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Mathematics of Origami

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Mathematics of Origami

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Report

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Abstract

Mathematics of Origami

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This report examines the mathematics of paper folding. One can solve cubic polynomials by folding a common tangent to two distinct parabolas. This then leads to constructions that cannot be done with a straightedge and compass such as angle trisection and doubling a cube.

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Chapter 1: Introduction

Origami is a traditional Japanese art of paper folding. Typically, origami artists use a piece of square, bicolored paper, and after a series of carefully calculated folds, the paper is transformed into birds, flowers, animals and other familiar shapes. Origami is a very popular hobby among Japanese children and its popularity spread to other countries around the world. Even with its growing popularity, many people think of origami as simple paper folding or childish activity. Recently, however, growing understanding of the mathematical and computational aspect of origami has allowed expert folders to create shapes that are incredibly complex and realistic from a single sheet of paper [**2**, p.168].

The connection between origami and Euclidean geometry is quite obvious. Creases and edges of paper represent lines, and intersections of creases and edges represent points. Some origami constructions are more intuitive and easier to perform than constructions with straightedge and compass. For example, finding a midpoint or constructing a perpendicular bisector of a segment is much easier with paper folding. Thus paper folding can be used in geometry classes to introduce constructions of points, lines and angles.

The types of folding operations that are possible have been enumerated by several mathematicians. A list of origami operations is presented in the following chapters. Based on these operations, paper folders can trisect angles and double the volume of a cube which cannot be done with a straightedge and compass [**5**].

Folding and unfolding of two- or three-dimensional objects has application well beyond paper folding. A foldable telescope lens is being developed so it can be sent into space. Car manufacturing companies use origami to test and produce safer ways to fold airbags [**9**].

Chapter 2: Folding Binary Fractions

In paper folding, *folding* a fraction, $\frac{a}{b}$ where a and b are positive integers, is defined as locating a point $\frac{a}{b}$ of the way along an edge. For example, folding a fraction $\frac{3}{8}$ is equivalent to locating a point that is $\frac{3}{8}$ of the side length away from the bottom edge. Often in practical paper folding, there is a need to fold a fraction $\frac{1}{n}$, where n is a natural number. The simplest example of this is when $n = 2$, i.e. folding a square paper in half. It is evident that by repeatedly folding a piece of paper in half, one can divide a side into proportion of $\frac{1}{2^n}$ where n is some positive integer. Then it follows that one can also fold a fraction of the form $\frac{m}{2^n}$, for some positive integers m and n, and $m \leq 2^n$, by dividing a side into 2^n equal parts and counting up m creases from the bottom. But this method of folding takes $2^n - 1$ folds and leaves the paper with many unnecessary crease lines. There is a more efficient method of constructing a *binary fraction*, a rational fraction whose denominator is a power of two. Binary fraction is first converted to binary notation (e.g. $\frac{25}{32} = .11001$). Then, starting from the least significant digit, which is always 1, fold the top of the paper down. For each remaining digit, fold the top down to the previous crease for 1 and unfold, and fold the bottom up to the previous crease and unfold for 0. This method is explained by writing the binary notation as a nested series.

$$.11001 = \frac{1}{2} \times \left(1 + \frac{1}{2} \times \left(1 + \frac{1}{2} \times \left(0 + \frac{1}{2} \times \left(0 + \frac{1}{2} \times (1)\right)\right)\right)\right) \tag{1}$$

The nested series is a string of operations of either adding 1 and multiplying by $\frac{1}{2}$ or adding 0 and multiplying by $\frac{1}{2}$. The former is equivalent to folding the top down and the latter is equivalent to folding the bottom up to the previous crease as shown in Figure 1 [**3** pp. 5-9]. In Figure 1, x is the distance from the previous crease to the bottom edge, and the dotted lines are the new fold lines.

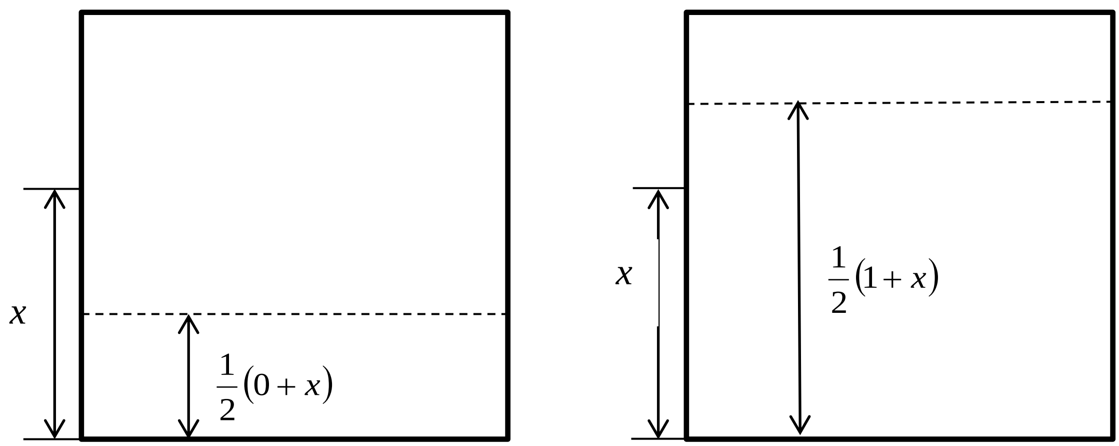

Figure 1. Binary Folding [**3**, p. 10].

Folding the top down to the previous crease is adding half of $1 - x$ to $1 - x$. Thus,

$$x + \frac{1}{2}(1 - x) = \frac{1}{2}(1 + x). \tag{2}$$

A binary fraction, when reduced to its simplest form, will have n digits in its binary notation. Therefore, it is clear that for a fraction $\frac{m}{2^n}$, the method of repeatedly folding in half and counting up takes $2^n - 1$ folds, whereas the binary folding method takes n folds.

Chapter 3: Folding Other Rational Numbers

Constructing the number $\frac{1}{3}$, or trisecting a segment, can be done with a straightedge and compass. It can also be done in origami by taking a vertex of a square and folding it onto the midpoint of a nonadjacent side as shown in Figure 2. This simple and elegant fold is Haga's First Theorem fold.

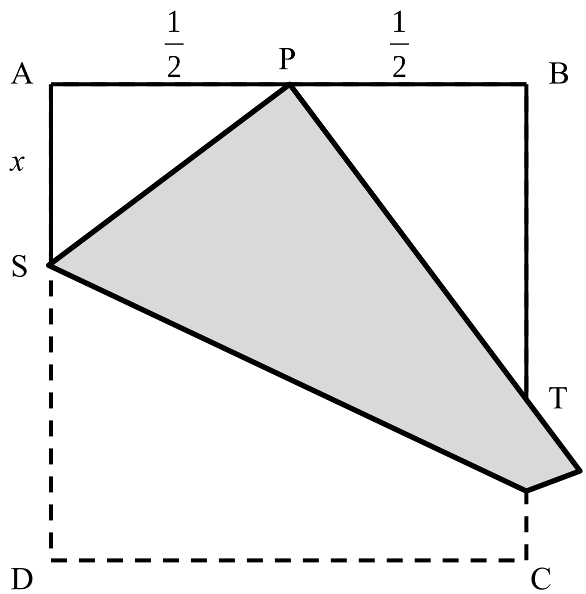

Figure 2. Haga's First Theorem fold [**1**, p. 3].

Since p is the midpoint of AB, the length of AP and BP are both $\frac{1}{2}$. Letting the length of AS be equal to x and using the Pythagorean Theorem yields the following equation.

$$(1-x)^2 = x^2 + \left(\frac{1}{2}\right)^2 \qquad (3)$$

Solving equation (3) for x yields $\frac{3}{8}$. Since $\triangle ASP$ is similar to $\triangle BPT$ the length of BT can be found by solving $\frac{\frac{3}{8}}{\frac{1}{2}} = \frac{\frac{1}{2}}{BT}$, which yields $BT = \frac{2}{3}$. Thus, point T trisects the segment BC. Another interesting result of Haga's fold is that $\triangle ASP$ is a *Pythagorean triangle*, a right triangle whose sides are integers. The ratio of sides of $\triangle ASP$ is $\frac{3}{8}$ to $\frac{1}{2}$ to $\frac{5}{8}$. And therefore $\triangle ASP$ is a 3-4-5 triangle [**1**, pp. 4-5].

By loosening the restriction on Haga's First Theorem fold, one can generalize the fold and find other rational numbers. Instead of folding one of the vertices to the midpoint of a nonadjacent side, it is folded to a different reference point, namely distances of binary fractions.

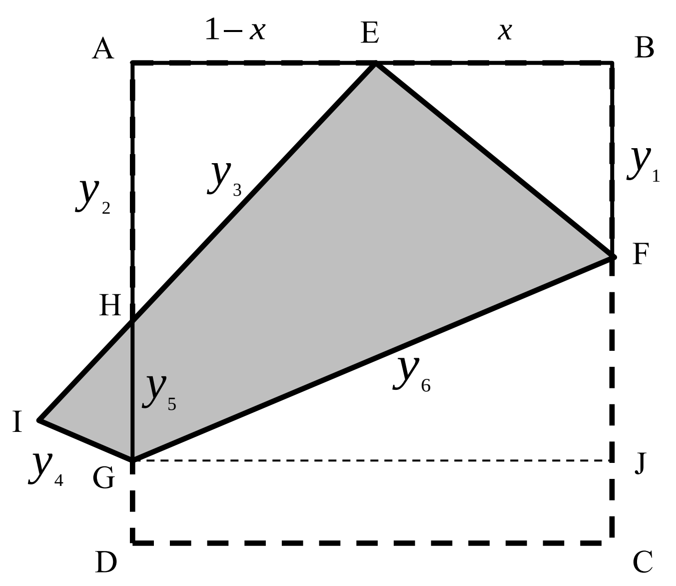

Figure 3. Generalization of Haga's First Theorem fold [**1**, p. 9].

Depending on the value of the distance x, a generalized Haga's fold yields different y_i values. Table 1 shows rational numbers constructed by Haga's fold with various initial reference points.

Table 1. Rational numbers constructed from generalized Haga's first theorem fold

x	½	1/4	3/4	1/8	3/8	5/8	7/8
y1	3/8	15/32	7/32	63/128	55/128	39/128	15/128
y2	2/3	2/5	6/7	2/9	6/11	10/13	14/15
y3	5/6	17/20	25/28	65/72	73/88	89/104	113/120
y4	1/8	9/32	1/32	49/128	25/128	9/128	1/128
y5	5/24	51/160	25/224	47/119	7/27	56/349	1/17
1-y1	5/8	17/32	25/32	65/128	73/128	89/128	113/128
1-y2	1/3	3/5	1/7	7/9	5/11	3/13	1/15
1-y3	1/6	3/20	3/28	7/72	15/88	15/104	7/120
1-y4	7/8	23/32	31/32	79/128	103/128	119/128	127/128

There are other ways to start from a binary fraction and construct fractions whose denominator is not a power of 2. One such method uses two crossing diagonals. Starting from a unit square, two bottom corners are connected to points on the opposite sides, as shown in Figure 4.

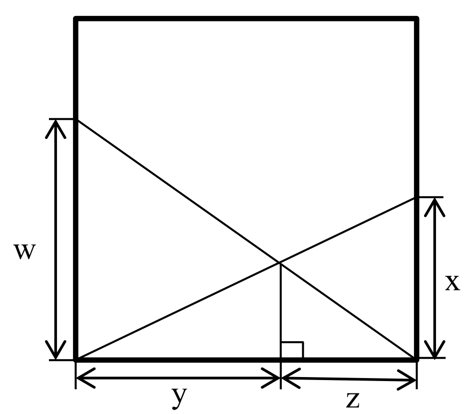

Figure 4. Crossing diagonals construction [**3**, p. 15].

The two diagonals have different slopes and thus intersect. Coordinates of the intersection can be found by solving a system of equation. From Figure 4,

$$y = \frac{w}{w + x}, z = \frac{x}{w + x} \tag{4}$$

The idea is to choose reference points that are easily constructible, i.e. binary fractions. Let $w = \frac{m}{p}$, and $x = \frac{n}{p}$, where m and n are integers and p is a power of 2. Then,

$$y = \frac{m}{m+n}, z = \frac{n}{m+n} \tag{5}$$

Table 2 shows some of constructible rational numbers by the two crossing diagonals method [**3**, pp.15- 16].

Table 2. Construction of rational numbers by crossing diagonals.

w	½	¼	1/8	1/8	1/8	1/16
x	1	1	5/8	3/4	1	9/16
y	1/3	1/5	1/6	1/7	1/9	1/10
z	2/3	4/5	5/6	6/7	8/9	9/10

Chapter 4: Axiomatic Origami

In practical paper folding, a few basic procedures are repeated in different combinations to create crease lines, and as a final product, origami figures. In modern origami, crease patterns are incredibly complex, and it is necessary to break down the pattern into basic steps that are easily understood. After all, the point of creating a crease pattern is to communicate folding procedures with others. Mathematicians have agreed upon a set of basic folding operations. In addition to being a communication tool, this set of operations helps mathematicians to compare origami operations with Euclidean ones. This set of operations define the following list.

(O1) Given two intersecting lines, a unique point of intersection $P = l_1 \cap l_2$ can be determined.

(O2) Given two parallel lines l_1 and l_2, line m parallel to and equidistant from them can be constructed.

(O3) Given two intersecting lines l_1 and l_2, an angle bisector can be constructed

(O4) Given two non-identical points P and Q, a line connecting the two can be folded.

(O5) Given two non-identical points P and Q, a unique perpendicular bisector of the segment PQ can be folded.

(O6) Given a point P and a line l, a unique line l' perpendicular to l and containing P can be folded.

(O7) Given two points P and Q and a line l, P can be folded onto l such that Q lies on the fold.

(O7*) Given two points P_1 and P_2 and two lines l_1 and l_2, one can simultaneously fold P_1 onto l_1 and P_2 onto l_2. [**5**]

Any Euclidean constructions can be done with a combination of basic origami operations, i.e. origami procedures (O1) – (O7). And any origami constructions done by origami procedures (O1) – (O7) can be achieved by Euclidean methods [**5,** pp. 362 - 365]. In that sense, Euclidean constructions are equivalent to constructions done by (O1) through (O7) of the origami procedures.

There are a few differences between origami operations and the Euclidean procedures. For instance, the basic entity of origami is a line, and a point is defined as an intersection of two lines. In Euclidean geometry, however, a point is the basic entity, and two points define a line. What really sets origami constructions apart from Euclidean construction is the origami procedure (O7*). The procedure (O7) describes folding a point onto a line in a specific way. A crease line resulting from such fold is a perpendicular bisector of a point P_1 and a point of a line l_1. Then, the set of these crease lines is precisely the set of tangents to the parabola with focus P_1 and the directrix l_1. As shown in Figure 5, (O7*) is a procedure for folding two points onto two lines, and thus it is equivalent to finding a simultaneous tangent of two parabolas with foci P_1 and P_2 and directrices l_1 and l_2 respectively [**5**].

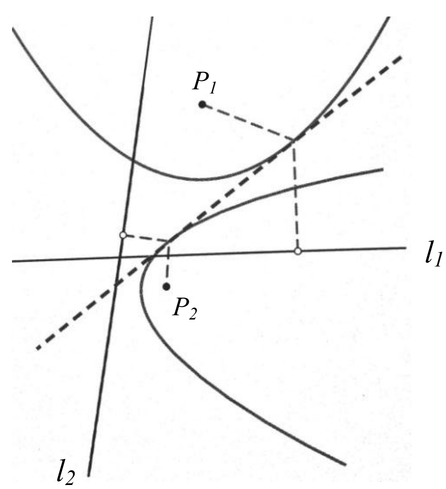

Figure 5. Common tangent of two parabolas [**2**, p. 288].

Finding a common tangent of two parabolas is equivalent to solving a cubic polynomial. This is impossible to do by the straightedge-and-compass method which can only solve quadratics [**3**]. Since (O7*) is the operation that allows for such a solution, it is reasonable to believe that (O7*) can be used to find cube roots [**5**].

Chapter 5: Solving A General Cubic Equation

Suppose two parabolas exist with equations

$$p_1 : (y-n)^2 = 2a(x-m) \tag{6}$$

and,

$$p_2 : x^2 = 2by, \tag{7}$$

where $a, b, x, m,$ and n are real numbers.

The common tangent of p_1 and p_2 is

$$y = cx + d, \tag{8}$$

where c and d are real numbers.

Let P_1 (x_1, y_1) be the tangent point of (6) and (8). The tangent line can also be represented as

$$(y-n)(y_1-n) = a(x-m) + a(x_1-m) \tag{9}$$

and solving (9) for y yields

$$y = \frac{a}{y_1-n} \cdot x + n + \frac{ax_1 - 2am}{y_1-n}. \tag{10}$$

Therefore, the slope of the line c is

$$c = \frac{a}{y_1-n}, \tag{11}$$

and the y-intercept d is

$$d = n + \frac{ax_1 - 2am}{y_1 - n}. \tag{12}$$

Solving (11) for y_1 and (12) for x_1 yields,

$$y_1 = \frac{a + nc}{c} \quad and \quad x_1 = \frac{d - n}{c} + 2m. \tag{13}$$

Substituting (13) in (6) and solving for a yields,

$$a = 2c\left(d - n + cm\right). \tag{14}$$

Suppose $P_2(x_2, y_2)$ is the point where (8) is tangent to (7). Then (8) can be expressed as

$$xx_2 = by + by_2 \tag{15}$$

and solving for y yields,

$$y = \frac{x_2}{b} \cdot x - y_2. \tag{16}$$

The slope of the line tangent to parabola p_2 is

$$c = \frac{x_2}{b}, \tag{17}$$

and the y-intercept is

$$d = -y_2. \tag{18}$$

Solving (17) for x_2 and (18) for $y_2$yields,

$$x_2 = bc \quad and \quad y_2 = -d. \tag{19}$$

Substituting x_2 for x and y_2 for y in (7), and solving for d yields,

$$d = -\frac{bc^2}{2} \tag{20}$$

Combining (20) and (14) produces

$$a = 2c\left(-\frac{bc^2}{2} - n + cm\right). \tag{21}$$

From (21), distributing *2c* and subtracting *a* from both sides then dividing both sides of the equation by *b* yields

$$c^3 - \frac{2m}{b} \cdot c^2 + \frac{2n}{b} \cdot c + \frac{a}{b} = 0. \tag{22}$$

Therefore, the slope of the tangent line *c* is the solution of the cubic equation in (22). Note that cubic polynomials can be written as

$$x^3 + px^2 + qx + r = 0, \tag{23}$$

where *p*, *q* and *r* are real numbers. Without loss of generality, let *b* in (7) be the unit length 1. Then

$$m = -\frac{p}{2}, n = \frac{q}{2} \text{ and } a = r. \tag{24}$$

Thus, (23) can be written as (22). Once m, n and r are calculated, one can substitute these values in (6) and find the focus and directrix of one of the parabolas. The focus, F_1, and the directrix, l_1, of parabola p_1 is given by

$$F_1\left(-\frac{p}{2} + \frac{r}{2}, \frac{q}{2}\right) \text{ and } x = -\frac{p}{2} - \frac{r}{2}. \tag{25}$$

The focus F_2 of p_2 and its directrix l_2 is

$$F_2\left(0, \frac{1}{2}\right) \text{ and } y = -\frac{1}{2}. \tag{26}$$

Simultaneously folding foci of parabolas p_1 and $p_2$onto directrices l_1 and l_2 respectively, gives the common tangent to the two parabolas p_1 and p_2 whose slope is the solution of the given cubic equation in (23) [**5**, pp. 366 – 369].

Chapter 6: Trisecting angles

Using the previously defined set of operations for origami, trisecting an angle can be done by solving the Chebychev equation [4]. The equation

$$4x^3 - 3x = \cos(3\theta) \tag{27}$$

is equivalent to

$$x^3 - \frac{3}{4}x - \frac{1}{4}\cos(3\theta) = 0. \tag{28}$$

Then, assuming $\cos(3\theta)$ is known, and using (25), one can fold the focus

$$F_1\left(-\frac{1}{8}\cos(3\theta), -\frac{3}{8}\right) \tag{29}$$

onto the directrix

$$x = \frac{1}{8}\cos(3\theta). \tag{30}$$

(26) is used for the focus and the directrix of the second parabola [**5**, p370]. This method does violate Euclidean postulates relating to constructions. A purely origami construction is given by Abe [**7**] and is shown in Figure 6 and 7.

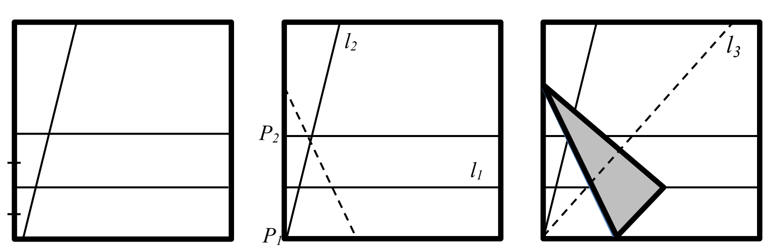

Figure 6. Abe's angle trisection method [**7**].

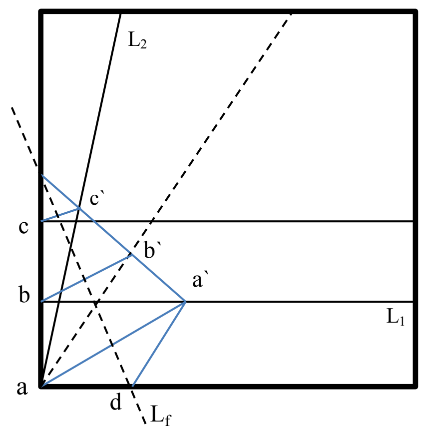

Figure 7. Abe's trisection explained [**2**, p. 287].

From Figure 7, since b is the midpoint of segment ac by construction and a' is on L_1, $\triangle aa'c$ is an isosceles triangle and segment $a'b$ bisects the $\angle\, aa'c$. Note that $\triangle\, a'ac'$ is a reflection of the $\triangle\, aa'c$ across the crease line L_f, and thus $\triangle\, a'ac'$ is also an isosceles triangle and segment ab' bisects the $\angle\, a'ac'$. Therefore $\angle\, c'ab'$ and $\angle\, a'ab'$ are congruent. Let the $m\angle\, c'ab'$ and the $m\angle\, a'ab'$ be α. Since segment $a'c'$ is perpendicular to segment ab', and $\angle\, c'a'd$ is a 90° angle, $m\angle\, aa'd$ is also α. But segment ad and segment $a'd$ are congruent, triangle ada' is an isosceles triangle making the $m\angle\, a'ad$ equal to α. Therefore,

$$m\angle c`ab` = m\angle b`aa` = m\angle a`ad = \alpha . \tag{31}$$

Chapter 7: Doubling a cube

It can be shown that doubling a cube is a matter of solving a rather simple cubic equation

$$x^3 - 2 = 0 \tag{32}$$

It can be done by constructing a common tangent between two parabolas. One with focus $F_1(-1, 0)$, and directrix x=1, and the other parabola with focus and directrix as in (26). This is displayed in Figure 8.

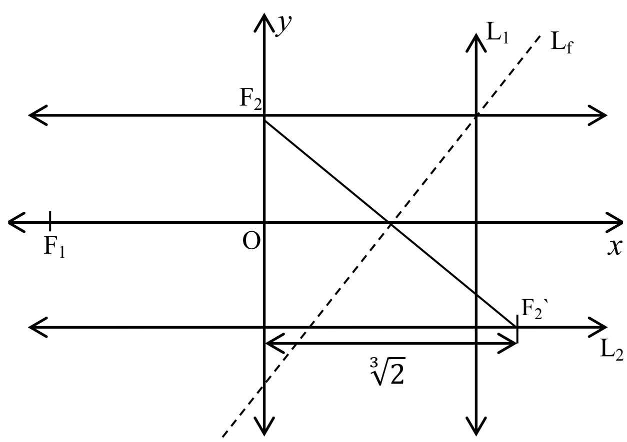

Figure 8. Construction of $\sqrt[3]{2}$.

The slope of the crease line L_f is $\sqrt[3]{2}$. Since segment F_2F_2' is perpendicular to L_f, its slope is opposite reciprocal of $\sqrt[3]{2}$. Therefore, F_2' marks the distance $\sqrt[3]{2}$ from the y-axis.

Chapter 8: Conclusion

The connection between paper folding and its applications to mathematics have been shown. Efficient methods of locating a reference point in the form of a binary fraction were presented. It was also shown that other rational numbers can be folded. There were many different folds that can construct rational numbers. Haga's First theorem and the crossing diagonals method were discussed. Surprisingly, Haga's fold can construct so many rational numbers with only one crease. The crossing diagonals method, which uses the fact that two intersecting lines have different slope, has shown to fold rational numbers $\frac{1}{n}$ for natural numbers n such that $n \leq 10$.

Axioms, or allowed operations, of origami were presented. Operations (O7) and (O7*) show surprising connection between origami and tangents of conic sections. It was seen that (O7*) is equivalent to finding a common tangent to two parabolas, and is analytically a cubic problem. Thus, it was shown that origami can solve cubic problems that cannot be solved by straightedge and compass. Solutions to two of the three such problems, namely trisecting an angle, and doubling a cube, were also presented which shows that paper folding can be a more powerful tool than simply using Euclidean construction with straightedge and compass.

Paper folding can be used in classrooms to introduce concepts in geometry. Some constructions are much easier to do with paper folding. For example, finding a midpoint, constructing a perpendicular bisector of a segment and constructing an angle bisector are matters of folding a point onto a point or a line onto a line. Also, parallel lines and transversals can be studied using paper folding.

Another benefit of teaching geometry with origami is that students already have experience folding paper. Everyone has folded a piece of paper before, and no one is intimidated with folding paper, which is not the case with straightedge and compass. In that sense, paper folding levels the playing field for all learners inviting all to participate. Paper folding also encourages students to make conjectures about a certain fold and convince others why it is so, which is a great way to introduce the mathematical concept of proof.

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