The Report Committee for Katherine Rose Garrett
Certifies that this is the approved version of the following report:

A Brief Introduction into Fair Division

APPROVED BY
SUPERVISING COMMITTEE:

Supervisor:

Efraim Armendariz

Mark Daniels

A Brief Introduction into Fair Division

by

Katherine Rose Garrett, BA

Report

Presented to the Faculty of the Graduate School of

The University of Texas at Austin

in Partial Fulfillment

of the Requirements

for the Degree of

Master of Arts

The University of Texas at Austin

August 2011

Dedication

I would like to dedicate this paper to my parents for always believing in me and always being proud of my accomplishments, to my husband for pushing me when I did not want to be pushed, and to my students who will someday realize that understanding mathematics is worth the struggle to get there.

Acknowledgements

I would like to thank Andrea Perurena for her mathematical mind, her ability to be a “good” teacher when I needed to be taught, her patience and her willingness to help me time and time again; and to my friends Carmen Finch, Stephanie Foster, Darlene Sugarek, and Jodi Wheeler whose support, encouragement, kindness and help through the often frustrating world of math I depended on, thank you.

August 2011

Abstract

A Brief Introduction into Fair Division

Katherine Rose Garrett, MA

The University of Texas at Austin, 2011

Supervisor: Efraim Armendariz

This report discusses the theory of fair division from its history to its uses in social sciences. Fair division goes beyond the envy free solution of the cake cutting problem to how people divide chores and rent and allocate assets in a divorce. Fair division can also potentially be used to solve social problems as with voting irregularities.

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Introduction

How does one divide an object so that everyone receiving an allocation is happy with what they have received? Is it even possible?

This dilemma of fair division can be found in the Bible (Numbers 33:54), when Moses was told to distribute the land by giving a larger group a larger inheritance, and to a smaller group a smaller one. Allocating an object fairly has long been an issue. Often people are envious of the portion someone else has received.

This paper explores some of the applications of the fair division algorithm. Dubins and Spanier describe *fair division* as the problem of dividing an object among a finite number of people so that each is satisfied that he has received his fair share [**6**, p.1].

One can use fair division to divide an object among *n* number of people or in the case of conflict resolution. Fair division has been used to allocate assets in a divorce, dividing housework, and how much one pays in rent. The future of fair division likely lies in the social sciences and applications to help solve problems of welfare and voting. For fair division to be used in dispute resolutions there needs to be further collaboration between mathematicians and social scientists. The two need to be able to work together to produce joint research.

Fair Division of Cake

ENVY FREE AND PROPORTIONAL MEASURES

Disagreements often occur when people are asked to divide an object. Usually, there is envy that one person's piece is bigger than another. Is there a way to end the disagreement and guarantee that everyone receives a fair share? Yes, but two properties need to exist: envy freeness and proportionality. *Proportionality* is a division of a good whereby each player (person receiving good) gets what he or she believes is $\frac{1}{n}th$ of the object between *n* players or of average value. The next condition needed for fairness is for the division to be *envy free*, or the thought that each player believes her piece is the largest or tied for the largest piece or is at least as valuable as any other player's. *Efficiency*, or *pareto-optimal*, is defined as the division where no other division increases the shares of one player and reduces the share of another [**1**, p. 268].

Let C be an entire cake, where an allocation $\langle A_1, A_2, ..., A_n \rangle$ is an *n*-tuple of mutually disjoint pieces. Assume there are *n* players, therefore, A_1 is Player 1's piece, A_2 is Player 2's piece, …, and A_n being Player *n*'s piece. These pieces are measured m_1, m_2, …, m_n to evaluate the pieces of cakes distributed. In terms of measures, the allocations are defined as *proportional*, if for each i = 1, 2, …, n, $m_i,(A_i) \geq \frac{1}{n}$; *envy-free* if, for all i, j = 1, 2, …, n, $m_i(A_i) \geq m_i(A_j)$, and *efficient* if there does not exist an allocation $\langle B_1, B_2, ..., B_n \rangle$ for each i = 1, 2, …, n, $m_i(B_i) \geq m_i(A_i)$ with at least one of the inequalities being strict [**1**, p. 269].

A CASE FOR 2

In the book, *Perplexing Puzzles and Tantalizing Teasers*, Martin Gardner poses a scenario about siblings Henry and Henrietta splitting a piece of cake [**9**, p. 21]. Each child thinks the other will cut it to give a larger portion to the cutter. The children's dad makes a suggestion on how to cut the cake so that each child is satisfied with the piece received. So what was Dad's suggestion?

Dad suggests that the children use the cut and choose method to divide the cake. This involves one of the children cutting the cake and the other choosing the first piece. The method proves to be fair because the cutter does not have a chance of receiving more than half the cake unless she is willing to risk receiving less than half if the chooser takes the larger piece [**7**, p.1]. Using the cut and choose method guarantees that both Henry and Henrietta will either be satisfied with the piece received or reduces the envy experienced at the other person having a larger piece.

A CASE FOR 3

Although the suggestion from Dad answers the question of how to divide a cake fairly and proportionally between two people, the next question would be, can it be divided between 3 people? There are several variations of the answer to this question.

An Extension of Cut and Choose

The first and most simple answer is an extension of the cut-and-choose method from Robertson and Webb [**15**, p. 112]

Step 1: *A* cuts the cake into what she believes is 1/3 of the cake and 2/3 of the cake.

Step 2: *B* cuts the remaining 2/3 of the cake in half.

Step 3: *C* chooses which piece she believes is the largest. *A* chooses from the remaining two pieces, and then *B* takes the last piece.

The problem with using the cut and choose method for 3 people is that *B* may not be satisfied with her piece, either because she may not be happy with the first cut or is forced to take a piece she does not want [**4**, p. 11]. One would say that this method does not provide a solution that is either envy-free or proportional.

The Lone Divider Method

Steinhaus claims to have solved the problem himself during World War II [**7**, p. 2]. Steinhaus was the first person to introduce trimming the cake to assure that each piece is proportional [**15**, p. 114]. The Lone Divider method is illustrated in Figure 1:

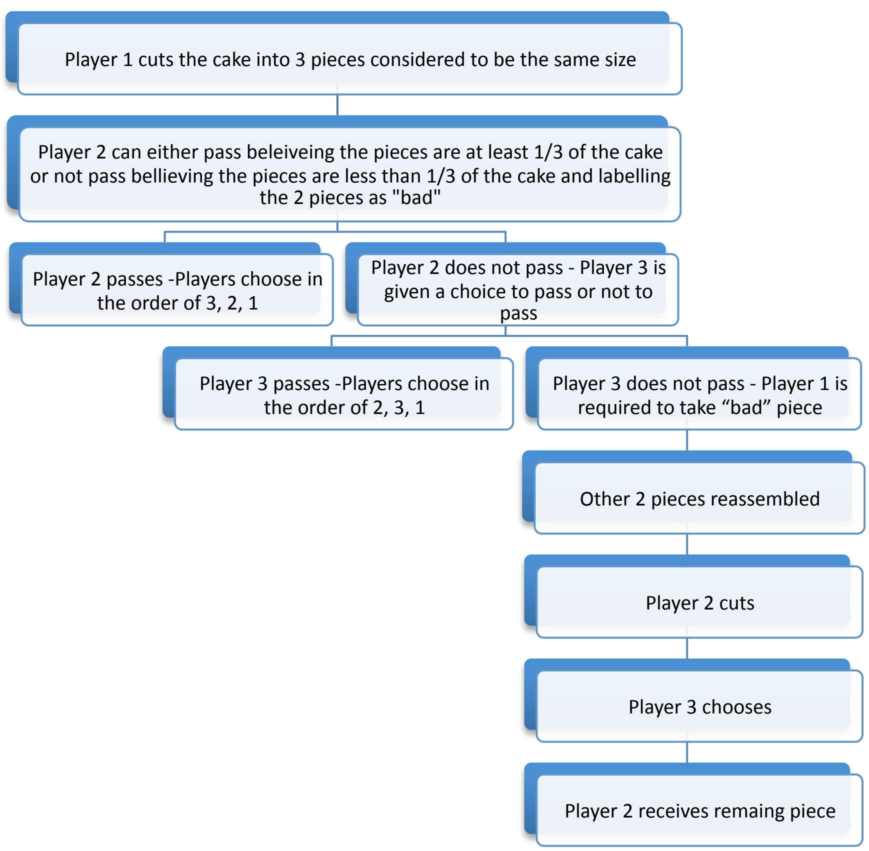

Figure 1. Steinhaus Division for n =3.

Although Steinhaus creates a solution that is proportional, it is not necessarily envy-free. [**4**, p. 12].

The Selfridge-Conway Method

The Selfridge-Conway method also uses the idea of trimming to allocate the pieces of cake. First, player *A* cuts the cake into 3 pieces $\{X_1, X_2, X_3\}$ she considers to be the same size.

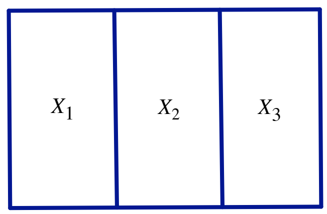

Figure 2. First Cut by Player *A*.

Second, Player *B* must decide to pass or trim to create a tie for the largest. In this example, Player *B* decides to trim the cake.

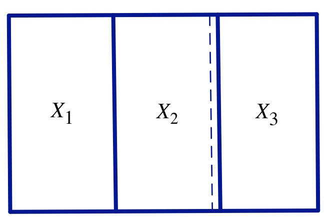

Figure 3. Player B Trimming.

The trimming, *T*, is set aside then Player *C* chooses a piece. If Player *B* trimmed a piece, she is required to choose the piece trimmed. Player *A* takes the remaining piece. However, at this point, only part of the cake has been allocated. This does create an envy free partition of the cake for Player *C* because she chose first, and for Player *B* because her cut made two of the pieces tied for the largest, and also for Player *A* because her initial cut made each piece 1/3 and the trimmed piece was either chosen by Player *C* or required to have been taken by Player *B*. Had Player *B* passed in the second step the allocation of the cake would be complete, however, since this example has Player *B* trim, the player receiving the trimmed piece (Player *B*) must now cut *T* into pieces {*T*1, *T*2, *T*3} [**4**, pp. 13-15].

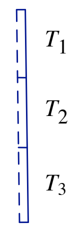

Figure 4. Trimming of *T*.

Player *C* chooses the first trimming, next Player *A*, and finally Player *B*. Again, leaving all players envy free, and feeling as if they each received 1/3 of the cake thus creating a proportional allocation of cake as well.

Dubins and Spanier Moving Knife Solutions

Although there are different versions of the classic moving-knife procedure, the Dubins and Spanier model goes as follows: A knife is slowly moving at a constant speed parallel to itself over the cake.

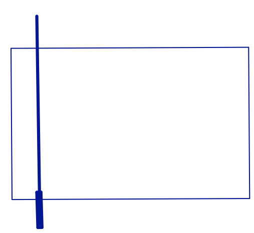

Figure 5. Moving Knife Procedure.

As time passes the potential slice of cake increases from 0% to 100% of the cake. The first person satisfied with the slice would receive that piece. If two or more people indicate satisfaction with the slice it can be given to any of them. The process would continue with the remaining people and the remaining part of the cake. The drawback with this type of allocation is whether the first person continues to believe that she received a fair piece after seeing the remaining pieces being cut [**7**, p. 2].

A CASE FOR *N*

After Steinhaus answered the question of how to divide a cake among three people, the question was extended to $n > 3$. However, Steinhaus was unable to make the solution work, and therefore the procedure came from Knaster and Banach. [**5**, p. 35].

The Last Diminisher Method

Knaster and Banach solve this problem by stating that an individual has two choices when dividing the cake, either receiving $\frac{1}{n}th$ of the cake or sharing in the remaining $\frac{n-1}{n}th$ of the cake [**7**, p. 2]. Although proportional, the problem that arises is when an individual feels that someone else got more than the "fair" share, which brings into question the envy freeness using this method.

Brams and Taylor Method

The question of whether or not there could be an envy free division among n people was left unanswered until Brams and Taylor presented their model of fair division. The central feature of the Brams and Taylor method is trimming the pieces of cake. However, Brams and Taylor believe that one needs to start the trimming and choosing process with more pieces than there are players, which will lead to envy free allocations. [**4**, p. 14]

Brams and Taylor illustrate their method of an envy free protocol for an arbitrary *n*, in this version $n = 4$, to show that full allocation of the cake can be accomplished in a finite number of steps. [**4**, pp. 15-16].

While Player 2 cuts the cake into 4 pieces, she keeps one piece and hands another piece to each of the other three players, who are asked whether or not she objects to this allocation. If no one objects, each player keeps the piece given and the procedure is done. If there is an objection the objecting player chooses another player's piece, called *A*, with the original piece being *B*. The other two pieces are reassembled and will be reallocated later. Those two players will now go through a series of trim and choose among all the players to obtain six total pieces. Each player will now choose a piece with special requirements being placed on the trimmer. A chosen player will now take the first leftover piece, L_1, and begin the trim and choose process with that piece, ending in a partial allocation of the cake. This sequence is repeated for the other leftover piece, L_2 and it repeated until the entire cake has been allocated [**4**, pp. 15 – 17].

Fair Division in Conflict Resolution

Fair Division in Divorce

One of the best uses of fair division is in divorce cases. This is where Brams' adjusted-winner procedure can be used to allocate assets. Brams describes the *adjusted-winner* (*AW*) *procedure* as "a surprisingly simple algorithm of fair division" [**11**, p. 2]. Suppose there are two sets of counting numbers: $a_1, \ldots, a_N$ and $b_1, \ldots, b_N$. The valuations of participant *A* and *B* are $\left(\sum_{i=1}^{N} a_i = \sum_{i=1}^{N} b_i = 100\right)$ or reordered as $\frac{a_1}{b_1} \geq \frac{a_2}{b_2} \geq \frac{a_N}{b_N}$.

In his interview with Jones, Brams describes a demonstration of the adjusted winner procedure in a case modeled after the Donald and Ivana Trump divorce case. The procedure gave exactly the same solution as the two lawyers who negotiated for 45 minutes [**9**, p. 265]! An example of the adjusted winner can be found in Table 1.

Table 1. Trump Divorce Point Distribution Table. [**11**]

Asset	Donald's Points	Ivana's Points
Connecticut Mansion	10	38
Palm Beach Mansion	40	20
Trump Plaza Apartment	10	30
Trump Tower Triplex	38	10
Cash and Jewelry	2	2
Total	100	100

Initially, whoever assigns the highest value of an asset retains the asset. If $a_1 > \sum_{i=2}^{N} b_i$, then *A* receives *x*, where $x = \frac{100}{(a_1 + b_1)}$, share of the first item or xa_1 and *B* receives $(1-x)$ portion of the first item and all the other items 2, …, *N*, or $(1-x)b_1 + \sum_{i=2}^{N} b_i$. However, if $\sum_{i=1}^{N-1} a_i \le b_N$, then *A* receives all the items 1, …, $N-1$ and *x*, where $x = \frac{1-100}{(a_N + b_N)}$ share of item *N*; and *B* receives $(1-x)$ share of item *N*, as shown in the table below.

Table 2. Trump Asset Allocation. [**11**]

Donald's Assets		Ivana's Assets	
Palm Beach Mansion	40	Connecticut Mansion	38
Trump Tower Triplex	38	Trump Plaza Apartment	30
Total	78		68

Starting with the first item, increase *i* up to some value of *r* which will satisfy either one of the conditions

$$\sum_{i=1}^{r-1} a_i \le \sum_{i=r}^{N} b_i$$

(1a)

$$\sum_{i=1}^{r} a_i > \sum_{i=r+1}^{N} b_i \,.$$

(1b)

The initial tallying the points does show a clear “loser.” One party will clearly be dissatisfied and claim the other has more than their fair share. Adding in the Cash and Jewelry still only gives Ivana 70 points to Donald’s 78, again, the claim made is that Donald has more.

To make sure that each party has an equal amount of points, the point ratio is calculated and shown in Table 3.

Table 3. Asset Point Ratio. [**11**]

Asset	Donald's Points	Ivana's Points	Point Ratio
Palm Beach Mansion	40	20	2
Trump Tower Triplex	38	10	3.8

If there is equality in (1a) then A receives items $1, 2, \ldots, r-1$, and B receives $r, r+1, \ldots, N$. However, if there is inequality in (1a), a will receive items $1, 2, \ldots, r-1$ and x, where $x = \frac{\left(\sum_{i=r}^{N} b_i - \sum_{i=1}^{r-1} a_i\right)}{\left(a_r + b_r\right)}$, share of item r, and B will receive $(1-x)$ share of item r, and items $r+1, \ldots, N$, or $(1-x)b_r + \sum_{i=r+1}^{N} b_i$.

Table 4. Points After Asset Transfer. [**11**]

Donald's Points After Transfer	Ivana's Points After Transfer
38 + 40(1 - x*)	38 + 30 + 2 + 20(x*)

Since $x = \frac{2}{15}$, Donald would start transferring part of the Palm Beach Mansion asset to Ivana so that both parties would have an equal percentage of the assets, in this case each party would have 72.67% of the points value, which both believe is clearly equitable and fair as well.

FAIR DIVISION OF CHORES

One of the most common issues that arise between people that live together is the division of household chores. Unless both parties are dividing every task equally, conflict can erupt between the parties resulting in one person doing more of the housekeeping or no one doing anything.

Unlike the division of cake or another desirable object, household chores are seen as an undesirable [**12**, p. 117]. When dividing a cake, each player wants the largest piece. In an undesirable, like chore division, each player now wants the smallest piece. However, division of chores can be solved the same way as cutting cake.

Chores for Two

Martin Gardner describes a scenario in the *aha! Insight* book. The problem Gardner describes is between Janet and Buster Jones who are trying to divide a list of chores. The two are arguing about the situation until Buster's mother figures out a way for neither person to be saddled with only the dirty chores. [**8**, p. 123]. Much like Dad in the cake cutting problem Mother suggests that Buster divide the chores into two lists and have Janet choose which list she would like to do.

Chores for Three

At the end of Gardner's puzzle, Mother moves in and now the dilemma is determining how to split the chores between three people [**8**, p. 123].

A three-person division of chores can be divided with one person taking a fair amount of a third of the chores and having the other two people divide the remaining chores in half via the cut and choose method.

Peterson and Su devise a three person chore division procedure that is similar to previous attempts of the moving knife algorithm. The key difference lies in the fact that this new method divides the chores into six pieces rather than three. Each player is then assigned two pieces that she feels are as small as each pair received by the other players [**12**, p. 118]. First, divide the chores into three portions i, j, k using any three person, envy free cake cutting procedure (see Figure 2). Each portion is then labeled i, j, k, with regards to the assigned player. Second, let player i divide portion i into two pieces and assign those pieces to the other two players, j and k. Third, repeat the second step for each player j and k. Once all portions have been cut and distributed the procedure has ended. [**12**, p.119]

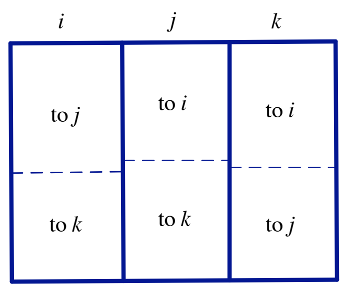

Figure 6. An envy free assignment of chores for three.

Chores for Four

The 4-person chore division procedure is also a moving knife procedure that draws ideas from the envy free moving knife procedure for cakes [**12**, p. 119]. The protocol begins with the assumption that there are four players, Alice, Betty, Carl, and Debbie. First, let Alice and Betty divide the chores into four pieces $\{X_1, X_2, X_3, X_4\}$ they agree are equal. Assume one piece is smaller than the others, in this case X_4. Each person believes the following:

Alice: $X_1 = X_2 = X_3 = X_4$

Betty: $X1 = X_2 = X_3 = X_4$

Carl: $X_4 < X_1, X_2, X_3$

Debbie: $X_4 < X_1, X_2, X_3$

Second, let Carl and Debbie mark X_1, X_2, and X_3, where their cuts create a tie with X_4 as the smallest piece [**12**, p. 119].

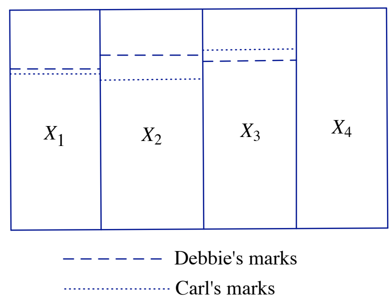

Figure 7. Possible markings of Step 2.

Third, let Betty add back to one piece of the corresponding trimming to create a two-way tie for smallest piece. Next, the players will choose a piece in the following order Alice, Betty, Carl, Debbie; with Betty required to take the added back piece if Alice did not. Carl is required to choose a piece at his marking if one is available. Thus all pieces will have been allocated in an envy free fashion, therefore, the last step is divide the trimmings.

FAIR DIVISION OF RENT

One of the most practical questions that arises, which mathematics can answer, is how to determine which room and for what amount should a housemate pay when the rooms have varying sizes and features [**17**, p. 930].

Before this question can be answered, some background information is needed. First, one must understand the use of Sperner's lemma and how, combined with Simmons' approach to cake cutting the two can be adapted to rent partitioning.

An example of a Sperner labelled triangle is shown (Figure 8), whereby each of the main vertices has a different label, and the label of the vertex along an edge has to match one of the main vertices of the edge. Based on the labelling, 3 elementary simplicies, or triangles labelled with all three labels are shown; now Sperner's lemma can now be applied.

Sperner's lemma. *Any Sperner-labelled triangulation of a n-simplex must contain an odd number of fully labeled elementary n-simplicies. In particular, there is at least one.*

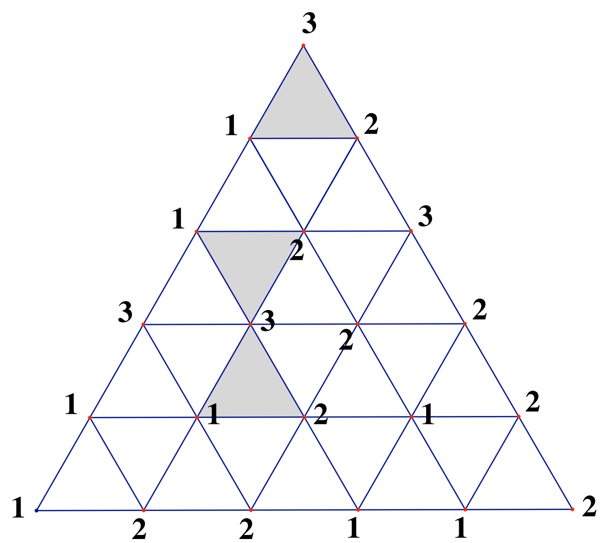

Figure 8. A Sperner labelled 2-simplex with elementary triangles marked.

Su then discusses Simmons' approach to cake cutting. Simmons' approach gives the player a preference of a piece if he thinks any of the other pieces are better given the *cut-set*, or the partition of the cake by a set of cuts. Simmons' approach also gives two assumptions based on the cut-set; similar assumptions will later be used for rental harmony [**17**, p. 935].

With the combination of the lemma and the approach, Su gives a simple algorithm solution to the rent partition problem based on a special case of the chore division problem. The dilemma is whether or not there is a fair way to divide the rooms among the rent. The difference between the cake cutting dilemma and the division of rent is that the rooms, unlike the cake, cannot be divided and reassembled. Very similar to chore division, rent partition includes dividing a bad (rent) instead of a good (cake), but since each person has to pay rent, it rules out the envy free of chore division. [**17**, p. 938]. In order for each roommate to feel that his rent is fair or equitable, the *Rental Harmony Theorem* must be applied, as well as Simmons' approach of cake cutting with cut-set assumptions.

> ***Rental Harmony Theorem***. *Suppose that n housemates in an n-bedroom house seek to decide who gets which room and for what part of the total rent. Also, suppose that the following conditions hold:*
>
> *(1)* (*Good House*) *In any partition of the rent, each person finds some room acceptable. (2)* (*Miserly Tenants*) *Each person always prefers a free room (one that costs no rent) to a non-free room.*
>
> *(3)* (*Closed Preference Sets*) *A person who prefers a room for a convergent sequence of prices prefers that room at the limiting price.*
>
> *Then there exists a partition of the rent so that each person prefers a different room.*

With the addition of the Simmons approach, Sperner's lemma comes in to play after the supposition of *n* renters and *n* rooms. The rooms to be assigned are numbered 1,

…, n with the price of the ith room, x_i, and the total rent being equal to 1. Therefore, $x_1 + x_2 + ... + x_n = 1$ and $x_i \geq 0$.

A triangulation of this simplex with each roommate at the vertex being asked which room would be chosen based on the conditions. The vertex is now labelled by the room choice. In this triangulation there exists at least one pricing scheme where each roommate prefers a different room, thus following Sperner's lemma!

FAIR DIVISION OF DRAFTS

The current draft system used has teams with the worse win-loss record picking first from the new crop of players. This method is supposed to make the worse teams more competitive the following season [**3**, p. 81]. However, there is no feasible way to make the draft outcome pareto-optimal with sincere choices and a series of trades among multiple teams.

Although extending the draft to multiple teams has not yet been discovered, Dawson has found a way to minimize disadvantage between two teams. When choosing players between two teams there are two methods of choice, the *regular draft* and the *modified draft*. The regular draft gives the owners an option to flip a coin, with the winning owner, assume O_1 deciding whether to choose first or let O_2 choose first. After that decision is made, each owner takes turns choosing a player. In the modified draft the winner still decides who chooses first, but the pattern for choosing has changed. Assume O_1 wins the toss and decides to choose first, players picked will follow the pattern O_1, O_2, O_2, O_1, O_1, …, O_2 until no more players remain. The problem owners face when using either of these two draft choices is O_1 choosing first thus having an advantage over O_2 who picks the weaker players from the pool [**6**, p. 82].

Dawson uses a classic cake cutting method to apply a modified draft among owners. With the regular draft one owner has a distinct advantage over the other owner. Dawson's cut and choose method minimizes the disadvantage that the second owner has in the talent pool [**6**, p.82]. The draft begins the same way, with the owners flipping a coin and the winner choosing role I or role II. The winner, in this case O_I, forms two teams from the available pool. Then O_{II} picks one of the teams and leaves the other team to O_I. The last step lets the owners take turns exchanging players in the waiver pool and ends when neither owner wants to make a change. In the cut and choose protocol, it is suggested that the winner always take the role of O_{II} to guarantee the maximal advantage by using the waiver pool [**6**, p. 84]. Dawson admits that the cut-and-choose method gives a result that is at least as good as a result as any other method gives, but does note that this method gives the owners a perception that the draft is more equitable [**6**, p 87].

Conclusion

The allocation of goods to a specified recipient is an issue that has been considered from many different perspectives. This issue arises in many real-world situations from divorce settlements and inheritances to congressional redistricting and airport traffic management. Fair division is the procedure that walks the fine balance to many of these sensitive subjects.

This paper described the mathematical problem of cake cutting that started it all; with Steinhaus asking if it could be done, and for how many, and could it be fair? That question launched a series of exploration from many different people solving it from the most logical to topological with elements of calculus and set theory in between. This problem has been solved for a case for 2 people, and three people, and more than three people. The problem of fair division has been solved with simple cuts and existence theorems, moving knife procedures, protocols, and matrices on computer.

In the case of divorce, it is found that a social scientist played a part in creating the adjusted winner procedure to allocate assets more simply than the negotiating lawyers. This method is used over and over again, not only in divorce proceedings, but also to help other disputes requiring assets being allocated.

The method to divide a cake fairly can also be applied to determine how to divide chores, or rent, or players on team. The question that remains is, among how many people can this object list be divided by? Although there was a solution for 4, can this be extended to 5 or more? Again, going back to Steinhaus' original question of, is there a way; can it be done?

What becomes apparent is that regardless of the situation, the same basic premises are repeated. The underlying question in these applications is whether or not an allocation is truly fair? Does the recipient feel as though she has received a proportional allocation of the good or is envy present?

With regard to teaching, fair division can be looked upon to not only clarify but to examine the concept of division more in depth. One of the most difficult concepts for elementary and middle school students to understand is that of division. With whole number division students can see an item being dividing equally among a group, as long as the group is a multiple of the item. However, the problem that occurs is when the item is being divided by a group not of that multiple. For example, most students understand $4 \div 4$, but have a hard time understanding the importance of the remainder in $5 \div 4$. Once a student understands that the remaining piece has to be divided in a way that no other person can dispute, and each person is happy with, there exists a fair division.

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Vita

Katherine Rose Garrett earned her B.A. in Mathematics at the University of the Incarnate Word in San Antonio, Texas. Teaching experience includes elementary and secondary level at both private schools and public schools. She is especially interested in ways to improve the teaching methodology for at-risk learners.

Email Address: katherinergarrett@gmail.com

This report was typed by the author.