_{L^q(B(2M+2))} \cdot \|w(t,\cdot)\|^{1-\theta}_{L^{3q/(3-q)}(B(2M+2))}$

where $q' = q/(q-1)$ and $\theta = (4q-6)/q$.

Note that due to $12/8 < q < 2$, we have $0 < \theta < 1$. So, for any $M \geq 2^k$, we get

$$\begin{aligned}
&\|w\|_{L^2(-4,0;L^{q'}(B(2M+2)))} \\
&\quad \leq \|w\|^{\theta}_{L^2(-4,0;L^q(B(2M+2)))} \cdot \|w\|^{1-\theta}_{L^2(-4,0;L^{3q/(3-q)}(B(2M+2)))} \\
&\leq C \cdot (M^{1+(3/q)})^{\theta}(M^{3/q})^{1-\theta} = C \cdot M^{4-\frac{3}{q}}.
\end{aligned}$$

5. From (3.82), for any $M \geq 2^k$, we get

$$\|\nabla u(t,\cdot)\|_{L^q(B(2M+2))} \leq C \cdot M^{3/q} \cdot \|\mathcal{M}(|\nabla u|^q)(t,\cdot)\|^{1/q}_{L^1(B(2))}.$$

So, for any $M \geq 2^k$, from (3.12), we obtain

$$\begin{aligned}
\|\nabla u\|_{L^2(-4,0;L^q(B(2M+2)))} &\leq C \cdot M^{3/q} \cdot \|\|\mathcal{M}(|\nabla u|^q)\|^{1/q}_{L^1_x(B(2))}\|_{L^2_t(-4,0)} \\
&\leq C \cdot M^{3/q} \cdot \|\mathcal{M}(|\nabla u|^q)\|^{1/q}_{L^{2/q}(-4,0;L^1(B(2)))} \leq C \cdot M^{3/q}.
\end{aligned}$$

Using the above results **(1, $\cdots$, 5)** all together, we have, for any $M \geq 2^k$,

$$\begin{aligned}
(III) &\leq C \cdot \frac{1}{M^{3+d}} \cdot \|w\|_{L^2(-4,0;L^{q'}(B(2M+2)))}\|\nabla u\|_{L^2(-4,0;L^q(B(2M+2)))} \\
&\leq C \cdot \frac{1}{M^{3+d}} \cdot M^{4-(3/q)} \cdot M^{3/q} = C \cdot M^{1-d}.
\end{aligned}$$

It proves the above third claim (3.79).

Finally we combine three claims (3.77), (3.78) and (3.79):

$$
\begin{aligned}
&\|(-\Delta)^{\frac{\alpha}{2}}\nabla^d u\|_{L^\infty(-(1/6)^2,0;L^1(B((1/6))))}\\
&\quad\le \|C\Big(1+\Big|\int_{|y|\ge(1/6)}\frac{\nabla^d u(\cdot_t,\cdot_x-y)}{|y|^{3+\alpha}}dy\Big|\Big)\|_{L^\infty(-(1/6)^2,0;L^1(B((1/6))))}\\
&\quad\le C+C\sum_{j=k}^{\infty}(\frac{1}{2^\alpha})^j\cdot\|\|((h^\alpha)_{2^j}*\nabla^d u)(\cdot_t,\cdot_x)\|\|_{L^\infty(-(1/6)^2,0;L^1(B((1/6))))}\\
&\quad\le C+C\sum_{j=k}^{\infty}(\frac{1}{2^\alpha})^j\cdot(2^j)^{1-d}\le C+C\sum_{j=k}^{\infty}(\frac{1}{2^{d+\alpha-1}})^j\le C
\end{aligned}
$$

thanks to $d+\alpha-1>0$ from $d\ge 1$ and $\alpha>0$.

By the exact same way, we can also prove that

$$\|(-\Delta)^{\frac{\alpha}{2}}\nabla^m u\|_{L^\infty(-(1/6)^2,0;L^1(B((1/6))))}\le C$$

for $m=d+1,...,d+4$. By repeated uses of Sobolev's inequality, we get

$$\|(-\Delta)^{\frac{\alpha}{2}}\nabla^d u\|_{L^\infty(-(1/6)^2,0;L^\infty(B((1/6))))}\le C(d,\alpha).$$

It finishes the proof of Proposition 3.1.2. □

3.4 Proof of the main theorem 1.1.1

We begin this section by presenting one lemma about pivot quantities. After that, Subsection 3.4.2 will cover the part (II) for $\alpha=0$ while Subsection 3.4.3 does the part (II) for $0<\alpha<2$. Finally the part (I) for $0\le\alpha<2$ will be proved in Subsection 3.4.4.

3.4.1 L^1-pivot quantities

The following lemma says that L^1 space-time norm of our pivot quantities can be controlled by L^2 space norm of the initial data. These quantities have the best scaling as $|\nabla u|^2$ and $|\nabla^2 P|$ have among all other *a priori* quantities from L^2 initial data (also see (1.4)).

Lemma 3.4.1. *There exist constant $C > 0$ and $C_{d,\alpha}$ for integer $d \geq 1$ and real $\alpha \in (0,2)$ with the following property:*

If (u, P) is a solution of (Problem I-n) for some $1 \leq n \leq \infty$, then we have

$$\int_0^\infty \int_{\mathbb{R}^3} \big(|\nabla u(t,x)|^2 + |\nabla^2 P(t,x)| + |\mathcal{M}(|\nabla u|)(t,x)|^2\big) dx dt \leq C \|u_0\|^2_{L^2(\mathbb{R}^3)}$$

and

$$\begin{aligned}\int_0^\infty \int_{\mathbb{R}^3} \Big(&|\mathcal{M}(\mathcal{M}(|\nabla u|))|^2 + |\mathcal{M}(|\nabla u|^q)|^{2/q} + |\mathcal{M}(|\mathcal{M}(|\nabla u|)|^q)|^{2/q} \\ &+ \sum_{m=d}^{d+4} \sup_{\delta>0} (|(\nabla^{m-1} h^\alpha)_\delta * \nabla^2 P|) \Big) dx dt \leq C_{d,\alpha} \|u_0\|^2_{L^2(\mathbb{R}^3)}\end{aligned}$$

for any integer $d \geq 1$ and any real $\alpha \in (0,2)$ where $q = q(\alpha)$ is defined by $12/(\alpha+6)$.

Remark 3.4.1. The definitions of h^α and $(\nabla^{m-1} h^\alpha)_\delta$ can be found in (3.9).

Remark 3.4.2. In the proof, we will see that every quantity in the left hand sides of the above two estimates can be controlled by dissipation of energy $\|\nabla u\|^2_{L^2((0,\infty)\times\mathbb{R}^3)}$ only. It explains the latter part of Remark 1.1.2.

Proof. From (3.4), we have

$$\|\nabla u\|^2_{L^2(0,\infty;L^2(\mathbb{R}^3))} \leq \|u_0 * \phi_{\frac{1}{n}}\|^2_{L^2(\mathbb{R}^3)} \leq \|u_0\|^2_{L^2(\mathbb{R}^3)}.$$

For the pressure term, we use boundedness of the Riesz transform on Hardy space $\mathcal{H}$ and the compensated compactness result in Coifman, Lions, Meyer and Semmes [22]:

$$\begin{aligned}\|\nabla^2 P\|_{L^1(0,\infty;L^1(\mathbb{R}^3))} &\le \|\nabla^2 P\|_{L^1(0,\infty;\mathcal{H}(\mathbb{R}^3))} \le C\|\Delta P\|_{L^1(0,\infty;\mathcal{H}(\mathbb{R}^3))} \\ &= \|\operatorname{div}\operatorname{div}\Big((u*\phi_{1/n})\otimes u\Big)\|_{L^1(0,\infty;\mathcal{H}(\mathbb{R}^3))} \\ &\le C\cdot\|\nabla(u*\phi_{1/n})\|_{L^2(0,\infty;L^2(\mathbb{R}^3))}\|\nabla u\|_{L^2(0,\infty;L^2(\mathbb{R}^3))} \\ &\le C\cdot\|\nabla u\|^2_{L^2(0,\infty;L^2(\mathbb{R}^3))} \le C\|u_0\|^2_{L^2(\mathbb{R}^3)}.\end{aligned} \tag{3.83}$$

For Maximal functions, we have

$$\begin{aligned}\|\mathcal{M}(\mathcal{M}(|\nabla u|))\|^2_{L^2(0,\infty;L^2(\mathbb{R}^3))} &\le C\cdot\|\mathcal{M}(|\nabla u|)\|^2_{L^2(0,\infty;L^2(\mathbb{R}^3))} \\ &\le C\cdot\|\nabla u\|^2_{L^2(0,\infty;L^2(\mathbb{R}^3))} \\ &\le C\cdot\|u_0\|^2_{L^2(\mathbb{R}^3)}.\end{aligned}$$

Let $d \ge 1$ and $0 < \alpha < 2$ and take $q = 12/(\alpha+6)$. Due to $1 < (2/q) < (4/3)$, we get

$$\begin{aligned}\|\mathcal{M}(|\nabla u|^q)\|^{2/q}_{L^{2/q}(0,\infty;L^{2/q}(\mathbb{R}^3))} &\le C\cdot\||\nabla u|^q\|^{2/q}_{L^{2/q}(0,\infty;L^{2/q}(\mathbb{R}^3))} \\ &= C\cdot\|\nabla u\|^2_{L^2(0,\infty;L^2(\mathbb{R}^3))} \\ &\le C\cdot\|u_0\|^2_{L^2(\mathbb{R}^3)}\end{aligned}$$

and

$$\begin{aligned}\|\mathcal{M}(|\mathcal{M}(|\nabla u|)|^q)\|^{2/q}_{L^{2/q}(0,\infty;L^{2/q}(\mathbb{R}^3))} &\le C\cdot\||\mathcal{M}(|\nabla u|)|^q\|^{2/q}_{L^{2/q}(0,\infty;L^{2/q}(\mathbb{R}^3))} \\ &\le C\cdot\|\mathcal{M}(|\nabla u|)\|^2_{L^2(0,\infty;L^2(\mathbb{R}^3))} \\ &\le C\cdot\|\nabla u\|^2_{L^2(0,\infty;L^2(\mathbb{R}^3))} \\ &\le C\cdot\|u_0\|^2_{L^2(\mathbb{R}^3)}\end{aligned}$$

where C depends only on α.

Thanks to the property of Hardy space (2.2) with (3.83), we have

$$\begin{aligned}\sum_{m=d}^{d+4}\|\sup_{\delta>0}(|(\nabla^{m-1}h^{\alpha})_{\delta}*\nabla^2P|)\|_{L^1(0,\infty;L^1(\mathbb{R}^3))} &\leq \sum_{m=d}^{d+4}C\|\nabla^2P\|_{L^1(0,\infty;\mathcal{H}(\mathbb{R}^3))}\\ &\leq C\|u_0\|^2_{L^2(\mathbb{R}^3)}\end{aligned}$$

where the above C depends only on d and α. □

We are ready to prove the main theorem 1.1.1.

Remark 3.4.3. In the following subsection 3.4.2 and 3.4.3, we consider solutions of (Problem I-n) for positive integers n. However it will be clear that every computation in these subsections can also be verified for the case $n=\infty$ once we assume that the smooth solution u of the Navier-Stokes exists. The case $n=\infty$ case (the original Navier-Stokes) will be covered in Subsection 3.4.4.

We focus on the $\alpha=0$ case of the part (II) first.

3.4.2 Proof of Theorem 1.1.1, part (II) for the $\alpha=0$ case

Proof of Theorem 1.1.1, part (II) for the $\alpha=0$ case.

Let any initial data u_0 of (1.2) be given. From the Leray's construction, there exists the C^∞ solution sequence $\{u_n\}_{n=1}^{\infty}$ of (Problem I-n) on $(0,\infty)$ with corresponding pressures $\{P_n\}_{n=1}^{\infty}$. From now on, our goal is to make an estimate for $\nabla^d u_n$ which is uniform in n.

For each n, $\epsilon > 0$, $t > 0$ and $x \in \mathbb{R}^3$, define a new flow $X_{n,\epsilon}(\cdot, t, x)$ by solving

$$\frac{\partial X_{n,\epsilon}}{\partial s}(s,t,x) = u_n * \phi_{\frac{1}{n}} * \phi_\epsilon(s, X_{n,\epsilon}(s,t,x)) \quad \text{for } s \in [0,t],$$
$$X_{n,\epsilon}(t,t,x) = x.$$

For convenience, we define $F_n(t,x)$ and $g_n(t)$.

$$F_n(t,x) = \big(|\nabla u_n|^2 + |\nabla^2 P_n| + |\mathcal{M}(\nabla u_n)|^2\big)(t,x), \quad g_n(t) = \int_{\mathbb{R}^3} F_n(t,x)dx.$$

We define, for n, $t > 0$ and ϵ such that $0 < 4\epsilon^2 \leq t$,

$$\Omega_{n,\epsilon,t} = \{x \in \mathbb{R}^3 \quad | \quad \frac{1}{\epsilon}\int_{t-4\epsilon^2}^{t}\int_{B(2\epsilon)} F_n(s, X_{n,\epsilon}(s,t,x)+y)dyds \leq \bar{\eta}\}$$

where $\bar{\eta}$ comes from Proposition 3.1.2. We estimate the size of $(\Omega_{n,\epsilon,t})^C$:

$$\begin{aligned}
|(\Omega_{n,\epsilon,t})^C| &= |\{x \in \mathbb{R}^3 \quad | \quad \frac{1}{\epsilon}\int_{t-4\epsilon^2}^{t}\int_{B(2\epsilon)} F_n(s, X_{n,\epsilon}(s,t,x)+y)dyds > \bar{\eta}\}| \\
&\leq \frac{1}{\bar{\eta}}\int_{\mathbb{R}^3}\Big(\frac{1}{\epsilon}\int_{t-4\epsilon^2}^{t}\int_{B(2\epsilon)} F_n(s, X_{n,\epsilon}(s,t,x)+y)dyds\Big)dx \\
&= \frac{1}{\bar{\eta}\epsilon}\Big(\int_{B(2\epsilon)}\int_{-4\epsilon^2}^{0}\int_{\mathbb{R}^3} F_n(t+s, X_{n,\epsilon}(t+s,t,x)+y)dxdsdy\Big) \\
&= \frac{1}{\bar{\eta}\epsilon}\Big(\int_{B(2\epsilon)}\int_{-4\epsilon^2}^{0}\int_{\mathbb{R}^3} F_n(t+s, z+y)dzdsdy\Big) \\
&\leq \frac{1}{\bar{\eta}\epsilon}\Big(\int_{B(2\epsilon)} 1dy\Big)\Big(\int_{-4\epsilon^2}^{0}\int_{\mathbb{R}^3} F_n(t+s, \bar{z})d\bar{z}ds\Big) \\
&\leq \frac{C\epsilon^2}{\bar{\eta}}\Big(\int_{-4\epsilon^2}^{0}\int_{\mathbb{R}^3} F_n(t+s, \bar{z})d\bar{z}ds\Big) \\
&\leq C\frac{\epsilon^4}{\bar{\eta}}\Big(\frac{1}{4\epsilon^2}\int_{-4\epsilon^2}^{0} g_n(t+s)ds\Big) \leq \epsilon^4\mathcal{M}^{(t)}(\frac{C}{\bar{\eta}}g_n \cdot \mathbf{1}_{(0,\infty)})(t) = \epsilon^4\tilde{g}_n(t)
\end{aligned} \tag{3.84}$$

where $\tilde{g}_n = \mathcal{M}^{(t)}(\frac{C}{\bar{\eta}}g_n \cdot \mathbf{1}_{(0,\infty)})$ and $\mathcal{M}^{(t)}$ is the Maximal function in $\mathbb{R}^1$. For the third inequality, we used the fact that $X_{n,\epsilon}(\cdot, t, x)$ is incompressible. From the fact that the Maximal operator is bounded from L^1 to $L^{1,\infty}$ together with Lemma 3.4.1, we get $\|\tilde{g}_n(\cdot)\|_{L^{1,\infty}(0,\infty)} \leq \frac{C}{\bar{\eta}}\|g_n(\cdot)\|_{L^1(0,\infty)} \leq \frac{C}{\bar{\eta}}\|u_0\|^2_{L^2(\mathbb{R}^3)}$.

Now we fix n, t, ϵ and x with $n \geq 1$, $0 < t < \infty$, $0 < 4\epsilon^2 \leq t$ and $x \in \Omega_{n,\epsilon,t}$. We define v, Q on $(-4,\infty)\times\mathbb{R}^3$ by using the Galilean invariance:

$$\begin{aligned} v(s,y) =& \epsilon u_n(t+\epsilon^2 s, X_{n,\epsilon}(t+\epsilon^2 s,t,x)+\epsilon y) \\ & -\epsilon(u_n * \phi_\epsilon)(t+\epsilon^2 s, X_{n,\epsilon}(t+\epsilon^2 s,t,x)) \\ Q(s,y) =& \epsilon^2 P_n(t+\epsilon^2 s, X_{n,\epsilon}(t+\epsilon^2 s,t,x)+\epsilon y) \\ & +\epsilon y\partial_s[(u_n * \phi_\epsilon)(t+\epsilon^2 s, X_{n,\epsilon}(t+\epsilon^2 s,t,x))]. \end{aligned} \tag{3.85}$$

Remark 3.4.4. This specially designed ϵ-scaling will give the mean zero property to both the velocity and the convective velocity of the resulting equation (3.86).

Let us denote $\Box$ and $\Diamond$ by $\Box = \big(t+\epsilon^2 s, X_{n,\epsilon}(t+\epsilon^2 s,t,x)+\epsilon y\big)$ and $\Diamond = \big(t+\epsilon^2 s, X_{n,\epsilon}(t+\epsilon^2 s,t,x)\big)$, respectively. Then the chain rule gives us

$$\begin{aligned} &\partial_s v(s,y) = \epsilon^3\partial_t(u_n)(\Box) + \epsilon^3\big((u_n * \phi_{\frac{1}{n}} * \phi_\epsilon)(\Diamond)\cdot\nabla\big)u_n(\Box) - \epsilon\partial_s[(u_n*\phi_\epsilon)(\Diamond)], \\ &\big(v *_y \phi_{\frac{1}{n\epsilon}}\big)(s,y) = \epsilon(u_n * \phi_{\frac{1}{n}})(\Box) - \epsilon(u_n*\phi_\epsilon)(\Diamond), \\ &\int_{\mathbb{R}^3}\big(v *_y \phi_{\frac{1}{n\epsilon}}\big)(s,z)\phi(z)dz = \epsilon(u_n * \phi_{\frac{1}{n}} * \phi_\epsilon)(\Diamond) - \epsilon(u_n*\phi_\epsilon)(\Diamond), \\ &\bigg(\Big(\big(v *_y \phi_{\frac{1}{n\epsilon}}\big)(s,y) - \int_{\mathbb{R}^3}\big(v *_y \phi_{\frac{1}{n\epsilon}}\big)(s,z)\phi(z)dz\Big)\cdot\nabla\bigg)v(s,y) = \\ &\epsilon^3\Big((u_n*\phi_{\frac{1}{n}})(\Box)\cdot\nabla\Big)u_n(\Box) - \epsilon^3\Big((u_n*\phi_{\frac{1}{n}}*\phi_\epsilon)(\Diamond)\cdot\nabla\Big)u_n(\Box), \\ &-\Delta_y v(s,y) = -\epsilon^3\Delta_y u_n(\Box) \text{ and} \\ &\nabla_y Q(s,y) = \epsilon^3\nabla P_n(\Box) + \epsilon\partial_s[(u_n*\phi_\epsilon)(\Diamond))]. \end{aligned}$$

Thus, for $(s,y)\in(-4,\infty)\times\mathbb{R}^3$, we get

$$\Big[\partial_s v + \bigg(\Big(\big(v*\phi_{\frac{1}{n\epsilon}}\big) - \int\big(v*\phi_{\frac{1}{n\epsilon}}\big)\phi\Big)\cdot\nabla\bigg)v + \nabla Q - \Delta v\Big](s,y) = 0. \tag{3.86}$$

As a result, $(v(\cdot_s,\cdot_y), Q(\cdot_s,\cdot_y))$ is a solution of (Problem II-$\frac{1}{n\epsilon}$).

From the definition of the Maximal function, we can verify that $|\mathcal{M}(\nabla v)|^2$ has the right scaling as $|\nabla v|^2$ has in the following sense:

$$\begin{aligned}\mathcal{M}(\nabla v)(s,y) &= \sup_{M>0} \frac{C}{M^3}\int_{B(M)} \epsilon^2(\nabla u_n)\big(t+\epsilon^2 s, X_{n,\epsilon}(t+\epsilon^2 s,t,x)+\epsilon(y+z)\big)dz\\ &= \sup_{\epsilon M>0} \frac{C}{\epsilon^3 M^3}\int_{B(\epsilon M)} \epsilon^2(\nabla u_n)\big(t+\epsilon^2 s, X_{n,\epsilon}(t+\epsilon^2 s,t,x)+\epsilon y+\bar{z}\big)d\bar{z}\\ &= \epsilon^2\mathcal{M}(\nabla u_n)(\square).\end{aligned} \tag{3.87}$$

As a result, we have

$$\begin{aligned}&\int_{-4}^{0}\int_{B(2)} \big(|\nabla v(s,y)|^2+|\nabla^2 Q(s,y)|+|\mathcal{M}(\nabla v)(s,y)|^2\big)dyds\\ &= \epsilon^4\int_{-4}^{0}\int_{B(2)} \Big[|\nabla u_n|^2+|\nabla^2 P_n|+|\mathcal{M}(\nabla u_n)|^2\Big](\square)dyds\\ &= \epsilon^4\int_{-4}^{0}\int_{B(2)} F_n(\square)dyds\\ &= \epsilon^{-1}\int_{t-4\epsilon^2}^{t}\int_{B(2\epsilon)} F_n\big(s, X_{n,\epsilon}(s,t,x)+y\big)dyds \le \bar{\eta}\end{aligned}$$

where the first equality comes from the definition of (v,Q), and the second one follows the change of variable $\big(t+\epsilon^2 s, \epsilon y\big) \to (s,y)$. Moreover, it satisfies

$$\int_{\mathbb{R}^3} \phi(z)v(s,z)dz = 0, \quad -4 < s < 0. \tag{3.88}$$

So (v,Q) satisfies all conditions of (3.10) and (3.11) in Proposition 3.1.2 with $r = 1/(n\epsilon) \in [0,\infty)$.

The conclusion of Proposition 3.1.2 implies that if $x \in \Omega_{n,\epsilon,t}$ for some n,t and ϵ such that $4\epsilon^2 \le t$ then $|\nabla^d v(0,0)| \le C_d$. As a result, by using $\nabla^d v(0,0) =$

$\epsilon^{d+1}\nabla^d u_n(t,x)$ for any integer $d \geq 1$, we have

$$|\{x \in \mathbb{R}^3| \quad |\nabla^d u_n(t,x)| > \frac{C_d}{\epsilon^{d+1}}\}| \leq |\Omega^C_{n,\epsilon,t}| \leq \epsilon^4 \cdot \tilde{g}_n(t).$$

Let K be any open bounded subset in $\mathbb{R}^3$. Also we define $p = 4/(d+1)$. Then for any $t > 0$, we have

$$\beta^p \cdot \Big|\{x \in K : |(\nabla^d u_n)(t,x)| > \beta\}\Big| \leq \begin{cases} \beta^p \cdot |K|, & \text{if } \beta \leq C \cdot t^{-2/p} \\ C \cdot \tilde{g}_n(t), & \text{if } \beta > C \cdot t^{-2/p}. \end{cases}$$

It implies

$$\|(\nabla^d u_n)(t,\cdot)\|^p_{L^{p,\infty}(K)} \leq C \cdot \max\big(\tilde{g}_n(t), \frac{|K|}{t^2}\big).$$

We pick any $t_0 > 0$. If we take $L^{1,\infty}(t_0,T)$-norm to the both sides of the above inequality, then we obtain

$$\begin{aligned} \|\nabla^d u_n\|^p_{L^{p,\infty}(t_0,\infty;L^{p,\infty}(K))} &\leq C\Big(\|\tilde{g}_n\|_{L^{1,\infty}(0,\infty)} + |K| \cdot \|\frac{1}{|\cdot|^2}\|_{L^{1,\infty}(t_0,\infty)}\Big) \\ &\leq C\Big(\|u_0\|^2_{L^2(\mathbb{R}^3)} + \frac{|K|}{t_0}\Big) \end{aligned} \tag{3.89}$$

where C depends only on $d \geq 1$.

We observe that the above estimate is uniform in n. It is well known that both ∇u and $\nabla^2 u$ are locally integrable functions if u is a suitable weak solution u which can be obtained by a limiting argument of u_n (e.g. see [59]). Thus, the above estimate (3.89) holds even for u with $d = 1, 2$.

Remark 3.4.5. In fact, for the case $d = 1$, the above estimate says $\nabla u \in L^{2,\infty}_{loc}$, which is useless because we know a better estimate $\nabla u \in L^2$.

Remark 3.4.6. For $d \geq 3$, the above estimate (3.89) does not give us any direct information about higher derivatives $\nabla^d u$ of a weak solution u because full regularity of weak solutions is still open, so $\nabla^d u$ may not be locally integrable for $d \geq 3$. Instead, the only thing we can say is that, for $d \geq 3$, higher derivatives $\nabla^d u_n$ of a Leray's approximation u_n have $L^{4/(d+1),\infty}_{loc}$ bounds which are uniform in $n \geq 1$.

□

From now on, we will prove the $0 < \alpha < 2$ case of the part (II).

3.4.3 Proof of Theorem 1.1.1, part (II) for the $0 < \alpha < 2$ case

Proof of Theorem 1.1.1, part (II) for the $0 < \alpha < 2$ case.

We fix $d \geq 1$ and $0 < \alpha < 2$. Then, for any positive integer n, any $t > 0$ and $x \in \mathbb{R}^3$, we denote $F_n(t,x)$ by:

$$F_n(t,x) = \Big(|\nabla u_n(t,x)|^2 + |\nabla^2 P_n(t,x)| + |\mathcal{M}(\nabla u_n)(t,x)|^2 + |\mathcal{M}(\mathcal{M}(|\nabla u_n|))|^2 + (\mathcal{M}(|\mathcal{M}(|\nabla u_n|)|^q))^{2/q} + |\mathcal{M}(|\nabla u_n|^q)|^{2/q} + \sum_{m=d}^{d+4} \sup_{\delta > 0} (|(\nabla^{m-1} h^\alpha)_\delta * \nabla^2 P|) \Big).$$

We define g_n, $\tilde{g}_n$, $X_{n,\epsilon}$ and $\Omega_{n,\epsilon,t}$ in a similar way as we did in the previous section 3.4.2. Note that we have $\|\tilde{g}_n\|_{L^{1,\infty}(0,\infty)} \leq \frac{C_{d,\alpha}}{\bar{\eta}} \cdot \|u_0\|^2_{L^2(\mathbb{R}^3)}$ from Lemma 3.4.1.

Now we pick any $x \in \Omega_{n,\epsilon,t}$ and any ϵ such that $4\epsilon^2 \leq t$, and define v and Q as the previous section 3.4.2 (see (3.85)).

In order to follow the same way we did in the previous subsection 3.4.2, we need to verify if all quantities in $F_n(t,x)$ have the right scaling after the transform (3.85). For Maximal of Maximal functions,

$$
\begin{aligned}
&\mathcal{M}(\mathcal{M}(|\nabla v|))(s,y)\\
&= \sup_{M>0} \frac{C}{M^3} \int_{B(M)} \mathcal{M}(|\nabla v|)(s, y+z)dz\\
&= \sup_{M>0} \frac{C}{M^3} \int_{B(M)} \epsilon^2 \mathcal{M}(|\nabla u_n|)\big(t+\epsilon^2 s, X_{n,\epsilon}(t+\epsilon^2 s, t, x) + \epsilon(y+z)\big)dz\\
&= \epsilon^2 \mathcal{M}(\mathcal{M}(|\nabla u_n|))(\square).
\end{aligned}
$$

where $\square = \big(t+\epsilon^2 s, X_{n,\epsilon}(t+\epsilon^2 s, t, x) + \epsilon y\big)$ and we used the idea of (3.87) for the second and third equalities. Likewise, we get $\mathcal{M}(|\nabla v|^q)(s,y) = \epsilon^{2q} \cdot \mathcal{M}(|\nabla u_n|^q)(\square)$ and $\mathcal{M}(|\mathcal{M}(|\nabla v|)|^q)(s,y) = \epsilon^{2q} \cdot \mathcal{M}(|\mathcal{M}(|\nabla u_n|)|^q)(\square)$.

Also, we have, for any function $\mathcal{G} \in C_0^\infty$,

$$
\begin{aligned}
\sup_{\delta>0}(|\mathcal{G}_\delta * \nabla^2 Q|)(s,y) &= \sup_{\delta>0} \Big| \int_{\mathbb{R}^3} \frac{1}{\delta^3} \mathcal{G}(\frac{z}{\delta}) \cdot (\nabla^2 Q)(s, y-z)dz \Big|\\
&= \sup_{\delta>0} \Big| \int_{\mathbb{R}^3} \frac{\epsilon^4}{\delta^3} \mathcal{G}(\frac{z}{\delta}) \cdot (\nabla^2 P_n)\big(t+\epsilon^2 s, X_{n,\epsilon}(t+\epsilon^2 s, t, x) + \epsilon(y-z)\big)dz \Big|\\
&= \sup_{\delta>0} \Big| \int_{\mathbb{R}^3} \frac{\epsilon^4}{\epsilon^3\delta^3} \mathcal{G}(\frac{z}{\epsilon\delta}) \cdot (\nabla^2 P_n)\big(t+\epsilon^2 s, X_{n,\epsilon}(t+\epsilon^2 s, t, x) + \epsilon y - z\big)dz \Big|\\
&= \sup_{\epsilon\delta>0} \Big| \int_{\mathbb{R}^3} \epsilon^4 \mathcal{G}_{\epsilon\delta}(z) \cdot (\nabla^2 P_n)\big(t+\epsilon^2 s, X_{n,\epsilon}(t+\epsilon^2 s, t, x) + \epsilon y - z\big)dz \Big|\\
&= \sup_{\epsilon\delta>0} \epsilon^4 \Big| \Big(\mathcal{G}_{\epsilon\delta} * (\nabla^2 P_n)\Big)\big(t+\epsilon^2 s, X_{n,\epsilon}(t+\epsilon^2 s, t, x) + \epsilon y\big) \Big|\\
&= \epsilon^4 \sup_{\delta>0} \Big| \mathcal{G}_\delta * (\nabla^2 P_n) \Big| (\square).
\end{aligned}
$$

Thus, by taking $\mathcal{G} = (\nabla^{m-1} h^\alpha)$, we have

$$
\sup_{\delta>0}(|(\nabla^{m-1}h^\alpha)_\delta * \nabla^2 Q|)(s,y) = \epsilon^4 \sup_{\delta>0} \Big| (\nabla^{m-1}h^\alpha)_\delta * (\nabla^2 P_n) \Big| (\square).
$$

As a result, we have

$$\begin{aligned}
&\int_{-4}^{0}\int_{B(2)}\Big[|\nabla v|^2+|\nabla^2 Q|+|\mathcal{M}(\nabla v)|^2+\\
&\quad+|\mathcal{M}(\mathcal{M}(|\nabla v|))|^2+|\mathcal{M}(|\mathcal{M}(|\nabla v|)|^q)|^{q/2}\\
&\quad+|\mathcal{M}(|\nabla v|^q)|^{2/q}+\sum_{m=d}^{d+4}\sup_{\delta>0}(|(\nabla^{m-1}h^\alpha)_\delta*\nabla^2 Q|)\Big](s,y)dyds\\
&=\epsilon^4\int_{-4}^{0}\int_{B(2)}F_n(\square)dyds\\
&=\epsilon^{-1}\int_{t-4\epsilon^2}^{t}\int_{B(2\epsilon)}F_n\big(s,X_{n,\epsilon}(s,t,x)+y\big)dyds\le\bar{\eta}.
\end{aligned}$$

Then (v,Q) satisfies (3.12) as well as (3.10) and (3.11) of Proposition 3.1.2 with $r=1/(n\epsilon)\in[0,\infty)$. In sum if $x\in\Omega_{n,\epsilon,t}$ and if $4\epsilon^2\le t$, then we get

$$|(-\Delta)^{\alpha/2}\nabla^d v(0,0)|\le C_{d,\alpha}.$$

Because u_n is a smooth solution of (Problem I-n), the fractional derivative $(-\Delta)^{\alpha/2}\nabla^d u_n$ is not only a distribution but also a locally integrable function. Indeed, from a boot-strapping argument, it is easy to show that $\nabla^d u_n(t)$ has a desirable behavior at infinity which is required in order to use the integral representation (2.4) pointwise. For example, $(C^2\cap W^{2,\infty})$ is enough (for a better approach, see Silvestre [71]). Also it can be easily verified that the resulting function $(-\Delta)^{\alpha/2}[\nabla^d u_n(t,\cdot)](x)$ from the integral representation (2.4) satisfies the definition in Remark 1.1.1.

As a result, it makes sense to talk about pointwise values of $(-\Delta)^{\alpha/2}\nabla^d u_n$. Note that, for any integer $d\ge 1$ and any real $0<\alpha<2$,

$$(-\Delta)^{\alpha/2}\nabla^d v(0,0)=\epsilon^{d+\alpha+1}(-\Delta)^{\alpha/2}\nabla^d u_n(t,x).$$

Thus we can deduce the following set inclusion:

$$\{x \in \mathbb{R}^3| \quad |(-\Delta)^{\frac{\alpha}{2}}\nabla^d u_n(t,x)| > \frac{C_{d,\alpha}}{\epsilon^{d+\alpha+1}}\} \qquad \subset \qquad \Omega^C_{n,\epsilon,t}. \tag{3.90}$$

It implies, for any $0 < t < \infty$ and for any $0 < 4\epsilon^2 \le t$,

$$|\{x \in \mathbb{R}^3| \quad |(-\Delta)^{\frac{\alpha}{2}}\nabla^d u_n(t,x)| > \frac{C_{d,\alpha}}{\epsilon^{d+\alpha+1}}\}| \le |\Omega^C_{n,\epsilon,t}| \le \epsilon^4 \cdot \tilde{g}_n(t).$$

We define $p = 4/(d+\alpha+1)$. As we did for case $\alpha = 0$, we obtain

$$\|(-\Delta)^{\frac{\alpha}{2}}\nabla^d u_n\|^p_{L^{p,\infty}(t_0,\infty;L^{p,\infty}(K))} \le C\Big(\|u_0\|^2_{L^2(\mathbb{R}^3)} + \frac{|K|}{t_0}\Big)$$

for any integer $n, d \ge 1$, for any real $\alpha \in (0,2)$, for any bounded open subset K of $\mathbb{R}^3$, and for any $t_0 \in (0,\infty)$ where C depends only on d and α.

If we restrict further $(d+\alpha) < 3$, then $p = \frac{4}{d+\alpha+1} > 1$. This implies $(-\Delta)^{\alpha/2}\nabla^d u_n \in L^q_{loc}((t_0,\infty)\times K)$ for every q between 1 and p, and the norm is uniformly bounded in n. Thus, from the weak-compactness of L^q for $q > 1$, we conclude that if u is a suitable weak solution obtained by a limiting argument of u_n, then any higher derivatives $(-\Delta)^{\alpha/2}\nabla^d u$, which is defined by following Remark 1.1.1, lie in L^1_{loc} as long as $(d+\alpha) < 3$ with the same estimate

$$\|(-\Delta)^{\frac{\alpha}{2}}\nabla^d u\|^p_{L^{p,\infty}(t_0,\infty;L^{p,\infty}(K))} \le C_{d,\alpha}\Big(\|u_0\|^2_{L^2(\mathbb{R}^3)} + \frac{|K|}{t_0}\Big). \tag{3.91}$$

□

3.4.4 Proof of Theorem 1.1.1, part (I)

Proof of Theorem 1.1.1, part (I). Suppose that (u, P) is a smooth solution of the Navier-Stokes equations (1.1) on $(0, T)$ with (1.2). Then it satisfies all conditions of (Problem I-n) for $n = \infty$ on $(0, T)$. As we mentioned at Remark 3.4.3, we follow every steps in Subsection 3.4.2 and 3.4.3 except each last arguments which require $d < 3$ or $(d + \alpha) < 3$. Indeed, under the scaling (3.85), the resulting function (v, Q) is a solution for (Problem II-r) for $r = 0$.

Recall that u is smooth by assumption. As a result, we do NOT have any restriction like $d < 3$ or $(d+\alpha) < 3$ at this time because we do not need any limiting argument any more which requires the weak-compactness of L^q for $q > 1$. Thus, we obtain (3.91) for any integer $d \geq 1$, for any real $\alpha \in [0, 2)$ and for any $t_0 \in (0, T)$. It finishes the proof of the part (I) of the main theorem 1.1.1.

□

Chapter 4

Persistence of Hölder continuity for non-local integro-differential equations

4.1 Introduction and the main result with its applications

Let $N \geq 1$ be any dimension. Any constants in this chapter may depend on N. We consider the following evolution equation

$$\partial_t w(t,x) = \int_{\mathbb{R}^N} [w(t,y) - w(t,x)] K(t,x,y) dy \tag{4.1}$$

where K satisfies the *weak*-($*$)-kernel condition, which will be given in Definition 4.1.3. The above integral is understood in the sense of principal value. More precisely, we denote the integral operator T_t^K and $(T_t^K)_\epsilon$ for $\epsilon > 0$ corresponding to any given kernel K at time t by

$$\begin{aligned} (T_t^K)_\epsilon(f)(x) &= \int_{|x-y|\geq\epsilon} [f(x) - f(y)] K(t,x,y) dy \quad \text{and} \\ (T_t^K)(f)(x) &= \lim_{\epsilon\to 0} (T_t^K)_\epsilon(f)(x). \end{aligned}$$

Then, (4.1) is equivalent to $(\partial_t w)(t,x) + T_t^K(w(t,\cdot))(x) = 0$. Related to the above singular integral, there have been many interests recently, not only from the field of analysis, but also from the field of probability (e.g. Caffarelli and Silvestre [12], Schwab [67], Bass and Levin [2], Jacob, Potrykus, and Wu [46], and Chen, Kim, and Kumagai [18]).

Definition 4.1.1. We denote Sobolev spaces by $W^{k,p}$ and $H^k := W^{k,2}$ for integers $k \geq 0$ and for $p \in [1, \infty]$ in the usual way. In addition, the symbol $\mathcal{S}$ is used to represent the Schwartz space in $\mathbb{R}^N$.

For any dimension d, we say that a function f lies in $C^k(\mathbb{R}^d)$ for an integer $k \geq 0$ if f is k-times differentiable in $\mathbb{R}^d$ and all derivatives up to k order are continuous, while f lies in $C^k(\overline{\mathbb{R}^d})$ if $f \in C^k(\mathbb{R}^d)$ and if $\nabla^l f$ are bounded for all integer l such that $0 \leq l \leq k$. In other words, $C^k(\overline{\mathbb{R}^d}) = C^k(\mathbb{R}^d) \cap W^{k,\infty}(\mathbb{R}^d)$. Also, $C^\infty(\overline{\mathbb{R}^d}) := \cap_{k=1}^{\infty} C^k(\overline{\mathbb{R}^d})$.

We say that a bounded f lies in $C^\beta(\mathbb{R}^d)$ for $0 < \beta < 1$ if $\sup_{x,y} \frac{|f(x)-f(y)|}{|x-y|^\beta}$ is finite and we define the semi-norm $[f]_{C^\beta} := \sup_{x,y} \frac{|f(x)-f(y)|}{|x-y|^\beta}$ and the norm $\|f\|_{C^\beta} := \|f\|_{L^\infty} + [f]_{C^\beta}$. We also define the space $C^{k,\beta}(\mathbb{R}^d)$ by the norm $\|f\|_{C^{k,\beta}} := \|f\|_{W^{k,\infty}} + [\nabla^k f]_{C^\beta}$.

$B_r(x), B_r$ represent the balls in $\mathbb{R}^N$ with a radius r centered at x and 0, respectively.

Our main goal is to obtain *a priori* estimate for solutions of (4.1). The aim is to prove the result [10] of Caffarelli, Chan, and Vasseur with different techniques (a similar result for the stationary case was obtained by Kassmann in [47]). In particular, we prove persistence of Hölder continuity in $L^\infty(0, \infty; C^\beta(\mathbb{R}^N))$, which is a new result, by observing the evolution of a dual class of test functions. This class, which appears in the work of Kiselev and Nazarov [48], plays a similar role of the dual space of C^β. They obtained, in [48], Hölder regularity for solutions of the critical surface quasi-geostrophic (SQG) equation. It is interesting to compare this method with that of Caffarelli and Vasseur [13]. In [13], the estimate $C^\beta([t,\infty)\times\mathbb{R}^N))$ for any $t>0$ was proved by using a De Giorgi iteration technique (for other different proofs, we refer to Kiselev, Nazarov, and Volberg [49] and Constantin and Vicol [25]).

We define the $(*)$-kernel condition on the kernel K.

Definition 4.1.2. Let $0<\alpha<2$, $0<\zeta\leq\infty$, $0\leq\omega<\alpha$ and $1\leq\Lambda<\infty$ (for the case $\alpha\geq 1$, three more parameters ν, s_0 and τ, which are satisfying $(\alpha-1)<\nu<1$, $0<s_0\leq\infty$, $0\leq\tau<\infty$ and $\nu+\omega<\min\{N,\alpha\}$, are needed). Then we say that a measurable function $K:[0,T]\times\mathbb{R}^N\times\mathbb{R}^N\to[0,\infty)$ satisfies the $(*)$-kernel condition on $[0,T]$ for the parameter set $\{\alpha,\zeta,\omega,\Lambda\}$ (if $\alpha\geq 1$, for the parameter set $\{\alpha,\zeta,\omega,\Lambda,\nu,s_0,\tau\}$) if the following conditions hold for all finite $t\in[0,T]$:

$$\odot \text{ Symmetry in } x,y\text{: } K(t,x,y)=K(t,y,x), \text{ for } x,y\in\mathbb{R}^N. \tag{4.2}$$

$$\odot \text{ Bounds: } \Lambda^{-1}\cdot\mathbf{1}_{|x-y|\leq\zeta}\leq K(t,x,y)|x-y|^{N+\alpha}\leq\Lambda\cdot(1+|x-y|^\omega) \quad \text{for } x,y\in\mathbb{R}^N. \tag{4.3}$$

For convenience, we define the associated function k by $k(t,x,z)=K(t,x,x+z)|z|^{N+\alpha}$. Then the above two conditions are equivalent to $k(t,x,y-x)=k(t,y,x-y)$ and $\Lambda^{-1}\cdot\mathbf{1}_{|z|\leq\zeta}\leq k(t,x,z)\leq\Lambda\cdot(1+|z|^\omega)$, respectively.

Only when $\alpha \geq 1$, we assume one more condition:

$$\odot \text{ Local Hölder continuity: } \sup_{|z-\tilde{z}|\leq s_0, |z|\leq s_0} \frac{|k(t,x,z) - k(t,x,\tilde{z})|}{|z-\tilde{z}|^\nu} \leq \tau, \text{ for } x \in \mathbb{R}^N. \tag{4.4}$$

We present the definition of the *weak*-$(*)$-kernel condition, which is slightly weaker than the above $(*)$-kernel condition in Definition 4.1.2.

Definition 4.1.3. Under the same setting of the parameters in Definition 4.1.2, we say that K satisfies the *weak*-$(*)$-kernel condition on $[0, T]$ if K satisfies (4.2) and (4.3) for the case $\alpha < 1$. If $\alpha \geq 1$, we ask K to hold the following condition (4.5) as well as (4.2) and (4.3).

$$\odot \text{ Cancellation: } \left| \int_{S^{N-1}} k(t,x,s\sigma)\sigma d\sigma \right| \leq \tau \cdot s^\nu \text{ for } s \in (0, s_0) \text{ and for } x \in \mathbb{R}^N \tag{4.5}$$

where σ is a surface element on the unit sphere $S^{N-1} \subset \mathbb{R}^N$.

Remark 4.1.1. The $(*)$-kernel condition in Definition 4.1.2 implies the *weak*-$(*)$-kernel condition in Definition 4.1.3. Indeed, for the case $\alpha < 1$, they are exactly same. If $\alpha \geq 1$, then the only difference between them is that the $(*)$-kernel condition needs (4.4) while the *weak*-$(*)$-kernel condition requires (4.5). Also, it is easy to verify that (4.4) implies (4.5) up to a constant by the following argument: for any $s \in (0, s_0/2)$,

$$\begin{aligned} \left| \int_{S^{N-1}} k(t,x,s\sigma)\sigma d\sigma \right| &= \left| \int_{S_+^{N-1}} k(t,x,s\sigma)\sigma d\sigma + \int_{S_-^{N-1}} k(t,x,s\sigma)\sigma d\sigma \right| \\ &\leq \int_{S_+^{N-1}} \Big| k(t,x,s\sigma) - k(t,x,-s\sigma) \Big| d\sigma. \end{aligned}$$

where S_+^{N-1} and S_-^{N-1} are the upper and the lower hemispheres, respectively. Then, thanks to (4.4), we have

$$\leq \int_{S_+^{N-1}} \tau \Big|2s\sigma\Big|^{\nu} d\sigma \leq (C\tau) \cdot s^{\nu}.$$

Remark 4.1.2. In the work of [10], the upper bound for k is just Λ while, in this thesis, we have $\Lambda \cdot (1 + |x-y|^{\omega})$ in (4.3), which is slightly more general than that of [10]. Some example from pseudo-differential operator with this upper bound (4.3) for $\omega = \alpha/2$ can be found in Example 2 in Section 4 of Komatsu [51].

Remark 4.1.3. The purpose of the condition (4.5) with $(\alpha - 1) < \nu$ is to consider $T_t^K(f)(\cdot)$ not only as a distribution but also as a locally integrable function. In general, without such an additional cancellation condition, if $\alpha \geq 1$, then $T_t^K(f)(x)$ is not well-defined even for $f \in C_c^{\infty}$. In Lemma 4.2.2 and Lemma 4.2.1, it will be shown that as long as the corresponding kernel K satisfies the *weak*-$(*)$-kernel condition, the operator T_t^K is well defined, and $T_t^K(f)$ is a locally integrable function for some class of functions f.

Remark 4.1.4. Let the kernel K satisfy the *weak*-$(*)$-kernel condition for some $\alpha \geq 1$. Then we can combine the two conditions (4.3) and (4.5) in order to get an estimate of the integral in (4.5) for all $s \in (0, \infty)$. Indeed, the condition (4.3) implies that, for $s \in [s_0, \infty)$,

$$\int_{S^{N-1}} \Big|k(t,x,s\sigma)\Big| d\sigma \leq C\Lambda \cdot (1 + s^{\omega}) \leq (C\Lambda \cdot s_0^{-\nu}) \cdot s^{\nu} \cdot (1 + s^{\omega}).$$

Thus, together with the condition (4.5), we have, for $s \in (0, \infty)$,

$$\Big| \int_{S^{N-1}} k(t,x,s\sigma)\sigma d\sigma \Big| \leq \bar{\tau} \cdot s^{\nu} \cdot (1 + s^{\omega}) \tag{4.6}$$

where $\bar{\tau} := \max\{\tau, (C\Lambda \cdot s_0^{-\nu})\}$.

Remark 4.1.5. We present some typical examples satisfying either the $(*)$-kernel condition or the *weak*-$(*)$-kernel condition.

(I) For the simplest example, if $K := c_\alpha/|x-y|^{N+\alpha}$ (i.e. $k := c_\alpha$), then the equation (4.1) becomes the fractional heat equation (some regularity results can be found in Caffarelli and Figalli [11]). This kernel satisfies the $(*)$-kernel condition. Indeed, (4.2) and (4.3) are trivial. For $\alpha \geq 1$, since k is a constant function, (4.4) holds for any $\nu \in (\alpha - 1, 1)$ with $\tau = 0$ and $s_0 = \infty$.

(II) One may assume that the kernel has the form not of $K(t, x, y)$ but of $K(t, x - y)$ (for more general cases, we refer to Silvestre [70]). Then the natural symmetry we would impose to the kernel is $K(t, x - y) = K(t, y - x)$, which implies (4.2) directly. This K holds the *weak*-$(*)$-kernel condition for any $\alpha \in (0, 2)$ once we assume the bounds condition (4.3). Indeed, for $\alpha \geq 1$, the integral in (4.5) is always zero due to the cancellation from the symmetry (i.e. any $\nu \in (\alpha - 1, 1)$ with $\tau = 0$ and $s_0 = \infty$ works).

Here is our main theorem about persistence of Hölder continuity.

Theorem 4.1.1. *Let* $\{\alpha, \zeta, \omega, \Lambda\}$ *(for the case* $\alpha \geq 1$*,* $\{\alpha, \zeta, \omega, \Lambda, \nu, s_0, \tau\}$*) be a set of the parameters in Definition 4.1.2. Then there exist two constants* $\beta > 0$ *and*

$C > 0$ with the following two properties (I) and (II):

(I). Let $w_0 \in (L^1 \cap L^\infty)(\mathbb{R}^N)$ be a given function and let $0 < T \leq \infty$. Let K satisfy the $(*)$-kernel condition on $[0,T]$ (see Definition 4.1.2). Then there exists a weak solution w of (4.1) on $(0,T)$ satisfying the following estimates for a.e. $t \in (0,T)$:

$$\|w(t,\cdot)\|_{C^\beta(\mathbb{R}^N)} \leq C \cdot \|w_0\|_{C^\beta(\mathbb{R}^N)} \quad \text{if } w_0 \in C^\beta(\mathbb{R}^N), \tag{4.7}$$

$$\|w(t,\cdot)\|_{C^\beta(\mathbb{R}^N)} \leq C \cdot \max\{1, \frac{1}{t^{\beta/\alpha}}\} \cdot \|w_0\|_{L^\infty(\mathbb{R}^N)}, \quad \text{and} \tag{4.8}$$

$$\|w(t,\cdot)\|_{C^\beta(\mathbb{R}^N)} \leq C\Big(\|w_0\|_{L^\infty(\mathbb{R}^N)} + \max\{1, \frac{1}{t^{(N+\beta)/\alpha}}\} \cdot \|w_0\|_{L^1(\mathbb{R}^N)}\Big). \tag{4.9}$$

(II). Let $w_0 \in C^\infty(\overline{\mathbb{R}^N}) \cap L^1(\mathbb{R}^N)$ be a given function and let $0 < T \leq \infty$. Let K satisfy the weak-$(*)$-kernel condition on $[0,T]$ (see Definition 4.1.3). In addition, we assume

$$k(\cdot,\cdot,\cdot) \in C^\infty_{t,x,z}\Big([0,T] \times \overline{\mathbb{R}^N} \times \overline{\mathbb{R}^N}\Big). \tag{4.10}$$

Suppose that $w \in L^\infty(0,T;(L^1 \cap L^\infty)(\mathbb{R}^N))$ is a smooth solution of (4.1) on $[0,T) \times \mathbb{R}^N$ for the initial data w_0. Then, w satisfy (4.7), (4.8), and (4.9) for any $t \in (0,T)$.

We concentrate our effort first to prove the part (II) of the above theorem in Section 4.2, 4.3, and 4.4. In fact, we will show the part (II) carefully to ensure that the two constants C and β in the conclusion of the part (II) depend only on the parameters in Definition 4.1.2. Thus, these two constants C and β depend neither on T nor on any actual norms coming from the smoothness assumption (4.10). As a result, the part (I), which will be proved in Appendix, follows the part (II) by a limit argument. Unfortunately, if $\alpha \geq 1$, then we need the condition (4.4), which is more restrictive than (4.5).

Remark 4.1.6. More precisely, the conclusion of the part (I) follows once we regularize the function k in a proper way, which should keep all the parameters. In short, since k may not be bounded due to (4.3), we make it bounded first. Then take a convolution with a mollifier. This process does not hurt the parameter set essentially if $\alpha < 1$. However for the case $\alpha \geq 1$, the cancellation condition (4.5) is not preserved during the process. That is the reason we impose the $(*)$-kernel condition to the part (I) of Theorem 4.1.1 instead of the *weak*-$(*)$-kernel condition.

Remark 4.1.7. Thanks to the symmetry condition (4.2), we use the following weak formulation of (4.1):

$$\int_{\mathbb{R}^N} (\partial_t w)(t,x)\eta(x)dx + \frac{1}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N} [w(t,x) - w(t,y)][\eta(x) - \eta(y)]K(t,x,y)dydx = 0$$

for $\eta(\cdot) \in C_c^\infty(\mathbb{R}^N)$ and for a.e. t (e.g. see [10] or [47]).

As in [10], we show how our result can be applied to a fully non-linear problem. We introduce the following non-linear evolution problem:

$$\partial_t\theta(t,x) - \int_{\mathbb{R}^N} \phi'(\theta(t,y) - \theta(t,x))G(y-x)dy = 0. \tag{4.11}$$

This equation can be considered as the evolution problem coming from the Euler-Lagrange equation for the variational integral

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N} \phi(\theta(t,y) - \theta(t,x))G(y-x)dydx$$

(for more detailed explanation, see [10]). This non-linear problem can be found in Giacomin, Lebowitz, and Presutti [39], or in the field of image processing (e.g. see Gilboa and Osher [41], Lou, Zhang, Osher, and Bertozzi [60]).

We impose the following conditions to the equation (4.11).

Let $0 < \alpha < 2$, $0 < \zeta \le \infty$, $0 \le \omega < \alpha$ and $1 \le \Lambda < \infty$ (for the case $\alpha \ge 1$, we need two more parameters ν and M such that $(\alpha - 1) < \nu < 1$, $0 \le M < \infty$, and $\nu + \omega < \min\{N, \alpha\}$). Let $\phi : \mathbb{R} \to [0, \infty)$ be an even function of class C^2 satisfying

$$\phi(0) = 0 \quad \text{and} \quad \sqrt{\Lambda^{-1}} \le \phi''(x) \le \sqrt{\Lambda}, \quad x \in \mathbb{R}.$$

We assume that the kernel $G : \mathbb{R}^N/\{0\} \to [0, \infty)$ satisfies $G(-x) = G(x)$ and

$$\sqrt{\Lambda^{-1}} \cdot \mathbf{1}_{|x|\le\zeta} \le G(x) \cdot |x|^{N+\alpha} \le \sqrt{\Lambda} \cdot (1 + |x|^{\omega}) \text{ for } x \in \mathbb{R}^N/\{0\}. \tag{4.12}$$

Let $\theta_0 \in (W^{1,1} \cap W^{1,\infty})(\mathbb{R}^N)$ be a given function. For the case $\alpha \ge 1$, we assume further

$$\phi'' \in C^{\nu}(\mathbb{R}) \text{ with } [\phi'']_{C^{\nu}(\mathbb{R})} \cdot \|\nabla\theta_0\|^{\nu}_{L^{\infty}(\mathbb{R}^N)} \le M.$$

Remark 4.1.8. The upper bound (4.12) for $G(\cdot)$ is $\sqrt{\Lambda} \cdot (1 + |x|^{\omega})$ and it is more flexible than that of [10], where just $\sqrt{\Lambda}$ was used as the upper bound.

Following the approach of [10], we present the following important consequence of the part (II) of Theorem 4.1.1.

Theorem 4.1.2. *We have two constants $\beta > 0$ and $C > 0$ which depend only on the above parameters, and there exists a global-time weak solution θ of the equation* (4.11) *with the following estimates for a.e. $t \in (0, \infty)$:*

$$\|\nabla\theta(t,\cdot)\|_{C^\beta(\mathbb{R}^N)} \leq C \cdot \|\nabla\theta_0\|_{C^\beta(\mathbb{R}^N)} \quad \textit{if} \quad \nabla\theta_0 \in C^\beta(\mathbb{R}^N),$$

$$\|\nabla\theta(t,\cdot)\|_{C^\beta(\mathbb{R}^N)} \leq C \cdot \max\{1, \frac{1}{t^{\beta/\alpha}}\} \cdot \|\nabla\theta_0\|_{L^\infty(\mathbb{R}^N)}, \quad \textit{and}$$

$$\|\nabla\theta(t,\cdot)\|_{C^\beta(\mathbb{R}^N)} \leq C\Big(\|\nabla\theta_0\|_{L^\infty(\mathbb{R}^N)} + \max\{1, \frac{1}{t^{(N+\beta)/\alpha}}\} \cdot \|\nabla\theta_0\|_{L^1(\mathbb{R}^N)}\Big).$$

The main idea of the above theorem 4.1.2 is the following: First, we regularize θ_0, G, and ϕ in a proper way so that we obtain a sequence of smooth solutions of (4.11). Then, we take a derivative ($w := D_e\theta$) to the non-linear equation (4.11), and we freeze some coefficients. As a result of this process, we obtain the linear equation (4.1) together with the *weak*-($*$)-kernel condition on the K satisfying (4.10). Thus, we can use the conclusion of the part (II) of Theorem 4.1.1. Finally, we extract a weak solution by a limit argument. This proof will be given in Appendix.

As the last application, we present persistence of Hölder regularity for drift diffusion equations with supercritical fractional diffusion

$$\partial_t w + (b \cdot \nabla)w + (-\Delta)^{\alpha/2} w = 0 \tag{4.13}$$

under some additional assumptions on the drift velocity b. One mathematical issue is to find a minimal condition on b to ensure certain regularity of a solution w. From the paper [13], we have the global regularity for $\alpha = 1$ and for $b \in L_t^\infty BMO_x$ with div $b = 0$ (or see [48] for the same setting, [49] for 2D-SQG equation).

Here we follow the framework of [48] in order to show a similar result for $\alpha \in (0,1)$ and for $b \in L^\infty_t C^{1-\alpha}_x$ with div $b = 0$. Even though it has been known that $b \in L^\infty_t C^{1-\alpha}_x$ gives a solution Hölder regularity (see Silvestre [72] without div $b = 0$, Constantin and Wu [27] for the 2D-SQG equation), we think it is worth including Theorem 4.1.3 since it is interesting to know whether the method, which we use in this chapter, can be applied even to the equation (4.13) for $\alpha < 1$. Moreover, its proof is almost free once we have Theorem 4.1.1, and the proof reveals clearly why the scaling of $b(t) \in C^{1-\alpha}$ is crucial. Its proof will be given in Section 4.5.

Theorem 4.1.3. *Let $\alpha \in (0,1)$ and $B \in [0,\infty)$. Then, there exist C and $\beta > 0$ with the following property:*

Let $T > 0$. Suppose $b \in C^\infty(\overline{\mathbb{R}^N} \times [0,T])$ and $b(t) \in C^{1-\alpha}$ with $[b(t)]_{C^{1-\alpha}} \le B$ and div $b(t,\cdot) = 0$ for all $t \in [0,T]$. Assume $w \in L^\infty(0,T;(L^1 \cap L^\infty)(\mathbb{R}^N))$ is a smooth solution of (4.13) with a smooth initial data $w_0 \in (L^1 \cap L^\infty)(\mathbb{R}^N)$. Then, we have (4.7) if $w_0 \in C^\beta$, (4.8) and (4.9) for any $t \in [0,T]$.

Remark 4.1.9. The above constants C and β depend only on B and α.

Remark 4.1.10. If we have slightly stronger assumptions on b (or w), then higher regularity for a solution w beyond Hölder regularity can be obtained. For example, we refer to the papers of Silvestre [73]($b \in L_t^\infty C_x^{1-\alpha+\epsilon}$ without div $b = 0$), Constantin and Wu [26]($w \in L_t^\infty C_x^{1-\alpha+\epsilon}$ for 2D-SQG), Dong and Pavlović [32] ($b \in C_t C_x^{1-\alpha}$ for 2D-SQG). For many other models, we refer to Constantin, Iyer, and Wu [24], Friedlander and Vicol [38], Miao and Xue [61], Chae, Constantin, and Wu, [15], and references therein.

Now we want to explain the main idea of the part (II) of Theorem 4.1.1, which is the heart of this chapter. As mentioned earlier, our proof follows the spirit of the paper [48]. First, thanks to the duality of the equation (4.1) from the symmetry condition (4.2), we can focus only on the evolution of $\mathfrak{U}_r$, a class of test functions, which is related to the dual space of the Hölder space C^β (see Definition 4.2.1 of $\mathfrak{U}_r$, Lemma 4.2.4, and Lemma 4.2.5). In this chapter, we take the same definition of the class $\mathfrak{U}_r$ from the paper [48] while another class can be found in Dabkowski [29]. In particular, the class introduced in [29] is quite different from that of [48] and it was successfully used to obtain eventual regularity of the super-critical surface quasi-geostrophic (SQG) equation.

Second, we prove the short-time evolution of test functions (Proposition 4.3.1). In order to obtain it, we need to manage the competition (refer to Remark 4.3.4) between the L^p condition and the concentration condition. The former condition, which can be proved from the lower bound $\Lambda^{-1} \cdot \mathbf{1}_{|z|\le\zeta}$, of the kernel has a regularization effect (Lemma 4.3.4, Lemma 4.3.5) as a diffusion term in usual PDEs does. However, the latter condition comes from the upper bound $\Lambda \cdot (1 + |z|^\omega)$ of the kernel and this upper bound plays a similar role as a source term in usual PDEs

(Lemma 4.3.3).

In addition, since the length of the time interval coming from the conclusion of Proposition 4.3.1 is proportional to r^α where r is the parameter of $\mathcal{U}_r$, it should be verified that we can repeat the short-time evolution (Proposition 4.3.1) as many times as we want in order to reach any fixed time (refer to Remark 4.4.1).

For the case $\alpha < 1$, the main difficulty is to handle both lower and upper bounds (4.3) of the kernel: in particular, both the finite size ζ of support of the lower bound and the term $(1 + |z|^\omega)$ of the upper bound cause some troubles. In order to cover the case $\alpha \geq 1$, we use the cancellation condition (4.5), which is designed to cancel desirable amount of singularity at $x = y$ of the kernel. Then we can interpret $T_t^K(f)$ as locally integrable functions for some class of functions (see Lemma 4.2.1, Lemma 4.2.2). This fact will be crucial to prove the concentration condition (Lemma 4.3.3).

We want to mention a few articles related to the integral operator T_t^K corresponding a kernel K. For smooth bounded kernels, we may use a theory of pseudo differential operators (e.g. Kumano-go [53], Komatsu [50]), while for measurable kernels, there exists a fundamental solution (see [51]). Also, we refer to [47] and Barlow, Bass, Chen, and Kassmann [1]. Recently, in Dyda and Kassmann [34], assumptions of kernels have been extended in some geometrical sense. If we focus on non-divergence case, we refer to [12].

As mentioned before, the following three sections 4.2, 4.3, and 4.4 are dedicated to the proof of the part (II) of Theorem 4.1.1. More precisely, in Section 4.2, we introduce some definitions and few important lemmas. After that, we present and prove the main proposition 4.3.1 in Section 4.3. Finally, the proof of the part (II) of Theorem 4.1.1 ends in Section 4.4. Section 4.5 is dedicated to Theorem 4.1.3. At the end of this thesis, Appendix contains the proofs of the part (I) of Theorem 4.1.1 and Theorem 4.1.2.

4.2 Background lemmas

From now on, we fix a parameter set $\{\alpha, \zeta, \omega, \Lambda\}$, which appears in Definition 4.1.2 (for the case $\alpha \geq 1$, $\{\alpha, \zeta, \omega, \Lambda, \nu, s_0, \tau\}$). Also, suppose that K satisfies the *weak*-$(*)$-kernel condition in Definition 4.1.3 on the parameter set together with the smoothness assumption (4.10). For the case $\alpha < 1$, we define and fix a constant γ such that $0 < \gamma < (\alpha - \omega)$ while, for the case $\alpha \geq 1$, we take γ to be $\max\{(\alpha - N), 0\} < \gamma < (\alpha - (\omega + \nu)))$.

Before considering a general $\zeta \in (0, \infty]$, we will prove first the conclusion of the part (II) of Theorem 4.1.1 for a fixed $\zeta = \zeta_0$ where

$$\zeta_0 := \max\{\Big(\frac{8}{V_N}\Big)^{1/N}, 2 \cdot (11)^{1/\gamma}\} \tag{4.14}$$

(V_N is the volume of the unit ball in $\mathbb{R}^N$). This definition of ζ_0 will help us to obtain enough regularization directly so that the proof becomes more straightforward.

Once we prove the part (II) of Theorem 4.1.1 with $\zeta = \zeta_0$, a general proof for any value $\zeta \in (0, \infty]$ will follow a scaling argument.

Remark 4.2.1. Indeed, we assume that the part (II) of Theorem 4.1.1 with $\zeta = \zeta_0$ is true for a moment. Then, the case $\zeta > \zeta_0$ is included in the case $\zeta = \zeta_0$ because of $\{|x-y| \le \zeta_0\} \subset \{|x-y| \le \zeta\}$. On the other hand, for the case $\zeta \in (0, \zeta_0)$, we take $\epsilon := \zeta/\zeta_0 < 1$ and define a scaling: $w^\epsilon(t,x) = w(\epsilon^\alpha t, \epsilon x)$ and $K^\epsilon(t,x,y) = \epsilon^{N+\alpha} \cdot K(\epsilon^\alpha t, \epsilon x, \epsilon y)$. Suppose that w satisfies (4.1) on $[0,T]$ for a kernel K satisfying the *weak*-$(*)$-kernel condition for a parameter set $\{\alpha, \zeta, \omega, \Lambda\}$ (for $\alpha \ge 1$, $\{\alpha, \zeta, \omega, \Lambda, \nu, s_0, \tau\}$). Observe first that

$$(\partial_t w^\epsilon)(t,x) = \epsilon^\alpha (\partial_t w)(\epsilon^\alpha t, \epsilon x) = \epsilon^\alpha \int_{\mathbb{R}^N} [w(\epsilon^\alpha t, y) - w(\epsilon^\alpha t, \epsilon x)] K(\epsilon^\alpha t, \epsilon x, y) dy$$
$$= \epsilon^\alpha \int [w^\epsilon(t, \frac{y}{\epsilon}) - w^\epsilon(t,x)] K(\epsilon^\alpha t, \epsilon x, y) dy$$
$$= \epsilon^{N+\alpha} \int [w^\epsilon(t,y) - w^\epsilon(t,x)] K(\epsilon^\alpha t, \epsilon x, \epsilon y) dy = \int [w^\epsilon(t,y) - w^\epsilon(t,x)] K^\epsilon(t,x,y) dy.$$

For the parameter set for K^ϵ, we have, due to $\epsilon < 1$,

$$K^\epsilon(t,x,y)|x-y|^{N+\alpha} = K(\epsilon^\alpha t, \epsilon x, \epsilon y)|\epsilon x - \epsilon y|^{N+\alpha}$$
$$\le \Lambda \cdot (1 + |\epsilon x - \epsilon y|^\omega) \le \Lambda \cdot (1 + |x-y|^\omega).$$

Likewise, we have

$$K^\epsilon(t,x,y)|x-y|^{N+\alpha} \ge \Lambda^{-1} \cdot \mathbf{1}_{|\epsilon x - \epsilon y| \le \zeta} = \Lambda^{-1} \cdot \mathbf{1}_{|x-y| \le \zeta_0}.$$

If $\alpha \ge 1$, then we get

$$\Big| \int_{S^{N-1}} K^\epsilon(t,x,x+s\sigma) s^{N+\alpha} \sigma d\sigma \Big| = \Big| \int_{S^{N-1}} K(\epsilon^\alpha t, \epsilon x, \epsilon x + (\epsilon s)\sigma) \cdot (\epsilon s)^{N+\alpha} \sigma d\sigma \Big|$$
$$= \Big| \int_{S^{N-1}} k(\epsilon^\alpha t, \epsilon x, (\epsilon s)\sigma) \sigma d\sigma \Big| \le \tau \cdot (\epsilon s)^\nu = (\tau \epsilon^\nu) \cdot s^\nu \text{ if } \epsilon s \le s_0 \text{ (or } s \le \frac{s_0}{\epsilon} \text{).}$$

Thanks to $\epsilon < 1$, we have $s_0 \le \frac{s_0}{\epsilon}$ and $(\tau\epsilon^\nu) \le \tau$. Thus w^ϵ is a solution on $[0, T/\epsilon^\alpha]$ for the kernel K^ϵ satisfying the *weak*-$(*)$-kernel condition for the set $\{\alpha, \zeta_0, \omega, \Lambda\}$ (for $\alpha \ge 1$, $\{\alpha, \zeta_0, \omega, \Lambda, \nu, s_0, \tau\}$) so that there exist two constants C and β satisfying (4.7), (4.8) and (4.9) for w^ϵ on $[0, \epsilon^{-\alpha}T]$. As a result, we obtain

$$\begin{aligned}
&\|w(t,\cdot)\|_{C^\beta(\mathbb{R}^N)} = \|w(t,\cdot)\|_{L^\infty} + [w(t,\cdot)]_{C^\beta} = \|w^\epsilon(\frac{t}{\epsilon^\alpha},\cdot)\|_{L^\infty} + \frac{1}{\epsilon^\beta}[w^\epsilon(\frac{t}{\epsilon^\alpha},\cdot)]_{C^\beta} \\
&\le \frac{1}{\epsilon^\beta}\|w^\epsilon(\frac{t}{\epsilon^\alpha},\cdot)\|_{C^\beta(\mathbb{R}^N)} \le C\frac{1}{\epsilon^\beta}\|w^\epsilon(0,\cdot)\|_{C^\beta(\mathbb{R}^N)} \\
&= C\frac{1}{\epsilon^\beta}\Big(\|w_0\|_{L^\infty} + \epsilon^\beta[w_0]_{C^\beta}\Big) \le \Big[C\frac{1}{\epsilon^\beta}\Big]\cdot\|w_0\|_{C^\beta(\mathbb{R}^N)} \quad \text{ if } w_0 \in C^\beta(\mathbb{R}^N),
\end{aligned}$$

$$\begin{aligned}
&\|w(t,\cdot)\|_{C^\beta(\mathbb{R}^N)} \le \frac{1}{\epsilon^\beta}\|w^\epsilon(\frac{t}{\epsilon^\alpha},\cdot)\|_{C^\beta(\mathbb{R}^N)} \le C\frac{1}{\epsilon^\beta}\cdot\max\{1, \frac{\epsilon^\beta}{t^{\beta/\alpha}}\}\cdot\|w^\epsilon(0,\cdot)\|_{L^\infty(\mathbb{R}^N)} \\
&\le \Big[C\frac{1}{\epsilon^\beta}\Big]\cdot\max\{1, \frac{1}{t^{\beta/\alpha}}\}\cdot\|w_0\|_{L^\infty(\mathbb{R}^N)}, \quad \text{ and} \\
&\|w(t,\cdot)\|_{C^\beta(\mathbb{R}^N)} \le \Big[C\frac{1}{\epsilon^{N+\beta}}\Big]\cdot\Big(\|w_0\|_{L^\infty(\mathbb{R}^N)} + \max\{1, \frac{1}{t^{(N+\beta)/\alpha}}\}\cdot\|w_0\|_{L^1(\mathbb{R}^N)}\Big).
\end{aligned}$$

Therefore, it is enough to show the part (II) of Theorem 4.1.1 only for $\zeta = \zeta_0$.

It will be shown in Lemma 4.2.2 that the operator $T_t^K(f)$ is well-defined pointwise for $f \in (C^2 \cap L^1)(\mathbb{R}^N)$. Moreover the operator can be extended to more general spaces. For example, if f is locally integrable and $\int \frac{|f|}{1+|x|^{N+\alpha-\omega}}dx < \infty$, then we can define $T_t^K(f)$ as an element of $\mathcal{S}'$ where $\mathcal{S}'$ is the dual of Schwartz space $\mathcal{S}$ (see also Silvestre [71]). We will make use of the following Lemma 4.2.1, which says that $T_t^K(|\cdot|^\gamma)$ is not only an element of $\mathcal{S}'$ but also a locally integrable function with a desirable estimate. This fact will be used to obtain the concentration condition (Lemma 4.3.3) for the evolution of $\mathcal{U}_r$, which will be introduced in Definition 4.2.1.

Lemma 4.2.1. *We have an estimate*

$$\Big|T_t^K\Big(|\cdot|^\gamma\Big)(x)\Big| \le \begin{cases} C\cdot|x|^{\gamma-\alpha}\cdot(1+|x|^{\omega}), & \textit{if } 0<\alpha<1, \\ C\cdot|x|^{\gamma-\alpha}\cdot(1+|x|^{\nu+\omega}), & \textit{if } 1\le\alpha<2. \end{cases}$$

Remark 4.2.2. Recall that γ is a fixed constant such that $0<\gamma<\alpha-\omega$ (for $\alpha\ge 1$, $\max\{(\alpha-N),0\}<\gamma<(\alpha-(\omega+\nu))$).

Proof.

$$\begin{aligned}
&\int_{\mathbb{R}^N}\Big[|x|^\gamma-|y|^\gamma\Big]K(t,x,y)dy=\int_{\mathbb{R}^N}\Big[|x|^\gamma-|x+z|^\gamma\Big]K(t,x,x+z)dz\\
&=\int_{\mathbb{R}^N}\frac{|x|^\gamma-|x+z|^\gamma}{|z|^{N+\alpha}}k(t,x,z)dz\\
&=|x|^\gamma\int_{\mathbb{R}^N}\frac{1-|\frac{x}{|x|}+\frac{z}{|x|}|^\gamma}{|z|^{N+\alpha}}k(t,x,z)dz.
\end{aligned}$$

Then we use the change of variables $\frac{z}{|x|}=\bar{z}$ and the polar coordinate to get

$$\begin{aligned}
&=|x|^{\gamma-\alpha}\int_{\mathbb{R}^N}\frac{1-|\frac{x}{|x|}+\bar{z}|^\gamma}{|\bar{z}|^{N+\alpha}}k(t,x,|x|\bar{z})d\bar{z}\\
&=|x|^{\gamma-\alpha}\int_{S^{N-1}}\int_0^\infty\frac{1-|\frac{x}{|x|}+s\sigma|^\gamma}{s^{1+\alpha}}k(t,x,|x|s\sigma)dsd\sigma\\
&=|x|^{\gamma-\alpha}\Big(\int_{S^{N-1}}\int_0^{1/2}\cdots dsd\sigma+\int_{S^{N-1}}\int_{1/2}^\infty\cdots dsd\sigma\Big)=|x|^{\gamma-\alpha}\Big((I)+(II)\Big).
\end{aligned}$$

From the condition $\gamma+\omega<\alpha$, we have

$$\Big|(II)\Big|\le\Lambda\int_{S^{N-1}}\int_{1/2}^\infty\frac{(2+s^\gamma)(1+|x|^\omega s^\omega)}{s^{1+\alpha}}dsd\sigma\le C(1+|x|^\omega).$$

On the other hand, from Taylor expansion $1-|\frac{x}{|x|}+s\sigma|^\gamma=-\gamma(\frac{x}{|x|}\cdot\sigma)s+R_\sigma(s)$ with an error estimate $|R_\sigma(s)| \le Cs^2$ for $s \in [0,1/2]$ and $\sigma \in S^{N-1}$, we have

$$\begin{aligned}
\Big|(I)\Big| &\le \Big|\int_{S^{N-1}}\int_0^{1/2}\gamma(\frac{x}{|x|}\cdot\sigma)\frac{s}{s^{1+\alpha}}k(t,x,|x|s\sigma)dsd\sigma\Big| \\
&\qquad + \Lambda\int_{S^{N-1}}\int_0^{1/2}\frac{|R_\sigma(s)|(1+|x|^\omega s^\omega)}{s^{1+\alpha}}dsd\sigma \\
&\le C\int_0^{1/2}\frac{1}{s^\alpha}\Big|\int_{S^{N-1}}k(t,x,|x|s\sigma)\sigma d\sigma\Big|ds + C\cdot\frac{1+|x|^\omega}{2-\alpha}\cdot\Lambda \\
&= C\cdot(III) + C\cdot(1+|x|^\omega).
\end{aligned}$$

For the case $\alpha < 1$, (III) is bounded above by $C\cdot(1+|x|^\omega)$.

If $\alpha \ge 1$, we can use the condition (4.6), which is obtained from (4.5) and (4.3), together with $\nu > (\alpha-1)$:

$$\begin{aligned}
(III) &\le \int_0^{1/2}\frac{1}{s^\alpha}\bar{\tau}|x|^\nu s^\nu(1+|x|^\omega s^\omega)ds \le \bar{\tau}|x|^\nu\cdot(1+|x|^\omega)\int_0^{1/2}\frac{1}{s^{\alpha-\nu}}ds \\
&\le C\cdot(1+|x|^\omega)\cdot|x|^\nu.
\end{aligned}$$

□

We give, in the following lemma, some properties of the integral operator T_t^K and the related evolution equation (4.1).

Lemma 4.2.2. *For any $f \in C^2(\mathbb{R}^N)\cap L^1(\mathbb{R}^N)$, $T_t^K(f)$ is well-defined pointwise. Moreover, the following properties hold:*

(I). Duality of T:

$$\int f(x)T_t^K(g)(x)dx = \int T_t^K(f)(x)g(x)dx,$$

for $f \in C^2(\overline{\mathbb{R}^N})\cap L^1(\mathbb{R}^N)$ and either $g \in C^2(\overline{\mathbb{R}^N})\cap L^1(\mathbb{R}^N)$ or $g(x) = |x|^\gamma$.

(II). Mean zero of T:

$$\int T_t^K(f)(x)dx = 0,$$

for $f \in C^2(\overline{\mathbb{R}^N})\cap L^1(\mathbb{R}^N)$.

Proof. Let $f \in C^2(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$. Let $x \in \mathbb{R}^N$. Then, we have

$$T_t^K(f)(x) = \int (f(x) - f(y))K(t,x,y)dy = \int \frac{f(x) - f(x+z)}{|z|^{N+\alpha}} k(t,x,z)dz$$
$$= \int_0^1 \int_{S^{N-1}} \frac{(\nabla f)(x) \cdot (r\sigma) + R_f(x, r\sigma)}{r^{1+\alpha}} k(t,x,r\sigma)d\sigma dr$$
$$+ \int_{|z|\geq 1} \frac{f(x) - f(x+z)}{z^{N+\alpha}} k(t,x,z)dz = (a) + (b).$$

where we used the Taylor expansion of f in the first integral.

For (b), we use the upper bound of (4.3):

$$|(b)| \leq \Lambda \int_{|z|\geq 1} \Big(|f(x)| + |f(x+z)|\Big) \frac{1 + |z|^\omega}{|z|^{N+\alpha}} dz$$
$$\leq C|f(x)| \int_{|z|\geq 1} \frac{1}{|z|^{N+\alpha-\omega}} dz + C \int_{|z|\geq 1} |f(x+z)| dz$$
$$\leq C|f(x)| + \|f\|_{L^1}.$$

For (a), if $\alpha < 1$, we use $|R_f(x, r\sigma)| \leq C \cdot \|\nabla^2 f\|_{L^\infty(B_1(x))} \cdot r^2$ from the Taylor error estimate where $B_x(r)$ is the ball of radius r centered at x:

$$|(a)| \leq C|\nabla f(x)| \cdot \int_0^1 \frac{r}{r^{1+\alpha}} + C\|\nabla^2 f\|_{L^\infty(B_1(x))} \cdot \int_0^1 \frac{r^2}{r^{1+\alpha}}$$
$$\leq C\Big(|\nabla f(x)| + \|\nabla^2 f\|_{L^\infty(B_1(x))}\Big).$$

If $\alpha \geq 1$, we use the condition (4.6) with the assumption $(\alpha - 1) < \nu$:

$$(a) \leq C|\nabla f(x)| \cdot \int_0^1 \frac{1}{r^\alpha} \Big| \int_{S^{N-1}} k(t,x,r\sigma)\sigma d\sigma \Big| dr + C\|\nabla^2 f\|_{L^\infty(B_1(x))}$$
$$\leq C|\nabla f(x)| \cdot \int_0^1 \frac{1}{r^\alpha} r^\nu dr + C\|\nabla^2 f\|_{L^\infty(B_1(x))}$$
$$\leq C\Big(|\nabla f(x)| + \|\nabla^2 f\|_{L^\infty(B_1(x))}\Big).$$

Now we can easily verify that $T_t^K(f)$ is well-defined pointwise with the estimate:

$$|T_t^K(f)(x)| \leq C\Big(\|f\|_{L^1(\mathbb{R}^N)} + \|f\|_{W^{2,\infty}(B_1(x))}\Big).$$

Note that if $f \in C^2(\overline{\mathbb{R}^N}) \cap L^1(\mathbb{R}^N)$, then the above argument implies $T_t^K(f) \in L^\infty(\mathbb{R}^N)$. Then, the proof of (I) follows the symmetry in x, y of K. Indeed, if $f, g \in C^2(\overline{\mathbb{R}^N}) \cap L^1(\mathbb{R}^N)$, then we have

$$\begin{aligned}\int f(x)T_t^K(g)(x)dx &= \int\int f(x)(g(x)-g(y))K(t,x,y)dydx \\ &= \frac{1}{2}\int\int (f(x)-f(y))(g(x)-g(y))K(t,x,y)dydx \\ &= \int\int (f(x)-f(y))g(x)K(t,x,y)dydx \\ &= \int T_t^K(f)(x)g(x)dx.\end{aligned} \tag{4.15}$$

In addition, for the case $f \in C^2(\overline{\mathbb{R}^N}) \cap L^1(\mathbb{R}^N)$ with $g(x) = |x|^\gamma$, then the integral $\int |f(x)T_t^K(g)(x)|dx$ is bounded due to the assumption $f \in L^\infty \cap L^1$ with Lemma 4.2.1. Indeed, Lemma 4.2.1 implies that $T_t^K(|\cdot|^\gamma)$ is integrable in the unit ball containing the origin and is bounded outside of the ball. Then, together with $f \in L^\infty \cap L^1$, we obtain $f \cdot T_t^K(|\cdot|^\gamma) \in L^1$. Thus all equalities of (4.15) can be justified via a limit argument.

To prove (II), we take $g \in C_c^\infty$ such that $g = 1$ in $B_1(0)$ and $supp(g) \subset B_2(0)$ and define g_n by $g_n(\cdot) := g(\cdot/n)$. Then, thanks to the property (I) with g_n, the conclusion follows by taking a limit $n \to \infty$. □

In the following lemma, we present a maximum principle for solutions of (4.1).

Lemma 4.2.3. *Suppose that $w \in L_t^\infty(L_x^\infty \cap L_x^1)$ is a smooth solution of* (4.1). *Then, the following properties hold:*

(I). If a convex function $\eta : \mathbb{R} \to \mathbb{R}$ *is* C^1*, then we have*

$$\partial_t(\eta(w)) + T_t^K(\eta(w)) \leq 0. \tag{4.16}$$

Moreover, (4.16) *holds even for* $\eta(x) = |x|$.

(II). L^p-*norm is non-increasing for* $1 \leq p \leq \infty$*:*

$$\|w(t, \cdot)\|_{L^p(\mathbb{R}^N)} \leq \|w(s, \cdot)\|_{L^p(\mathbb{R}^N)} \text{ for any } s < t.$$

Remark 4.2.3. Also we assume that the solutions are smooth. However the estimate of the result does not depend on this smoothness.

Proof. To prove (I) for convex $\eta \in C^1$, we multiply $\eta'(w)$ to the equation (4.1) to get $\partial_t(\eta(w)) + \eta'(w)T_t^K(w) = 0$. Then it is enough to show $\eta'(w)T_t^K(w) - T_t^K(\eta(w)) \geq 0$. Using the integral representation of T_t^K, we have

$$\begin{aligned}
&\eta'(w(x))\Big(T_t^K(w)\Big)(x) - \Big(T_t^K(\eta(w))\Big)(x) = \\
&\int (w(x) - w(y))K(x,y)\eta'(w(x))dy - \int (\eta(w(x)) - \eta(w(y)))K(x,y)dy \\
&= \int \Big(\eta'(w(x))(w(x) - w(y)) - (\eta(w(x)) - \eta(w(y)))\Big)K(x,y)dy \geq 0
\end{aligned}$$

because $\eta'(a)(a-b) - (\eta(a) - \eta(b)) \geq 0$ from convexity of η. The case $\eta(x) = |x|$ follows a limit argument by taking a sequence of C^1-convex functions which converge to η.

For the part (II) (non-increasing of L^p-norm for $p \in [1,\infty]$), we refer to the paper of Córdoba and Córdoba [28].

Remark 4.2.4. Without a major modification, their proof works for smooth solutions w of (4.1) as well as for smooth solutions w of (4.13) as long as div $b = 0$.

□

Now we adopt the notion of the class $\mathfrak{U}_r$ of test functions following the paper [48]. Let $A \geq 1$ be a constant which will be chosen later.

Definition 4.2.1. We say that a measurable function $\varphi(\cdot)$ on $\mathbb{R}^N$ lies in $\mathfrak{U}_r$ for some $r \in (0,\infty)$ if φ satisfies the following four conditions:

$$\int_{\mathbb{R}^N} \varphi(x)dx = 0 \qquad \text{the mean zero-condition,}$$

$$\int_{\mathbb{R}^N} |\varphi(x)||x-x_0|^\gamma dx \leq r^\gamma \quad \text{for some } x_0 \in \mathbb{R}^N \quad \text{the concentration-condition,}$$

$$\|\varphi\|_{L^\infty(\mathbb{R}^N)} \leq \frac{A}{r^N} \qquad \text{the } L^\infty\text{-condition, and}$$

$$\|\varphi\|_{L^1(\mathbb{R}^N)} \leq 1 \qquad \text{the } L^1\text{-condition.}$$

In addition, we say that φ lies in $a\mathfrak{U}_r$ for some $a > 0$ when $(1/a)\varphi \in \mathfrak{U}_r$. We call x_0 a center of φ.

The following lemma connects between C^β space and $\mathfrak{U}_r$, which tells us that $r^{-\beta}\mathfrak{U}_r$ plays a similar role of the dual space of C^β.

Lemma 4.2.4. *Let β be any constant such that $0 < \beta \leq \gamma$.*

(I) Then we have

$$\Big|\int_{\mathbb{R}^N} w(x)\varphi(x)dx\Big| \leq r^\beta [w]_{C^\beta(\mathbb{R}^N)}$$

for any $w \in C^\beta(\mathbb{R}^N)$, for any $0 < r < \infty$, and for any $\varphi \in \mathcal{U}_r$.

(II) Conversely, we have a constant C such that

if a bounded function w satisfies $\sup_{\varphi\in\mathcal{S}\cap\mathcal{U}_r, 0<r\le 1} r^{-\beta}\Big|\int_{\mathbb{R}^3} w(x)\varphi(x)dx\Big| < \infty$,

then $w \in C^\beta$ and

$$\|w\|_{C^\beta(\mathbb{R}^N)} \le C\cdot\Big(\|w\|_{L^\infty} + \sup_{\varphi\in\mathcal{S}\cap\mathcal{U}_r, 0<r\le 1} r^{-\beta}\Big|\int_{\mathbb{R}^3} w(x)\varphi(x)dx\Big|\Big). \tag{4.17}$$

Proof. For the part *(I)*, let x_0 be a center of φ. Then, from the mean zero property,

$$\Big|\int_{\mathbb{R}^N} w(x)\varphi(x)dx\Big| \le \Big|\int_{\mathbb{R}^N}\Big(w(x)-w(x_0)\Big)\varphi(x)dx\Big|$$
$$\le [w]_{C^\beta(\mathbb{R}^N)}\int_{\mathbb{R}^N}|x-x_0|^\beta|\varphi(x)|dx$$
$$\le [w]_{C^\beta(\mathbb{R}^N)}\Big(\int_{\mathbb{R}^N}|x-x_0|^\gamma|\varphi(x)|dx\Big)^{\beta/\gamma}\Big(\int_{\mathbb{R}^N}|\varphi(x)|dx\Big)^{(\gamma-\beta)/\gamma} \le r^\beta[w]_{C^\beta(\mathbb{R}^N)}.$$

For the part *(II)*, we recall Littlewood-Paley projections Δ_j, which is defined by $\Delta_j(w) = w * \Psi_{2^{-j}}$ where $\Psi_t(x) = t^{-N}\Psi(x/t)$ and $\hat{\Psi}(\xi) = \eta(\xi) - \eta(2\xi)$ with $\eta \in C_0^\infty, 0 \le \eta(\xi) \le 1, \eta = 1$ for $|\xi| \le 1$ and $\eta = 0$ for $|\xi| \ge 2$. We use the characterization of C^β in terms of Littlewood-Paley projections (see Stein [75]). Indeed, if a bounded function w in $\mathbb{R}^N$ satisfies

$$\sup_{j=1,2,3,\cdots} 2^{\beta j}\|\Delta_j(w)\|_{L^\infty(\mathbb{R}^N)} < \infty$$

then w lies in $C^\beta(\mathbb{R}^N)$ and it has the estimate

$$\|w\|_{C^\beta(\mathbb{R}^N)} \le C_1(\|w\|_{L^\infty(\mathbb{R}^N)} + \sup_{j=1,2,3,\cdots} 2^{\beta j}\|\Delta_j(w)\|_{L^\infty(\mathbb{R}^N)})$$

where C_1 depends only on β, N and the choice of Ψ. In order to show (4.17), it is enough to find $0 < a < \infty$ such that $\Psi_{2^{-j}} \in a\mathcal{U}_{2^{-j}}$ for all $j \ge 1$ because

$\Delta_j(w)(x) = \int_{\mathbb{R}^3} w(y)\Psi_{2^{-j}}(x-y)dy$ and $\mathcal{U}_r$ is translation invariant.

It is clear that Ψ is a Schwartz function from the fact $\eta \in C_0^\infty$. Thus we can take $a := \|\Psi\|_{L^\infty(\mathbb{R}^3)} + \|\Psi\|_{L^1(\mathbb{R}^3)} + \int_{\mathbb{R}^N} |\Psi(x)||x|^\gamma dx < \infty$. Then, for any $r > 0$, we have

$$\int (1/a)\Psi_r = (1/a)\int \Psi = (1/a)\hat{\Psi}(0) = 0,$$

$$\|(1/a)\Psi_r\|_{L^\infty} \le (1/a)r^{-N}\|\Psi\|_{L^\infty} \le r^{-N} \le \frac{A}{r^N},$$

$$\|(1/a)\Psi_r\|_{L^1} \le (1/a)\|\Psi\|_{L^1} \le 1, \text{ and}$$

$$\int_{\mathbb{R}^N} |(1/a)\Psi_r(x)||x|^\gamma dx = (1/a)r^\gamma \cdot \int_{\mathbb{R}^N} |\Psi(x)||x|^\gamma dx \le r^\gamma.$$

Thus (4.17) follows with $C = C_1 \cdot \max\{1, a\}$.

□

We define the backward kernel $K^{(\overline{T})}$ corresponding to any finite time $\overline{T} < T$ and to the kernel K by

$$K^{(\overline{T})}(s,x,y) = K(\overline{T} - s, x, y). \tag{4.18}$$

Then it is easy to see $T_t^K = T_{\overline{T}-t}^{K^{(\overline{T})}}$ and they share the *weak*-$(*)$-kernel condition with the same parameter set.

Lemma 4.2.5. *Let $w, \varphi \in L^\infty(0,\overline{T};(L^1\cap L^\infty)(\mathbb{R}^N))$ be two smooth solutions of* (4.1) *with $\overline{T} < \infty$ for each smooth initial data $w_0, \varphi_0 \in (L^1 \cap L^\infty)(\mathbb{R}^N)$ and for each associated kernels K and $K^{(\overline{T})}$, respectively. In addition, we assume $\varphi_0 \in \mathcal{U}_r \cap \mathcal{S}$ for some $r \in (0,1]$. Then, we have*

$$\int_{\mathbb{R}^3} w_0(x)\varphi(\overline{T},x)dx = \int_{\mathbb{R}^3} w(\overline{T},x)\varphi_0(x)dx.$$

Proof. Let $t \in [0, \overline{T}]$. Then, we have

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^3} w(t,x)\varphi(\overline{T}-t,x)dx \\
&= \int_{\mathbb{R}^3} (\partial_t w)(t,x)\varphi(\overline{T}-t,x)dx - \int_{\mathbb{R}^3} w(t,x)(\partial_t \varphi)(\overline{T}-t,x)dx \\
&= -\int_{\mathbb{R}^3} T_t^K(w(t,\cdot))(x)\varphi(\overline{T}-t,x)dx + \int_{\mathbb{R}^3} w(t,x)T_{\overline{T}-t}^{K(\overline{T})}(\varphi(\overline{T}-t,\cdot))(x)dx.
\end{aligned}$$

Then, we use Lemma 4.2.2 and the fact $T_t^K = T_{\overline{T}-t}^{K(\overline{T})}$ to get

$$\begin{aligned}
&= -\int_{\mathbb{R}^3} T_t^K(w(t,\cdot))(x)\varphi(\overline{T}-t,x)dx + \int_{\mathbb{R}^3} T_{\overline{T}-t}^{K(\overline{T})}(w(t,\cdot))(x)\varphi(\overline{T}-t,x)dx \\
&= -\int_{\mathbb{R}^3} T_t^K(w(t,\cdot))(x)\varphi(\overline{T}-t,x)dx + \int_{\mathbb{R}^3} T_t^K(w(t,\cdot))(x)\varphi(\overline{T}-t,x)dx = 0.
\end{aligned}$$

As a result, we conclude that $\int_{\mathbb{R}^3} w(t,x)\varphi(\overline{T}-t,x)dx$ is constant in t. Then put $t := 0$ and $t := \overline{T}$.

□

4.3 The main proposition and its proof

We are ready to present the main proposition about the evolution of test functions in a short time interval, whose length is proportional to r^α. Roughly speaking, if $\varphi_0 \in \mathcal{U}_r$, then there exist $z = z(r,s)$ and β such that $\varphi(s) \in \left(\frac{r}{z}\right)^\beta \mathcal{U}_z$ for $s \in [0, \delta r^\alpha]$.

Proposition 4.3.1. *There exist constants $A \geq 1$, $\delta > 0$, $L > 0$ and $\beta > 0$ with the following property:*

Let $0 < r \leq 1$ and $\varphi_0 \in \mathcal{U}_r \cap \mathcal{S}$. Then, there exist a smooth solution $\varphi \in L^\infty(0,T;(L^1 \cap L^\infty)(\mathbb{R}^N))$ of (4.1) with the initial condition $\varphi(0) = \varphi_0$. Also, *for any* $s \in [0, \min\{\delta r^{\alpha}, T\}]$*, we have*

$$\varphi(s) \in \Big(\frac{r}{z(r,s)}\Big)^{\beta} \mathcal{U}_{z(r,s)} \tag{4.19}$$

where $z(r,s)$ *is defined by* $z(r,s) = r(1 + L\frac{s}{r^{\alpha}})$.

Moreover, if $r = 1$*, then*

$$\varphi(s) \in (1 + Ls)\mathcal{U}_1 \quad \text{for any } s \in [0, T). \tag{4.20}$$

Proof. Let $\varphi_0 \in \mathcal{U}_r \cap \mathcal{S}$ for some $0 < r \leq 1$. Then there exists a weak solution φ corresponding to the initial data φ_0 (this can be proved by following [51]. Or refer to the approximation scheme in [10]). Moreover this solution is smooth, and it lies in $L^{\infty}_t(H^b_x)$ for every integer $b \geq 0$ due to the smoothness assumption (4.10) of k (it can be proved by using a standard energy argument).

First we state the following elementary inequalities without proof.

Lemma 4.3.2. *(I).* $(1 - x) \leq \frac{1}{1+x}$ *for* $x \geq 0$.

(II). $(1 + \frac{\eta}{2}x) \leq (1 + x)^{\eta}$ *for any* $0 \leq x \leq 1$ *if* $0 \leq \eta \leq 1$.

(III). $(1 + x)^{\eta} \leq (1 + \eta x)$ *for any* $x \geq 0$ *if* $0 \leq \eta \leq 1$.

(IV). $(1 + x)^{\eta} \leq (1 + 2\eta x)$ *for any* $0 < x < C_{\eta}$ *if* $\eta \geq 1$.

In order to obtain (4.19), we need to verify the mean zero, the concentration, the L^{∞}, and the L^1 conditions. First the mean-zero condition is easily verified in **STEP 1**. Second, we derive some estimates for remained three other conditions in **STEP 2-4**. Then, in **STEP 5** , we combine all the estimates we obtained in **STEP 2-4** to finish the proof. Without loss of generality, we can assume that a center of

φ_0 is the origin (i.e. $x_0 = 0$).

STEP 1. Mean zero-condition.

From (II) of Lemma 4.2.2, we have, for any $t \in (0, T)$,

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}^N} \varphi(t,x)dx &= \int_{\mathbb{R}^N} \Big(\frac{\partial}{\partial t}\varphi\Big)(t,x)dx \\ &= -\int_{\mathbb{R}^N} T_t^K(\varphi(t,\cdot))(x)dx = 0. \end{aligned}$$

STEP 2. Concentration-condition.

Lemma 4.3.3. *There exists a constant $C_{conc} > 0$ such that, for any $s \in (0, T)$, we have*

$$\int_{\mathbb{R}^N} |\varphi(s,x)||x|^\gamma dx \le r^\gamma (1 + C_{conc} A^{\frac{\alpha-\gamma}{N}} \frac{s}{r^\alpha}) \tag{4.21}$$

where C_{conc} does not depend on A as long as $A \ge 1$.

Remark 4.3.1. This lemma says that test functions lose their concentration with certain rate as time goes on. In **STEP 5**, it will be shown that the rate can be absorbed into the regularization effect from the L^1 and the L^∞ conditions.

Proof.

$$\begin{aligned} \frac{d}{ds}\int_{\mathbb{R}^N} |\varphi(s,x)||x|^\gamma dx &= \int_{\mathbb{R}^N} \frac{\partial}{\partial s}\Big(|\varphi(s,x)|\Big)|x|^\gamma dx \\ &\le \int_{\mathbb{R}^N} -T_s\Big(|\varphi(s,x)|\Big)\cdot |x|^\gamma dx \\ &= \int_{\mathbb{R}^N} -T_s\Big(|x|^\gamma\Big)\cdot |\varphi(s,x)|dx \\ &\le \int_{\mathbb{R}^N} \Big|T_s\Big(|x|^\gamma\Big)\Big|\cdot |\varphi(s,x)|dx = (I) \end{aligned}$$

where we used Lemma 4.2.3 and Lemma 4.2.2.

First, consider the case $\alpha < 1$. Then, thanks to Lemma 4.2.1, we have

$$\begin{aligned}(I) \le& C \int_{\mathbb{R}^N} |x|^{\gamma-\alpha} \cdot (1 + |x|^{\omega}) \cdot |\varphi(s,x)| dx \\ =& C\Big(\int_{B(A^{-1/N} r)} |x|^{\gamma-\alpha} \cdot (1 + |x|^{\omega}) \cdot |\varphi(s,x)| dx \\ &+ \int_{B(A^{-1/N} r)^C} |x|^{\gamma-\alpha} \cdot |\varphi(s,x)| dx \\ &+ \int_{B(A^{-1/N} r)^C} |x|^{\gamma-\alpha+\omega} \cdot |\varphi(s,x)| dx \Big).\end{aligned}$$

From the condition $\gamma < (\alpha - \omega)$, we have that the functions $|\cdot|^{\gamma-\alpha}$ and $|\cdot|^{\gamma-\alpha+\omega}$ are decreasing. Also, note that L^∞ and L^1 norms of φ are decreasing from Lemma 4.2.3 and $A^{-1/N} \cdot r \le 1$ from $A \ge 1$ and $r \le 1$. Thus we have

$$\begin{aligned}\le& C\Big((A^{-1/N} r)^{\gamma-\alpha+N} \cdot (1 + (A^{-1/N} r)^{\omega}) \cdot \|\varphi(s)\|_{L^\infty} \\ &+ (A^{-1/N} r)^{\gamma-\alpha} \|\varphi(s)\|_{L^1} + (A^{-1/N} r)^{\gamma-\alpha+\omega} \|\varphi(s)\|_{L^1} \Big) \\ \le& C A^{\frac{\alpha-\gamma}{N}} r^{\gamma-\alpha}.\end{aligned}$$

Likewise, for the case $\alpha \ge 1$, Lemma 4.2.1 with $\gamma < \alpha - (\nu + \omega)$ gives us the same conclusion. Then, we have (4.21) thanks to the initial condition $\int_{\mathbb{R}^N} |\varphi(0,x)||x|^{\gamma} dx \le r^{\gamma}$. □

STEP 3. L^∞-condition.

Lemma 4.3.4. *There exist two constants $\delta_{L^\infty} > 0$ and $C_{L^\infty} > 0$ such that, for any $s \in [0, \min\{\delta_{L^\infty} r^{\alpha}, T\}]$, we have*

$$\|\varphi(s,\cdot)\|_{L^\infty(\mathbb{R}^N)} \le \frac{A}{r^N}(1 - C_{L^\infty} A^{\frac{\alpha}{N}} \frac{s}{r^{\alpha}}) \tag{4.22}$$

where C_{L^∞} does not depend on A as long as $A \ge 1$.

Remark 4.3.2. This lemma is proved by using the lower bound of (4.3), which gives us some regularization effect. We follow a similar argument of Theorem 4.1 in the paper [28], which showed a L^∞ decay for smooth solutions of the 2D surface QG equation.

Proof. First, we define $M(t) := \|\varphi(t,\cdot)\|_{L^\infty}$. We claim that there exist $\delta_1 > 0$ and $C_1 > 0$ such that for any $t \in [0, \delta_1 r^\alpha]$ satisfying $M(t) \geq \frac{1}{2}\frac{A}{r^N}$, we have

$$M(t) \leq \frac{A}{r^N}(1 - C_1 A^{\frac{\alpha}{N}} \frac{t}{r^\alpha}). \tag{4.23}$$

To prove the above claim (4.23), first pick any $t \in (0, T)$ such that

$$M(t) \geq \frac{1}{2} \cdot \frac{A}{r^N}.$$

Then we know $M(t^*) \geq \frac{1}{2}\frac{A}{r^N}$ for all $t^* < t$ from Lemma 4.2.3. It can be easily proved that there exists a point x_t such that $|\varphi(t, x_t)| = M(t)$. Indeed, because our kernel lies in $C^\infty_{t,x,y}\Big([0,T] \times \overline{\mathbb{R}^N} \times \overline{\mathbb{R}^N})\Big)$ with $\varphi_0 \in \mathcal{S}$, we can show $\varphi \in L^\infty_t H^d$ for every integer $d \geq 0$ by standard energy estimates. In particular, $\varphi(t,\cdot) \in H^b$ for some integer $b > (N/2)$ for every time. Then, $\varphi(t,\cdot)$ vanishes at the infinity thanks to a Fourier transform argument. Since $\varphi(t,\cdot)$ is continuous, there exists a maximum (or minimum) point.

Then, for almost every time $t \in (0, T)$, there exist a point $\tilde{x}_t$ such that $|\varphi(t, \tilde{x}_t)| = M(t)$ with the following inequality:

$$\frac{d}{dt}M(t) \leq \begin{cases} \Big(\frac{\partial\varphi}{\partial t}\Big)(t, \tilde{x}_t) & \text{if } \varphi(t, \tilde{x}_t) = M(t), \\ -\Big(\frac{\partial\varphi}{\partial t}\Big)(t, \tilde{x}_t) & \text{if } \varphi(t, \tilde{x}_t) = -M(t) \end{cases}$$

(this can be proved by following the argument of [28]).

We assume the first case $\varphi(t, \tilde{x}_t) = M(t) > 0$ (the other one can be dealt in similar fashion). Then

$$\begin{aligned}\frac{d}{dt}M(t) &\leq \Big(\frac{\partial\varphi}{\partial t}\Big)(t, \tilde{x}_t) \leq -T_t^K(\varphi(t,\cdot))(x) \\ &= -\int_{\mathbb{R}^N} \Big(\varphi(t, \tilde{x}_t) - \varphi(t, y)\Big) K(t, \tilde{x}_t, y) dy \\ &\leq -\Lambda^{-1} \int_{|\tilde{x}_t - y| \leq \zeta} \frac{\varphi(t, \tilde{x}_t) - \varphi(t, y)}{|\tilde{x}_t - y|^{N+\alpha}} dy = -\Lambda^{-1} \cdot (*).\end{aligned} \tag{4.24}$$

We used $K \geq 0$ and $\varphi(t, \tilde{x}_t) - \varphi(t, y) \geq 0$ with the lower bound of the kernel (4.3).

Let R be any number between 0 and ζ, which will be chosen soon. We separate the ball $B_R(\tilde{x}_t)$ into two disjoint regions Ω_1 and Ω_2 by the following way: $\Big(\varphi(t, \tilde{x}_t) - \varphi(t, y)\Big) > \frac{1}{2}\varphi(t, \tilde{x}_t)$ implies $y \in \Omega_1$. Otherwise, $y \in \Omega_2$. Then we have the following upper bound of measure of Ω_2:

$$|\Omega_2| = \frac{2}{M(t)} \int_{\Omega_2} \frac{M(t)}{2} dy \leq \frac{2}{M(t)} \int_{\Omega_2} \varphi(t, y) dy \leq \frac{2}{M(t)} \|\varphi(t, \cdot)\|_{L^1(\mathbb{R}^N)} \leq \frac{2}{M(t)}.$$

As a result, from $R < \zeta$, we have

$$\begin{aligned}(*) &\geq \int_{|\tilde{x}_t - y| \leq R} \frac{\varphi(t, \tilde{x}_t) - \varphi(t, y)}{|\tilde{x}_t - y|^{N+\alpha}} dy \geq \int_{\Omega_1} \frac{\varphi(t, \tilde{x}_t) - \varphi(t, y)}{|\tilde{x}_t - y|^{N+\alpha}} dy \\ &\geq \frac{M(t)}{2R^{N+\alpha}} |\Omega_1| = \frac{M(t)}{2R^{N+\alpha}} (|B_R(\tilde{x}_t)| - |\Omega_2|) \geq \frac{M(t)}{2R^{N+\alpha}} (V_N R^N - \frac{2}{M(t)}).\end{aligned}$$

Now we choose R by $V_N R^N = \frac{2}{M(t)} \cdot 2$. Then, it is clear that $R \leq \zeta$ because $M(t) \geq \frac{1}{2}\frac{A}{r^N} \geq \frac{1}{2}\frac{1}{r^N} \geq \frac{1}{2}$ and $R = \Big(\frac{4}{M(t)V_N}\Big)^{1/N} \leq \Big(\frac{8}{V_N}\Big)^{1/N} \leq \zeta$ by (4.14). Coming back to (4.24), we have

$$\frac{d}{dt}M(t) \leq -\Lambda^{-1}\frac{M(t)}{2R^{N+\alpha}}(V_N R^N - \frac{2}{M(t)}) = -\Lambda^{-1}\frac{M(t)}{2R^{N+\alpha}} \cdot \frac{2}{M(t)} = -\Lambda^{-1}\frac{1}{R^{N+\alpha}}$$
$$= -\Lambda^{-1}\Big(\frac{M(t)V_N}{4}\Big)^{(N+\alpha)/N} = -C \cdot M(t)^{1+\frac{\alpha}{N}}.$$

Solving this differential inequality, we obtain

$$M(t) \leq M(0)\Big(1 + C \cdot M(0)^{\alpha/N} \cdot t\Big)^{-N/\alpha}$$

From the fact $\frac{1}{2}\frac{A}{r^N} \leq M(t) \leq M(0) \leq \frac{A}{r^N}$, we have

$$\leq \frac{A}{r^N}\Big(1 + C \cdot \Big(\frac{1}{2} \cdot \frac{A}{r^N}\Big)^{\alpha/N} \cdot t\Big)^{-N/\alpha}$$

For any $p > 0$, it is easy to see $(1+x)^{-p} \leq (1 - \frac{1}{2}px)$ for $0 \leq x \leq C_p$. Thus, we have

$$\leq \frac{A}{r^N}\Big(1 - C_1 \cdot A^{\alpha/N} \cdot \frac{t}{r^\alpha}\Big)$$

as long as $t \leq C_1^{-1}A^{-\alpha/N}r^\alpha$. By taking $\delta_1 := C_1^{-1}A^{-\alpha/N}$, we proved the claim (4.23) under the assumption $M(t) \geq \frac{1}{2}\frac{A}{r^N}$.

Thanks to (4.23), the whole case (4.22) can be achieved easily by taking $\delta_{L^\infty} := \frac{1}{2}\delta_1$ and $C_{L^\infty} := C_1$. Indeed, if $M(t) \leq \frac{1}{2}\frac{A}{r^N}$, then we have

$$M(t) \leq \frac{1}{2}\frac{A}{r^N} \leq \frac{A}{r^N}(1 - C_1 A^{\frac{\alpha}{N}}\frac{t}{r^\alpha})$$

as long as $t \leq \frac{1}{2}C_1^{-1}A^{-\alpha/N}r^\alpha = \frac{1}{2}\delta_1 r^\alpha$.

□

STEP 4. L^1-condition.

Lemma 4.3.5. *There exist two constants* $\delta_{L^1} > 0$ *and* $C_{L^1} > 0$ *such that, for any* $s \in [0, \min\{\delta_{L^1} r^\alpha, T\}]$, *we have*

$$\|\varphi(s,\cdot)\|_{L^1(\mathbb{R}^N)} \le (1 - C_{L^1} \cdot \frac{s}{r^\alpha})$$

where C_{L^1} *does not depend on* A *as long as* $A \ge 1$.

Remark 4.3.3. In this time, we obtain L^1 decay by using the lower bound of the kernel (4.3). In general, without the mean zero property, we do not expect L^1 decay (refer to [28]). However, with the mean zero property, we can manage certain amount of cancellation of the L^1-norm. This idea comes from the argument in [48] where L^1 decay for mean-zero solutions for the 2D-SQG equation in a periodic setting was obtained.

Proof. First, by using (4.21), we can find δ_2 such that $\int_{\mathbb{R}^N} |\varphi(s,x)||x|^\gamma dx \le \frac{11}{10} r^\gamma$ for all $t \in [0, \delta_2 r^\alpha]$. i.e. we take δ_2 so small that $C_{conc} A^{\frac{\alpha-\gamma}{N}} \delta_2 \le \frac{1}{10}$.

We claim that there exists a constant $C_2 > 0$ such that for any $t \in [0, \delta_2 r^\alpha]$ satisfying $\|\varphi(t)\|_{L^1(\mathbb{R}^N)} \ge \frac{9}{10}$, we have

$$\|\varphi(t,\cdot)\|_{L^1(\mathbb{R}^N)} \le (1 - C_2 \cdot \frac{t}{r^\alpha}). \tag{4.25}$$

To prove (4.25), let $t \in [0, \delta_2 r^\alpha]$ satisfy $\|\varphi(t,\cdot)\|_{L^1(\mathbb{R}^N)} \ge \frac{9}{10}$. For simplicity, we define $a := (11)^{1/\gamma}$. Then, from (4.14), we know

$$2a \le \zeta \tag{4.26}$$

and the following estimates hold:

$$\int_{B_{ar}} |\varphi(t,x)|dx \geq \frac{8}{10}, \int_{B_{ar}} \varphi^+(t,x)dx \geq \frac{3}{10}, \text{ and } \int_{B_{ar}} \varphi^-(t,x)dx \geq \frac{3}{10}. \quad (4.27)$$

where $f^+ := \max\{f, 0\}$ and $f^- := \max\{-f, 0\}$.

Indeed, from the concentration condition, we obtain the following upper bound of L^1-norm of φ outside of the ball B_{ar}:

$$\int_{(B_{ar})^C} |\varphi(t,x)|dx = \int_{(B_{ar})^C} |\varphi(t,x)||x|^{\gamma}|x|^{-\gamma}dx \leq \frac{11}{10} r^{\gamma}(ar)^{-\gamma} = \frac{1}{10}.$$

Then, thanks to the mean-zero property, we get the following lower bounds of $\|\varphi\|_{L^1(B_{ar})}$, $\|\varphi^+\|_{L^1(B_{ar})}$ and $\|\varphi^-\|_{L^1(B_{ar})}$:

$$\int_{B_{ar}} |\varphi(t,x)|dx = \|\varphi(t,\cdot)\|_{L^1(\mathbb{R}^N)} - \int_{(B_{ar})^C} |\varphi(t,x)|dx \geq \frac{9}{10} - \frac{1}{10} = \frac{8}{10},$$

$$\begin{aligned}\int_{B_{ar}} \varphi^{\pm}(t,x)dx &= \int_{\mathbb{R}^N} \varphi^{\mp}(t,x)dx - \int_{(B_{ar})^C} \varphi(t,x)^{\pm}dx \\ &= \frac{1}{2}\|\varphi(t,\cdot)\|_{L^1(\mathbb{R}^N)} - \int_{(B_{ar})^C} \varphi(t,x)^{\pm}dx \\ &\geq \frac{1}{2}\|\varphi(t,\cdot)\|_{L^1(\mathbb{R}^N)} - \int_{(B_{ar})^C} |\varphi(t,x)|dx \geq \frac{9}{20} - \frac{1}{10} > \frac{3}{10}.\end{aligned}$$

We denote symbols D^s_+, D^s_- and S^s by

$$D^s_{\pm} = \{x \in \mathbb{R}^N | \quad \pm\varphi(s,x) \geq 0\} \text{ and}$$

$$S^s = \{x \in \mathbb{R}^N | \quad \varphi(s,x) = 0\}.$$

Then, we have

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^N} |\varphi(s,x)|dx = \int_{\mathbb{R}^N} \frac{\partial}{\partial t}|\varphi(s,x)|dx = \int_{\mathbb{R}^N} \Big(1_{D^s_+}(x) - 1_{D^s_-}(x)\Big)(\partial_t\varphi)(s,x)dx \\ &= -\int_{\mathbb{R}^N} \Big(1_{D^s_+}(x) - 1_{D^s_-}(x)\Big)T^K_s(\varphi(s,\cdot))(x)dx \\ &= -\int_{\mathbb{R}^N} \Big(1_{D^s_+}(x) - 1_{D^s_-}(x)\Big)\int_{\mathbb{R}^N}[\varphi(s,x) - \varphi(s,y)]K(s,x,y)dydx \\ &= -\frac{1}{2}\iint \underbrace{\Big[\Big(1_{D^s_+}(x) - 1_{D^s_-}(x)\Big) - \Big(1_{D^s_+}(y) - 1_{D^s_-}(y)\Big)\Big]\Big(\varphi(s,x) - \varphi(s,y)\Big)\cdot K}_{(*)}\, dydx.\end{aligned}$$

Then, we split the above integral into 9 components:

$$
\begin{aligned}
&= -\frac{1}{2}\Big[\int_{D_+^s}\int_{D_+^s}(*)dydx + \int_{D_+^s}\int_{S^s}(*)dydx + \int_{D_+^s}\int_{D_-^s}(*)dydx \\
&\quad + \int_{S^s}\int_{D_+^s}(*)dydx + \int_{S^s}\int_{S^s}(*)dydx + \int_{S^s}\int_{D_-^s}(*)dydx \\
&\quad + \int_{D_-^s}\int_{D_+^s}(*)dydx + \int_{D_-^s}\int_{S^s}(*)dydx + \int_{D_-^s}\int_{D_-^s}(*)dydx\Big] \\
&= -\frac{1}{2}\Big[(I) + (II) + \cdots + (IX)\Big].
\end{aligned}
$$

We will prove the inequality: $\frac{1}{2}\Big[(I) + (II) + \cdots + (IX)\Big] \geq C_2 \cdot r^{-\alpha}$, which will imply the claim (4.25) later. First, we observe that $(I) = (V) = (IX) = 0$ by the definition of $(*)$.

Second, we have $(II) = (IV)$ by symmetry of the kernel. Indeed,

$$
\begin{aligned}
(II) &= \int_{D_+^s}\int_{S^s}(*)dydx = \int_{D_+^s}\int_{S^s}\varphi(s,x)\cdot K(s,x,y)dydx \\
&= \int_{S^s}\int_{D_+^s}\varphi(s,x)\cdot K(s,x,y)dxdy = \int_{S^s}\int_{D_+^s}\varphi(s,y)\cdot K(s,y,x)dydx \\
&= \int_{S^s}\int_{D_+^s}\varphi(s,y)\cdot K(s,x,y)dydx = (IV).
\end{aligned}
$$

Likewise, we have $(VI) = (VIII)$ and $(III) = (VII)$. Thus, we have

$$
\begin{aligned}
&\Big[(I) + (II) + \cdots + (IX)\Big] = 2\Big[(IV) + (VI) + (III)\Big] \\
&= 2\Big[\int_{S^s}\int_{D_+^s}(*)dydx + \int_{S^s}\int_{D_-^s}(*)dydx + \int_{D_+^s}\int_{D_-^s}(*)dydx\Big] \\
&= 2\Big[\int_{S^s}\int_{D_+^s\cup D_-^s}(*)dydx + \int_{D_+^s}\int_{D_-^s}(*)dydx\Big] \\
&= 2\Big[\int_{S^s}\int_{\mathbb{R}^N}(*)dydx + \int_{D_+^s}\int_{D_-^s}(*)dydx\Big] = 2[(D) + (E)]
\end{aligned}
$$

where the third equality follows $\int_{S^s}\int_{S^s}(*)dydx = (V) = 0$.

In order to use the lower bound of the kernel (4.3), we need to restrict the above integral on a subset of $\{|x-y| \le \zeta\}$. For this purpose, we define the subsets $\tilde{D}^s_+, \tilde{D}^s_-$ and $\tilde{S}^s$ by $\tilde{D}^s_\pm = D^s_\pm \cap B_{ar}$ and $\tilde{S}^s = S_s \cap B_{ar}$. Then, if $x, y \in B_{ar}$, then $|x-y| \le 2ar \le 2a \le \zeta$ from (4.26). Thus, from the lower bound of the kernel (4.3), we have

$$
\begin{aligned}
(D) &= \int_{S^s}\int_{\mathbb{R}^N}(*)dydx \\
&= \int_{S^s}\int_{\mathbb{R}^N}\Big[(0-0) - \Big(1_{D^s_+}(y) - 1_{D^s_-}(y)\Big)\Big]\Big(-\varphi(s,y)\Big)\cdot K(s,x,y)dydx \\
&= \int_{S^s}\int_{\mathbb{R}^N}|\varphi(s,y)|\cdot K(s,x,y)dydx \\
&\ge \Lambda^{-1}\int_{\tilde{S}^s}\int_{B_{ar}}\frac{|\varphi(s,y)|}{|x-y|^{N+\alpha}}dydx \ge \Lambda^{-1}(2ar)^{-(N+\alpha)}\int_{\tilde{S}^s}\|\varphi(s,\cdot)\|_{L^1(B_{ar})}dx \\
&= \Lambda^{-1}\cdot(2ar)^{-(N+\alpha)}\cdot|\tilde{S}^s|\cdot\|\varphi(s,\cdot)\|_{L^1(B_{ar})} \ge \Lambda^{-1}\cdot(2ar)^{-(N+\alpha)}\cdot|\tilde{S}^s|\cdot\frac{8}{10}
\end{aligned}
$$

where, for the last equality, the estimate (4.27) was used.

Also, we have

$$
\begin{aligned}
(E) &= \int_{D^s_+}\int_{D^s_-}(*)dydx \\
&= \int_{D^s_+}\int_{D^s_-}\Big[\Big(1_{D^s_+}(x) - 0\Big) - \Big(0 - 1_{D^s_-}(y)\Big)\Big]\Big(\varphi(s,x) - \varphi(s,y)\Big)\cdot K(s,x,y)dydx \\
&= 2\int_{D^s_+}\int_{D^s_-}\Big(\varphi(s,x) - \varphi(s,y)\Big)\cdot K(s,x,y)dydx \\
&= 2\Big[\int_{D^s_+}\int_{D^s_-}\varphi(s,x)\cdot K(s,x,y)dydx + \int_{D^s_+}\int_{D^s_-}-\varphi(s,y)\cdot K(s,x,y)dydx\Big] \\
&= 2[(E_1) + (E_2)].
\end{aligned}
$$

We can obtain the following lower bound of (E_1) by the following way:

$$
\begin{aligned}
(E_1) &= \int_{D_+^s}\int_{D_-^s} \varphi(s,x)\cdot K(s,x,y)dydx = \int_{D_-^s}\int_{D_+^s} \varphi(s,x)\cdot K(s,x,y)dxdy \\
&\geq \Lambda^{-1}\int_{\tilde{D}_-^s}\int_{\tilde{D}_+^s} \frac{\varphi(s,x)}{|x-y|^{N+\alpha}}\cdot dxdy \geq \Lambda^{-1}(2ar)^{-(N+\alpha)}\int_{\tilde{D}_-^s} \|\varphi^+(s,\cdot)\|_{L^1(B_{ar})}dy \\
&\geq \Lambda^{-1}(2ar)^{-(N+\alpha)}\cdot|\tilde{D}_-^s|\cdot\frac{3}{10}.
\end{aligned}
$$

Likewise, for (E_2), we have $(E_2) \geq \Lambda^{-1}\cdot(2ar)^{-(N+\alpha)}\cdot|\tilde{D}_+^s|\cdot\frac{3}{10}$.

Now we have a desirable estimate for $\Big[(I)+(II)+\cdots+(IX)\Big]$:

$$
\begin{aligned}
&\frac{1}{2}\Big[(I)+(II)+\cdots+(IX)\Big] = \Big[(IV)+(VI)+(III)\Big] = [(D)+(E)] \\
&\geq \Big[\Lambda^{-1}\cdot(2ar)^{-(N+\alpha)}\cdot|\tilde{S}^s|\cdot\frac{8}{10} \\
&\qquad + 2\Big(\Lambda^{-1}\cdot(2ar)^{-(N+\alpha)}\cdot|\tilde{D}_-^s|\cdot\frac{3}{10} + \Lambda^{-1}\cdot(2ar)^{-(N+\alpha)}\cdot|\tilde{D}_+^s|\cdot\frac{3}{10}\Big)\Big] \\
&\geq \Lambda^{-1}\cdot(2ar)^{-(N+\alpha)}\cdot\frac{6}{10}\cdot\Big(|\tilde{S}^s|+|\tilde{D}_-^s|+|\tilde{D}_+^s|\Big) \\
&= \Lambda^{-1}\cdot(2ar)^{-(N+\alpha)}\cdot\frac{6}{10}\cdot\Big((ar)^N\cdot V_N\Big) = C_2\cdot r^{-\alpha}.
\end{aligned}
$$

It proves the claim (4.25), under the assumption $\|\varphi(t)\|_{L^1(\mathbb{R}^N)} \geq \frac{9}{10}$, because

$$
\begin{aligned}
\|\varphi(t)\|_{L^1(\mathbb{R}^N)} &= \|\varphi(0)\|_{L^1(\mathbb{R}^N)} + \int_0^t \frac{d}{ds}\|\varphi(s,\cdot)\|_{L^1(\mathbb{R}^N)}ds \\
&\leq 1 + C_2\cdot r^{-\alpha}\cdot t \quad \text{ for any } t\in[0,\delta_2 r^\alpha].
\end{aligned}
$$

On the other hand, if $\|\varphi(t)\|_{L^1(\mathbb{R}^N)} \leq \frac{9}{10}$, then we have

$$
\|\varphi(t)\|_{L^1(\mathbb{R}^N)} \leq \frac{9}{10} \leq (1 - C_2\cdot\frac{t}{r^\alpha})
$$

as long as $t \leq \frac{1}{10}C_2^{-1}r^\alpha$. Therefore, by taking $\delta_{L^1} := \min\{\delta_2, \frac{1}{10}C_2^{-1}\}$ and $C_{L^1} := C_2$, we finish the proof of Lemma 4.3.5.

□

STEP 5. Combining all conditions.

Now we are ready to finish the proof of the main proposition 4.3.1. In **STEP 2-4**, we proved that

$$\int_{\mathbb{R}^N} |\varphi(s,x)||x|^\gamma dx \le r^\gamma(1 + C_{conc}A^{\frac{\alpha-\gamma}{N}}\frac{s}{r^\alpha}) \quad \text{for } s \in [0,T), \tag{4.28}$$

$$\|\varphi(s,\cdot)\|_{L^\infty(\mathbb{R}^N)} \le \frac{A}{r^N}(1 - C_{L^\infty}A^{\frac{\alpha}{N}}\frac{s}{r^\alpha}) \quad \text{for } s \in [0, \min\{\delta_{L^\infty}\cdot r^\alpha, T\}], \text{ and} \tag{4.29}$$

$$\|\varphi(s,\cdot)\|_{L^1(\mathbb{R}^N)} \le (1 - C_{L^1}\cdot\frac{s}{r^\alpha}) \quad \text{for } s \in [0, \min\{\delta_{L^1}\cdot r^\alpha, T\}]. \tag{4.30}$$

Note that the constants C_{L^1}, C_{L^∞}, and C_{conc} are independent of A as long as $A \ge 1$ while δ_{L^1} and δ_{L^∞} depend on A. We define $\delta_3 := \min\{\delta_{L^1}, \delta_{L^\infty}\}$ so that the above three estimates (4.28), (4.29), and (4.30) hold at the same time for all $s \in [0, \min\{\delta_3 r^\alpha, T\}]$. Without loss of generality, we can assume $C_{L^1} = C_{L^\infty} \le C_{conc}$.

Recall that we are looking for $\beta > 0$ and $z(r,s)$ such that $\varphi(s) \in \left(\frac{r}{z}\right)^\beta \mathcal{U}_z$. Thus, from Definition 4.2.1 of $\mathcal{U}_r$ and from the above three estimates (4.28), (4.29), and (4.30), we need the followings:

$$\begin{aligned} r^\gamma(1 + C_{conc}A^{\frac{\alpha-\gamma}{N}}\frac{s}{r^\alpha}) &\le \left(\frac{r}{z}\right)^\beta z^\gamma, \\ \frac{A}{r^N}(1 - C_{L^\infty}A^{\frac{\alpha}{N}}\frac{s}{r^\alpha}) &\le \left(\frac{r}{z}\right)^\beta \cdot \frac{A}{z^N}, \text{ and} \\ (1 - C_{L^1}\cdot\frac{s}{r^\alpha}) &\le \left(\frac{r}{z}\right)^\beta. \end{aligned} \tag{4.31}$$

Remark 4.3.4. (4.31) is equivalent to

$$r(1 + C_{conc}A^{\frac{\alpha-\gamma}{N}}\frac{s}{r^{\alpha}})^{1/(\gamma-\beta)} \le z, \tag{4.32}$$

$$z \le r(1 - C_{L^\infty}A^{\frac{\alpha}{N}}\frac{s}{r^{\alpha}})^{-1/(N+\beta)}, \text{ and} \tag{4.33}$$

$$z \le r(1 - C_{L^1} \cdot \frac{s}{r^{\alpha}})^{-1/\beta}. \tag{4.34}$$

The power of A in (4.32) is strictly less than that of A in (4.33). This fact is crucial because we can choose A large enough to hold (4.32) and (4.33) at the same time. Then we can make β small enough to hold (4.34), too. We will now give all the details.

We take any $A \ge 1$ large enough to satisfy the inequality:

$$\frac{8}{\gamma}(N + (1/2)) \cdot C_{conc}A^{\frac{\alpha-\gamma}{N}} \le C_{L^\infty}A^{\frac{\alpha}{N}}.$$

In addition, we take any $\beta \in (0, \gamma/2]$ so small that the following inequality holds:

$$\frac{4}{\gamma}\beta \cdot C_{conc}A^{\frac{\alpha-\gamma}{N}} \le C_{L^1}.$$

Finally, we define a constant L by

$$L := \frac{2}{\gamma - \beta} \cdot C_{conc}A^{\frac{\alpha-\gamma}{N}}$$

and a function $z(r, s)$ by

$$z(r, s) := r(1 + L\frac{s}{r^{\alpha}}).$$

For the Concentration-condition, from (II) of Lemma 4.3.2, we have

$$\begin{aligned}\int_{\mathbb{R}^N} |\varphi(s,x)||x|^\gamma dx &\le r^\gamma (1 + C_{conc} A^{\frac{\alpha-\gamma}{N}} \frac{s}{r^\alpha}) = r^\gamma \Big[1 + \frac{\gamma-\beta}{2} \cdot L \cdot \frac{s}{r^\alpha}\Big] \\ &\le r^\gamma \Big[1 + L \cdot \frac{s}{r^\alpha}\Big]^{\gamma-\beta} = \left(\frac{r}{z}\right)^\beta \cdot z^\gamma\end{aligned}$$

where the last inequality holds as long as for $s \le (1/L) r^\alpha$. We define $\delta_4 := \min\{\delta_3, (1/L)\}$.

On the other hand, from $0 < \beta \le \gamma/2 < 1/2$, we observe the followings:

$$2 \cdot (N+\beta) \cdot L \le \frac{8}{\gamma}(N + (1/2)) \cdot C_{conc} A^{\frac{\alpha-\gamma}{N}} \le C_{L^\infty} A^{\frac{\alpha}{N}} \quad \text{and} \tag{4.35}$$

$$\beta \cdot L \le \frac{4}{\gamma} \beta \cdot C_{conc} A^{\frac{\alpha-\gamma}{N}} \le C_{L^1}. \tag{4.36}$$

For the L^∞-condition, from (4.35) and from (I) and (IV) of Lemma 4.3.2, we have

$$\begin{aligned}\|\varphi(s,\cdot)\|_{L^\infty(\mathbb{R}^N)} &\le \frac{A}{r^N}(1 - C_{L^\infty} A^{\frac{\alpha}{N}} \frac{s}{r^\alpha}) \le \frac{A}{r^N} \cdot \Big[1 - 2\cdot(N+\beta) \cdot L \cdot \frac{s}{r^\alpha}\Big] \\ &\le \frac{A}{r^N} \cdot \Big[1 + 2 \cdot (N+\beta) \cdot L \cdot \frac{s}{r^\alpha}\Big]^{-1} \le \frac{A}{r^N} \cdot \Big[1 + L \cdot \frac{s}{r^\alpha}\Big]^{-(N+\beta)} = \left(\frac{r}{z}\right)^\beta \cdot \frac{A}{z^N}\end{aligned}$$

for $s \in [0, \min\{\delta_5 r^\alpha, T\}]$ where $\delta_5 := \min\{\delta_4, (1/L) \cdot C\}$.

For the L^1-condition, from (4.36) and (I) and (III) of Lemma 4.3.2, we have

$$\|\varphi(s,\cdot)\|_{L^1(\mathbb{R}^N)} \le (1 - C_{L^1} \frac{s}{r^\alpha}) \le 1 - \beta \cdot L \cdot \frac{s}{r^\alpha} = \left(\frac{r}{z}\right)^\beta$$

for $s \in [0, \min\{\delta_5 r^\alpha, T\}]$.

Together with the mean zero property of φ in **STEP 1**, we proved for any $s \in [0, \min\{\delta_5 r^\alpha, T\}]$ with $r \in (0,1]$ and for any $\varphi_0 \in \mathcal{U}_r$, we have the evolution estimate

$$\varphi(s) \in \Big(\frac{r}{r(1+L\frac{s}{r^\alpha})}\Big)^\beta \mathcal{U}_{r(1+L\frac{s}{r^\alpha})} = \Big(\frac{r}{z}\Big)^\beta \mathcal{U}_z.$$

which proves (4.19).

It remains to prove (4.20). Let $r = 1$ (i.e. $\varphi_0 \in \mathcal{U}_1$). Note that Lemma 4.3.3 holds for all time $s \in [0, T)$ and L^p norm is decreasing all time $s \in [0, T)$ and for any $1 \le p \le \infty$ from (II) of Lemma 4.2.3. Thus we have $\varphi(s) \in (1 + Ls)\mathcal{U}_1$ for all $s \in [0, T)$. This is the end of the proof of Proposition 4.3.1.

□

4.4 Proof of the part (II) of Theorem 4.1.1

Proof of the part (II) of Theorem 4.1.1. Let t be any time between 0 and T. Thanks to (II) of Lemma 4.2.4 and (II) of Lemma 4.2.3, the only thing we need to do is to find an estimate on $r^{-\beta}\Big|\int_{\mathbb{R}^3} w(t,x)\varphi_0(x)dx\Big|$ for $\varphi_0 \in \mathcal{S} \cap \mathcal{U}_r$ with $0 < r \le 1$. From Proposition 4.3.1, we have a smooth solution φ on $[0, t]$ correspoding to the initial data φ_0 with the kernel $K^{(t)}$, which is defined by $K^{(t)}(s) := K(t - s)$ (see the definition (4.18)). From Lemma 4.2.5, we want a control on $r^{-\beta}\Big|\int_{\mathbb{R}^3} w_0(x)\varphi(t,x)dx\Big|$.

Indeed,

$$
\begin{aligned}
\|w(t,\cdot)\|_{C^\beta(\mathbb{R}^N)} &\le C\cdot\Big(\|w(t,\cdot)\|_{L^\infty} + \sup_{\varphi_0\in\mathcal{S}\cap\mathcal{U}_r,0<r\le 1} r^{-\beta}\Big|\int_{\mathbb{R}^3} w(t,x)\varphi_0(x)dx\Big|\Big)\\
&\le C\cdot\Big(\|w_0\|_{L^\infty} + \sup_{\varphi_0\in\mathcal{S}\cap\mathcal{U}_r,0<r\le 1} r^{-\beta}\Big|\int_{\mathbb{R}^3} w_0(x)\varphi(t,x)dx\Big|\Big).
\end{aligned}
\tag{4.37}
$$

Remark 4.4.1. The main idea is to repeat (4.19) as many times as we want until the time evolution reaches the given time $t\in(0,T)$. For example, as long as $s_1\le\delta r^\alpha$, $s_2\le\delta z(r,s_1)^\alpha$, and $z(r,s_1)\le 1$, we can repeat Proposition 4.3.1 twice to get the following time evolution:

$$
\begin{aligned}
\varphi(0)\in\mathcal{U}_r \Rightarrow \varphi(s_1)\in\Big(\frac{r}{z(r,s_1)}\Big)^\beta\mathcal{U}_{z(r,s_1)}\\
\Rightarrow\varphi(s_1+s_2)\in\Big(\frac{r}{z(r,s_1)}\Big)^\beta\times\Big(\frac{z(r,s_1)}{z(z(r,s_1),s_2)}\Big)^\beta\mathcal{U}_{z(z(r,s_1),s_2)}\\
=\Big(\frac{r}{z(z(r,s_1),s_2)}\Big)^\beta\mathcal{U}_{z(z(r,s_1),s_2)}.
\end{aligned}
$$

However, when $z(r,s)$ reaches 1 before the given time t, then we cannot use (4.19) any more. Instead, we need to use (4.20), which grows as time increases. For this reason, we obtain only (4.38) first which depends on t. This defect is overcome by investigating the evolution of the L^1 norm of $\varphi(s)$ (see (4.39)). Since this examination requires a careful estimate (4.4) for repetitions of (4.19), we present a rigorous argument below. As a result, the final estimate is independent of the length of time interval (see (4.41)).

Define a constant $\eta := (1+L\cdot\delta)>1$. For each $r\in(0,1]$, we define the integer $k=k(r)\ge 1$ such that $r\cdot\eta^{k-1}\le 1<r\cdot\eta^k$. Also define $z_n=z_n(r)$ for

$n = 0, 1, 2, \cdots, k-1, k$ by

$$z_n = \begin{cases} r \cdot \eta^n & \text{if } 0 \le n \le (k-1), \\ 1 & \text{if } n = k. \end{cases}$$

Note that $r = z_0 < z_1 < \cdots < z_{k-1} \le 1 = z_k$.

We find $\tilde{t} = \tilde{t}(r) \in [0, \delta r^\alpha (\eta^\alpha)^{k-1}) = [0, \delta (z_{k-1})^\alpha)$ such that $z_{k-1}(1 + L\frac{\tilde{t}}{(z_{k-1})^\alpha}) = 1$, which is always possible because $z_{k-1} \le 1 < z_{k-1} \cdot (1 + L\frac{\delta(z_{k-1})^\alpha}{(z_{k-1})^\alpha}) = z_{k-1} \cdot \eta$.

Also define $t_n = t_n(r)$ for $n = 0, 1, 2, \cdots, k-1, k, k+1$ by

$$t_n = \begin{cases} \delta \cdot r^\alpha \Big(\frac{(\eta^\alpha)^n - 1}{\eta^\alpha - 1} \Big) & \text{if } 0 \le n \le (k-1), \\ (t_{k-1} + \tilde{t}) & \text{if } n = k, \\ \infty & \text{if } n = k+1. \end{cases}$$

Note that, for $1 \le n \le (k-1)$,

$$\begin{aligned} t_n &= \delta r^\alpha \Big(1 + \eta^\alpha + (\eta^\alpha)^2 + \cdots + (\eta^\alpha)^{n-1} \Big) \\ &= \delta r^\alpha + \delta r^\alpha \eta^\alpha + \delta r^\alpha (\eta^\alpha)^2 + \cdots + \delta r^\alpha (\eta^\alpha)^{n-1} \\ &= \delta (z_0)^\alpha + \delta (z_1)^\alpha + \cdots + \delta (z_{n-1})^\alpha \text{ and} \end{aligned}$$

$$t_n - t_{n-1} = \delta (z_{n-1})^\alpha.$$

Now we make a partition of $[0, \infty) \subset \mathbb{R}^1$ by

$$[0, \infty) = [t_0, t_1) \cup [t_1, t_2) \cup \cdots \cup [t_{k-2}, t_{k-1}) \cup [t_{k-1}, t_k) \cup [t_k, t_{k+1})$$

where these union are disjoint.

Finally, we are ready to apply the main proposition 4.3.1 as many times as we want. Indeed, if $t \in [t_n, t_{n+1})$ with $0 \le n \le (k-1)$, then we can repeat the main proposition 4.3.1 so that we obtain

$$\begin{aligned}\varphi(t) \in & \Big(\frac{r}{z_1}\Big)^{\beta} \times \Big(\frac{z_1}{z_2}\Big)^{\beta} \times \cdots \times \Big(\frac{z_n}{z_n(1+L\cdot\frac{t-t_n}{(z_n)^{\alpha}})}\Big)^{\beta} \mathcal{U}_{z_n(1+L\cdot\frac{t-t_n}{(z_n)^{\alpha}})} \\ = & \Big(\frac{r}{z_n(1+L\cdot\frac{t-t_n}{(z_n)^{\alpha}})}\Big)^{\beta} \mathcal{U}_{z_n(1+L\cdot\frac{t-t_n}{(z_n)^{\alpha}})}.\end{aligned}$$

Moreover, because

$$\varphi(t_k) \in \Big(\frac{r}{z_k}\Big)^{\beta} \mathcal{U}_{z_k} = \Big(\frac{r}{1}\Big)^{\beta} \mathcal{U}_1,$$

we get, for the case $t \in [t_k, t_{k+1}) = [t_k, \infty)$,

$$\varphi(t) \in \Big(\frac{r}{1}\Big)^{\beta} \cdot (1 + L(t-t_k)) \cdot \mathcal{U}_1.$$

From the above argument, for any fixed $r \in (0,1]$, we can extend the function $z = z(r,s)$ of Proposition 4.3.1 up to all $s \in [0,\infty)$ by

$$z(r,s) = \begin{cases} z_n(1+L\cdot\frac{s-t_n}{(z_n)^{\alpha}}) & \text{for } s \in [t_n, t_{n+1}) \text{ with } 0 \le n \le (k-1), \\ 1 \quad \text{for } s \in [t_k, t_{k+1}) = [t_k, \infty). \end{cases}$$

In terms of the function z, we obtained

$$\varphi(t) \in \Big(\frac{r}{z}\Big)^{\beta} \cdot (1 + L \cdot (t-t_k)^{+}) \cdot \mathcal{U}_z.$$

As a result, we have

$$\begin{aligned}\Big| \int_{\mathbb{R}^3} w_0(x)\varphi(t,x)dx \Big| \le & \Big(\frac{r}{z}\Big)^{\beta} \cdot (1 + L\cdot t) \cdot z^{\beta} \cdot \|w_0\|_{C^{\beta}(\mathcal{R}^N)} \\ = & r^{\beta} \cdot (1 + L \cdot t) \cdot \|w_0\|_{C^{\beta}(\mathcal{R}^N)}.\end{aligned}$$

From the observation (4.37), we have proved, for any $t \in [0,T)$,

$$\|w(t,\cdot)\|_{C^{\beta}(\mathbb{R}^N)} \le C \cdot (1 + L \cdot t) \cdot \|w_0\|_{C^{\beta}(\mathbb{R}^N)} \tag{4.38}$$

where C does not depend on t. Note that this estimate blows up as t goes to infinity.

We can overcome the above blow-up defect by obtaining the evolution of L^1-norm of $\varphi(s)$. Indeed, for the case $t < t_k$, i.e. for $t \in [t_n, t_{n+1})$ with $0 \le n \le (k-1)$, the function $z(r,t)$ is bounded below by $C \cdot t^{1/\alpha}$ where C does not depend on $r \in (0,1]$. Indeed,

$$\Big(z(r,t)\Big)^{\alpha} = \Big(z_n(1 + L \cdot \frac{t - t_n}{(z_n)^{\alpha}})\Big)^{\alpha} \ge (z_n)^{\alpha} = (r \cdot \eta^n)^{\alpha} = r^{\alpha} \cdot (\eta^{\alpha})^n$$
$$\ge \Big(\frac{\eta^{\alpha} - 1}{\eta\delta}\Big)\delta r^{\alpha} \cdot \frac{((\eta^{\alpha})^{n+1} - 1)}{(\eta^{\alpha} - 1)} \ge \Big(\frac{\eta^{\alpha} - 1}{\eta\delta}\Big) t_{n+1} \ge \Big(\frac{\eta^{\alpha} - 1}{\eta\delta}\Big) t.$$

Recall that $\varphi(t) \in (r/z)^{\beta} \cdot \mathcal{U}_z$ for any $t < t_k$. Thus, thanks to (4.4), we have the evolution of L^1-norm $\|\varphi(t)\|_{L^1(\mathbb{R}^N)} \le (r/z)^{\beta} \le C \cdot r^{\beta} \cdot t^{-\beta/\alpha}$.

On the other hand, from (4.4), we have $\|\varphi(t_k)\|_{L^1(\mathbb{R}^N)} \le r^{\beta}$. Thus, from (II) of Lemma 4.2.3, we get $\|\varphi(t)\|_{L^1(\mathbb{R}^N)} \le r^{\beta}$ as long as $t \ge t_k$. Therefore, we have a control for any $t \in (0,T)$:

$$\begin{aligned}\Big|\int_{\mathbb{R}^3} w_0(x)\varphi(t,x)dx\Big| &\le \|w_0\|_{L^{\infty}(\mathcal{R}^N)} \cdot \|\varphi(t)\|_{L^1(\mathbb{R}^N)} \\ &\le C \cdot r^{\beta} \cdot \max\{1, \frac{1}{t^{\beta/\alpha}}\} \cdot \|w_0\|_{L^{\infty}(\mathcal{R}^N)}.\end{aligned} \tag{4.39}$$

Thus, from (4.37), we have

$$\|w(t,\cdot)\|_{C^{\beta}(\mathbb{R}^N)} \le C \cdot \max\{1, \frac{1}{t^{\beta/\alpha}}\} \cdot \|w_0\|_{L^{\infty}(\mathbb{R}^N)} \tag{4.40}$$

where C does not depend on t. Now we can combine (4.38) with (4.40) to get

$$\begin{aligned}\|w(t,\cdot)\|_{C^{\beta}(\mathbb{R}^N)} &\le C \cdot \min\{\max\{1, \frac{1}{t^{\beta/\alpha}}\}, (1 + L \cdot t)\} \cdot \|w_0\|_{C^{\beta}(\mathbb{R}^N)} \\ &\le C \cdot \|w_0\|_{C^{\beta}(\mathbb{R}^N)} \text{ for any } t \in [0,T).\end{aligned} \tag{4.41}$$

Similarly, we can prove $\|\varphi(t)\|_{L^\infty(\mathbb{R}^N)} \le C \cdot r^\beta \cdot \max\{1, \frac{1}{t^{(N+\beta)/\alpha}}\}$. As a result, from (4.37), we have

$$\|w(t,\cdot)\|_{C^\beta(\mathbb{R}^N)} \le C\Big(\|w_0\|_{L^\infty(\mathbb{R}^N)} + \max\{1, \frac{1}{t^{(N+\beta)/\alpha}}\} \cdot \|w_0\|_{L^1(\mathbb{R}^N)}\Big).$$

□

4.5 Drift diffusion equations with supercritical fractional diffusion

Proof of Theorem 4.1.3. As mentioned earlier, the proof follows the framework of [48]. Let $\alpha \in (0,1)$ and $B \in [0,\infty)$ be fixed. Fix γ such that $0 < \gamma < \alpha$. Suppose that b is a smooth divergent free drift velocity such that $\sup_{t\in[0,T]} [b(t)]_{C^{1-\alpha}} \le B$. We define K by $k(t,x,y) := c_\alpha$ to make $T_t^K = (-\Delta)^{\alpha/2}$.

For any $\bar{T} > 0$, we define the dual problem of (4.13) by

$$\partial_t \varphi + (b^{\bar{T}} \cdot \nabla)\varphi + (-\Delta)^{\alpha/2}\varphi = 0 \tag{4.42}$$

where $b^{\bar{T}}(t,x) := -b(\bar{T}-t,x)$. Due to div $b(s,\cdot) = 0$, we have the same conclusion of Lemma 4.2.5 for this pair of two problems (4.13),(4.42). Thus, once we get the same conclusion of Proposition 4.3.1 for the problem (4.42), then we can repeat Section 4.4 (the proof of the part (II) of Theorem 4.1.1) to obtain Theorem 4.1.3.

From now on, our goal is to prove Proposition 4.3.1 for the problem (4.42). Let $0 < r \le 1$ and $\varphi_0 \in \mathcal{U}_r \cap \mathcal{S}$. Let x_0 be a center of φ_0. Suppose φ is a smooth solution of $\partial_t\varphi + (b\cdot\nabla)\varphi + (-\Delta)^{\alpha/2}\varphi = 0$ with the initial data φ_0 with a center x_0.

Then we define the curve $X(\cdot)$ by solving

$$\dot{X}(s) = (b(s,\cdot))_{X(s),(A^{-1/N}r)} \quad \text{with} \quad X(0) = x_0$$

where $(f)_{x,l}$ is defined by $\frac{1}{|B_l|}\int_{B_l(x)} f(y)dy$ for any L^1_{loc} function f.

Remark 4.5.1. The existence of a weak solution φ for given any time interval $[0,T]$ can be proved by using the usual vanishing viscosity method: Add artificial viscosity term $\epsilon\Delta\varphi$, then take the limit to get a weak solution φ. Moreover this weak solution φ is smooth thanks to $b \in C^\infty(\overline{\mathbb{R}^N} \times [0,T])$ (e.g. see [73]).

Remark 4.5.2. In order to control the drift term, we will use $X(s)$ as a moving center of $\varphi(s)$ (see Lemma 4.5.2) while, in the proof of Proposition 4.3.1, we used a fixed center in time (see Lemma 4.3.3).

We present some simple properties of C^β space for $0 < \beta < 1$.

Lemma 4.5.1. *If $f \in C^\beta$ for $\beta \in (0,1)$, then we have, for any $l \in (0,\infty)$, for any $p \in [1,\infty)$, for any integer $k \geq 1$,*

$$\Big(\frac{1}{|B_l|}\int_{B_l(x)} |f(y) - (f)_{x,l}|^p dy\Big)^{1/p} \leq 2^\beta \cdot [f]_{C^\beta} \cdot l^\beta \quad \textit{and}$$

$$|(f)_{x,l} - (f)_{x,2^k\cdot l}| \leq 2^{N+\beta} \cdot \Big(\sum_{j=1}^{k} (2^\beta)^j\Big) \cdot [f]_{C^\beta} \cdot l^\beta.$$

Remark 4.5.3. In [48] (for the case $\alpha = 1$), the following analogous properties of BMO functions was used:

$$\Big(\frac{1}{|B_l|}\int_{B_l(x)} |f(y) - (f)_{x,l}|^p dy\Big)^{1/p} \leq C \cdot \|f\|_{BMO} \quad \text{and}$$

$$|(f)_{x,l} - (f)_{x,2^k\cdot l}| \leq C \cdot k \cdot \|f\|_{BMO}.$$

Proof of Lemma 4.5.1. We observe that, for $y \in B_l(x)$,

$$|f(y) - (f)_{x,l}| \leq \frac{1}{|B_l|} \int_{B_l(x)} |f(y) - f(z)| dz \leq [f]_{C^\beta} \cdot (2 \cdot l)^\beta$$

thanks to $|y - z| \leq (2 \cdot l)$. Thus we have the first property.

For the second property, we observe first that for any $l_0 \in (0, \infty)$,

$$\begin{aligned} |(f)_{x,l_0} - (f)_{x,2\cdot l_0}| &\leq \frac{1}{|B_{l_0}|} \int_{B_{l_0}(x)} |f(z) - (f)_{x,2\cdot l_0}| dz \\ &\leq \frac{2^N}{|B_{2\cdot l_0}|} \int_{B_{2\cdot l_0}(x)} |f(z) - (f)_{x,2\cdot l_0}| dz \\ &\leq 2^N \cdot 2^\beta \cdot [f]_{C^\beta} \cdot (2 \cdot l_0)^\beta = 2^{N+2\beta} \cdot [f]_{C^\beta} \cdot (l_0)^\beta. \end{aligned}$$

Thus, we have, for any $l \in (0, \infty)$

$$\begin{aligned} |(f)_{x,l} - (f)_{x,2^k\cdot l}| &\leq \sum_{j=1}^{k} |(f)_{x,2^{j-1}\cdot l} - (f)_{x,2^j\cdot l}| \leq \sum_{j=1}^{k} 2^{N+2\beta} \cdot [f]_{C^\beta} \cdot (2^{j-1} \cdot l)^\beta \\ &\leq 2^{N+\beta} \cdot \sum_{j=1}^{k} (2^\beta)^j \cdot [f]_{C^\beta} \cdot l^\beta. \end{aligned}$$

□

Thanks to div $b = 0$ and the maximum principle of [28], we can repeat **STEP 1**, **STEP 3**, **STEP 4** without an essential modification (for details, refer to [48]). However, for **STEP 2**-Concentration condition, we need the following lemma instead of Lemma 4.3.3.

Lemma 4.5.2. *There exists a constant $\hat{C}_{conc} > 0$ such that, for any $s \in (0, T)$, we have*

$$\int_{\mathbb{R}^N} |\varphi(s,x)||x - X(s)|^\gamma dx \leq r^\gamma \Big[1 + \hat{C}_{conc} \cdot (1 + B) \cdot A^{\frac{\alpha-\gamma}{N}} \cdot \frac{s}{r^\alpha}\Big]$$

where $\hat{C}_{conc}$ is independent of B and A as long as $A \geq 1$.

Proof. Since $\frac{\varphi}{|\varphi|}(b\cdot\nabla)\varphi = (b\cdot\nabla)|\varphi|$ a.e., we have

$$\begin{aligned}
&\frac{d}{ds}\int|\varphi(s,x)||x-X(s)|^\gamma dx = \int\Big(\frac{\partial}{\partial s}|\varphi|\Big)|x-X(s)|^\gamma dx + \int_{\mathbb{R}^N}|\varphi|\frac{\partial}{\partial s}|x-X(s)|^\gamma dx\\
&\le \int(-(b\cdot\nabla)|\varphi| - (-\Delta)^{\alpha/2}|\varphi|)\cdot|x-X(s)|^\gamma dx - \int|\varphi|\dot X(s)\cdot\nabla\Big(|x-X(s)|^\gamma\Big)dx\\
&\le \int|\varphi|(b-\dot X(s))\cdot\nabla\Big(|x-X(s)|^\gamma\Big)dx + \Big|\int(-\Delta)^{\alpha/2}(|x-X(s)|^\gamma)\cdot|\varphi|dx\Big|\\
&\le C\int\underbrace{|\varphi|\cdot|b-(b(s,\cdot))_{X(s),(A^{-1/N}r)}|\cdot|x-X(s)|^{\gamma-1}}_{(*)}dx + C\int|x-X(s)|^{\gamma-\alpha}\cdot|\varphi|dx.
\end{aligned}$$

As before (Lemma 4.3.3), the second integral is bounded by $CA^{\frac{\alpha-\gamma}{N}}r^{\gamma-\alpha}$ as long as $A\ge 1$. For the first integral, we split it into two integrals $\int_{B_{(A^{-1/N}r)}(X(s))}(*)dx$ and $\int_{\mathbb{R}^N - B_{(A^{-1/N}r)}(X(s))}(*)dx$.

First we observe that

$$\begin{cases}
\|\varphi(s)\|_{L^q(\mathbb{R}^N)} \le \Big(\frac{A}{r^N}\Big)^{1-(1/q)} \text{ for } q\in[1,\infty],\\
\|b-(b(s,\cdot))_{X(s),l}\|_{L^z(B_l(X(s)))} \le CB\cdot l^{(1-\alpha)+(N/z)} \text{ for } z\in[1,\infty) \text{ and for } l>0,\\
\||x-X(s)|^{\gamma-1}\|_{L^p(B_{(A^{-1/N}r)}(X(s)))} \le C\cdot[A^{-1/N}r]^{(\gamma-1)+\frac{N}{p}} \text{ for } p\in[1,\frac{N}{1-\gamma})
\end{cases}$$

where the first inequality comes from an interpolation between L^∞ and L^1 with the maximum principle of [28] while the second one follows Lemma 4.5.1.

We take $p := \frac{1+(N/(1-\gamma))}{2} \in (1,\frac{N}{1-\gamma})$ and define $z_1\in[1,\infty)$ by $\frac{1}{p}+\frac{1}{z_1}=1$ to obtain

$$\begin{aligned}
&\int_{B_{(A^{-1/N}r)}(X(s))}(*)dx \le \|\varphi(s)\|_{L^\infty}\cdot\|b-(b(s,\cdot))_{X(s),(A^{-1/N}r)}\|_{L^{z_1}(B_{(A^{-1/N}r)}(X(s)))}\\
&\qquad\qquad\cdot\||x-X(s)|^{\gamma-1}\|_{L^p(B_{(A^{-1/N}r)}(X(s)))}\\
&\le C\cdot\frac{A}{r^N}\cdot B\cdot[A^{-1/N}r]^{(1-\alpha)+(N/z_1)}\cdot[A^{-1/N}r]^{(\gamma-1)+\frac{N}{p}}\\
&\le C\cdot B\cdot A^{\frac{\alpha-\gamma}{N}}\cdot r^{\gamma-\alpha}
\end{aligned}$$

where the above constant is independent of B and A.

For the rest integral, we take $z_2 := \frac{2N}{\alpha-\gamma} \in [1,\infty)$ and define q by by $\frac{1}{q}+\frac{1}{z_2} = 1$. Then we have

$$\int_{\mathbb{R}^N - B_{(A^{-1/N}r)}(X(s))} (*)dx = \sum_{k=1}^{\infty} \int_{B_{(2^k\cdot A^{-1/N}r)}(X(s)) - B_{(2^{k-1}\cdot A^{-1/N}r)}(X(s))} (*)dx$$

$$\leq \sum_{k=1}^{\infty} (2^{k-1}\cdot A^{-1/N}r)^{\gamma-1} \int_{B_{(2^k\cdot A^{-1/N}r)}(X(s))} |\varphi|\cdot|b-(b(s,\cdot))_{X(s),(A^{-1/N}r)}|dx$$

$$\leq CA^{\frac{1-\gamma}{N}} r^{\gamma-1} \sum_{k=1}^{\infty} (2^{\gamma-1})^k \Big[\int_{B_{(2^k\cdot A^{-1/N}r)}(X(s))} |\varphi|\cdot|b-(b(s,\cdot))_{X(s),(2^k\cdot A^{-1/N}r)}|dx$$

$$+ \int_{B_{(2^k\cdot A^{-1/N}r)}(X(s))} |\varphi|\cdot|(b(s,\cdot))_{X(s),(2^k\cdot A^{-1/N}r)} - (b(s,\cdot))_{X(s),(A^{-1/N}r)}|dx\Big]$$

$$\leq CA^{\frac{1-\gamma}{N}} r^{\gamma-1} \sum_{k=1}^{\infty} (2^{\gamma-1})^k \Big[\|\varphi\|_{L^q}\cdot\|b-(b(s,\cdot))_{X(s),(2^k\cdot A^{-1/N}r)}\|_{L^{z_2}(B_{(2^k\cdot A^{-1/N}r)}(X(s)))}$$

$$+ \|\varphi\|_{L^1}\cdot 2^{N+(1-\alpha)}\cdot \sum_{j=1}^{k} (2^{1-\alpha})^j \cdot B\cdot (A^{-1/N}r)^{1-\alpha}\Big]$$

$$\leq C\cdot A^{\frac{1-\gamma}{N}} r^{\gamma-1}\cdot \sum_{k=1}^{\infty} (2^{\gamma-1})^k \Big[\Big(\frac{A}{r^N}\Big)^{1-(1/q)}\cdot C\cdot B\cdot (2^k\cdot A^{-1/N}r)^{(1-\alpha)+(N/z_2)}$$

$$+ C\cdot (2^{1-\alpha})^k\cdot B\cdot (A^{-1/N}r)^{1-\alpha}\Big]$$

$$\leq C\cdot B\cdot A^{\frac{\alpha-\gamma}{N}} r^{\gamma-\alpha}\cdot \Big[\sum_{k=1}^{\infty} (2^{\frac{\gamma-\alpha}{2}})^k + \sum_{k=1}^{\infty} (2^{\gamma-\alpha})^k\Big]$$

$$\leq C\cdot B\cdot A^{\frac{\alpha-\gamma}{N}} r^{\gamma-\alpha}$$

where the above constant is independent of B and A. Thus we proved Lemma 4.5.2.

□

Now we can repeat **STEP 5** in Proposition 4.3.1 once we replace the constant C_{conc}, which appears in **STEP 5**, with $\Big[\hat{C}_{conc}\cdot(1+B)\Big]$ of Lemma 4.5.2. Thus we proved Proposition 4.3.1 for the problem (4.42), which gives us Theorem 4.1.3. □

Appendices

Appendix A

Miscellaneous Proofs for Chapter 3

A.1 Proof for Lemma 3.3.1

Proof for Lemma 3.3.1. We fix (n, a, b, p) such that $n \geq 2$, $0 < b < a < 1$ and $1 < p < \infty$. Let α be any multi index such that $|\alpha| = n$ and $D^\alpha = \partial_{\alpha_1}\partial_{\alpha_2}D^\beta$ where β is a multi index with $|\beta| = n - 2$.

Observe that, from $\mathrm{div}(v_2) = 0$ and $\mathrm{div}(v_1) = 0$, we have

$$\begin{aligned} -\Delta(D^\alpha P) &= \mathrm{div}\,\mathrm{div}\, D^\alpha(v_2 \otimes v_1) \\ &= D^\alpha\Big(\sum_{ij}(\partial_j v_{2,i})(\partial_i v_{1,j})\Big) \\ &= \partial_{\alpha_1}\partial_{\alpha_2}H \end{aligned}$$

where $H = D^\beta\Big(\sum_{ij}(\partial_j v_{2,i})(\partial_i v_{1,j})\Big)$ and $v_k = (v_{k,1}, v_{k,2}, v_{k,3})$ for $k = 1, 2$. Then, for any (p_1, p_2) such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, we get

$$\|H\|_{L^p(B(a))} \leq C\|v_2\|_{W^{n-1,p_2}(B(a))} \cdot \|v_1\|_{W^{n-1,p_1}(B(a))}$$

where C is independent of choice of p_1 and p_2 and

$$\|H\|_{W^{1,\infty}(B(a))} \leq C\|v_2\|_{W^{n,\infty}(B(a))} \cdot \|v_1\|_{W^{n,\infty}(B(a))}.$$

We fix a function $\psi \in C^\infty(\mathbb{R}^3)$ satisfying:

$$\psi = 1 \quad \text{in } B(b + \frac{a-b}{3}), \quad \psi = 0 \quad \text{in } (B(b + \frac{2(a-b)}{3}))^C \text{ and } 0 \leq \psi \leq 1.$$

We decompose $D^\alpha P$ by using ψ:

$$\begin{aligned} -\Delta(\psi D^\alpha P) &= -\psi \Delta D^\alpha P - 2\operatorname{div}((\nabla\psi)(D^\alpha P)) + (D^\alpha P)\Delta\psi \\ &= \psi \partial_{\alpha_1}\partial_{\alpha_2} H - 2\operatorname{div}((\nabla\psi)(D^\alpha P)) + (D^\alpha P)\Delta\psi \\ &= -\Delta Q_1 - \Delta Q_2 - \Delta Q_3 \end{aligned}$$

where

$$\begin{aligned} -\Delta Q_1 &= \partial_{\alpha_1}\partial_{\alpha_2}(\psi H), \\ -\Delta Q_2 &= -\partial_{\alpha_2}[(\partial_{\alpha_1}\psi)(H)] - \partial_{\alpha_1}[(\partial_{\alpha_2}\psi)(H)] + (\partial_{\alpha_1}\partial_{\alpha_2}\psi)(H) \quad \text{and} \\ -\Delta Q_3 &= -2\operatorname{div}((\nabla\psi)(D^\alpha P)) + (D^\alpha P)\Delta\psi. \end{aligned}$$

Here Q_2 and Q_3 are defined by the representation formula $(-\Delta)^{-1}(f) = \frac{1}{4\pi}(\frac{1}{|x|} * f)$ while Q_1 is defined by the Riesz transforms.

Then, by the Riesz transform, we get

$$\begin{aligned} \|Q_1\|_{L^p(B(b))} &\le C\|\psi H\|_{L^p(\mathbb{R}^3)} \le C\|H\|_{L^p(B(a))} \\ &\le C\|v_2\|_{W^{n-1,p_2}(B(a))} \cdot \|v_1\|_{W^{n-1,p_1}(B(a))}. \end{aligned}$$

Moreover, using Sobolev's inequality, we have

$$\begin{aligned} \|Q_1\|_{L^\infty(B(b))} &\le C\Big(\|Q_1\|_{L^4(B(b))} + \|\nabla Q_1\|_{L^4(B(b))}\Big) \\ &\le C\|H\|_{W^{1,4}(B(a))} \le C\|H\|_{W^{1,\infty}(B(a))} \\ &\le C\|v_2\|_{W^{n,\infty}(B(a))} \cdot \|v_1\|_{W^{n,\infty}(B(a))}. \end{aligned}$$

For $x \in B(b)$, we compute

$$|Q_2(x)| = \left| \frac{1}{4\pi} \int_{(B(b+\frac{2(a-b)}{3})-B(b+\frac{a-b}{3}))} \frac{1}{|x-y|} \Big(\partial_{\alpha_2}[(\partial_{\alpha_1}\psi)(H)](y) \right.$$

$$\left. - \partial_{\alpha_1}[(\partial_{\alpha_2}\psi)(H)](y) + (\partial_{\alpha_1}\partial_{\alpha_2}\psi)(H)(y) \Big) dy \right|$$

$$\leq 2\|\nabla\psi\|_{L^\infty} \cdot \sup_{y \in B(b+\frac{a-b}{3})^C} (|\nabla_y \frac{1}{|x-y|}|) \cdot \|H\|_{L^1(B(a))}$$

$$+ \|\nabla^2\psi\|_{L^\infty} \cdot \sup_{y \in B(b+\frac{a-b}{3})^C} (|\frac{1}{|x-y|}|) \cdot \|H\|_{L^1(B(a))}$$

$$\leq C \cdot \|H\|_{L^1(B(a))}$$

because $|x-y| \geq (a-b)/3$. Likewise, for $x \in B(b)$, we get

$$|Q_3(x)| \leq C \Big(\sum_{k=0}^{n} \|\nabla^{k+1}\psi\|_{L^\infty} \Big) \cdot \Big(\sum_{k=0}^{n} \sup_{y \in B(b+\frac{a-b}{3})^C} |\nabla_y^{k+1} \frac{1}{|x-y|}| \Big) \cdot \|P\|_{L^1(B(a))}$$

$$+ C \Big(\sum_{k=0}^{n} \|\nabla^{k+2}\psi\|_{L^\infty} \Big) \cdot \Big(\sum_{k=0}^{n} \sup_{y \in B(b+\frac{a-b}{3})^C} |\nabla_y^{k} \frac{1}{|x-y|}| \Big) \cdot \|P\|_{L^1(B(a))}$$

$$\leq C \cdot \|P\|_{L^1(B(a))}.$$

Finally, we obtain

$$\|\nabla^n P\|_{L^p(B(b))} \leq \|Q_1\|_{L^p(B(b))} + C\||Q_2| + |Q_3|\|_{L^\infty(B(b))}$$

$$\leq C \cdot \|H\|_{L^p(B(a))} + C \cdot \|H\|_{L^1(B(a))} + C \cdot \|P\|_{L^1(B(a))}$$

$$\leq C_{a,b,p,n} \Big(\|v_2\|_{W^{n-1,p_2}(B(a))} \cdot \|v_1\|_{W^{n-1,p_1}(B(a))} + \cdot \|P\|_{L^1(B(a))} \Big)$$

and

$$\|\nabla^n P\|_{L^\infty(B(b))} \leq \||Q_1| + |Q_2| + |Q_3|\|_{L^\infty(B(b))}$$

$$\leq C \cdot \|H\|_{W^{1,\infty}(B(a))} + C \cdot \|H\|_{L^1(B(a))} + C \cdot \|P\|_{L^1(B(a))}$$

$$\leq C_{a,b,n} \Big(\|v_2\|_{W^{n,\infty}(B(a))} \cdot \|v_1\|_{W^{n,\infty}(B(a))} + \|P\|_{L^1(B(a))} \Big).$$

□

A.2 Proof for Lemma 3.3.3

Proof for Lemma 3.3.3. We fix (n, a, b) such that $n \geq 0$ and $0 < b < a < 1$ and let α be a multi index with $|\alpha| = n$. Then, by taking D^α to (3.5), we have

$$\begin{aligned} 0 =& \partial_t(D^\alpha v_1) + \sum_{\beta \leq \alpha, |\beta| > 0} \binom{\alpha}{\beta} ((D^\beta v_2) \cdot \nabla)(D^{\alpha-\beta} v_1) + (v_2 \cdot \nabla)(D^\alpha v_1) \\ & + \nabla(D^\alpha P) - \Delta(D^\alpha v_1). \end{aligned} \tag{A.1}$$

We define $\Phi(t,x) \in C^\infty$ by $0 \leq \Phi \leq 1, \Phi = 1$ on Q_b and $\Phi = 0$ on Q_a^C. We observe that, for $p \geq \frac{1}{2}$ and for $f \in C^\infty$,

$$(p + \frac{1}{2})|f|^{p-\frac{3}{2}} f \cdot \partial_x f = \partial_x |f|^{p+\frac{1}{2}} \text{ and } (p + \frac{1}{2})|f|^{p-\frac{3}{2}} f \cdot \Delta f \leq \Delta(|f|^{p+\frac{1}{2}}).$$

which can be verified directly due to the fact $|\nabla f| \geq |\nabla |f||$.

Now we multiply $(p + \frac{1}{2})\Phi \frac{D^\alpha v_1}{|D^\alpha v_1|^{(3/2)-p}}$ to (A.1), use the above observation and integrate in x to get: for any $p \geq \frac{1}{2}$,

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}^3} \Phi(t,x) |D^\alpha v_1(t,x)|^{p+\frac{1}{2}} dx \\ & \leq \int_{\mathbb{R}^3} (|\partial_t \Phi(t,x)| + |\Delta \Phi(t,x)|) |D^\alpha v_1(t,x)|^{p+\frac{1}{2}} dx \\ & \quad + (p + \frac{1}{2}) \int_{\mathbb{R}^3} |\nabla D^\alpha P(t,x)| |D^\alpha v_1(t,x)|^{p-\frac{1}{2}} dx \\ & \quad + (p + \frac{1}{2}) \sum_{\beta \leq \alpha, |\beta| > 0} \binom{\alpha}{\beta} \int_{\mathbb{R}^3} \Big| (D^\beta v_2(t,x) \cdot \nabla) D^{\alpha-\beta} v_1(t,x) \Big| |D^\alpha v_1(t,x)|^{p-\frac{1}{2}} dx \\ & \quad - \int_{\mathbb{R}^3} \Phi(t,x)(v_2(t,x) \cdot \nabla)(|D^\alpha v_1(t,x)|^{p+\frac{1}{2}}) dx \end{aligned}$$

$$
\begin{aligned}
&\le C\| |\nabla^n v_1(t,\cdot)|^{p+\frac{1}{2}} \|_{L^1(B(a))} \\
&\quad + C\|\nabla^{n+1} P(t,\cdot)\|_{L^{2p}(B(a))} \cdot \| |\nabla^n v_1(t,\cdot)|^{p-\frac{1}{2}} \|_{L^{\frac{2p}{2p-1}}(B(a))} \\
&+ C\|v_2(t,\cdot)\|_{W^{n,\infty}(B(a))} \cdot \|v_1(t,\cdot)\|_{W^{n,p+\frac{1}{2}}(B(a))} \cdot \| |\nabla^n v_1(t,\cdot)|^{p-\frac{1}{2}} \|_{L^{\frac{p+\frac{1}{2}}{p-\frac{1}{2}}}(B(a))} \\
&- \int_{\mathbb{R}^3} \Phi(t,x) \operatorname{div} \Big(v_2(t,x) \otimes |D^\alpha v_1(t,x)|^{p+\frac{1}{2}} \Big) dx \\
&\quad \le C\|v_1(t,\cdot)\|^{p+\frac{1}{2}}_{W^{n,p+\frac{1}{2}}(B(a))} \\
&\quad\quad + C\|\nabla^{n+1} P(t,\cdot)\|_{L^{2p}(B(a))} \cdot \|\nabla^n v_1(t,\cdot)\|^{p-\frac{1}{2}}_{L^p(B(a))} \\
&\quad\quad + C\|v_2(t,\cdot)\|_{W^{n,\infty}(B(a))} \cdot \|v_1(t,\cdot)\|^{p+\frac{1}{2}}_{W^{n,p+\frac{1}{2}}(B(a))} \\
&\quad\quad + C\|v_2(t,\cdot)\|_{L^\infty(B(a))} \cdot \|\nabla^n v_1(t,\cdot)\|^{p+\frac{1}{2}}_{L^{p+\frac{1}{2}}(B(a))}.
\end{aligned}
$$

Then the integration on $[-a^2, t]$ for any $t \in [-b^2, 0]$ gives us

$$
\begin{aligned}
&\|D^\alpha v_1\|^{p+\frac{1}{2}}_{L^\infty(-(b)^2,0;L^{p+\frac{1}{2}}(B(b)))} \\
&\le C\|v_1\|^{p+\frac{1}{2}}_{L^{p+\frac{1}{2}}(-(a)^2,0;W^{n,p+\frac{1}{2}}(B(a)))} \\
&\quad + C\|\nabla^{n+1} P\|_{L^1(-(a)^2,0;L^{2p}(B(a)))} \cdot \|\nabla^n v_1\|^{p-\frac{1}{2}}_{L^\infty(-(a)^2,0;L^p(B(a)))} \\
&\quad + C\|v_2\|_{L^2(-(a)^2,0;W^{n,\infty}(B(a)))} \cdot \|v_1\|^{p+\frac{1}{2}}_{L^{2p+1}(-(a)^2,0;W^{n,p+\frac{1}{2}}(B(a)))} \\
&\quad + C\|v_2\|_{L^2(-(a)^2,0;L^\infty(B(a)))} \cdot \|\nabla^n v_1\|^{p+\frac{1}{2}}_{L^{2p+1}(-(a)^2,0;L^{p+\frac{1}{2}}(B(a)))}
\end{aligned}
$$

Thus for the case $p = 1/2$, we have

$$
\begin{aligned}
&\|D^\alpha v_1\|_{L^\infty(-(b)^2,0;L^1(B(b)))} \\
&\le C\Big[\Big(\|v_2\|_{L^2(-a^2,0;W^{n,\infty}(B(a)))} + 1 \Big) \cdot \|v_1\|_{L^2(-a^2,0;W^{n,1}(B(a)))} \\
&\qquad\qquad + \|\nabla^{n+1} P\|_{L^1(-a^2,0;L^1(B(a)))} \Big]
\end{aligned}
$$

while, for the case $p \geq 1$, we have

$$
\begin{aligned}
&\|D^\alpha v_1\|^{p+\frac{1}{2}}_{L^\infty(-(b)^2,0;L^{p+\frac{1}{2}}(B(b)))} \\
&\leq C\Big(\|v_2\|_{L^2(-a^2,0;W^{n,\infty}(B(a)))} + 1\Big) \\
&\qquad \cdot \Big(\|v_1\|^{\frac{1}{p+\frac{1}{2}}}_{L^2(-(a)^2,0;W^{n,2p}(B(a)))} \cdot \|v_1\|^{1-\frac{1}{p+\frac{1}{2}}}_{L^\infty(-(a)^2,0;W^{n,p}(B(a)))}\Big)^{p+\frac{1}{2}} \\
&\qquad + C\|\nabla^{n+1}P\|_{L^1(-(a)^2,0;L^{2p}(B(a)))} \cdot \|v_1\|^{p-\frac{1}{2}}_{L^\infty(-(a)^2,0;W^{n,p}(B(a)))} \\
&\leq C_{a,b,n,p}\Big[\Big(\|v_2\|_{L^2(-a^2,0;W^{n,\infty}(B(a)))} + 1\Big) \cdot \|v_1\|_{L^2(-(a)^2,0;W^{n,2p}(B(a)))} \\
&\qquad\qquad + \|\nabla^{n+1}P\|_{L^1(-(a)^2,0;L^{2p}(B(a)))}\Big] \cdot \|v_1\|^{p-\frac{1}{2}}_{L^\infty(-(a)^2,0;W^{n,p}(B(a)))}.
\end{aligned}
$$

□

A.3 Proof for Lemma 3.3.4

Proof for Lemma 3.3.4. Let $M_0 > 0$ and $1 \leq p < \infty$. Then, for any $M \geq M_0$ and for any $f \in C^1(\mathbb{R}^3)$ such that $\int_{\mathbb{R}^3} \phi(x)f(x)dx = 0$, we have

$$
\begin{aligned}
&\|f\|_{L^p(B(M))} = \Big(\int_{B(M)} \Big|\int_{\mathbb{R}^3} (f(x) - f(y))\phi(y)dy\Big|^p dx\Big)^{1/p} \\
&\leq C\Big(\int_{B(M)} \Big(\int_{B(1)} \Big|f(x) - f(y)\Big|dy\Big)^p dx\Big)^{1/p} \\
&\leq C\Big(\int_{B(M)} \Big(\int_{B(1)} \int_0^1 \Big|(\nabla f)((1-t)x + ty) \cdot (x-y)\Big|dtdy\Big)^p dx\Big)^{1/p} \\
&\leq C(M+1)\Big(\int_{B(M)} \Big(\int_{B(1)} \int_0^1 \Big|(\nabla f)((1-t)x + ty)\Big|dtdy\Big)^p dx\Big)^{1/p} \\
&\leq C(M+1)\Big(\int_{B(M)} \Big(\int_{B(1)} \int_0^{\frac{M}{M+1}} \Big|(\nabla f)((1-t)x + ty)\Big|dtdy\Big)^p dx\Big)^{1/p} \\
&\quad + C(M+1)\Big(\int_{B(M)} \Big(\int_{B(1)} \int_{\frac{M}{M+1}}^1 \Big|(\nabla f)((1-t)x + ty)\Big|dtdy\Big)^p dx\Big)^{1/p} \\
&= (I) + (II)
\end{aligned}
$$

where we used $x \in B(M)$ and $y \in B(1)$.

For (I), we have

$$
\begin{aligned}
(I) &\le C_{M_0}\Big(\int_{B(1)}\int_0^{\frac{M}{M+1}}\Big(\int_{B(M)}\Big|(\nabla f)((1-t)x+ty)\Big|^p dx\Big)^{1/p} dtdy\Big) \\
&\le C_{M_0}\cdot M\int_0^{\frac{M}{M+1}}\frac{1}{(1-t)^{3/p}}\Big(\int_{B((1-t)M+1)}\Big|(\nabla f)(z)\Big|^p dz\Big)^{1/p} dt \\
&\le C_{M_0}\cdot M\int_0^{\frac{M}{M+1}}\frac{1}{(1-t)^{3/p}}\Big(\int_{B(1)}\int_{B((1-t)M+2)}\Big|(\nabla f)(z+u)\Big|^p dzdu\Big)^{1/p} dt \\
&\le C_{M_0}\cdot M\int_0^{\frac{M}{M+1}}\frac{((1-t)M+2)^{3/p}}{(1-t)^{3/p}}\Big(\int_{B(1)}\mathcal{M}(|\nabla f|^p)(u)du\Big)^{1/p} dt \\
&\le C_{M_0,p}\cdot M\cdot\|\mathcal{M}(|\nabla f|^p)\|_{L^1(B(1))}^{1/p}\int_0^{\frac{M}{M+1}}\Big(M^{3/p}+\frac{1}{(1-t)^{3/p}}\Big)dt \\
&\le C_{M_0,p}\cdot M\cdot\|\mathcal{M}(|\nabla f|^p)\|_{L^1(B(1))}^{1/p}\Big(M^{3/p}+\int_{\frac{1}{M+1}}^1\frac{1}{s^{3/p}}ds\Big) \\
&\le C_{M_0}\cdot M\cdot\|\mathcal{M}(|\nabla f|^p)\|_{L^1(B(1))}^{1/p}\Big(M^{3/p}+(M+1)^{3/p}\Big) \\
&\le C_{M_0,p}\cdot M^{1+\frac{3}{p}}\cdot\|\mathcal{M}(|\nabla f|^p)\|_{L^1(B(1))}^{1/p}
\end{aligned}
$$

where we used the integral version of the Minkoski's inequality and the fact $(1+M) \le C_{M_0}\cdot M$ from $M \ge M_0$ for the first inequality.

For (II), we observe that if $\frac{M}{M+1} \le t \le 1$, then $0 \le 1-t \le \frac{1}{M+1}$ and

$$|(1-t)x+ty| \le (1-t)\cdot|x| + t|y| \le \frac{M}{M+1} + 1 \le 2$$

due to $x \in B(M)$ and $y \in B(1)$. Thus, we have

$$\begin{aligned}
(II) &\le C_{M_0} \cdot M\Big(\int_{B(M)} \Big(\int_{\frac{M}{M+1}}^{1} \frac{1}{t^3} \int_{B(2)} \Big|(\nabla f)(z)\Big| dz dt\Big)^p dx\Big)^{1/p} \\
&\le C_{M_0} \cdot M \cdot M^{3/p} \cdot \int_{B(2)} |(\nabla f)(z)| dz \cdot \int_{\frac{M}{M+1}}^{1} \frac{1}{t^3} dt \\
&\le C_{M_0} M^{1+\frac{3}{p}} \cdot \|\nabla f\|_{L^1(B(2))}.
\end{aligned}$$

□

Appendix B

Miscellaneous Proofs for Chapter 4

B.1 Proof of the part (I) of Theorem 4.1.1

Proof of the part (I) *of Theorem 4.1.1.* In this subsection, we suppose that the kernel K satisfies not the *weak*-$(*)$-kernel condition in Definition 4.1.3 but the $(*)$-kernel condition in Definition 4.1.2 (we recall that the latter condition implies the former one). Note that the kernel K does not need to satisfy (4.10) any more. Thus, we first construct a family of kernels K_ϵ keeping all the parameters of the $(*)$-kernel condition uniformly in $\epsilon > 0$, and satisfying (4.10). Then we use the conclusion of the part (II) of Theorem 4.1.1.

We define Φ by $\Phi(t,x,y) := \Phi^1(t)\Phi^2(x)\Phi^2(y)$ for $t \in \mathbb{R}$ and $x, y \in \mathbb{R}^N$ and $\Phi_\epsilon(\cdot) := \epsilon^{-(2N+1)}\Phi(\cdot/\epsilon)$ where Φ^1 and Φ^2 are standard C_c^∞ mollifiers in $\mathbb{R}$ and $\mathbb{R}^N$, respectively. Let $(w_0)_\epsilon := w_0 * \Phi^2$ and $h(t,x,y) := k(t,x,y-x)$. Then we define a family of kernels by $h_\epsilon := h^\epsilon *_{t,x,y} \Phi_\epsilon$ where

$$h^\epsilon(t,x,y) := \begin{cases} h(t,x,y) & \text{for } |x-y| < (1/\epsilon) \text{ with } t \in [0, \min\{(1/\epsilon), T\}], \\ \Lambda^{-1} & \text{otherwise.} \end{cases}$$

Since $|h^\epsilon(t,x,y)| \leq \Lambda(1+\epsilon^{-\omega}) < \infty$ for all t, x and y, we observe that $k_\epsilon(t,x,z) := h_\epsilon(t,x,x+z)$ satisfies the condition (4.10).

For each $\epsilon \ll \zeta/2$, the associated kernel $K_\epsilon(t,x,y) := k_\epsilon(t,x,y-x) \cdot |y-$

$x|^{-(N+\alpha)}$ satisfies the $(*)$-kernel condition on the same parameter set of the original kernel K except $\zeta_\epsilon := \zeta/2$ and $\Lambda_\epsilon := 2\Lambda$ (for $\alpha \geq 1$, we assume further $\epsilon \ll s_0/2$ and $(s_0)_\epsilon := s_0/2$). Then, we can construct a weak solution w_ϵ corresponding the kernel K_ϵ and the initial data $(w_0)_\epsilon$, and this solution w_ϵ is smooth since k_ϵ satisfy (4.10) (for existence, see [51] or refer the approximation scheme in [10] while smoothness is a consequence of a standard energy argument). Thanks to the part (II) of Theorem 4.1.1, these solutions satisfy (4.7), (4.8), and (4.9). As a result, we can extract a limit function w, which is a weak solution for the original kernel K and the initial data w_0.

□

B.2 Proof of Theorem 4.1.2

Proof of Theorem 4.1.2. For convenience, we define a function g by $g(x) = G(x) \cdot |x|^{N+\alpha}$. In addition to all the assumptions of Theorem 4.1.2, we assume further

$$\theta_0 \in C^\infty(\overline{\mathbb{R}^N}), \quad g \in C^\infty(\overline{\mathbb{R}^N}), \text{ and } \phi \in C^\infty(\overline{\mathbb{R}}). \tag{B.1}$$

Then there exists a weak solution θ of (4.11) in global time and it is smooth. Indeed, for existence issue, we refer to Benilan and Brezis [5] or the appendix in the paper [10]. Smoothness follows a difference quotient argument.

We will show that the conclusions of Theorem 4.1.2 hold for this smooth solution θ. Moreover, it will be clear that the constants C and β depend only on the parameters in the hypotheses of Theorem 4.1.2 and they are independent of the actual norms coming from the above additional assumption (B.1). Thus the conclusions of Theorem 4.1.2 without (B.1) follows by a limit argument.

Remark B.2.1. Indeed, if we do not have (B.1), then we regularize θ_0, g, and ϕ first:

$$(\theta_0)_\epsilon := \theta_0 * \Phi^2_\epsilon, \quad g_\epsilon := g^\epsilon * \Phi^2_\epsilon, \quad \text{ and } \phi_\epsilon := \phi * \Phi^1_\epsilon$$

where Φ^1 and Φ^2 are mollifiers in $\mathbb{R}^1$ and $\mathbb{R}^N$, respectively, and g^ϵ is defined by $g^\epsilon(x) := \begin{cases} g(x) \text{ if } |x| \le (1/\epsilon), \\ 0 \text{ otherwise.} \end{cases}$ As a result, we obtain (B.1) for $(\theta_0)_\epsilon, g_\epsilon$, and ϕ_ϵ. Moreover, for any $\epsilon \le (\zeta/2)$, all the assumptions (the parameters) of Theorem 4.1.2 still work for for $(\theta_0)_\epsilon, g_\epsilon$, and ϕ_ϵ except we need to replace the original ζ by $\zeta/2$ for the condition (4.12).

We take a derivative ($D_e\theta := w$) on the equation (4.11) so that we get the following equation

$$\partial_t w(t,x) - \int_{\mathbb{R}^N} (w(t,y) - w(t,x))\phi''(\theta(t,y) - \theta(t,x))G(y-x)dy = 0.$$

By putting $K(t,x,y) := \phi''(\theta(t,y) - \theta(t,x))G(y-x)$, this function $w(= D_e\theta)$ solves the linear equation (4.1). Moreover, it is easy to see that this new kernel K satisfies (4.2), (4.3), and (4.10) directly (a rigorous proof can be completed by using the difference quotient argument, which is contained in [10]). Then, Theorem 4.1.2 for the case $\alpha < 1$ follows once we apply the part (II) of Theorem 4.1.1 to w.

For the the case $\alpha \ge 1$, we need to verify the cancellation condition (4.5) to get the *weak*-($*$)-kernel condition. Let $s \in (0,1)$, $t \in [0,T]$ and $x \in \mathbb{R}^N$. Then, we have

$$\Big|\int_{S^{N-1}} k(t,x,s\sigma)\sigma d\sigma\Big| = \Big|\int_{S^{N-1}} K(t,x,x+s\sigma)|s\sigma|^{N+\alpha}\sigma d\sigma\Big|$$
$$= \Big|\int_{S_+^{N-1}} \phi''(\theta(t,x+s\sigma)-\theta(t,x))G(s\sigma)s^{N+\alpha}\sigma d\sigma$$
$$+ \int_{S_-^{N-1}} \phi''(\theta(t,x+s\sigma)-\theta(t,x))G(s\sigma)s^{N+\alpha}\sigma d\sigma\Big|$$

where S_+^{N-1} and S_-^{N-1} are upper and lower hemispheres, respectively. Then, by symmetry of $G(\cdot)$,

$$= \Big|\int_{S_+^{N-1}} \Big[\phi''(\theta(t,x+s\sigma)-\theta(t,x)) - \phi''(\theta(t,x-s\sigma)-\theta(t,x))\Big]G(s\sigma)s^{N+\alpha}\sigma d\sigma.$$

We use the assumption $\phi'' \in C^\nu$:

$$\leq \int_{S_+^{N-1}} [\phi'']_{C^\nu(\mathbb{R})} \cdot \Big|\theta(t,x+s\sigma)-\theta(t,x-s\sigma)\Big|^\nu G(s\sigma)s^{N+\alpha}|\sigma| d\sigma$$
$$\leq \int_{S_+^{N-1}} [\phi'']_{C^\nu(\mathbb{R})} \cdot \|\nabla\theta(t)\|_{L_x^\infty}^\nu \Big|2s\sigma\Big|^\nu G(s\sigma)s^{N+\alpha} d\sigma$$
$$\leq C[\phi'']_{C^\nu(\mathbb{R})} \cdot \|\nabla\theta_0\|_{L^\infty}^\nu \cdot \sqrt{\Lambda}\cdot s^\nu \cdot \int_{S_+^{N-1}} (1+s^\omega) d\sigma$$
$$\leq C\cdot M\cdot\sqrt{\Lambda}\cdot s^\nu\cdot(1+s^\omega) \leq C\cdot M\cdot\sqrt{\Lambda}\cdot s^\nu$$

where the proof of non-increasing of $\|\nabla\theta(t)\|_{L_x^\infty}$ is in the part (II) of Lemma 4.2.3. By putting $\tau := C\cdot M\cdot\sqrt{\Lambda}$ with $s_0 := 1$, we get the condition (4.5). Then, we apply the part (II) of Theorem 4.1.1 to w.

□

Bibliography

[1] Martin T. Barlow, Richard F. Bass, Zhen-Qing Chen, and Moritz Kassmann. Non-local Dirichlet forms and symmetric jump processes. *Trans. Amer. Math. Soc.*, 361(4):1963–1999, 2009.

[2] Richard F. Bass and David A. Levin. Transition probabilities for symmetric jump processes. *Trans. Amer. Math. Soc.*, 354(7):2933–2953 (electronic), 2002.

[3] J. T. Beale, T. Kato, and A. Majda. Remarks on the breakdown of smooth solutions for the 3-D Euler equations. *Comm. Math. Phys.*, 94(1):61–66, 1984.

[4] H. Beirão da Veiga. A new regularity class for the Navier-Stokes equations in $\mathbf{R}^n$. *Chinese Ann. Math. Ser. B*, 16(4):407–412, 1995. A Chinese summary appears in Chinese Ann. Math. Ser. A **16** (1995), no. 6, 797.

[5] P. Benilan and H. Brezis. Solutions faibles d'équations d'évolution dans les espaces de Hilbert. *Ann. Inst. Fourier (Grenoble)*, 22(2):311–329, 1972.

[6] Jöran Bergh and Jörgen Löfström. *Interpolation spaces. An introduction.* Springer-Verlag, Berlin, 1976. Grundlehren der Mathematischen Wissenschaften, No. 223.

[7] Luigi C. Berselli and Giovanni P. Galdi. Regularity criteria involving the pressure for the weak solutions to the Navier-Stokes equations. *Proc. Amer.*

Math. Soc., 130(12):3585–3595 (electronic), 2002.

[8] Clayton Bjorland and Alexis Vasseur. Weak in space, log in time improvement of the Ladyženskaja-Prodi-Serrin criteria. *Journal of Mathematical Fluid Mechanics*. To appear, DOI:10.1007/s00021-009-0020-3.

[9] L. Caffarelli, R. Kohn, and L. Nirenberg. Partial regularity of suitable weak solutions of the Navier-Stokes equations. *Comm. Pure Appl. Math.*, 35(6):771–831, 1982.

[10] Luis Caffarelli, Chi Hin Chan, and Alexis Vasseur. Regularity theory for parabolic nonlinear integral operators. *J. Amer. Math. Soc.*, 24(3):849–869, 2011.

[11] Luis Caffarelli and Alessio Figalli. Regularity of solutions to the parabolic fractional obstacle problem. *arXiv:1101.5170v1*, 2011.

[12] Luis Caffarelli and Luis Silvestre. Regularity theory for fully nonlinear integro-differential equations. *Comm. Pure Appl. Math.*, 62(5):597–638, 2009.

[13] Luis A. Caffarelli and Alexis Vasseur. Drift diffusion equations with fractional diffusion and the quasi-geostrophic equation. *Ann. of Math. (2)*, 171(3):1903–1930, 2010.

[14] Luis A. Caffarelli and Alexis F. Vasseur. The De Giorgi method for regularity of solutions of elliptic equations and its applications to fluid dynamics. *Discrete Contin. Dyn. Syst. Ser. S*, 3(3):409–427, 2010.

[15] Dongho Chae, Peter Constantin, and Jiahong Wu. Inviscid models generalizing the two-dimensional Euler and the surface quasi-geostrophic equations. *Arch. Ration. Mech. Anal.*, 202(1):35–62, 2011.

[16] Dongho Chae, Kyungkeun Kang, and Jihoon Lee. On the interior regularity of suitable weak solutions to the Navier-Stokes equations. *Comm. Partial Differential Equations*, 32(7-9):1189–1207, 2007.

[17] Chi Hin Chan. Smoothness criterion for Navier-Stokes equations in terms of regularity along the streamlines. *Methods Appl. Anal.*, 17(1):81–103, 2010.

[18] Zhen-Qing Chen, Panki Kim, and Takashi Kumagai. Global heat kernel estimates for symmetric jump processes. *Trans. Amer. Math. Soc.*, 363(9):5021–5055, 2011.

[19] A. Cheskidov and R. Shvydkoy. The regularity of weak solutions of the 3D Navier-Stokes equations in $B^{-1}_{\infty,\infty}$. *Archive for Rational Mechanics and Analysis*, 195:159–169, 2010. 10.1007/s00205-009-0265-2.

[20] Kyudong Choi. Persistence of Hölder continuity for non-local integro-differential equations. *preprint, arXiv:1112.6064v1*, 2011.

[21] Kyudong Choi and Alexis Vasseur. Estimates on fractional higher derivatives of weak solutions for the navier-stokes equations. *preprint, arXiv:1105.1526v1*, 2011.

[22] R. Coifman, P.-L. Lions, Y. Meyer, and S. Semmes. Compensated compactness and Hardy spaces. *J. Math. Pures Appl. (9)*, 72(3):247–286, 1993.

[23] Peter Constantin. Navier-Stokes equations and area of interfaces. *Comm. Math. Phys.*, 129(2):241–266, 1990.

[24] Peter Constantin, Gautam Iyer, and Jiahong Wu. Global regularity for a modified critical dissipative quasi-geostrophic equation. *Indiana Univ. Math. J.*, 57(6):2681–2692, 2008.

[25] Peter Constantin and Vlad Vicol. Nonlinear maximum principles for dissipative linear nonlocal operators and applications. *arXiv:1110.0179v1*, 2011.

[26] Peter Constantin and Jiahong Wu. Regularity of Hölder continuous solutions of the supercritical quasi-geostrophic equation. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 25(6):1103–1110, 2008.

[27] Peter Constantin and Jiahong Wu. Hölder continuity of solutions of supercritical dissipative hydrodynamic transport equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 26(1):159–180, 2009.

[28] Antonio Córdoba and Diego Córdoba. A maximum principle applied to quasi-geostrophic equations. *Comm. Math. Phys.*, 249(3):511–528, 2004.

[29] Michael Dabkowski. Eventual regularity of the solutions to the supercritical dissipative quasi-geostrophic equation. *Geom. Funct. Anal.*, 21(1):1–13, 2011.

[30] Ennio De Giorgi. Sulla differenziabilità e l'analiticità delle estremali degli integrali multipli regolari. *Mem. Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. (3)*, 3:25–43, 1957.

[31] Hongjie Dong and Dapeng Du. On the local smoothness of solutions of the Navier-Stokes equations. *J. Math. Fluid Mech.*, 9(2):139–152, 2007.

[32] Hongjie Dong and Nataša Pavlović. Regularity criteria for the dissipative quasi-geostrophic equations in Hölder spaces. *Comm. Math. Phys.*, 290(3):801–812, 2009.

[33] G. F. D. Duff. Derivative estimates for the Navier-Stokes equations in a three-dimensional region. *Acta Math.*, 164(3-4):145–210, 1990.

[34] Bartlomiej Dyda and Moritz Kassmann. Comparability and regularity estimates for symmetric nonlocal dirichlet forms. *arXiv:1109.6812v1*, 2011.

[35] L. Escauriaza, G. A. Serëgin, and V. Šverák. $L_{3,\infty}$-solutions of Navier-Stokes equations and backward uniqueness. *Uspekhi Mat. Nauk*, 58(2(350)):3–44, 2003.

[36] C. Fefferman and E. M. Stein. H^p spaces of several variables. *Acta Math.*, 129(3-4):137–193, 1972.

[37] Ciprian Foiaş, Colette Guillopé, and Roger Temam. New a priori estimates for Navier-Stokes equations in dimension 3. *Comm. Partial Differential Equations*, 6(3):329–359, 1981.

[38] Susan Friedlander and Vlad Vicol. Global well-posedness for an advection-diffusion equation arising in magneto-geostrophic dynamics. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 28(2):283–301, 2011.

[39] Giambattista Giacomin, Joel L. Lebowitz, and Errico Presutti. Deterministic and stochastic hydrodynamic equations arising from simple microscopic model systems. In *Stochastic partial differential equations: six perspectives*, volume 64 of *Math. Surveys Monogr.*, pages 107–152. Amer. Math. Soc., Providence, RI, 1999.

[40] Yoshikazu Giga and Okihiro Sawada. On regularizing-decay rate estimates for solutions to the Navier-Stokes initial value problem. In *Nonlinear analysis and applications: to V. Lakshmikantham on his 80th birthday. Vol. 1, 2*, pages 549–562. Kluwer Acad. Publ., Dordrecht, 2003.

[41] Guy Gilboa and Stanley Osher. Nonlocal operators with applications to image processing. *Multiscale Model. Simul.*, 7(3):1005–1028, 2008.

[42] David Goldberg. A local version of real Hardy spaces. *Duke Math. J.*, 46(1):27–42, 1979.

[43] Loukas Grafakos. *Classical and modern Fourier analysis.* Pearson Education, Inc., Upper Saddle River, NJ, 2004.

[44] Stephen Gustafson, Kyungkeun Kang, and Tai-Peng Tsai. Interior regularity criteria for suitable weak solutions of the Navier-Stokes equations. *Comm. Math. Phys.*, 273(1):161–176, 2007.

[45] Eberhard Hopf. Über die Anfangswertaufgabe für die hydrodynamischen Grundgleichungen. *Math. Nachr.*, 4:213–231, 1951.

[46] Niels Jacob, Alexander Potrykus, and Jiang-Lun Wu. Solving a non-linear stochastic pseudo-differential equation of Burgers type. *Stochastic Process. Appl.*, 120(12):2447–2467, 2010.

[47] Moritz Kassmann. A priori estimates for integro-differential operators with measurable kernels. *Calc. Var. Partial Differential Equations*, 34(1):1–21, 2009.

[48] A. Kiselev and F. Nazarov. A variation on a theme of Caffarelli and Vasseur. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)*, 370(Kraevye Zadachi Matematicheskoi Fiziki i Smezhnye Voprosy Teorii Funktsii. 40):58–72, 220, 2009.

[49] A. Kiselev, F. Nazarov, and A. Volberg. Global well-posedness for the critical 2D dissipative quasi-geostrophic equation. *Invent. Math.*, 167(3):445–453, 2007.

[50] Takashi Komatsu. Continuity estimates for solutions of parabolic equations associated with jump type Dirichlet forms. *Osaka J. Math.*, 25(3):697–728, 1988.

[51] Takashi Komatsu. Uniform estimates for fundamental solutions associated with non-local Dirichlet forms. *Osaka J. Math.*, 32(4):833–860, 1995.

[52] Igor Kukavica. The fractal dimension of the singular set for solutions of the Navier-Stokes system. *Nonlinearity*, 22(12):2889–2900, 2009.

[53] Hitoshi Kumano-go. *Pseudodifferential operators.* MIT Press, Cambridge, Mass., 1981. Translated from the Japanese by the author, Rémi Vaillancourt and Michihiro Nagase.

[54] O. A. Ladyženskaja. Uniqueness and smoothness of generalized solutions of Navier-Stokes equations. *Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 5:169–185, 1967.

[55] O. A. Ladyženskaja, V. A. Solonnikov, and N. N. Ural′ceva. *Linear and quasi-linear equations of parabolic type.* Translated from the Russian by S. Smith. Translations of Mathematical Monographs, Vol. 23. American Mathematical Society, Providence, R.I., 1967.

[56] P. G. Lemarié-Rieusset. *Recent developments in the Navier-Stokes problem*, volume 431 of *Chapman & Hall/CRC Research Notes in Mathematics.* Chapman & Hall/CRC, Boca Raton, FL, 2002.

[57] Jean Leray. Sur le mouvement d'un liquide visqueux emplissant l'espace. *Acta Math.*, 63(1):193–248, 1934.

[58] Fanghua Lin. A new proof of the Caffarelli-Kohn-Nirenberg theorem. *Comm. Pure Appl. Math.*, 51(3):241–257, 1998.

[59] Pierre-Louis Lions. *Mathematical topics in fluid mechanics. Vol. 1*, volume 3 of *Oxford Lecture Series in Mathematics and its Applications.* The Clarendon Press Oxford University Press, New York, 1996. Incompressible models, Oxford Science Publications.

[60] Yifei Lou, Xiaoqun Zhang, Stanley Osher, and Andrea Bertozzi. Image recovery via nonlocal operators. *J. Sci. Comput.*, 42(2):185–197, 2010.

[61] Changxing Miao and Liutang Xue. On the regularity of a class of generalized quasi-geostrophic equations. *J. Differential Equations*, 251(10):2789–2821, 2011.

[62] Giovanni Prodi. Un teorema di unicità per le equazioni di Navier-Stokes. *Ann. Mat. Pura Appl. (4)*, 48:173–182, 1959.

[63] James C. Robinson and Witold Sadowski. Decay of weak solutions and the singular set of the three-dimensional Navier-Stokes equations. *Nonlinearity*, 20(5):1185–1191, 2007.

[64] Laurent Saloff-Coste. *Aspects of Sobolev-type inequalities*, volume 289 of *London Mathematical Society Lecture Note Series.* Cambridge University Press, Cambridge, 2002.

[65] Vladimir Scheffer. The Navier-Stokes equations on a bounded domain. *Comm. Math. Phys.*, 73(1):1–42, 1980.

[66] Maria E. Schonbek and Michael Wiegner. On the decay of higher-order norms of the solutions of Navier-Stokes equations. *Proc. Roy. Soc. Edinburgh Sect. A*, 126(3):677–685, 1996.

[67] Russell W. Schwab. Periodic homogenization for nonlinear integro-differential equations. *SIAM J. Math. Anal.*, 42(6):2652–2680, 2010.

[68] G. A. Seregin. A new version of the Ladyzhenskaya-Prodi-Serrin condition. *Algebra i Analiz*, 18(1):124–143, 2006.

[69] James Serrin. The initial value problem for the Navier-Stokes equations. In *Nonlinear Problems (Proc. Sympos., Madison, Wis*, pages 69–98. Univ. of Wisconsin Press, Madison, Wis., 1963.

[70] Luis Silvestre. Hölder estimates for solutions of integro-differential equations like the fractional Laplace. *Indiana Univ. Math. J.*, 55(3):1155–1174, 2006.

[71] Luis Silvestre. Regularity of the obstacle problem for a fractional power of the Laplace operator. *Comm. Pure Appl. Math.*, 60(1):67–112, 2007.

[72] Luis Silvestre. Hölder estimates for advection fractional-diffusion equations. *arXiv:1009.5723*, 2010.

[73] Luis Silvestre. On the differentiability of the solution to an equation with drift and fractional diffusion. *arXiv:1012.2401*, 2010.

[74] V. A. Solonnikov. A priori estimates for solutions of second-order equations of parabolic type. *Trudy Mat. Inst. Steklov.*, 70:133–212, 1964.

[75] Elias M. Stein. *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, volume 43 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1993.

[76] Michael Struwe. On partial regularity results for the Navier-Stokes equations. *Comm. Pure Appl. Math.*, 41(4):437–458, 1988.

[77] Mitchell H. Taibleson and Guido Weiss. The molecular characterization of certain Hardy spaces. In *Representation theorems for Hardy spaces*, volume 77 of *Astérisque*, pages 67–149. Soc. Math. France, Paris, 1980.

[78] Alexis Vasseur. Higher derivatives estimate for the 3D Navier-Stokes equation. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 27(5):1189–1204, 2010.

[79] Alexis F. Vasseur. A new proof of partial regularity of solutions to Navier-Stokes equations. *NoDEA Nonlinear Differential Equations Appl.*, 14(5-6):753–785, 2007.

[80] Jörg Wolf. A direct proof of the Caffarelli-Kohn-Nirenberg theorem. In *Parabolic and Navier-Stokes equations. Part 2*, volume 81 of *Banach Center Publ.*, pages 533–552. Polish Acad. Sci. Inst. Math., Warsaw, 2008.

Vita

Kyudong Choi was born in Seoul, South Korea. He graduated from the Joong-ang High School in Seoul, South Korea in 1999, and he received a Bachelor of Science degree for Computer Science and for Mathematics, which is a double major, from the Korea University, in Seoul, South Korea in 2006. He entered the Graduate School of the University of Texas at Austin, in Texas, U.S. for Mathematics in 2007.

E-mail address: kchoi@math.utexas.edu

This dissertation was typeset with LaTeX[†] by the author.

[†]LaTeX is a document preparation system developed by Leslie Lamport as a special version of Donald Knuth's TeX Program.