The Dissertation Committee for Yingwu Zhao
certifies that this is the approved version of the following dissertation:

Stochastic Equilibria in a General Class of Incomplete Brownian Market Environments

Committee:

Gordan Žitković, Supervisor

Mihai Sîrbu

Thaleia Zariphopoulou

Luis Caffarelli

Lexing Ying

Stathis Tompaidis

Stochastic Equilibria in a General Class of Incomplete Brownian Market Environments

by

Yingwu Zhao, B.S.

DISSERTATION

Presented to the Faculty of the Graduate School of

The University of Texas at Austin

in Partial Fulfillment

of the Requirements

for the Degree of

DOCTOR OF PHILOSOPHY

THE UNIVERSITY OF TEXAS AT AUSTIN

May 2012

To my wife Li and my son Henry

Acknowledgments

I would like to express my deepest gratitude to my supervisor, Dr. Gordan Žitković, for his insightful guidance and substantial advice throughout this research. It has been an amazing experience working with him. In particular, I am extremely thankful to Gordan, for introducing me into this wonderful area of Mathematical Finance, inspiring me with countless brilliant ideas, and granting me generous financial support.

I am grateful to Dr. Thaleia Zariphopoulou for initially inspiring my interest in the area of Mathematical Finance, through her wonderfully taught classes. I own my knowledge in the areas of Probability, Stochastic Analysis, Stochastic Control, SDEs, BSDEs, etc, and their applications in mathematical finance to Dr. Thaleia Zariphopoulou, Dr. Gordan Žitković, Dr. Mihai Sîrbu, and Dr. Gerard Brunick. I would also like to thank Dr. Alexis Vasseur, Dr. Panagiotis Souganidis and Dr. Todd Arbogast for inspiring my interest in PDEs and Applied Mathematics. From their contributions, I have gained invaluable knowledge and insights which are absolutely essential to my research.

I would like to express my sincere gratitude to Dr. Lexing Ying for introducing me to various computational techniques as well as his kind support during my graduate study and research. I would also like to thank Dr. Luis Caffarelli, Dr. Alexis Vasseur, and Dr. Alessio Figalli for spending their precious time providing me with helpful research comments and suggestions. I am truly grateful to all my dissertation committee members, who spent their valuable time in reading this dissertation and providing valuable comments.

I would like to gratefully acknowledge the financial support from Department of Mathematics. The completion of my research would not have been possible without it. I would like to thank Dr. Bruce Palka, Dr. Lorenzo Sadun, and Dr. Dan Knopf for their generous advice and kind encouragement during my graduate study in Mathematics Department. My appreciation also goes to Dr. Mark Daniels, who has kindly coached me in teaching techniques and advised me about teaching in general.

Furthermore, I would like to express my most profound appreciation to my parents for their years of selfless dedication, support and consistent encouragement along the way I grew up. I would also like to thank my dear beautiful wife, Li, who has filled my heart with love and happiness.

Finally, I would like to thank my God, my Lord and Savior, Christ Jesus, for guiding me with His abundant grace and unsearchable wisdom, especially along my path of spiritual and theological exploration. I thank Him for giving me the knowledge, hope, strength, peace and freedom that surpass my understanding. All praise and glory be to Him, till the day I shall meet Him face to face.

Stochastic Equilibria in a General Class of Incomplete Brownian Market Environments

Publication No. _______________

Yingwu Zhao, Ph.D.
The University of Texas at Austin, 2012

Supervisor: Gordan Žitković

This dissertation is a contribution to the equilibrium theory in incomplete financial markets. It shows that, under appropriate conditions, an equilibrium exists and is unique in a general class of incomplete Brownian market environments either composed of exponential-utility-maximizing agents or populated by a class of convex-risk-measure-minimizing agents.

We first use the Dynamic Programming Principle to deduce the Hamilton-Jacobi-Bellman (HJB) equation for each agent, and solve the individual optimization problem, to identify the optimal control. Using the optimal portfolio, we establish the equivalence between the existence of a stochastic equilibrium in an incomplete Brownian market and solvability of a non-linearly coupled parabolic PDE system with a homogeneously-quadratic non-linear structure.

To solve this PDE system, we work mainly in anisotropic Hölder spaces. There, we construct a proper class of Hölder subspaces, where potential solu-

tions to the equilibrium PDE system are expected to "live". These turn out to be convex and compact under the uniform topology, thanks to the help of an Arzelá-Ascoli-type theorem for unbounded domains. We then define an appropriate functional on the subspace, and show that, if we choose the parameters associated with the subspace carefully, this functional maps the subspace back to itself. After that, we apply Schauder's fixed point theorem on a constructed subset of the subspace, and establish the existence of solutions to the PDE system, therefore equivalently, the existence of market equilibria in these general incomplete Brownian market environments.

To prove the uniqueness of the solution to the parabolic PDE system, we utilize classical L^2-type energy estimates and the Gronwall's inequality. This way, we also establish the uniqueness of a market equilibrium within a class of smooth Markovian markets.

Table of Contents

Acknowledgments **v**

Abstract **vii**

Chapter 1. Introduction **1**

Chapter 2. The Problem Statement **6**

2.1 The Market Environment 6
2.1.1 The Information Structure 6
2.1.2 Completeness Constraints 6
2.1.3 Utility-Maximizing Agents 7
2.1.4 Risk-Measure-Minimizing Agents 9
2.1.5 Behavior of Agents 11
2.2 The Problem Statement 13
2.2.1 Market-Clearing Conditions 13
2.2.2 The Problem Statement 14
2.3 Assumptions and Conventions 14

Chapter 3. Market Equilibria with Utility-Maximizing Agents **18**

3.1 Individual Optimization Problem 18
3.1.1 The Value Function and the HJB Equation 18
3.1.2 Solution to HJB and Verification 19
3.2 Market Equilibria with Utility-Maximizing Agents 22
3.2.1 The Equilibrium PDE System 22
3.2.2 A Market Populated by a Single Agent 23
3.2.3 Hedgeable Terminal Random Endowments 25
3.2.4 “Totally Nonhedgeable” Terminal Random Endowments 26
3.2.5 A Linear Combination of Hedgeable and “Totally Nonhedgeable” Terminal Random Endowments 26

3.2.6 Existence and Uniqueness of an Equilibrium in the General Case . . . 27

Chapter 4. Market Equilibria with Risk-Measure-Minimizing Agents **31**

4.1 Individual Optimization Problem . . . 31

4.1.1 Value Function and the HJB . . . 31

4.1.2 Solution to HJB and Verification . . . 33

4.2 Market Equilibria with Risk-Measure-Minimizing Agents . . . 38

4.2.1 The Equilibrium PDE System . . . 38

4.2.2 Existence and Uniqueness of an Equilibrium with Risk-Measure-Minimizing Agents . . . 39

Chapter 5. A System of BSDEs **43**

Chapter 6. A Solution to the PDE System (6.1.1) **46**

6.2 The Function Space $\mathcal{B}$ and the Map $\mathcal{H}$ . . . 47

6.3 Existence and Uniqueness of the Solution to PDE System (6.1.1) 53

Chapter 7. Summary **57**

Appendices **59**

Appendix A. Several Technical Lemmas **60**

A.1 On the Function h . . . 60

A.2 A Few Convolution-Related Computations . . . 64

A.3 On The Penalty Function f . . . 65

A.4 A Version of the Arzelá-Ascoli Theorem . . . 66

Appendix B. Anisotropic Hölder Spaces **69**

B.1 Classical Anisotropic Hölder Spaces. . . . 69

B.2 Isotropic Hölder Spaces. . . . 71

B.3 Functions that Vanish at Infinity . . . 72

B.4 The Sobolev Space $H^1(Q)$ . . . 72

B.5 Interpolation Inequalities . . . 73

B.6 A Linear Cauchy Problem . . . 74

Bibliography 75

Vita 79

Chapter 1

Introduction

The competitive equilibrium, a class of price-determination models based on a balance of demand and supply, has been an active research area in Economics for more than a century. Perhaps the oldest work on this subject is that of Leon Walras [24] in 1874. Later, a mathematical approach, addressing the question of existence and uniqueness of Walras's equations, was given by Wald [23] in 1936. The first complete and mathematically rigorous existence proof of an equilibrium in an economy with multiple agents and finitely many assets was accomplished by Arrow and Debreu [2] in 1954. Bewley's paper [4] at 1972 is frequently cited as the classical reference on competitive equilibrium with an infinite-dimensional commodity space.

The issue of existence and uniqueness of equilibrium in *complete* continuous-time stochastic models with heterogeneous agents has been subject to active research in the latter half of the twentieth century, and has made significance advances - see, for example, [1], [6], [10], [11], [12], [15], [16], [17], [25] as well as Chapter 4 of [18]. Among these references, the central idea of finding an equilibrium is the representative-agent approach, which is closely related to market completeness. Using this idea, one tries to assign weights to different agents to form a representative agent, and thereby reduces the problem to one of the determination of the proper weights. As a result, it turns the infinite-dimensional problem of finding an equilibrium process into a finite-dimensional problem of finding a finite set of real weights. Furthermore, when viewed from a PDE perspective, the same idea can be used to reduce a parabolic PDE system (formed by the Hamilton-Jacobi-Bellman (HJB) equations induced by each agent) into one single HJB PDE (corresponding to the representative agent).

When the market model is *incomplete*, the equilibrium analysis becomes much more difficult, mainly due to the fact that the classical reduction described above will not work. While some authors successfully treated classes of degenerate markets using this idea, the representative agent will not exist in a generic incomplete model. Therefore, one faces a difficult, necessarily infinite-dimensional problem, or equivalently from the PDE's perspective, one confronts a fully coupled non-linear parabolic PDE system, with a highly non-trivial structure. So far, to the best of our knowledge, the only paper in continuous time where a fully-incomplete market structure is analyzed and existence of equilibria is established, is by Gordan Žitković [26]. In his work, the market is assumed to have a single stock whose price dynamics is driven by a single Brownian Motion while the terminal payoff further depends on an independent one-jump Poisson process, and the author reduced the problem into a semi-linear PDE system and solved it.

In this thesis, we are interested in the existence and uniqueness of stochastic equilibria in a general class of fully incomplete continuous-time financial market environments where the market participants are either heterogeneous exponential-utility maximizers or convex-risk-measure minimizers with random endowments that are generally not hedgeable, due to the incompleteness feature of the market model. We use the Dynamic Programming Principle to derive the Hamilton-Jacobi-Bellman equation for each agent, and relate the problem of finding a market equilibrium to a quasi-linear PDE system. After that, we solve it using a combination of old and new techniques.

In Chapter 2, we introduce a class of incomplete financial market models containing multiple non-redundant assets, as well as a group of financial agents with non-hedgeable terminal random endowments (or liabilities). These agents come from two classes, the first one being a formal, but not conceptual, subclass of the second. The first one contains classical exponential-utility maximizers who adjust their portfolios by dynamically trading so as to *maximize* their expected terminal utilities. The other class of agents, instead of maximizing utilities, try to *minimize* their terminal risks, which are measured by a class of convex risk measures. After that, we formulate the equilibrium problem in an appropriate mathematical framework. At the very end, we list the standing assumptions and notational conventions which are used throughout the rest of this dissertation.

In Chapter 3, we establish the existence and uniqueness of an equilibrium when the market is populated by exponential-utility-maximizing agents. We start by solving the optimal control problem for a single agent, and relate the optimal control to a solution of a quasilinear parabolic PDE. Using the newly-obtained form of the single-agent's optimal portfolio, we observe that the equilibrium condition can be interpreted as a special form of coupling of single-agent equations. This way, we reduce the problem to a system of quasi-linear parabolic PDEs, which exhibits non-trivial coupling and a quadratic non-linear structure. We start its analysis by examining several simple cases, which reduce the problem, in various ways, into complete market scenarios where we can construct the equilibrium in a fairly explicit form. These examples illustrate quite clearly the way in which the problem becomes dramatically more difficult when the market is incomplete. We conclude the chapter with our main theorem. It states that, under the appropriate smallness condition, the PDE system admits a unique solution in $C^{2,\alpha}(Q)$, thus establishing existence and uniqueness (within a certain class) of a stochastic incomplete-market equilibrium.

In Chapter 4, we answer a more general question, namely whether the equilibrium exists and whether it is unique when the market is formed by interacting convex-risk-measure-minimizing agents; the answer is, again, affirmative. Just like in Chapter 3, we solve the individual optimal control problem first, through an approach based on the Dynamic Programming Principle and the related HJB equation. The non-linear HJB equation is of Isaacs type, due to the fact that an agent minimizing his or her convex risk measure can be viewed as participating in a stochastic game, where the agent is playing against a particularly malicious nature. The structure of the optimal portfolio is, however, much less explicit in this case (compared to the case of an exponential agent). Nevertheless, it can still be used, together with the equilibrium condition, to transform the equilibrium problem into a quasi-linear PDE system, which takes a more general form than the one in the previous chapter. Moreover, the system turns out to be solvable in $C^{2,\alpha}(Q)$ under a set of conditions similar to that in the main theorem in Chapter 3.

In Chapter 5, we use our previous PDE-based results to establish a new existence result within the theory of Backward Stochastic Differential Equations (BSDE). We use a well-known relationship between the quadratic BSDE and quasilinear (quadratic) PDE, to show that a class of multi-dimensional quadratic BSDEs admit unique solutions.

Chapter 6 provides technical details related to our solution of a class of quasi-linear parabolic PDE systems, slightly more general than that appearing in the equilibrium PDE systems. Here, we work with the classical anisotropic Hölder spaces, perform various heat-kernel and convolution-based computations, and apply the Schauder's fixed point theorem to show the existence of solutions. Uniqueness is obtained by using energy-type estimates and the Gronwall's inequality.

Chapter 2

The Problem Statement

2.1 The Market Environment

2.1.1 The Information Structure

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space, where the filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$ satisfies the usual conditions of right-continuity and completeness. We assume further that $\mathbb{F}$ is the $\mathbb{P}$-completion of the filtration generated by an n-dimensional Brownian motion $\{\boldsymbol{B}_t\}_{t\in[0,T]} = \{B_t^{[1]}, \dots, B_t^{[n]}\}_{t\in[0,T]}$.

2.1.2 Completeness Constraints

On the probabilistic setup described above, we single out a family of financial markets, which will contain all possible market dynamics we allow the eventual equilibrium to take. We refer the reader to Appendix B for the function-space notation, such as $(C^{0,\alpha}(Q))^d$, where $0 < d < n$, and we use $Q := [0,T] \times \mathbb{R}^n$ to denote the domain.

Let $\boldsymbol{\sigma} = (\sigma_k^j)_{k\le n}^{j\le d}$ be a $d \times n$ matrix with the block form $\boldsymbol{\sigma} = \begin{bmatrix} \boldsymbol{I}_d & \boldsymbol{0} \end{bmatrix}$, where $\boldsymbol{I}_d$ is a $d \times d$ identity matrix and $\boldsymbol{0}$ is a $d \times n$ zero matrix. For $\boldsymbol{\lambda} = (\lambda^{[1]}, \dots, \lambda^{[d]}) \in (C^{0,\alpha}(Q))^d$, we define the d-dimensional Itô-process

$\{\boldsymbol{S}^{\boldsymbol{\lambda}}_t\}_{t\in[0,T]} = \{(S^{\boldsymbol{\lambda}})^{[1]}_t, \dots, (S^{\boldsymbol{\lambda}})^{[d]}_t\}_{t\in[0,T]}$ by

$$d\boldsymbol{S}^{\boldsymbol{\lambda}}_t = \boldsymbol{\lambda}(t, \boldsymbol{B}_t)\, dt + \boldsymbol{\sigma}\, d\boldsymbol{B}_t,\ t \in [0, T], \quad \boldsymbol{S}^{\boldsymbol{\lambda}}_0 = (0, 0, \dots, 0) \in \mathbb{R}^d, \tag{2.1.1}$$

where the values of all multi-dimensional stochastic processes are interpreted as column vectors. Written component-wise, the dynamics of $\boldsymbol{S}^{\boldsymbol{\lambda}}$ is given by

$$d(S^{\boldsymbol{\lambda}})^{[j]}_t = \lambda^{[j]}(t, \boldsymbol{B}_t)\, dt + \, dB^{[j]}_t,\ j = 1, \dots, d. \tag{2.1.2}$$

Remark 2.1.1. It is important to note that a specific form for the dynamics of the process $S^{\boldsymbol{\lambda}}$ is not important for our purposes. We only care about the market subspace it spans, i.e., the set of all admissible stochastic integrals with respect to it (see 2.1.3 below for precise definitions of admissibility). For that reason, we choose arithmetic dynamics and the simplest possible volatility structure. A linear change of variables is enough to ensure that all the results in this dissertation remain valid under the weaker assumption that $\boldsymbol{\sigma}$ is a general *full-rank* $d \times n$ matrix.

Thanks to Remark 2.1.1 above, we can (and do) interpret the process $\{\lambda^{[j]}(t, \boldsymbol{B}_t)\}_{t\in[0,T]}$ as the market price of risk of the j-th asset. In general, we will identify the d-dimensional process $\{\boldsymbol{\lambda}(t, \boldsymbol{B}_t)\}_{t\in[0,T]}$ and its Markov representative $\boldsymbol{\lambda} \in (C^{0,\alpha}(Q))^d$ and call it the **market price of risk**.

2.1.3 Utility-Maximizing Agents

We assume that there is a finite number $I \in \mathbb{N}$ of agents, all of whom actively participate in trading in all available assets, and they belong to one of two classes. The first class is composed of those with exponential-utilities. In the description of this class we adopt the Alt-von Neumann-Morgenstern expected-utility paradigm and assume that the behavior of each agent in this class is fully specified by the following two ingredients:

1. the *utility function* given by $U^{[i]}(x) = -\exp(-\gamma_i x)$, $x \in \mathbb{R}$, for $\gamma_i > 0$,

2. the *random endowment*, i.e., a random variable $\mathcal{E}^{[i]} \in \mathbb{L}^\infty(\mathcal{F}_T)$ of the form

$$\mathcal{E}^{[i]} = g^{[i]}(\boldsymbol{B}_T) \text{ for } g^{[i]} \in C^0(\mathbb{R}^n).$$

Remark 2.1.2.

1. It is important to note that the agents' random endowments depend on all components of the n-dimensional "factor" process $\{\boldsymbol{B}_t\}_{t\in[0,T]}$. It is precisely this property that makes the situation truly incomplete. No matter what the prevailing market price of risk $\boldsymbol{\lambda}$ happens to be, the market will (generically) not be able to span all $\mathcal{E}^{[i]}$, for $i = 1, \ldots, I$.

2. It is implicitly assumed that all agents assess the likelihood of future events according to the same probability $\mathbb{P}$. This is, however, not a significant assumption, thanks to the exponential nature of the utility functions. Indeed, using the identity

$$\mathbb{E}^{\mathbb{P}^{[i]}}[-\exp(-\gamma_i(X + \mathcal{E}^{[i]}))] = \mathbb{E}[-\exp(-\gamma_i(X + \tilde{\mathcal{E}}^{[i]}))],$$

where $\tilde{\mathcal{E}}^{[i]} = \mathcal{E}^{[i]} - \frac{1}{\gamma_i}\log(\frac{d\mathbb{P}^{[i]}}{d\mathbb{P}})$, we can easily "absorb" different subjective probabilities into the random endowment, if the appropriate regularity conditions are met.

2.1.4 Risk-Measure-Minimizing Agents

The second class of agents are those who strive to minimize their risk, as measured by a convex risk measure, through dynamical trading in all assets. Coherent risk measures were introduced by Artzner et al. [3] in finite sample spaces, and later by Delbaen [7] in general probability spaces. They were later extended by Föllmer and Schied [13] and Frittelli and Rosazza Gianin [14] to the class of convex risk measures:

Definition 2.1.3. A map $\rho : \mathbb{L}^\infty(\mathbb{P}) \to \mathbb{R}$ is called a convex risk measure if it satisfies the following three conditions, $\forall\, X, Y \in \mathbb{L}^\infty(\mathbb{P})$:

1. Convexity: $\rho(\lambda X + (1-\lambda)Y) \leq \lambda\rho(X) + (1-\lambda)\rho(Y), \forall\, \lambda \in [0,1]$.
2. Monotonicity: If $X \geq Y$, then $\rho(X) \leq \rho(Y)$.
3. Translation Invariance: If $m \in \mathbb{R}$, then $\rho(X+m) = \rho(X) - m$.

The class of convex risk measures, $\rho^{[i]}$ where $i = 1, \ldots, I$, which we assume that agents use, are further restricted by the following regularity assumption:

Assumption 2.1.4. There exist convex continuous functions $f^{[i]}(\boldsymbol{x})$ where $i = 1, \dots, I$, such that $\rho^{[i]}$ is represented by the following:

$$\rho^{[i]}(X) = \sup_{\boldsymbol{\nu} \in \mathcal{N}} \mathbb{E}\Big[Z_T^{\boldsymbol{\nu}} \left(-X - \int_0^T f^{[i]}(\boldsymbol{\nu}_s)\, ds \right) \Big] \tag{2.1.3}$$

where $\mathcal{N}$ denote the class of d-dimensional $\mathbb{F}$-progressively measurable processes $\{\boldsymbol{\nu}_t\}_{t \in [0,T]}$, such that:

$$\mathcal{N} := \Big\{ \{\boldsymbol{\nu}_t\}_{t \in [0,T]} : \forall\, c > 0,\, \mathbb{E}\Big[\exp\left(c \int_0^T \|\boldsymbol{\nu}_t\|^2 dt \right) \Big] < \infty \Big\}$$

and $\{Z_t^{\boldsymbol{\nu}}\}_{t \in [0,T]}$ is the stochastic exponential of $\boldsymbol{\nu}_t$, i.e.

$$Z_t^{\boldsymbol{\nu}} := \exp\left(- \int_0^t \boldsymbol{\nu}_s d\boldsymbol{B}_s - \tfrac{1}{2} \int_0^t \|\boldsymbol{\nu}_s\|^2 ds \right)$$

Remark 2.1.5.

1. The representation (2.1.3) of a convex risk measure is not uncommon. In fact, Delbaen, Peng and Rosazza Gianin [9] have shown that, under minimal conditions, for a given convex risk measure ρ, there always exists proper, convex, upper semi-continuous function f, such that ρ admits a representation similar to above.

2. Note that $\{Z_t\}_{t \in [0,T]}$, defined by:

$$Z_t := \exp\left(- \int_0^t \boldsymbol{\nu}_s d\boldsymbol{B}_s - \tfrac{1}{2} \int_0^t \|\boldsymbol{\nu}_s\|^2 ds \right)$$

is a square-integrable martingale. Furthermore, it has a finite n-th moment $\forall\, n \in \mathbb{N}$.

3. In the definition of $\mathcal{N}$ above, the requirement that it has finite exponential moments for all $c > 0$ is not necessary. In fact, it is enough to require merely that it has finite exponential moment up to some constant $c_0 > 0$, which can be explicitly computed in the proof of Theorem 4.1.2, and as a result, $\{Z_t\}_{t\in[0,T]}$ has finite n-th moment up to certain $n_0 > 0$, which is enough to achieve our result.

Finally, the behavior of each agent in the second class is fully specified by the following two ingredients:

1. the *convex risk measure* given by Assumption 2.1.4.
2. the *random endowment*, i.e., a random variable $\mathcal{E}^{[i]} \in \mathbb{L}^\infty(\mathcal{F}_T)$ of the form
$$\mathcal{E}^{[i]} = g^{[i]}(\boldsymbol{B}_T) \text{ for } g^{[i]} \in C^0(\mathbb{R}^n).$$

2.1.5 Behavior of Agents

Let us now focus on the case when the set of tradeable assets consists of d risky assets whose dynamics are fixed and given by $\{S_t^{\boldsymbol{\lambda}}\}_{t\in[0,T]}$, for some $\boldsymbol{\lambda} \in (C^{0,\alpha}(Q))^d$ (the existence of the trivial numéraire asset with constant value 1 is assumed throughout). Agent i uses a dynamic self-financing portfolio strategy which maximizes the expected utility or minimizes the risk measure from total terminal wealth. More precisely, let $\mathcal{A}_i$ and $\widetilde{\mathcal{A}}_i$ denote families of

d-dimensional $\mathbb{F}$-progressively measurable processes $\{\boldsymbol{\pi}_t\}_{t\in[0,T]}$, such that:

$$\mathcal{A}_i := \left\{ \{\boldsymbol{\pi}_t\}_{t\in[0,T]} \,:\, \exists\, b_i > 1, \text{ s.t. } \mathbb{E}\Big[\exp\Big(\int_0^T \tfrac{1}{2} b_i \gamma_i^2 \|\boldsymbol{\pi}_t\|^2 dt\Big)\Big] < \infty \right\}$$
$$\widetilde{\mathcal{A}}_i := \left\{ \{\boldsymbol{\pi}_t\}_{t\in[0,T]} \,:\, \forall\, c > 0,\, \mathbb{E}\Big[\exp\Big(c\int_0^T \|\boldsymbol{\pi}_t\|^2 dt\Big)\Big] < \infty \right\} \tag{2.1.4}$$

so that, for given initial wealth $\xi^{[i]} \in \mathbb{R}$, where $i = 1, \ldots, I$, actions of agents with exponential-utilities are determined by the following optimization problem:

$$\mathbb{E}\Big[U^{[i]}\big(\xi^{[i]} + \textstyle\int_0^T \boldsymbol{\pi}_u \, dS_u^{\boldsymbol{\lambda}} + \mathcal{E}^{[i]}\big)\Big] \to \max \text{ over } \boldsymbol{\pi} \in \mathcal{A}_i. \tag{2.1.5}$$

and actions of agents with convex risk measures, specified in Assumption 2.1.4, are determined by the optimization problem below:

$$\rho^{[i]}\big(\xi^{[i]} + \textstyle\int_0^T \boldsymbol{\pi}_u \, dS_u^{\boldsymbol{\lambda}} + \mathcal{E}^{[i]}\big) \to \min \text{ over } \boldsymbol{\pi} \in \widetilde{\mathcal{A}}_i. \tag{2.1.6}$$

Remark 2.1.6.

1. Due to the regularity of some of the ingredients, one does not need the sophistication encountered in general semimartingale models and the weaker notions of admissibility typically used there (see, e.g., the classes Θ_i, $i = 1, 2, 3, 4$ in [8] or the notion of permissibility in [21]).

2. In the definition of $\widetilde{\mathcal{A}}$ above, the finite-exponential-moment requirement for all $c > 0$ is not necessary. In fact, it is enough to require merely an exponential moment up to some order $c_0 > 0$, exists, which can be explicitly computed in the proof of Theorem 4.1.2.

2.2 The Problem Statement

2.2.1 Market-Clearing Conditions

A fundamental economic paradigm states that the prevailing market dynamics must have the following property: the demand and supply for each tradeable asset must offset each other at each time and in each state of the world. More precisely, we have the following definition:

Definition 2.2.1. Given a fixed $\boldsymbol{\lambda} := (\lambda^{[1]}(t, \boldsymbol{x}), \ldots, \lambda^{[d]}(t, \boldsymbol{x}))$, the process $\{\boldsymbol{S}_t^{\boldsymbol{\lambda}}\}_{t \in [0,T]}$ is said to have an **equilibrium price dynamics** or, to be an **equilibrium price**, if

1. **(Rationality)**

   (a) when the market is composed of utility-maximizing agents, there exist processes $\{\boldsymbol{\pi}_t^{(\boldsymbol{\lambda},i)}\}_{t\in[0,T]} := \{(\pi_t^{(\boldsymbol{\lambda},i,1)}, \ldots, \pi_t^{(\boldsymbol{\lambda},i,d)})\}_{t\in[0,T]} \in \mathcal{A}_i$, $i = 1, \ldots, I$, such that for all $\boldsymbol{\pi} \in \mathcal{A}_i$ and all $i = 1, \ldots, I$:

   $$\mathbb{E}[U^{[i]}(\int_0^T \boldsymbol{\pi}_u^{(\boldsymbol{\lambda},i)}\, d\boldsymbol{S}_u^{\boldsymbol{\lambda}} + \mathcal{E}^{[i]})] \geq \mathbb{E}[U^{[i]}(\int_0^T \boldsymbol{\pi}_u\, d\boldsymbol{S}_u^{\boldsymbol{\lambda}} + \mathcal{E}^{[i]})]$$

   (b) when the market is populated by risk-measure-minimizing agents, there exist processes $\{\boldsymbol{\pi}_t^{(\boldsymbol{\lambda},i)}\}_{t\in[0,T]} := \{(\pi_t^{(\boldsymbol{\lambda},i,1)}, \ldots, \pi_t^{(\boldsymbol{\lambda},i,d)})\}_{t\in[0,T]} \in \widetilde{\mathcal{A}}_i$, $i = 1, \ldots, I$, such that for all $\boldsymbol{\pi} \in \widetilde{\mathcal{A}}_i$ and all $i = 1, \ldots, I$:

   $$\rho^{[i]}(\int_0^T \boldsymbol{\pi}_u^{(\boldsymbol{\lambda},i)}\, d\boldsymbol{S}_u^{\boldsymbol{\lambda}} + \mathcal{E}^{[i]}) \leq \rho^{[i]}(\int_0^T \boldsymbol{\pi}_u\, d\boldsymbol{S}_u^{\boldsymbol{\lambda}} + \mathcal{E}^{[i]})$$

2. **(Market Clearing)** $\sum_{i=1}^I \pi_t^{(\boldsymbol{\lambda},i,j)} = 0$, for all $t \in [0,T]$, a.s., and all $j = 1, \ldots, d$.

2.2.2 The Problem Statement

We are mainly interested in the following problem: does there exist an equilibrium market price of risk $\boldsymbol{\lambda}(t, \boldsymbol{x})$? Is it unique?

In the rest of this dissertation, we will show, under appropriate assumptions, that the answers to both of the questions above are affirmative, both for the market populated by agents with exponential-utilities and the market composed of agents who minimize risk measures.

2.3 Assumptions and Conventions

Here is a list of assumptions we will use for various results in the rest of the dissertation.

Assumption 2.3.1. For $i = 1, \dots, I$:

1. $g^{[i]}$ belongs to the isotropic Hölder space $C^{2,\alpha}(\mathbb{R}^n)$.

2. There exists a positive decreasing radial function $h : \mathbb{R}^n \to \mathbb{R}^+$, i.e. $h(\boldsymbol{x}) = R(\|\boldsymbol{x}\|)$ for some decreasing function $R : \mathbb{R}^+ \to \mathbb{R}^+$, with $h \in \mathbb{L}^1(\mathbb{R}^n)$, such that:

$$\left|g^{[i]}(\boldsymbol{x})\right|, \; \left|g^{[i]}_{x_j}(\boldsymbol{x})\right| \leq C_g h(\boldsymbol{x})$$

for some constant $C_g > 0$, and all $\boldsymbol{x} \in \mathbb{R}^n$.

Remark 2.3.2.

1. Thanks to Lemma A.1.1, we will assume, without loss of generality, that function h satisfies the condition (A.1.1) for some constant $B_0 > 1$. If we apply Lemma A.1.2 (2), we get the following useful inequalities:

$$\left|g^{[i]}(\boldsymbol{x})\right|, \left|g^{[i]}_{x_j}(\boldsymbol{x})\right| \leq C_g M_0(B_0, D)\phi_D(\boldsymbol{x}), \quad \forall \boldsymbol{x} \in \mathbb{R}^n,$$

where $\phi_D(\boldsymbol{x}) := \exp\left(-\frac{\|\boldsymbol{x}\|^2}{D}\right) * h(\boldsymbol{x})$.

2. Furthermore, without loss of generality, we assume that $|h|_{\mathbb{L}^1} = 1$. For example, one can take h to be one of the following:

   (a) $h(\boldsymbol{x}) := \frac{\left(\sqrt{1+\|\boldsymbol{x}\|^2}\right)^{-a}}{\left|\left(\sqrt{1+\|\boldsymbol{x}\|^2}\right)^{-a}\right|_{\mathbb{L}^1}}$ where $a > n$.

   (b) $h(\boldsymbol{x}) := \frac{\left(\|\boldsymbol{x}\|^n(\ln\|\boldsymbol{x}\|)^b\right)^{-1}}{\left|\left(\|\boldsymbol{x}\|^n(\ln\|\boldsymbol{x}\|)^b\right)^{-1}\right|_{\mathbb{L}^1}}$ where $b > 1$.

Assumption 2.3.3. For any $i = 1, \ldots, I$, we assume that the function $f^{[i]} : \mathbb{R}^n \to \mathbb{R}$ in Assumption 2.1.4, satisfies the following conditions:

1. $f^{[i]}$ is separable in the following sense:

$$f^{[i]}(x_1, \ldots, x_n) = f^{[i]}_1(x_1, \ldots, x_d) + f^{[i]}_2(x_{d+1}, \ldots, x_n)$$

   where functions $f^{[i]}_1 : \mathbb{R}^d \to \mathbb{R}$ and $f^{[i]}_2 : \mathbb{R}^{n-d} \to \mathbb{R}$ are proper, strictly convex, and continuously differentiable.

2. Functions $f^{[i]}_1$ and $f^{[i]}_2$ satisfy the following quadratic growth conditions:

$$\left|f^{[i]}_1(\boldsymbol{x})\right| \leq L_2\|\boldsymbol{x}\|^2, \quad \text{and} \quad L_1\|\boldsymbol{x}\|^2 \leq f^{[i]}_2(\boldsymbol{x}) \leq L_2\|\boldsymbol{x}\|^2, \quad \forall \boldsymbol{x} \in \mathbb{R}^n,$$

   for some constants $0 < L_1 < L_2$.

3. There exists a positive constant $L^{[i]}$, such that the following conditions holds:

$$\left\|Df_1^{[i]}(\boldsymbol{x})\right\| \geq \frac{1}{L^{[i]}}\|\boldsymbol{x}\|, \quad \forall\, \boldsymbol{x} \in \mathbb{R}^n.$$

Furthermore, the sum $\sum_{i=1}^{I} Df_1^{[i]}$ of $Df_1^{[i]}$ is assumed to satisfy the following inequalities:

$$\left\|\sum_{i=1}^{I} Df_1^{[i]}(\boldsymbol{x}) - \sum_{i=1}^{I} Df_1^{[i]}(\boldsymbol{y})\right\| \geq \frac{1}{L_3(\|\boldsymbol{x}\|, \|\boldsymbol{y}\|)}\|\boldsymbol{x}-\boldsymbol{y}\|$$

$$\left\|\sum_{i=1}^{I} Df_1^{[i]}(\boldsymbol{x})\right\| \geq \frac{1}{L_3}\|\boldsymbol{x}\|$$

$\forall\, \boldsymbol{x} \in \mathbb{R}^n$, and for some positive constants L_3 and locally bounded function $L_3 : \mathbb{R}_+^2 \to \mathbb{R}_+$.

Remark 2.3.4. Note that in the assumption (3) above, the seemingly artificial requirements admit an economic interpretation. For example, in the prototypical case of an entropic risk measure, $f^{[i]}(\boldsymbol{x}) = \frac{1}{2\gamma_i}\|\boldsymbol{x}\|^2$, where γ_i has to be positive to make $f^{[i]}(\boldsymbol{x})$ a valid penalty function. Seen from another perspective, γ_i corresponds to the risk aversion of the exponential-utility the entropic risk measure is equivalent to, in the sense that minimizing the entropic risk measure produces the same optimal strategy as maximizing the exponential-utility with risk aversion γ_i. Loosely speaking, the conditions above can then be understood as the conditions that keep the risk-measure analogue of the risk aversion coefficient positive and bounded away from 0.

Lastly, throughout this thesis, we will use the following convention:

Convention 1. *For a vector* $X \in \mathbb{R}^n$ *and a matrix* $Z \in \mathbb{R}^{I \times n}$*, we denote by* $\|X\|$ *the Euclidean norm and by* $\|Z\|$ *the Frobenius (Hilbert-Schmidt) norm* $\left[tr\left(ZZ^T\right)\right]^{1/2}$*. For the following functions (where* $0 < d < n$ *and the usual argument names are included for the readers convenience),*

$$\begin{aligned} V(t,\xi,\boldsymbol{x}) :&[0,T] \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}, \\ v(t,\xi,z,\boldsymbol{x}) :&[0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}, \\ u(t,\boldsymbol{x}) :&Q \to \mathbb{R}, \text{ and} \\ \boldsymbol{u}(t,\boldsymbol{x}) :&= (u^{[1]}(t,\boldsymbol{x}),\ldots,u^{[I]}(t,\boldsymbol{x})) : Q \to \mathbb{R}^I \end{aligned}$$

we use the following shortcuts:

$$\begin{aligned} DV &:= (V_{x_1},\ldots,V_{x_n})^T \quad D_dV := (V_{x_1},\ldots,V_{x_d})^T \\ Dv &:= (v_{x_1},\ldots,v_{x_n})^T \quad D_dv := (v_{x_1},\ldots,v_{x_d})^T \\ Du &:= (u_{x_1},\ldots,u_{x_n})^T \quad D\boldsymbol{u} := \left(u^{[i]}_{x_j}\right)_{I\times n} \in \mathbb{R}^{I\times n} \\ D_du &:= (u_{x_1},\ldots,u_{x_d})^T \quad D_{n-d}u := \left(u_{x_{d+1}},\ldots,u_{x_n}\right)^T \end{aligned}$$

With $*$ *denoting convolution, we set* $\phi_D(\boldsymbol{x}) := \exp\left(-\frac{\|\boldsymbol{x}\|^2}{D}\right) * h(\boldsymbol{x})$*, as well as*

$$\begin{aligned} H &:= |h|_{\mathbb{L}^\infty} \quad G := \max_{i=1,\ldots,I}\left(\left|g^{[i]}\right|_{2,\alpha}\right) \\ \boldsymbol{\gamma} &:= (\gamma_1,\ldots,\gamma_I) \in \mathbb{R}^I_+ \quad \overline{\gamma} := \left(\sum_{i=1}^{I}\gamma_i^{-1}\right)^{-1} \end{aligned}$$

The following constants (where ε_0 *appears in Lemma A.1.1 and A.1.2)*

$$\alpha,\ B_0,\ C_g,\ D,\ G,\ H,\ L^{[i]},\ L_1,\ L_2,\ L_3,\ T,\ n,\ d,\ I,\ \varepsilon_0,\ \boldsymbol{\gamma},$$

or any functions thereof are called "Generic Constants".

Chapter 3

Market Equilibria with Utility-Maximizing Agents

3.1 Individual Optimization Problem

In this section, we discuss the optimization problem for an individual exponential-utility-maximizing agent. To simplify the notation, we will drop the upper index i in this section.

3.1.1 The Value Function and the HJB Equation

One can characterize the optimal portfolio by a solution to a quasi-linear PDE, i.e., a Hamilton-Jacobi-Bellman equation (HJB). Existing characterizations of this type under various conditions, in case of exponential-utility, are too numerous to list (see, for example, the references in [5]). We start by defining the value function $V^* : [0,T] \times \mathbb{R}^{n+1} \to \mathbb{R}$, as the following:

$$V^*(t, \xi, \boldsymbol{x}) := \max_{\boldsymbol{\pi} \in \mathcal{A}} \mathbb{E}\Big[-\exp\Big\{ -\gamma \Big(\xi + \textstyle\int_t^T \boldsymbol{\pi}_s^T d\boldsymbol{S}_s^{\boldsymbol{\lambda}} + g(\boldsymbol{B}_T) \Big) \Big\} \Big] \tag{3.1.1}$$

where we assume that the Brownian Motion $\boldsymbol{B}_s$, $t \leq s \leq T$, satisfies $\boldsymbol{B}_t = \boldsymbol{x}$. We can formally deduce the Hamilton-Jacobi-Bellman equation through the

Dynamic Programming Principle, as follows:

$$\begin{cases} V_t + \max_{\boldsymbol{\pi}\in\mathbb{R}^d}\left(\frac{1}{2}V_{\xi\xi}||\boldsymbol{\sigma}^T\boldsymbol{\pi}||^2 + \left(V_\xi\boldsymbol{\lambda}^T + D_dV_\xi^T\right)\boldsymbol{\pi}\right) + \frac{1}{2}\Delta V = 0 \\ V(T,\xi,\boldsymbol{x}) = -\exp\{-\gamma\left(\xi + g(\boldsymbol{x})\right)\} \end{cases} \tag{3.1.2}$$

Thanks to the special structure of the exponential-utility, we can guess that the solution V takes the following form:

$$V(t,\xi,\boldsymbol{x}) := -\exp\{-\gamma(\xi + u(t,\boldsymbol{x}))\}.$$

Indeed, that leads to the following quasilinear equation for u:

$$\begin{cases} u_t + \frac{1}{2}\Delta u - \boldsymbol{\lambda}^T D_d u + \frac{1}{2\gamma}\|\boldsymbol{\lambda}\|^2 - \frac{\gamma}{2}\|D_{n-d}u\|^2 = 0 \\ u(T,\boldsymbol{x}) = g(\boldsymbol{x}) \end{cases} \tag{3.1.3}$$

The following equivalent form will also come in handy:

$$\begin{cases} u_t + \frac{1}{2}\Delta u + \frac{\gamma}{2}\left\|\frac{1}{\gamma}\boldsymbol{\lambda} - D_d u\right\|^2 - \frac{\gamma}{2}\|Du\|^2 = 0 \\ u(T,\boldsymbol{x}) = g(\boldsymbol{x}) \end{cases}$$

3.1.2 Solution to HJB and Verification

Theorem 3.1.1. *Assuming that $g \in C^{2,\alpha}(\mathbb{R}^n)$ and $\boldsymbol{\lambda} \in C^{0,\alpha}(Q)$, the HJB equation* (3.1.3) *has a unique solution $u \in C^{2,\alpha}(Q)$. Furthermore, the portfolio $\{\boldsymbol{\pi}_t^{(\boldsymbol{\lambda})}\}_{t\in[0,T]}$, given by,*

$$\boldsymbol{\pi}_t^{(\boldsymbol{\lambda})} := \frac{1}{\gamma}\boldsymbol{\lambda}\left(t,\boldsymbol{B}_t\right) - D_d u(t,\boldsymbol{B}_t),$$

is admissible and optimal.

Proof. The first part of the theorem is a direct consequence of Theorem 8.1 of chapter V from [20]. Now let us verify that the portfolio given above is indeed admissible and optimal. The admissibility of $\boldsymbol{\pi}^{(\boldsymbol{\lambda})}$ is obvious, since both $\boldsymbol{\lambda}$ and $D_d u$ are uniformly bounded. For optimality, set

$$V(t,\boldsymbol{x}) := -\exp\{-\gamma\,(\xi + u(t,\boldsymbol{x}))\}$$

where $u \in C^{2,\alpha}(Q)$ is the unique solution to HJB equation (3.1.3), then one can verify that V is the solution to the formal HJB equation (3.1.2), and it inherits its regularity from u. For any fixed admissible portfolio $\boldsymbol{\pi}_t$, we define stopping times τ_n, $\forall\, n \in \mathbb{N}$, as follows:

$$\tau_n := \inf\{s \geq t \,:\, |X_s^{\boldsymbol{\pi}}| \geq n\}.$$

Itô formula, applied to $V(s\wedge\tau_n, X^{\boldsymbol{\pi}}_{s\wedge\tau_n}, \boldsymbol{B}_{s\wedge\tau_n})$, yields:

$$\begin{aligned} V(T\wedge\tau_n, X^{\boldsymbol{\pi}}_{T\wedge\tau_n}, \boldsymbol{B}_{T\wedge\tau_n}) &= V(t,\xi,\boldsymbol{x}) \\ &\quad + \int_t^{T\wedge\tau_n} \left[V_\xi \boldsymbol{\pi}_s^T \boldsymbol{\sigma} + DV\right](s, X_s^{\boldsymbol{\pi}}, \boldsymbol{B}_s)\, d\boldsymbol{B}_s \\ &\quad + \int_t^{T\wedge\tau_n} \mathcal{L}^{\boldsymbol{\pi}_s} V(s, X_s^{\boldsymbol{\pi}}, \boldsymbol{B}_s)\, ds, \end{aligned} \tag{3.1.4}$$

where

$$\mathcal{L}^{\pi} V = V_t + \tfrac{1}{2} V_{\xi\xi} \left\|\boldsymbol{\sigma}^T \pi\right\|^2 + \left(V_\xi \boldsymbol{\lambda}^T + D_d V_\xi^T\right)\pi + \tfrac{1}{2}\Delta V.$$

Notice that the first term on the right hand side is a martingale due to the uniform boundedness of $X_s^{\boldsymbol{\pi}}$ on $\chi_{\{s\leq\tau_n\}}$ and the admissibility of $\boldsymbol{\pi}$. We recall that $V(t,\xi,\boldsymbol{x})$ is the solution to HJB equation (3.1.2), and take expectations of both sides to get:

$$\begin{aligned} \mathbb{E}\Big[V(T\wedge\tau_n, X^{\boldsymbol{\pi}}_{T\wedge\tau_n}, \boldsymbol{B}_{T\wedge\tau_n})\Big] - V(t,\xi,\boldsymbol{x}) &= \mathbb{E}\Big[\int_t^{T\wedge\tau_n} \mathcal{L}^{\boldsymbol{\pi}} V(s, X_s^{\boldsymbol{\pi}}, \boldsymbol{B}_s)\, ds\Big] \\ &\leq \mathbb{E}\Big[\int_t^{T\wedge\tau_n} \max_{\pi\in\mathbb{R}^d} \mathcal{L}^{\pi} V(s, X_s^{\boldsymbol{\pi}}, \boldsymbol{B}_s)\, ds\Big] = 0 \end{aligned} \tag{3.1.5}$$

Our next claim is that the family $\big(V(T\wedge\tau_n, X^{\boldsymbol{\pi}}_{T\wedge\tau_n}, \boldsymbol{B}_{T\wedge\tau_n})\big)_{n\in\mathbb{N}}$ is uniformly integrable. To see this, observe that by the admissibility condition (2.1.4), there exists $b>1$, such that

$$
\begin{aligned}
&\mathbb{E}\Big[\,\big|V(T\wedge\tau_n, X^{\boldsymbol{\pi}}_{T\wedge\tau_n}, \boldsymbol{B}_{T\wedge\tau_n})\big|^{\sqrt{b}}\,\Big]\\
\le&\mathbb{E}\Big[\exp\Big(-\gamma\sqrt{b}\,\big(X^{\boldsymbol{\pi}}_{T\wedge\tau_n}+u(T\wedge\tau_n,\boldsymbol{B}_{T\wedge\tau_n})\big)\Big)\Big]\\
\le&\exp\Big(\gamma\sqrt{b}\,|u|_0\Big)\,\mathbb{E}\Big[\exp\Big(-\gamma\sqrt{b}\Big(\xi+\int_t^{T\wedge\tau_n}\boldsymbol{\pi}_u^T\boldsymbol{\lambda}_u du+\int_t^{T\wedge\tau_n}\boldsymbol{\pi}_u^T\boldsymbol{\sigma}d\boldsymbol{B}_u\Big)\Big)\Big]\\
\le&\exp\left(\gamma\sqrt{b}\left(|u|_0+|\xi|+|\boldsymbol{\lambda}|_0\,H\sqrt{dT}\left(\int_0^T\|\boldsymbol{\pi}_u\|^2du\right)^{1/2}\right)\right)\times\\
&\quad\mathbb{E}\Big[\exp\Big(-\gamma\sqrt{b}\int_t^{T\wedge\tau_n}\boldsymbol{\pi}_u^T\boldsymbol{\sigma}d\boldsymbol{B}_u\Big)\Big]\\
\le&\exp\left(\gamma\sqrt{b}\left(|u|_0+|\xi|+|\boldsymbol{\lambda}|_0\,H\sqrt{dT}\left(\int_0^T\|\boldsymbol{\pi}_u\|^2du\right)^{1/2}\right)\right)\times\\
&\quad\mathbb{E}\Big[\exp\Big(-\gamma\sqrt{b}\int_t^{T\wedge\tau_n}\boldsymbol{\pi}_u^T\boldsymbol{\sigma}d\boldsymbol{B}_u-\tfrac{1}{2}\gamma^2b\int_t^{T\wedge\tau_n}\|\boldsymbol{\pi}_u\|^2du+\tfrac{1}{2}\gamma^2b\int_t^{T\wedge\tau_n}\|\boldsymbol{\pi}_u\|^2du\Big)\Big]\\
\le&\exp\left(\gamma\sqrt{b}\left(|u|_0+|\xi|+|\boldsymbol{\lambda}|_0\,H\sqrt{dT}\left(\int_0^T\|\boldsymbol{\pi}_u\|^2du\right)^{1/2}\right)\right)\times\\
&\quad\mathbb{E}\Big[\exp\Big(-\gamma\sqrt{b}\int_t^{T\wedge\tau_n}\boldsymbol{\pi}_u^T\boldsymbol{\sigma}d\boldsymbol{B}_u-\tfrac{1}{2}\gamma^2b\int_t^{T\wedge\tau_n}\|\boldsymbol{\pi}_u\|^2du\Big)\Big]\times\\
&\quad\mathbb{E}\Big[\exp\Big(\tfrac{1}{2}\gamma^2b\int_t^{T\wedge\tau_n}\|\boldsymbol{\pi}_u\|^2du\Big)\Big]\\
\le&\exp\left(\gamma\sqrt{b}\left(|u|_0+|\xi|+|\boldsymbol{\lambda}|_0\,H\sqrt{dT}\left(\int_0^T\|\boldsymbol{\pi}_u\|^2du\right)^{1/2}\right)\right)\times\\
&\quad\mathbb{E}\Big[\exp\Big(\tfrac{1}{2}\gamma^2b\int_0^{T}\|\boldsymbol{\pi}_u\|^2du\Big)\Big]<\infty
\end{aligned}
$$

We can now let $n\to\infty$ in equation (3.1.4) to conclude that:

$$\mathbb{E}\Big[V(T,X^{\boldsymbol{\pi}}_T,\boldsymbol{B}_T)\Big]\le V(t,\xi,\boldsymbol{x}).$$

Notice that $V(T, X_T^{\boldsymbol{\pi}}, \boldsymbol{B}_T) = -\exp\{-\gamma\left(\xi + \int_t^T \boldsymbol{\pi}_s^T d\boldsymbol{S}_s^{\boldsymbol{\lambda}} + g(\boldsymbol{B}_T)\right)\}$, so that:

$$\mathbb{E}\Big[-\exp\{-\gamma\left(\xi + \int_t^T \boldsymbol{\pi}_s^T d\boldsymbol{S}_s^{\boldsymbol{\lambda}} + g(\boldsymbol{B}_T)\right)\}\Big] \leq V(t, \xi, \boldsymbol{x}) \tag{3.1.6}$$

As a result $V^* \leq V$, i.e.:

$$V^*(t, \xi, \boldsymbol{x}) := \max_{\boldsymbol{\pi} \in \mathcal{A}} \mathbb{E}\Big[-\exp\{-\gamma\left(\xi + \int_t^T \boldsymbol{\pi}_s^T d\boldsymbol{S}_s^{\boldsymbol{\lambda}} + g(\boldsymbol{B}_T)\right)\}\Big] \leq V(t, \xi, \boldsymbol{x})$$

On the other hand, since $V_{\xi\xi} < 0$ and because of the quadratic structure of the Hamiltonian of HJB (3.1.2), if we let $\boldsymbol{\pi}_s = \boldsymbol{\pi}^{(\boldsymbol{\lambda})}(s, \boldsymbol{B}_s) := \frac{1}{\gamma}\boldsymbol{\lambda}(s, \boldsymbol{B}_s) - D_d u(s, \boldsymbol{B}_s)$, all the inequalities in (3.1.5) and (3.1.6) become equalities. As a result:

$$V^*(t, \xi, \boldsymbol{x}) = V(t, \xi, \boldsymbol{x})$$

Therefore, we have proved that the solution $V(t, \xi, \boldsymbol{x})$ to HJB (3.1.2) is indeed the value function, and the optimal utility is achieved by the portfolio $\boldsymbol{\pi}^{(\boldsymbol{\lambda})}(s, \boldsymbol{B}_s)$. □

3.2 Market Equilibria with Utility-Maximizing Agents

3.2.1 The Equilibrium PDE System

Recall the market clearing condition of Definition 2.2.1 and the optimal portfolio formula of Theorem 3.1.1. If an equilibrium exists, we have that, for all agents $i = 1, \dots, I$, and $j = 1, \dots, d$:

$$\sum_{i=1}^{I} \pi^{(\boldsymbol{\lambda}, i, j)}(t, \boldsymbol{x}) = \sum_{i=1}^{I} \left(\frac{1}{\gamma_i}\lambda^{[j]}(t, \boldsymbol{x}) - u_{x_j}^{[i]}(t, \boldsymbol{x})\right) = 0$$

equivalently:

$$\boldsymbol{\lambda}(t, \boldsymbol{x}) = \overline{\gamma} \sum_{i=1}^{I} D_d u^{[i]}(t, \boldsymbol{x}). \tag{3.2.1}$$

where we recall that $\overline{\gamma} := \left(\sum_{i=1}^{I} \gamma_i^{-1} \right)^{-1}$. If we insert it into the HJB equation (3.1.3) for each agent, we have that $\boldsymbol{u} := (u^{[1]}, \dots, u^{[I]})$ solves the following PDE system (due to the optimality of $\boldsymbol{\pi}_t^{(\boldsymbol{\lambda},i)}$, for all $i = 1, \dots, I$):

$$\begin{cases} u_t^{[i]} + \frac{1}{2}\Delta u^{[i]} - \left(D_d u^{[i]}\right)^T \left(\overline{\gamma} \sum_{k=1}^{I} D_d u^{[k]} \right) \\ \qquad + \dfrac{1}{2\gamma_i} \left\| \overline{\gamma} \sum_{k=1}^{I} D_d u^{[k]} \right\|^2 - \dfrac{\gamma_i}{2} \left\| D_{n-d} u^{[i]} \right\|^2 = 0 \\ u^{[i]}(T, \boldsymbol{x}) = g^{[i]}(\boldsymbol{x}) \end{cases} \tag{3.2.2}$$

Equivalently, we can rewrite it into the following form:

$$\begin{cases} u_t^{[i]} + \frac{1}{2}\Delta u^{[i]} + \frac{1}{2\gamma_i} \left\| \overline{\gamma} \sum_{k=1}^{I} D_d u^{[k]} - \gamma_i D_d u^{[i]} \right\|^2 - \frac{\gamma_i}{2} \left\| D u^{[i]} \right\|^2 = 0 \\ u^{[i]}(T, \boldsymbol{x}) = g^{[i]}(\boldsymbol{x}) \end{cases}$$

On the other hand, suppose that the PDE system (3.2.2) has a solution. Then, obviously, (3.2.1) defines an equilibrium price dynamic and finding an equilibrium becomes equivalent to solving (3.2.2). However, before we attempt to solve the problem in its general setting, let us first look at a few special cases.

3.2.2 A Market Populated by a Single Agent

In this case, the market equilibrium simply means there is zero volume of trading, thus by Theorem 3.1.1, we have $\boldsymbol{\lambda}(t, \boldsymbol{x}) = \gamma D_d u(t, \boldsymbol{x})$. If we insert this equality into the HJB equation, we get:

$$\begin{cases} u_t + \frac{1}{2}\Delta u - \frac{\gamma}{2}\|Du\|^2 = 0 \\ u(T, \boldsymbol{x}) = g(\boldsymbol{x}) \end{cases} \tag{3.2.3}$$

Now we can solve this equation above explicitly. Indeed, let $U(t, \boldsymbol{x}) := \exp(-\gamma u(t, \boldsymbol{x}))$. Then we have:

$$\begin{aligned} U_t &= -\gamma u_t U \\ U_{x_j} &= -\gamma u_{x_j} U \\ U_{x_j x_j} &= \left(\gamma^2 u_{x_j}^2 - \gamma u_{x_j x_j}\right) U \\ \Delta U &= \left(\gamma^2 \|Du\|^2 - \gamma \Delta u\right) U \end{aligned}$$

If we multiply equation (3.2.3) by $-\gamma U$, we get:

$$\begin{cases} U_t + \frac{1}{2}\Delta U = 0 \\ U(T, \boldsymbol{x}) = \exp(-\gamma g(\boldsymbol{x})) \end{cases} \tag{3.2.4}$$

Note that the assumption that g is bounded allows us to use the explicit formula for U as the solution to the PDE above. Let $H(t, \boldsymbol{x}, \boldsymbol{y})$ be defined as follows:

$$H(t, \boldsymbol{x}, \boldsymbol{y}) := \left(\frac{1}{2\pi(T-t)}\right)^{n/2} \exp\left(-\frac{\|\boldsymbol{x} - \boldsymbol{y}\|^2}{2(T-t)}\right) \exp(-\gamma g(\boldsymbol{y})), \tag{3.2.5}$$

and note that:

$$U(t, \boldsymbol{x}) = \int_{\mathbb{R}^n} H(t, \boldsymbol{x}, \boldsymbol{y})\, d\boldsymbol{y}$$

Then the unique solution to the PDE is:

$$u(t, \boldsymbol{x}) = -\frac{1}{\gamma} \ln(U(t, \boldsymbol{x})) = -\frac{1}{\gamma} \ln\left(\int_{\mathbb{R}^n} H(t, \boldsymbol{x}, \boldsymbol{y})\, d\boldsymbol{y}\right),$$

and so,

$$\lambda_j^0(t,\boldsymbol{x}) = -\frac{U_{x_j}}{U} = \frac{\int_{\mathbb{R}^n} \frac{(x_j-y_j)}{T-t} H(t,\boldsymbol{x},\boldsymbol{y})\,d\boldsymbol{y}}{\int_{\mathbb{R}^n} H(t,\boldsymbol{x},\boldsymbol{y})\,d\boldsymbol{y}} \tag{3.2.6}$$

is the unique equilibrium market price of risk.

3.2.3 Hedgeable Terminal Random Endowments

By “hedgeable terminal random endowments”, we mean the following:

Assumption 3.2.1. For $i = 1, \ldots, I$, we assume that $g^{[i]}$ depends only on the first d variables.

Then the equilibrium problem is reduced to the case of a complete market, and one can easily find the “Representative Agent” and an explicit formula for the equilibrium price. Indeed, in this case, the solution to the HJB equation will also only depend on the first d elements of $\boldsymbol{x}$, and last $n-d$ Brownian Motions become irrelevant. Let $W(t,\boldsymbol{x}) := \sum_{i=1}^{I} u^{[i]}(t,\boldsymbol{x})$, and sum the equations in the PDE system (3.2.2) to get:

$$W_t + \tfrac{1}{2}\Delta W - \frac{\overline{\gamma}}{2}\|D_d W\|^2 = 0 \tag{3.2.7}$$

Therefore, we are looking at a representative agent with risk aversion $\overline{\gamma}$, and would like to have zero trading volume in the equilibrium. In the same way as above, we can define $H_{Rep}(t,\boldsymbol{x},\boldsymbol{y})$ and $U_{Rep}(t,\boldsymbol{x})$ as:

$$\begin{aligned} H_{Rep}(t,\boldsymbol{x},\boldsymbol{y}) &:= \left(\frac{1}{2\pi(T-t)}\right)^{n/2} \exp\left(-\frac{\|\boldsymbol{x}-\boldsymbol{y}\|^2}{2(T-t)}\right) \exp\left(-\overline{\gamma}\sum_{i=1}^{I} g^{[i]}(\boldsymbol{y})\right) \\ U_{Rep}(t,\boldsymbol{x}) &:= \int_{\mathbb{R}^n} H_{Rep}(t,\boldsymbol{x},\boldsymbol{y})\,d\boldsymbol{y}, \end{aligned} \tag{3.2.8}$$

then the unique solution W to the PDE (3.2.7) and the unique equilibrium market price of risk $\boldsymbol{\lambda}^{Rep}$ are given by:

$$\begin{aligned} W(t, \boldsymbol{x}) &= -\frac{1}{\bar{\gamma}} \ln\left(U_{Rep}(t, \boldsymbol{x})\right) = -\frac{1}{\bar{\gamma}} \ln\left(\int_{\mathbb{R}^n} H_{Rep}(t, \boldsymbol{x}, \boldsymbol{y})\, d\boldsymbol{y}\right) \\ \lambda_j^{Rep}(t, \boldsymbol{x}) &= -\frac{(U_{Rep})_{x_j}}{U_{Rep}} = \frac{\int_{\mathbb{R}^n} \frac{(x_j - y_j)}{T-t} H_{Rep}(t, \boldsymbol{x}, \boldsymbol{y})\, d\boldsymbol{y}}{\int_{\mathbb{R}^n} H_{Rep}(t, \boldsymbol{x}, \boldsymbol{y})\, d\boldsymbol{y}} \end{aligned} \tag{3.2.9}$$

3.2.4 "Totally Nonhedgeable" Terminal Random Endowments

By "totally nonhedgeable" terminal random endowments, we mean the following:

Assumption 3.2.2. For $i = 1, \ldots, I$, we assume that $g^{[i]}$ depends on only the last $n - d$ variables, i.e. $g^{[i]}(\boldsymbol{x}) = g^{[i]}(\boldsymbol{x}_{n-d})$, where $\boldsymbol{x}_{n-d} \in \mathbb{R}^{n-d}$.

Now observe that the equilibrium system (3.2.2) takes the following form:

$$\begin{cases} u_t^{[i]} + \frac{1}{2}\Delta u^{[i]} - \frac{\gamma_i}{2}\left\|D_{n-d}u^{[i]}\right\|^2 = 0 \\ u^{[i]}(T, \boldsymbol{x}) = g^{[i]}\left(\boldsymbol{x}_{n-d}\right) \end{cases}$$

which has unique explicit solutions, as shown above. However, the solutions all depend only on the last $n - d$ elements of the variable $\boldsymbol{x}$. As a result, by (3.2.1), one can easily see that the unique equilibrium market price of risk $\boldsymbol{\lambda}$ is $\boldsymbol{\lambda}(t, \boldsymbol{x}) = 0$, $\forall\, (t, \boldsymbol{x}) \in Q$.

3.2.5 A Linear Combination of Hedgeable and "Totally Nonhedgeable" Terminal Random Endowments

Now we assume that the terminal endowments can be expressed as linear combinations of the previous two special cases, i.e.:

Assumption 3.2.3. For $i = 1, \dots, I$, we assume that $g^{[i]}$ is separable, in the following sense:

$$g^{[i]}(\boldsymbol{x}) = g_1^{[i]}(x_1, \dots, x_d) + g_2^{[i]}(x_{d+1}, \dots, x_n)$$

Then it is not hard to see that the solution to the equilibrium PDE system (3.2.2), is also separable, i.e. $\boldsymbol{u}(t, \boldsymbol{x}) = \boldsymbol{u}_1(t, \boldsymbol{x}_d) + \boldsymbol{u}_2(t, \boldsymbol{x}_{n-d})$, where $\boldsymbol{x}_d \in \mathbb{R}^d$ and $\boldsymbol{x}_{n-d} \in \mathbb{R}^{n-d}$, and the system (3.2.2) decouples into the two special cases above. Therefore it becomes clear that, under this assumption, a result similar to (3.2.9) will hold.

3.2.6 Existence and Uniqueness of an Equilibrium in the General Case

Let $\mathcal{B}(\cdot, \cdot, \cdot, \cdot, \cdot, \cdot)$ be as in Definition 6.2.1.

Theorem 3.2.4. *There exists a positive generic constant C independent of C_g such that for $T \leq T_0 = C/C_g^2$, under the Assumption 2.3.1, the equilibrium PDE system* (3.2.2) *has a unique solution $u \in C^{2,\alpha}(Q)$. Moreover, there exist generic constants C_1, D_1, C_2, D_2, E, F and A_1, A_2, A_3, such that*

$$\boldsymbol{u} \in \mathcal{B}(C_1, D_1, C_2, D_2, E, F)$$

and, for all $(t, \boldsymbol{x}) \in Q$,

$$\begin{aligned} \|\boldsymbol{u}(t, \boldsymbol{x})\| &\leq A_1 \phi_{D_1}(\boldsymbol{x}) \\ \|D\boldsymbol{u}(t, \boldsymbol{x})\| &\leq A_2 \phi_{D_2}(\boldsymbol{x}) \\ |\boldsymbol{u}|_{2,\alpha} &\leq A_3 \end{aligned}$$

An equilibrium market price of risk, denoted by $\boldsymbol{\lambda}$*, is given by* (3.2.1)*. It is unique in* $(C^{1,\alpha}(Q))^d$ *and there exist generic constants* A_4 *and* A_5*, such that:*

$$\begin{aligned}\|\boldsymbol{\lambda}(t,\boldsymbol{x})\| \leq & A_4\phi_{D_2}(\boldsymbol{x}), \quad \forall\,(t,\boldsymbol{x})\in Q\\ |\boldsymbol{\lambda}|_{1,\alpha} \leq & A_5\end{aligned}$$

Finally, if the function h *of Assumption 2.3.1 further satisfies the condition* (A.1.2) *in Lemma A.1.1, there exist generic constants* $\widetilde{A}_1$*,* $\widetilde{A}_2$ *and* $\widetilde{A}_3$ *such that:*

$$\begin{aligned}\|\boldsymbol{u}(t,\boldsymbol{x})\| \leq & \widetilde{A}_1 h(\boldsymbol{x})\\ \|D\boldsymbol{u}(t,\boldsymbol{x})\| \leq & \widetilde{A}_2 h(\boldsymbol{x})\\ \|\boldsymbol{\lambda}(t,\boldsymbol{x})\| \leq & \widetilde{A}_3 h(\boldsymbol{x})\end{aligned}$$

Proof. We shall apply the results of Section 6.3. To show the existence of a solution, we need to verify that PDE system (3.2.2) satisfies the Assumption 6.1.5. Let $Z := (Z_{ij})_{I\times n} \in \mathbb{R}^{I\times n}$, denote $Z := (Z^{[1]},\ldots,Z^{[I]})$, where $Z^{[i]} = (Z_{ij})_{j=1}^n \in \mathbb{R}^n$, for $i = 1,\ldots,I$, are the row vectors of Z, and write:

$$\begin{aligned}Z^{[i]} :=& (Z_{i1},\ldots,Z_{id},Z_{id+1},\ldots,Z_{in})^T\\ :=& (Z^{[i]}_d, Z^{[i]}_{n-d})^T \in \mathbb{R}^l\times\mathbb{R}^{d-l}\times\mathbb{R}^{n-d}\end{aligned} \tag{3.2.10}$$

Also, we introduce the function $f^{[i]}$ by

$$f^{[i]}(t,\boldsymbol{x},Z) := -\left(Z^{[i]}_d\right)^T\left(\overline{\gamma}\sum_{k=1}^I Z^{[k]}_d\right) + \frac{1}{2\gamma_i}\left(\left\|\overline{\gamma}\sum_{k=1}^I Z^{[k]}_d\right\|^2\right) - \frac{\gamma_i}{2}\left\|Z^{[i]}_{n-d}\right\|^2$$

Then it is easy to see that:

$$\begin{aligned}\left|f^{[i]}(t,\boldsymbol{x},Z)\right| \leq & \overline{\gamma}\sum_{k=1}^I\left\|Z^{[k]}_d\right\|\left\|Z^{[i]}_d\right\| + \frac{\overline{\gamma}^2 I}{2\gamma_i}\left(\sum_{k=1}^I\left\|Z^{[k]}_d\right\|^2\right) + \frac{\gamma_i}{2}\left\|Z^{[i]}_{n-d}\right\|^2\\ \leq & P^{[i]}(\boldsymbol{\gamma},I,n)\|Z\|^2\end{aligned}$$

where $P^{[i]}$ is a generic constant that depends only on $\boldsymbol{\gamma}$, I, n. Furthermore, using the following inequality,

$$\begin{aligned}|uv|_\alpha &= |uv|_0 + [uv]_\alpha \\ &\le |uv|_0 + |u|_0\,[v]_\alpha + [u]_\alpha\,|v|_0\,,\end{aligned}$$

valid for $u, v \in C^{0,\alpha}(Q)$, it is not hard to verify the condition (3) in Assumption 6.1.5. Furthermore, for $Z, \widetilde{Z} \in \mathbb{R}^{I\times n}$, we set $Y := Z - \widetilde{Z}$, and use the notation in (3.2.10) so that

$$\begin{aligned}
&f^{[i]}(t,\boldsymbol{x},Z) - f^{[i]}(t,\boldsymbol{x},\widetilde{Z}) \\
=& -\overline{\gamma}\left(\left(Z_d^{[i]}\right)^T\sum_{k=1}^{I} Z_d^{[k]} - \left(\widetilde{Z}_d^{[i]}\right)^T\sum_{k=1}^{I}\widetilde{Z}_d^{[k]}\right) + \frac{\overline{\gamma}^2}{2\gamma_i}\left(\left\|\sum_{k=1}^{I} Z_d^{[k]}\right\|^2 - \left\|\sum_{k=1}^{I}\widetilde{Z}_d^{[k]}\right\|^2\right) \\
&\qquad - \frac{\gamma_i}{2}\left(\left\|Z_{n-d}^{[i]}\right\|^2 - \left\|\widetilde{Z}_{n-d}^{[i]}\right\|^2\right) \\
=& -\overline{\gamma}\left(\left(Y_d^{[i]}\right)^T\sum_{k=1}^{I} Z_d^{[k]} + \left(\widetilde{Z}_d^{[i]}\right)^T\sum_{k=1}^{I} Y_d^{[k]}\right) + \frac{\overline{\gamma}^2}{2\gamma_i}\left(\sum_{k,m=1}^{I}\left(Z_d^{[m]} + \widetilde{Z}_d^{[m]}\right)^T Y_d^{[k]}\right) \\
&\qquad - \frac{\gamma_i}{2}\left(\left(Z_{n-d}^{[i]} + \widetilde{Z}_{n-d}^{[i]}\right)^T Y_{n-d}^{[i]}\right) \\
:=& \sum_{k=1}^{I}\boldsymbol{h}_{ik}^T Y_d^{[k]} + \boldsymbol{r}_i^T Y_{n-d}^{[i]}
\end{aligned}$$

where $\boldsymbol{h}_{ik}$ and $\boldsymbol{r}_i$ are:

$$\begin{cases}\boldsymbol{h}_{ik} := -\overline{\gamma}\widetilde{Z}_d^{[i]} + \frac{\overline{\gamma}^2}{2\gamma_i}\sum_{m=1}^{I}\left(Z_d^{[m]} + \widetilde{Z}_d^{[m]}\right) - \delta_{ik}\left(\overline{\gamma}\sum_{m=1}^{I} Z_d^{[m]}\right)\\ \boldsymbol{r}_i := -\frac{\gamma_i}{2}\left(Z_{n-d}^{[i]} + \widetilde{Z}_{n-d}^{[i]}\right)\end{cases}$$

and δ_{ik} is the Kronecker delta function. Notice that the $\boldsymbol{h}_{ik}$ and $\boldsymbol{r}_i$ are polynomials, thus locally bounded functions. Consequently, Assumption 6.1.5 (4) is satisfied.

By Lemma 6.2.3, Theorem 6.3.1 and Corollary 6.3.2, there exist generic constants C_1, D_1, C_2, D_2, E, F and A_1, A_2, A_3, such that the equilibrium PDE system (3.2.2) has a unique solution, denoted by $\boldsymbol{u}$, which belongs to $\mathcal{B}(C_1, D_1, C_2, D_2, E, F)$, and it satisfies the first set of desired inequalities. If we further assume that the function h in Assumption 2.3.1, satisfies the condition (A.1.2) in Lemma A.1.1, we can apply Lemma A.1.2 to ensure the existence of generic constants $\widetilde{A}_1$ and $\widetilde{A}_2$ such that the last set of desired inequalities holds.

Finally, the market price of risk $\boldsymbol{\lambda}$ is given by the equation (3.2.1), and its uniqueness follows from the uniqueness of the solution to the equilibrium PDE system (3.2.2). It is not hard to see that, under condition (A.1.2), we can use the equation (3.2.1) and Lemma A.1.2 to show that there exist generic constants A_4, A_5 and $\widetilde{A}_3$, such that the desired estimates for $\boldsymbol{\lambda}$ hold. $\square$

Chapter 4

Market Equilibria with Risk-Measure-Minimizing Agents

4.1 Individual Optimization Problem

In this section, we discuss the optimization problem for an individual agent with a convex risk measure. To simplify the notation, we drop the upper index i throughout.

4.1.1 Value Function and the HJB

Recall that the risk measure is given in terms of its penalty function f, which satisfies the Assumption 2.3.3:

$$\rho(X) = \sup_{\boldsymbol{\nu} \in \mathcal{N}} \mathbb{E}\left[Z_T^{\boldsymbol{\nu}} \left(-X - \int_0^T f(\boldsymbol{\nu}_s)\, ds \right) \right]$$

As in the exponential case, one can characterize the optimal portfolio by a solution to a quasi-linear PDE, derived from the Hamilton-Jacobi-Bellman equation (HJB). We start by defining the value function $v^* : [0, T] \times \mathbb{R}^{n+2} \to \mathbb{R}$, as follows:

$$v^*(t, \xi, z, \boldsymbol{x}) := \inf_{\boldsymbol{\pi} \in \widetilde{\mathcal{A}}} \sup_{\boldsymbol{\nu} \in \mathcal{N}} \mathbb{E}^{t,\xi,z,\boldsymbol{x}} \Big[Z_T^{\boldsymbol{\nu}} (-g(\boldsymbol{B}_T) - Y_T) \Big] \tag{4.1.1}$$

under the following, stochastic, dynamics:

$$\begin{cases} dY_s = \boldsymbol{\pi}_s^T \boldsymbol{\lambda}_s ds + \boldsymbol{\pi}_s^T \boldsymbol{\sigma}_s d\boldsymbol{B}_s + f(\boldsymbol{\nu}_s)ds; \quad Y_t = \xi \\ dZ_s^{\boldsymbol{\nu}} = -Z_s^{\boldsymbol{\nu}} \boldsymbol{\nu}_s^T d\boldsymbol{B}_s; \quad Z_t^{\boldsymbol{\nu}} = z \\ d\boldsymbol{B}_s = d\boldsymbol{B}_s; \quad \boldsymbol{B}_t = \boldsymbol{x} \end{cases} \tag{4.1.2}$$

We can formally deduce the Hamilton-Jacobi-Bellman equation by applying Itô's formula to $v(t, \xi, z, \boldsymbol{x})$:

$$dv(s, Y_s, Z_s, \boldsymbol{B}_s) = (v_\xi \boldsymbol{\pi}_s^T \boldsymbol{\sigma} - v_z Z_s \boldsymbol{\nu}_s^T + Dv^T)d\boldsymbol{B}_s + (v_s + \tfrac{1}{2}\Delta v + v_\xi(\boldsymbol{\pi}_s^T \boldsymbol{\lambda}_s + f) +$$
$$D_d v_\xi^T \boldsymbol{\pi}_s - v_{\xi z} Z_s \boldsymbol{\pi}_s^T \boldsymbol{\sigma} \boldsymbol{\nu}_s - D v_z^T \boldsymbol{\nu}_s Z_s + \tfrac{1}{2} v_{\xi\xi} \|\boldsymbol{\pi}_s\|^2 + \tfrac{1}{2} v_{zz} Z_s^2 \|\boldsymbol{\nu}_s\|^2)ds$$

Thus the HJB take the form:

$$\begin{cases} v_t + \frac{1}{2}\Delta v + \inf_{\boldsymbol{\pi}\in\mathbb{R}^d} \sup_{\boldsymbol{\nu}\in\mathbb{R}^n} (v_\xi(\boldsymbol{\pi}^T \boldsymbol{\lambda} + f) + D_d v_\xi^T \boldsymbol{\pi} - v_{\xi z} z(\boldsymbol{\pi}^T \boldsymbol{\sigma} \boldsymbol{\nu}) - D v_z^T \boldsymbol{\nu} z \\ \qquad + \frac{1}{2} v_{\xi\xi} \|\boldsymbol{\pi}\|^2 + \frac{1}{2} v_{zz} z^2 \|\boldsymbol{\nu}\|^2) = 0 \\ v(T, \xi, z, \boldsymbol{x}) = -z\xi - zg(x) \end{cases} \tag{4.1.3}$$

Similarly as in the exponential-utility case, we can guess that the solution $v(t, \xi, z, \boldsymbol{x})$ takes the following form:

$$v(t, \xi, z, \boldsymbol{x}) := -z\xi - zu(t, \boldsymbol{x}).$$

After we insert it into the Hamilton-Jacobi-Bellman equation (4.1.3), we get the following quasi-linear PDE for the function $u(t, \boldsymbol{x})$:

$$\begin{cases} u_t + \frac{1}{2}\Delta u + \sup_{\boldsymbol{\pi}\in\mathbb{R}^d} \inf_{\boldsymbol{\nu}\in\mathbb{R}^n} (\boldsymbol{\pi}^T \boldsymbol{\lambda} + f(\boldsymbol{\nu}) - \boldsymbol{\pi}^T \boldsymbol{\sigma} \boldsymbol{\nu} - Du^T \boldsymbol{\nu}) = 0 \\ u(T, \boldsymbol{x}) = g(\boldsymbol{x}) \end{cases} \tag{4.1.4}$$

Let $\widetilde{f}(\boldsymbol{x})$ denote the convex conjugate of function $f(\boldsymbol{x})$, so that

$$\widetilde{f}(\boldsymbol{y}) := \sup_{\boldsymbol{\nu}\in\mathbb{R}^n} (\boldsymbol{y}^T \boldsymbol{\nu} - f(\boldsymbol{\nu})) = \widetilde{f}_1(\boldsymbol{y}_1) + \widetilde{f}_2(\boldsymbol{y}_2)$$

where $\boldsymbol{y}_1 \in \mathbb{R}^d$ and $\boldsymbol{y}_2 \in \mathbb{R}^{n-d}$. Then observe that:

$$\begin{aligned}
&\sup_{\boldsymbol{\pi}\in\mathbb{R}^d} \inf_{\boldsymbol{\nu}\in\mathbb{R}^n} (\boldsymbol{\pi}^T\boldsymbol{\lambda} + f(\boldsymbol{\nu}) - \boldsymbol{\pi}^T\boldsymbol{\sigma}\boldsymbol{\nu} - Du^T\boldsymbol{\nu}) \\
=&\sup_{\boldsymbol{\pi}\in\mathbb{R}^d} (\boldsymbol{\pi}^T\boldsymbol{\lambda} - \widetilde{f}(\boldsymbol{\sigma}^T\boldsymbol{\pi} + Du)) \\
=&\sup_{\boldsymbol{\pi}\in\mathbb{R}^d} (\boldsymbol{\pi}^T\boldsymbol{\lambda} - \widetilde{f}_1(\boldsymbol{\pi} + D_d u) - \widetilde{f}_2(D_{n-d}u)) \\
=&-\boldsymbol{\lambda}^T D_d u + f_1(\boldsymbol{\lambda}) - \widetilde{f}_2(D_{n-d}u)
\end{aligned}$$

Therefore, the HJB equation (4.1.3) turns into:

$$\begin{cases} u_t + \frac{1}{2}\Delta u - \boldsymbol{\lambda}^T D_d u + f_1(\boldsymbol{\lambda}) - \widetilde{f}_2(D_{n-d}u) = 0 \\ u(T,\boldsymbol{x}) = g(\boldsymbol{x}) \end{cases} \tag{4.1.5}$$

Remark 4.1.1. Comparing equation (4.1.5) above with the HJB equation from an exponential-utility-maximizing agent, i.e. equation (3.1.3), one can clearly see that (3.1.3) is nothing but a special case of the HJB (4.1.5) above. That is precisely what one would expect due to the formal equivalence between entropic risk measures and the exponential-utility.

4.1.2 Solution to HJB and Verification

Theorem 4.1.2. *For $g \in C^{2,\alpha}(\mathbb{R}^n)$ and $\boldsymbol{\lambda} \in C^{0,\alpha}(Q)$, under the Assumption 2.3.3, the HJB equation* (4.1.5) *has a unique solution $u \in C^{2,\alpha}(Q)$. Furthermore, the portfolio $\boldsymbol{\pi}^{(\boldsymbol{\lambda})}$, given by:*

$$\boldsymbol{\pi}_t^{(\boldsymbol{\lambda})} := Df_1(\boldsymbol{\lambda}(t,\boldsymbol{B}_t)) - D_d u(t,\boldsymbol{B}_t),$$

is admissible and optimal.

Proof. Once again, we apply Theorem 8.1 of chapter V from [20], to (4.1.5). It's not hard to check that the conditions required by the theorem are satisfied, thanks to Assumption 2.3.3. Now let us verify that the portfolio given above is indeed admissible and optimal. The admissibility of $\boldsymbol{\pi}^{(\boldsymbol{\lambda})}$ is obvious, since both $\boldsymbol{\lambda}(t,\boldsymbol{x})$ and $D_d u(t,\boldsymbol{x})$ are uniformly bounded and the function Df_1 is continuous. To show optimality, set:

$$v(t,\xi,z,\boldsymbol{x}) := -z\xi - zu(t,\boldsymbol{x}) \tag{4.1.6}$$

where $u \in C^{2,\alpha}(Q)$ is the unique solution to HJB equation (4.1.5), one can then easily verify that v is the solution to the formal HJB equation (4.1.3).

For any admissible $\boldsymbol{\pi}$, define $\boldsymbol{\nu}^{\boldsymbol{\pi}}$ as:

$$\boldsymbol{\nu}^{\boldsymbol{\pi}}_t := \left(D\widetilde{f}_1(\boldsymbol{\pi}_t + D_d u(t,\boldsymbol{B}_t)), D\widetilde{f}_2(D_{n-d}u(t,\boldsymbol{B}_t))\right)^T$$

where u is the solution to the HJB equation (4.1.4). Recall that, in Assumption 2.3.3, we assumed that: $\|Df_1(\boldsymbol{x})\| \geq (1/L)\|\boldsymbol{x}\|$, thus thanks to Proposition A.3.1, we have:

$$\left\|D\widetilde{f}_1(\boldsymbol{x})\right\| \leq L\|\boldsymbol{x}\|$$

Observe further, that $\|Du\|$ is bounded, thus it is easy to see that $\boldsymbol{\nu}^{\boldsymbol{\pi}}_t \in \mathcal{N}$, due to the requirement of $\boldsymbol{\pi}_t \in \widetilde{\mathcal{A}}$.

We insert $\boldsymbol{\pi}$ and $\boldsymbol{\nu}^{\boldsymbol{\pi}}$ in (4.1.2), to construct the corresponding processes $Y^{\boldsymbol{\pi}}$ and $Z^{\boldsymbol{\pi}}$, and insert them into the function $v(t,\xi,z,\boldsymbol{x})$ defined in (4.1.6). By Itô's formula:

$$dv(s,Y^{\boldsymbol{\pi}}_s,Z^{\boldsymbol{\pi}}_s,\boldsymbol{B}_s) = -Z^{\boldsymbol{\pi}}_s\left(\boldsymbol{\pi}^T_s\boldsymbol{\sigma} - (Y^{\boldsymbol{\pi}}_s + u)(\boldsymbol{\nu}^{\boldsymbol{\pi}}_s)^T + Du\right)d\boldsymbol{B}_s$$
$$-Z^{\boldsymbol{\pi}}_s(u_t + \tfrac{1}{2}\Delta u + \boldsymbol{\pi}^T_s\boldsymbol{\lambda} + f(\boldsymbol{\nu}^{\boldsymbol{\pi}}_s) - \boldsymbol{\pi}^T_s\boldsymbol{\sigma}\boldsymbol{\nu}^{\boldsymbol{\pi}}_s - Du^T\boldsymbol{\nu}^{\boldsymbol{\pi}}_s)ds$$

If the stopping times $\{T_n\}_{n\in\mathbb{N}}$ are defined by:

$$T_n := \inf_{s\geq t}\{s\,:\,\|Y_s^{\boldsymbol{\pi}}\| \geq n \text{ or } \|Z_s^{\boldsymbol{\pi}}\| \geq n\},$$

by the choice of $\boldsymbol{\nu}^{\boldsymbol{\pi}}$, we have

$$\begin{aligned}
&\mathbb{E}v(T\wedge T_n, Y^{\boldsymbol{\pi}}_{T\wedge T_n}, Z^{\boldsymbol{\pi}}_{T\wedge T_n}, \boldsymbol{B}_{T\wedge T_n}) - v(t,\xi,z,\boldsymbol{x})\\
=&\int_t^{T\wedge T_n} -Z_s^{\boldsymbol{\pi}}(u_t + \tfrac{1}{2}\Delta u + \boldsymbol{\pi}_s^T\boldsymbol{\lambda} + f(\boldsymbol{\nu}_s^{\boldsymbol{\pi}}) - \boldsymbol{\pi}_s^T\boldsymbol{\sigma}\boldsymbol{\nu}_s^{\boldsymbol{\pi}} - Du^T\boldsymbol{\nu}_s^{\boldsymbol{\pi}})ds\\
=&\int_t^{T\wedge T_n} \Big(-Z_s^{\boldsymbol{\pi}}(u_t + \tfrac{1}{2}\Delta u + \inf_{\boldsymbol{\nu}\in\mathbb{R}^n}\big(\boldsymbol{\pi}_s^T\boldsymbol{\lambda} + f(\boldsymbol{\nu}) - \boldsymbol{\pi}_s^T\boldsymbol{\sigma}\boldsymbol{\nu} - Du^T\boldsymbol{\nu})ds\big)\Big)\\
\geq&\int_t^{T\wedge T_n} \Big(-Z_s^{\boldsymbol{\pi}}(u_t + \tfrac{1}{2}\Delta u + \sup_{\boldsymbol{\pi}\in\mathbb{R}^d}\inf_{\boldsymbol{\nu}\in\mathbb{R}^n}\big(\boldsymbol{\pi}^T\boldsymbol{\lambda} + f(\boldsymbol{\nu}) - \boldsymbol{\pi}^T\boldsymbol{\sigma}\boldsymbol{\nu} - Du^T\boldsymbol{\nu})ds\big)\Big)\\
=&0
\end{aligned}$$

We claim that the family

$$v(T\wedge T_n, Y^{\boldsymbol{\pi}}_{T\wedge T_n}, Z^{\boldsymbol{\pi}}_{T\wedge T_n}, \boldsymbol{B}_{T\wedge T_n}) = -Z^{\boldsymbol{\pi}}_{T\wedge T_n}Y^{\boldsymbol{\pi}}_{T\wedge T_n} - Z^{\boldsymbol{\pi}}_{T\wedge T_n}u(T\wedge T_n, \boldsymbol{B}_{T\wedge T_n})$$

is uniformly integrable, with respect to index $n \in \mathbb{N}$. Since the function u is bounded and $Z^{\boldsymbol{\pi}}$ is square integrable, $(Z^{\boldsymbol{\pi}}_{T\wedge T_n}u(T\wedge T_N, \boldsymbol{B}_{T\wedge T_n}))_{n\in\mathbb{N}}$ is uniformly integrable. Now observe that:

$$Y_s^{\boldsymbol{\pi}} = \xi + \int_t^s \boldsymbol{\pi}_s^T\boldsymbol{\lambda}_s ds + \int_t^s \boldsymbol{\pi}_s^T\boldsymbol{\sigma}d\boldsymbol{B}_s + \int_t^s f(\boldsymbol{\nu}_s)ds$$

$$
\begin{aligned}
&\mathbb{E}\Big[(Z^{\boldsymbol{\pi}}_{T\wedge T_n})^2\left(\int_t^{T\wedge T_n}\boldsymbol{\pi}_u^T\boldsymbol{\lambda}_u du\right)^2\Big]+\mathbb{E}\Big[(Z^{\boldsymbol{\pi}}_{T\wedge T_n})^2\left(\int_t^{T\wedge T_n}f(\boldsymbol{\nu}_u)du\right)^2\Big]\\
\le&\left(\mathbb{E}\big[(Z^{\boldsymbol{\pi}}_{T\wedge T_n})^4\big]\right)^{\frac{1}{2}}\left(\mathbb{E}\Big[\int_t^{T\wedge T_n}\boldsymbol{\pi}_u^T\boldsymbol{\lambda}_u du\Big]^4\right)^{\frac{1}{2}}+L_2^2\mathbb{E}\Big[(Z^{\boldsymbol{\pi}}_{T\wedge T_n})^2\left(\int_t^{T\wedge T_n}\|\boldsymbol{\nu}_u\|^2du\right)^2\Big]\\
\le& T\,|\boldsymbol{\lambda}|_0^2\left(\mathbb{E}\big[(Z^{\boldsymbol{\pi}}_T)^4\big]\right)^{\frac{1}{2}}\left(\mathbb{E}\Big[\int_0^T\|\boldsymbol{\pi}_u\|^2du\Big]^2\right)^{\frac{1}{2}}+L_2^2\left(\mathbb{E}\big[(Z^{\boldsymbol{\pi}}_T)^4\big]\right)^{\frac{1}{2}}\left(\mathbb{E}\Big[\int_0^T\|\boldsymbol{\nu}_u\|^2du\Big]^4\right)^{\frac{1}{2}}\\
\le&\left(\mathbb{E}\big[(Z^{\boldsymbol{\pi}}_T)^4\big]\right)^{\frac{1}{2}}\left(NT\,|\boldsymbol{\lambda}|_0^2\,\mathbb{E}\Big[\exp\left(\int_0^T\|\boldsymbol{\pi}_t\|^2dt\right)\Big]+NL_2^2\mathbb{E}\Big[\exp\left(\int_0^T\|\boldsymbol{\nu}_t\|^2dt\right)\Big]\right)\\
<&\infty
\end{aligned}
$$

and for $1 < b < 2$:

$$
\begin{aligned}
&\mathbb{E}\left|Z^{\boldsymbol{\pi}}_{T\wedge T_n}\int_t^{T\wedge T_n}\boldsymbol{\pi}_u^T\boldsymbol{\sigma}d\boldsymbol{B}_u\right|^b\\
\le&\left(\mathbb{E}\left|Z^{\boldsymbol{\pi}}_{T\wedge T_n}\right|^{\frac{2b}{2-b}}\right)^{\frac{2-b}{2}}\left(\mathbb{E}\left(\int_t^{T\wedge T_n}\boldsymbol{\pi}_u^T\boldsymbol{\sigma}d\boldsymbol{B}_u\right)^2\right)^{b/2}\\
\le&\left(\mathbb{E}\left|Z^{\boldsymbol{\pi}}_T\right|^{\frac{2b}{2-b}}\right)^{\frac{2-b}{2}}\left(\mathbb{E}\int_0^T\|\boldsymbol{\pi}_u\|^2du\right)^{b/2}\\
\le& N\left(\mathbb{E}\left|Z^{\boldsymbol{\pi}}_T\right|^{\frac{2b}{2-b}}\right)^{\frac{2-b}{2}}\left(\mathbb{E}\Big[\exp\left(\int_0^T\|\boldsymbol{\pi}_t\|^2dt\right)\Big]\right)^{b/2}\\
<&\infty
\end{aligned}
$$

Therefore, we can let $n \to \infty$, and get the following:

$$
\mathbb{E}v(T, Y^{\boldsymbol{\pi}}_T, Z^{\boldsymbol{\pi}}_T, \boldsymbol{B}_T) \ge v(t, \xi, z, \boldsymbol{x})
$$

As a result, $\forall\, \boldsymbol{\pi} \in \widetilde{\mathcal{A}}$:

$$
\sup_{\boldsymbol{\nu}\in\mathcal{N}}\mathbb{E}v(T, Y^{\boldsymbol{\pi},\boldsymbol{\nu}}_T, Z^{\boldsymbol{\nu}}_T, \boldsymbol{B}_T) \ge v(t, \xi, z, \boldsymbol{x})
$$

Therefore:

$$v^*(t,\xi,z,\boldsymbol{x}) = \inf_{\boldsymbol{\pi}\in\widetilde{\mathcal{A}}} \sup_{\boldsymbol{\nu}\in\mathcal{N}} \mathbb{E} v(T, Y_T^{\boldsymbol{\pi}}, Z_T^{\boldsymbol{\pi}}, \boldsymbol{B}_T) \geq v(t,\xi,z,\boldsymbol{x})$$

On the other hand, if we insert $\boldsymbol{\pi}^{(\boldsymbol{\lambda})} := Df_1(\boldsymbol{\lambda}(t,\boldsymbol{B}_t)) - D_d u(t,\boldsymbol{B}_t)$ and any $\boldsymbol{\nu} \in \mathcal{N}$ into SDEs (4.1.2), we get process Y^* and Z^*. Then we insert them into function $v(t,\xi,z,\boldsymbol{x})$ defined in (4.1.6) so that:

$$dv(s,Y_s^*,Z_s^*,\boldsymbol{B}_s) = -Z_s^*\left((\boldsymbol{\pi}_s^{(\boldsymbol{\lambda})})^T\boldsymbol{\sigma} - (Y_s^*+u)(\boldsymbol{\nu}_s)^T + Du\right)d\boldsymbol{B}_s$$

$$-Z_s^*(u_t + \tfrac{1}{2}\Delta u + (\boldsymbol{\pi}_s^{(\boldsymbol{\lambda})})^T\boldsymbol{\lambda} + f(\boldsymbol{\nu}_s) - (\boldsymbol{\pi}_s^{(\boldsymbol{\lambda})})^T\boldsymbol{\sigma}\boldsymbol{\nu}_s - Du^T\boldsymbol{\nu}_s)ds$$

As before, we define the stopping times $T_n := \inf_{s\geq t}\{s : \|Y_s^*\| \geq n \text{ or } \|Z_s^*\| \geq n\}$. Thus by the choice of $\boldsymbol{\pi}^{(\boldsymbol{\lambda})}$, we get:

$$\begin{aligned}
&\mathbb{E}v(T\wedge T_n, Y^*_{T\wedge T_n}, Z^*_{T\wedge T_n}, \boldsymbol{B}_{T\wedge T_n}) - v(t,\xi,z,\boldsymbol{x})\\
=&\int_t^{T\wedge T_n} -Z_s^*(u_t + \tfrac{1}{2}\Delta u + (\boldsymbol{\pi}_s^{(\boldsymbol{\lambda})})^T\boldsymbol{\lambda} + f(\boldsymbol{\nu}_s) - (\boldsymbol{\pi}_s^{(\boldsymbol{\lambda})})^T\boldsymbol{\sigma}\boldsymbol{\nu}_s - Du^T\boldsymbol{\nu}_s)ds\\
\leq&\int_t^{T\wedge T_n} \sup_{\boldsymbol{\nu}\in\mathbb{R}^n}\left(-Z_s^{\boldsymbol{\pi}}(u_t + \tfrac{1}{2}\Delta u + (\boldsymbol{\pi}_s^{(\boldsymbol{\lambda})})^T\boldsymbol{\lambda} + f(\boldsymbol{\nu}) - (\boldsymbol{\pi}_s^{(\boldsymbol{\lambda})})^T\boldsymbol{\sigma}\boldsymbol{\nu} - Du^T\boldsymbol{\nu})ds\right)\\
=&\int_t^{T\wedge T_n} -Z_s^*(u_t + \tfrac{1}{2}\Delta u + (\boldsymbol{\pi}^{(\boldsymbol{\lambda})})^T\boldsymbol{\lambda} - \widetilde{f}_1(\boldsymbol{\pi}^{(\boldsymbol{\lambda})} + D_d u) - \widetilde{f}_2(D_{n-d}u))ds\\
=&\int_t^{T\wedge T_n} -Z_s^*(u_t + \tfrac{1}{2}\Delta u + -\boldsymbol{\lambda}^T D_d u + f_1(\boldsymbol{\lambda}) - \widetilde{f}_2(D_{n-d}u))ds\\
=&0
\end{aligned}$$

As before, it is not hard to show $v(T\wedge T_n, Y^*_{T\wedge T_n}, Z^*_{T\wedge T_n}, \boldsymbol{B}_{T\wedge T_n})$ is uniformly integrable, and then pass the limit, $n\to\infty$, to get:

$$\mathbb{E}v(T, Y_T^*, Z_T^*, \boldsymbol{B}_T) \leq v(t,\xi,z,\boldsymbol{x})$$

Hence, we have the following:

$$\sup_{\boldsymbol{\nu}\in\mathcal{N}} \mathbb{E}v(T, Y_T^*, Z_T^*, \boldsymbol{B}_T) \leq v(t,\xi,z,\boldsymbol{x})$$

Therefore:

$$v^*(t,\xi,z,\boldsymbol{x}) = \inf_{\boldsymbol{\pi}\in\widetilde{\mathcal{A}}} \sup_{\boldsymbol{\nu}\in\mathcal{N}} \mathbb{E}v(T, Y_T^{\boldsymbol{\pi}}, Z_T^{\boldsymbol{\pi}}, \boldsymbol{B}_T) \leq v(t,\xi,z,\boldsymbol{x})$$

All in all, we showed that:

$$v^*(t,\xi,z,\boldsymbol{x}) = v(t,\xi,z,\boldsymbol{x})$$

If we use $\boldsymbol{\pi}^{(\boldsymbol{\lambda})}$ and $\boldsymbol{\nu}_t^* := \Big(\boldsymbol{\lambda}(t,\boldsymbol{B}_t), D\widetilde{f}_2(D_{n-d}u)\Big)^T$ in the arguments above, all the inequalities become equalities, thus we see that $\boldsymbol{\pi}^{(\boldsymbol{\lambda})}$ is indeed optimal.

□

4.2 Market Equilibria with Risk-Measure-Minimizing Agents

4.2.1 The Equilibrium PDE System

The market clearing condition in Definition 2.2.1 and the optimal-portfolio formula in Theorem 4.1.2, state that, in an equilibrium, we have

$$\sum_{i=1}^{I} \boldsymbol{\pi}^{(\boldsymbol{\lambda},i)}(t,\boldsymbol{x}) = \sum_{i=1}^{I} \Big(Df_1^{[i]}(\boldsymbol{\lambda}(t,\boldsymbol{x})) - D_d u^{[i]}(t,\boldsymbol{x})\Big) = 0 \qquad (4.2.1)$$

Note that the function $\sum_{i=1}^{I} f_1^{[i]} : \mathbb{R}^d \to \mathbb{R}$, is proper, strictly convex and continuously differentiable. Thus, thanks to Proposition B.2.4 from [22], we know that its gradient admits an inverse function, which is the gradient of its conjugate, and we denote it by $F(\boldsymbol{y})$, i.e.,

$$F(\boldsymbol{y}) := \left(\sum_{i=1}^{I} Df_1^{[i]}(\boldsymbol{y})\right)^{-1}$$

Equivalently, (4.2.1) becomes

$$\boldsymbol{\lambda}(t,\boldsymbol{x}) = F(\sum_{i=1}^{I} D_d u^{[i]}(t,\boldsymbol{x})) \tag{4.2.2}$$

If we insert (4.2.2) into the HJB equation (4.1.5) for each agent, we have that $\boldsymbol{u} := (u^{[1]}, \dots, u^{[I]})$ solves the following PDE system, due to the optimality of $\boldsymbol{\pi}^{(\boldsymbol{\lambda},i)}$, for all $i = 1, \dots, I$.

$$\begin{cases} u_t^{[i]} + \frac{1}{2}\Delta u^{[i]} - \left(D_d u^{[i]}\right)^T \left(F(\sum_{k=1}^{I} D_d u^{[k]})\right) + f_1^{[i]}\left(F(\sum_{k=1}^{I} D_d u^{[k]})\right) \\ \qquad\qquad - \widetilde{f}_2^{[i]}(D_{n-d}u^{[i]}) = 0 \\ u^{[i]}(T,\boldsymbol{x}) = g^{[i]}(\boldsymbol{x}) \end{cases} \tag{4.2.3}$$

On the other hand, if the PDE system (4.2.3) has a solution, then obviously (4.2.2) leads to an equilibrium price dynamics. Therefore, finding an equilibrium becomes equivalent to solving (4.2.3).

Remark 4.2.1. Comparing the equilibrium PDE system (4.2.3) to the one for exponential-utility agents, i.e., equation (3.2.2), one can see, not surprisingly, that (3.2.2) is just a special case of (4.2.3).

4.2.2 Existence and Uniqueness of an Equilibrium with Risk-Measure-Minimizing Agents

Let the space $\mathcal{B}(\cdot,\cdot,\cdot,\cdot,\cdot,\cdot)$ be as in Definition 6.2.1.

Theorem 4.2.2. *There exists a positive constant C, independent of C_g such that for $T \leq T_0 = C/C_g^2$, and under the Assumption 2.3.1, the equilibrium PDE system* (4.2.3) *has a unique solution $u \in C^{2,\alpha}(Q)$. Moreover,* *there exist generic constants C_1, D_1, C_2, D_2, E, F and A_1, A_2, A_3, such that $\boldsymbol{u} \in \mathcal{B}(C_1, D_1, C_2, D_2, E, F)$ and*

$$\begin{aligned}\|\boldsymbol{u}(t,\boldsymbol{x})\| \leq & A_1\phi_{D_1}(\boldsymbol{x}) \\ \|D\boldsymbol{u}(t,\boldsymbol{x})\| \leq & A_2\phi_{D_2}(\boldsymbol{x}) \\ |\boldsymbol{u}|_{2,\alpha} \leq & A_3\end{aligned}$$

An equilibrium market price of risk $\boldsymbol{\lambda}$ is given by equation (3.2.1). Moreover, it is unique in $\left(C^{1,\alpha}(Q)\right)^d$, and there exist generic constants A_4, A_5, such that:

$$\begin{aligned}\|\boldsymbol{\lambda}(t,\boldsymbol{x})\| \leq & A_4\phi_{D_2}(\boldsymbol{x}) \\ |\boldsymbol{\lambda}|_{\alpha} \leq & A_5\end{aligned}$$

Finally, if the function h in Assumption 2.3.1 further satisfies the condition (A.1.2) in Lemma A.1.1, then there exist generic constants $\widetilde{A}_1$, $\widetilde{A}_2$ and $\widetilde{A}_3$ such that:

$$\begin{aligned}\|\boldsymbol{u}(t,\boldsymbol{x})\| \leq & \widetilde{A}_1 h(\boldsymbol{x}) \\ \|D\boldsymbol{u}(t,\boldsymbol{x})\| \leq & \widetilde{A}_2 h(\boldsymbol{x}) \\ \|\boldsymbol{\lambda}(t,\boldsymbol{x})\| \leq & \widetilde{A}_3 h(\boldsymbol{x})\end{aligned}$$

Proof. As above, we want to use the results in Section 6.3. To show that solutions to the PDE system (4.2.3) exist, we introduce the function $F^{[i]}$: $[0,T] \times \mathbb{R}^n \times \mathbb{R}^{I\times n}$ by

$$F^{[i]}(t,\boldsymbol{x},Z) = -\left(Z_d^{[i]}\right)^T \left(F(\sum_{k=1}^{I} Z_d^{[k]})\right) + f_1^{[i]}(F(\sum_{k=1}^{I} Z_d^{[k]})) - \widetilde{f}_2^{[i]}(Z_{n-d}^{[i]})$$

where we use the notation defined in (3.2.10). We need to check that $F^{[i]}$

satisfies the conditions in Assumption 6.1.5. By Assumption 2.3.3 we have:

$$\left\|\sum_{i=1}^{I} Df_1^{[i]}(\boldsymbol{x}_1)-\sum_{i=1}^{I} Df_1^{[i]}(\boldsymbol{x}_2)\right\| \geq \frac{1}{L_3(\|\boldsymbol{x}_1\|,\|\boldsymbol{x}_2\|)}\|\boldsymbol{x}_1-\boldsymbol{x}_2\|$$
$$\left\|\sum_{i=1}^{I} Df_1^{[i]}(\boldsymbol{x})\right\| \geq \frac{1}{L_3}\|\boldsymbol{x}\|$$

Therefore, as the inverse of $\sum_{i=1}^{I} Df_1^{[i]}$, the function F is locally Lipschitz and has at most linear growth, i.e.

$$\begin{aligned} \|F(\boldsymbol{y}_1)-F(\boldsymbol{y}_2)\| \leq& L_3(|F(\boldsymbol{y}_1)|,|F(\boldsymbol{y}_2)|)\|\boldsymbol{y}_1-\boldsymbol{y}_2\| \\ \|F(\boldsymbol{y})\| \leq& L_3\|\boldsymbol{y}\| \end{aligned} \tag{4.2.4}$$

where both L_3 and F are locally bounded. Further notice that both functions $f_1^{[i]}$ and $\widetilde{f}^{[i]}$ are local Lipschitz. Thus, the function $F^{[i]}$ is local Lipschitz w.r.t Z, i.e.

$$\left|F^{[i]}(Z^{(1)})-F^{[i]}(Z^{(2)})\right| \leq L_4(\left\|Z^{(1)}\right\|,\left\|Z^{(2)}\right\|)\left\|Z^{(1)}-Z^{(2)}\right\|$$

where $L_4 : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is some locally bounded (increasing) function. Now thanks to Assumption 2.3.3 and Proposition A.3.1, it's clear that $F^{[i]}$ satisfies the quadratic growth condition in Assumption 6.1.5 (2). Finally, in order to check Assumption 6.1.5 (3), we pick $Z \in \left(C^{0,\alpha}(Q)\right)^{I\times n}$, and compute $\left[F^{[i]}(Z(t,\boldsymbol{x}))\right]_\alpha$:

$$\left|F^{[i]}(Z(t,\boldsymbol{x}))-F^{[i]}(Z(s,\boldsymbol{y}))\right| \leq L_4(|Z|_0,|Z|_0)\|Z(t,\boldsymbol{x})-Z(s,\boldsymbol{y})\| \tag{4.2.5}$$

Therefore, we have:

$$\left[F^{[i]}(Z(t,\boldsymbol{x}))\right]_\alpha \leq L_4(|Z|_0,|Z|_0)\left[Z(t,\boldsymbol{x})\right]_\alpha$$

In addition, remember that the function $F^{[i]}$ is continuous, so there exists a constant $L(|Z|_0)$, such that:

$$\left|F^{[i]}(Z(t,\boldsymbol{x}))\right| \leq L(|Z|_0) \tag{4.2.6}$$

Combining (4.2.5) and (4.2.6), we have verified the condition (3) in Assumption 6.1.5.

We can, thus, apply Lemma 6.2.3, Theorem 6.3.1 and Corollary 6.3.2, to conclude that there exist generic constants C_1, D_1, C_2, D_2, E, F and A_1, A_2, A_3, such that the equilibrium PDE system (3.2.2) has a unique solution, denoted by $\boldsymbol{u}$, which belongs to $\mathcal{B}(C_1, D_1, C_2, D_2, E, F)$, and it satisfies the first set of desired inequalities. If we further assume that function h in Assumption 2.3.1, satisfies the condition (A.1.2) in Lemma A.1.1, by Lemma A.1.2, there exist generic constants $\widetilde{A}_1$ and $\widetilde{A}_2$ such that the last set of desired inequalities holds.

Finally, the market price of risk $\boldsymbol{\lambda}$ is given by the equation (4.2.2), and its uniqueness follows from the uniqueness of solution to equilibrium PDE system (4.2.3). It is not hard to see that there exist generic constants A_4, A_5 and $\widetilde{A}_3$ (under condition (A.1.2)), such that the desired estimates for $\boldsymbol{\lambda}$ hold by equation (3.2.1), thanks to Lemma A.1.2.

□

Chapter 5

A System of BSDEs

Consider the following system of Backward Stochastic Differential Equations (BSDE):

$$Y_t = \boldsymbol{g}(\boldsymbol{B}_T) + \int_t^T \boldsymbol{f}(s, \boldsymbol{B}_s, Z_s)\, ds - \int_t^T Z_s\, d\boldsymbol{B}_s \tag{5.1.1}$$

where $\boldsymbol{g} := (g^{[1]}, \dots, g^{[I]})^T$, $\boldsymbol{f} := (f^{[1]}, \dots, f^{[I]})^T$ and $Y_t := (Y^{[1]}, \dots, Y^{[I]})$ take values in $\mathbb{R}^I$, $Z_t \in \mathbb{R}^{I\times n}$ and $\boldsymbol{B}_t \in \mathbb{R}^n$.

Definition 5.1.3. A pair of adapted and pathwise square-integrable processes $(Y_t, Z_t) : [0,T] \times \Omega \to \mathbb{R}^n \times \mathbb{R}^{I\times n}$ is called a solution of the system (5.1.1), if they satisfy the system (5.1.1) $\mathbb{P}$-almost surely.

Let $\mathcal{B}(\cdot,\cdot,\cdot,\cdot,\cdot,\cdot)$ be as in Definition 6.2.1.

Theorem 5.1.4. *Under Assumption 6.1.5, we fix $T \le T_0$, where the constant T_0 is given by* (6.2.2) *or* (6.2.3). *Then, the system* (5.1.1) *has a solution (Y_t, Z_t), and there exists a function $\boldsymbol{u} \in \mathcal{B}(C_1, D_1, C_2, D_2, E, F)$ and generic constants C_1, D_1, C_2, D_2, E, F defined by Lemma 6.2.3, such that, $Y_t = \boldsymbol{u}(t, \boldsymbol{B}_t)$ and $Z_t = D\boldsymbol{u}(t, \boldsymbol{B}_t)$.*

In addition, such a solution is unique, within the class of pairs (Y, Z), where Z is uniformly bounded.

Proof. Thanks to Theorem 6.3.1, let $\boldsymbol{u} := (u^{[1]}, \ldots, u^{[I]}) \in C^{2,\alpha}(Q)$ be the solution to the PDE system (6.1.1). we apply Itô's formula to $u^{[i]}(s, \boldsymbol{B}_s)$, for $i = 1, \ldots, I$, to obtain:

$$\begin{aligned} du^{[i]}(s, \boldsymbol{B}_s) &= \left(u_s^{[i]}(s, \boldsymbol{B}_s) + \tfrac{1}{2}\Delta u^{[i]}(s, \boldsymbol{B}_s)\right) ds + Du^{[i]}(s, \boldsymbol{B}_s)^T d\boldsymbol{B}_s \\ &= - f^{[i]}(s, \boldsymbol{B}_s, D\boldsymbol{u}(s, \boldsymbol{B}_s))ds + Du^{[i]}(s, \boldsymbol{B}_s)d\boldsymbol{B}_s \end{aligned}$$

The processes $Y_t := \boldsymbol{u}(t, \boldsymbol{B}_t)$ and $Z_t := D\boldsymbol{u}(t, \boldsymbol{B}_t)$ obviously form an adapted solution to the BSDE (5.1.1). It remains to show that the solution is unique. Suppose that there is another solution $(\widetilde{Y}_t, \widetilde{Z}_t)$ to (5.1.1) with $\sup_{t\in[0,T]} |\widetilde{Z}_t| \leq \widetilde{N}$, $\mathbb{P}$-almost surely for some fixed constant $\widetilde{N} > 0$. Set $W_t := Y_t - \widetilde{Y}_t$ and $X_t := Z_t - \widetilde{Z}_t$. We then apply Itô's formula to $\left(W_t^{[i]}\right)^2$, take expectations, and observe that by Assumption 6.1.5, we have:

$$\begin{aligned} &\mathbb{E}\left|W_t^{[i]}\right|^2 + \int_t^T \mathbb{E}\left\|X_s^{[i]}\right\|^2 ds \\ =&\mathbb{E}\Big[\int_t^T 2W_s^{[i]}\left(f^{[i]}(s, \boldsymbol{B}_s Z_s) - f^{[i]}(s, \boldsymbol{B}_s, \widetilde{Z}_s)\right) ds\Big] \\ \leq&\mathbb{E}\Big[\int_t^T 2\left|W_s^{[i]}\right| \ \widetilde{N}_2(C_2, \widetilde{N})\|X_s\| \, ds\Big] \\ \leq& I\widetilde{N}_2^2(C_2, \widetilde{N})\int_t^T \mathbb{E}\left|W_s^{[i]}\right|^2 ds + \int_t^T \frac{1}{I}\mathbb{E}\|X_s\|^2 ds \end{aligned}$$

where $\widetilde{N}_2$ is a constant that depends on C_2, $\widetilde{N}$ and the function $N_1(x, y)$ of Assumption 6.1.5. If we sum both sides of the inequality above over $i = 1, \ldots, I$, we get:

$$\mathbb{E}\|W_t\|^2 + \int_t^T \mathbb{E}\|X_s\|^2 ds \leq I\widetilde{N}_2^2(C_2, \widetilde{N})\int_t^T \mathbb{E}\|W_s\|^2 ds + \int_t^T \mathbb{E}\|X_s\|^2 ds$$

After canceling the integral term on both sides, we obtain:

$$\mathbb{E}\|W_t\|^2 \leq I\widetilde{N}_2^2(C_2, \widetilde{N})\int_t^T \mathbb{E}\|W_s\|^2 ds$$

By Gronwall's inequality, we have that $W_t = 0$, $\mathbb{P}$-a.s. as a result, we showed that $(Y_t, Z_t) = (\widetilde{Y}_t, \widetilde{Z}_t)$, $\mathbb{P}$-a.s. on $[0, T]$, and the adapted solution to BSDE (5.1.1) is unique for bounded process Z_t.

$\square$

Chapter 6

A Solution to the PDE System (6.1.1)

We devote this section to the technical aspects of the proof of the existence and uniqueness of the solution to the PDE system (6.1.1), which is somewhat more general than the equilibrium PDE system of previous chapters:

$$\begin{cases} u_t^{[i]} + \frac{a}{2}\Delta u^{[i]} + f^{[i]}(t, \boldsymbol{x}, D\boldsymbol{u}) = 0, \ i = 1, \ldots, I, \\ u^{[i]}(T, \boldsymbol{x}) = g^{[i]}(\boldsymbol{x}), \ i = i, \ldots, I, \end{cases} \tag{6.1.1}$$

where $a > 0$. We start the analysis by describing the standing assumptions on functions $g^{[i]}$ and $f^{[i]}$:

Assumption 6.1.5. For $i = 1, \ldots, I$, we assume that:

1. $g^{[i]}$ satisfies Assumption 2.3.1, so that (see Remark 2.3.2):

$$\left|g^{[i]}(\boldsymbol{x})\right|, \left|g_{x_j}^{[i]}(\boldsymbol{x})\right| \leq C_g M_0(B_0, D)\phi_D(\boldsymbol{x})$$

   where the function $\phi_D(\boldsymbol{x})$ is defined by Convention 1 in Section 2.3 of Chapter 2.

2. There exists constants $C_p, C_q > 0$ and a function $0 \leq Q^{[i]}(t, \boldsymbol{x}) \leq C_q \phi_D(\boldsymbol{x})$, such that $\forall\, (t, \boldsymbol{x}) \in Q$, $\forall Z \in \mathbb{R}^{I \times n}$:

$$\left|f^{[i]}(t, \boldsymbol{x}, Z)\right| \leq C_p \|Z\|^2 + Q^{[i]}(t, \boldsymbol{x}). \tag{6.1.2}$$

3. For all $Z \in \left(C^{0,\alpha}(Q)\right)^{I\times n}$ we have $f^{[i]}(t,\boldsymbol{x},Z(t,\boldsymbol{x})) \in C^{0,\alpha}(Q)$, and:

$$\left|f^{[i]}(t,\boldsymbol{x},Z(t,\boldsymbol{x}))\right|_\alpha \leq N_1(|Z|_0)\left([Z]_\alpha + 1\right) \tag{6.1.3}$$

   where $N_1 : \mathbb{R}_+ \to \mathbb{R}_+$ is a locally bounded function.

4. $f^{[i]}$ is locally Lipschitz w.r.t. Z, i.e. $\forall\, Z^{(1)}, Z^{(2)} \in \mathbb{R}^{I\times n}$:

$$\left|f^{[i]}(t,\boldsymbol{x},Z^{(1)}) - f^{[i]}(t,\boldsymbol{x},Z^{(2)})\right| \leq N_1\left(\left\|Z^{(1)}\right\|, \left\|Z^{(2)}\right\|\right)\left\|Z^{(1)} - Z^{(2)}\right\| \tag{6.1.4}$$

   where $N_1 : \mathbb{R}_+^2 \mapsto \mathbb{R}_+$ is a locally bounded function.

Remark 6.1.6. For a function $f^{[i]}$ of the form $f^{[i]}(t,\boldsymbol{x},Z) = f^{[i]}(Z)$, the third assumption becomes unnecessary. Indeed, in this case, (6.1.3) is a direct consequence of (6.1.2) and (6.1.4).

6.2 The Function Space $\mathcal{B}$ and the Map $\mathcal{H}$

The idea behind our existence proof of a solution to the PDE system (6.1.1) is to use the Schauder's fixed point theorem on a proper subset of a subspace $\mathcal{B}$ of $\left(C^{1,\alpha}(Q)\right)^I$, introduced below:

Definition 6.2.1. For constants C_1, D_1, C_2, D_2, E, $F > 0$, we define a subspace of $C^{1,\alpha}(Q)$ as follows:

$$\mathcal{D}(C_1,D_1,C_2,D_2,E,F) = \{u \in C^{1,\alpha}(Q)\,:\,|u(t,\boldsymbol{x})| \leq C_1\phi_{D_1}(\boldsymbol{x}),\ [u]_\alpha \leq E, \text{ and}$$
$$\forall\, j=1,\ldots,n, \left|u_{x_j}(t,\boldsymbol{x})\right| \leq C_2\phi_{D_2}(\boldsymbol{x}),\ \left[u_{x_j}\right]_\alpha \leq F\}$$

The I-th Cartesian power $\mathcal{D}^I$ of $\mathcal{D}$ is denoted by $\mathcal{B}$.

Thanks to Lemma A.1.2, the inclusion $\mathcal{D} \subset H^1(Q) \cap C_v(Q)$ holds, so that $\mathcal{B} \subseteq \left(C^{1,\alpha}(Q)\right)^I \cap \left(H^1(Q)\right)^I \cap \left(C_v(Q)\right)^I$.

Lemma 6.2.2. *The space $\mathcal{D}(C_1, D_1, C_2, D_2, E, F)$ is closed in $C^0(Q)$. And, thus, the space $\mathcal{B}(C_1, D_1, C_2, D_2, E, F)$ is closed in $\left(C^0(Q)\right)^I$.*

Proof. To see this, let $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{D}$ be a sequence that converges uniformly to a function u: $\forall\, \varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that for $n \geq n_\varepsilon$

$$|u(t,\boldsymbol{x}) - u(s,\boldsymbol{y})| \leq |u(t,\boldsymbol{x}) - u_n(t,\boldsymbol{x})| + |u_n(t,\boldsymbol{x}) - u_n(s,\boldsymbol{y})| + |u_n(s,\boldsymbol{y}) - u(s,\boldsymbol{y})| \leq 2\varepsilon + E\Big(d_p((t,\boldsymbol{x}),\, (s,\boldsymbol{y}))\Big)^\alpha.$$

Consequently, $u \in C^{0,\alpha}(Q)$, and $[u]_\alpha \leq E$. In addition, we observe that sequences $\{(u_{x_j})_n\}_{n\in\mathbb{N}}$, for $j = 1, \dots, n$, are bounded by a function that vanishes at infinity, and are, hence, equicontinuous due to the uniform bound of their $[\cdot]_\alpha$ norms by F. We can, therefore, apply Proposition A.4.1 and extract from them uniformly convergent subsequences, for notational reasons still denoted by $\{(u_{x_j})_n\}_{n\in\mathbb{N}}$, for $j = 1, \dots, n$. Therefore $u_{x_j} \in C^0(Q)$ and $\{(u_{x_j})_n\}_{n\in\mathbb{N}}$ converge to u_{x_j} uniformly. In a similar way, for each $\varepsilon > 0$, we can find $n_\varepsilon \in \mathbb{N}$ such that for $n \geq n_\varepsilon$ we have

$$\left|u_{x_j}(t,\boldsymbol{x}) - u_{x_j}(s,\boldsymbol{y})\right| \leq \left|u_{x_j}(t,\boldsymbol{x}) - (u_{x_j})_n(t,\boldsymbol{x})\right| + \left|(u_{x_j})_n(t,\boldsymbol{x}) - (u_{x_j})_n(s,\boldsymbol{y})\right| + \left|(u_{x_j})_n(s,\boldsymbol{y}) - u(s,\boldsymbol{y})\right| \leq 2\varepsilon + F\Big(d_p((t,\boldsymbol{x}),\, (s,\boldsymbol{y}))\Big)^\alpha.$$

Therefore, $u \in C^{1,\alpha}(Q)$ and $\left[u_{x_j}\right]_\alpha \leq F$, and it follows that $u \in \mathcal{D}$. □

Thanks to Theorem B.6.1, for $\boldsymbol{v}$ in $\mathcal{B}(C_1,\, D_1,\, C_2,\, D_2,\, E,\, F)$ fixed, the following linear PDE system has a unique solution $\boldsymbol{u} \in \left(C^{2,\alpha}(Q)\right)^I$:

$$\begin{cases} u_t^{[i]} + \frac{a}{2}\Delta u^{[i]} + f^{[i]}(t,\boldsymbol{x},D\boldsymbol{v}) = 0 \\ u^{[i]}(T,\boldsymbol{x}) = g^{[i]}(\boldsymbol{x}) \end{cases} \tag{6.2.1}$$

One can, therefore, define a map $\mathcal{H} : \mathcal{B} \mapsto \left(C^{2,\alpha}(Q)\right)^I$ using the equation above; namely, $\mathcal{H}(\boldsymbol{v}) := \boldsymbol{u}$. In fact, if we choose the positive constants C_1, D_1, C_2, D_2, E and F properly, we can make sure that the range of the map $\mathcal{H}$, denoted by $\mathcal{R}(\mathcal{H})$, is still inside of $\mathcal{B}$.

Lemma 6.2.3. *Let the constant T_0 be defined by*

$$T_0 := \left(\frac{\sqrt{aenIC_pC_g^2M_0^2 + aeC_q} - \sqrt{aenIC_p}C_gM_0}{6\sqrt{2nIC_p}C_q} \right)^2, \text{ when } C_q > 0 \quad (6.2.2)$$

and

$$T_0 := \frac{ae}{288n^2I^2C_p^2C_g^2M_0^2}, \text{ when } C_q = 0 \quad (6.2.3)$$

where $M_0 := M_0(B_0, D)$ is as in Lemma A.1.2. Furthermore, for $T \leq T_0$, we choose the constants C_1, D_1, C_2, D_2, E, F so that:

$$\begin{aligned} D_2 &:= \max(2D,\, 6aT) \\ C_2 &:= \frac{\sqrt{ae}}{6\sqrt{2}nIC_p\sqrt{T}} \\ D_1 &:= 2aT + D_2/2 \\ C_1 &:= nIC_pTC_2^2 + C_qT + C_gM_0 \end{aligned}$$

and let $F > 0$ to be the unique solution to the following equation:

$$\widetilde{A}\left(F + \widetilde{B}\right)^{\frac{1+\alpha}{2+\alpha}} = F$$

where:

$$\begin{aligned} \widetilde{A} &:= N(C_1)^{\frac{1}{2+\alpha}} \left(N_0(Q,\alpha,T)N_1(\sqrt{I}C_2)\sqrt{nI}\right)^{\frac{1+\alpha}{2+\alpha}} \\ \widetilde{B} &:= \frac{G + N_1(\sqrt{I}C_2)}{N_1(\sqrt{I}C_2)\sqrt{nI}} \end{aligned}$$

where constant N comes from the Theorem B.5.2, constant N_0 comes from Theorem B.6.1 and function N_1 is from Assumption 6.1.5. Finally, let J and E be given by:

$$J := N_0(Q, \alpha, a, T)\left(G + N_1(\sqrt{I}C_2)\left(\sqrt{nI}F + 1\right)\right)$$
$$E := NJ^{\frac{\alpha}{2+\alpha}}\left(C_1\right)^{\frac{2}{2+\alpha}}$$

Then the range of the map $\mathcal{H}$, defined above, is inside of $\mathcal{B}$, i.e. $\mathcal{H} : \mathcal{B} \mapsto \mathcal{B}$. Furthermore, $\mathcal{R}(\mathcal{H})$ is bounded by J in $C^{2,\alpha}(Q)$. i.e. $\forall\, u \in \mathcal{R}(\mathcal{H})$, $|u|_{2,\alpha} \leq J$.

Proof. By Assumption 6.1.5, $f^{[i]} \in C^{0,\alpha}(Q)$, $\forall\, i = 1, \ldots, I$, and furthermore, by Lemma A.1.2 (1), we have:

$$\begin{aligned}\left|f^{[i]}(t, \boldsymbol{x}, D\boldsymbol{v})\right| \leq & C_p\|D\boldsymbol{v}(t, \boldsymbol{x})\|^2 + Q^{[i]}(t, \boldsymbol{x}) \\ \leq & C_p \sum_{i=1}^{I}\sum_{j=1}^{n}\left|v_{x_j}^{[i]}\right|^2 + C_q\phi_D(\boldsymbol{x}) \\ \leq & nIC_pC_2^2\phi_{D_2}^2(\boldsymbol{x}) + C_q\phi_D(\boldsymbol{x}) \\ \leq & \left(nIC_pC_2^2 + C_q\right)\phi_{D_2/2}(\boldsymbol{x})\end{aligned}$$

since $D \leq D_2/2$. In addition, by Assumption 6.1.5, we can also estimate the $C^{0,\alpha}(Q)$ norm of $f^{[i]}(t, \boldsymbol{x}, \boldsymbol{v}, D\boldsymbol{v})$:

$$\begin{aligned}\left|f^{[i]}(t, \boldsymbol{x}, D\boldsymbol{v})\right|_\alpha = & N_1(|D\boldsymbol{v}|_0)\left([D\boldsymbol{v}]_\alpha + 1\right) \\ = & N_1(\sqrt{I}C_2)\left(\left(\sum_{i=1}^{I}\sum_{j=1}^{n}\left[v_{x_j}^{[i]}\right]_\alpha^2\right)^{1/2} + 1\right) \\ \leq & N_1(\sqrt{I}C_2)\left(\sqrt{nI}F + 1\right)\end{aligned}$$

where N_1 is a constant that depends on the function N_1 and the constant C_2. In addition, we have the classical convolution formula to represent the solution (see [20], §4.1), for $i = 1, \dots, I$, we have:

$$u^{[i]}(t,\boldsymbol{x}) = \int_{\mathbb{R}^n} K_a(T-t,\boldsymbol{x}-\boldsymbol{y})\, g^{[i]}(\boldsymbol{y})\, d\boldsymbol{y} + \int_t^T \int_{\mathbb{R}^n} K_a(s-t,\boldsymbol{x}-\boldsymbol{y}) f^{[i]}(s,\boldsymbol{y},D\boldsymbol{v})\, d\boldsymbol{y} ds$$

where:

$$K_a(t,\boldsymbol{x}) := \begin{cases} \left(\frac{1}{2\pi a t}\right)^{n/2} \exp\left(-\frac{\|\boldsymbol{x}\|^2}{2at}\right) & \text{for } t > 0 \\ 0 & \text{for } t \le 0 \end{cases}$$

Now, we recall that $|g(t,\boldsymbol{x})| \le C_g \phi_D(\boldsymbol{x})$, and use Lemma A.2.1 and Corollary A.2.3 to estimate $\left|u^{[i]}\right|$:

$$\begin{aligned}
\left|u^{[i]}(t,\boldsymbol{x})\right| &= \left| K_a(T-t,\boldsymbol{x}) * g^{[i]}(\boldsymbol{x}) + \int_t^T K_a(s-t,\boldsymbol{x}) * f^{[i]}(s,\boldsymbol{x},D\boldsymbol{v})\, ds \right| \\
&\le K_a(T-t,\boldsymbol{x}) * C_g M_0(B_0,D)\phi_D(\boldsymbol{x}) \\
&\quad + \int_t^T K_a(s-t,\boldsymbol{x}) * \left(nIC_pC_2^2 + C_q\right)\phi_{D_2/2}(\boldsymbol{x})\, ds \\
&= C_g M_0(B_0,D)\left(\tfrac{1}{2\pi a(T-t)}\right)^{n/2}\left(\tfrac{2a(T-t)D\pi}{2a(T-t)+D}\right)^{n/2}\phi_{2a(T-t)+D}(\boldsymbol{x}) \\
&\quad + \left(nIC_pC_2^2 + C_q\right)\int_t^T \left(\tfrac{1}{2a\pi(s-t)}\right)^{n/2}\left(\tfrac{2a(s-t)D_2\pi}{2a(s-t)+D_2}\right)^{n/2}\phi_{2a(s-t)+D_2/2}(\boldsymbol{x})\, ds \\
&\le \left(nIC_pTC_2^2 + C_qT + C_gM_0(B_0,D)\right)\phi_{2aT+D_2/2}(\boldsymbol{x}) := C_1\phi_{D_1}(\boldsymbol{x})
\end{aligned}$$

Secondly, we observe that:

$$u^{[i]}_{x_j}(t,\boldsymbol{x}) = K_a(T-t,\boldsymbol{x}) * g^{[i]}_{x_j}(\boldsymbol{x}) + \int_t^T \frac{\partial}{\partial x_i} K_a(s-t,\boldsymbol{x}) * f^{[i]}_{\boldsymbol{v}}(s,\boldsymbol{y})\, ds$$

We estimate $\left|u^{[i]}_{x_j}\right|$ in a similar way:

$$
\begin{aligned}
\left|u^{[i]}_{x_j}(t,\boldsymbol{x})\right| \leq\ & K_a(T-t,\boldsymbol{x}) * C_g M_0 \phi_D(\boldsymbol{x}) \\
& + \int_t^T \left(\tfrac{1}{2\pi a(s-t)}\right)^{n/2} \tfrac{|x_i|}{a(s-t)} \exp\left(-\tfrac{\|\boldsymbol{x}\|^2}{2a(s-t)}\right) * \left(nIC_pC_2^2 + C_q\right) \phi_{D_2/2}(\boldsymbol{x})\, ds \\
\leq & C_g M_0 \left(\tfrac{1}{2\pi a(T-t)}\right)^{n/2} \left(\tfrac{2a(T-t)D\pi}{2a(T-t)+D}\right)^{n/2} \phi_{2a(T-t)+D}(\boldsymbol{x}) + \int_t^T \tfrac{nIC_pC_2^2+C_q}{a(s-t)} \\
& \left(\tfrac{1}{2\pi a(s-t)}\right)^{n/2} \tfrac{3\sqrt{2a(s-t)}}{2\sqrt{e}} \left(\tfrac{a(s-t)D_2\pi}{2a(s-t)+D_2/2}\right)^{n/2} \phi_{3aT+D_2/2}(\boldsymbol{x})\, ds \\
\leq & \left(C_g M_0 + \tfrac{3\sqrt{2}}{\sqrt{ae}} \left(nIC_pC_2^2 + C_q\right)\sqrt{T}\right) \phi_{3aT+D_2/2}(\boldsymbol{x}) \\
\leq & C_2 \phi_{D_2}(\boldsymbol{x})
\end{aligned}
$$

where one can readily check that the last inequality holds due to the choices of the constants C_2, D_2 and smallness of T in Lemma 6.2.3. In addition, using the parabolic interpolation and Theorem B.6.1, we can estimate $\left[u^{[i]}_{x_j}\right]_\alpha$ as the following:

$$
\begin{aligned}
\left[u^{[i]}_{x_j}\right]_\alpha \leq & N \left[u^{[i]}\right]_{2,\alpha}^{\frac{1+\alpha}{2+\alpha}} \left|u^{[i]}\right|_0^{\frac{1}{2+\alpha}} \\
\leq & N \left(C_1\right)^{\frac{1}{2+\alpha}} N_0^{\frac{1+\alpha}{2+\alpha}}(Q,\alpha,T) \left(\left|g^{[i]}\right|_{2,\alpha} + \left|f^{[i]}\right|_\alpha\right)^{\frac{1+\alpha}{2+\alpha}} \\
\leq & N \left(C_1\right)^{\frac{1}{2+\alpha}} N_0^{\frac{1+\alpha}{2+\alpha}} \left(G + N_1(\sqrt{I}C_2)\left(\sqrt{nI}F + 1\right)\right)^{\frac{1+\alpha}{2+\alpha}} \\
= & N(C_1)^{\frac{1}{2+\alpha}} \left(N_0 N_1 \sqrt{nI}\right)^{\frac{1+\alpha}{2+\alpha}} \left(F + \frac{G+N_1}{N_1\sqrt{nI}}\right)^{\frac{1+\alpha}{2+\alpha}} \\
:= & \widetilde{A}\left(F + \widetilde{B}\right)^{\frac{1+\alpha}{2+\alpha}} \\
= & F
\end{aligned}
$$

The last equality holds due to the choice of constant F in Lemma 6.2.3. Furthermore, we notice that:

$$
\left|u^{[i]}\right|_{2,\alpha} \leq N_0(Q,\alpha,T)\left(G + N_1(\sqrt{I}C_2)\left(\sqrt{nI}F + 1\right)\right) := J
$$

Finally, thanks to the parabolic interpolation, we have:

$$\left[u^{[i]}\right]_{\alpha} \leq N \left[u^{[i]}\right]_{2,\alpha}^{\frac{\alpha}{2+\alpha}} \left|u^{[i]}\right|_{0}^{\frac{2}{2+\alpha}} \leq N J^{\frac{\alpha}{2+\alpha}} (C_1)^{\frac{2}{2+\alpha}} := E$$

Therefore, we have shown that the unique solution $\boldsymbol{u}$ to system (6.2.1) also belongs to the space $\mathcal{B}$. □

6.3 Existence and Uniqueness of the Solution to PDE System (6.1.1)

Theorem 6.3.1. *Under Assumption 6.1.5 and with $T \leq T_0$, where T_0 is given by* (6.2.2) *or* (6.2.3)*, the PDE system* (6.1.1) *has a unique solution $\boldsymbol{u} \in C^{2,\alpha}(Q)$. Moreover, $\boldsymbol{u} \in \mathcal{B}(C_1, D_1, C_2, D_2, E, F)$, where constants C_1, D_1, C_2, D_2, E, F are defined as in Lemma 6.2.3.*

Corollary 6.3.2. *Under the same set of assumptions as in Theorem 6.3.1, there exist generic constants A_1, A_2 and A_3, such that the solution $\boldsymbol{u}$ satisfies:*

$$\begin{aligned}\|\boldsymbol{u}(t,\boldsymbol{x})\| \leq& A_1 \phi_{D_1}(\boldsymbol{x}) \\ \|D\boldsymbol{u}(t,\boldsymbol{x})\| \leq& A_2 \phi_{D_2}(\boldsymbol{x}) \\ |\boldsymbol{u}|_{2,\alpha} \leq& A_3\end{aligned}$$

In addition, if the function h in Assumption 2.3.1 further satisfies the condition (A.1.2) *in Lemma A.1.1, there exist generic constants $\widetilde{A}_1$ and $\widetilde{A}_2$ such that:*

$$\begin{aligned}\|\boldsymbol{u}(t,\boldsymbol{x})\| \leq& \widetilde{A}_1 h(\boldsymbol{x}) \\ \|D\boldsymbol{u}(t,\boldsymbol{x})\| \leq& \widetilde{A}_2 h(\boldsymbol{x})\end{aligned}$$

Proof. We simply combine Theorem 6.3.1, Lemma 6.2.3 and Lemma A.1.2. □

Proof of Theorem 6.3.1. By Lemma 6.2.3, the map $\mathcal{H} : \mathcal{B} \to \mathcal{B}$ is well-defined by $\mathcal{H}(\boldsymbol{v}) = \boldsymbol{u}$ through (6.2.1). We claim that the map $\mathcal{H} : \mathcal{B} \to \mathcal{B}$ is continuous with respect to the $\left(C^{1,0}(Q)\right)^I$ topology, where we recall

$$C^{1,0}(Q) := \{u(t, \boldsymbol{x}) \in C^0(Q) \,:\, |u|_{1,0} := |u|_0 + |Du|_0 < \infty\}.$$

To see this, pick $\boldsymbol{v}$, $\widetilde{\boldsymbol{v}} \in \mathcal{B}$, and set $\boldsymbol{u} := \mathcal{H}(\boldsymbol{v})$, $\widetilde{\boldsymbol{u}} := \mathcal{H}(\widetilde{\boldsymbol{v}})$, $\boldsymbol{p} := \boldsymbol{v} - \widetilde{\boldsymbol{v}}$, and $\boldsymbol{q} := \boldsymbol{u} - \widetilde{\boldsymbol{u}}$. Subtraction then yields:

$$\begin{cases} q_t^{[i]} + \frac{a}{2}\Delta q^{[i]} + \left(f^{[i]}(t, \boldsymbol{x}, D\boldsymbol{v}) - f^{[i]}(t, \boldsymbol{x}, D\widetilde{\boldsymbol{v}})\right) = 0 \\ q^{[i]}(T, \boldsymbol{x}) = 0 \end{cases}$$

Thanks to Assumption 6.1.5, we have that:

$$\begin{aligned} \left|q^{[i]}\right| \leq & \int_t^T \int_{\mathbb{R}^n} K_a(s-t, \boldsymbol{x}-\boldsymbol{y}) \left|f^{[i]}(s, \boldsymbol{y}, D\boldsymbol{v}(s, \boldsymbol{y})) - f^{[i]}(s, \boldsymbol{y}, D\widetilde{\boldsymbol{v}}(s, \boldsymbol{y}))\right| \, d\boldsymbol{y} ds \\ \leq & \int_t^T \int_{\mathbb{R}^n} K_a(s-t, \boldsymbol{x}-\boldsymbol{y}) N_1(\sqrt{I}C_2) \left|D\boldsymbol{v}(s, \boldsymbol{y}) - D\widetilde{\boldsymbol{v}}(s, \boldsymbol{y})\right| \, d\boldsymbol{y} ds \\ \leq & T N_1(\sqrt{I}C_2) \left|D\boldsymbol{p}\right|_0 \end{aligned}$$

Furthermore, observe that

$$\begin{aligned} \left|q_{x_j}^{[i]}\right| \leq & \int_t^T \int_{\mathbb{R}^n} \tfrac{|x_j - y_j|}{a(s-t)} K_a(s-t, \boldsymbol{x}-\boldsymbol{y}) \left|f^{[i]}(s, \boldsymbol{y}, D\boldsymbol{v}(s, \boldsymbol{y})) - f^{[i]}(s, \boldsymbol{y}, D\widetilde{\boldsymbol{v}}(s, \boldsymbol{y}))\right| \, d\boldsymbol{y} ds \\ \leq & N_1(\sqrt{I}C_2) \left|D\boldsymbol{p}\right|_0 \int_t^T \int_{\mathbb{R}^n} \frac{|x_j - y_j|}{a(s-t)} K_a(s-t, \boldsymbol{x}-\boldsymbol{y}) \, d\boldsymbol{y} ds \\ \leq & N_1(\sqrt{I}C_2) \left|D\boldsymbol{p}\right|_0 \int_0^T \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi a t}} \frac{|x|}{at} \exp(-\frac{x^2}{2at}) \, dx dt \\ = & \sqrt{\frac{8T}{a\pi}} N_1(\sqrt{I}C_2) \left|D\boldsymbol{p}\right|_0 \end{aligned}$$

which yields

$$|\boldsymbol{q}|_{1,0} \leq (T \wedge \sqrt{\frac{8T}{a\pi}}) \sqrt{I} N_1(\sqrt{I}C_2) \left|\boldsymbol{p}\right|_{1,0}$$

Therefore, we have shown that $\mathcal{H} : \mathcal{B} \to \mathcal{B}$ is indeed continuous in the $\left(C^{1,0}(Q)\right)^I$ norm. We also observe that $\forall\, \boldsymbol{u} \in \mathcal{R}(\mathcal{H})$:

$$[\boldsymbol{u}]_\alpha \leq \sqrt{n}E.$$

It follows that $\mathcal{R}(\mathcal{H})$ is an equicontinuous subset in $\left(C^0(Q)\right)^I$. We apply Proposition A.4.1, to conclude that $\overline{\mathcal{R}(\mathcal{H})}$ is compact in $\left(C^0(Q)\right)^I$, where the closure is taken in the $\left(C^0(Q)\right)^I$-topology. By Lemma 6.2.2, we have $\overline{\mathcal{R}(\mathcal{H})} \subset \mathcal{B}$. Furthermore, as in the proof of Lemma 6.2.2, it is easy to see that $\overline{\mathcal{R}(\mathcal{H})}$ is also compact in $(C^{1,0}(Q))^I$.

Thus, $\mathcal{H}$ maps $\mathcal{B}$ into itself continuously. Moreover, $\mathcal{B}$ is a convex subset of $(C^{1,0}(Q))^I$, and $\mathcal{R}(\mathcal{H})$ is contained in a $(C^{1,0}(Q))^I$-compact subset of $\mathcal{B}$, i.e., in $\overline{\mathcal{R}(\mathcal{H})}$. Therefore, by Schauder's fixed point theorem, $\mathcal{H}$ has a fixed point $\boldsymbol{u} \in \mathcal{B}$. Hence, $\boldsymbol{u}$ is a solution to the PDE system (6.1.1).

For uniqueness, recall that the solution belongs to space $\mathcal{B} \subset \left(C^{1,\alpha}(Q)\right)^I \cap \left(C_v(Q)\right)^I \cap \left(H^1(Q)\right)^I$, so to show that the solution is unique there, we only need to show the solution is unique in $\left(C_v(Q)\right)^I \cap \left(H^1(Q)\right)^I$. Suppose that $\boldsymbol{u}$, $\widetilde{\boldsymbol{u}} \in \left(C_v(Q)\right)^I \cap \left(H^1(Q)\right)^I$ are two solutions to (6.1.1) and set $\boldsymbol{w} := \boldsymbol{u} - \widetilde{\boldsymbol{u}}$. Then $\boldsymbol{w}$ satisfies the following system of PDE:

$$\begin{cases} w_t^{[i]} + \frac{1}{2}\Delta w^{[i]} + f^{[i]}(t, \boldsymbol{x}, D\boldsymbol{u}) - f^{[i]}(t, \boldsymbol{x}, D\widetilde{\boldsymbol{u}}) = 0 \\ w^{[i]}(T, \boldsymbol{x}) = 0 \end{cases} \tag{6.3.1}$$

By Assumption 6.1.5, the difference $f^{[i]}(t, \boldsymbol{x}, D\boldsymbol{u}) - f^{[i]}(t, \boldsymbol{x}, D\widetilde{\boldsymbol{u}})$ satisfies:

$$\begin{aligned} \left| f^{[i]}(t, \boldsymbol{x}, D\boldsymbol{u}) - f^{[i]}(t, \boldsymbol{x}, D\widetilde{\boldsymbol{u}}) \right| &\leq N_1(\|D\boldsymbol{u}\|, \|D\widetilde{\boldsymbol{u}}\|)\|D\boldsymbol{w}\| \\ &:= N_1\|D\boldsymbol{w}\| \end{aligned} \tag{6.3.2}$$

where N_1 is a bounded constant that depends on $|D\boldsymbol{u}|_0$, $D\,|\widetilde{\boldsymbol{u}}|_0$ and N_1 (assumed to be locally bounded). We want to show that PDE system (6.3.1) admits only the trivial solution. To do so, we multiply the above equation by $w^{[i]}$ and integrate over $\mathbb{R}^n$ to obtain the following:

$$\int_{\mathbb{R}^n} w_t^{[i]} w^{[i]} \, d\boldsymbol{x} + \int_{\mathbb{R}^n} \tfrac{1}{2}\Delta w^{[i]} w^{[i]} \, d\boldsymbol{x} + \int_{\mathbb{R}^n} \left(f_{\boldsymbol{u}}^{[i]} - f_{\widetilde{\boldsymbol{u}}}^{[i]} \right) w^{[i]} \, d\boldsymbol{x} = 0$$

Then by (6.3.2), we have:

$$\int_{\mathbb{R}^n} \left(f_{\boldsymbol{u}}^{[i]} - f_{\widetilde{\boldsymbol{u}}}^{[i]} \right) w^{[i]} \, d\boldsymbol{x} \leq \int_{\mathbb{R}^n} N_1 \|D\boldsymbol{w}\| \left| w^{[i]} \right| \, d\boldsymbol{x}$$

If we insert the inequality above into the previous equation and apply integration by part, we get:

$$-\frac{d}{dt} \left| w^{[i]} \right|_{\mathbb{L}^2}^2 + \left| D w^{[i]} \right|_{\mathbb{L}^2}^2 \leq N_1 \int_{\mathbb{R}^n} \|D\boldsymbol{w}\| \left| w^{[i]} \right| \, d\boldsymbol{x}$$

Summing the inequalities above for all $1 \leq i \leq I$, we obtain:

$$\begin{aligned}
&-\frac{d}{dt}\left(\sum_{i=1}^{I} \left| w^{[i]} \right|_{\mathbb{L}^2}^2 \right) + \sum_{i=1}^{I} \left| D w^{[i]} \right|_{\mathbb{L}^2}^2 = -\frac{d}{dt} |\boldsymbol{w}|_{\mathbb{L}^2}^2 + |D\boldsymbol{w}|_{\mathbb{L}^2}^2 \\
\leq& 2N_1 \int_{\mathbb{R}^n} \|D\boldsymbol{w}\| \sum_{i=1}^{I} \left| w^{[i]} \right| \, d\boldsymbol{x} \leq \int_{\mathbb{R}^n} \|D\boldsymbol{w}\|^2 + N_1^2 \left(\sum_{i=1}^{I} \left| w^{[i]} \right| \right)^2 \, d\boldsymbol{x} \\
\leq& \int_{\mathbb{R}^n} \|D\boldsymbol{w}\|^2 + I N_1^2 \sum_{i=1}^{I} \left| w^{[i]} \right|^2 \, d\boldsymbol{x} = |D\boldsymbol{w}|_{\mathbb{L}^2}^2 + I N_1^2 \, |\boldsymbol{w}|_{\mathbb{L}^2}^2
\end{aligned}$$

It follows that

$$\begin{cases} \frac{d}{dt} \left| \boldsymbol{w}((T-t), \cdot) \right|_{\mathbb{L}^2}^2 \leq I N_1^2 \left| \boldsymbol{w}((T-t), \cdot) \right|_{\mathbb{L}^2}^2 \\ \left| \boldsymbol{w}(T, \cdot) \right|_{\mathbb{L}^2}^2 = 0 \end{cases}$$

which, in turn, implies that $\boldsymbol{w} = 0$, by Gronwall's inequality. □

Chapter 7

Summary

We have studied market equilibria in a general class of Brownian market environments with two classes of financial agents. Agents in the first class dynamically trade risky assets to maximize their terminal exponential utilities and agents in the second class try to minimize their convex risk measures through dynamical portfolio adjustment. We transformed the equilibrium problem into a non-linear parabolic PDE system with a homogeneously-quadratic non-linear structure. Then we proved the existence and uniqueness of solutions to the PDE system, thus established the existence and uniqueness of market equilibria.

To the best of our knowledge, the present work is the only example that the existence and uniqueness of market equilibria are established in a general class of fully-incomplete continuous-time Brownian market models, where both the prices and the set of replicable claims are determined as part of the equilibrium.

Financial models which do not exhibit equilibrium phenomena are generally considered "ill-posed". Therefore, the implication from the existence and uniqueness of equilibria in this class of Brownian market models is that these models are economically “well-posed”. Furthermore, the techniques we used in solving the non-linear equilibrium PDE system may be adapted or generalized to solve other non-linear PDE systems that exhibit similar non-linear structures.

The main assumption of this work is the “smallness” condition we put on the terminal random endowments, agents risk aversions, etc, in order to solve the equilibrium PDE system with a short-time solution. However, we conjecture that the “smallness” condition is, in fact, not necessary and that it is possible to prove the existence and uniqueness of global solutions to the equilibrium PDE system, and consequently, the existence and uniqueness of market equilibria, for arbitrary time horizons. The subject shall be an interesting exploration for future research.

Appendices

Appendix A

Several Technical Lemmas

A.1 On the Function h

The following lemma justifies the claim that we can assume, without loss of generality, that the function h automatically satisfies conditions (A.1.1) and (A.1.2), in addition to our main Assumption 2.3.1.

Lemma A.1.1. *Suppose that $h : \mathbb{R}^n \to \mathbb{R}^+$ is a positive decreasing radial function, i.e. that there exists a decreasing function $R : \mathbb{R}^+ \to \mathbb{R}^+$ with $h(\boldsymbol{x}) = R(\|\boldsymbol{x}\|)$, and that $h \in \mathbb{L}^1(\mathbb{R}^n)$. Then for any constant $B_0 > 1$, there exists another decreasing radial function $\widetilde{h} \in \mathbb{L}^1(\mathbb{R}^n)$, such that, $\forall\, \boldsymbol{x} \in \mathbb{R}^n$: $h(\boldsymbol{x}) \leq \widetilde{h}(\boldsymbol{x})$ and*

$$\widetilde{h}(\boldsymbol{x}) \leq B_0\, \widetilde{h}\left(\boldsymbol{x} + \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}\right). \tag{A.1.1}$$

In addition, given $\varepsilon_0 > 0$ and $D > 0$, we can further choose $\widetilde{h}$ to satisfies the following inequality for some constant $C_0 > 0$:

$$\widetilde{h}(\boldsymbol{x}) \geq \frac{1}{C_0} \exp\left(-\tfrac{\|\boldsymbol{x}\|^2}{D(1+\varepsilon_0)}\right), \quad \forall\, \boldsymbol{x} \in \mathbb{R}^n \tag{A.1.2}$$

Proof. We denote by $D_m := B_{m+1}(0) \cap B_m^C(0)$, the ring from radius m to radius

$m+1$ in $\mathbb{R}^n$. and define the function f as follows:

$$f(\boldsymbol{x}) := \sum_{m=o}^{\infty} \Omega_n \big[(m+1)^n - m^n\big] \max\big(R(m),\, R(0)B_0^{-m}\big)\, \chi_{D_m}(\boldsymbol{x})$$
$$= \sum_{m=0}^{\infty} \Omega_n \big[\sum_{k=0}^{n-1} C_n^k m^k\big] \max\big(R(m),\, R(0)B_0^{-m}\big)\, \chi_{D_m}(\boldsymbol{x})$$

where $\Omega_n := \frac{\pi^{n/2}}{\Gamma(n/2+1)}$ is a generic constant that only depends on dimension n and $R(0) := |h|_0$. Since $h \in \mathbb{L}^1(\mathbb{R}^n)$ and $B_0 > 1$, it is easy to see that $f \in \mathbb{L}^1(\mathbb{R}^n)$. Moreover, one can readily check that f satisfies the condition (A.1.1). Notice that, for any fixed constants $\varepsilon_0 > 0$ and $D > 0$, the function $\left(\sqrt{1+\|\boldsymbol{x}\|^2}\right)^{-a}$ satisfies both conditions (A.1.1) and (A.1.2), when $a > n$, for some $C_0 > 0$ and $B_0 = 3^{a/2} > 1$. Finally, we can choose $\widetilde{h}(\boldsymbol{x}) := f(\boldsymbol{x}) + \left(\sqrt{1+\|\boldsymbol{x}\|^2}\right)^{-a}$. □

The following lemma illustrates the relationship between the function h and its mollification $\phi_D(\boldsymbol{x}) := \exp\left(-\frac{\|\boldsymbol{x}\|^2}{D}\right) * h(\boldsymbol{x})$, i.e. we have the following asymptotic equivalence result:

Lemma A.1.2. *Under the same assumptions for h as in the previous lemma, we have the following results:*

1. *For any positive constant C, we have:*

$$\phi_C^2(\boldsymbol{x}) \leq |h|_{\mathbb{L}^1}\, \phi_{C/2}(\boldsymbol{x})$$

In particular, $\phi_C \in \mathbb{L}^2(\mathbb{R}^n)$.

2. *If h satisfies the condition* (A.1.1)*, then there exists a constant $M_0(B_0, D) > 0$ such that:*

$$h(\boldsymbol{x}) \leq M_0(B_0, D)\phi_D(\boldsymbol{x})$$

3. *If we further assume that h satisfies the condition* (A.1.2)*, then there exists a constant $M_1(B_0, C_0, D, H, \varepsilon_0) > 0$ such that:*

$$\phi_D(\boldsymbol{x}) \leq M_1(B_0, C_0, D, H, \varepsilon_0)h(\boldsymbol{x})$$

Proof. (1) follows from a simple application of Hölder's inequality:

$$\begin{aligned}\left(\exp\left(-\tfrac{\|\boldsymbol{x}\|^2}{C}\right) * h(\boldsymbol{x})\right)^2 &= \left(\int_{\mathbb{R}^n} \exp\left(-\frac{\|\boldsymbol{x}-\boldsymbol{y}\|^2}{C}\right) h(\boldsymbol{y})^{\frac{1}{2}} h(\boldsymbol{y})^{\frac{1}{2}}\, d\boldsymbol{y}\right)^2 \\ &\leq |h|_{\mathbb{L}^1} \exp\left(-\tfrac{\|\boldsymbol{x}\|^2}{C/2}\right) * h(\boldsymbol{x})\end{aligned}$$

To prove (2), we recall the condition (A.1.1) on the function h in Lemma A.1.1, and simply observe that for $\boldsymbol{x} \neq 0$ we have

$$\begin{aligned}\exp\left(-\tfrac{\|\boldsymbol{x}\|^2}{D}\right) * h(\boldsymbol{x}) &= \int_{\mathbb{R}^n} \exp\left(-\frac{\|\boldsymbol{x}-\boldsymbol{y}\|^2}{D}\right) h(\boldsymbol{y})\, d\boldsymbol{y} \\ &\geq \int_{B_1(x)} \exp\left(-\frac{\|\boldsymbol{x}-\boldsymbol{y}\|^2}{D}\right) h(\boldsymbol{x} + \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|})\, d\boldsymbol{y} \\ &\geq \frac{h(\boldsymbol{x})}{B_0} \int_{B_1(0)} \exp\left(-\frac{\|\boldsymbol{y}\|^2}{D}\right)\, d\boldsymbol{y} \\ &:= \frac{1}{M_{01}(B_0, D)} h(\boldsymbol{x}),\end{aligned}$$

where $M_{01}(B_0, D) := \left(\frac{1}{B_0}\int_{B_1(0)}, \exp\left(-\frac{\|\boldsymbol{y}\|^2}{D}\right)\, d\boldsymbol{y}\right)^{-1}$. Further, one can easily observe that the inequality holds when $\boldsymbol{x} = 0$, with constant $M_{02} = h(0)\left(\int_{\mathbb{R}^n} \exp\left(-\frac{\|\boldsymbol{y}\|^2}{D}\right) h(\boldsymbol{y})\, d\boldsymbol{y}\right)^{-1}$. It suffices to take $M_0 = \max(M_{01},\, M_{02})$.

Finally, to show (3), we use both condition (A.1.1) and (A.1.2), and observe the following, for all $\|\boldsymbol{x}\| \geq 1$:

$$\frac{h(\boldsymbol{x})*\exp\left(-\frac{\|\boldsymbol{x}\|^2}{D}\right)}{h(\boldsymbol{x})} = \int_{\mathbb{R}^n} \frac{h(\boldsymbol{x}-\boldsymbol{y})}{h(\boldsymbol{x})} \exp\left(-\frac{\|\boldsymbol{y}\|^2}{D}\right) d\boldsymbol{y}$$

$$= \int_{B_{\|\boldsymbol{x}\|}} \frac{h(\boldsymbol{x}-\boldsymbol{y})}{h(\boldsymbol{x})} \exp\left(-\frac{\|\boldsymbol{y}\|^2}{D}\right) d\boldsymbol{y} + \int_{B^C_{\|\boldsymbol{x}\|}} \frac{h(\boldsymbol{x}-\boldsymbol{y})}{h(\boldsymbol{x})} \exp\left(-\frac{\|\boldsymbol{y}\|^2}{D}\right) d\boldsymbol{y}$$

$$\leq \int_{\mathbb{R}^n} \frac{B_0^{\|\boldsymbol{y}\|+1} h(\boldsymbol{x})}{h(\boldsymbol{x})} \exp\left(-\frac{\|\boldsymbol{y}\|^2}{D}\right) d\boldsymbol{y} + \int_{B^C_{\|\boldsymbol{x}\|}} \frac{|h|_0}{h(\boldsymbol{x})} \exp\left(-\frac{\|\boldsymbol{y}\|^2}{D}\right) d\boldsymbol{y}$$

$$\leq \int_{\mathbb{R}^n} \exp\left(-\frac{\|\boldsymbol{y}\|^2}{D} + \ln B_0\|\boldsymbol{y}\| + \ln B_0\right) d\boldsymbol{y} + \frac{|h|_0\omega_n}{h(\boldsymbol{x})} \int_{\|\boldsymbol{x}\|}^{\infty} \exp\left(\frac{-r^2}{D}\right) r^{n-1} dr$$

$$\leq \int_{\mathbb{R}^n} \exp\left(-\frac{\|\boldsymbol{y}\|^2}{D} + \ln B_0\|\boldsymbol{y}\| + \ln B_0\right) d\boldsymbol{y} + \frac{|h|_0\widetilde{C}(\varepsilon_0, D)\omega_n}{h(\boldsymbol{x})} \int_{\|\boldsymbol{x}\|}^{\infty} \exp\left(\frac{-r^2}{D(1+\varepsilon_0)}\right) 2r dr$$

$$\leq \int_{\mathbb{R}^n} \exp\left(-\frac{\|\boldsymbol{y}\|^2}{D} + \ln B_0\|\boldsymbol{y}\| + \ln B_0\right) d\boldsymbol{y} + \frac{|h|_0\widetilde{C}D(1+\varepsilon_0)\omega_n}{h(\boldsymbol{x})} \exp\left(-\frac{\|\boldsymbol{x}\|^2}{D(1+\varepsilon_0)}\right)$$

$$\leq \int_{\mathbb{R}^n} \exp\left(-\frac{\|\boldsymbol{y}\|^2}{D} + \ln B_0\|\boldsymbol{y}\| + \ln B_0\right) d\boldsymbol{y} + |h|_0\, C_0\widetilde{C}(\varepsilon_0, D) D\omega_n(1+\varepsilon_0)$$

$$:= M_{11}(B_0, C_0, D, H, \varepsilon_0) < \infty$$

where:

$$M_{11} := \int_{\mathbb{R}^n} \exp\left(-\frac{\|\boldsymbol{y}\|^2}{D} + \ln B_0\|\boldsymbol{y}\| + \ln B_0\right) d\boldsymbol{y} + |h|_0\, C_0\widetilde{C}(\varepsilon_0, D) D\omega_n(1+\varepsilon_0)$$

The inequality now holds for $\|\boldsymbol{x}\| < 1$, with the constant:

$$M_{12} = \max_{\boldsymbol{x}\in B_1} \left(h(\boldsymbol{x}) * \exp\left(-\frac{\|\boldsymbol{x}\|^2}{D}\right)\right) / h(\boldsymbol{x}).$$

It remains to set $M_1 := \max(M_{11},\, M_{12})$.

□

A.2 A Few Convolution-Related Computations

In this section, we state and prove several useful computational results involving convolution. These results are used in Chapter 6.

Lemma A.2.1. *For constants C, $D > 0$, and $\boldsymbol{x} \in \mathbb{R}^n$, we have:*

$$\exp\left(-\tfrac{\|\boldsymbol{x}\|^2}{C}\right) * \phi_D(\boldsymbol{x}) = \left(\frac{CD\pi}{C+D}\right)^{n/2} \phi_{C+D}(\boldsymbol{x})$$

Proof. This follows by direct computation. □

Lemma A.2.2. *For constant $b > 0$, the following inequality holds for all $\varepsilon > 0$, $t \in \mathbb{R}$:*

$$|t| \exp\left(-\frac{t^2}{b}\right) \leq \sqrt{\frac{b(b+\varepsilon)}{2e\varepsilon}} \exp\left(-\frac{t^2}{b+\varepsilon}\right) \tag{A.2.1}$$

In particular, for $\varepsilon = b/2$, we have:

$$|t| \exp\left(-\frac{t^2}{b}\right) \leq \sqrt{\frac{3b}{2e}} \exp\left(-\frac{t^2}{3b/2}\right)$$

Proof. We divide the left hand side of the equation (A.2.1) by its right hand side and observe that the maximum of the obtained expression is 1. □

Using the notation above, we obtain the following, useful, consequence of Lemma A.2.2:

Corollary A.2.3. $|x|_i \exp\left(-\frac{\|\boldsymbol{x}\|^2}{C}\right) * \phi_D(\boldsymbol{x}) \le \frac{3\sqrt{C}}{2\sqrt{e}} \left(\frac{CD\pi}{C+D}\right)^{n/2} \phi_{3C/2+D}(\boldsymbol{x})$

Proof. By Lemmas A.2.2 and A.2.1:

$$\begin{aligned}
\left(|t| \exp\left(-\frac{t^2}{C}\right)\right) * \exp\left(-\frac{t^2}{D}\right) \le \sqrt{\frac{3C}{2e}} \exp\left(-\frac{t^2}{3C/2}\right) * \exp\left(-\frac{t^2}{D}\right) \\
=& \sqrt{\frac{3C}{2e}} \left(\frac{3CD\pi}{3C+2D}\right)^{1/2} \exp\left(-\frac{t^2}{3C/2+D}\right) \\
\le& \frac{3\sqrt{C}}{2\sqrt{e}} \sqrt{\frac{CD\pi}{C+D}} \exp\left(-\frac{t^2}{3C/2+D}\right)
\end{aligned}$$

□

A.3 On The Penalty Function f

Proposition A.3.1. *If the function $f = f_1 + f_2$ satisfies Assumption 2.3.3, then we have the following properties:*

1. *the conjugate $\widetilde{f}_2(\boldsymbol{x})$ of $f_2(\boldsymbol{x})$ satisfies the following growth condition:*

$$\frac{1}{4L_2}\|\boldsymbol{x}\|^2 \le \widetilde{f}_2(\boldsymbol{x}) \le \frac{1}{4L_1}\|\boldsymbol{x}\|^2$$

2. *$\widetilde{f}(\boldsymbol{x})$ is strictly convex and differentiable.*

3. *The gradient $Df(\boldsymbol{x})$ of $f(\boldsymbol{x})$ is one-to-one from $\mathbb{R}^n$ to $\mathbb{R}^n$. Furthermore, for $i = 1, 2$, we have $Df_i(\boldsymbol{x}) = \left(D\widetilde{f}_i(\boldsymbol{x})\right)^{-1}$.*

Proof. The function $\boldsymbol{x} \mapsto \frac{1}{2}\|\boldsymbol{x}\|^2$ is invariant under the conjugate transform. We can use that fact to help a direct computation which yields 1. above, directly. Parts 2. and 3. follow directly from Proposition B.2.4 in [22]. □

A.4 A Version of the Arzelá-Ascoli Theorem

The following version of the Arzelá-Ascoli theorem is needed in Chapter 6:

Proposition A.4.1. *Let $C(Q) := C(Q; \mathbb{R}^I)$ be the Banach space of continuous vector functions from Q to $\mathbb{R}^I$, with the following norm:*

$$|f|_{C(Q)} := \left(\sum_{i=1}^{I} \left| f^{[i]} \right|_0^2 \right)^{1/2}, \text{ for } f = (f^{[1]}, \ldots, f^{[I]}). \tag{A.4.1}$$

Let $A \subset C(Q)$ be a subset that is equicontinuous, i.e: for $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$, such that:

$$\sup_{f \in A} \max_{d_p\left((t,\boldsymbol{x}),(s,\boldsymbol{y})\right) \leq \delta} \| f(t, \boldsymbol{x}) - f(s, \boldsymbol{y}) \| \leq \varepsilon \tag{A.4.2}$$

Furthermore, if there exists a positive bounded function $g = (g^{[1]}, \ldots, g^{[I]}) \in C(Q)$, which vanishes at infinity, i.e.

$$\lim_{\boldsymbol{x} \to \infty} \left(\sup_{t \in [0,T]} g^{[i]}(t, \boldsymbol{x}) \right) = 0, \ \forall\, i = 1, \ldots, I$$

such that:

$$\left| f^{[i]} \right| \leq g^{[i]}; \quad \forall\, (t, \boldsymbol{x}) \in Q, \text{ and } \forall\, i = 1, \ldots, I$$

Then A is relatively compact.

Proof. Since $C(Q)$ is a metric space, it suffices to show that an arbitrary sequence $\{\boldsymbol{f}_n\}_{n \in \mathbb{N}} = \{(f_n^{[1]}, \ldots, f_n^{[I]})\}_{n \in \mathbb{N}} \subset A$ has a convergent subsequence. We start by looking at the first-component sequence $\{f_n^{[1]}\}_{n \in \mathbb{N}}$.

For all $\varepsilon > 0$, there exists $M \geq 0$, such that $\left| g^{[1]}(t, \boldsymbol{x}) \right| \leq \varepsilon$, for $\forall\, (t, \boldsymbol{x}) \in Q$ with $d_p(0, (t, \boldsymbol{x})) \geq M$. Let $\{(t_m, \boldsymbol{x}_m)\}_{m \in \mathbb{N}}$ be a dense set in $B_M(0) :=$

$\{(t,\boldsymbol{x}) \in Q \,:\, d_p(0,(t,\boldsymbol{x})) \le M\}$. For each $(t_m,\boldsymbol{x}_m)$ fixed, $\{f_n^{[1]}(t_m,\boldsymbol{x}_m)\}_{n\in\mathbb{N}}$ is bounded by $\left|g^{[1]}\right|_0$, so it has a convergent subsequence. By diagonal argument, we can find a subsequence, denoted by $\{f_{n_k}^{[1]}\}_{k\in\mathbb{N}}$, which is convergent for all $(t_m,\boldsymbol{x}_m)$ in the dense set.

Furthermore, by equicontinuity, there exists $\delta > 0$ corresponding to ε via (A.4.2), and there exists a finite subset $\{(t_m,\boldsymbol{x}_m)\}_{m=1}^{N(\varepsilon)} \subset \{(t_m,\boldsymbol{x}_m)\}_{l\in\mathbb{N}}$, such that:

$$\bigcup_{m=1}^{N(\varepsilon)} B_\delta((t_m,\boldsymbol{x}_m)) \supset B_M(0)$$

since $B_M(0)$ is compact. In addition, there exist an integer $L \in \mathbb{N}$ big enough, such that for all integers $k_1,\, k_2 \ge L$, we have:

$$\max_{1\le m\le N(\varepsilon)} \left| f_{n_{k_1}}^{[1]}(t_m,\boldsymbol{x}_m) - f_{n_{k_2}}^{[1]}(t_m,\boldsymbol{x}_m) \right| \le \varepsilon.$$

Now for any fixed $(t,\boldsymbol{x}) \in Q$, if $d_p(0,(t,\boldsymbol{x})) \le M$, one can choose $(t_{m_0},\boldsymbol{x}_{m_0}) \in \{(t_m,\boldsymbol{x}_m)\}_{m=1}^{N(\varepsilon)}$, such that:

$$d_p\left((t,\boldsymbol{x}),(t_{m_0},\boldsymbol{x}_{m_0})\right) \le \delta$$

Then for any $k_1,\, k_2 \ge L$, by $(**)$, we have:

$$\begin{aligned}\left| f_{n_{k_1}}^{[1]}(t,\boldsymbol{x}) - f_{n_{k_2}}^{[1]}(t,\boldsymbol{x}) \right| &\le \left| f_{n_{k_1}}^{[1]}(t,\boldsymbol{x}) - f_{n_{k_1}}^{[1]}(t_{m_0},\boldsymbol{x}_{m_0}) \right| \\ &\quad + \left| f_{n_{k_1}}^{[1]}(t_{m_0},\boldsymbol{x}_{m_0}) - f_{n_{k_2}}^{[1]}(t_{m_0},\boldsymbol{x}_{m_0}) \right| \\ &\quad + \left| f_{n_{k_2}}^{[1]}(t_{m_0},\boldsymbol{x}_{m_0}) - f_{n_{k_2}}^{[1]}(t,\boldsymbol{x}) \right| \\ &\le 2\varepsilon + \max_{1\le m\le N(\varepsilon)} \left| f_{n_{k_1}}^{[1]}(t_m,\boldsymbol{x}_m) - f_{n_{k_2}}^{[1]}(t_m,\boldsymbol{x}_m) \right| \le 3\varepsilon\end{aligned}$$

On the other hand, if for the fixed $(t,\boldsymbol{x})$, $d_p(0,(t,\boldsymbol{x})) > M$, then:

$$\left| f_{n_{k_1}}^{[1]}(t,\boldsymbol{x}) - f_{n_{k_2}}^{[1]}(t,\boldsymbol{x}) \right| \le 2g^{[1]}(t,\boldsymbol{x}) \le 2\varepsilon$$

We have shown that $\{f_{n_k}^{[1]}\}_{k\in\mathbb{N}}$ is uniformly Cauchy, thus it converges uniformly. In the same way, we can choose a subsequence for $\{f_n^{[2]}\}_{n\in\mathbb{N}}$ from $\{f_{n_k}^{[2]}\}_{k\in\mathbb{N}}$ such that it is uniformly convergent. If we repeat this process, in the end, we will find a subsequence of $\{\boldsymbol{f}_n\}_{n\in\mathbb{N}}$, denoted by $\{\boldsymbol{f}_{n_r}\}_{r\in\mathbb{N}} = \{(f_{n_r}^{[1]}, \ldots, f_{n_r}^{[I]})\}_{r\in\mathbb{N}}$, such that, $\{f_{n_r}^{[i]}\}_{r\in\mathbb{N}}$ converges uniformly for each $i = 1, \ldots, I$. Thus $\{\boldsymbol{f}_{n_r}\}_{r\in\mathbb{N}}$ converges in the $C(Q)$ norm, $|\cdot|_{C(Q)}$, defined previously by (A.4.1). $\square$

Appendix B

Anisotropic Hölder Spaces

As already observed in [26], classical Hölder spaces provide a convenient environment for problems related to stability, and consequently, equilibrium in financial markets. This appendix provides a short overview of the notation, some basic definitions and some well-known results.

B.1 Classical Anisotropic Hölder Spaces.

We fix $n \in \mathbb{N}$ and let $Q = [0, T] \times \mathbb{R}^n$ denote our space-time. Let $C^0(Q)$ be the class of all bounded continuous functions $u : Q \to \mathbb{R}$ which is a Banach space under the norm

$$|u|_0 = \sup_{(t,\boldsymbol{x}) \in Q} |u(t, \boldsymbol{x})| \, .$$

It pays to re-metrize Q using the so-called **parabolic metric** $d_p : Q \times Q \to [0, \infty)$, defined by

$$d_p\Big((t_1, \boldsymbol{x}_1), (t_2, \boldsymbol{x}_2)\Big) = \sqrt{|t_2 - t_1|} + |\boldsymbol{x}_2 - \boldsymbol{x}_1| \, ,$$

for $(t_i, \boldsymbol{x}_i) \in Q$, $i = 1, 2$, where $|\cdot|$ denotes the Euclidean distance on $\mathbb{R}^n$.

For a function $u \in C^0(Q)$ and $\alpha \in (0, 1]$, we define the **α-Hölder**

constant $[u]_\alpha \in [0,\infty]$ by:

$$[u]_\alpha := \sup_{(t_1,\boldsymbol{x}_1)\neq(t_2,\boldsymbol{x}_2)\in Q} \frac{|u(t_2,\boldsymbol{x}_2)-u(t_1,\boldsymbol{x}_1)|}{\Big(d_p\big((t_1,\boldsymbol{x}_1),(t_2,\boldsymbol{x}_2)\big)\Big)^\alpha}$$

The functional $|\cdot|_\alpha$ given by:

$$|u|_\alpha = |u|_0 + [u]_\alpha$$

is a Banach norm on the space $C^{0,\alpha}(Q)$ of all functions $u \in C^0(Q)$ with $[u]_\alpha < \infty$.

We use the "analyst's" notation

$$Du = (u_{x_1},\ldots,u_{x_n})^\tau,\; D^2u = [u_{x_j x_k}]_{j=1,\ldots,n}^{k=1,\ldots,n},\; \Delta u = \operatorname{Tr} D^2 u = \sum_{j=1}^n u_{x_j x_j},$$

for spatial partial derivatives of sufficiently regular functions on Q, where ^{T} denotes transposition and $[\cdot]_{j=1,\ldots,n}^{k=1,\ldots,n}$ denotes an $n\times n$-matrix. The set of all functions $u \in C^0(Q)$ such that all components of Du are in $C^{0,\alpha}(Q)$ is denoted by $C^{1,\alpha}(Q)$, and we can turn it into a Banach space by adjoining to it the norm $|u|_{1,\alpha}$, defined by

$$|u|_{1,\alpha} = |u|_0 + |Du|_0 + [u]_{1,\alpha},$$

where

$$|Du|_0 = \sum_{j=1}^n \left|u_{x_j}\right|_0 \text{ and } [u]_{1,\alpha} = \sum_{j=1}^n \left[u_{x_j}\right]_\alpha.$$

Similarly, the space $C^{2,\alpha}(Q)$ contains all functions in $C^0(Q)$ all of whose first and second spatial *and the first temporal* partial derivatives exist and belong to the space $C^{0,\alpha}(Q)$. The Banach norm there is given by

$$|u|_{2,\alpha} = |u|_0 + |u_t|_0 + |Du|_0 + \left|D^2u\right|_0 + [u]_{2,\alpha},$$

where

$$\left|D^2 u\right|_0 = \sum_{j,k=1}^{n} \left|u_{x_j x_k}\right|_0 \text{ and } [u]_{2,\alpha} = [u_t]_\alpha + \sum_{j,k=1}^{n} \left[u_{x_j x_k}\right]_\alpha .$$

Lastly, the Banach space $C^{1,0}(Q)$ will also be used, and it is defined as follows:

$$C^{1,0}(Q) := \{u \in C^0(Q) \,:\, |u|_{1,0} := |u|_0 + |Du|_0 < \infty\}.$$

B.2 Isotropic Hölder Spaces.

Analogously, we can define Hölder spaces in the **isotropic** setting, i.e., for functions depending only on the spatial variables. $C^0(\mathbb{R}^n)$ denotes the class of all bounded continuous functions $u : \mathbb{R}^n \to \mathbb{R}$; it is a a Banach space under the norm

$$|u|_0 := \sup_{\boldsymbol{x} \in \mathbb{R}^n} |u(\boldsymbol{x})| \, .$$

We overload the notation and, for a function $u \in C^0(\mathbb{R}^n)$ and a $\alpha \in (0,1]$, we define the α**-Hölder constant** $[u]_\alpha \in [0,\infty]$ by:

$$[u]_\alpha := \sup_{\boldsymbol{x}_1 \neq \boldsymbol{x}_2 \in \mathbb{R}^n} \frac{|u(\boldsymbol{x}_2) - u(\boldsymbol{x}_1)|}{|\boldsymbol{x}_2 - \boldsymbol{x}_1|^\alpha}$$

The functional $|\cdot|_\alpha$ given by: $|u|_\alpha = |u|_0 + [u]_\alpha$ is a Banach norm on the space $C^{0,\alpha}(\mathbb{R}^n)$ of all functions $u \in C^0(\mathbb{R}^n)$ with $[u]_\alpha < \infty$. The set of all functions $u \in C^0(\mathbb{R}^n)$ such that all components of Du are in $C^{0,\alpha}(\mathbb{R}^n)$ is denoted by $C^{1,\alpha}(\mathbb{R}^n)$, and we can turn it into a Banach space by adjoining to it the norm $|u|_{1,\alpha}$, defined by

$$|u|_{1,\alpha} = |u|_0 + |Du|_0 + [u]_{1,\alpha} \, ,$$

where

$$\left|Du\right|_0 = \sum_{j=1}^{n} \left|u_{x_j}\right|_0 \ \text{and} \ [u]_{1,\alpha} = \sum_{j=1}^{n} \left[u_{x_j}\right]_\alpha .$$

Similarly, the space $C^{2,\alpha}(\mathbb{R}^n)$ contains all functions in $C^0(\mathbb{R}^n)$ all of whose first and second spatial partial derivatives exist and belong to the space $C^{0,\alpha}(\mathbb{R}^n)$. The Banach norm there is given by

$$|u|_{2,\alpha} = |u|_0 + |Du|_0 + \left|D^2u\right|_0 + [u]_{2,\alpha} ,$$

where

$$\left|D^2u\right|_0 = \sum_{j,k=1}^{n} \left|u_{x_j x_k}\right|_0 \ \text{and} \ [u]_{2,\alpha} = \sum_{j,k=1}^{n} \left[u_{x_j x_k}\right]_\alpha .$$

B.3 Functions that Vanish at Infinity

Furthermore, we shall use function space $C_v(Q)$ in our main results. As a subspace of $C^0(Q)$, it is defined by the following:

$$C_v(Q) := \{u \in C^0(Q) \ : \lim_{d_p\left((t,\boldsymbol{x}),0\right)\to\infty} u(t,\boldsymbol{x}) = 0\} \tag{B.3.1}$$

B.4 The Sobolev Space $H^1(Q)$

The Sobolev space, $H^1(Q)$, which we will define as:

$$H^1(Q) := \{u(t,\boldsymbol{x}) \ : \ \forall\, t \in [0,T], \int_{\mathbb{R}^n} \left(|u|^2 + \|Du\|^2\right) d\boldsymbol{x} < \infty\} \tag{B.4.1}$$

will also be used.

B.5 Interpolation Inequalities

We rephrase (and minimally adjust) the statements of the following well-known results about the anisotropic Hölder spaces.

Theorem B.5.1 (Parabolic interpolation - additive form - [19] Theorem 8.8.1, p. 124.)**.** *There exist a constant $N = N(n, T) > 0$, such that for any $\epsilon > 0$, and $u \in C^{2,\alpha}(Q)$ we have:*

$$[u]_\alpha \leq \epsilon\,[u]_{2,\alpha} + N\epsilon^{-\alpha/2}\,|u|_0$$

$$|Du|_0 \leq \epsilon\,[u]_{1,\alpha} + N\epsilon^{-1/(1+\alpha)}\,|u|_0$$

$$[Du]_\alpha \leq \epsilon\,[u]_{1,\alpha} + N\epsilon^{-(1+\alpha)}\,|u|_0$$

Theorem B.5.2 (Parabolic interpolation - multiplicative form - [19], Exercise 8.8.2, p. 125.)**.** *There exist a constant $N = N(n) > 0$, such that for any $u \in C^{2,\alpha}(Q)$ we have:*

$$[u]_\alpha^{2+\alpha} \leq N\,[u]_{2,\alpha}^{\alpha}\,|u|_0^2$$

$$|Du|_0^{2+\alpha} \leq N\,[u]_{2,\alpha}\,|u|_0^{1+\alpha}$$

$$[Du]_\alpha^{2+\alpha} \leq N\,[u]_{2,\alpha}^{1+\alpha}\,|u|_0$$

Proposition B.5.3 (Parabolic interpolation - Hölder embedding)**.** *For any $0 < \alpha < \beta \leq 1$, there exist a constant N such that, for any $u \in C^{0,\alpha}(Q)$, we have:*

$$[u]_\alpha \leq N\,|u|_0^{1-\alpha/\beta}\,[u]_\beta^{\alpha/\beta}$$

Proof. It follows immediately from the following observation:

$$\left(\frac{|u(t,\boldsymbol{x})-u(s,\boldsymbol{y})|}{\Big(d_p\big((t,\boldsymbol{x}),(s,\boldsymbol{y})\big)\Big)^{\alpha}}\right)^{\beta/\alpha} = \frac{|u(t,\boldsymbol{x})-u(s,\boldsymbol{y})|}{\Big(d_p\big((t,\boldsymbol{x}),(s,\boldsymbol{y})\big)\Big)^{\beta}}\,|u(t,\boldsymbol{x})-u(s,\boldsymbol{y})|^{\beta/\alpha-1}$$
$$\leq 2^{\beta/\alpha-1}\,|u|_0^{\beta/\alpha-1}\,[u]_\beta$$

□

B.6 A Linear Cauchy Problem

Theorem B.6.1. *Let $g \in C^{2,\alpha}(Q)$, $f \in C^{0,\alpha}(Q)$ and $h \in \left(C^{0,\alpha}(Q)\right)^I$, with $|h|_\alpha \leq K$, then the following PDE has unique solution $u \in C^{2,\alpha}(Q)$:*

$$\begin{cases} u_t + \frac{1}{2}\Delta u + h^T Du + f = 0 \\ u(T,\boldsymbol{x}) = g(\boldsymbol{x}) \end{cases}$$

Furthermore, we have the following estimate:

$$|u|_{2,\alpha} \leq N_0(Q,\alpha,T,K)(|g|_{2,\alpha} + |f|_\alpha)$$

Proof. This is a direct corollary from [19] Exercises 9.1.3, p.140 and [26] Lemma 3.1. □

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Vita

Yingwu Zhao was born in Handan, Hebei Province in China in 1983, the son of Mr. Runxiang Zhao and Mrs. Xiuping Hao. In 2001, he entered Sichuan University, in Chengdu, where he studied Mathematics and got his Bachelor of Science degree in 2005. In August of the same year, he was admitted into the Graduate School of University of Texas at Austin for the PhD program of Mathematics, and later joined the research group of Mathematical Finance in Department of Mathematics. He is expecting to complete the degree of Doctor of Philosophy in Mathematics from The University of Texas at Austin in May of 2012.

Email: yzhao@math.utexas.edu

This dissertation was typeset with LaTeX[†] by the author.

[†]LaTeX is a document preparation system developed by Leslie Lamport as a special version of Donald Knuth's TeX Program.