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Arithmetic Reflection Groups and Congruence Subgroups

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Arithmetic Reflection Groups and Congruence Subgroups

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This thesis investigates the geometric and topological constraints placed on the quotient space of a Fuchsian or Kleinian group by requiring that the group admits a fundamental domain which is simultaneously a Ford domain and a Dirichlet domain. In the case of Fuchsian groups, a direct correspondence with reflection groups is proved, and this result is used to first find explicitly the 23 non-cocompact arithmetic maximal hyperbolic reflection groups in $\mathrm{Isom}(\mathbb{H}^2)$, and subsequently to test whether these groups are all congruence. In the case of Kleinian groups, similar results are shown, and some examples of reflection groups are considered.

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Chapter 1

Introduction

The modular group (P)$\mathrm{SL}_2(\mathbb{Z})$ is a well known example of a Fuchsian group acting on the upper half-plane model for the hyperbolic plane $\mathbb{H}^2$. This action admits a triangular fundamental domain P with vertices located at $\rho = -\frac{1}{2} + \frac{\sqrt{-3}}{2}$, $-\overline{\rho}$ and ∞. This domain arises out of two common constructions of fundamental domains for Fuchsian groups: it is both a *Ford domain* and a *Dirichlet domain* for the action of $\mathrm{PSL}_2(\mathbb{Z})$. One can also view P as the union of the triangle T with angles $\frac{\pi}{2}$, $\frac{\pi}{3}$ and 0 located at i, ρ and ∞ respectively with its reflection in the imaginary axis; thus, $\mathrm{PSL}_2(\mathbb{Z})$ can be viewed as the orientation-preserving index two subgroup of the group generated by reflections in the sides of P.

In this thesis, it is shown that these facts are not unrelated: in fact, there is a bijective correspondence between Fuchsian groups which admit such a fundamental domain and discrete hyperbolic reflection groups, which follows from this result.

Theorem 3.0.1. *A Fuchsian group Γ admits a fundamental domain which is simultaneously a Dirichlet domain and a Ford domain if and only if it is the orientation-preserving index two subgroup of a reflection group Γ_{ref}.*

This theorem gives a way of identifying reflection groups via their orientation-preserving subgroups. In particular, given a specific fundamental domain with the stated property, one obtains a specific polygon in $\mathbb{H}^2$ which defines the reflection group. In this thesis, this correspondence will be exploited in order to determine explicitly the polygons defining certain reflection groups.

Discrete reflection groups of spherical, Euclidean and hyperbolic space have been the subject of much study. In the present work, we will focus on finite covolume hyperbolic reflection groups. It was shown by Prokhorov [36] (in the non-cocompact case) and Vinberg [45] (in the cocompact case) that there are no finite covolume hyperbolic reflection groups above a certain dimension (respectively, 996 and 30).

It is possible to specialize further, and consider only *arithmetic* reflection groups, which necessarily have finite covolume. A program of work that includes Nikulin [29, 30, 31], Long–Maclachlan–Reid [27], and Agol [1], recently resulted in the following theorem, shown independently by Agol–Belolipetsky–Storm–Whyte [2] and Nikulin [33].

Theorem. *There are finitely many arithmetic maximal hyperbolic reflection groups.*

This result is obtained from the bounds for the dimensions in which examples may exist, combined with a bound on the possible number in each dimension. Given this result, the question naturally arises of how many such groups exist. This has been partially answered in dimension 2 by Nikulin [32] (and see also Allcock's enumeration [3]), where there are 122 defined over $\mathbb{Q}$. To this end, Agol, Belolipetsky, Storm and Whyte (see also Belolipetsky [7]) remark that the counting process could be made more effective under certain hypotheses, and, with this in mind, posed the question:

Question. *Is each arithmetic maximal hyperbolic reflection group also congruence?*

This thesis answers this question in the case of non-cocompact groups in dimension two. More precisely, we prove:

Theorem 4.3.3. *Of the* 23 *non-cocompact arithmetic maximal reflection groups belonging to* $\mathrm{Isom}(\mathbb{H}^2)$*, fifteen are congruence, and eight are not.*

It is possible to consider all of the above in the case of dimension three, and Kleinian groups. It will be shown that the analogue of Theorem 3.0.1 does not hold in full, and in particular that a wider variety of Kleinian groups can admit such a fundamental domain. However, it will be demonstrated that the techniques used to prove all of the above can still be brought to bear to consider examples of non-cocompact arithmetic maximal reflection groups belonging to $\mathrm{Isom}(\mathbb{H}^3)$. This will lead to the following.

Theorem 5.0.2. *There exists an arithmetic maximal reflection group in* $\mathrm{Isom}(\mathbb{H}^3)$ *which is not congruence.*

This thesis is organized as follows. Chapter 2 collects various background material and preliminaries which will be relevant. Chapter 3 is devoted to the proof of Theorem 3.0.1 above, and this theorem is applied to the non-cocompact arithmetic maximal reflection groups of $\mathrm{Isom}(\mathbb{H}^2)$ in Chapter 4, leading to Theorem 4.3.3. In Chapter 5, various results pertaining to the above concepts in dimension three are stated and proved, including Theorem 5.0.2.

Chapter 2

Background

In this chapter, we will review some preliminaries on hyperbolic geometry and arithmetic reflection groups.

2.1 Hyperbolic Space

By n-dimensional hyperbolic space $\mathbb{H}^n$ we mean the unique simply-connected Riemannian n-manifold with constant sectional curvature -1. We commonly study this space by using standard models: the upper half-space, Poincaré ball, hyperboloid, and projective or Klein models are perhaps most prevalent (see Ratcliffe [37], Chapter 4, for more). In the following, we will predominantly use the upper half-space model.

2.1.1 The Upper Half-Space Model

Let U^n denote the subset of $\mathbb{R}^n$ of points whose nth coordinate is positive, that is,

$$U^n := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n > 0\}.$$

We endow U^n with the metric

$$ds^2 = \frac{dx_1^2 + \ldots + dx_n^2}{x_n^2}.$$

It is a standard fact that (U^n, ds) is isometric to $\mathbb{H}^n$. The boundary $\partial\mathbb{H}^n = \mathbb{R}^{n-1} \cup \{\infty\}$ is homeomorphic to S^{n-1} and will be referred to as the *boundary sphere*. This model is conformal, and geodesic k-planes are vertical or the northern hemispheres of k-spheres S^k with their equators $S^{k-1} \subset \mathbb{R}^{n-1}$. Such planes meet the boundary sphere at right angles.

A horizontal plane $\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n = c\}$ is called a *horosphere based at* ∞ (or, in dimension two, a *horocycle based at* ∞). It is the boundary of the corresponding *horoball based at* ∞. It is well known that on a horosphere or horocycle the hyperbolic metric on $\mathbb{H}^n$ restricts to a Euclidean metric on $\mathbb{R}^{n-1}$.

2.1.2 Isometries

We may generate the group of isometries $\mathrm{Isom}(\mathbb{H}^n)$ by reflections in geodesic $(n-1)$-dimensional hyperplanes. Since these do not preserve orientation, the group $\mathrm{Isom}^+(\mathbb{H}^n)$ of orientation-preserving isometries is generated by products of pairs of reflections. In the case of $\mathrm{Isom}^+(\mathbb{H}^2)$, this group is the group of real linear fractional transformations, and can be identified with $\mathrm{PSL}_2(\mathbb{R})$ via the correspondence

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \longleftrightarrow \quad z \longmapsto \frac{az+b}{cz+d}.$$

In the same way, we have that $\mathrm{Isom}^+(\mathbb{H}^3) \cong \mathrm{PSL}_2(\mathbb{C})$. In this case, we identify $\mathbb{H}^3$ with the set of Hamiltonian quaternions

$$\left\{x_0 + x_1 i + x_2 j + x_3 k \mid x_0, x_1, x_2, x_3 \in \mathbb{R}, i^2 = j^2 = k^2 = ijk = -1\right\}$$

with $x_3 = 0$ and $x_2 > 0$. The action above extends to this setting, where by dividing by $(cz+d)$ is interpreted as multiplying by the quaternionic inverse. One checks that it preserves the upper half-space and all distances.

Isometries belonging to $\mathrm{PSL}_2(\mathbb{R})$ and $\mathrm{PSL}_2(\mathbb{C})$ fall into three distinct categories, based on their fixed points. These categories can be distinguished by the trace of the matrix. Let γ be a non-trivial element of $\mathrm{PSL}_2(\mathbb{R})$ or of $\mathrm{PSL}_2(\mathbb{C})$.

- If the action of γ fixes a point of hyperbolic space, then γ is called *elliptic*. In this case, the trace is real and $|\mathrm{tr}\,\gamma| < 2$.
- If the action of γ fixes no points of hyperbolic space and exactly one point of the boundary sphere, then γ is called *parabolic*. In this case, $\mathrm{tr}\,\gamma = 2$.
- In all other cases, γ fixes no points of hyperbolic space and two distinct points of the boundary sphere. In this case, γ is called *hyperbolic*. Here the trace, if real, satisfies $|\mathrm{tr}\,\gamma| > 2$. If $\gamma \in \mathrm{PSL}_2(\mathbb{C})$ we sometimes distinguish between elements with real trace and those with non-zero imaginary part; the latter are sometimes called *loxodromic*.

2.2 Fuchsian Groups

A *Fuchsian group* Γ is a discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$. The action of Γ on $\mathbb{H}^2$ then admits a connected fundamental domain. If this has finite area with respect to the hyperbolic metric, then we say Γ has *finite coarea*. It is known [20] that such a group is then also finitely presented, and admits a fundamental domain with finitely many sides.

For a given finitely generated Fuchsian group Γ, the *signature* $(g; n_1, \ldots, n_t; m; f)$ of Γ records the topology of the quotient space $\mathbb{H}^2/\Gamma$, where g is the genus, t is the number of cone points of orders $n_1, \ldots, n_t$ respectively, m is the number of cusps, and f is the number of infinite area funnels.

2.2.1 Fundamental Domains

Given a finite-sided fundamental domain P for the Fuchsian group Γ, one can recover a presentation for the group, and hence information about its signature, from it by using the Poincaré Polyhedron Theorem. To apply it, we require the following information: the number of sides m of P; the way in which the sides of P are paired; and the angle at each vertex of P. Given this we divide the vertices of P into equivalence classes, where two vertices are equivalent if they are identified by a side-pairing g_i or sequence of side-pairings. Each equivalence class is called a *vertex cycle*, and each vertex cycle gives rise to some word $w = g_k \dots g_1$ in the side-pairings with the property that, for some vertex v, w fixes v, and the sum of the angles in the cycle is $\dfrac{2\pi}{n_w}$ for $n_w \in \mathbb{N} \cup \{\infty\}$. A presentation for Γ is then

$$\Gamma \cong \langle g_1, \dots, g_m \mid w^{n_w} = 1 \text{ for each word } w \rangle .$$

If $n_w = \infty$, then the vertex is ideal, and this corresponds to a cusp.

2.2.2 Dirichlet Domains

Though there is no "canonical" choice of a fundamental domain for a given Fuchsian group Γ, there are some standard methods to construct one. Given a point $z_0 \in \mathbb{H}^2$ not fixed by the action of any non-trivial element of Γ, the *Dirichlet domain* for Γ centered at z_0 is defined to be

$$\left\{ x \in \mathbb{H}^2 \mid d(x, z_0) \leq d(x, \alpha(z_0)) \ \forall\ 1 \neq \alpha \in \Gamma \right\} .$$

It is an intersection of closed half-spaces. We generically expect the resulting domain to change with the choice of z_0 [16].

One can give an alternative definition of Dirichlet domain [6]. Given the same center z_0 and a non-trivial $\gamma \in \Gamma$, there is a unique decomposition $\gamma = \gamma_2\gamma_1$, where γ_i denotes reflection in the geodesic L_i, and we require that $\gamma_2(z_0) = z_0$, or $z_0 \in L_2$. Then the assumption that $\gamma(z_0) \neq z_0$ means that $z_0 \notin L_1$, so we may define H_γ to be the half-plane bounded by L_1 and containing z_0. The Dirichlet domain is then the intersection of all half-planes H_γ.

2.2.3 Ford Domains

Suppose now that Γ contains parabolic elements. The above allows us to define a *generalized Dirichlet domain* by taking our center to be a parabolic fixed point on the boundary $\partial\mathbb{H}^2$. We will typically conjugate Γ in $\mathrm{PSL}_2(\mathbb{R})$ so that this center is placed at ∞ in the upper half-plane. In this case, the reflections γ_1 and γ_2 are well-defined when $\gamma(\infty) \neq \infty$, and then the line L_1 is called the *isometric circle* of γ, which we will denote by S_γ. However, the reflections are not uniquely determined for any parabolic isometry fixing ∞. To account for this, we define a *Ford domain* [18] to be the intersection of the region exterior to all isometric circles with a fundamental domain for the action of the parabolic subgroup stabilizing ∞, $\Gamma_\infty < \Gamma$.

Given an isometry $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $c \neq 0$, the isometric circle S_γ can be read off from the matrix representation: it has center $-\dfrac{d}{c}$ and radius $\dfrac{1}{|c|}$.

The following result is well known as Shimizu's Lemma [43].

Theorem 2.2.1 (Shimizu)**.** *Suppose that a Fuchsian group contains the elements*

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, X = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

with $c \neq 0$. Then $|c| \geq 1$.

This result is proved by assuming that $0 < |c| < 1$ and exhibiting a sequence of elements which converge to the identity, thereby violating discreteness. Note that Shimizu's

Lemma shows that in any Fuchsian group containing A, the radius of an isometric circle cannot exceed 1.

2.3 Kleinian Groups

The group of orientation-preserving isometries $\mathrm{Isom}^+(\mathbb{H}^3)$ of the upper half-space model of $\mathbb{H}^3$ can be identified with $\mathrm{PSL}_2(\mathbb{C})$. A *Kleinian group* is a discrete subgroup of this isometry group. The definitions of Dirichlet domain and Ford domain carry over to this situation. A necessary, but not sufficient, requirement for the Ford domain to be cofinite is that Γ_∞ contain a copy of $\mathbb{Z}^2$.

If a Kleinian group Γ is non-cocompact but of finite covolume, then the action of Γ on the boundary sphere $\mathbb{C} \cup \{\infty\}$ gives rise to at least one point which is fixed by two non-conjugate (in Γ) parabolic isometries. Conjugating Γ in $\mathrm{PSL}_2(\mathbb{C})$ so that this point is moved to ∞, we see that the subgroup Γ_∞ of elements of Γ fixing ∞ is an orientation-preserving discrete group of isometries of the Euclidean plane $\mathbb{R}^2$. If Γ_∞ is torsion-free then it is isomorphic to $\mathbb{Z}^2$, and corresponds to a torus boundary component of the quotient space $\mathbb{H}^3/\Gamma$ which contributes non-trivial rational homology. If Γ_∞ has torsion then it corresponds to a cusp which is properly covered by a torus; there are four possibilities for such a cusp: a sphere with four cone points of order 2 $S^2(2,2,2,2)$; or three spheres with three cone points $S^2(2,3,6)$, $S^2(2,4,4)$, and $S^2(3,3,3)$. Each of these four cusps corresponds to a spherical boundary component of the quotient orbifold, which contributes no rational homology.

2.4 Arithmetic Groups

There is a substantial literature on the theory of arithmetic subgroups of algebraic groups, and in particular of the isometry groups of hyperbolic space $\mathbb{H}^n$. For the purposes of this thesis, only dimensions two and three will be discussed in detail, and so our groups will belong to $\mathrm{PSL}_2(\mathbb{R})$ and $\mathrm{PSL}_2(\mathbb{C})$. Additionally, all arithmetic groups will be non-cocompact. As such, in this section we introduce only those definitions and results that pertain to non-cocompact arithmetic Fuchsian and Kleinian groups.

2.4.1 Arithmetic Fuchsian Groups

A non-cocompact Fuchsian group $\Gamma < \mathrm{PSL}_2(\mathbb{R})$ is called *arithmetic* if it is commensurable with the group $\mathrm{PSL}_2(\mathbb{Z})$; that is, after possibly conjugating by some $\alpha \in \mathrm{PSL}_2(\mathbb{R})$, the intersection $\alpha\Gamma\alpha^{-1} \cap \mathrm{PSL}_2(\mathbb{Z})$ has finite index in both $\mathrm{PSL}_2(\mathbb{Z})$ and $\alpha\Gamma\alpha^{-1}$. Notice that if Γ is arithmetic, it is necessarily of finite covolume.

Such a group is then called *congruence* if it contains some *principal congruence subgroup*

$$\Gamma(N) = \mathrm{P}\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid a \equiv d \equiv 1, b \equiv c \equiv 0 \bmod N \right\}.$$

If this is the case, then the group is said to have *level* N. Note that the group $\Gamma(N)$ is the kernel of the natural projection

$$\varphi_N : \mathrm{PSL}_2(\mathbb{Z}) \to \mathrm{PSL}_2(\mathbb{Z}/N\mathbb{Z})$$

given by reducing each entry modulo N. As such, each principal congruence subgroup is normal in $\mathrm{PSL}_2(\mathbb{Z})$, and has finite index.

The following related groups are also frequently considered. Let $\Gamma_0(N)$ denote the preimage under φ_N of the group of upper-triangular matrices in $\mathrm{PSL}_2(\mathbb{Z}/N\mathbb{Z})$; i.e.,

$$\Gamma_0(N) = \mathrm{P}\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid c \equiv 0 \bmod N \right\}.$$

Note that we have the inclusion $\Gamma(N) < \Gamma_0(N)$. Since we will consider Ford domains in what follows, it is useful to note that Ford domains for each group $\Gamma_0(N)$ were found by Lascurain Orive [26].

In contrast to other groups (such as $(\mathrm{P})\mathrm{SL}_n(\mathbb{Z})$ for $n \geq 3$), it is known that not every arithmetic group commensurable with $\mathrm{PSL}_2(\mathbb{Z})$ is congruence. It is possible to invoke a number of results in order to test a given group for congruence. In order to do so we widen the definition of level to make sense for non-congruence groups.

Definition. A finite index subgroup $G < \mathrm{PSL}_2(\mathbb{Z})$ has *level* N if it contains the normal closure of the element

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^N = \begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix}$$

in $\mathrm{PSL}_2(\mathbb{Z})$, and N is minimal with this property. Equivalently, letting $\mathrm{PSL}_2(\mathbb{Z})$ act on the coset decomposition induced by G, one obtains a representation

$$\varphi_G : \mathrm{PSL}_2(\mathbb{Z}) \to S_k$$

of $\mathrm{PSL}_2(\mathbb{Z})$ into the symmetric group S_k on k letters, where $k = [\mathrm{PSL}_2(\mathbb{Z}) : G]$. The level of G is then defined to be the order of the element

$$\varphi_G\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right)$$

in S_k.

Given this wider definition, the following theorem of Wohlfahrt [49] provides a test for congruence.

Theorem 2.4.1 (Wohlfahrt)**.** *The group G is congruence if and only if it contains the principal congruence subgroup $\Gamma(N)$, where N is the level of G.*

Note that there is a simple formula to calculate the index $[\mathrm{PSL}_2(\mathbb{Z}) : \Gamma(N)]$; it is

$$[\mathrm{PSL}_2(\mathbb{Z}) : \Gamma(N)] = \begin{cases} \frac{N^3}{2} \prod_{p|N} \left(1 - \frac{1}{p^2}\right) & \text{for } N > 2, \\ 6 & \text{for } N = 2, \end{cases}$$

where p ranges over primes dividing N. We remark also that in this setting, Hsu [24] gives an algorithm that one can run to test a group for congruence.

Remark. An alternative definition of a congruence subgroup is that a group is congruence if it contains some $\mathrm{PSL}_2(\mathbb{R})$-conjugate $\alpha\Gamma(N)\alpha^{-1}$ of a principal congruence subgroup $\Gamma(N)$. For groups commensurable with $\mathrm{PSL}_2(\mathbb{Z})$, we must have that $\alpha \in \mathrm{PGL}_2(\mathbb{Q})$, the commensurator of $\mathrm{PSL}_2(\mathbb{Z})$. The following lemma shows that in the present situation, this is equivalent to the definition given above.

Lemma 2.4.2. *Any $\mathrm{PGL}_2(\mathbb{Q})$-conjugate of $\Gamma(N)$ contains some principal congruence subgroup $\Gamma(N')$.*

Proof. We refer to Chapter III of Lang [25] for the outline of the argument. If $\alpha \in \mathrm{GL}_2(\mathbb{Q})$, then we may clear denominators if necessary to ensure that $\alpha \in \mathrm{PGL}_2(\mathbb{Q})$ has integer entries. We set the determinant $\det \alpha = D \in \mathbb{Z}$. If we write

$$\alpha = \begin{pmatrix} x & y \\ z & w \end{pmatrix},$$

and let $A \in \Gamma(DN)$, so we have that each entry of $A - I$ is 0 mod DN, then it can be checked that each entry of

$$\alpha(A - I)\alpha^{-1} = \frac{1}{D} \begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} aDN & bDN \\ cDN & dDN \end{pmatrix} \begin{pmatrix} w & -y \\ -z & x \end{pmatrix}$$

is 0 mod N. As such, we have that $\alpha A \alpha^{-1} \in \Gamma(N)$, and hence $\alpha\Gamma(DN)\alpha^{-1} \subset \Gamma(N)$. From this it follows that the inverse conjugation $\alpha^{-1}\Gamma(N)\alpha$ contains $\Gamma(DN)$, as required. $\square$

It will be useful to understand the maximal groups in which arithmetic groups can be contained. Define Γ to be a *maximal arithmetic* group if it is not properly contained in another (arithmetic) Fuchsian group. It is known by Borel [8] that there are infinitely many maximal arithmetic Fuchsian groups in the commensurability class of $\mathrm{PSL}_2(\mathbb{Z})$; these groups have the form $N(\Gamma_0(n))$, where N denotes the normalizer in $\mathrm{PSL}_2(\mathbb{R})$, and n is a square-free integer. It is due to Ogg [34] that only finitely many of these maximal groups are of genus zero. The elements of $N(\Gamma_0(n))$ are well understood by work of Helling ([22, 23]); they are of the form

$$\begin{pmatrix} a\sqrt{q} & \frac{b}{\sqrt{q}} \\ \frac{cn}{\sqrt{q}} & d\sqrt{q} \end{pmatrix},$$

where $a, b, c, d \in \mathbb{Z}$, $q \geq 1$ is an integer dividing n, and $adq - \dfrac{bcn}{q} = 1$.

There are deep relations between the number theory of an arithmetic Fuchsian group and its action on the hyperbolic plane (and therefore the geometry of the quotient space). For example, if λ_1 denotes the first non-zero eigenvalue of the discrete spectrum of the Laplacian operator, then it is known [44] that the purely arithmetic condition of being congruence implies that

$$\lambda_1 \geq \frac{3}{16}.$$

On the other hand, Buser and Sarnak constructed examples of congruence Fuchsian groups with index two subgroups having λ_1 arbitrarily small [11].

The value of λ_1 is often difficult to calculate explicitly. As such, it is useful to estimate it via Cheeger's isoperimetric constant. This is defined for a finite area surface M as

$$h(M) = \inf_S \frac{\text{Length}(S)}{\min(\text{Area}(A), \text{Area}(B))}$$

where S is a union of geodesic segments which separates M into two connected components A and B. We have the following bounds, due respectively to Cheeger [14] and Buser [12]:

$$\frac{h^2(M)}{4} \leq \lambda_1(M) \leq 2h(M) + 10h^2(M).$$

2.4.2 Arithmetic Kleinian Groups

The construction of arithmetic Kleinian groups is analogous to that of Fuchsian groups. Let d be a square-free integer, $k = \mathbb{Q}(\sqrt{-d})$ be a number field, and $\mathcal{O}_d$ be the ring of integers of k. Then the corresponding analogue of $\mathrm{PSL}_2(\mathbb{Z})$ is the Bianchi group

$$\mathrm{PSL}_2(\mathcal{O}_d) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d, \in \mathcal{O}_d, ad - bc = 1 \right\}.$$

A non-cocompact Kleinian group $\Gamma < \mathrm{PSL}_2(\mathbb{C})$ is called *arithmetic* if it is commensurable with some Bianchi group $\mathrm{PSL}_2(\mathcal{O}_d)$. It is known that the Bianchi groups have finite covolume, and so again we find that non-cocompact arithmetic Kleinian groups must have finite covolume.

An arithmetic Kleinian group Γ is called *congruence* if it contains some principal congruence subgroup

$$\Gamma(I) = \mathrm{P}\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathcal{O}_d) \mid a \equiv d \equiv 1, b \equiv c \equiv 0 \bmod I \right\},$$

where $I < \mathcal{O}_d$ is an ideal. These groups are again normal and of finite index. We similarly define the related groups

$$\Gamma_0(I) = \mathrm{P}\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathcal{O}_d) \mid c \equiv 0 \bmod I \right\}.$$

As in the case of $\mathrm{PSL}_2(\mathbb{Z})$, there exist non-congruence arithmetic groups commensurable with the Bianchi groups. We extend the notion of level thus (see Petersen [35]):

Definition. Suppose that we are given a fixed Bianchi group $\mathrm{PSL}_2(\mathcal{O}_d)$ and a finite index subgroup $G < \mathrm{PSL}_2(\mathcal{O}_d)$. Then G has $\mathbb{Z}$-level N if G contains the normal closure of the group generated by matrices of the form

$$\left\{ \begin{pmatrix} 1 & Nx \\ 0 & 1 \end{pmatrix} \mid x \in \mathcal{O}_d \right\}$$

in $\mathrm{PSL}_2(\mathcal{O}_d)$, and N is the minimal positive integer with this property. We say that G has $\mathcal{O}_d$-level I (for an ideal $I < \mathcal{O}_d$) if I is maximal with the property that the normal closure of the group generated by

$$\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix},$$

for all $y \in I$, is contained in G.

We have the following extension of Wohlfahrt's Theorem (Theorem 2.4.1) which appears in Petersen [35]. We also note that Scarth's thesis proves it for a wider class of groups ([42], Corollary 5.2.3).

Theorem 2.4.3. *Given an imaginary quadratic number field $\mathbb{Q}(\sqrt{-d})$ of class number one and a finite index subgroup $G < \mathrm{PSL}_2(\mathcal{O}_d)$, then G is congruence if and only if it contains the principal congruence subgroup $\Gamma(I)$, where I is the $\mathcal{O}_d$-level of G.*

As in the Fuchsian case, we are able to compute a representation

$$\varphi_G : \mathrm{PSL}_2(\mathcal{O}_d) \to S_h,$$

where $h = [\mathrm{PSL}_2(\mathcal{O}_d) : G]$, and the orders of the images $\varphi_G(A)$ and $\varphi_G(B)$. Although this does not necessarily give the $\mathcal{O}_d$-level of G, it is enough to tell us the $\mathbb{Z}$-level, which will suffice for our purposes.

Define Γ to be a *maximal arithmetic* Kleinian group if it is not properly contained in another Kleinian group. Given some Bianchi group $\mathrm{PSL}_2(\mathcal{O}_d)$, there are infinitely many maximal arithmetic Kleinian groups in its commensurability class, and these have the structure $N(\Gamma_0(I))$, where N denotes normalizer in $\mathrm{PSL}_2(\mathbb{C})$, and I is a square-free ideal ([8], [15]).

A Fuchsian group has genus zero if and only if it is normally generated by parabolic elements (which correspond to cusps) and elliptic elements (which correspond to cone points); as such, quotienting out parabolics and abelianizing the result leaves only elements of finite order. This process can also be carried out for a Kleinian group: following Baker [5], we define a Kleinian group Γ to have *trivial cuspidal cohomology* if the rank

$$\dim_{\mathbb{Q}}((\Gamma/U_\Gamma)^{\mathrm{ab}} \otimes_{\mathbb{Z}} \mathbb{Q})$$

is equal to zero, where U_Γ is the group generated by parabolic elements of Γ. Topologically, this is the rank of the rational homology which is not contributed by the boundary of the manifold or orbifold $\mathbb{H}^3/\Gamma$. It is known ([5], [21], [41], [46], [50]) that finitely many of the Bianchi groups have trivial cuspidal cohomology.

2.5 Reflection Groups

A *hyperbolic reflection group* $\Gamma_{\mathrm{ref}} < \mathrm{Isom}(\mathbb{H}^n)$ is the group generated by reflections in a hyperbolic polyhedron $Q \subset \mathbb{H}^n$. It is assumed such groups are discrete. We say Γ_{ref} is of *finite covolume* if Q has finite volume. A reflection group Γ_{ref} is not orientation-preserving, but it admits an index two subgroup Γ which is, for example by considering the group generated by products of pairs of the generating reflections; this subgroup is often called the *rotation subgroup* of Γ_{ref}.

In the case of dimension two, it is known that the requirement that Γ_{ref} be discrete is equivalent to the polygon Q having all angles equal to submultiples of π (where the convention is that $0 = \dfrac{\pi}{\infty}$, and a vertex with this angle is an ideal vertex). In higher dimensions, this condition on the dihedral angles between the bounding hyperplanes is a necessary, but not sufficient, condition for discreteness. For example, Andreev's Theorem [4] describes the picture for compact polyhedra in three dimensions (see also [40] for an alternative proof and Rivin [39] for the non-compact case):

Theorem 2.5.1 (Andreev)**.** *If P is a compact, finite-sided hyperbolic polyhedron with dihedral angle α_i at each edge e_i, then the following conditions hold:*

1. *For each i, $\alpha_i > 0$;*

2. *If the three edges e_i, e_j, and e_k meet at a vertex, then $\alpha_i + \alpha_j + \alpha_k > \pi$;*

3. *If there exists a prismatic 3-circuit intersecting e_i, e_j, and e_k, then $\alpha_i + \alpha_j + \alpha_k < \pi$;*

4. *If there exists a prismatic 4-circuit intersecting e_i, e_j, e_k and e_l, then $\alpha_i + \alpha_j + \alpha_k + \alpha_l < 2\pi$;*

5. *For a quadrilateral face with edges enumerated successively e_1, e_2, e_3, e_4, and e_{12}, e_{23}, e_{34}, e_{41} are such that e_{12} is the third edge meeting at the vertex where e_1 and e_2 intersect (and similarly for other e_{ij}), then*

   (a) $\alpha_1 + \alpha_3 + \alpha_{12} + \alpha_{23} + \alpha_{34} + \alpha_{41} < 3\pi$; and

(b) $\alpha_2 + \alpha_4 + \alpha_{12} + \alpha_{23} + \alpha_{34} + \alpha_{41} < 3\pi$.

Moreover, any abstract polyhedron satisfying the above can be realized as a compact hyperbolic polyhedron in $\mathbb{H}^3$*, and any* P *satisfying the conditions is unique up to isometries of* $\mathbb{H}^3$.

We define an *arithmetic* hyperbolic reflection group to be a reflection group belonging to $\mathrm{Isom}(\mathbb{H}^n)$ which is commensurable with a discrete, arithmetic group of orientation-preserving isometries. Such a group is *maximal* if it is not properly contained in another such group. It was proved by Agol, Belolipetsky, Storm, and Whyte [2] and Nikulin [32] that there are only finitely many arithmetic maximal hyperbolic reflection groups; it is of interest to ask how many there are in total.

In their paper, Agol, Belolipetsky, Storm and Whyte argue as follows. Each maximal reflection group corresponds to a maximal arithmetic isometry group, obtained by including in the group the symmetries of the polyhedron Q. It is known that above dimension two, the covolumes of these groups are discrete, and that there are only finitely many below a given bound. Therefore an upper bound for the volume of the polyhedron gives an upper bound on the number of possible reflection groups in the given dimension.

The existence of this bound implies the finiteness of the set of groups in question. However, in order to compute the number, it would be helpful to improve the current bound. If the groups were congruence, the additional information one would obtain regarding the first eigenvalue of the Laplacian discussed above would be one step in this direction, since this would provide a lower bound on λ_1 in all dimensions by Burger-Sarnak [10]. It is known that if the group is maximal as an arithmetic group (and not just as a reflection group), then it is congruence. It is therefore reasonable to ask whether all arithmetic maximal hyperbolic reflection groups are congruence, and, if not, whether one can find a universal lower bound on their values of λ_1.

Chapter 3

Dirichlet–Ford Domains

The main goal of this chapter is to prove the following theorem.

Theorem 3.0.1. *A Fuchsian group* Γ *admits a fundamental domain which is simultaneously a Dirichlet domain and a Ford domain if and only if it is the rotation subgroup of a reflection group* Γ_{ref}.

3.1 Preliminaries

Let Γ be a non-cocompact Fuchsian group. We begin by stating the definition of Dirichlet–Ford domain.

Definition. A *Dirichlet–Ford domain*, or a *DF domain*, is a fundamental domain for Γ which is both a Dirichlet domain for some center z_0 and a Ford domain for some choice of a fundamental region for Γ_∞.

To see that DF domains exist, consider the example of $\mathrm{PSL}_2(\mathbb{Z})$. Constructing the Dirichlet domain centered at $z_0 = 2i$ (or in fact any point yi on the imaginary axis for $y > 1$) gives rise to the well-known fundamental domain for $\mathrm{PSL}_2(\mathbb{Z})$ which is an ideal triangle P with vertices at $\rho = -\dfrac{1}{2} + \dfrac{\sqrt{-3}}{2}$, $-\overline{\rho} = \dfrac{1}{2} + \dfrac{\sqrt{-3}}{2}$ and ∞. This fundamental domain P can also be obtained as a Ford domain: it suffices to consider the isometric circles of radius 1, centered at each integer on the real line, and take the vertical strip between $-\dfrac{1}{2}$ and $\dfrac{1}{2}$ as a fundamental region for

$$\Gamma_\infty = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle.$$

Thus P is a DF domain for $\mathrm{PSL}_2(\mathbb{Z})$.

We will make use of the following two standard theorems (see Ratcliffe [37], Chapter 7).

Theorem 3.1.1. *Let G be a discrete reflection group with respect to the polygon Q. Then all dihedral angles of Q are integer submultiples of π, and if g_S and g_T are reflections in adjacent sides S and T of Q meeting with angle $\dfrac{\pi}{k}$, then $g_S g_T$ has order k.*

Theorem 3.1.2. *Let Q be a finite-sided convex hyperbolic polygon of finite volume, all of whose angles are integer submultiples of π. Then the group G generated by reflections of $\mathbb{H}^2$ in the sides of Q is a discrete reflection group.*

3.2 Reflection Groups

We begin by proving the following, one direction of Theorem 3.0.1.

Theorem 3.2.1. *If Γ_{ref} is the discrete group generated by reflections in a finite-sided, finite area, non-compact polygon Q, then the rotation subgroup Γ of Γ_{ref} admits a DF domain.*

Proof. Since Q is not compact, it has at least one ideal vertex. Suppose this vertex is placed at ∞. Then Q has two vertical sides which meet at ∞, contained in vertical geodesics L and M respectively, and a finite number k of non-vertical sides $S_1, \ldots, S_k$. If σ_i, $1 \le i \le k$, denotes the reflection in the geodesic $\tilde{S}_i$ in which the ith side is contained, and σ_L and σ_M denote reflections in L and M respectively, then

$$\{\sigma_1, \ldots, \sigma_k, \sigma_L, \sigma_M\}$$

constitutes a generating set for Γ_{ref}. Let $P = Q \cup \sigma_M(Q)$ denote the union of Q and its reflection in the vertical side M. We wish to show that P is a fundamental domain for Γ, the rotation subgroup of Γ_{ref}, and further, that P is a DF domain.

The rotation subgroup Γ consists of all of the elements of Γ_{ref} which preserve orientation; i.e. those elements which consist of an even number of reflections. It is generated by products of two reflections. Consider the set

$$\mathcal{S} = \{\sigma_M\sigma_1, \ldots, \sigma_M\sigma_k, \sigma_M\sigma_L\}.$$

This is a generating set for Γ, because, given any pair of reflections σ_i, σ_j, we have

$$\sigma_i\sigma_j = \sigma_i(\sigma_M\sigma_M)\sigma_j = (\sigma_i\sigma_M)(\sigma_M\sigma_j) = (\sigma_M\sigma_i)^{-1}(\sigma_M\sigma_j).$$

For each i including $i = L$, the generator $\sigma_M\sigma_i$ identifies S_i with $\sigma_M(S_i)$, because σ_i fixes S_i pointwise. Thus the set $\mathcal{S}$ forms a set of side-pairings of P. Since $\sigma_M \notin \Gamma$, no two points of $\mathring{P}$ can be identified by elements of Γ, but each side of P is identified with another. Thus P is a fundamental domain for Γ.

Lemma 3.2.2. *The polygon P is a Dirichlet domain for Γ.*

Proof. Let $z_0 \in M \cap \mathring{P}$ be any point interior to P which lies on the vertical geodesic M. Fix some side S_i of P. Then $\tilde{S}_i$ bisects z_0 and $\sigma_i(z_0)$, and so $\sigma_M(\tilde{S}_i)$ bisects $\sigma_M(z_0) = z_0$ and $\sigma_M\sigma_i(z_0)$. But, by construction, $\sigma_M(S_i)$ is itself a side of P. Note that this holds equally for $i = L$. This shows that P contains a Dirichlet fundamental domain for Γ; but since we know that P is a fundamental domain, it shows that P is itself a Dirichlet domain. $\square$

Lemma 3.2.3. *The polygon P is a Ford domain for Γ.*

Proof. For each i, $1 \leq i \leq k$, the side S_i is paired with $\sigma_M(S_i)$ by the generator $\sigma_M \sigma_i$. Since σ_i fixes $\tilde{S}_i$ pointwise and σ_M is a Euclidean isometry, $\sigma_M \sigma_i$ acts as a Euclidean isometry on $\tilde{S}_i$. Furthermore, $\sigma_M \sigma_i$ sends $\sigma_i(\infty) \neq \infty$ to ∞, and thus is not itself a Euclidean isometry. This is enough for us to conclude that $\tilde{S}_i$ is the isometric circle of $\sigma_M \sigma_i$. A similar argument shows that $\sigma_M(\tilde{S}_i)$ is the isometric circle for $(\sigma_M \sigma_i)^{-1} = \sigma_i \sigma_M$. The parabolic element $\sigma_M \sigma_L$ pairs L with $\sigma_M(L)$. Thus P is defined by isometric circles, and so P contains a Ford domain for Γ. Since P is a fundamental domain, it must be a Ford domain. $\square$

The combination of Lemmas 3.2.2 and 3.2.3 completes the proof of Theorem 3.2.1.

$\square$

Remark. Notice that this construction, when combined with the Poincaré Polygon Theorem, provides a presentation for the group Γ. If we suppose that S_1 is the side adjacent to the vertical geodesic M (and so S_k is adjacent to L), that the angle between S_i and S_{i-1} is $\dfrac{\pi}{n_i}$ (where, for present purposes, $M = S_0$ and $L = S_{k+1}$), and for simplicity we denote $\tau_i = \sigma_M \sigma_i$, then

$$\Gamma \cong \left\langle \tau_1, \ldots, \tau_k, \tau_L \mid (\tau_1)^{n_1}, (\tau_2 \tau_1^{-1})^{n_2}, \ldots, (\tau_k \tau_{k-1})^{n_k}, (\tau_L \tau_k^{-1})^{n_{k+1}} \right\rangle,$$

where if $n_i = \infty$ (and thus the corresponding vertex is ideal), we delete the relation $(\tau_i \tau_{i-1}^{-1})^{\infty}$.

We now prove the other direction of Theorem 3.0.1.

Theorem 3.2.4. *If the Fuchsian group Γ admits a DF domain, then it is the rotation subgroup of the group generated by reflections in the sides of a polygon Q.*

Before commencing the proof, we will prove two elementary but important lemmas. The first is stated as an exercise in Beardon [6], section 9.6.

Lemma 3.2.5. *Any vertex cycle on the boundary of a Ford domain P is contained within a horocycle based at ∞.*

Proof. Fix a vertex v. By construction of P, v lies on or exterior to all isometric circles, and necessarily lies on at least one. We first consider a $\gamma \in \Gamma$ such that $v \notin S_\gamma$. Then v lies exterior to S_γ. Recalling the decomposition of $\gamma = \gamma_2\gamma_1$ into two reflections, where γ_1 is reflection in S_γ and γ_2 fixes ∞, we observe that γ_1 sends v to a point interior to S_γ. Then, since γ_2 sends S_γ to $S_{\gamma^{-1}}$, it sends $\gamma_1(v)$ to a point interior to $S_{\gamma^{-1}}$ (see Figure 3.1). It follows that $\gamma(v)$ cannot be a vertex of P.

Now suppose that $v \in S_\gamma$. Then γ_1 fixes S_γ pointwise, and hence fixes v. Since γ_2 is reflection in a vertical line, it necessarily preserves the imaginary part of v, proving the lemma. $\square$

Remark. From the argument above, it follows that when we wish to find the images of a vertex v of P under side-pairings of P, it suffices to consider those side-pairings γ such that $v \in S_\gamma$.

Remark. The lemma holds for any point on the boundary of the Ford domain P. For our purposes, it will be enough to have it for the vertices of P.

The second lemma can be found in, for example, Greenberg [20], p. 203. Since the proof is simple application of the definition of a Dirichlet domain, we include it here.

Lemma 3.2.6. *Let P be a Dirichlet domain for Γ with center z_0. Let $1 \neq \gamma \in \Gamma$ and suppose that z, $\gamma(z) \in \partial P \cap \mathbb{H}^2$. Then $d_{\mathbb{H}}(z, z_0) = d_{\mathbb{H}}(\gamma(z), z_0)$.*

Proof. This is an application of the definition of a Dirichlet domain stated above. Specifically, setting $x = z$ and $\alpha = \gamma^{-1}$ yields the inequality

$$d(z, z_0) \leq d(z, \gamma^{-1}(z_0)) = d(\gamma(z), z_0),$$

the latter equality holding because γ is an isometry. Setting $x = \gamma(z)$ and $\alpha = \gamma$ now gives

$$d(\gamma(z), z_0) \leq d(\gamma(z), \gamma(z_0)) = d(z, z_0).$$

Combining these two inequalities gives the required equality. □

We will now use these two lemmas to prove Theorem 3.2.4.

Proof. Suppose P is a DF domain for Γ. We will show that P has reflective symmetry about a vertical line M, and furthermore that the side-pairings of P pair points of ∂P with their reflections in M.

Suppose that we have conjugated Γ so that Γ_∞ is generated by

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Since P is a DF domain, it is contained in a Dirichlet fundamental region for Γ_∞, which is a vertical strip

$$\left\{ z \in \mathbb{H}^2 \mid x_0 \leq \mathrm{Re}(z) \leq x_0 + 1 \right\}$$

for some $x_0 \in \mathbb{R}$. By the assumption that Γ be finitely generated, we see that above a certain height, there are no sides of P besides the vertical lines with real parts x_0 and

$x_0 + 1$ respectively. Shimizu's Lemma (see Theorem 2.2.1) tells us that the radii of the isometric circles S_γ cannot exceed 1, and hence that this height is at most 1. Consider a point $z = x_0 + iy \in \partial P$. Choosing $\gamma = A$, we see that $A(z) = x_0 + 1 + iy$ is another boundary point of P. We apply Lemma 3.2.6 to z and $A(z)$, and deduce that the Dirichlet center z_0 of P lies equidistant from these points. Since they have the same height y, the locus of equidistant points is the vertical line bisecting them. We conclude that $\mathrm{Re}(z_0) = x_0 + \frac{1}{2}$.

Now suppose that $v \in \mathbb{H}^2$ is a vertex of P, and $\gamma \in \Gamma$ is a side-pairing such that $\gamma(v)$ is another vertex of P. By the argument in the proof of Lemma 3.2.5, $v \in S_\gamma$. Then Lemma 3.2.5 tells us that $\mathrm{Im}(\gamma(v)) = \mathrm{Im}(v)$, and Lemma 3.2.6 tells us that $d_{\mathbb{H}}(\gamma(v), z_0) = d_{\mathbb{H}}(v, z_0)$. We consider the two sets

$$\left\{z \in \mathbb{H}^2 \mid \mathrm{Im}(z) = \mathrm{Im}(v)\right\}$$

and

$$\left\{z \in \mathbb{H}^2 \mid d_{\mathbb{H}}(z, z_0) = d_{\mathbb{H}}(v, z_0)\right\}.$$

The former is the horizontal line through v, and the latter a circle with Euclidean center located vertically above z_0. In particular, the picture is symmetrical in the vertical line $\{\mathrm{Re}(z) = x_0 + \frac{1}{2}\}$ (see Figure 3.2). It follows that either $\gamma(v) = v$ or $\gamma(v) = v^*$, where v^* is the reflection of v in the line

$$\left\{\mathrm{Re}(z) = x_0 + \frac{1}{2}\right\}.$$

Suppose first that $\gamma(v) = v$. Because γ_1 fixes v, it must be that γ_2 also fixes v, and thus that γ_2 is reflection in the vertical line through v (see Figure 3.3). Since γ is a side-pairing, and P is locally finite, we may find a point $w \in S_\gamma \cap \partial P$, not equal to v, such that $\gamma(w) \in \partial P$. Now $w \in S_\gamma$, so w is fixed by γ_1, and γ_2 preserves imaginary parts, so $\mathrm{Im}(\gamma(v)) = \mathrm{Im}(v)$.

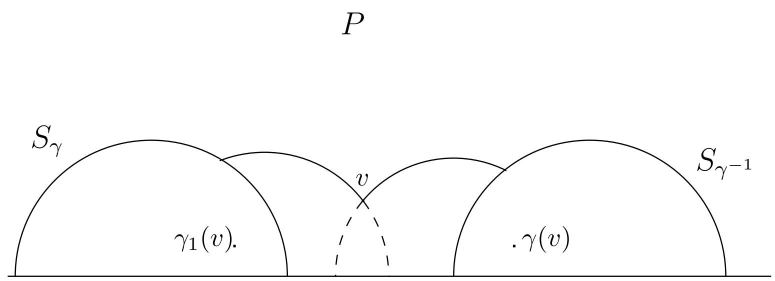


Figure 3.1: The case $v \notin S_\gamma$.

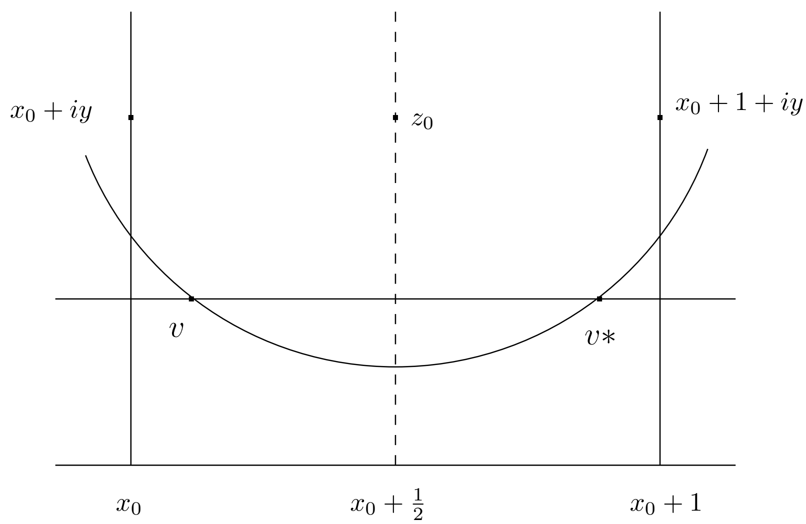


Figure 3.2: The circle and horocycle intersect in at most two points.

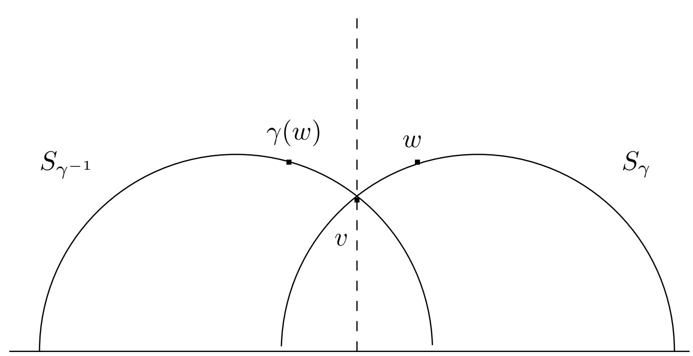


Figure 3.3: The isometry γ fixes the vertex v.

Again applying Lemma 3.2.6, we deduce that z_0 must lie equidistant from w and $\gamma(w)$. Referring again to Figure 3, this locus is precisely the vertical line through v. It therefore follows that $\mathrm{Re}(v) = \mathrm{Re}(z_0) = x_0 + \frac{1}{2}$. Thus we have shown that if v is fixed by γ, then v necessarily lies directly below the Dirichlet center z_0. Bearing in mind the above discussion, the contrapositive of this states that if $\mathrm{Re}(v) \neq x_0 + \frac{1}{2}$, then any side-pairing γ pairing v with a vertex of P must send v to v^*.

We now turn our attention to ideal vertices. Let $v \in \partial\mathbb{H}^2$ be a vertex of P. Then two isometric circles meet at v. Fix one such circle S, which is the isometric circle S_γ of some element $\gamma \in \Gamma$. This isometric circle S_γ contains a side of P adjacent to v, and we pick two points of S_γ, w_1, $w_2 \in \partial P \cap \mathbb{H}^2$ (see Figure 3.4). All three of the points v, w_1, w_2 are fixed by γ_1. By Lemma 3.2.6, γ_2 must send both w_1 and w_2 to points the same respective distance from z_0. Since γ_2 preserves imaginary parts, we see that for each i, w_i is either fixed or sent to its reflection in the line $\{\mathrm{Re}(z) = \mathrm{Re}(z_0)\}$. If w_1 were fixed, w_2 would neither be fixed nor sent to its reflection, and vice-versa if w_2 were fixed. Thus we conclude that γ_2 is reflection in the line $\{\mathrm{Re}(z) = \mathrm{Re}(z_0)\}$.

The above arguments show that if any side-pairing of P, $\gamma \in \Gamma \setminus \Gamma_\infty$, is written $\gamma = \gamma_2\gamma_1$, where γ_1 is reflection in S_γ and γ_2 is a reflection fixing ∞, then γ_2 is reflection σ_M in the line $M = \{z \mid \mathrm{Re}(z) = x_0 + \frac{1}{2}\}$. Letting $L = \{z \mid \mathrm{Re}(z) = x_0\}$ and σ_L denote reflection in L, we may take $A = \sigma_M\sigma_L$ to see that this also holds for Γ_∞. We summarize this in the following result.

Proposition 3.2.7. *Suppose the Fuchsian group Γ admits a DF domain P. Then P has reflective symmetry about a vertical line M. Furthermore, the side-pairings of P each have the form $\sigma_M\sigma_1$, where σ_M is reflection in M and σ_1 is reflection in a side of P.*

We now use Proposition 3.2.7 to complete the proof of Theorem 3.2.4. We do this by showing that the polygon Q obtained by dividing P in half along M satisfies the hypotheses of Theorem 3.1.2.

By Proposition 3.2.7, we have a generating set for Γ of the form

$$\{\sigma_M\sigma_1, \ldots, \sigma_M\sigma_k, \sigma_M\sigma_L\}.$$

By the above discussion, each vertex cycle contains exactly two vertices, with the exception of the point $L \cap \mathring{P}$ directly below z_0. By the Poincaré Polygon Theorem, each vertex cycle sums to $\dfrac{2\pi}{n_i}$ for some $n_i \in \mathbb{N} \cup \{\infty\}$. By the reflectional symmetry of P, the angle at each vertex of Q is $\dfrac{\pi}{n_i}$. We therefore conclude that Q is a convex, finite-sided polygon with all angles integer submultiples of π. We see that the reflections

$$\{\sigma_1, \ldots, \sigma_k, \sigma_M, \sigma_L\}$$

generate a reflection group Γ_{ref} which contains Γ as an orientation-preserving subgroup of index two, or, in other words, its rotation subgroup. This concludes the proof of Theorem 3.2.4, and hence of Theorem 3.0.1. $\square$

Remark. We established that $\text{Re}(z_0) = x_0 + \frac{1}{2}$, but did not deduce anything about the imaginary part of z_0. This is because, by Lemma 3.2.2, we may take the Dirichlet center of P to be any point of the interior of P on this vertical line. This observation will be explored in the subsequent section.

3.3 Double Dirichlet Domains

In this section, we show that there is a similar result to Theorem 3.0.1 for cocompact reflection groups of $\text{Isom}(\mathbb{H}^2)$. Since the definition of a Ford fundamental domain is only valid for non-cocompact Fuchsian groups, we require a slightly different definition as our starting point. However, we note that, as observed in the previous section, if a group admits a DF domain P, then P arises as the Dirichlet domain for various choices of Dirichlet center z_0. This motivates the following definition.

Definition. A *Double Dirichlet Domain*, P, for a Fuchsian group Γ is a fundamental domain for Γ which arises as the Dirichlet domain centered at two distinct points z_0 and z_1.

Notice that every DF domain satisfies this definition. We prove that possessing a Double Dirichlet domain places constraints on a Fuchsian group which are similar to those of DF domains.

Theorem 3.3.1. *A cofinite Fuchsian group Γ admits a Double Dirichlet domain if and only if it is the rotation subgroup of a reflection group Γ_{ref}.*

The proof of this result will follow a similar path to that of Theorem 3.0.1. We first prove a technical lemma. In the previous section, we made an assumption on Γ and Γ_∞ which implied that the two vertical sides were identified by a side-pairing. It was then demonstrated in the subsequent arguments that, given this, there was a unique way that the remaining sides could be identified. In the absence of a similar assumption on the group Γ in the present setting, we prove explicitly that there can be only one way to pair the sides of P.

Lemma 3.3.2. *If $P = P_0 = P_1$ is the Dirichlet domain centered at z_0 and at z_1, then the sides of P are identified the same way in each case.*

Proof. Suppose, for the sake of contradiction, that this is not the case. Any side of a Dirichlet domain bisects the domain's center and its image under some isometry. Here, we have a side A of P which is the bisector of both the pair z_0 and $\gamma_0^{-1}(z_0)$ and the pair z_1 and $\gamma_1^{-1}(z_1)$, where $\gamma_0 \neq \gamma_1$ are the isometries defining that side of P. It follows that γ_0 pairs A with some side B, and γ_1 pairs A with some other side $C \neq B$. Let $d := d(z_0, z_1)$ be the distance between the two centers z_0 and z_1. Since $\gamma_0^{-1}(z_0)$ and $\gamma_1^{-1}(z_1)$ are the reflections of each in A, we see that

$$d(\gamma_0^{-1}(z_0), \gamma_1^{-1}(z_1)) = d.$$

Applying the isometry γ_1 to both points, this gives that

$$d(\gamma_1(\gamma_0^{-1}(z_0)), z_1) = d.$$

Now, if $\gamma_1(\gamma_0^{-1}(z_0)) = z_0$, then the isometries γ_0 and γ_1 both send $\gamma_0^{-1}(z_0)$ to z_0 and $\gamma_1^{-1}(z_1)$ to z_1. Since they also both preserve orientation, this implies that $\gamma_0 = \gamma_1$, a contradiction. Thus $\gamma_1(\gamma_0^{-1}(z_0)) \neq z_0$. But then $\gamma_1(\gamma_0^{-1}(z_0))$ is a point in the orbit of z_0, and thus the construction of P_0 involves the half-space $\{x \in \mathbb{H}^2 \mid d(x, z_0) \leq d(x, \gamma_1(\gamma_0^{-1}(z_0)))\}$. As we saw above,

$$d(\gamma_1(\gamma_0^{-1}(z_0)), z_1) = d(z_0, z_1) = d.$$

Hence z_1 is equidistant from z_0 and $\gamma_1(\gamma_0^{-1}(z_0))$. Thus z_1 cannot be in the interior of P_0, contradicting the assumption that $P_0 = P_1$. □

We are now able to prove Theorem 3.3.1.

Proof. Suppose Γ_{ref} is generated by reflections in the polygon Q. Conjugate Γ such that some side M of Q is vertical. We then remark that the proof of Theorem 3.2.1 can be applied, with the modification that L is no longer a vertical side. Then the proof of Lemma

3.2.2 shows that any point of $M \cap \mathring{P}$ is a Dirichlet center; thus, P arises as the Dirichlet domain for at least two centers.

Now suppose that P is a Double Dirichlet domain. Then there exist two distinct points z_0 and z_1 which are the corresponding Dirichlet centers. Let M be the geodesic passing through these points, and suppose for simplicity that we have conjugated so that M is vertical. As in the proof of Theorem 3.2.4, we aim to show that P has reflective symmetry about M, and that the side-pairings of P pair points with their reflections in M.

Consider one vertex v of P lying on M. If v is ideal, then we can apply Theorem 3.2.4 and get the required conclusion, so suppose v is finite. Then, by Lemma 3.2.6, v must be identified with another vertex v' such that $d(v, z_0) = d(v', z_0)$ and $d(v, z_1) = d(v', z_1)$. Since v, z_0 and z_1 are all distinct points lying on the same geodesic M, it follows that the intersection of the two relevant loci is exactly one point, v. Thus v is fixed by a rotation which pairs the two sides adjacent to v, and furthermore, M bisects the angle at v.

Given this starting point, it now follows by the proof of Theorem 3.2.4 that all of the side-pairings of P consist of products $\sigma_m \sigma_1$ of reflections in sides composed with reflection in M. One again observes that all but at most two of the vertex cycles contain exactly two vertices, and applies the Poincaré Polygon Theorem to see that we may add in the reflection σ_M without violating discreteness. Thus Γ is the rotation subgroup of a reflection group, as required. $\square$

We next apply Theorems 3.0.1 and 3.3.1 to show that given the signature of a sphere, with cone points and/or punctures, such that its fundamental group is hyperbolic, we may find a Fuchsian group Γ, with quotient space of the given signature, which admits a Double Dirichlet domain, or a DF domain if the signature has at least one puncture.

Corollary 3.3.3. *Given the signature* $(0; n_1, \dots, n_t; m)$ *of a (non-trivial) sphere with* $m \geq 0$ *punctures and* $t \geq 0$ *cone points of orders* $n_i \in \mathbb{N}$*, for* $1 \leq i \leq t$*, there exists a Fuchsian group* Γ *such that* Γ *admits a Double Dirichlet domain (and a DF domain if* $m > 0$*) and* $\mathbb{H}^2/\Gamma$ *is a sphere of the given signature.*

Proof. With the discussion following the proof of Theorem 3.2.1 in mind, this result follows immediately from the fact that a convex polygon with the required angles may be constructed in the hyperbolic plane. Specifically, we require a polygon Q with t finite vertices with angle $\dfrac{\pi}{n_i}$ as required, and with m ideal vertices. □

Remark. If $m > 0$ above, then there is a certain amount of freedom in our choice of the polygon Q. For example, we do not necessarily have to place one of the ideal vertices of Q at ∞. We do so in order to ensure that we obtain a DF domain for Γ. Instead, we could have all of the ideal vertices lie in $\mathbb{R}$, thereby placing the line of symmetry L away from any of the ideal vertices. Similarly, if $m > 1$, we could construct Q so that L meets only one of the m ideal vertices, instead of 2 in the construction above. We also do not have to construct Q so that each angle is bisected by a vertical line; we only do so in order to demonstrate that it is possible to find the required polygon.

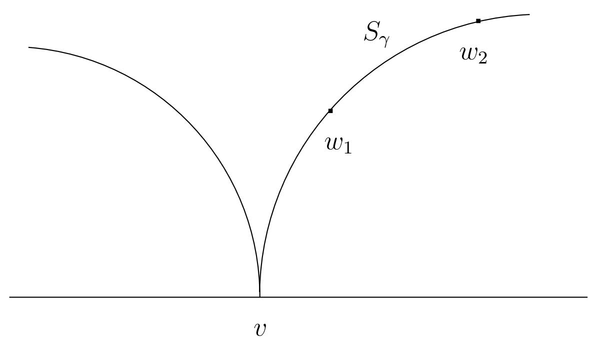


Figure 3.4: The case $v \in \partial\mathbb{H}^2$.

Chapter 4

Arithmetic Maximal Reflection Groups

In this chapter, we apply Theorem 3.0.1 to find all of the non-cocompact arithmetic maximal reflection groups of $\mathrm{Isom}(\mathbb{H}^2)$, and generators in $\mathrm{PSL}_2(\mathbb{R})$ for their respective rotation subgroups. Using this information, we test these groups for congruence.

4.1 Preliminaries

From work of Nikulin [32], we know that there are 122 arithmetic maximal reflection groups in $\mathrm{Isom}(\mathbb{H}^2)$. Such a group is non-cocompact if and only if the corresponding polygon has at least one ideal vertex; this information can be determined from Nikulin's tables. Alternatively, Allcock gives angle information, and ideal vertices are represented by the symbol ∞ in his tables. In this way, we find that 23 of the 122 groups are non-cocompact. Furthermore, we find that thirteen of these have exactly one ideal vertex, and the remaining ten have two ideal vertices.

If Γ_{ref} is a cofinite arithmetic reflection group in $\mathrm{Isom}(\mathbb{H}^2)$, then its rotation subgroup Γ is a cofinite arithmetic Fuchsian group of genus zero. Since the quotient space $\mathbb{H}^2/\Gamma$ has underlying space a punctured sphere, it can only cover other punctured spheres; thus, we see that if Γ is properly contained in some maximal Fuchsian group M, then M must also be of genus zero. The maximal arithmetic Fuchsian groups of genus zero are understood [27]; in particular, there are finitely many such groups. In the non-cocompact case, there are 45; they are $\mathrm{PSL}_2(\mathbb{Z})$ and the normalizers $N(\Gamma_0(n))$ (see Section 2.4) for n belonging to the set

$$\{2, 3, 5, 6, 7, 10, 11, 13, 14, 15, 17, 19, 21, 22, 23, 26, 29, 30, 31, 33, 34, 35, 38, 39,$$

$$41, 42, 46, 47, 51, 55, 59, 62, 66, 69, 70, 71, 78, 87, 94, 95, 105, 110, 119, 141\}.$$

So each non-cocompact arithmetic maximal reflection group gives rise to a rotation subgroup which is contained in one of these normalizers, and which admits a DF domain. Each of these groups is one-cusped, and contains the corresponding $\Gamma_0(n)$ as a finite index subgroup. Notice that, given some maximal arithmetic Fuchsian group M, if M admits a DF domain then, by Theorem 3.0.1, it is itself one of these rotation subgroups, and further that it corresponds to a reflection group with one ideal vertex.

4.2 Examples with One Ideal Vertex

In this section, we find generators in $\mathrm{PSL}_2(\mathbb{R})$, and DF domains, for (the rotation subgroups of) the thirteen non-cocompact arithmetic maximal reflection groups with one ideal vertex. In the process, we will prove the following result.

Proposition 4.2.1. *There are thirteen maximal arithmetic Fuchsian groups of genus zero which admit a DF domain; they are* $\mathrm{PSL}_2(\mathbb{Z})$ *and the normalizers* $N(\Gamma_0(n))$ *for values of* n *in the set*

$$\{2, 3, 5, 6, 7, 10, 13, 14, 21, 30, 34, 39\}.$$

These groups are precisely the rotation subgroups of the thirteen non-cocompact arithmetic maximal reflection groups with one ideal vertex.

It follows immediately from the construction that all thirteen of these examples are congruence.

4.2.1 Construction of Ford domains

In each of the following examples, we compute a Ford domain as follows. Each example contains the element $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, and so we take our fundamental region for Γ_∞ to be between $-\frac{1}{2}$ and $\frac{1}{2}$ on the real line. Since the set of possible entries of matrices in each group is a discrete subset of $\mathbb{R}$, there is a unique smallest such entry x_1; finding all matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with lower-left entry $c = x_1$ is equivalent to finding all isometric circles of maximal radius; notice that we may always pre- and post-multiply by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ if necessary to ensure that the centers $-\frac{d}{c}$ and $\frac{a}{c}$ of the isometric circle and its inverse respectively are between $-\frac{1}{2}$ and $\frac{1}{2}$ inclusive. We then proceed to find all matrices with the next smallest lower-left entry x_2 in the same way, where we may ignore those whose isometric circles (and those of their inverses) are completely contained within those already found. We continue this process until we have a polygon defined by the isometric circles and the lines $\Re(z) = -\frac{1}{2}$ and $\Re(z) = \frac{1}{2}$ which has the required area, which is given to us by knowing the signature of the corresponding quotient surface. Since all these groups are cofinite, the process will always terminate in finite time. We give as an example a detailed explanation of how this algorithm applies to one of the present cases.

We will calculate in detail a Ford domain for the group $N(\Gamma_0(34))$. By Long, Maclachlan and Reid [27], this group has signature $(0; 2, 2, 2, 2, 2, 4; 1)$ and coarea $\frac{9\pi}{2}$. We begin by including the element $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and taking a fundamental region for its action bounded by the lines $\Re(z) = -\frac{1}{2}$ and $\Re(z) = \frac{1}{2}$ (see Figure 4.1). We note that the smallest lower left entry of an element of $\Gamma_0(34)$ is equal to 34; with this in mind, we next seek to minimize the lower left entry of

$$\begin{pmatrix} a\sqrt{q} & \dfrac{b}{\sqrt{q}} \\ \dfrac{34c}{\sqrt{q}} & d\sqrt{q} \end{pmatrix},$$

where $a, b, c, d \in \mathbb{Z}$, $q \geq 1$ is an integer dividing 34, and $adq - \dfrac{34bc}{q} = 1$. We see that $\dfrac{34c}{\sqrt{q}}$ is minimized by minimizing c and maximizing q; in other words, taking $c = 1$ and $q = 34$. We look for ways to complete the matrix

$$\begin{pmatrix} a\sqrt{34} & \dfrac{b}{\sqrt{34}} \\ \sqrt{34} & d\sqrt{34} \end{pmatrix}$$

such that $\dfrac{a\sqrt{34}}{\sqrt{34}}$ and $\dfrac{-d\sqrt{34}}{\sqrt{34}}$ are between $-\dfrac{1}{2}$ and $\dfrac{1}{2}$. This can only be achieved if $a = d = 0$, which then implies that $b = -1$, and we have the matrix

$$\begin{pmatrix} 0 & -\dfrac{1}{\sqrt{34}} \\ \sqrt{34} & 0 \end{pmatrix},$$

which we add to our generating set. We also add the isometric circle of this matrix to our Ford domain, and note that the isometric circle of the inverse is the same isometric circle. This circle has center 0 and radius $\dfrac{1}{\sqrt{34}}$ (see Figure 4.2).

We now seek the next smallest lower left entry; this is obtained by setting $c = 1$ and $q = 17$. This makes the standard matrix

$$\begin{pmatrix} a\sqrt{17} & \dfrac{b}{\sqrt{17}} \\ 2\sqrt{17} & d\sqrt{17} \end{pmatrix}.$$

Here if either a or d were equal to 0, then the determinant would be $-2b$, for which no choice of integer b makes the determinant equal to 1. Thus if $a = d = 1$, we have $b = 8$ and we obtain the matrix

$$\begin{pmatrix} \sqrt{17} & \dfrac{8}{\sqrt{17}} \\ 2\sqrt{17} & \sqrt{17} \end{pmatrix},$$

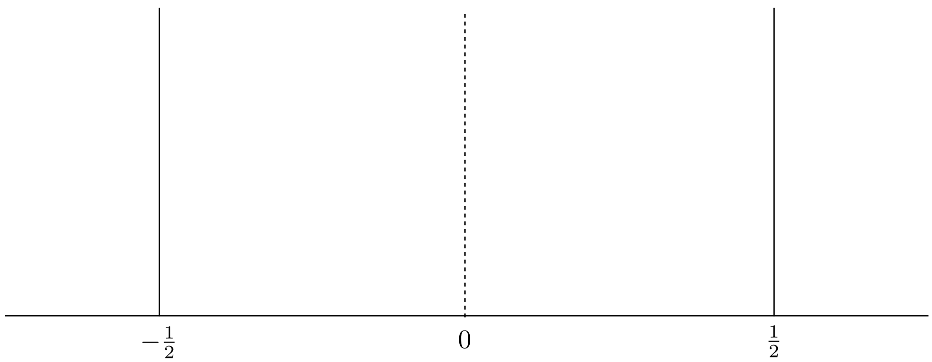


Figure 4.1: First step in the construction of a Ford domain for $N(\Gamma_0(34))$

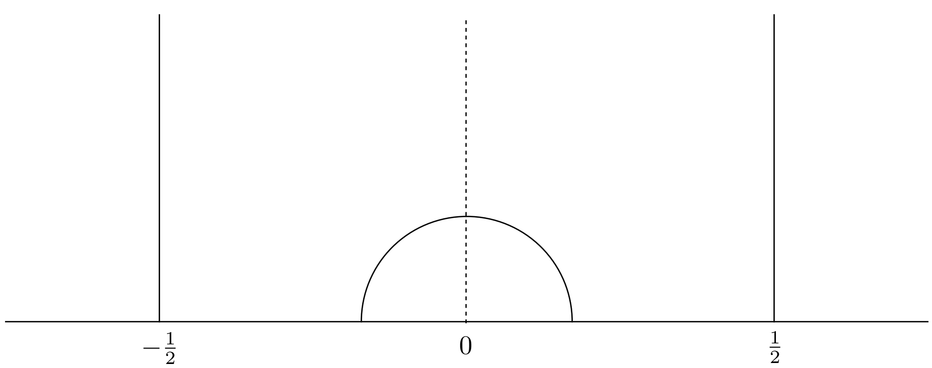


Figure 4.2: Second step in the construction of a Ford domain for $N(\Gamma_0(34))$

which gives us two isometric circles (for the element and its inverse) centered at $-\dfrac{1}{2}$ and $\dfrac{1}{2}$ respectively, and of radius $\dfrac{1}{2\sqrt{17}}$ (see Figure 4.3).

We next come to the case of $c = 2$, $q = 34$. But we notice that the matrix

$$\begin{pmatrix} a\sqrt{34} & \dfrac{b}{\sqrt{34}} \\ 2\sqrt{34} & d\sqrt{34} \end{pmatrix}$$

has determinant $34ad - 2b$, and thus no choices of $a, b, d \in \mathbb{Z}$ make this determinant equal to 1. We therefore move on to the case $c = 2$, $q = 17$. Here $\dfrac{2.34}{\sqrt{17}} = 4\sqrt{17}$, so we seek matrices of the form

$$\begin{pmatrix} a\sqrt{17} & \dfrac{b}{\sqrt{17}} \\ 4\sqrt{17} & d\sqrt{17} \end{pmatrix}.$$

If either a or d is even then the determinant must be even. If $a = d = 1$ we find $b = 4$ and obtain the matrix

$$\begin{pmatrix} \sqrt{17} & \dfrac{4}{\sqrt{17}} \\ 4\sqrt{17} & \sqrt{17} \end{pmatrix}.$$

We see that the isometric circles for this element and its inverse are not contained inside the ones we have already found, and so we add these circles to our Ford domain, and the matrix to our list (see Figure 4.4).

We move on to the case of $c = 3$, $q = 34$. This gives a matrix of the form

$$\begin{pmatrix} a\sqrt{34} & \dfrac{b}{\sqrt{34}} \\ 3\sqrt{34} & d\sqrt{34} \end{pmatrix}$$

and again we see that if a or d is 0 then the determinant cannot be equal to 1. This leads us to the matrix

$$\begin{pmatrix} \sqrt{34} & \dfrac{11}{\sqrt{34}} \\ 3\sqrt{34} & \sqrt{34} \end{pmatrix}.$$

The isometric circles of this element and its inverse are not covered by our existing circles, and so we add these circles to our Ford domain, and the element to our generating set (see Figure 4.5).

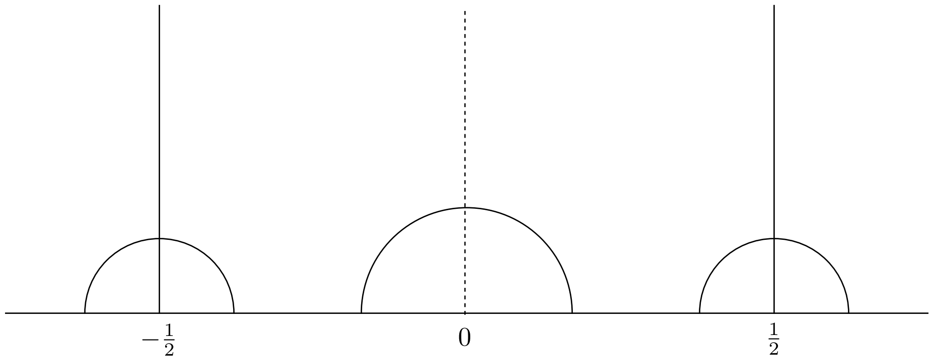


Figure 4.3: Third step in the construction of a Ford domain for $N(\Gamma_0(34))$

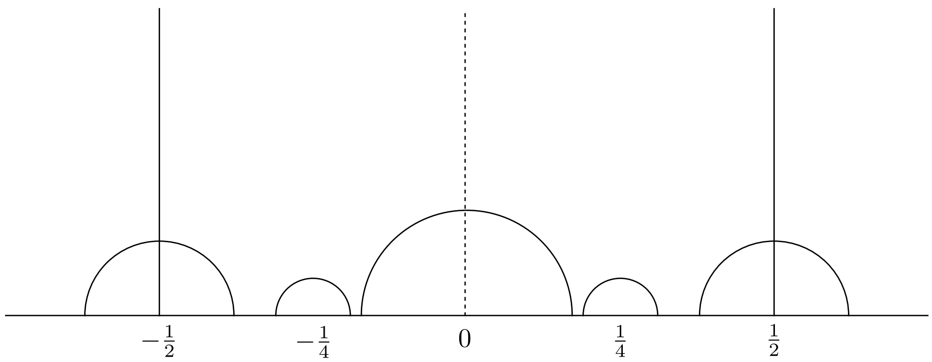


Figure 4.4: Fourth step in the construction of a Ford domain for $N(\Gamma_0(34))$

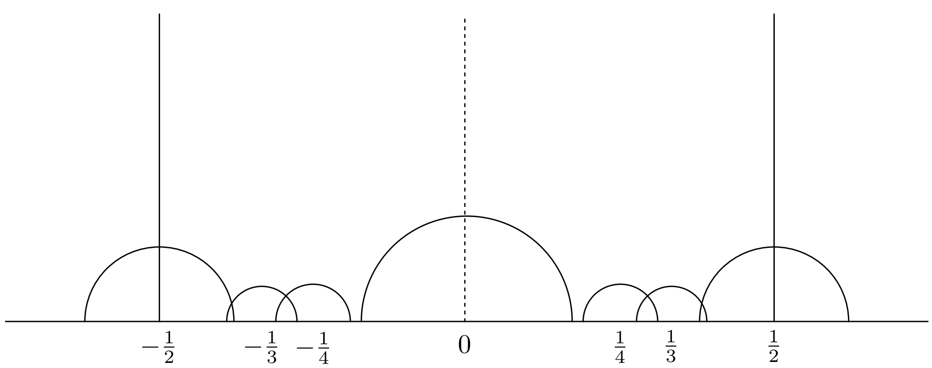


Figure 4.5: Fifth step in the construction of a Ford domain for $N(\Gamma_0(34))$

The next case to consider is that of $c = 4$, $q = 34$, and therefore matrices of the form

$$\begin{pmatrix} a\sqrt{34} & \dfrac{b}{\sqrt{34}} \\ 4\sqrt{34} & d\sqrt{34} \end{pmatrix};$$

however, such a matrix must have even determinant. We therefore move on to the case of $c = 1$, $q = 2$, and matrices of the form

$$\begin{pmatrix} a\sqrt{2} & \dfrac{b}{\sqrt{2}} \\ 17\sqrt{2} & d\sqrt{2} \end{pmatrix}.$$

We seek values of a and d between -8 and 8 inclusive such that $2ad - 17b = 1$. One solution is to let $a = d = 3$ and $b = 1$; the resulting matrix

$$\begin{pmatrix} 3\sqrt{2} & \dfrac{1}{\sqrt{2}} \\ 17\sqrt{2} & 3\sqrt{2} \end{pmatrix}.$$

has isometric circles not covered by those we have, and so we add these circles to our Ford domain and the matrix to our generating set. At this point we observe that our Ford domain is now of finite area, and in fact has area equal to $\dfrac{9\pi}{2}$, the coarea of the group. We also check that all of the isometric circles which correspond to other possible matrices of the present form are covered by the existing isometric circles. Notice also that if we did not know in advance the coarea of our group, we would have to continue this algorithm, and it would terminate when we reach a point where the radii of the isometric circles is less than the imaginary part of the lowest point of our Ford domain.

We therefore conclude that the matrices

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -\frac{1}{\sqrt{34}} \\ \sqrt{34} & 0 \end{pmatrix}, \begin{pmatrix} \sqrt{17} & \frac{8}{\sqrt{17}} \\ 2\sqrt{17} & \sqrt{17} \end{pmatrix},$$

$$\begin{pmatrix} \sqrt{17} & \frac{4}{\sqrt{17}} \\ 4\sqrt{17} & \sqrt{17} \end{pmatrix}, \begin{pmatrix} \sqrt{34} & \frac{11}{\sqrt{34}} \\ 3\sqrt{34} & \sqrt{34} \end{pmatrix}, \begin{pmatrix} 3\sqrt{2} & \frac{1}{\sqrt{2}} \\ 17\sqrt{2} & 3\sqrt{2} \end{pmatrix},$$

constitute a generating set for $N(\Gamma_0(34))$. We observe that for each matrix in this list, the two entries on the diagonal are equal to each other. This means that each side of our domain is paired with its reflection in the imaginary axis; in other words, this will be a DF domain (see Figure 4.6).

4.2.2 Results

Below we list, in each of the thirteen cases we consider, the matrices required to define a Ford domain by considering their isometric circles, and those of their inverses in the case they are not of order two. These are obtained using the same algorithm described above. These matrices also define a DF domain; in the setup described, this is equivalent to each generator having equal diagonal entries, that is, $a = d$. The DF domains themselves will be drawn.

Example 1. $\mathrm{PSL}_2(\mathbb{Z})$ is generated by the two matrices

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Example 2. $N(\Gamma_0(2))$ is generated by

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -\frac{1}{\sqrt{2}} \\ \sqrt{2} & 0 \end{pmatrix}.$$

Example 3. $N(\Gamma_0(3))$ is generated by

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -\frac{1}{\sqrt{3}} \\ \sqrt{3} & 0 \end{pmatrix}.$$

Example 4. $N(\Gamma_0(5))$ is generated by

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -\frac{1}{\sqrt{5}} \\ \sqrt{5} & 0 \end{pmatrix}, \begin{pmatrix} \sqrt{5} & \frac{2}{\sqrt{5}} \\ 2\sqrt{5} & \sqrt{5} \end{pmatrix}.$$

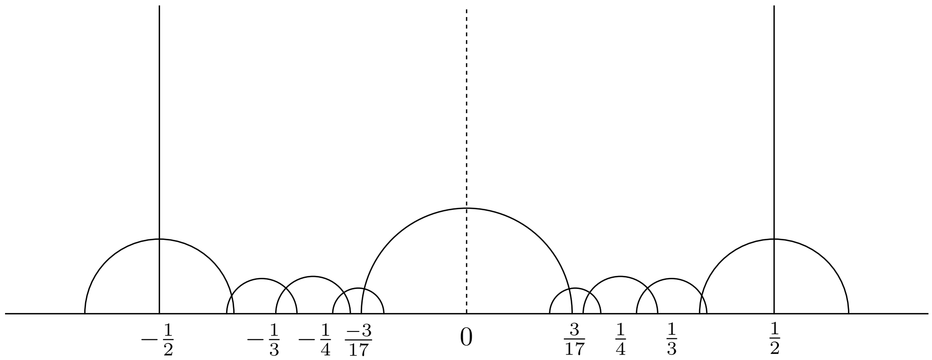


Figure 4.6: Final result of the construction of a Ford domain for $N(\Gamma_0(34))$

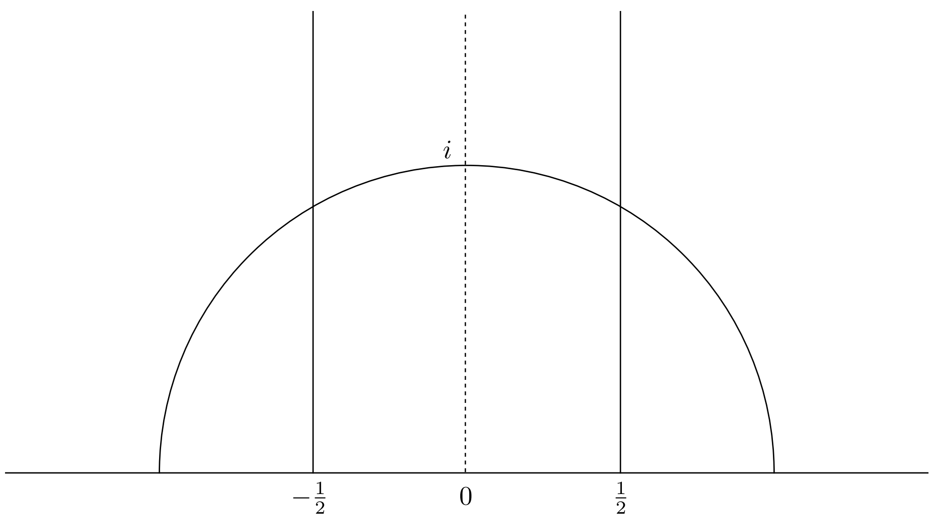


Figure 4.7: Ford domain for $\mathrm{PSL}_2(\mathbb{Z})$

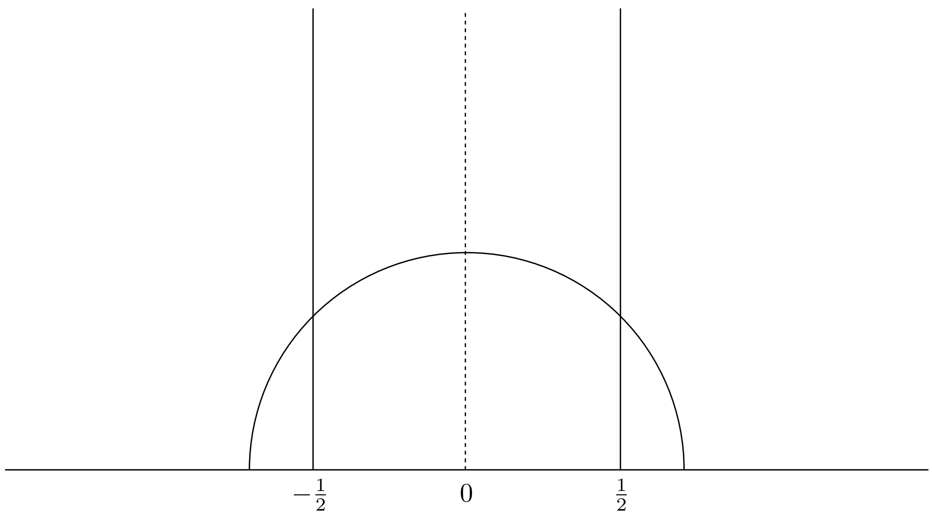


Figure 4.8: Ford domain for $N(\Gamma_0(2))$

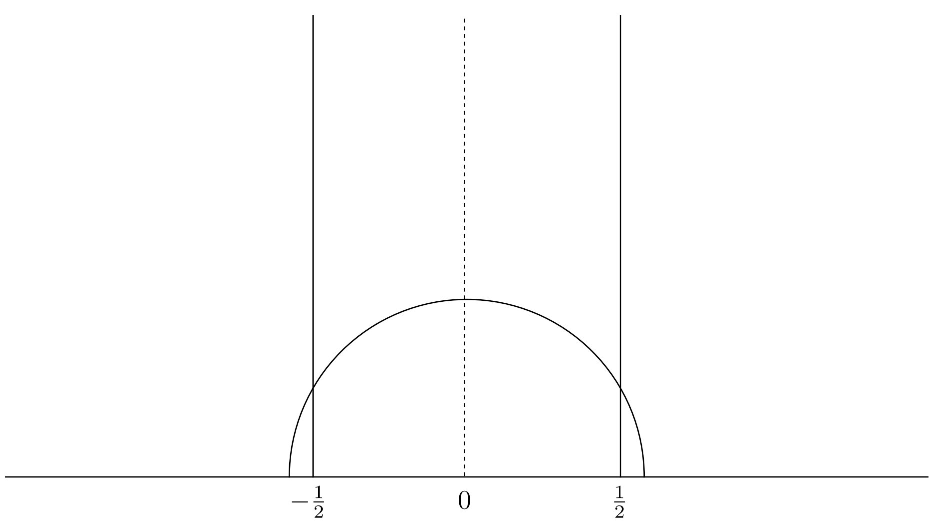


Figure 4.9: Ford domain for $N(\Gamma_0(3))$

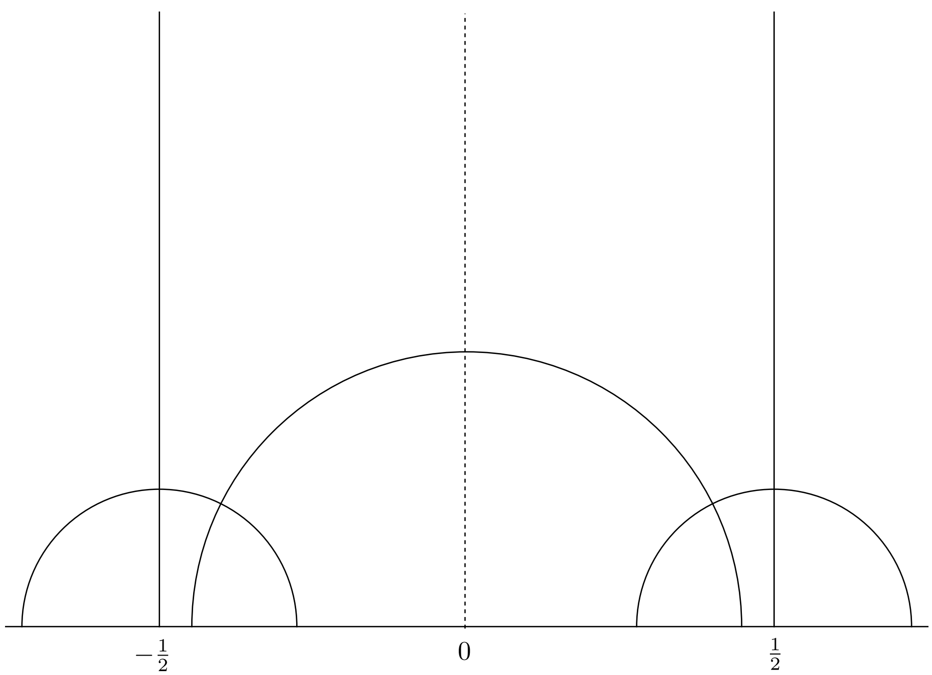


Figure 4.10: Ford domain for $N(\Gamma_0(5))$

Example 5. $N(\Gamma_0(6))$ is generated by

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -\frac{1}{\sqrt{6}} \\ \sqrt{6} & 0 \end{pmatrix}, \begin{pmatrix} \sqrt{3} & \frac{1}{\sqrt{3}} \\ 2\sqrt{3} & \sqrt{3} \end{pmatrix}.$$

Example 6. $N(\Gamma_0(7))$ is generated by

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -\frac{1}{\sqrt{7}} \\ \sqrt{7} & 0 \end{pmatrix}, \begin{pmatrix} \sqrt{7} & \frac{3}{\sqrt{7}} \\ 2\sqrt{7} & \sqrt{7} \end{pmatrix}.$$

Example 7. $N(\Gamma_0(10))$ is generated by

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -\frac{1}{\sqrt{10}} \\ \sqrt{10} & 0 \end{pmatrix}, \begin{pmatrix} \sqrt{5} & \frac{2}{\sqrt{5}} \\ 2\sqrt{5} & \sqrt{5} \end{pmatrix}.$$

Example 8. $N(\Gamma_0(13))$ is generated by

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -\frac{1}{\sqrt{13}} \\ \sqrt{13} & 0 \end{pmatrix}, \begin{pmatrix} \sqrt{13} & \frac{6}{\sqrt{13}} \\ 2\sqrt{13} & \sqrt{13} \end{pmatrix}, \begin{pmatrix} \sqrt{13} & \frac{4}{\sqrt{13}} \\ 3\sqrt{13} & \sqrt{13} \end{pmatrix}.$$

Example 9. $N(\Gamma_0(14))$ is generated by

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -\frac{1}{\sqrt{14}} \\ \sqrt{14} & 0 \end{pmatrix}, \begin{pmatrix} \sqrt{7} & \frac{3}{\sqrt{7}} \\ 2\sqrt{7} & \sqrt{7} \end{pmatrix}, \begin{pmatrix} 2\sqrt{2} & \frac{1}{\sqrt{2}} \\ 7\sqrt{2} & 2\sqrt{2} \end{pmatrix}.$$

Example 10. $N(\Gamma_0(21))$ is generated by

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -\frac{1}{\sqrt{21}} \\ \sqrt{21} & 0 \end{pmatrix}, \begin{pmatrix} \sqrt{21} & \frac{10}{\sqrt{21}} \\ 2\sqrt{21} & \sqrt{21} \end{pmatrix}, \begin{pmatrix} \sqrt{7} & \frac{2}{\sqrt{7}} \\ 3\sqrt{7} & \sqrt{7} \end{pmatrix}.$$

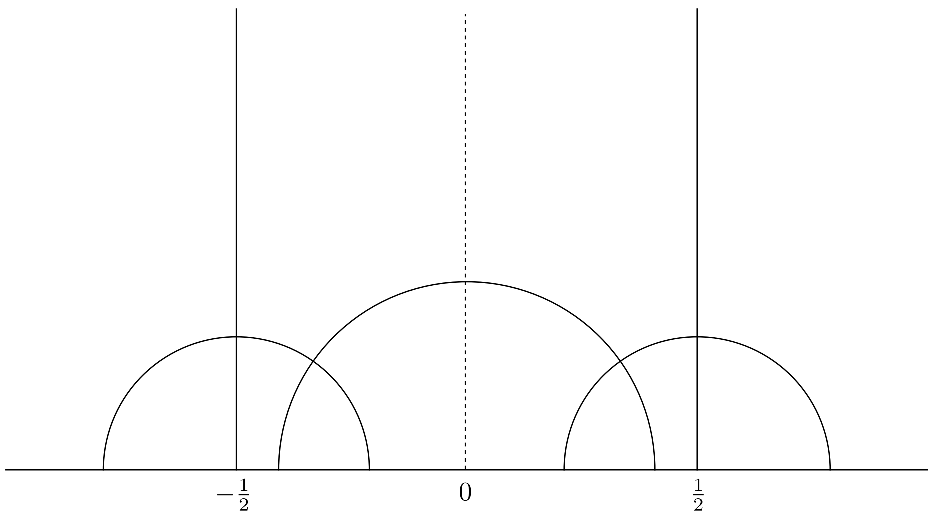


Figure 4.11: Ford domain for $N(\Gamma_0(6))$

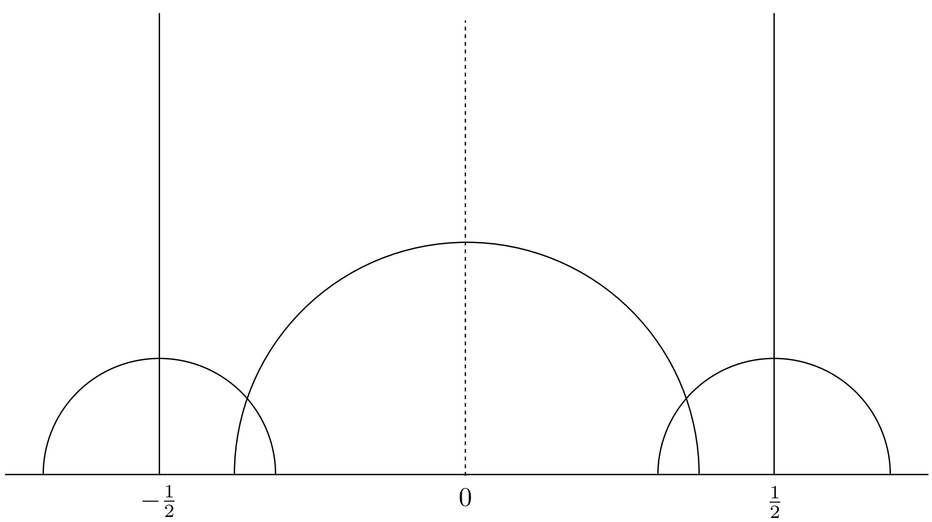


Figure 4.12: Ford domain for $N(\Gamma_0(7))$

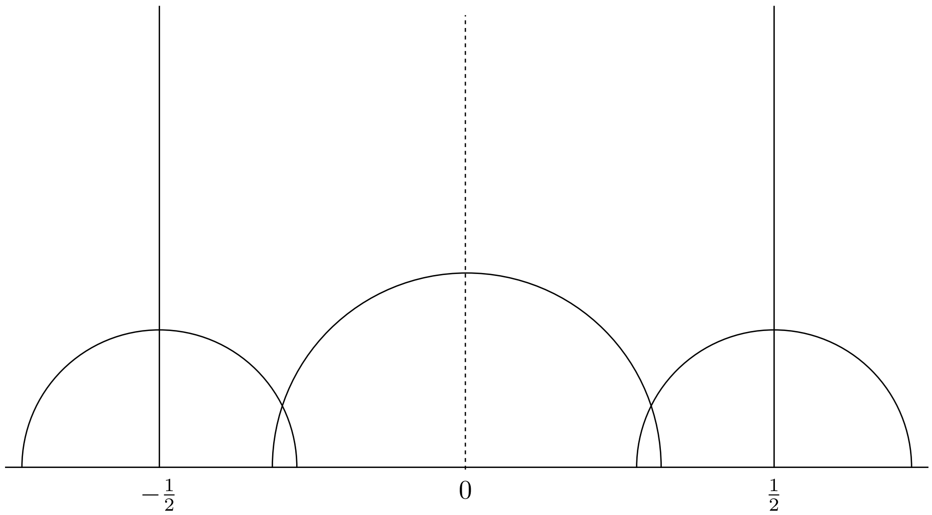


Figure 4.13: Ford domain for $N(\Gamma_0(10))$

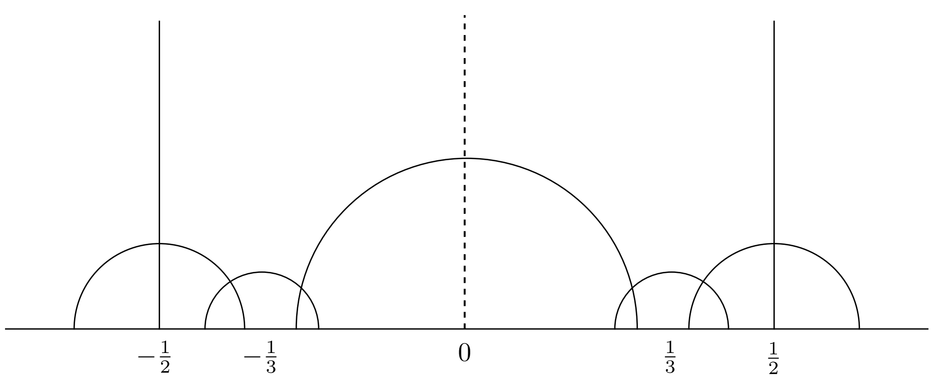


Figure 4.14: Ford domain for $N(\Gamma_0(13))$

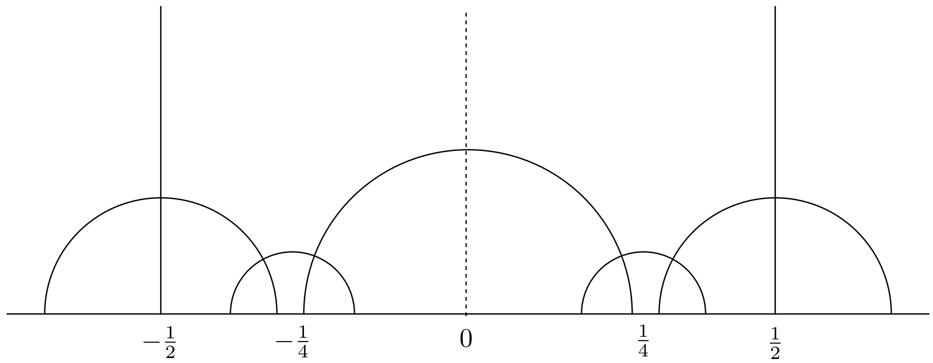


Figure 4.15: Ford domain for $N(\Gamma_0(14))$

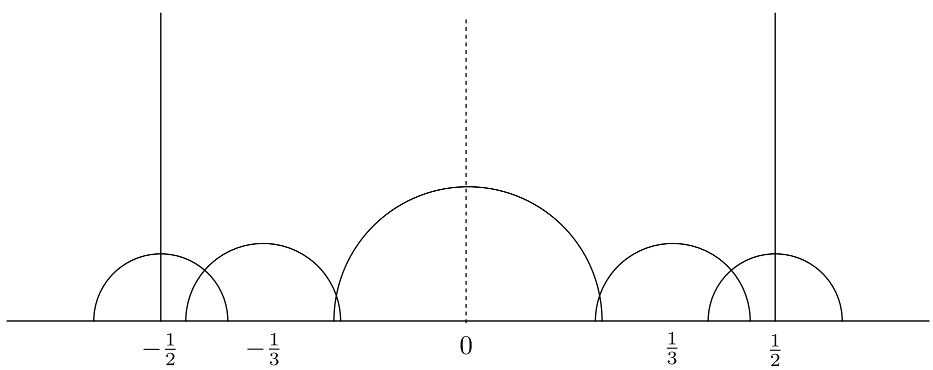


Figure 4.16: Ford domain for $N(\Gamma_0(21))$

Example 11. $N(\Gamma_0(30))$ is generated by

$$\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}, \begin{pmatrix}0 & -\frac{1}{\sqrt{30}}\\ \sqrt{30} & 0\end{pmatrix}, \begin{pmatrix}\sqrt{15} & \frac{7}{\sqrt{15}}\\ 2\sqrt{15} & \sqrt{15}\end{pmatrix},$$

$$\begin{pmatrix}\sqrt{10} & \frac{3}{\sqrt{10}}\\ 3\sqrt{10} & \sqrt{10}\end{pmatrix}, \begin{pmatrix}\sqrt{6} & \frac{1}{\sqrt{6}}\\ 5\sqrt{6} & \sqrt{6}\end{pmatrix}.$$

Example 12. $N(\Gamma_0(34))$ is generated by

$$\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}, \begin{pmatrix}0 & -\frac{1}{\sqrt{34}}\\ \sqrt{34} & 0\end{pmatrix}, \begin{pmatrix}\sqrt{17} & \frac{8}{\sqrt{17}}\\ 2\sqrt{17} & \sqrt{17}\end{pmatrix},$$

$$\begin{pmatrix}\sqrt{17} & \frac{4}{\sqrt{17}}\\ 4\sqrt{17} & \sqrt{17}\end{pmatrix}, \begin{pmatrix}\sqrt{34} & \frac{11}{\sqrt{34}}\\ 3\sqrt{34} & \sqrt{34}\end{pmatrix}, \begin{pmatrix}3\sqrt{2} & \frac{1}{\sqrt{2}}\\ 17\sqrt{2} & 3\sqrt{2}\end{pmatrix}.$$

Example 13. $N(\Gamma_0(39))$ is generated by

$$\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}, \begin{pmatrix}0 & -\frac{1}{\sqrt{39}}\\ \sqrt{39} & 0\end{pmatrix}, \begin{pmatrix}\sqrt{39} & \frac{19}{\sqrt{39}}\\ 2\sqrt{39} & \sqrt{39}\end{pmatrix},$$

$$\begin{pmatrix}\sqrt{13} & \frac{4}{\sqrt{13}}\\ 3\sqrt{13} & \sqrt{13}\end{pmatrix}, \begin{pmatrix}3\sqrt{3} & \frac{2}{\sqrt{3}}\\ 13\sqrt{3} & 3\sqrt{3}\end{pmatrix}, \begin{pmatrix}\sqrt{13} & \frac{2}{\sqrt{13}}\\ 6\sqrt{13} & \sqrt{13}\end{pmatrix}.$$

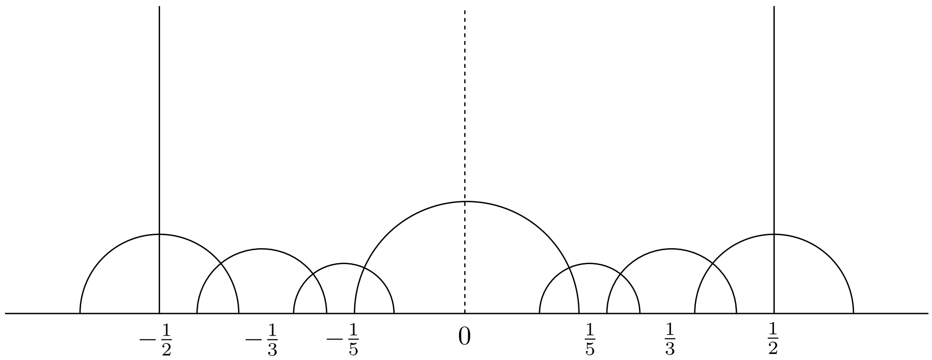


Figure 4.17: Ford domain for $N(\Gamma_0(30))$

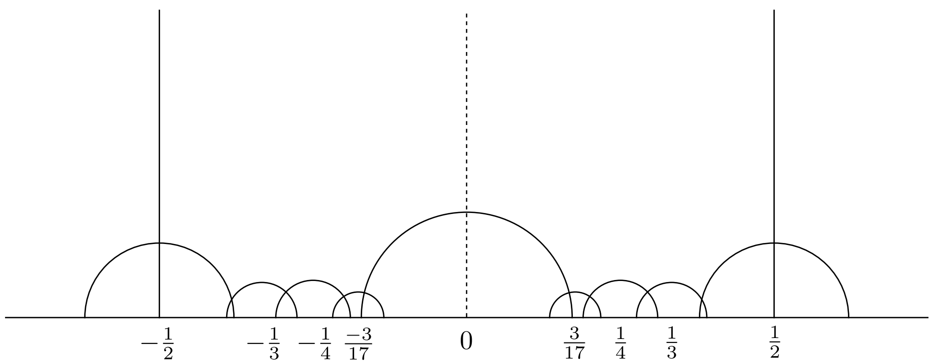


Figure 4.18: Ford domain for $N(\Gamma_0(34))$

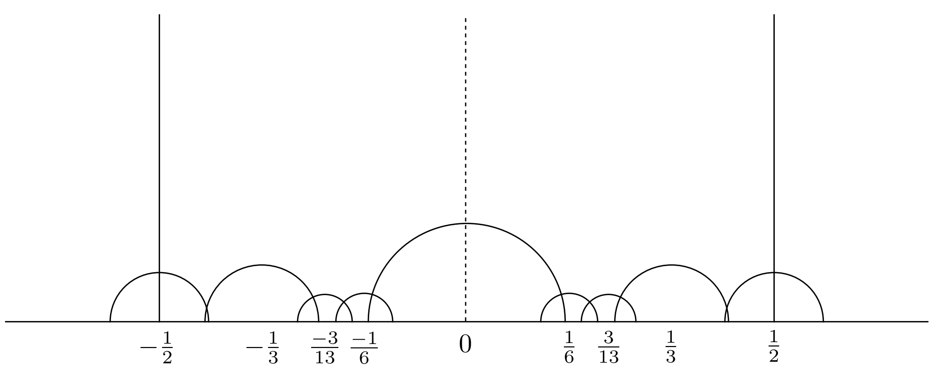


Figure 4.19: Ford domain for $N(\Gamma_0(39))$

4.3 Examples with Two Ideal Vertices

In this section, we find generators in $\mathrm{PSL}_2(\mathbb{R})$, and DF domains, for (the rotation subgroups of) the ten non-cocompact arithmetic maximal reflection groups with two ideal vertices.

To find these groups, the following definition will be useful.

Definition. Suppose Γ is a cofinite Fuchsian group with Γ_∞ consisting of parabolic elements, and which does not admit a DF domain. We will say that Γ admits an *almost-DF domain* if it admits a Ford domain which is symmetric about a vertical line L, such that all but two of the side-pairings are symmetric about L, and the remaining two side-pairings are involutions about fixed points on their respective sides.

In the process of finding the ten examples, we will prove the following result.

Proposition 4.3.1. *Of the* 32 *maximal arithmetic Fuchsian groups of genus zero which do not admit a DF domain, precisely ten admit an almost-DF domain; they are* $N(\Gamma_0(n))$ *for* n *belonging to the set*

$$\{11, 15, 17, 19, 22, 26, 33, 42, 55, 66\}.$$

Each of these ten groups contains an index two subgroup which admits a DF domain, and which corresponds to an arithmetic maximal reflection group with two ideal vertices.

Since these groups are proper subgroups of the maximal groups, it is not immediate that they are congruence. Testing these groups for congruence will lead to the following result.

Theorem 4.3.2. *Of the ten arithmetic maximal reflection groups with two ideal vertices,* *two are congruence, and eight are not. Hence, of the* 23 *non-cocompact arithmetic maximal reflection groups in* $\text{Isom}(\mathbb{H}^2)$*, fifteen are congruence, and eight are not.*

In summary, we have the following.

Theorem 4.3.3. *Of the* 23 *non-cocompact arithmetic maximal reflection groups contained in* $\text{Isom}(\mathbb{H}^2)$*, fifteen are congruence, and eight are not.*

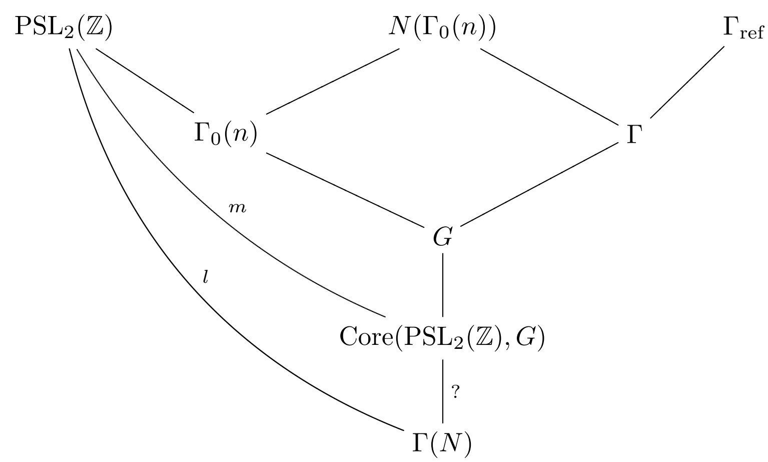
4.3.1 Proofs

Before considering the individual examples, we first describe how to test each of the groups for congruence. Recall that a non-cocompact arithmetic Fuchsian group Γ is congruence if and only if, after possible conjugation in $\text{PSL}_2(\mathbb{R})$, it contains some principal congruence subgroup $\Gamma(N)$. Suppose Γ is the rotation subgroup of a reflection group Γ_{ref}, and that $\Gamma < N(\Gamma_0(n))$ for some fixed n. Since all principal congruence subgroups are contained in $\text{PSL}_2(\mathbb{Z})$, to test Γ (and hence Γ_{ref}) for congruence, it suffices to test $G = \Gamma \cap \text{PSL}_2(\mathbb{Z}) = \Gamma \cap \Gamma_0(n)$. A set of generators for G can be obtained in terms of generators for $N(\Gamma_0(n))$ by using a computer algebra system such as Magma [9] to find the intersection of the two subgroups $\Gamma_0(n)$ and Γ. This can then be expressed as a set of generators for G as a subgroup of $\text{PSL}_2(\mathbb{Z})$ by using the Euclidean algorithm on each generator.

Given these generators and knowledge of the index $[\text{PSL}_2(\mathbb{Z}) : G]$, the following method determines whether G is congruence. Suppose G is congruence. Then it contains some principal congruence subgroup $\Gamma(N)$. More precisely, by Wohlfahrt's theorem (Theorem 2.4.1), it must contain $\Gamma(N)$ for N equal to the *level* of G. Since $\Gamma(N)$ is a normal subgroup of $\text{PSL}_2(\mathbb{Z})$, it is then contained in the *normal core* of G in $\text{PSL}_2(\mathbb{Z})$, which is defined as the intersection

$$\mathrm{Core}(\mathrm{PSL}_2(\mathbb{Z}), G) := \bigcap_{\gamma \in \mathrm{PSL}_2(\mathbb{Z})} \gamma G \gamma^{-1},$$

and is the unique maximal normal subgroup of $\mathrm{PSL}_2(\mathbb{Z})$ contained within G; that is, every normal subgroup of $\mathrm{PSL}_2(\mathbb{Z})$ which is contained in G is contained in this core. We have the following diagram.



Note that we may compute the two indices

$$l = [\mathrm{PSL}_2(\mathbb{Z}) : \Gamma(N)]$$

and

$$m = [\mathrm{PSL}_2(\mathbb{Z}) : \mathrm{Core}(\mathrm{PSL}_2(\mathbb{Z}), G)]\,,$$

the former by way of a standard formula (see Section 2.4), and the latter by using Magma. If m divides l, then this analysis tells us nothing, as it does not confirm or preclude the possibility that G contains $\Gamma(N)$. However, if $m > l$, or m fails to divide l, then we have a contradiction, which allows us to conclude that G, and hence Γ and Γ_{ref}, is not congruence. The results are summarized in Table 4.1 at the end of the section.

Remark. In all cases, we use the algorithm of Hsu [24] to check whether G is in fact congruence. This serves as a check in the case G is not congruence, and provides an answer when G is congruence, when the method above gave no conclusive answer. We develop the method above in order that we may apply it to other situations where Hsu's algorithm is not immediately available to us; in particular, we will use this method in Chapter 5 to treat groups contained in $\text{Isom}(\mathbb{H}^3)$.

Example 14. $N(\Gamma_0(11))$ is generated by the matrices

$$\gamma_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \gamma_2 = \begin{pmatrix} 0 & -\frac{1}{\sqrt{11}} \\ \sqrt{11} & 0 \end{pmatrix}, \gamma_3 = \begin{pmatrix} \sqrt{11} & \frac{5}{\sqrt{11}} \\ 2\sqrt{11} & \sqrt{11} \end{pmatrix},$$

$$\gamma_4 = \begin{pmatrix} -\sqrt{11} & -\frac{4}{\sqrt{11}} \\ 3\sqrt{11} & \sqrt{11} \end{pmatrix}, \gamma_5 = \begin{pmatrix} \sqrt{11} & -\frac{4}{\sqrt{11}} \\ 3\sqrt{11} & -\sqrt{11} \end{pmatrix}.$$

A Ford domain defined by these elements is given in Figure 4.20. The subgroup Γ generated by γ_1, γ_2, γ_3,

$$\gamma_5\gamma_1\gamma_4 = \begin{pmatrix} 10 & 3 \\ 33 & 10 \end{pmatrix}, \gamma_5\gamma_4 = \begin{pmatrix} 23 & 8 \\ 66 & 23 \end{pmatrix}$$

has index two in $N(\Gamma_0(11))$ and admits a DF domain (shown in Figure 4.21). Thus Γ is the rotation subgroup of some arithmetic reflection group Γ_{ref}.

Claim. *Γ_{ref} is a maximal reflection group.*

Proof. If Γ_{ref} were not maximal, it would be properly contained in another reflection group H_{ref}, which is therefore also arithmetic. Let $H < H_{\text{ref}}$ denote the rotation subgroup. Note that then we have $\Gamma < H$. Since Γ and H are both arithmetic Fuchsian groups of genus zero, they are contained in a common maximal, arithmetic, genus zero Fuchsian group M

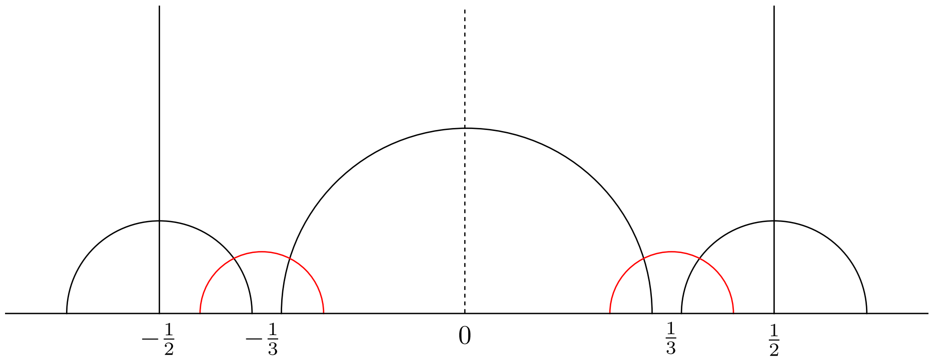


Figure 4.20: Ford domain for $N(\Gamma_0(11))$

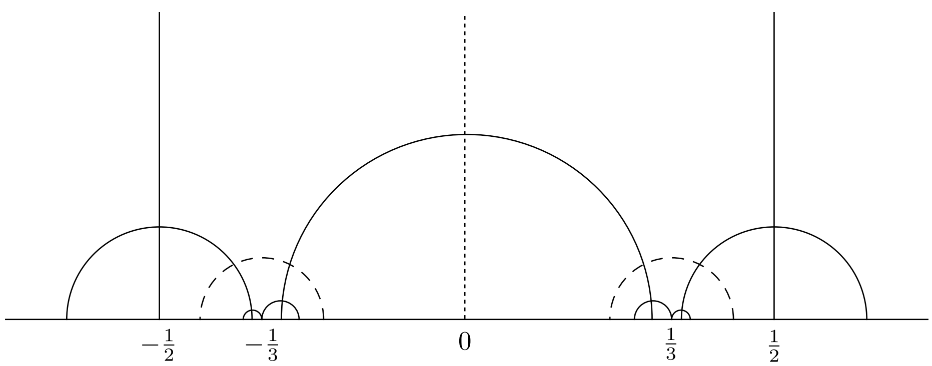


Figure 4.21: Ford domain for $\Gamma < N(\Gamma_0(11))$

from the appropriate list in Long–Maclachlan–Reid [27]. By construction, Γ is contained in the normalizer $N(\Gamma_0(11))$, and by area considerations we find that $[N(\Gamma_0(11)) : \Gamma] = 2$. Further, Γ cannot be contained in any other of these maximal arithmetic groups; to see this, observe that if $n \neq 11$ then, if we pick some non-zero integer b coprime to n, we may find integers a, d such that $\begin{pmatrix} a & b \\ n & d \end{pmatrix} \in \Gamma_0(n)$. We then have

$$\gamma_2 \begin{pmatrix} a & b \\ n & d \end{pmatrix} \gamma_2 = \begin{pmatrix} 0 & -\frac{1}{\sqrt{11}} \\ \sqrt{11} & 0 \end{pmatrix} \begin{pmatrix} a & b \\ n & d \end{pmatrix} \begin{pmatrix} 0 & -\frac{1}{\sqrt{11}} \\ \sqrt{11} & 0 \end{pmatrix}$$
$$= \begin{pmatrix} -d & \frac{n}{11} \\ 11b & -a \end{pmatrix}.$$

We wish to show that this does not belong to $\Gamma_0(n)$. If n is not divisible by 11 this is clear, so suppose $n \geq 22$ is a multiple of 11. Then, by construction, b is coprime to 11, and so $11b$ is not divisible by n. This shows that γ_2 cannot belong to any normalizer $N(\Gamma_0(n))$ except $N(\Gamma_0(11))$.

It remains to verify that we cannot have $H = M = N(\Gamma_0(11))$. But if this were the case, then $N(\Gamma_0(11))$ would admit a DF domain, and we would have a fourteenth arithmetic maximal reflection group with one ideal vertex, a contradiction. □

Thus the group Γ is the rotation subgroup of an arithmetic maximal reflection group. We find that the subgroup $G = \Gamma \cap \Gamma_0(11)$ is generated by the elements

$$\gamma_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \gamma_2\gamma_1^{-1}\gamma_2 = \begin{pmatrix} 1 & 0 \\ 11 & 1 \end{pmatrix}, \quad \gamma_2\gamma_3 = \begin{pmatrix} -2 & -1 \\ 11 & 5 \end{pmatrix},$$
$$\gamma_2\gamma_3^{-1} = \begin{pmatrix} 2 & -1 \\ 11 & -5 \end{pmatrix}, \quad \gamma_4 = \begin{pmatrix} 10 & 3 \\ 33 & 10 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} 23 & 8 \\ 66 & 23 \end{pmatrix}.$$

It has index two in both Γ and $\Gamma_0(11)$, and index 24 in $\mathrm{PSL}_2(\mathbb{Z})$. It has level $N = 11$, so we test for whether G contains $\Gamma(11)$. We find that

$$[\mathrm{PSL}_2(\mathbb{Z}) : \Gamma(11)] = 660$$

but

$$[\mathrm{PSL}_2(\mathbb{Z}) : \mathrm{Core}(\mathrm{PSL}_2(\mathbb{Z}), G)] = 1351680,$$

allowing us to conclude that G is not congruence.

In this case, checking our findings using Hsu's algorithm, we find

$$L = (2\ 4\ 9\ 15\ 8\ 5\ 11\ 13\ 7\ 3\ 6)(10\ 17\ 21\ 23\ 22\ 19\ 14\ 12\ 18\ 20\ 16)$$

and

$$R = (1\ 2\ 5\ 12\ 14\ 7\ 4\ 10\ 16\ 8\ 3)(9\ 17\ 19\ 13\ 11\ 18\ 21\ 24\ 22\ 20\ 15)$$

are both of order 11, also giving that the level of G is 11. Hsu's test is then that G is congruence if and only if $(R^2L^{-\frac{1}{2}})^3 = 1$, where $\frac{1}{2}$ is the multiplicative inverse of 2 mod 11, in this case equal to 6. We find that R^2L^{-6} has order 6, and so G is not congruence.

Example 15. $N(\Gamma_0(15))$ is generated by

$$\gamma_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \gamma_2 = \begin{pmatrix} 0 & -\frac{1}{\sqrt{15}} \\ \sqrt{15} & 0 \end{pmatrix}, \gamma_3 = \begin{pmatrix} \sqrt{15} & \frac{7}{\sqrt{15}} \\ 2\sqrt{15} & \sqrt{15} \end{pmatrix},$$

$$\gamma_4 = \begin{pmatrix} -\sqrt{5} & -\frac{2}{\sqrt{5}} \\ 3\sqrt{5} & \sqrt{5} \end{pmatrix}, \gamma_5 = \begin{pmatrix} \sqrt{5} & -\frac{2}{\sqrt{5}} \\ 3\sqrt{5} & -\sqrt{5} \end{pmatrix}.$$

The corresponding Ford domain is given in Figure 4.22, and includes the dashed isometric circle. In this case, Γ is generated by γ_1, γ_2, γ_3, along with

$$\gamma_5\gamma_4 = \begin{pmatrix} 11 & 4 \\ 30 & 11 \end{pmatrix}, \gamma_5\gamma_1\gamma_4 = \begin{pmatrix} 4 & 1 \\ 15 & 4 \end{pmatrix}.$$

The DF domain for Γ is also given in Figure 4.22, and comprises only the solid circles. Computation reveals that

$$[N(\Gamma_0(15)) : \Gamma_0(15)] = 4$$

and

$$[N(\Gamma_0(15)) : \Gamma \cap \Gamma_0(15)] = 4,$$

from which it follows that $G = \Gamma \cap \Gamma_0(15) = \Gamma_0(15)$. Thus the rotation subgroup Γ contains $\Gamma_0(15)$, and hence $\Gamma(15)$. This reflection group Γ_{ref} is congruence.

Example 16. $N(\Gamma_0(17))$ is generated by

$$\gamma_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \gamma_2 = \begin{pmatrix} 0 & -\frac{1}{\sqrt{17}} \\ \sqrt{17} & 0 \end{pmatrix}, \gamma_3 = \begin{pmatrix} \sqrt{17} & \frac{8}{\sqrt{17}} \\ 2\sqrt{17} & \sqrt{17} \end{pmatrix},$$

$$\gamma_4 = \begin{pmatrix} -\sqrt{17} & -\frac{6}{\sqrt{17}} \\ 3\sqrt{17} & \sqrt{17} \end{pmatrix}, \gamma_5 = \begin{pmatrix} \sqrt{17} & -\frac{6}{\sqrt{17}} \\ 3\sqrt{17} & -\sqrt{17} \end{pmatrix}, \gamma_6 = \begin{pmatrix} \sqrt{17} & \frac{4}{\sqrt{17}} \\ 4\sqrt{17} & \sqrt{17} \end{pmatrix}.$$

The corresponding Ford domain is given in Figure 4.23. In this case, Γ is generated by γ_1, γ_2, γ_3, γ_6, along with

$$\gamma_5\gamma_4 = \begin{pmatrix} 35 & 12 \\ 102 & 35 \end{pmatrix}, \gamma_5\gamma_1\gamma_4 = \begin{pmatrix} 16 & 5 \\ 51 & 16 \end{pmatrix}, \gamma_5\gamma_2\gamma_4 = \begin{pmatrix} 3\sqrt{17} & \frac{19}{\sqrt{17}} \\ 8\sqrt{17} & 3\sqrt{17} \end{pmatrix}.$$

The level of $G = \Gamma \cap \Gamma_0(17)$ is $N = 17$. We find

$$[\mathrm{PSL}_2(\mathbb{Z}) : \Gamma(17)] = [\mathrm{PSL}_2(\mathbb{Z}) : \mathrm{Core}(\mathrm{PSL}_2(\mathbb{Z}), G)] = 2448,$$

from which we are unable to conclude anything. We therefore apply Hsu's test, which tells us that this example is congruence.

Example 17. $N(\Gamma_0(19))$ is generated by

$$\gamma_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \gamma_2 = \begin{pmatrix} 0 & -\frac{1}{\sqrt{19}} \\ \sqrt{19} & 0 \end{pmatrix}, \gamma_3 = \begin{pmatrix} \sqrt{19} & \frac{9}{\sqrt{19}} \\ 2\sqrt{19} & \sqrt{19} \end{pmatrix},$$

$$\gamma_4 = \begin{pmatrix} \sqrt{19} & \frac{6}{\sqrt{19}} \\ 3\sqrt{19} & \sqrt{19} \end{pmatrix}, \gamma_5 = \begin{pmatrix} -\sqrt{19} & -\frac{5}{\sqrt{19}} \\ 4\sqrt{19} & \sqrt{19} \end{pmatrix}, \gamma_6 = \begin{pmatrix} \sqrt{19} & -\frac{5}{\sqrt{19}} \\ 4\sqrt{19} & -\sqrt{19} \end{pmatrix}.$$

Here Γ is generated by γ_1, γ_2, γ_3, γ_4, along with

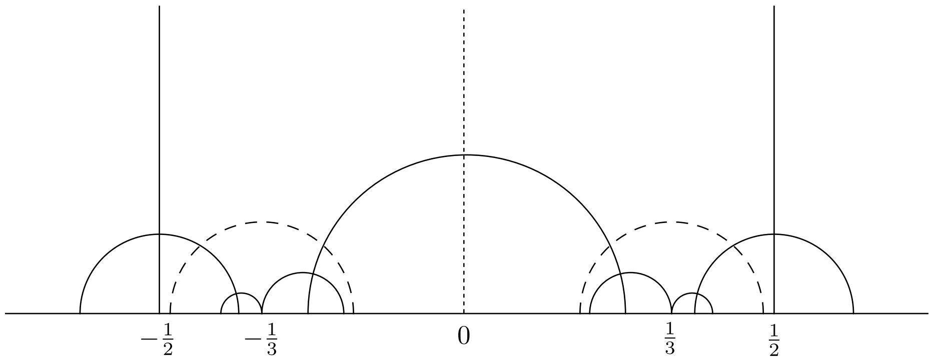


Figure 4.22: Ford domain for $N(\Gamma_0(15))$

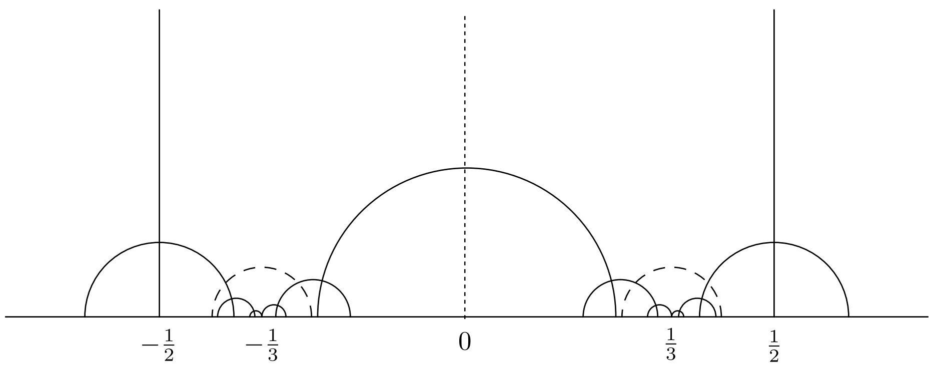


Figure 4.23: Ford domain for $N(\Gamma_0(17))$

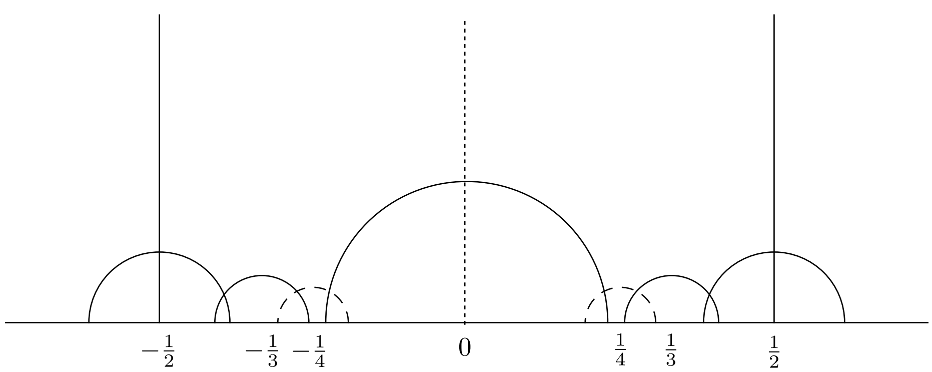


Figure 4.24: Ford domain for $N(\Gamma_0(19))$

$$\gamma_6\gamma_5 = \begin{pmatrix} 39 & 10 \\ 152 & 39 \end{pmatrix}, \gamma_6\gamma_1\gamma_5 = \begin{pmatrix} 37 & 9 \\ 152 & 37 \end{pmatrix}, \gamma_6\gamma_3\gamma_5 = \begin{pmatrix} 7\sqrt{19} & \frac{31}{\sqrt{19}} \\ 30\sqrt{19} & 7\sqrt{19} \end{pmatrix}.$$

A DF domain for Γ is obtained by rotating each half of the Ford domain in Figure 4.24 by γ_5 and γ_6 respectively. The level of $G = \Gamma \cap \Gamma_0(19)$ is $N = 19$. We find

$$[\mathrm{PSL}_2(\mathbb{Z}) : \Gamma(19)] = 3420,$$

and

$$[\mathrm{PSL}_2(\mathbb{Z}) : \mathrm{Core}(\mathrm{PSL}_2(\mathbb{Z}), G)] = 1793064960.$$

This example is not congruence.

Example 18. $N(\Gamma_0(22))$ is generated by

$$\gamma_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \gamma_2 = \begin{pmatrix} 0 & -\frac{1}{\sqrt{22}} \\ \sqrt{22} & 0 \end{pmatrix}, \gamma_3 = \begin{pmatrix} \sqrt{11} & \frac{5}{\sqrt{11}} \\ 2\sqrt{11} & \sqrt{11} \end{pmatrix},$$

$$\gamma_4 = \begin{pmatrix} \sqrt{22} & \frac{7}{\sqrt{22}} \\ 3\sqrt{22} & \sqrt{22} \end{pmatrix}, \gamma_5 = \begin{pmatrix} -\sqrt{11} & -\frac{3}{\sqrt{11}} \\ 4\sqrt{11} & \sqrt{11} \end{pmatrix}, \gamma_6 = \begin{pmatrix} \sqrt{11} & -\frac{3}{\sqrt{11}} \\ 4\sqrt{11} & -\sqrt{11} \end{pmatrix}.$$

Here Γ is generated by γ_1, γ_2, γ_3, γ_4, along with

$$\gamma_6\gamma_5 = \begin{pmatrix} 23 & 6 \\ 88 & 23 \end{pmatrix}, \gamma_6\gamma_1\gamma_5 = \begin{pmatrix} 21 & 5 \\ 88 & 21 \end{pmatrix}, \gamma_6\gamma_3\gamma_5 = \begin{pmatrix} 3\sqrt{11} & \frac{7}{\sqrt{11}} \\ 14\sqrt{11} & 3\sqrt{11} \end{pmatrix}.$$

The level of $G = \Gamma \cap \Gamma_0(22)$ is $N = 22$. We find

$$[\mathrm{PSL}_2(\mathbb{Z}) : \Gamma(22)] = 3960,$$

and

$$[\mathrm{PSL}_2(\mathbb{Z}) : \mathrm{Core}(\mathrm{PSL}_2(\mathbb{Z}), G)] = 34016140984320.$$

This example is not congruence.

Example 19. $N(\Gamma_0(26))$ is generated by

$$\gamma_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \gamma_2 = \begin{pmatrix} 0 & -\frac{1}{\sqrt{26}} \\ \sqrt{26} & 0 \end{pmatrix}, \gamma_3 = \begin{pmatrix} \sqrt{13} & \frac{6}{\sqrt{13}} \\ 2\sqrt{13} & \sqrt{13} \end{pmatrix},$$

$$\gamma_4 = \begin{pmatrix} -\sqrt{26} & -\frac{9}{\sqrt{26}} \\ 3\sqrt{26} & \sqrt{26} \end{pmatrix}, \gamma_5 = \begin{pmatrix} \sqrt{26} & -\frac{9}{\sqrt{26}} \\ 3\sqrt{26} & -\sqrt{26} \end{pmatrix}, \gamma_6 = \begin{pmatrix} \sqrt{13} & \frac{3}{\sqrt{13}} \\ 4\sqrt{13} & \sqrt{13} \end{pmatrix}.$$

Here Γ is generated by γ_1, γ_2, γ_3, γ_6, along with

$$\gamma_5\gamma_4 = \begin{pmatrix} 53 & 18 \\ 156 & 53 \end{pmatrix}, \gamma_5\gamma_1\gamma_4 = \begin{pmatrix} 25 & 8 \\ 78 & 25 \end{pmatrix}, \gamma_5\gamma_2\gamma_4 = \begin{pmatrix} 6\sqrt{26} & \frac{55}{\sqrt{26}} \\ 17\sqrt{26} & 6\sqrt{26} \end{pmatrix}.$$

The level of $G = \Gamma \cap \Gamma_0(26)$ is $N = 26$. We find

$$[\mathrm{PSL}_2(\mathbb{Z}) : \Gamma(26)] = 6552,$$

and

$$[\mathrm{PSL}_2(\mathbb{Z}) : \mathrm{Core}(\mathrm{PSL}_2(\mathbb{Z}), G)] = 439697276928.$$

This example is not congruence.

Example 20. $N(\Gamma_0(33))$ is generated by

$$\gamma_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \gamma_2 = \begin{pmatrix} 0 & -\frac{1}{\sqrt{33}} \\ \sqrt{33} & 0 \end{pmatrix}, \gamma_3 = \begin{pmatrix} \sqrt{33} & \frac{16}{\sqrt{33}} \\ 2\sqrt{33} & \sqrt{33} \end{pmatrix}, \gamma_4 = \begin{pmatrix} -\sqrt{11} & -\frac{4}{\sqrt{11}} \\ 3\sqrt{11} & \sqrt{11} \end{pmatrix},$$

$$\gamma_5 = \begin{pmatrix} \sqrt{11} & -\frac{4}{\sqrt{11}} \\ 3\sqrt{11} & -\sqrt{11} \end{pmatrix}, \gamma_6 = \begin{pmatrix} \sqrt{33} & \frac{8}{\sqrt{33}} \\ 4\sqrt{33} & \sqrt{33} \end{pmatrix}, \gamma_7 = \begin{pmatrix} 2\sqrt{3} & \frac{1}{\sqrt{3}} \\ 11\sqrt{3} & 2\sqrt{3} \end{pmatrix}.$$

Here Γ is generated by γ_1, γ_2, γ_3, γ_6, γ_7, along with

$$\gamma_5\gamma_4 = \begin{pmatrix} 23 & 8 \\ 66 & 23 \end{pmatrix}, \gamma_5\gamma_1\gamma_4 = \begin{pmatrix} 10 & 3 \\ 33 & 10 \end{pmatrix},$$

$$\gamma_5\gamma_2\gamma_4 = \begin{pmatrix} 3\sqrt{33} & \frac{37}{\sqrt{33}} \\ 8\sqrt{33} & 3\sqrt{33} \end{pmatrix}, \gamma_5\gamma_7\gamma_4 = \begin{pmatrix} 9\sqrt{3} & \frac{11}{\sqrt{3}} \\ 22\sqrt{3} & 9\sqrt{3} \end{pmatrix}.$$

The level of $G = \Gamma \cap \Gamma_0(33)$ is $N = 33$. We find

$$[\mathrm{PSL}_2(\mathbb{Z}) : \Gamma(33)] = 15840,$$

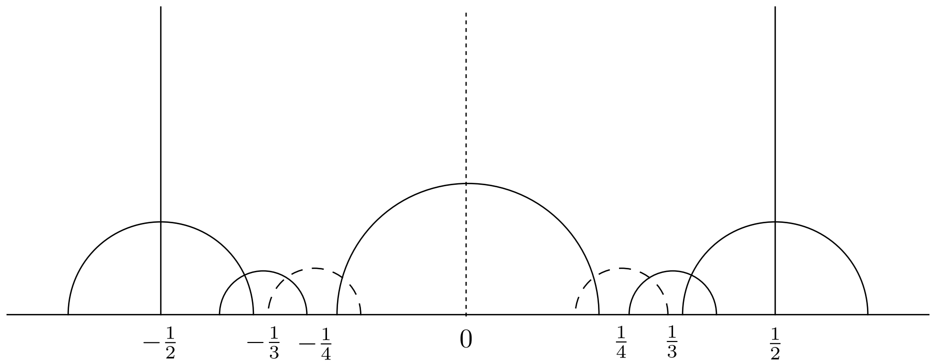


Figure 4.25: Ford domain for $N(\Gamma_0(22))$

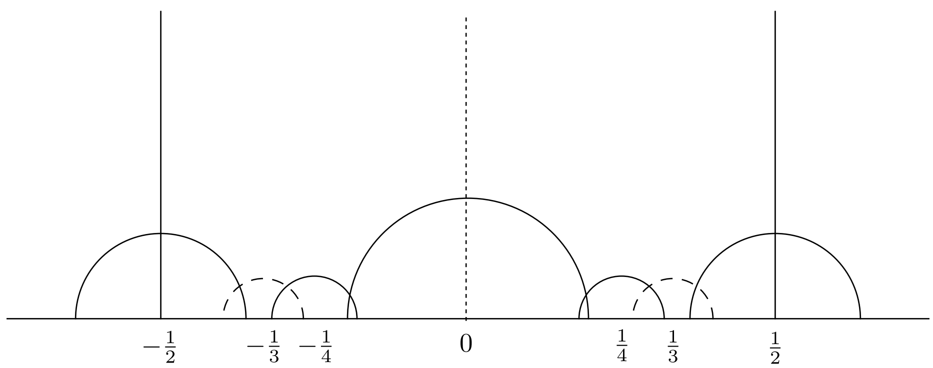


Figure 4.26: Ford domain for $N(\Gamma_0(26))$

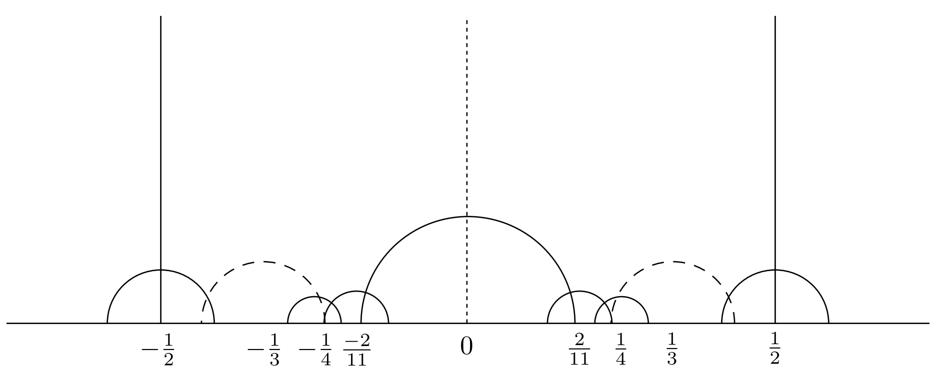


Figure 4.27: Ford domain for $N(\Gamma_0(33))$

and

$$[\mathrm{PSL}_2(\mathbb{Z}) : \mathrm{Core}(\mathrm{PSL}_2(\mathbb{Z}), G)] = 139330113471774720.$$

This example is not congruence.

Example 21. $N(\Gamma_0(42))$ is generated by

$$\gamma_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \gamma_2 = \begin{pmatrix} 0 & -\frac{1}{\sqrt{42}} \\ \sqrt{42} & 0 \end{pmatrix}, \gamma_3 = \begin{pmatrix} \sqrt{21} & \frac{10}{\sqrt{21}} \\ 2\sqrt{21} & \sqrt{21} \end{pmatrix}, \gamma_4 = \begin{pmatrix} -\sqrt{14} & -\frac{5}{\sqrt{14}} \\ 3\sqrt{14} & \sqrt{14} \end{pmatrix},$$

$$\gamma_5 = \begin{pmatrix} \sqrt{14} & -\frac{5}{\sqrt{14}} \\ 3\sqrt{14} & -\sqrt{14} \end{pmatrix}, \gamma_6 = \begin{pmatrix} \sqrt{21} & \frac{5}{\sqrt{21}} \\ 4\sqrt{21} & \sqrt{21} \end{pmatrix}, \gamma_7 = \begin{pmatrix} \sqrt{7} & \frac{1}{\sqrt{7}} \\ 6\sqrt{7} & \sqrt{7} \end{pmatrix}.$$

Here Γ is generated by γ_1, γ_2, γ_3, γ_6, γ_7, along with

$$\gamma_5\gamma_4 = \begin{pmatrix} 29 & 10 \\ 84 & 29 \end{pmatrix}, \gamma_5\gamma_1\gamma_4 = \begin{pmatrix} 13 & 4 \\ 42 & 13 \end{pmatrix},$$

$$\gamma_5\gamma_2\gamma_4 = \begin{pmatrix} 4\sqrt{42} & \frac{61}{\sqrt{42}} \\ 11\sqrt{42} & 4\sqrt{42} \end{pmatrix}, \gamma_5\gamma_7\gamma_4 = \begin{pmatrix} 7\sqrt{7} & \frac{19}{\sqrt{7}} \\ 18\sqrt{7} & 7\sqrt{7} \end{pmatrix}.$$

The level of $G = \Gamma \cap \Gamma_0(42)$ is $N = 42$. We find

$$[\mathrm{PSL}_2(\mathbb{Z}) : \Gamma(42)] = 24192,$$

and

$$[\mathrm{PSL}_2(\mathbb{Z}) : \mathrm{Core}(\mathrm{PSL}_2(\mathbb{Z}), G)] = 53198770598313984.$$

This example is not congruence.

Example 22. $N(\Gamma_0(55))$ is generated by

$$\gamma_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \gamma_2 = \begin{pmatrix} 0 & -\frac{1}{\sqrt{55}} \\ \sqrt{55} & 0 \end{pmatrix}, \gamma_3 = \begin{pmatrix} \sqrt{55} & \frac{27}{\sqrt{55}} \\ 2\sqrt{55} & \sqrt{55} \end{pmatrix},$$

$$\gamma_4 = \begin{pmatrix} \sqrt{55} & \frac{18}{\sqrt{55}} \\ 3\sqrt{55} & \sqrt{55} \end{pmatrix}, \gamma_5 = \begin{pmatrix} 3\sqrt{5} & \frac{4}{\sqrt{5}} \\ 11\sqrt{5} & 3\sqrt{5} \end{pmatrix}, \gamma_6 = \begin{pmatrix} \sqrt{11} & \frac{2}{\sqrt{11}} \\ 5\sqrt{11} & \sqrt{11} \end{pmatrix},$$

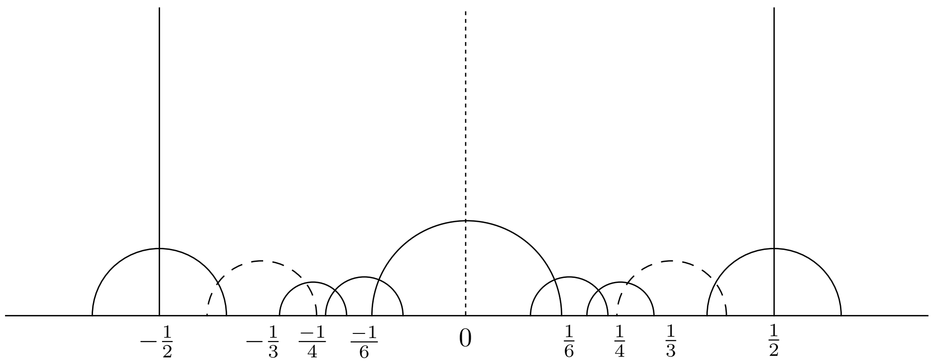


Figure 4.28: Ford domain for $N(\Gamma_0(42))$

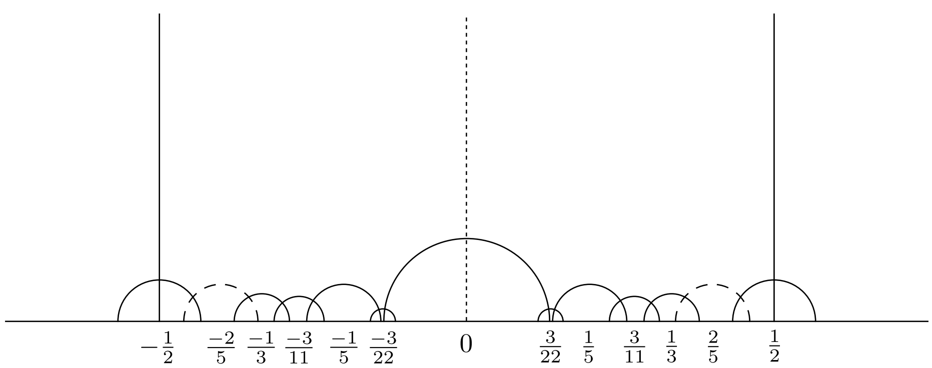


Figure 4.29: Ford domain for $N(\Gamma_0(55))$

$$\gamma_7 = \begin{pmatrix} 3\sqrt{5} & \frac{2}{\sqrt{5}} \\ 22\sqrt{5} & 3\sqrt{5} \end{pmatrix}, \gamma_8 = \begin{pmatrix} -2\sqrt{11} & -\frac{9}{\sqrt{11}} \\ 5\sqrt{11} & 2\sqrt{11} \end{pmatrix}, \gamma_9 = \begin{pmatrix} 2\sqrt{11} & -\frac{9}{\sqrt{11}} \\ 5\sqrt{11} & -2\sqrt{11} \end{pmatrix}.$$

Here Γ is generated by γ_1, γ_2, γ_3, γ_4, γ_5, γ_6, γ_7, along with

$$\gamma_9\gamma_8 = \begin{pmatrix} 89 & 36 \\ 220 & 89 \end{pmatrix}, \gamma_9\gamma_1\gamma_8 = \begin{pmatrix} 21 & 8 \\ 55 & 21 \end{pmatrix},$$

$$\gamma_9\gamma_2\gamma_8 = \begin{pmatrix} 16\sqrt{55} & \frac{361}{\sqrt{55}} \\ 39\sqrt{55} & 16\sqrt{55} \end{pmatrix}, \gamma_9\gamma_5\gamma_8 = \begin{pmatrix} 19\sqrt{5} & \frac{41}{\sqrt{5}} \\ 44\sqrt{5} & 19\sqrt{5} \end{pmatrix},$$

$$\gamma_9\gamma_6\gamma_8 = \begin{pmatrix} 21\sqrt{11} & \frac{97}{\sqrt{11}} \\ 50\sqrt{11} & 21\sqrt{11} \end{pmatrix}, \gamma_9\gamma_7\gamma_8 = \begin{pmatrix} 173\sqrt{5} & \frac{358}{\sqrt{5}} \\ 418\sqrt{5} & 173\sqrt{5} \end{pmatrix}.$$

The level of $G = \Gamma \cap \Gamma_0(55)$ is $N = 55$. We find

$$[\mathrm{PSL}_2(\mathbb{Z}) : \Gamma(55)] = 79200,$$

and

$$[\mathrm{PSL}_2(\mathbb{Z}) : \mathrm{Core}(\mathrm{PSL}_2(\mathbb{Z}), G)] = 29219642612755929759744 00.$$

This example is not congruence.

Example 23. $N(\Gamma_0(66))$ is generated by

$$\gamma_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \gamma_2 = \begin{pmatrix} 0 & -\frac{1}{\sqrt{66}} \\ \sqrt{66} & 0 \end{pmatrix}, \gamma_3 = \begin{pmatrix} \sqrt{33} & \frac{16}{\sqrt{33}} \\ 2\sqrt{33} & \sqrt{33} \end{pmatrix},$$

$$\gamma_4 = \begin{pmatrix} \sqrt{22} & \frac{7}{\sqrt{22}} \\ 3\sqrt{22} & \sqrt{22} \end{pmatrix}, \gamma_5 = \begin{pmatrix} \sqrt{33} & \frac{8}{\sqrt{33}} \\ 4\sqrt{33} & \sqrt{33} \end{pmatrix}, \gamma_6 = \begin{pmatrix} 9\sqrt{3} & \frac{11}{\sqrt{3}} \\ 22\sqrt{3} & 9\sqrt{3} \end{pmatrix},$$

$$\gamma_7 = \begin{pmatrix} -\sqrt{11} & -\frac{2}{\sqrt{11}} \\ 6\sqrt{11} & \sqrt{11} \end{pmatrix}, \gamma_8 = \begin{pmatrix} \sqrt{11} & -\frac{2}{\sqrt{11}} \\ 6\sqrt{11} & -\sqrt{11} \end{pmatrix}.$$

Here Γ is generated by γ_1, γ_2, γ_3, γ_4, γ_5, γ_6, along with

$$\gamma_8\gamma_7 = \begin{pmatrix} 23 & 4 \\ 132 & 23 \end{pmatrix}, \gamma_8\gamma_1\gamma_7 = \begin{pmatrix} 43 & 7 \\ 264 & 43 \end{pmatrix},$$

$$\gamma_8\gamma_2\gamma_7 = \begin{pmatrix} \sqrt{66} & \frac{13}{\sqrt{66}} \\ 5\sqrt{66} & \sqrt{66} \end{pmatrix}, \gamma_8\gamma_3\gamma_7 = \begin{pmatrix} 13\sqrt{33} & \frac{68}{\sqrt{33}} \\ 82\sqrt{33} & 13\sqrt{33} \end{pmatrix},$$

$$\gamma_8\gamma_4\gamma_7 = \begin{pmatrix} 4\sqrt{22} & \frac{13}{\sqrt{22}} \\ 27\sqrt{22} & 4\sqrt{22} \end{pmatrix}, \gamma_8\gamma_5\gamma_7 = \begin{pmatrix} \sqrt{33} & \frac{4}{\sqrt{33}} \\ 8\sqrt{33} & \sqrt{33} \end{pmatrix},$$

$$\gamma_8\gamma_6\gamma_7 = \begin{pmatrix} 79\sqrt{3} & \frac{37}{\sqrt{3}} \\ 506\sqrt{3} & 79\sqrt{3} \end{pmatrix}.$$

The level of $G = \Gamma \cap \Gamma_0(66)$ is $N = 66$. We find

$$[\mathrm{PSL}_2(\mathbb{Z}) : \Gamma(66)] = 95040,$$

and

$$[\mathrm{PSL}_2(\mathbb{Z}) : \mathrm{Core}(\mathrm{PSL}_2(\mathbb{Z}), G)] = 2587234892173279325404729815225220065342259\!20.$$

This example is not congruence.

4.3.2 Estimates

Since we found examples of arithmetic maximal reflection groups which are not congruence, and which therefore do not necessarily have a lower bound on their first eigenvalue of $\frac{3}{16}$, it is reasonable to ask whether one can produce useful bounds on this quantity for these examples. Here we provide one rough estimate for the spectral gap of the rotation subgroup Γ via the Cheeger constant (see Section 2.4). We do this by observing that in each two-vertex example, we have a pair of involutions α and β in $N(\Gamma_0(n))$ which do not belong to the subgroup Γ. The isometric circles of α and β descend to two geodesic arcs s_α, s_β in the quotient, and these arcs have common endpoints. Furthermore, they separate the quotient into two connected components of equal area, thereby maximizing the denominator in the definition of the Cheeger constant. We therefore calculate the upper bounds for h and λ_1 arising from this decomposition and the inequalities described in Chapter 2. This data is listed in Table 4.2. We note that this data does not prove the existence of a maximal arithmetic reflection group with $\lambda_1 < \dfrac{3}{16} = 0.1875$.

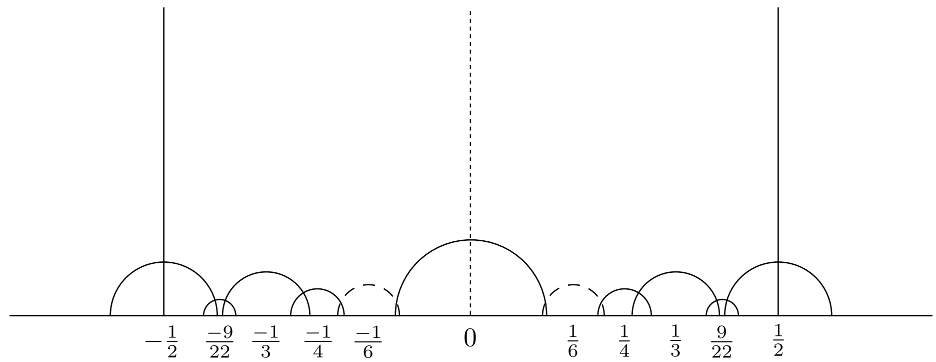


Figure 4.30: Ford domain for $N(\Gamma_0(66))$

n	$[\mathrm{PSL}_2(\mathbb{Z}) : \Gamma(n)]$	$[\mathrm{PSL}_2(\mathbb{Z}) : \mathrm{Core}(\mathrm{PSL}_2(\mathbb{Z}), G)]$	Congruence?
11	660	1351680	No
15	1440	720	Yes
17	2448	2448	Yes
19	3420	1793064960	No
22	3960	34016140984320	No
26	6552	439697276928	No
33	15840	139330113471774720	No
42	24192	53198770598313984	No
55	79200	2921964261275592975974400	No
66	95040	258723489217327932540472981522522006534225920	No

Table 4.1: Results for two ideal vertices

n	Estimate for h	Upper Bound for λ_1
11	0.310382	1.58413
19	0.189393	0.737484
22	0.177715	0.671254
26	0.234439	1.0185
33	0.476386	3.22221
42	0.351283	1.93656
55	0.208312	0.850564
66	0.317591	1.64382

Table 4.2: Upper bounds for the spectral gap of $\mathbb{H}^2/\Gamma$

Chapter 5

Dirichlet–Ford Domains and Kleinian Groups

This chapter discusses what constraints are placed on Kleinian groups which admit a Dirichlet–Ford domain. Throughout, Γ will be a non-cocompact Kleinian group, conjugated in $\mathrm{PSL}_2(\mathbb{C})$ such that the parabolic fixed point at ∞ has stabilizer Γ_∞ which acts cocompactly on $\mathbb{C}$.

Definition. A *Dirichlet–Ford domain*, or *DF domain*, P, for the Kleinian group Γ is a Dirichlet domain for some center z_0 which is also a Ford domain for some choice of fundamental region for Γ_∞.

It will be shown that there is no direct correspondence between DF domains and reflection groups as there is in the case of Fuchsian groups. In particular, we will prove the following theorem.

Theorem 5.0.1. *The rotation subgroup of any non-cocompact reflection group of* $\mathrm{Isom}(\mathbb{H}^3)$ *admits a Dirichlet–Ford domain.*

It will also be shown that the converse of Theorem 5.0.1 does not hold; we will exhibit examples of Kleinian groups which do not admit DF domains. However, it is still possible to extend the methods of Chapter 4 to test arithmetic maximal reflection groups for congruence. We will show the following.

Theorem 5.0.2. *There exists an arithmetic maximal reflection group in* $\mathrm{Isom}(\mathbb{H}^3)$ *which is not congruence.*

The discussion will be restricted to DF domains; as the discussion of Section 3.3 demonstrates, it is not unreasonable to suppose that Double Dirichlet domains share many similar properties.

5.1 Dirichlet–Ford Domains

In this section, it will be shown that only one direction (the analogue of Theorem 3.2.1) of Theorem 3.0.1 holds when we consider Kleinian groups in the place of Fuchsian groups. This is because the added dimension gives new possibilities for the shape of the domains in question; in particular, they no longer have to glue up in a completely symmetrical way, although some symmetry remains. Examples will be given to demonstrate this flexibility, which extends as far as having non-trivial cuspidal cohomology.

We begin by proving a result analogous to Theorem 3.2.1.

Theorem 5.1.1. *Let* $Q \subset \mathbb{H}^3$ *be a finite-sided, convex, non-compact hyperbolic polyhedron satisfying the hypotheses of Andreev's Theorem 2.5.1, and let* G *be the discrete group of reflections in* Q*. Then* G *contains an index* 2 *Kleinian subgroup which admits a DF domain.*

Proof. Suppose that Q is placed in upper half-space $\mathbb{H}^3$ such that one of its faces M is contained in a vertical plane. Let

$$G = \langle \tau_1, \ldots, \tau_k, \tau_M \rangle$$

be a generating set for G. Let

$$\Gamma = \langle \tau_M \tau_1, \ldots, \tau_M \tau_k \rangle$$

be the index 2 subgroup. Let $P = Q \cup \tau_M Q$. Let $w_0 = x_0 + y_0 i + z_0 j \in \mathring{M}$, for $z_0 > 0$. The claim is that w_0 is a Dirichlet center for Γ. Fix a generator $\gamma_i = \tau_M \tau_i$. Then the plane P_i fixed by τ_i bisects w_0 and $\tau_i(w_0)$, and so $\tau_L(P_i)$, which by construction is a face of P, bisects w_0 and $\gamma_i(w_0)$. Furthermore, P_i is the isometric sphere of γ_i, and so P is a Ford domain. □

The next result provides a family of counterexamples to the analogue of Theorem 3.2.4 by exhibiting Kleinian groups which admit DF domains and do not have index 2 in a reflection group.

Proposition 5.1.2. *Let Q be an all-right hyperbolic polyhedron, with a vertex at ∞, and all vertices ideal. Let G be the group of reflections in Q. Then G contains a subgroup of index 4 which admits a DF domain.*

Proof. Since Q is all-right, the link of each vertex is a rectangle. Rotate Q in $\mathbb{H}^3$ so that the four vertical sides, which meet at the vertex at ∞, each lie above vertical or horizontal lines in $\mathbb{C}$. Let H be a vertical side, V a horizontal side, and τ_H and τ_V the respective reflections. Let $P = (Q \cup \tau_H Q) \cup \tau_V(Q \cup \tau_H Q)$. Then P is the union of 4 copies of Q. Looking down from ∞ on the floor of P, label by A the non-vertical face adjacent to the top-left vertex and to the vertical face opposite H. Label any non-vertical faces adjacent to this face B. Proceed to label every non-vertical face A or B, with no two adjacent faces sharing the same label. The symmetry of P implies that this labeling is symmetric in both horizontal and vertical directions. Define the subgroup Γ as follows. Given a non-vertical side P_i of P, if P_i has label A, let the element $\tau_H \tau_i$ belong to Γ; if P_i has label B, let $\tau_V \tau_i$ belong to Γ. If

H' is the face opposite H, and V' opposite V, let $\tau_H\tau_{H'}$ and $\tau_V\tau_{V'}$ belong to Γ. Then P is a DF domain for Γ. $\square$

Remark. Given a group Γ constructed as in the above proof, note that Γ is not an index 2 subgroup of the group of reflections in the polyhedron $(Q \cup \tau_H Q)$. This is because the reflection τ_H will be absent from this group, preventing the construction of elements of Γ of the form $\tau_H\tau_i$. The same is valid for the group of reflections in the polyhedron $(Q \cup \tau_V Q)$. Furthermore, Γ is not the rotation subgroup of any reflection group, as can be seen from the fact that Γ possesses torus cusps.

Remark. The following is an alternative definition pointed out by Allcock. Given the polyhedron Q, color the faces white or black so that adjacent faces have different colors. Then define a map $\psi : G \to \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ that sends all white reflections to $(1,0)$ and all black reflections to $(0,1)$. Then the subgroup Γ is the kernel of ψ.

Since there is no direct analogue of Theorem 3.2.4 for Kleinian groups, the question arises as to what, if anything, is implied about a Kleinian group by it having a DF domain. For example, one might ask whether such groups must have trivial cuspidal cohomology. The following example gives a Kleinian group which admits a DF domain, but which has non-trivial cuspidal cohomology; that is, there exists a non-peripheral homology class of infinite order in the first homology of the quotient space.

Example. Let $\Gamma < \mathrm{PSL}_2(\mathbb{C})$ be generated by the matrices

$$\begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 5i \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & \frac{-1}{\sqrt{2}} \\ \sqrt{2} & 0 \end{pmatrix}, \begin{pmatrix} -\sqrt{2} & \frac{i}{\sqrt{2}} \\ -i\sqrt{2} & -\sqrt{2} \end{pmatrix},$$

and

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & \frac{-1}{\sqrt{2}} \\ \sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} 1 & \bar{a} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \sqrt{2}a & \sqrt{2}a\bar{a} - \frac{1}{\sqrt{2}} \\ \sqrt{2} & \sqrt{2}\bar{a} \end{pmatrix}$$

for each $a \in \{1, 2, 1+i, 2+i, 2i, 1+2i, 2+2i, 1-i, 2-i, -2i, 1-2i, 2-2i\}$, where $\bar{a}$ is the complex conjugate of a. Then the isometric spheres of these matrices have centers at the Gaussian integers $\{x + iy \mid x, y \in \mathbb{Z}\}$ and radius $\frac{1}{\sqrt{2}}$. The square with vertices at $\pm\frac{5}{2} \pm \frac{5}{2}i$ is a Dirichlet domain for the action of Γ_∞. Let P be the intersection of the exterior of all these isometric spheres with the chimney above the given rectangle. Then P is a DF domain for Γ, with Dirichlet center any point of $\mathring{P}$ above 0. Every dihedral angle of P is $\frac{\pi}{2}$. The quotient space $\mathbb{H}^3/\Gamma$ has 14 boundary components; the cusp at ∞ gives a boundary torus, and each of the 13 cusp cycles in $\mathbb{C}$ gives a $(2,2,2,2)$ or a $(2,4,4)$ sphere (see Section 2.3). Thus the peripheral homology has rank 1. Computation using Gap [19] gives that $H_1(\mathbb{H}^3/\Gamma)$ has $\mathbb{Q}$-rank 2, so there is infinite non-peripheral homology.

Remark. The cuspidal cohomology of this example has rank 1, but it can be modified to give examples where this rank is arbitrarily high.

Remark. This example is arithmetic. To see this, observe that Γ belongs to the normalizer $N(\Gamma_0(2))$ of $\Gamma_0(2) < \mathrm{PSL}_2(\mathcal{O}_1)$; this follows from the fact that

$$\gamma = \begin{pmatrix} 0 & \frac{-1}{\sqrt{2}} \\ \sqrt{2} & 0 \end{pmatrix}$$

belongs to this normalizer, and each of the generators can be written as the product of γ with an element of $\Gamma_0(2)$. Since Γ has finite covolume, it must be a finite index subgroup of an arithmetic group, and so it is itself arithmetic.

Remark. The quotient space of $\mathbb{H}^3$ by this group is not a manifold, so one can thus ask whether there exists another example which has non-trivial cuspidal cohomology, and which is additionally torsion-free.

Although there does not appear to be a specific condition for a Kleinian group which is equivalent to having a DF domain, we can say something about a group which admits a DF domain. We cannot always decompose an orientation-preserving isometry of $\mathbb{H}^3$ into the composition of two reflections, but Carathéodory [13] shows that we need at most four. If $\gamma \notin \Gamma_\infty$, these can be taken to be $\gamma = \gamma_4 \circ \gamma_3 \circ \gamma_2 \circ \gamma_1$, where γ_1 is reflection in the isometric sphere S_γ, γ_2 in the vertical plane R_γ bisecting S_γ and $S_{\gamma^{-1}}$, and $\gamma_4 \circ \gamma_3$ is rotation around the vertical axis through the North pole of $S_{\gamma^{-1}}$.

Theorem 5.1.3. *Suppose the Kleinian group Γ admits a DF domain P. Then the planes R_γ, for side-pairings $\gamma \in \Gamma \setminus \Gamma_\infty$ of P, all intersect in a vertical axis. Furthermore, for each such γ, $\gamma_4 \circ \gamma_3 = 1$, and so each element of the corresponding generating set for Γ has real trace.*

Proof. Let P be a Ford domain. Suppose there is some side-pairing γ such that $\gamma_4 \circ \gamma_3 \neq 1$. By considering the North pole of S_γ and its image, the North pole of $S_{\gamma^{-1}}$, we see that if P were a Dirichlet domain, its center w_0 would have to be in the plane R_γ. But given any such choice of w_0, one can find a point $w \in P \cap S_\gamma$ such that $d(w_0, w) \neq d(w_0, \gamma(w))$. Thus P is not a Dirichlet domain. Since each $\gamma \in \Gamma \setminus \Gamma_\infty$ is then simply the composition of two reflections, it is the conjugate in $\mathrm{PSL}_2(\mathbb{C})$ of an element of $\mathrm{PSL}_2(\mathbb{R})$. It thus has real trace. Since it is assumed that any element of Γ_∞ is parabolic, these too have real trace.

Next suppose that the planes R_γ do not have a common intersection. Since we know that $\gamma_4 \circ \gamma_3 = 1$, for a given γ, the plane R_γ represents the set of potential Dirichlet centers. If there is no common such center, P is not a Dirichlet domain. Thus if P is a DF domain, the planes R_γ have a common intersection. □

The examples given earlier in this section give a flavor of the particular case with only two distinct, perpendicular planes R_γ. It is therefore possible for DF domains to be more complicated than this. This theorem provides a useful criterion for having a DF domain, which can be used to check known Ford domains. Observe that the vertical axis of intersection of the planes R_γ must correspond to a Dirichlet center for the action of Γ_∞. Thus we see that the figure-8 knot group [38], as well as the Whitehead link group and the group of the Borromean rings [47] do not admit DF domains. Furthermore, the groups obtained from a standard Ford domain in [48] cannot admit DF domains. Although in some cases, with the right choice of Ford domain, one can generate congruence subgroups of Bianchi groups using elements of real trace, the sides of the domain are identified in a way similar to the corresponding Fuchsian congruence subgroup, and so these groups seldom admit a DF domain.

5.2 Reflection Groups

In this section, we prove Theorem 5.0.2 by exhibiting an example of a maximal arithmetic reflection group in $\mathrm{Isom}(\mathbb{H}^3)$ which is not congruence. This construction is very much informed by the nature of the non-congruence examples found in Chapter 4; we first recall how those are constructed, without reference to DF domains.

We may summarize the method for constructing non-congruence arithmetic maximal hyperbolic reflection groups in $\mathbb{H}^2$ as follows: begin by considering maximal non-cocompact arithmetic Fuchsian groups; restrict attention to those of genus zero, and which are not themselves reflection groups, but which are, in some sense, “almost” reflection groups; construct a subgroup of index 2 which is a reflection group, and maximal by the small index; test this group for congruence.

Though we have less information about hyperbolic reflection groups in $\mathbb{H}^3$, there is no obstruction to carrying out a similar method there. In particular, we seek a maximal non-cocompact arithmetic Kleinian group which admits a Ford domain with the following property: all by two of the faces are paired with their reflection in a fixed vertical plane R; and the remaining two sides are paired with themselves via order two rotations.

We begin by recalling that the analogues of $\mathrm{PSL}_2(\mathbb{Z})$ which we will consider are the Bianchi groups $\mathrm{PSL}_2(\mathcal{O}_d)$, where d is a positive square-free integer, and $\mathcal{O}_d$ is the ring of integers of the number fields $\mathbb{Q}(\sqrt{-d})$. It is known that the non-cocompact maximal arithmetic Kleinian groups are precisely the normalizers $N(\Gamma_0(J)) < \mathrm{PSL}_2(\mathbb{C})$ for (square-free) ideals $J < \mathcal{O}_d$ (see Borel [8], Chinburg–Friedman [15], and Chapter 11 of Maclachlan–Reid [28]).

Consider the group $\Gamma_0(5) < \mathrm{PSL}_2(\mathcal{O}_2)$, where $\mathcal{O}_2 = \mathbb{Z}[\sqrt{-2}]$. It consists of the matrices

$$\Gamma_0(5) = \left\{ \begin{pmatrix} a & b \\ 5c & d \end{pmatrix} \right\} < \mathrm{PSL}_2(\mathcal{O}_2).$$

In this case $\mathcal{O}_2$ is a principal ideal domain, and the ideal (5) is prime, so the normalizer $N(\Gamma_0(5))$ is generated by $\Gamma_0(5)$ and the involutions

$$\begin{pmatrix} 0 & -\frac{1}{\sqrt{5}} \\ \sqrt{5} & 0 \end{pmatrix}, \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix},$$

where $i = \sqrt{-1}$. We now construct a Ford domain for this normalizer.

We begin by selecting a fundamental region for the stabilizer of ∞ acting on $\mathbb{C}$; we choose the rectangle with vertices at $\pm\frac{1}{2}$ and $\pm\frac{1}{2} + \frac{\sqrt{-2}}{2}$. Sides of this are paired by the three isometries

$$\gamma_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \gamma_2 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \gamma_3 = \begin{pmatrix} -i & \sqrt{2} \\ 0 & i \end{pmatrix}.$$

Next we seek isometric spheres in this region; we find that the following isometries suffice:

$$\gamma_4 = \begin{pmatrix} 0 & -\frac{1}{\sqrt{5}} \\ \sqrt{5} & 0 \end{pmatrix}, \gamma_5 = \begin{pmatrix} \sqrt{5} & \frac{2}{\sqrt{5}} \\ 2\sqrt{5} & \sqrt{5} \end{pmatrix}, \gamma_6 = \begin{pmatrix} -i\sqrt{5} & \frac{2i\sqrt{-2}}{\sqrt{5}} \\ i\sqrt{5}\sqrt{-2} & i\sqrt{5} \end{pmatrix},$$

$$\gamma_7 = \begin{pmatrix} (1+\sqrt{-2})\sqrt{5} & \frac{7}{\sqrt{5}} \\ 2\sqrt{5} & (1-\sqrt{-2})\sqrt{5} \end{pmatrix}, \gamma_8 = \begin{pmatrix} 2+\sqrt{-2} & 1 \\ 5 & 2-\sqrt{-2} \end{pmatrix},$$

$$\gamma_9 = \begin{pmatrix} -\sqrt{5} & \frac{\frac{-2(1-\sqrt{-2})}{\sqrt{5}}}{\sqrt{5}} \\ (1+\sqrt{-2})\sqrt{5} & \sqrt{5} \end{pmatrix}, \gamma_9' = \begin{pmatrix} \sqrt{5} & \frac{\frac{-2(1+\sqrt{-2})}{\sqrt{5}}}{-\sqrt{5}} \\ (1-\sqrt{-2})\sqrt{5} & -\sqrt{5} \end{pmatrix}.$$

Observe that γ_4, γ_5, γ_6, γ_7 and γ_8 pair their isometric spheres with their reflection in the plane above the y-axis (so $x = 0$) - this can be seen most readily by observing that in each case, the trace is real, and the diagonal entries are complex conjugates. In contrast, γ_9 and γ_9' are involutions whose axes are as indicated in Figure 5.1 below.

Denote by Q the polyhedron which is that portion of the above Ford domain lying above the rectangle with vertices at $-\frac{1}{2}$, $-\frac{1}{2}+\frac{\sqrt{-2}}{2}$, $\frac{\sqrt{-2}}{2}$ and $0 \in \mathbb{C}$. Note that not all of the dihedral angles of Q are submultiples of π. Then let $Q' = Q \cup \gamma_9 Q$. As above, we use the relations of the side-pairings in $N(\Gamma_0(5))$ to check the dihedral angles of Q' and find that here they are all submultiples of π. We thus consider the group $\Gamma_{Q'}^{\mathrm{ref}}$ of reflections in the side of Q', and its orientation-preserving index 2 subgroup $\Gamma_{Q'}$.

We find that $\Gamma_{Q'}$ is generated by the following elements:

$$\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_7, \gamma_8, \gamma_9'\gamma_9, \gamma_9'\gamma_2 \circ \gamma_9, \gamma_9'\gamma_3\gamma_9, \gamma_9'\gamma_5\gamma_9.$$

Each isometry on this list is the composition of reflection in the corresponding face of Q' with reflection in the vertical plane $x = 0$. The claim now is that $\Gamma_{Q'}$, which is an arithmetic hyperbolic reflection group, is maximal and not congruence.

To see that $\Gamma_{Q'}$ is maximal, observe that it contains $\gamma_4 = \begin{pmatrix} 0 & -\frac{1}{\sqrt{5}} \\ \sqrt{5} & 0 \end{pmatrix}$, and that this cannot belong to any other normalizer $N(\Gamma_0(J))$ for an ideal $J < \mathcal{O}_2$, and hence $\Gamma_{Q'}$

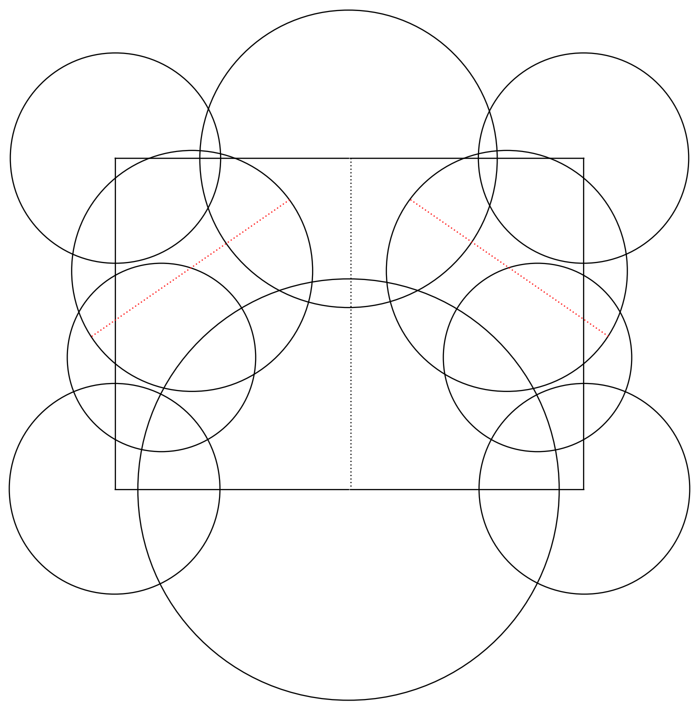


Figure 5.1: A Ford domain for $N(\Gamma_0(5))$

cannot be contained in any maximal arithmetic Kleinian group but $N(\Gamma_0(5))$. We observe that this maximal group is not a reflection group, and hence that $\Gamma_{Q'}^{\mathrm{ref}}$ is maximal.

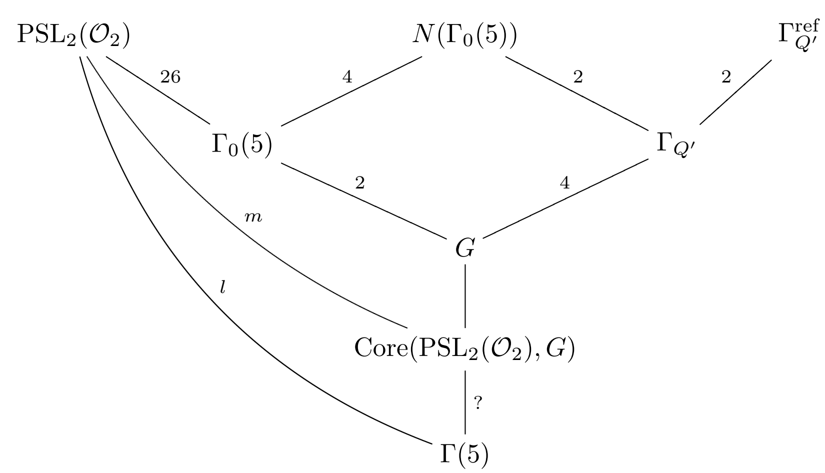
To examine congruence, we set up the same diagram as we had for Fuchsian groups in Chapter 3. Letting $G = \mathrm{PSL}_2(\mathcal{O}_2) \cap \Gamma_{Q'} = \Gamma_0(5) \cap \Gamma_{Q'}$, we find that G has index 52 in $\mathrm{PSL}_2(\mathcal{O}_2)$, and we obtain a representation

$$\varphi_G : \mathrm{PSL}_2(\mathcal{O}_2) \to S_{52}.$$

This enables us to calculate that the orders of the images of the matrices

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & \sqrt{-2} \\ 0 & 1 \end{pmatrix}$$

under the map φ_G are both equal to 5, an thus that the $\mathbb{Z}$-level of G is 5. Since the ideal (5) is prime in $\mathcal{O}_2$ it follows that the $\mathcal{O}_2$-level of G is also (5). By the analogue of Wohlfahrt's Theorem given in Theorem 2.4.3, G is congruence if and only if G contains the principal congruence subgroup $\Gamma(5)$. We have the following diagram.



We are given a formula for the index $[\mathrm{PSL}_2(\mathcal{O}_2) : \Gamma(5)]$ by Fine [17] and find that this index is equal to $l = 1550$. By using Magma [9] we determine that the index of the kernel of φ_G

induced by the action on cosets is $m = 63897600$. We therefore conclude that G, and hence $\Gamma_{Q'}^{\mathrm{ref}}$, is not congruence.

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