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RANK GRADIENT IN CO-FINAL TOWERS OF CERTAIN KLEINIAN GROUPS

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Dedicated to my wife Ivy, and my sons, Murilo and Alvaro.

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RANK GRADIENT IN CO-FINAL TOWERS OF CERTAIN KLEINIAN GROUPS

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This dissertation provides the first known examples of finite co-volume Kleinian groups which have co-final towers of finite index subgroups with positive rank gradient. We prove that if the fundamental group of an orientable finite volume hyperbolic 3-manifold has finite index in the reflection group of a right-angled ideal polyhedron in $\mathbb{H}^3$ then it has a co-final tower of finite sheeted covers with positive rank gradient. The manifolds we provide are also known to have co-final towers of covers with zero rank gradient. We also prove that the reflection groups of compact right-angled hyperbolic polyhedra satisfying mild conditions have co-final towers of finite sheeted covers with positive rank gradient.

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Chapter 1

Introduction

Let G be a finitely generated group. The *rank* of G is the minimal cardinality of a generating set, and is denoted by $\mathrm{rk}(G)$. If G_j is a finite index subgroup of G, the Reidemeister-Schreier process ([LS]) gives an upper bound on the rank of G_j.

$$\mathrm{rk}(G_j) - 1 \leq [G : G_j](\mathrm{rk}(G) - 1)$$

Given a finitely generated group G and a collection $\{G_j\}$ of finite index subgroups, the *rank gradient* of the pair $(G, \{G_j\})$ is defined by (see [La1])

$$\mathrm{rgr}(G, \{G_j\}) = \lim_{j\to\infty} \frac{\mathrm{rk}(G_j) - 1}{[G : G_j]}$$

We say that the collection of finite index subgroups $\{G_j\}$ is *co-final* if $\cap_j G_j = \{1\}$, and we call it a *tower* if $G_{j+1} < G_j$. Note that if $\{G_j\}$ is a tower then the sequence $\left\{\dfrac{\mathrm{rk}(G_j) - 1}{[G : G_j]}\right\}$ is non-increasing and therefore converges.

An important line of research in low dimensional topology is the study of the behavior of the topology, geometry and algebra of the finite sheeted covers of a finite volume hyperbolic 3-manifold M. One problem of particular interest is the behavior of the rank of the fundamental groups of the finite sheeted covers of such manifolds. In some particular cases it is easy to determine the rank gradient for families of finite covers, but in genaral this is a very hard problem. For instance

Question 1. *Does there exist an orientable finite volume hyperbolic 3-manifold M with a co-final family of covers $\{M_j\}$ such that* $\operatorname{rgr}(\pi_1(M_1), \pi_1(M_j)) > 0$*?*

This is the main focus of this thesis. In Chapters 2 and 3 we provide what seems to be the first examples of orientable finite volume hyperbolic 3-manifolds which have co-final towers of finite sheeted covers with positive rank gradient. The manifolds we provide are those whose fundamental group have finite index in the group of reflections of certain hyperbolic right-angled polyhedra. In Chapter 4 we relate our results to other outstanding problems in 3-manifold topology.

1.1 Preliminary material

1.1.1 Examples for rank gradient

In some particular cases it is easy to determine rank gradient, for example:

Example 1. When G is a free group, the rank gradient of any pair $(G, \{G_j\})$ is positive. This follows since the Reidemeister-Schreier process produces an equality for free groups.

Example 2. The same is true if G is the fundamental group of a closed surface S with $\chi(S) < 0$. Let S be a surface with genus g such that $\chi(S) = 2-2g < 0$. Note that $\operatorname{rk}(\pi_1(S)) = 2g$. Let $S' \longrightarrow S$ a covering of degree d and assume

S' has genus g' so that $\mathrm{rk}(\pi_1(S')) = 2g'$. From $\chi(S') = d \cdot \chi(S)$ we get $2g' = d(2g) - 2d + 2$. Since $2g > 2$ we see that the number of generators of the fundamental groups of such surfaces grows linearly with their genera.

Example 3. If $\phi : G \twoheadrightarrow F_2$, where F_2 is the free group on two generators then, using example 1, one can find a families of subgroups with positive rank gradient. These familes are given by the finite index subgroups $G_n < G$ such that $\phi|_{G_n} : G_n \twoheadrightarrow F_n$, where F_n is the free group on n letters. We remark that these families are not co-final, as $\ker(\phi)$ is a subgroup of each G_n.

Example 4. A group G is called virtually abelian if it has a finite index abelian subgroup. Let $H < G$ be a finite index abelian subgroup of rank h. If $\{H_i\}$ is a tower of finite index subgroups of H, then $\mathrm{rk}(H_i) \leq h$. It is then easy to see that the pair $(G, \{H_i\})$ has zero rank gradient.

Example 5. A 3-manifold M is called virtually fibered if it has a fibered finite sheeted cover M'. This means that

$$M' \cong \frac{S \times [0,1]}{(x,0) \sim (\phi(x),1)}$$

where S is the fiber and $\phi : S \longrightarrow S$ is an orientation preserving homeomorphism. We see that $\pi_1(M')$ is a HNN-extension of $\pi_1(S)$ and thus $\mathrm{rk}(\pi_1(M')) \leq \mathrm{rk}(\pi_1(S)) + 1$. Choosing $\{M_i' \longrightarrow M'\}$ to be a tower of covers dual to the fiber S, i.e., surface bundles associated to ϕ^{p_i}, we see that $\mathrm{rk}(\pi_1(M')) \leq \mathrm{rk}(\pi_1(S)) + 1$ as well. Therefore the pair $(\pi_1(M), \{\pi_1(Mi)\})$ has zero rank gradient.

Example 6. For each $k \in \mathbb{Z}$, consider the reduction map $\mathrm{SL}(n,\mathbb{Z}) \longrightarrow \mathrm{SL}(n,\mathbb{Z}/k\mathbb{Z})$. The kernel of this map is a *congruence subgroup* of $\mathrm{SL}(n,\mathbb{Z})$.

When $n > 2$ then $\mathrm{SL}(n, \mathbb{Z})$, has zero rank gradient with respect to towers of congruence subgroups (J. Tits [Ti]).

1.1.2 Hyperbolic geometry

We review some basic facts and terminology from hyperbolic geometry. For more details about hyperbolic space and its isometries see [Ra].

Hyperbolic space

The hyperbolic space is defined by

$$\mathbb{H}^3 = \{(x, y, z) \in \mathbb{R}^3 | z > 0\}$$

with metric

$$ds^2 = \frac{dx^2 + dy^2 + dz^2}{z^2}$$

We will also often use the *Poincaré conformal ball model* for $\mathbb{H}^3$. This is useful for visualizing hyperbolic polyhedra. It is defined as

$$\mathbb{B}^3 = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 < 1\}$$

with metric

$$ds^2 = 4\frac{dw^2}{(1 - w^2)^2}$$

where $w^2 = x^2 + y^2 + z^2$ and $dw^2 = dx^2 + dy^2 + dz^2$.

With this metric, the geodesic planes in $\mathbb{H}^3$ correspond to vertical Euclidean planes and hemispheres in $\{z > 0\}$ perpendicular to the plane $\{z = 0\}$.

Geodesic lines correspond to the intersection of two geodesic planes, i.e., vertical lines and semi-circles perpendicular to $\{z = 0\}$.

In the conformal ball model $\mathbb{B}^3$, planes correspond to the intersection of $\mathbb{B}^3$ with spheres and planes in $\mathbb{R}^3$ perpendicular to $\partial\mathbb{B}^3$. Geodesic lines are obtained as the intersection of geodesic planes. Therefore they correspond to circular arcs and lines perpendicular to $\partial\mathbb{B}^3$.

The group of orientation preserving isometries of $\mathbb{H}^3$ can be identified to $PSL_2(\mathbb{C})$. The elements $\gamma \in PSL_2(\mathbb{C})$ are classified as *elliptic, parabolic* or *hyperbolic* according to the traces of their lifts to $\mathrm{SL}_2(\mathbb{C})$. Denote a lift of γ to $\mathrm{SL}_2(\mathbb{C})$ by γ': γ is elliptic if $|tr(\gamma')| < 2$, parabolic if $|tr(\gamma')| = 2$ and hyperbolic if $|tr(\gamma')| > 2$.

1.1.3 Discrete groups and fundamental domains

Here we discuss some basic facts about discrete groups of isometries of $\mathbb{H}^3$. We remark that all the theorems below hold for hyperbolic, Euclidean or spherical spaces of any dimension. For a more detailed treatment of discrete groups refer to [Ra].

Definition 1. A *discrete group* is a topological group Γ whose points are open.

Definition 2. A group G acts *discontinuously* on $\mathbb{H}^3$ if and only if G acts on $\mathbb{H}^3$ and for each compact subset K of $\mathbb{H}^3$, the set $K \cap gK$ is nonempty for only finitely many g in G.

The main point of these definitions is that in our context they are equivalent (see [Ra]):

Theorem 1.1.1. *A group Γ of isometries of $\mathbb{H}^3$ is discrete if and only if Γ acts discontinuously on $\mathbb{H}^3$.*

A subset R of $\mathbb{H}^3$ is called a *fundamental region* for a group Γ of isometries of $\mathbb{H}^3$ if

(1) the set R is open;

(2) the members of $\{gR|g \in \Gamma\}$ are mutually disjoint; and

(3) $\mathbb{H}^3 = \cup\{g\bar{R}|g \in \Gamma\}$.

When Γ is a discrete group of isometries of $\mathbb{H}^3$, a *convex fundamental polyhedron* for Γ is a convex polyhedron P in $\mathbb{H}^3$ whose interior is a fundamental domain for Γ. P is called *exact* if for each side S of P there is an element g of Γ such that $S = P \cap gP$. It is known that every discrete group Γ has an exact convex fundamental polyhedron. The element g is called a *side-pairing.*

The main result we need concerning convex fundamental polyhedra for discrete groups is (see [Ra])

Theorem 1.1.2. *Let S be a side of an exact convex polyhedron for a discrete group Γ of isometries of $\mathbb{H}^3$. Then there is a unique element $g \in \Gamma$ such that $S = P \cap gP$. Moreover, $g^{-1}S$ is also a side of P and Γ is generated by the set*

$$\Phi = \{g \in \Gamma | P \cap gP, \ \textit{is a side of} \ \ P\}$$

1.1.4 Reflection groups

We now discuss refletion groups of convex polyhedra in $\mathbb{H}^3$. Again, all the results below hold for hyperbolic, Euclidean or spherical spaces of any dimension. For a more details on reflections groups please refer to [Ra].

Let S be a side of an n-dimensional convex polyhedron P in $\mathbb{H}^3$. The reflection of $\mathbb{H}^3$ in the side S is the reflection of $\mathbb{H}^3$ in the hyperplane $\langle S\rangle$ spanned by S. The group G generated by reflections of $\mathbb{H}^3$ in the sides of P is called *reflection group of P*.

Theorem 1.1.3. *Let G be the reflection group of a convex polyhedron P in $\mathbb{H}^3$ of finite volume. Then*

$$\mathbb{H}^3 = \{gP | g \in G\}$$

Let P be an exact convex fundamental polyhedron for a discrete group Γ of isometries of $\mathbb{H}^3$. Then for each side S of P, there is a unique element g such that $S = P \cap gP$. We say Γ is a *discrete reflection group* with respect to P when g is the reflection in the hyperplane $\langle S\rangle$.

Our main interest is in discrete reflection groups. These are very common, as shown on the theorem below (see [Ra]).

Theorem 1.1.4. *Let P be a finite sided convex polyhedron in $\mathbb{H}^3$ of finite volume all of whose dihedral angles are submultiples of π. Then the group Γ generated by reflections in the sides of P is a discrete reflection group with respect to the polyhedron P.*

1.1.5 Combinatorial description of hyperbolic right-angled polyhedra

In this section we provided a combinatorial description of right-angled hyperbolic polyhedra. This is given by Andreev's Theorem and is one of the main tools in the proof of our main results.

An *abstract polyhedron* $\mathcal{P}_1$ is a cell complex on S^2 which can be realized by a convex Euclidean polyhedron. A *labeling* of $\mathcal{P}_1$ is a map

$$\Theta : \mathrm{Edges}(\mathcal{P}_1) \longrightarrow (0, \pi/2]$$

The pair $(\mathcal{P}_1, \Theta)$ is a labeled abstract polyhedron. A labeled abstract polyhedron is said to be *realizable* as a hyperbolic polyhedron if there exists a hyperbolic polyhedron P_1 such that there is a label preserving graph isomorphism between the 1-skeleton of P_1 with edges labeled by dihedral angles and the 1-skeleton of $\mathcal{P}_1$ with edges labeled by Θ.

By a right-angled polyhedron we mean a polyhedron whose all of its dihedral angles are $\pi/2$. Let P_1 be a totally geodesic right-angled polyhedron in $\mathbb{H}^3$ (that is, faces of P_1 are contained in hyperplanes). We call a vertex of P_1 *ideal* if it lies in the boundary at infinity S^2_∞, where we here we consider the ball model for $\mathbb{H}^3$.

We consider the 1-skeleton of P_1 as a graph $\Gamma_1 \subset S^2$ with labels $\theta_e = \pi/2$. Let Γ_1^* be its dual graph, i.e., vertices of Γ_1^* correspond to faces of P_1 and two vertices are joined by an edge if their corresponding faces in P_1 share a common edge. A *k-circuit* is a simple closed curve composed of k edges in Γ_1^*. A *prismatic k-circuit* is a k-circuit γ so that no two edges of Γ_1 which correspond to edges traversed by γ share a vertex. Andreev's theorem for right-angled polyhedra in $\mathbb{H}^3$ ([An], see also [At]) can be stated as:

Theorem 1.1.5 (Andreev). *Let $\mathcal{P}_1$ be an abstract polyhedron. Then $\mathcal{P}_1$ is realizable as a hyperbolic right-angled polyhedron P_1 if and only if*

(1) P_1 has at least 6 faces;

(2) Vertices have valence 3 or 4;

(3) For any triple of faces of P_1, (f_i, f_j, f_k), such that $f_i \cap f_j$ and $f_j \cap f_k$ are edges of P_1 with distinct endpoints, $f_i \cap f_k = \emptyset$;.

(4) There are no prismatic 4-circuits.

Furthermore, each vertex of valence 3 in P_1 corresponds to a finite vertex in $\mathcal{P}_1$, each vertex of valence 4 in P_1 corresponds to an ideal vertex in $\mathcal{P}_1$, and the realization is unique up to isometry.

1.1.6 Kleinian groups, hyperbolic manifolds and orbifolds

By a *Kleinian group* Γ we mean a discrete subgroup of $PSL_2(\mathbb{C})$. This is equivalent to saying that the action of Γ in $\mathbb{H}^3$ is properly discontinuos (see section 1.1.3 for definition).

When Γ is torsion free (i.e., has no non-trivial elements of finite order) the quotient $M_\Gamma = \mathbb{H}^3/\Gamma$ is a hyperbolic 3-manifold with metric induced from $\mathbb{H}^3$. When Γ has torsion elements we call the quotient, $O_\Gamma = \mathbb{H}^3/\Gamma$, a hyperbolic *orbifold*.

When M_Γ has finite volume, then it is the interior of a compact manifold with (possibly empty) toroidal boundary. Each of these toroidal components correspond to a *cusp* of M_Γ, where a cusp is topologically of the form $T^2 \times [0, \infty)$. These are obtained as the quotient of a set of the form $B = \{(x, y, z) | z > 1\}$ by a subgroup of Γ isomorphic to $\mathbb{Z} \times \mathbb{Z}$, consisting of parabolic elements.

In the case that Γ contains elliptic elements, the quotient of their fixed point set is called *singular locus* of the orbifold O_Γ. When Γ is finitely generated, Selberg's lemma (see [Ra], page 331) implies that every finite volume orbifold O_Γ has a finite sheeted covering $M \longrightarrow O_\Gamma$, where M is a manifold.

We will say a Kleinian group Γ has finite co-volume if the corresponding manifold or orbifold has finite volume.

Chapter 2

Main theorem

The contents of this chapter are mostly those that appear in [Gi].

If M_1 is an orientable finite volume hyperbolic 3-manifold, we call the family of covers $\{M_j \longrightarrow M_1\}$ *co-final* (resp. a *tower*) if $\{\pi_1(M_j)\}$ is co-final (resp. a tower). By rank gradient of the the pair $(M_1, \{M_j\})$, $\mathrm{rgr}(M_1, \{M_j\})$, we mean the rank gradient of $(\pi_1(M_1), \{\pi_1(M_j)\})$.

Our main result is:

Theorem 2.0.6. *Let M_1 be an orientable finite volume hyperbolic 3-manifold whose fundamental group has finite index in the reflection group of a right-angled ideal polyhedron P_1 in $\mathbb{H}^3$. Then there exists a co-final tower of finite sheeted covers $\{M_j \longrightarrow M\}$ for which* $\mathrm{rgr}(M_1, \{M_j\}) > 0$.

The main idea of the proof of Theorem 2.0.6 is as follows: given P_1 as in the theorem, construct a collection of polyhedra $\{P_j\}$ whose reflection groups have finite index 2^{j-1} in the reflection group of P_1. If one is given an orientable hyperbolic 3-manifold M_1 whose fundamental group has finite index in the reflection group of P_1 then M_1 has at least as many cusps as the number of vertices of P_1. We may find manifold covers $M_j \longrightarrow M_1$ so that M_j is a

2^{j-1}-sheeted covering and has at least as many cusps as the number of ideal vertices of P_j. We then show that the P_j can be chosen so that the number of its vertices is of the same magnitude as 2^j.

This chapter will be organized as follows: In section 2.1 we use the characterization of right-angled ideal polyhedra given by Andreev's theorem ([An]) to show how the construction of the family $\{P_j\}$ will be done. In section 2.2 we prove Theorem 2.0.6. In section 2.3 we prove all the technical results we need to estimate $\mathrm{rk}(\pi_1(M_j))$. In section 2.4 we show how to construct $\{P_j\}$ so that the family $\{M_j\}$ is co-final. The idea for this appears in [Ag] (Theorem 2.2) and we include a proof here for completeness.

2.1 Construction of the family $\{P_j\}$

Andreev's theorem implies that, in the present setting, the 1-skeleton of P_1 is a 4-valent graph. The faces can therefore be checkerboard colored. Reflecting P_1 along a face f_1 gives a polyhedron P_2 which is also right-angled, ideal and totally geodesic with checkerboard colored faces (see figure below). We construct a sequence of polyhedra $P_1, P_2, ..., P_j, ...$ recursively, whereby P_{j+1} is obtained from P_j by reflection along a face f_j. The faces of P_{j+1} are colored accordingly with the coloring of the faces of P_j.

The notation for the remainder of this work is as follows: the number of vertices in the face f_j is denoted by S_{f_j} and ϕ_{f_j} denotes the reflection along f_j. B_j and W_j represent the maximal number of ideal vertices on a black or white face of the polyhedron P_j, respectively. V_j denotes the total number of

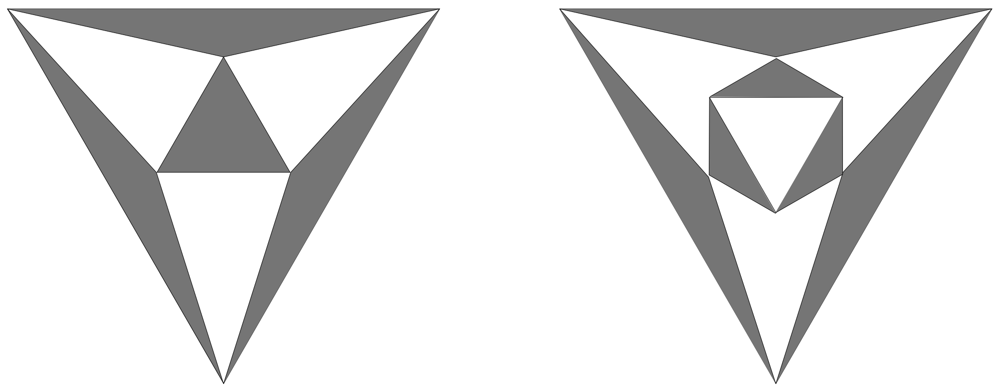

Figure 2.1: Polyhedron P_1 reflected along central black face yields P_2

vertices on P_j.

Throughout, the construction of the polyhedra P_j will be done in an alternating fashion with respect to the color of the faces: P_{2j} is obtained from P_{2j-1} by refection along a black face and P_{2j+1} is obtained from P_{2j} by reflection along a white face.

2.2 The Proof (construction of $\{M_j\}$)

Our construction of the family $\{M_j\}$ was inspired by the proof of Theorem 2.2 of Agol's paper [Ag]. The proof that this family can be made co-final is given in section 2.4 (following [Ag]).

Proof of Theorem 2.0.6. Consider the family of polyhedra $\{P_j\}$ obtained from P_1 as decribed above. Denote by G_j the reflection group of P_j and observe that G_{j+1} is a subgroup of G_j of index 2. G_1 acts on $\mathbb{H}^3$ with fundamental domain P_1. The orbifold $\mathbb{H}^3/G_1$ is non-orientable, and may be viewed as P_1 with its faces mirrored. The singular locus is the 2-skeleton of P_1. Each ideal vertex of P_1 corresponds to a cusp of $\mathbb{H}^3/G_1$.

Let M_1 be an orientable cusped hyperbolic 3-manifold such that $\pi_1(M_1)$ has finite index in G_1. Let $M_j \longrightarrow M_1$ be the cover of M_1 whose fundamental group is $\pi_1(M_j) = \pi_1(M_1) \cap G_j$. Since $[G_j : G_{j+1}] = 2$, we must have $[\pi_1(M_j) : \pi_1(M_{j+1})] \leq 2$. Also note that since $\text{vol}(P_j) = 2^{j-1}\text{vol}(P_1)$, for all but finitely many j (at most $[G_1 : \pi_1(M_1)]$) we must have $[\pi_1(M_j) : \pi_1(M_{j+1})] = 2$. We may thus assume that $[\pi_1(M_j) : \pi_1(M_{j+1})] = 2$. By mirroring the faces of P_j, it may be regarded as a non-orientable finite volume orbifold (as described before). This implies that $M_j \longrightarrow P_j$ is an orientable finite sheeted cover for $j = 1, 2, ...$

Note that $[\pi_1(M_1) : \pi_1(M_j)] = 2^{j-1}$. Thus to show that the family $\{M_j \longrightarrow M_1\}$ has positive rank gradient we will establish that $\text{rk}(\pi_1(M_j))$ grows with the same magnitude as 2^j.

By "half lives half dies"lemma (see [Ha], Theorem 3.5), an easy lower bound on the rank of the fundamental group of an orientable finite volume hyperbolic 3-manifold is the number of its cusps. Since the cusps of P_j correspond to its ideal vertices and the number of cusps does not go down under finite sheeted covers, it must be that M_j has at least as many cusps as the number of ideal vertices of P_j.

Recall that B_j and W_j are the maximal number of ideal vertices on a black or white face of the polyhedron P_j, respectively, and V_j is the total number of vertices on P_j. The claims below (proved in section 2.3) gives us the estimates we need for V_j in terms of V_1, B_1 and W_1.

Claim 1. $V_1 \geq B_1 + W_1 - 1$

Claim 2. For any $j \geq 6$,

$$V_j \geq 2^{j-1}V_1 - 2^{j-1}(B_1 + W_1) + 2^{j-1} + 2^{j-2}$$

Given these, we argue as follows:

$$\operatorname{rgr}(M_1, \{M_j\}) = \lim_{j\to\infty} \frac{\operatorname{rk}(\pi_1(M_j)) - 1}{[\pi_1(M_1) : \pi_1(M_j)]} \geq$$

$$\lim_{j\to\infty} \frac{V_j - 1}{2^{j-1}} \geq \lim_{j\to\infty} \frac{2^{j-1}V_1 - 2^{j-1}(B_1 + W_1) + 2^{j-1} + 2^{j-2} - 1}{2^{j-1}} \geq$$

$$\lim_{j\to\infty} \frac{2^{j-1}(B_1 + W_1 - 1) - 2^{j-1}(B_1 + W_1) + 2^{j-1} + 2^{j-2} - 1}{2^{j-1}} \geq$$

$$\lim_{j\to\infty} \frac{2^{j-2} - 1}{2^{j-1}} = \frac{1}{2}$$

which proves the theorem. □

2.3 Lower bounds on the number of ideal vertices of P_j

We now proceed to prove Claims 1 and 2. This requires several preliminary results.

Lemma 2.3.1. *Let P_{j+1} be obtained from P_j by reflection along a face f_j. Then $V_{j+1} = 2V_j - S_{f_j}$.*

Proof. Here we abuse notation and write $v \in f_j$ if v is an ideal vertex of the face f_j and write $v \notin f_j$ otherwise. Note that if $v \notin f_j$, then v yields two vertices on P_{j+1}, namely, v and $\phi_{f_j}(v)$. If $v \in f_j$, then it yields a single vertex (v itself).

If $v \notin f_j$, then, by the observation above, v yields two ideal vertices on P_{j+1}. Since a total of S_{f_j} ideal vertices lie in f_j and $V_j - S_{f_j}$ do not, it must be that that

$$V_{j+1} = 2(V_j - S_{f_j}) + S_{f_j} = 2V_j - S_{f_j}$$

□

Recall also that the construction of the family of polyhedra $\{P_j\}$ is made in an alternating fashion with respect to the color of the faces: P_{2j} is obtained from P_{2j-1} by refection along a black face and P_{2j+1} is obtained from P_{2j} by reflection along a white face.

Corollary 2.3.2. *For* $j \geq 1$

(1) $V_{2j} \geq 2V_{2j-1} - B_{2j-1}$

(2) $V_{2j+1} \geq 2V_{2j} - W_{2j}$

Proof. P_{2j} is obtained from P_{2j-1} by refection along a black face f_{2j-1}, thus $S_{f_{2j-1}} \leq B_{2j-1}$. By the lemma, $V_{2j} = 2V_{2j-1} - S_{f_{2j-1}}$ and therefore $V_{2j} \geq 2V_{2j-1} - B_{2j-1}$. The second inequality is similar. □

With the notation established above we now find lower bounds for the V_j in terms of V_1, B_1 and W_1. First we need to find upper bounds for B_j and

W_j in terms of B_1 and W_1. To do this in a way that will fit our purposes we establish two properties of the family $\{P_j\}$. As before, denote by ϕ_{f_j} the reflection along the face f_j.

Lemma 2.3.3. *(1) If P_j is reflected along a white (resp. black) face f_j, all black faces f_* (resp. white faces f_*) adjacent to f_j yield new black faces $\tilde{f}_*$ (resp. white faces $\tilde{f}_*$) on P_{j+1}. The number $S_{\tilde{f}_*}$ (resp. $S_{\tilde{f}_*}$) of ideal vertices on $\tilde{f}_*$ (resp. $\tilde{f}_*$) is $2S_{f_*}-2$ (resp. $2S_{f_*}-2$).*

(2) A face f_* not adjacent to f_j yield two new faces, f_* itself and $\phi_f(f_*)$, both with S_{f_*} vertices.

Proof. For the first property, reflecting f_* along f_j gives a face $\phi_{f_j}(f_*)$ in P_{j+1} adjacent to f_*. The dihedral angle between f_* and $\phi_f(f_*)$ is π. Thus, on P_{j+1}, they correspond to a single face denoted by $\tilde{f}_*$. The number of ideal vertices on $\tilde{f}_*$ is exactly $2S_{f_*}-2$. The second property should be clear. See figure 1 for an ilustration of these properties. □

As an immediate consequence we have

Corollary 2.3.4.

(1) $\begin{cases} B_{2j} = B_{2j-1} \\ W_{2j} \leq 2W_{2j-1} - 2 \end{cases}$

(2) $\begin{cases} B_{2j+1} \leq 2B_{2j} - 2 \\ W_{2j+1} = W_{2j} \end{cases}$

We are now in position to estimate the values B_j and W_j in terms of B_1 and W_1.

Theorem 2.3.5. *With the notation as before we have*

(1) $W_{2j+1} = W_{2j} \leq 2^j W_1 - \sum_{l=1}^{j} 2^l$

(2) $B_{2j+2} = B_{2j+1} \leq 2^j B_1 - \sum_{l=1}^{j} 2^l$

Proof. We procced by induction. By corollary 2.3.4 these statements are true for $j = 1$. Suppose it is also true for $j \leq n$. We now want to estimate $B_{2n+3} = B_{2n+4}$ and $W_{2n+2} = W_{2n+3}$. The hypothesis is that

$$W_{2j+1} = W_{2j} \leq 2^n W_1 - \sum_{l=1}^{n} 2^l$$

$$B_{2n+2} = B_{2n+1} \leq 2^n B_1 - \sum_{l=1}^{n} 2^l$$

P_{2n+2} is obtained from P_{2n+1} by reflection along a black face, denoted by f. White faces on P_{2n+1} adjacent to f yield new white faces on P_{2n+2} with at most $2W_{2n+1} - 2$ vertices, by Corollary 2.3.4. But

$$2W_{2n+1} - 2 \leq 2[2^n W_1 - \sum_{l=1}^{n} 2^l] - 2 = 2^{(n+1)} W_1 - \sum_{l=1}^{n+1} 2^l$$

which gives the desired result for W_{2n+2} and W_{2n+3}. Finally, P_{2n+3} is obtained from P_{2n+2} by a reflection along a white face, again denoted by f. Since black faces of P_{2n+2} have at most $B_{2n+2} (= B_{2n+1})$ vertices, black faces of P_{2n+3} will have at most $2B_{2n+1} - 2$ vertices, again by corollary 2.3.4. But

$$2B_{2n+1} - 2 \leq 2[2^n B_1 - \sum_{l=1}^{n} 2^l] - 2 = 2^{(n+1)} B_1 - \sum_{l=1}^{n+1} 2^l$$

vertices. This establishes the result for B_{2n+3} and B_{2n+4}. $\square$

Theorem 2.3.6. *With the notation as before, and for $j \geq 3$,*

(1) $V_{2j} \geq 2^{2j-1}V_1 - B_1 \sum_{l=j-1}^{2j-2} 2^l - W_1 \sum_{l=j}^{2j-2} 2^l + \sum_{l=j+2}^{2j-1} 2^l + 2^j + 2$

(2) $V_{2j+1} \geq 2^{2j}V_1 - B_1 \sum_{l=j}^{2j-1} 2^l - W_1 \sum_{l=j}^{2j-1} 2^l + \sum_{l=j+2}^{2j} 2^l + 2$

Proof. Lower bounds estimates for $V_1, ..., V_7$ are found recursively. V_1, V_2, V_3, V_4 and V_5 do not fit these formulas but V_6 and V_7 do. The statement is then true for $j = 3$. We now proceed by induction, using the previous proposition and corollary 2.3.2. Suppose it is true for $j \leq n, n \geq 3$. We want to show this implies true for $j = n+1$. By corollary 2.3.2, $V_{2n+2} \geq 2V_{2n+1} - B_{2n+1}$. The hypothesis is that

$$V_{2n+1} \geq 2^{2n}V_1 - B_1 \sum_{l=n}^{2n-1} 2^l - W_1 \sum_{l=n}^{2n-1} 2^l + \sum_{l=n+2}^{2n} 2^l + 2$$

We also know that

$$B_{2n+1} \leq 2^n B_1 - \sum_{l=1}^{n} 2^l$$

Thus

$$V_{2n+2} \geq 2V_{2n+1} - B_{2n+1} \geq$$

$$2[2^{2n}V_1 - B_1 \sum_{l=n}^{2n-1} 2^l - W_1 \sum_{l=n}^{2n-1} 2^l + \sum_{l=n+2}^{2n} 2^l + 2] - [2^n B_1 - \sum_{l=1}^{n} 2^l] =$$

$$2^{2n+1}V_1 - B_1 \sum_{l=n}^{2n-1} 2^{l+1} - W_1 \sum_{l=n}^{2n-1} 2^{l+1} + \sum_{l=n+2}^{2n} 2^{l+1} + 2^2 + \sum_{l=1}^{n} 2^l =$$

$$2^{2n+1}V_1 - B_1 \sum_{l=n}^{2n} 2^l - W_1 \sum_{l=n+1}^{2n} 2^l + \sum_{l=n+3}^{2n+1} 2^l + 2^{n+1} + 2$$

which establishes (1) for $2(n+1) = 2n+2$.

We use the exact same idea to and the estimate for V_{2n+2} to establish (2) for $2(n+1)+1 = 2n+3$. $\square$

Corollary 2.3.7. *For any $j \geq 6$,*

$$V_j \geq 2^{j-1}V_1 - 2^{j-1}(B_1 + W_1) + 2^{j-1} + 2^{j-2}$$

Hence Claim 2 in the proof of Theorem 2.0.6 is proved. We now prove

Claim 1. $V_1 \geq B_1 + W_1 - 1$

Proof. Let f_b and f_w be black and white faces of P_1 with maximal number of vertices, i.e., $S_{f_b} = B_1$ and $S_{f_w} = W_1$.

Case 1: The faces f_b and f_w are not adjacent

Here we get $V_1 \geq B_1 + W_1$ and the claim follows.

Case 2: The faces f_b and f_w are adjacent.

Since f_b and f_w share exactly 2 vertices we see that $V_1 \geq B_1 + W_1 - 2$. Suppose we have equality. Then every vertex of P_1 must be a vertex of either f_b or f_w. Recall that we can visualize the 1-skeleton of P_1 as lying in S^2. Label the vertices of P_1 by $\{v_1, ..., v_k\}$. The assumption is that all these vertices lie in the boundary of the disk $D = \overline{(f_b \cup f_w)} \subset S^2$. By Andreev's theorem (refer back to 1.1.5), P_1 has at least 6 faces, every face is at least 3-sided and all vertices are 4-valent. Denoting by F_1 and E_1 the number of faces and edges of P_1 respectively we have the relation $V_1 - E_1 + F_1 = 2$. Since vertices are 4-valent we also have $E_1 = 2V_1$. From these relations and $F_1 \geq 6$, we get $V_1 \geq 4$. At two of the vertices, say v_1 and v_2, three of the emanating edges lie in D and one does not. Denote the ones that do not lie in D by e_1 and e_2, respectively. At all other v_i we have two edges that lie in D and two that do not. Denote the latter by e_i, e_i'. We have a total of $2(k-2)+2 = 2k-2$ edges not in D. The problem we have now is combinatorial:

Proposition 2.3.8. *Consider the disk $D' = \overline{S^2 - D}$ and the points $v_1, ..., v_k \in \partial D'$, $k \geq 4$. Then it is not possible to subdivide D' by $2k-2$ edges in a way that exactly one edge emanates from both v_1 and v_2 and exactly two edges emanate from $v_3, ..., v_k$ in such a way that no pair of edges intersect and every face on the subdivision of D' is at least 3-sided (here we also consider sides coming from the boundary).*

This completes the proof of the claim. □

Proof of Proposition. Orient the boundary of D' counterclockwise. Starting at v_1, draw the edge e_1 emanating from it. The other endpoint of e_1 is some vertex v_{i_1}. Consider the vertices contained in the segment $[v_1, v_{i_1}] \subset \partial D'$ in the given orientation. If there are no vertices at all, then we must have a 2-sided face, which is not possible. Therefore, by relabeling, we may assume v_2 is the the first vertex between v_1 and v_{i_1}. Observe that the edges emanating from v_2 are trapped between the edge e_1 and $\partial D'$. Draw an edge e_2 emanating from

v_2 with the second endpoint v_{i_2}. It must be that v_{i_2} also lies in $[v_1, v_{i_1}]$, or else we find a pair of intersecting edges. As above, there must be a vertex in the segment $[v_2, v_{i2}]$. By repeating the above argument eventually we find a 2-sided face, which is not possible. Therefore it must be that $V_1 > B_1 + W_1 - 2$. $\square$

2.4 Co-finalness

In this section we provide a way of choosing the black or white faces on the polyhedra P_j along which it is reflected in such a way that the resulting family $\{M_j\}$ of manifolds is cofinal. The main result of this section, Theorem 2.4.1, appears as part of the proof of Theorem 2.2 of [Ag]. We include a proof here for completeness. To better describe this construction we need to change notation slightly by adding another index.

Start with P_1 and relabel it P_{11}. Reflect along a black face f_{11} obtaining P_{12}. Let $\phi_{f_{11}}$ represent such reflection. Observe that if f is adjacent to f_{11}, then $f \cup \phi_{f_{11}}(f)$ corresponds to a single face on P_{12}. We call f and $\phi_{f_{11}}(f)$ *subfaces* of $f \cup \phi_{f_{11}}(f)$. Next reflect P_{12} along a white face f_{12}, which is also a face of P_{11} or contais a face of P_{11} as a subface, obtaining P_{13}. We construct a subcollection $P_{11}, ..., P_{1k_1}$ of polyhedra such that

(i) If P_{1j} is obtained from $P_{1(j-1)}$ by reflection along a white (black) face then $P_{1(j+1)}$ is obtained from P_{1j} by reflection along a black (white) face.

(ii) Whenever possible, the face f_{1j} must be a face of P_{11} or contain a face of P_{11} as a subface.

(iii) No faces of P_{11} are subfaces of P_{1k_1}.

Now set $P_{1k_1} := P_{21}$. Suppose P_{n1} has been constructed. Construct the subcollection of polyhedra $P_{n1}, ..., P_{nk_n}$ such that

(i) The reflections were performed in a alternating fashion with respect to the color of the faces;

(ii) Whenever possible, the face f_{nj} must be a face of P_{n1} or contain a face of P_{n1} as a subface.

(iii) No faces of P_{n1} are subfaces of P_{nk_n}.

Now set $P_{nk_n} := P_{(n+1)1}$. Inductively we obtain a collection of polyhedra

$$P_{11}, P_{12}, ..., P_{1k_1} := P_{21}, ..., P_{2k_2} := P_{31}, ..., P_{nk_n} := P_{(n+1)1}, ...$$

satisfying (i), (ii) and (iii) above.

Let G_{ij} be the reflection group of P_{ij} and let M_{ij} be the cover of M_{11} whose fundamental group is $\pi_1(M_{ij}) = \pi_1(M_{11}) \cap G_{ij}$. Co-finalness of the family $\{M_{ij} \longrightarrow M_{11}\}$ is an immediate consequence of

Theorem 2.4.1. *Let G_{ij} be as above. Then $\cap_{ij} G_{ij} = \{1\}$.*

In order to prove this theorem we consider the base point for the fundamental group of each P_{ij} (viewed as orbifolds with their faces mirrored) to be the barycenter x_0 of P_{11}.

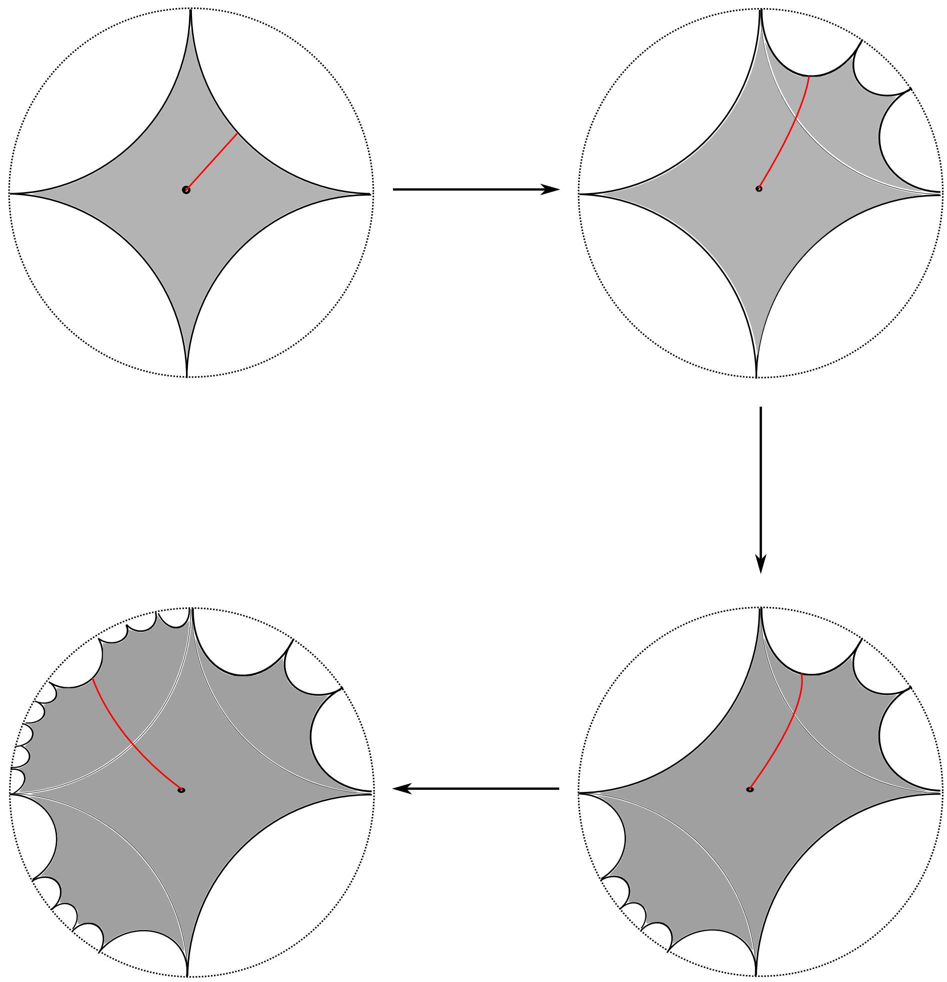

Figure 2.2: Construction of the family $\{P_{ij}\}$

Proof of Theorem. Set $R_{ij} = \inf_{\gamma}\{\ell(\gamma)\}$, where γ is an arc with endpoints in faces (possibly edges) of P_{ij} going through x_0. Note that, by construction, $\lim_{i\to\infty} R_{ij} = \infty$. For a non-trivial element $g \in G_{11}$ set $R_g = \inf_{[\alpha]=g}\{\ell(\alpha)\}$, where α is a loop in P_{11} based at x_0 and $[\alpha]$ represents its homotopy class. Let α_g be a loop in P_{11} based at x_0 such that $[\alpha_g] = g$ and $\ell(\alpha_g) \leq R_g + 1$.

We claim that for sufficiently large i one cannot have $g \in G_{ij}$. In fact, if α_{ij} is any loop in P_{ij} based at x_0, then this loop bounces off faces of P_{ij}, yielding an arc γ_{ij} throught x_0. Therefore $\ell(\alpha_{ij}) \geq \ell(\gamma_{ij}) \geq R_{ij}$. Since covering maps preserve length of curves, this implies that if i is large enough no such α_{ij} maps to α_g. Thus it is not possible to find a loop representative for g in P_{ij}. □

2.5 Remarks

2.5.1 A related theorem

An inportant result, closely related to our work is the following

Theorem 2.5.1. *Let M be a virtually fibered oriented hyperbolic 3-manifold of finite volume. Then there exists a co-final tower of regular finite sheeted covers $\{M'_j \longrightarrow M\}$ such that* $\mathrm{rgr}(M, \{M'_j\}) = 0$.

For the proof of this theorem we need the notion of *residual finiteness.*

Definition 3. A group G is residually finite if the intersection of all its subgroups of finite index is trivial.

It is known that the fundamental group of a 3-manifold is residually finite (see [He]).

The following lemma will also be used.

Lemma 2.5.2. *Let G be a finitely generated group, $\{G_j\}$ and $\{H_j\}$ be two collections of finite index subgroups such that $H_j < G_j$ and $[G_j : H_j] < \infty$. Then* $\mathrm{rgr}(G, \{H_j\}) \leq \mathrm{rgr}(G, \{G_j\})$

Proof. Just note that,

$$\mathrm{rgr}(G, \{H_j\}) = \lim_{j\to\infty} \frac{\mathrm{rk}(H_j) - 1}{[G : H_j]} \leq \lim_{j\to\infty} \frac{[G_j : H_j](\mathrm{rk}(G_j) - 1)}{[G : H_j]}$$

$$= \lim_{j\to\infty} \frac{[G_j : H_j](\mathrm{rk}(G_j) - 1)}{[G : G_j][G_j : H_j]} = \lim_{j\to\infty} \frac{\mathrm{rk}(G_j) - 1}{[G : G_j]} = \mathrm{rgr}(G, \{G_j\})$$

□

Proof of theorem. If M is virtually fibered then it is possible to find a tower of finite sheeted covers $\{\Gamma_j\}$ with $\mathrm{rgr}(\pi_1(M), \{\Gamma_j\}) = 0$ (refer back to section 1.1.1). Consider the *core* of Γ_j in $\pi_1(M)$ (i.e., $\mathrm{core}(\Gamma_j) = \cap_{g\in\pi_1(M)} g\Gamma_j g^{-1}$). Since Γ_j is a finite index subgroup, there are only finitely many of its conjugacy classes in $\pi_1(M)$ and thus $[\Gamma_j : \mathrm{core}(\Gamma_j)] < \infty$. The above lemma implies that the tower of normal subgroups $\{\mathrm{core}(\Gamma_j)\}$ also has zero rank gradient. This tower may not be co-final. Using residual finiteness we get a co-final tower $\{\tilde{\Gamma}_j\}$ of finite index subgroups of $\pi_1(M)$. Another application of the above lemma will give us the desired co-final tower with zero rank gradient. It is given by the covers $\{M'_j \longrightarrow M\}$ corresponding to the subgroups $\{\tilde{\Gamma}_j \cap \mathrm{core}(\Gamma_j)\}$. □

Remark 1. By the work of Agol ([Ag]), if M_1 is as in Theorem 2.0.6 then it virtually fibers. The above theorem shows that these manifolds also have towers with zero rank gradient. We discuss this further in Chapter 4.

2.5.2 Examples with large rank gradient

It is also easy to give examples of families $\{M_j \longrightarrow M_1\}$ with arbitrarily large rank gradient. Using the methods above it suffices to provide examples of polyhedra P_1 for which the difference $V_1 - (B_1 + W_1)$ is arbitrarily large.

Below we illustrate some cases in which this happens: consider the right-angled ideal polyhedron P_0 pictured below, viewed as lying in S^2.

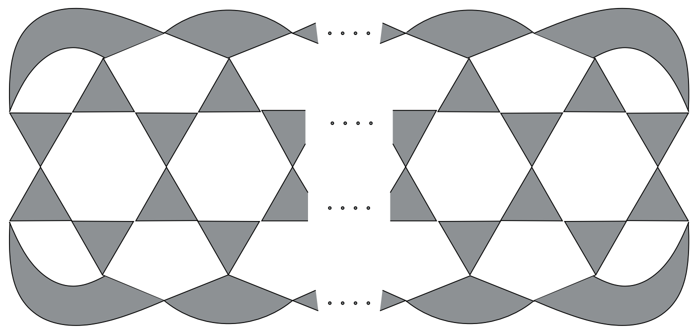

Figure 2.3: Polyhedron P_0

Note that, by Andreev's theorem, this polyhedron can be realized as a totally geodesic right-angled ideal polyhedron in $\mathbb{H}^3$. Reflecting P_0 along the white face containing the point at infinity of S^2 will give us a polyhedron P_1. Since P_1 is obtained from two copies of P_0 by gluing together the white faces containing the point at infinity, we have a maximum of 6 ideal vertices per white face of P_1 and a maximum of 4 per black faces. Obviously this construction can be made so that P_1 has arbitrarily many ideal vetices. Thus, given any $C > 0$ we may find P_1 such that for the family $\{M_j \longrightarrow M_1\}$ as above

$$\lim_{j\to\infty} \frac{\mathrm{rk}(\pi_1(M_j)) - 1}{[\pi_1(M_1) : \pi_1(M_j)]} \geq \lim_{j\to\infty} \frac{2^{j-1}(V_1 - (B_1 + W_1)) - 1}{2^{j-1}} > C$$

2.5.3 Number of vertices versus volume

There is a strong relationship between the volume of a (ideal) right-angled polyhedron and its number of vertices. Let O denote the right-angled ideal octahedron. Atkinson ([At]) proved that if P is a right-angled ideal polyhedron with V ideal vertices then

$$(V-2)\cdot\frac{\mathrm{vol}(O)}{4}\leq \mathrm{vol}(P)\leq (V-4)\frac{\mathrm{vol}(O)}{2}$$

A key ingredient in the proof of theorem 2.0.6 was the fact that the number of ideal vertices of the polyhedra in the family $\{P_j\}$ grows with magnitude 2^j. One may at first suspect that the growth we need follows directly from Atkinson's inequality. Note, however, this is not the case.

Chapter 3

The compact case

In this chapter we provide a result similar to that of Theorem 2.0.6 but for orbifolds arising as compact hyperbolic right-angled polyhedra. Recall that all vertices in these polyhedra are 3-valent. Therefore such a polyhedron can have its faces 4-colored. Here we denote these colors by *black (B), white (W), red (R)* and *yellow (Y)*. For a collection $\{P_j\}$ of polyhedra, denote by B_j, W_j, R_j, W_j the maximum number of vertices in a black, white, red or yellow face of P_j respectively. Again, let G_j denote the reflection group of P_j. Let r_j denote the number of faces of P_j. It is not hard to see that $\mathrm{rk}(G_j) = r_j$. In fact, $H_1(G_j, \mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z})^{r_j}$. This gives $\mathrm{rk}(G_j) \geq r_j$. Now observe that G_j is generated by reflections along faces of P_j. Thus we also have $\mathrm{rk}(G_j) \leq r_j$.

For a compact hyperbolic right-angled polyhedron P, let v, e and f denote its number of vertices, edges and faces respectively. By Andreev's theorem (refer back to Theorem 1.1.5), all vertices are 3 valent. Since the boundary of this polyhedron is topologically a sphere, an Euler characteristic argument gives us

$$f = \frac{1}{2}v + 2$$

and therefore, in order to estimate the rank of the reflection group of such polyhedra, one only needs to estimate their number of vertices.

3.1 A theorem for compact polyhedra

Theorem 3.1.1. *Let P_1 be a compact hyperbolic right-angled polyhedra, G_1 its reflection group and V_1 the number of vertices in P_1. If $2V_1 \geq 2(B_1+W_1+R_1+Y_1)$ then there exists a co-final tower of finite index subgroups $\{G_j\}$ of G_1 such that* $\operatorname{rgr}(G_1\{G_j\}) > 0$.

An example of a such polyhedron is obtained from a compact hyperbolic dodecahedron D (see Figure 3.1 below). D has 20 vertices and all the faces have 5 vertices. Therefore D does not satisfy the hypothesis of the theorem. However, we can obtain a polyhedron P_1 from D by performing certain refections.

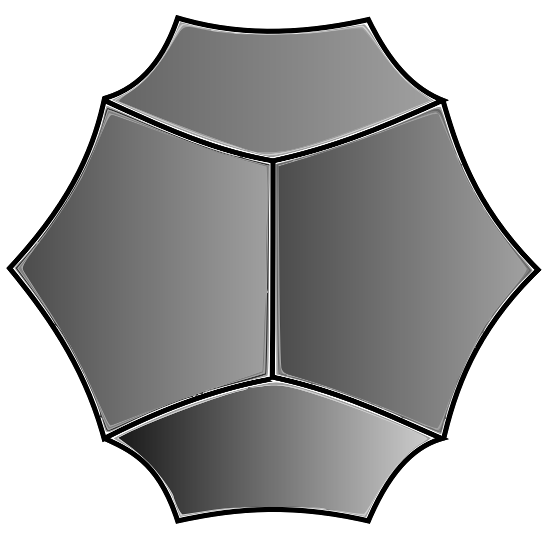

Figure 3.1: Hyperbolic right-angled dodecahedron.

Another way to visualize D is as an abstract polyhedron on S^2, which we identify with the extended plane.

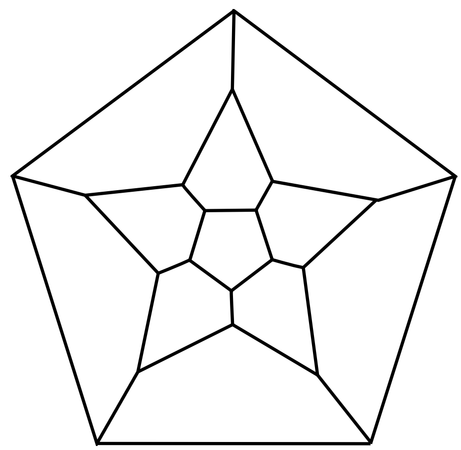

Figure 3.2: Dodecahedron visualized in the extended plane.

Let f_1 be the central face of D (as seen in figure 3.2). Reflect D along f_1 obtaining a polyhedron P_0.

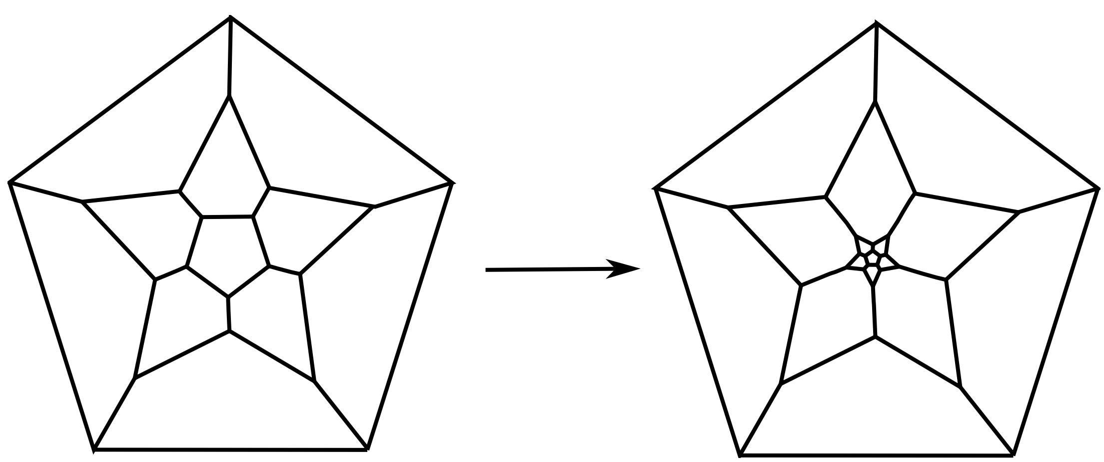

Figure 3.3: Left: dodecahedron D. Right: reflect along central face obtaining P_0.

P_0 has 30 vertices and all faces have either 5 or 6 vertices (this follows from Lemmas 3.2.1 and 3.2.2 below). Thus this polyhedron does not satisfy the hypothesis of the theorem either.

Finally reflect P_0 along the outter face (corresponding to the unbounded region of the plane) obtaining P_1.

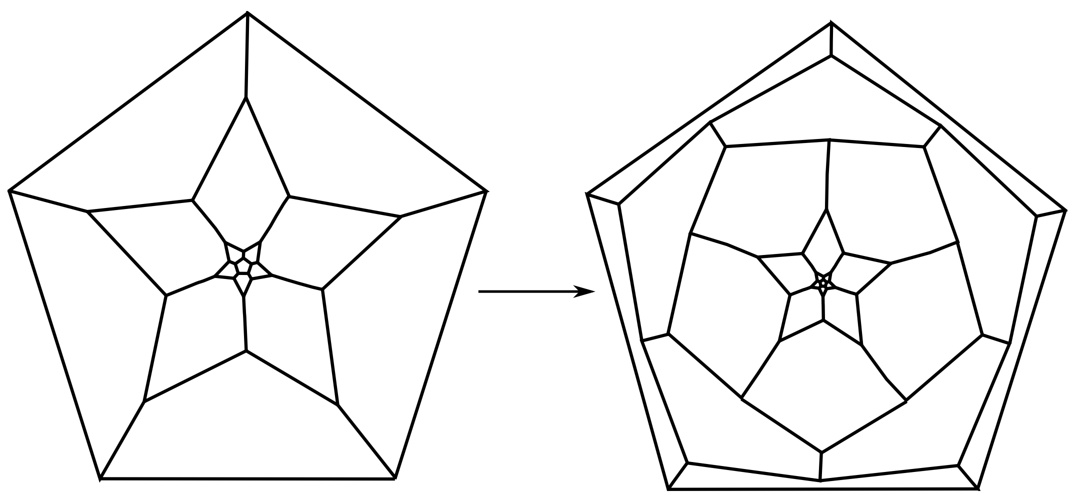

Figure 3.4: Left: polyhedron P_0. Right: reflect along outter face obtaining P_1.

P_1 has 50 vertices and again all the faces have either 5 or 6 vertices. Therefore, no matter how one colors this polyhedron, the coloring will satisfy the hypothesis of the theorem.

The proof of this theorem is similar to the proof of Theorem 2.0.6. The groups G_j arise as reflection groups of polyhedra P_j obtained from P_1 by the same type of construction as in the case of polyhedron with all vertices ideal. Here we perform reflections in an alternating fashion, but now with respect to the colors black, white, red and yellow, in this order. Recall that in the proof of Theorem 2.0.6 a key point was Claim 2. The corresponding result for compact polyhedra is

Lemma 3.1.2. *For $n \geq 1$ we have*

$$V_{4n+1} \geq 2^{4n} V_1 - 2^{4n+1}(B_1 + W_1 + R_1 + Y_1) + 2^{4n+2}$$

We ommit the tecnical results needed for this lemma for now and proceed to the proof of Theorem 3.1.1:

Proof of Theorem 3.1.1. From the observation above, it suffices to show that the number of vertices the polyhedra in the $\{P_{4_j+1}\}$ grows linearly with the degree $[G_1 : G_{4j+1}] = 2^{4n}$. We have

$$\lim_{j\to\infty} \frac{V_{4j+1}}{2^{4j}} \geq \lim_{j\to\infty} \frac{2^{4n} V_1 - 2^{4n+1}(B_1 + W_1 + R_1 + Y_1) + 2^{4n+2}}{2^{4j}} \geq$$

If $2V_1 \geq 2(B_1 + W_1 + R_1 + Y_1)$ then

$$\lim_{j\to\infty} \frac{2^{4n} V_1 - 2^{4n+1}(B_1 + W_1 + R_1 + Y_1) + 2^{4n+2}}{2^{4j}} \geq$$

$$\lim_{j\to\infty} \frac{2^{4j+2}}{2^{4j}} = 4$$

which proves that this family has positive rank gradient.

The proof that this family can be made cofinal is the same as the proof of Theorem 5. $\square$

Remark 2. Without the requirement $V_1 \geq 2(B_1 + W_1 + R_1 + Y_1)$ the construction in the proof of Theorem 3.1.1 does not work for general compact right-angled polyhedra. Consider for instance the right-angled Euclidean cube in $\mathbb{R}^3$. No matter how one performs reflections, in each step on the construction of $\{P_j\}$ we have a polyhedron (a parallelepiped) with exactly 6 faces and 8 vertices.

3.2 Lower bounds on the number of vertices

The key ingredients in the proof of Lemma 3.1.2 are the lemmas below. Recall that we perform reflections in a alternating fashion with respect to the colors black, white, red and yellow.

Using the fact that we perform reflections in a alternating fashion with respect to the colors black, white, red and yellow, in this order, we obtain

Lemma 3.2.1.

$$\begin{cases} V_{4j+1} \geq 2V_{4j} - 2Y_{4j} \\ V_{4j+2} \geq 2V_{4j+1} - 2B_{4j+1} \\ V_{4j+3} \geq 2V_{4j+2} - 2W_{4j+2} \\ V_{4(j+1)} \geq 2V_{4j+3} - 2R_{4j+3} \end{cases}$$

Proof. Given the polyhedron P_k, note that when we perform reflection along a face f_k, each vertex not in f_k generates two new vertices in P_{k+1}. Vertices in f_k do not yield any new vertices in P_{k+1}. Depending on the color of the face the reflection is performed, we have at most B_k, W_k, R_k or Y_k vertices in such a face. The inequalities folow easily from these observations. □

Since we build the family of polyhedra in an alternating fashion with respect to the 4 colors, we also obtain

Lemma 3.2.2. *For $j \geq 1$ we have*

$$(1)\quad \begin{cases} B_{4j+1} \leq 2^{3j}B_1 - (2^{3j} + \cdots + 2^2 + 2) \\ W_{4j+1} \leq 2^{3j}W_1 - (2^{3j} + \cdots + 2^2 + 2) \\ R_{4j+1} \leq 2^{3j}R_1 - (2^{3j} + \cdots + 2^2 + 2) \\ Y_{4j+1} = Y_{4j} \leq 2^{3j}Y_1 - (2^{3j} + \cdots + 2^2 + 2) \end{cases}$$

$$\textit{(2)}\ \begin{cases} B_{4j+2} = B_{4j+1} \leq 2^{3j} B_1 - (2^{3j} + \cdots + 2^2 + 2) \\ W_{4j+2} \leq 2^{3j+1} W_1 - (2^{3j+1} + \cdots + 2^2 + 2) \\ R_{4j+2} \leq 2^{3j+1} R_1 - (2^{3j+1} + \cdots + 2^2 + 2) \\ Y_{4j+2} \leq 2^{3j+1} Y_1 - (2^{3j+1} + \cdots + 2^2 + 2) \end{cases}$$

$$\textit{(3)}\ \begin{cases} B_{4j+3} \leq 2^{3j+1} B_1 - (2^{3j+1} + \cdots + 2^2 + 2) \\ W_{4j+3} = W_{4j+2} \leq 2^{3j+1} W_1 - (2^{3j+1} + \cdots + 2^2 + 2) \\ R_{4j+3} \leq 2^{3j+2} R_1 - (2^{3j+2} + \cdots + 2^2 + 2) \\ Y_{4j+3} \leq 2^{3j+2} Y_1 - (2^{3j+2} + \cdots + 2^2 + 2) \end{cases}$$

$$\textit{(4)}\ \begin{cases} B_{4(j+1)} \leq 2^{3j+2} B_1 - (2^{3j+2} + \cdots + 2^2 + 2) \\ W_{4(j+1)} \leq 2^{3j+2} W_1 - (2^{3j+2} + \cdots + 2^2 + 2) \\ R_{4(j+1)} = R_{4j+3} \leq 2^{3nj+2} R_1 - (2^{3j+2} + \cdots + 2^2 + 2) \\ Y_{4(j+1)} \leq 2^{3j+3} Y_1 - (2^{3j+3} + \cdots + 2^2 + 2) \end{cases}$$

Proof. The arguments here are very similar to those of Corollary 2.3.4 and Theorem 2.3.5. The only difference is that, starting with P_1, we perform reflections in an alternating fashion with respect to the colors black, white, red and yellow, in this order. By construction, one can easily verify all the inequalities above for $j = 1$.

Suppose they hold for $j = k$. The last set of inequalities for $j = k$ is

$$\begin{cases} B_{4(k+1)} \leq 2^{3k+2} B_1 - (2^{3k+2} + \cdots + 2^2 + 2) \\ W_{4(k+1)} \leq 2^{3j+2} W_1 - (2^{3k+2} + \cdots + 2^2 + 2) \\ R_{4(k+1)} = R_{4j+3} \leq 2^{3k+2} R_1 - (2^{3k+2} + \cdots + 2^2 + 2) \\ Y_{4(k+1)} \leq 2^{3k+3} Y_1 - (2^{3k+3} + \cdots + 2^2 + 2) \end{cases}$$

We now reflect $P_{4(k+1)}$ along a yellow face, obtaining $P_{4(k+1)+1}$. We have

$$\begin{cases} B_{4(k+1)+1} \leq 2B_{4(k+1)} - 2 \leq 2[2^{3k+2} B_1 - (2^{3k+2} + \cdots + 2^2 + 2)] \\ W_{4(k+1)+1} \leq 2W_{4(k+1)} - 2 \leq 2[2^{3k+2} W_1 - (2^{3k+2} + \cdots + 2^2 + 2)] \\ R_{4(k+1)+1} \leq 2R_{4(k+1)} - 2 \leq 2[2^{3k+2} R_1 - (2^{3k+2} + \cdots + 2^2 + 2)] \\ Y_{4(k+1)+1} = Y_{4(k+1)} \leq 2^{3k+3} Y_1 - (2^{3k+3} + \cdots + 2^2 + 2) \end{cases}$$

which gives the first inequalitie of the lamma for $j = k + 1$. The other three inequalities are obtained similarly by reflecting along black, white and yelow faces. $\square$

Lemma 3.1.2 follows directly from the following inequality:

Lemma 3.2.3. *Set* $S = B_1 + W_1 + R_1 + Y_1$. *Then, for* $n \geq 1$, *we have*

$$V_{4n+1} \geq 2^{4n} V_1 - S \sum_{j=0}^{n-1} 2^{4n-j} + 2^{4n+2}$$

Proof. One easily finds

$$V_5 \geq 2^4 V_1 - 2^4 S + 2^6$$

Therefore the statment is true for $n = 1$. Suppose now it is true for $n = k$. We wish to use induction and Lemmas 3.2.2 and 3.2.1 to prove it is true for $n = k + 1$.

Induction hypothesis gives

$$V_{4k+1} \geq 2^{4k} V_1 - S \sum_{j=0}^{k-1} 2^{4k-j} + 2^{4k+2}$$

and Lemmas 3.2.2 and 3.2.1 give

$$\begin{aligned} V_{4k+2} &\geq 2V_{4k+1} - 2B_{4k+1} \\ &\geq 2[2^{4k} V_1 - S \sum_{j=0}^{k-1} 2^{4k-j} + 2^{4k+2}] - 2[2^{3k} B_1 - \sum_{j=1}^{3k} 2^j] \\ &= 2^{4k+1} V_1 - S \sum_{j=0}^{k-1} 2^{4k-j+1} + 2^{4k+3} - 2^{3k+1} B_1 + \sum_{j=1}^{3k} 2^{j+1} \end{aligned}$$

The above estimate and the lemmas give

$$V_{4k+3} \geq 2V_{4k+2} - 2W_{4k+2}$$

$$\geq 2[2^{4k+1}V_1 - S\sum_{j=0}^{k-1} 2^{4k-j+1} + 2^{4k+3} - 2^{3k+1}B_1 + \sum_{j=1}^{3k} 2^{j+1}] - 2[2^{3k+1}W_1 - \sum_{j=1}^{3k+1} 2^j]$$

$$= 2^{4k+2}V_1 - S\sum_{j=0}^{k-1} 2^{4k-j+2} + 2^{4k+4} - 2^{3k+2}(B_1 + W_1) + \sum_{j=1}^{3k} 2^{j+2} + \sum_{j=1}^{3k+1} 2^{j+1}$$

$$\geq 2^{4k+2}V_1 - S\sum_{j=0}^{k-1} 2^{4k-j+2} + 2^{4k+4} - 2^{3k+2}(B_1 + W_1)$$

Again, the estimates above and the lemmas give

$$V_{4(k+1)} = V_{4k+4} \geq 2V_{4k+3} - 2R_{4k+3}$$

$$V_{4k+4} \geq 2[2^{4k+2}V_1 - S\sum_{j=0}^{k-1} 2^{4k-j+2} + 2^{4k+4} - 2^{3k+2}(B_1 + W_1)] - 2[2^{3k+2}R_1 - \sum_{j=1}^{3k+2} 2^j]$$

$$\geq 2^{4k+3}V_1 - S\sum_{j=0}^{k-1} 2^{4k-j+3} + 2^{4k+5} - 2^{3k+3}(B_1 + W_1 + R_1) + \sum_{j=1}^{3k+2} 2^{j+1}$$

$$\geq 2^{4k+3}V_1 - S\sum_{j=0}^{k-1} 2^{4k-j+3} + 2^{4k+5} - 2^{3k+3}(B_1 + W_1 + R_1)$$

Finally

$$V_{4(k+1)+1} \geq 2V_{4(k+1)} - 2Y_{4(k+1)}$$

$$\geq 2[2^{4k+3}V_1 - S\sum_{j=0}^{k-1} 2^{4k-j+3} + 2^{4k+5} - 2^{3k+3}(B_1 + W_1 + R_1)] - 2[2^{3(k+1)}Y_1 - \sum_{j=1}^{3(k+1)} 2^j]$$

$$\geq 2^{4k+4}V_1 - S\sum_{j=0}^{k-1} 2^{4k-j+4} + 2^{4k+6} - 2^{3(k+1)+1}(B_1 + W_1 + R_1 + Y_1) + \sum_{j=1}^{3(k+1)} 2^{j+1}$$

$$\geq 2^{4(k+1)}V_1 - S\sum_{j=0}^{k-1} 2^{4(k+1)-j} + 2^{4(k+1)+2} - 2^{3(k+1)+1}S$$

$$= 2^{4(k+1)}V_1 - S\sum_{j=0}^{k} 2^{4(k+1)-j} + 2^{4(k+1)+2}$$

which is the desired inequality for $n = k + 1$. $\square$

3.3 Remarks

3.3.1 Number of vertices versus volume

Atkinson ([At]) estimates the volume of a hyperbolic compact right-angled polyhedra in terms of its number of vertices. Let O denote the right-angled ideal octahedron, T denote the ideal tetrahedron, P a hyperbolic compact right-angled polyhedron and V the number of vertices in P. Then

$$(V-8)\frac{\mathrm{vol}(O)}{32} \leq \mathrm{vol}(P) \leq (V-10)\frac{5 \cdot \mathrm{vol}(T)}{8}$$

Again one sees that our estimates for the growth of the number of vertices in the family $\{P_j\}$ do not follow from Atkinson's estimate.

3.3.2 Further generalizations

It should be clear that a similar result as that of theorems 2.0.6 and 3.1.1 exists for general hyperbolic right-angled polyhedra (those with both types of vertices). The faces of such polyhedra can be 4-colored. Informally, we may regard compact polyhedra as the worst case scenario one may have and ideal polyhedron as the best. Note that, by reflecting a polyhedron P_j along a face f_j , all ideal vertices of f_j will be vertices of P_{j+1} and all non-ideal vertices disappear. Estimates for the number of vertices in a family $\{P_j\}$ of such polyhedra should therefore lie between the estimates for the families of ideal and compact polyhedra.

Chapter 4

Final Remarks

4.1 Heegaard genus and Heegaard gradient

Here we summarize what we need on Heegaard splittings. For details see [Sc].

A *handlebody* is a 3-manifold with boundary constructed as follows: begin with the 3-ball B^3 and in its boundary pick out two disjoint 2-disks D_0 and D_1. Using these disks, attach to B^3 a *handle*, that is, a copy of $D^2 \times I$, by identifying $D^2 \times \{i\}$ with D_i, $i = 0, 1$. One can continue to add more handles in a similar way. The result of adding g handles is a handlebody of genus g. Note that g is precisely the genus of the boundary surface.

Every closed 3-manifold M admits a decomposition into two handlebodies, i.e., there exist handlebodies H_1, H_2 such that M is obtained by attaching H_1 to H_2 by a homeomorphism of their boundaries (this implies that H_1 and H_2 have the same genus). Write $M = H_1 \cup_S H_2$, where S is the surface ∂H_i.

There is an analogous decomposition for 3-manifolds with boundary. A *compression body* is a connected 3-manifold obtained from a closed surface (not necessarily connected), denoted by $\partial_- H$. Consider $\partial_- H \times I$. Attach 1-handles to $\partial_- H \times \{1\}$ in such a way that we obtain a connected manifold.

The resulting manifold is called a compression body. Let H be a compression body. Its boundary is subdivided as $\partial_- H$ and the remaining is denoted $\partial_+ H$. The genus of H is the genus of $\partial_+ H$.

Every 3-manifold with boundary M can be decomposed into two compression bodies, i.e., there exists compression bodies H_1, H_2 such that M is obtaining by attaching H_1 to H_2 by a homeomorphism of their boundarie $\partial_+ H_i$. It is interesting to notice that, given any partion of ∂M as the disjoint union of two sets of connected components ∂M_1 and ∂M_2, we find a decomposition of M into compression bodies H_1, H_2 such that $\partial_- H_i = \partial M_i$.

It is conventional to consider a handlebody as a compression body in which $\partial_- H = \emptyset$.

This type of decomposition of a 3-manifold is called a *Heegaard decomposition*. The Heegaard genus of M, $\mathrm{Hg}(M)$, is defined as $\min_S\{g(S)\}$, where $S = \partial_+ H$ and H is a compression body (or handlebody) in a decomposition of M. The surfaces S is called a *Heegaard surfaces* for M. The disks D in the 1-handles $D \times I$ of a compression body H are called *meridian disks*. A collection of meridian disks is called *complete* if each of its complementary components is either a 3-ball or $\partial_- H \times I$.

If we restrict ourselves to Heegaard decompositions in which one of the compression bodies is a handlebody then we can obtain upper bounds for $\mathrm{rk}(\pi_1(M))$ in terms of $\mathrm{Hg}(M)$. To see this consider a minimal genus Heegaard decomposition for M, say, $M = H_1 \cup_S H_2$, where H_1 is a handlebody and $g(S) = s$. It is obvious that $\pi_1(H_1)$ is a free group on s letters. The attaching map of $\partial_+ H_1$ and $\partial_+ H_2$ provides us with a presentation for $\pi_1(M)$ as follows: consider a complete disk system $\{D_1, ..., D_k\}$ for H_2. Each curve ∂D_j, when viewed as (a conjugacy class) in $\pi_1(H_1)$ is a relator for the fundamental group. $\pi_1(M)$ has a presentation given by

$$\langle x_1, ..., x_r | r_1, ..., r_k \rangle$$

where the x_j are the generators for $\pi_1(H_1)$ and the r_i are the relators described above.

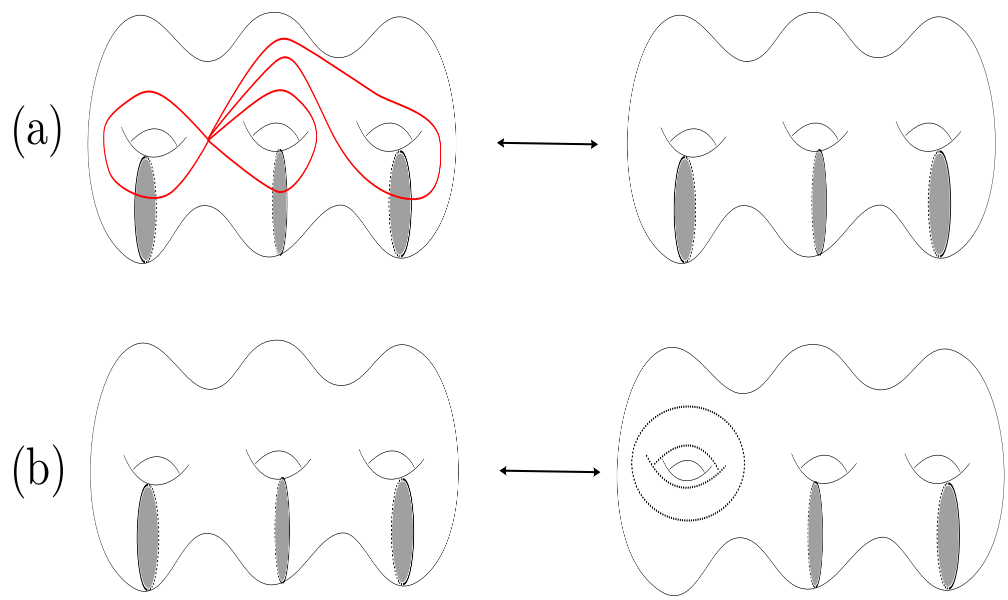

Figure 4.1: (a) Manifolds without boundary; (b) Manifolds with boundary

In 1960's Waldhausen ([Wa]) asked whether or not $\mathrm{rk}(\pi_1(M)) = \mathrm{Hg}(M)$. In the early 1980's Boileau-Zieschang ([BZ]) provided the first examples where strict inequality holds. These examples were Seifert fibered spaces. Other examples of closed 3-manifolds where strict inequality holds were given by

Schultens-Weidmann ([SW]). This problem remained opened until very recently for hyperbolic manifolds. Tao Li has recently announced counterexamples for closed hyperbolic 3-manifolds ([Li]). However, the exact relationship between rank of fundamental groups and Heegaard genus (of finite volume hyperbolic 3-manifolds) still remains unknown. For instance

Question 2. *Is there a universal constant C such that if M is a finite volume hyperbolic 3-manifold then* $\mathrm{Hg}(M) \leq C \cdot \mathrm{rk}(\pi_1(M))$*?*

Another concept due to Lackenby is that of *Heegaard gradient* ([La2]). Given a orientable 3-manifold M and a family $\{M_j\}$ of finite sheeted covers, we define the Heegaard grandient of $\{M_j \longrightarrow M\}$ by

$$\mathrm{Hgr}(M, \{M_j\}) = \lim_{j \to \infty} \frac{-\chi(S_j)}{d_j}$$

where d_j is the degree of the cover $M_j \longrightarrow M$ and S_j is a minimal genus Heegaard surface for M_j. By the above discussion on Heegaard genus we have that if $\mathrm{rgr}(M, \{M_j\}) > 0$, then $\mathrm{Hgr}(M, \{M_j\}) > 0$.

In [LLR] Long–Lubotzky–Reid prove that every orientable finite volume hyperbolic 3-manifold M has a co-final family of regular finite sheeted covers $\{M_j\}$ for which the Heegaard gradient is positive.

An important open problem for finite volume hyperbolic 3-manifolds associated to the work of Long–Lubotzky–Reid is

Question 3 (Rank vs. Heegaard gradient)**.** *Let M be a finite volume hyperbolic 3-manifold and $\{M_i \longrightarrow M\}$ a family of finite sheeted covers. Is it true that* $\mathrm{rgr}(M, \{M_i\}) > 0$ *if and only if* $\mathrm{Hgr}(M, \{M_i\}) > 0$*?*

These were also motivation for our work. Our results provide examples for which this question has positive answer.

4.2 Relation to the cost of group actions

Our work also relates to the work of Abért and Nikolov ([AN]), and in particular to a question about *cost of group actions* ([Ga]). For a more general treatment refer to [AN] and [Fa].

Question 4. *Let G be a finitely generated group and $\{G_j\}$ be a co-final tower of finite index normal subgroups of G. Does $\mathrm{rgr}(G, \{G_j\})$ depend on the tower $\{G_j\}$?*

Given the result of Theorem 2.0.6 one may naturally ask

Question 5. *Is it possible, in the setting of Theorem 2.0.6, to obtain a co-final tower of regular covers $\{M_j \longrightarrow M_1\}$ with positive rank gradient?*

A positive answer to this would be very relevant, as it implies that Question 4 has a positive answer. However, the tower constructed in Theorem 2.0.6 cannot consist of normal subgroups. To see this we argue as follows: as a particular case of the main theorem in [Ma] we have

Theorem 4.2.1. *Let P be a finite volume right-angled polyhedron in $\mathbb{H}^3$ and let G be its reflection group. Then $\mathrm{injrad}(G) < \cosh^{-1}(7) = 2.6639...$, where $\mathrm{injrad}(G)$ denotes half of the shortest translation length among hyperbolic elements of G.*

Therefore we can find a sequence $\{\gamma_j\}$ of hyperbolic elements, $\gamma_j \in G_j$, whose translation lengths are bounded above by 2.634. Since there exist at most finitely many conjugacy classes of hyperbolic elements of bounded translation length in G_1, it must be that an infinite subsequence $\{\gamma_{j_k}\}$ lie in the same conjugacy class in G_1. Let γ be a representative of this class and $g_{j_k} \in G_1$ be such that $\gamma_{j_k} = g_{j_k}\gamma g_{j_k}^{-1}$. If the tower $\{G_j\}$ consists of normal subgroups, then $\gamma \in G_{j_k}$ for all j_k, contradicting the fact that $\{G_{j_k}\}$ is co-final.

These covers are actually far from being normal: Lück Approximation Theorem ([Lu1]) implies these covers do not even satify a weaker condition (called Farber). See [Fa] for details.

Question 5 is relevant also because of the following result (see [AN]):

Theorem (Abért-Nikolov)**.** *If Question 4 has a negative answer then both the Rank vs. Heegaard gradient problem (see Question 3) and Question 2 above have a negative answers.*

4.3 More on right-angled polyhedra

As remarked before, the proof of Theorem 3.1.1 does not apply to an Euclidean cube. Questions that arises naturaly are:

Question 6. *Can the estimates in Lemma 3.1.2 be improved so that Theorem 3.1.1 is true for every compact hyperbolic polyhedra?*

Question 7. *Can one find estimates for general right-angled hyperbolic polyhedra (those possibly having both types of vertices) so that results similar to those of Theorem 3.1.1 are valid for every such polyhedra?*

Let M_1 be an orientable finite volume hyperbolic 3-manifold such that $\pi_1(M_1)$ has finite index in the reflection group of a hyperbolic right-angled polyhedron P_1.

Question 8. *Does M_1 have a co-final tower of finite sheeted covers $\{M_j \longrightarrow M_1\}$ such that $\mathrm{rgr}(M_1, \{M_j\}) > 0$?*

We also expect a much broader generalization of the results in this thesis.

Conjecture. Let M_1 be an orientable finite volume hyperbolic 3-manifold that decomposes into right-angled polyhedra. Then M_1 has a co-final tower of finite sheeted towers $\{M_j \longrightarrow M_1\}$ such that $\mathrm{rgr}(M_1, \{M_j\}) > 0$.

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