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Pythagorean Theorem Extensions

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Pythagorean Theorem Extensions

by

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Abstract

Pythagorean Theorem Extensions

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This report expresses some of the recent research surrounding the Pythagorean Theorem and Pythagorean triples. Topics discussed include applications of the Pythagorean Theorem relating to recursion methods, acute and obtuse triangles, Pythagorean triangles in squares, as well as Pythagorean boxes. A short discussion on the depth of the Pythagorean Theorem taught in secondary schools is also included.

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Chapter 1: An Introduction

Although the Pythagorean Theorem is named after the mathematician Pythagoras who lived around the 500 B.C. period, it was discovered much earlier by the Babylonian, Chinese, and Egyptian civilizations dating as far back as 1800-1500 B.C. Pythagoras, however, was likely the first to formally state the formula that had been found independently by several others. Egyptian peasants, according to a well known story, used a rope with evenly spaced knots to find that a right triangle could have lengths of 3, 4, and 5, known to some as the "Egyptian triangle" [**5**, p. 261]. It is documented that Pythagoras realized the property while waiting to be received by the tyrant Polycrates in the palace of Samos. Pythagoras confirmed the property in his mind while staring at the square tiles on the palace floor [**5**, p. 260].

The Pythagorean Theorem is learned in and reinforced throughout grade school. However, most students are only exposed to the surface of the concept: the sum of the squares of the legs of a right triangle is equal to the square of the hypotenuse, otherwise known as

$$a^2 + b^2 = c^2$$

where $a, b, c \in \mathbb{R}^+$. Figure 1 depicts the theorem in picture form, using the area of the square attached to each side of the right triangle.

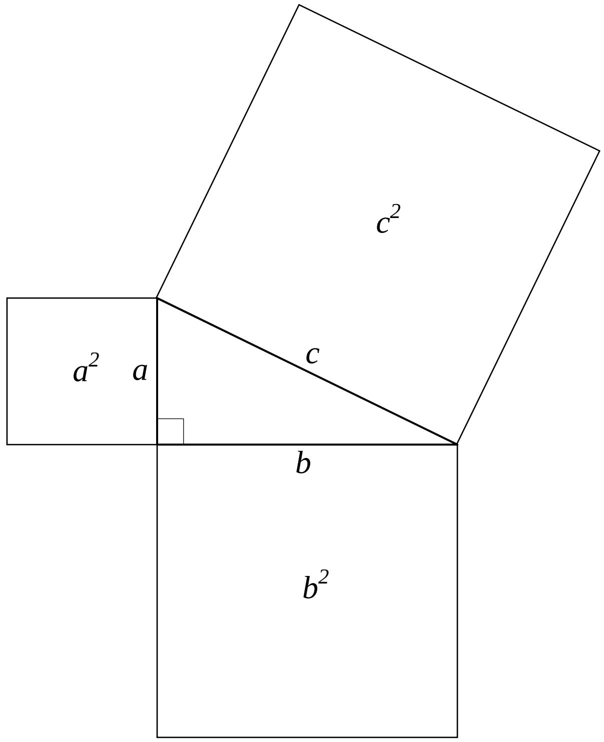

Figure 1. The Pythagorean Theorem [**5**, p. 260].

A special case of the Pythagorean Theorem is referred to as *Pythagorean triples* which are defined as three positive integers that satisfy the Pythagorean Theorem. Students are encouraged to memorize the first few *primitive*, or relatively prime, triples such as (3, 4, 5) and (5, 12, 13) when practicing and applying the theorem. These triples can be multiplied by constants which produce new non-primitive triples.

Although the Pythagorean Theorem was first proven long ago, new applications are still being made specifically focusing around the use of Pythagorean triples. It has been known for some time that there are infinitely many primitive triples. A recently created recursion method identifies all Pythagorean triples in a more efficient way than that of the Babylonians. Another new development focuses on the fact that Heronian triangles can be identified using a parametrization of Pythagorean triples. The question concerning how many Pythagorean triangles a square is able to be dissected into was answered as well. Taking the concept into three dimensions, Pythagorean boxes and quadruples also have their own parametrization. The extension of Pythagorean triples is obviously vast and perhaps still mostly unknown. In the following, some of these recognized findings are expounded upon and investigated.

Chapter 2: Pythagorean Triple Recursions

The first few primitive Pythagorean triples are rather easy to memorize and use when necessary. However, the question of how one might find all the other triples arises. The Wade father-son team produced a recursion method that produces all Pythagorean triples [**6**, p. 98]. While the following formula discovered by the Babylonians has been known for at least four thousand years and was proven by Fibonacci, it produces all of the primitive triples as well as some unreduced triples:

$$a = m^2 - n^2, b = 2mn, c = m^2 + n^2 \tag{1}$$

where $m, n \in \mathbb{Z}^+$. The Wades, on the other hand, found a formula that was capable of producing all primitive and all unreduced triples.

To begin, assume that $a < b < c$. Define the *distance* between two triples (a,b,c) and (d,e,f) by $D = |d - a|$ and the *height* of a triple (a,b,c) by $H = |c - b|$. For any H such that $H \in \mathbb{Z}^+$, there exist recursions that construct all triples of height H. In the specific case where $H = 1$,

$$1 = H = |c - b| = |m^2 + n^2 - 2mn| = |(m-n)^2|$$

or more generally, $1 = (m-n)^2$; thus, $m = n + 1$. After substituting into formula (1):

$$a_n = m^2 - n^2 = (n+1)^2 - n^2 = 2n + 1 \tag{2}$$

and

$$b_n = 2mn = 2(n+1)n = 2n^2 + 2n \tag{3}$$

which generates all primitive triples such that $H = 1$. Transforming equations (2) and (3) into a recursion requires the application of the facts that $D = 2$ and $b_{n+1} - b_n = 2a_n + 2$, granting a recursion that produces all reduced and unreduced Pythagorean triples with a height of one:

$$a_{k+1} = a_k + 2, b_{k+1} = 2a_k + b_k + 2, c_{k+1} = 2a_k + c_k + 2$$

where $a_0 = 3, b_0 = 4, c_0 = 5$ and $k = 0,1,2,\ldots$. After observing this first case, the following theorem can be stated.

Theorem 1. Let (a_0, b_0, c_0) be a Pythagorean triple of height H, D be a positive integer, and $\beta = \frac{D}{H}$. If βa_0 and $\frac{\beta D}{2}$ are integers, then the recursion

$$a_{k+1} = a_k + D, b_{k+1} = \beta a_k + b_k + \frac{\beta}{2} D, c_{k+1} = \beta a_k + c_k + \frac{\beta}{2} D$$

generates Pythagorean triples of height H (for $k = 0,1,2,\ldots$), and the distance between each consecutive pair (a_k, b_k, c_k) and $(a_{k+1}, b_{k+1}, c_{k+1})$ is precisely D [**6**, p. 99].

Theorem 1 can be proven by mathematical induction on k. However, the proof only holds for solutions of height H such that H is either a perfect square or *double square*, a perfect square multiplied by two. This leads to the next necessary theorem.

Theorem 2. Suppose H is neither a perfect square nor a double square, and that H has no divisors except 1. If $(a_0, b_0, c_0) = (3H, 4H, 5H)$ and $D = 2H$, then the recursion in Theorem 1 generates all Pythagorean triples of height H [**6**, p. 99].

The proof of Theorem 2 first shows that every triple of height H is a multiple of a triple with lower height H_0, since no triple of height H can be generated by (1) that satisfies the assumptions of Theorem 2. Therefore, $(a,b,c) = p(A,B,C)$ where $p \in \mathbb{Z}^+$. Disregarding when $H_0 = 1$ since it is a perfect square, notice that the height is pH_0, and H_0 must divide H. (A,B,C) must then be a multiple of a triple with a lower height than H_0 leading to conclude that (a,b,c) must be a multiple of a triple with height 1 and that the recursion in Theorem 1 generates all triples of height H when $D = 2H$ and $(a_0, b_0, c_0) = (3H, 4H, 5H)$ [**6**, p. 100].

Chapter 3: Triangles and $A = mP$

Markov discovered an algorithm for finding all Heronian triangles for which the area, A, is an integer multiple ($m \in \mathbb{N}$) of the perimeter, P [**3**, p. 115]. *Heronian* describes triangles that have integer sides and integer area. Throughout the 20th century, variations of this problem have been introduced and some solved. For example, it was shown by Whitworth and Biddle in 1904 that there only exist five triangles that satisfy the property $A = P$. Subbarao established that integer-sided triangles where $A = \frac{1}{\lambda}P$ for $\lambda \geq 3$ do not exist. Goehl was the first to state the problem involving $A = mP$ and subsequently solved this problem for the special case of right triangles only in 1985. Markov took Goehl's solution and expanded the procedure to the other cases of acute and obtuse triangles [**3**, p. 115].

This expanded procedure is achieved by beginning with a parametrization of the primitive Pythagorean triples proven by Beauregard and Suryanarayan [**3**, p. 114]. This parametrization is expressed as

$$\left(\frac{u^2 - v^2}{2}, uv, \frac{u^2 + v^2}{2}\right) \tag{4}$$

where u and v are integers which are relatively prime and odd. Recall that Heron's formula states

$$A = \sqrt{s(s-a)(s-b)(s-c)} \tag{5}$$

with $s=\frac{a+b+c}{2}$. Note that (5) can be written and rewritten in the following ways using algebraic manipulation and factoring:

$$4A=\sqrt{(a+b+c)(a+b-c)(a+c-b)(b+c-a)}\,,$$

and

$$[c^2-(a^2+b^2)]^2+(4A)^2=(2ab)^2\,, \tag{6}$$

and

$$[c^2-(a^2+b^2)]^2+[4m(a+b+c)]^2=(2ab)^2\ . \tag{7}$$

Equation (6) appears similar to the form of the Pythagorean Theorem. Therefore, combining (4) and (6), the problem can be simplified by saying that a triangle will only be Heronian if its components are even; if two of the components are even, then the third must also be. Note that $4A$ and $2ab$ are even, making $c^2-(a^2+b^2)$ even as well. Using this implication and the primitive triple parametrization (4), one can state that any non-primitive Pythagorean triple will be a multiple of a primitive triple and even. Thus yielding the following parametrizations:

$$\left(2n\frac{u^2-v^2}{2},2nuv,2n\frac{u^2+v^2}{2}\right)$$

for $n\in\mathbb{N}$, and

$$(k(u^2-v^2),2kuv,k(u^2+v^2)) \tag{8}$$

for $k\in\mathbb{N}$. All triples of the form in (8) must be considered in order to solve the problem. The expressions in (7) and (8) must be respectively equivalent as well, culminating in a system of three equations in six unknowns with m being a fixed value:

$$\begin{cases} \pm[c^2-(a^2+b^2)]=k(u^2-v^2) \\ 4m(a+b+c)=2kuv \\ 2ab=k(u^2+v^2) \end{cases}.$$

To solve the system, Markov proves that for any divisor d of the quantity $\left(\frac{16m^2}{u}\right)^2(u^2+v^2)$, there corresponds a value k for the obtuse case: $c^2-(a^2+b^2)>0$.

Similarly, a corresponding k exists for any divisor d of the quantity $\left(\frac{16m^2}{v}\right)^2(u^2+v^2)$ for the acute case: $c^2-(a^2+b^2)<0$; the only difference being u and v are interchanged [**3**, pp. 115-119]. Solving the system requires several steps of algebraic manipulation resulting in the following formulas for the sides of an obtuse triangle:

$$\begin{cases} a=\dfrac{(16m^2+d)u^2+16m^2v^2}{8muv} \\ b=\dfrac{2m(16m^2+d)(u^2+v^2)}{duv} \\ c=\dfrac{(16m^2+d)^2u^2+(16m^2)^2v^2}{8mduv} \end{cases}. \tag{9}$$

The formulas for the acute case are the same with the exception that u and v are naturally interchanged. It is possible to generate noninteger solutions, which must be excluded for the purpose of this problem. Markov summarizes his findings as an algorithm using the greatest integer function as in Theorem 3 below:

Theorem 3. The following algorithm solves the problem $A=mP$:

1. For a fixed m, find all divisors u of $2m$.

2. For each u, find all v relatively prime to u such that $1 \le v \le \lfloor \sqrt{3}u \rfloor$, where $\lfloor \ \rfloor$ indicates the greatest integer function.

3. To find the obtuse-triangle solutions: Select $v < u$ for every pair u, v and find all divisors d of $\left(\frac{16m^2}{u} \right)^2 (u^2 + v^2)$ such that $d \le \left\lfloor \frac{16m^2}{u} \sqrt{u^2 + v^2} \right\rfloor$. Then for every u, v, d, determine a, b, c from (9).

4. To find the acute-triangle solutions: Select $u < v \le \lfloor \sqrt{3}u \rfloor$ for every pair u, v and find all divisors d of $\left(\frac{16m^2}{u} \right)^2 (u^2 + v^2)$ such that $\frac{16m^2}{u^2}(v^2 - u^2) \le d \le \left\lfloor \frac{16m^2}{u} \sqrt{u^2 + v^2} \right\rfloor$. Then for every u, v, d, determine the sides a, b, c from (9).

5. To find the right-triangle solutions: Let $u = v = 1$ and find all divisors d of $2(16m^2)^2$ such that $d \le \lfloor 16m^2 \sqrt{2} \rfloor$. For every d, determine a, b, c from

$$\begin{cases} a = \dfrac{32m^2 + d}{8m} \\ b = \dfrac{4m(16m^2 + d)}{d} \\ c = \dfrac{(16m^2 + d)^2 + (16m^2)^2}{8md} \end{cases}$$

which were reduced from the equations in (9).

6. Discard the non-integer solutions [**3**, pp. 119-120].

Chapter 4: Squares and Pythagorean Triangles

Laczkovich concluded that a square could not be split into 30°-60°-90° triangles as well as that a square cannot be divided into triangles whose angles have even degree measures in 1990 [**2**, p. 284]. This led to the question of how many *Pythagorean triangles*, which are right triangles with integer sides, are possible in a dissected square.

It was found that two and three triangles are impossible mainly because $\sqrt{2}$ is irrational. Four triangles are impossible which turns out to be a remarkable case. Five triangles are feasible as well as every integer greater than five as stated in Theorem 4:

Theorem 4. Given a positive integer m, there exists a square that can be dissected into m Pythagorean triangles if and only if $m \geq 5$ [**2**, p. 284].

In order to prove that five Pythagorean triangles can be created, a square must first be dissected into four triangles that simply have integer sides, but not necessarily be right triangles, as seen in the following figure.

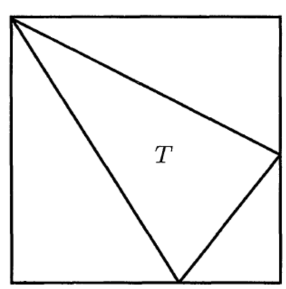

Figure 2. A square dissected into four triangles with integer sides [**2**, p. 285].

Note that T does not have to be a right triangle in Figure 2. To make five Pythagorean triangles in the square, T must be split into two right triangles.

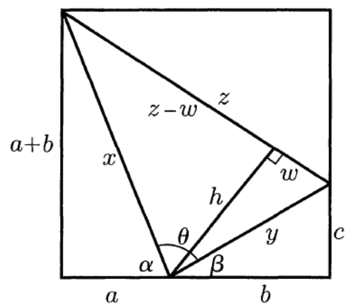

Figure 3. A square dissected into five right triangles [**2**, p. 285].

From Figure 3, it must be shown that legs h, w, and $z-w$ of the two new triangles are rational and ultimately possible integers, so the law of sines and trigonometry must be employed. The area of triangle T can be stated as:

$$A = \frac{1}{2} xy \sin\theta .$$

After substitution of the following and the fact that $\theta = \pi - \alpha - \beta = \pi - (\alpha + \beta)$,

$$\sin\theta = \sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta = (\frac{a+b}{x} \cdot \frac{b}{y}) + (\frac{a}{x} \cdot \frac{c}{y}),$$

then

$$A = \frac{1}{2}[(a+b)b + ac]. \tag{10}$$

An alternative area formula is

$$A = \frac{1}{2} zh, \tag{11}$$

thus determining a rational number for h using equations (10) and (11):

$$h = \frac{1}{z}[(a+b)b + ac].$$

The Pythagorean Theorem equations from the two triangles are

$$h^2 + w^2 = y^2$$

and

$$h^2 + (z-w)^2 = x^2,$$

yielding a rational number for w and, consequently, $z-w$ as well:

$$w = \frac{z^2 + y^2 - x^2}{2z}.$$

In order to reach the desired result of having integer sides, h, w, and $z-w$ must all be multiplied by the denominators' common multiple. An example of a square with five Pythagorean triangles is shown in the following figure.

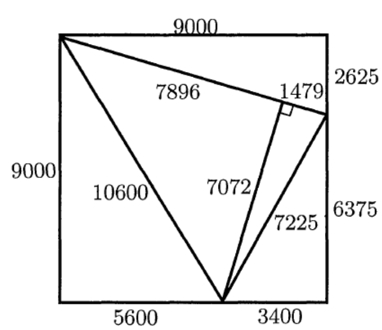

Figure 4. Five Pythagorean triangles [**2**, p. 285].

The next step in the proof considers further dissection of the square into $m+1$ triangles. Using the fact that any right triangle can be split into two smaller right triangles, where all three are similar to each other, the existence of $m+1$ triangles in a square is easily demonstrated, ending the proof for the time being.

Jepsen and Yang return to the case where $m=4$ to fully prove Theorem 4 [**2**, p. 286]. There are three possible arrangements for a square to be dissected into four Pythagorean triangles as depicted in Figure 5.

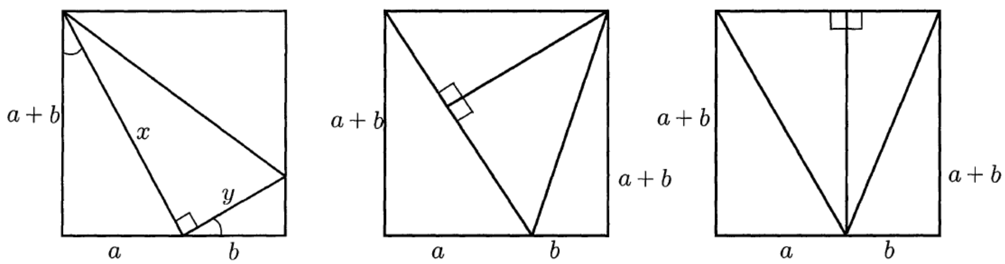

Figure 5. Possible arrangements for four right triangles [**2**, p. 286].

In all three configurations, the pairs a, $a+b$ and b, $a+b$ must be the legs of Pythagorean triangles. Jepsen and Yang state the conclusion that these pairs of legs do not exist using a proof by contradiction. Suppose that a, b, m, and n are positive integers with a and b relatively prime such that:

$$\begin{cases} a^2+(a+b)^2=m^2 \\ b^2+(a+b)^2=n^2 \end{cases}. \tag{12}$$

After substitution and the introduction of several other variables, an elliptic curve is produced where the rational points all occur when $y=0$:

$$y^2=(x+1)(x+2)(x-3)=x^3-7x-6.$$

It can be deduced that the nonexistence of rational points (x, y) where $y \neq 0$ implies that there are no integer solutions to the equations in (12) [**2**, p. 288]. The inference can now be made that a dissection of a square into four Pythagorean triangles is impossible, completing the proof of Theorem 4 where the number of triangles must be five or greater.

Chapter 5: Pythagorean Boxes

Consider adding a fourth variable to the Pythagorean Theorem and exploring the 3-dimensional aspect of the concept. Beauregard and Suryanarayan investigated the idea of Pythagorean boxes when compared to Pythagorean triples [**1**, p. 222]. A *Pythagorean box* is described as a rectangular prism whose edges and inside diagonals are integers represented by ordered, integer quadruples (x, y, z, w) and satisfy the equation

$$x^2 + y^2 + z^2 = w^2 \ , \tag{13}$$

where $w > 0$ [**1**, p. 222] and is the diagonal of the box. Beauregard and Suryanarayan's goal was to examine the geometric and algebraic properties of Pythagorean boxes while comparing these properties with those of Pythagorean triples. Resembling triples and the unit circle, every box correlates to a rational point on the unit sphere. The Babylonian Pythagorean triple parametrization can be written as

$$(n^2 - m^2, 2nm, n^2 + m^2)$$

where $n, m \in \mathbb{Z}^+$; boxes have analogous parametrizations as well.

Similar to Pythagorean triples, quadruples can be generated by multiplying each term by the same constant, in addition to being formed by a product of primitive boxes. Thus, the following operation $*$ defined by Beauregard and Suryanarayan can be used to construct nonprimitive boxes:

$$(x_1, y_1, z_1, w_1) * (x_2, y_2, z_2, w_2) = (x_1 w_2 + w_1 x_2, y_1 z_2 + z_1 y_2, z_1 z_2 - y_1 y_2, x_1 x_2 + w_1 w_2).$$

The equation in (13) can be described by the matrix:

$$\begin{bmatrix} w & x & 0 & 0 \\ x & w & 0 & 0 \\ 0 & 0 & z & y \\ 0 & 0 & -y & z \end{bmatrix}.$$

The matrix's determinant is nonzero, therefore allowing an inverse to exist which is consistent with the following quadruple representation otherwise known as a rational box:

$$\frac{1}{y^2+z^2}(-x,-y,z,w),$$

giving way to identifying more Pythagorean boxes.

Beauregard and Suryanarayan define $\sim$ as a relation where, if there exists $n,m \in \mathbb{Z}^+$ such that

$$(nx_1, ny_1, nz_1, nw_1) = (mx_2, my_2, mz_2, mw_2),$$

then

$$(x_1, y_1, z_1, w_1) \sim (x_2, y_2, z_2, w_2).$$

This relation identifies all boxes of the form (nx, ny, nz, nw). Another way to identify a Pythagorean box is devised by considering the fact that the square of an odd integer reduces to 1 (mod 8). Therefore, at least two variables on the left side of equation (13) must be even, namely y and z, without loss of generality. It follows that x and w must be odd if (x, y, z, w) is primitive and nondegenerate ($xyz \neq 0$) [**1**, p. 224]. The quadruple can thus be defined by

$$x = \frac{a^2+b^2-c^2}{c}, y = 2a, z = 2b, w = \frac{a^2+b^2+c^2}{c}$$

where $a, b, c \in \mathbb{Z},\ c > 0$, and a and b are not both 0. Since this is a quadruple, it can be conversely stated that a, b, c can be formulated using the equations

$$a = \frac{y}{2}, b = \frac{z}{2}, c = \frac{(w - x)}{2}$$

yielding another way to calculate and identify Pythagorean boxes.

The positive integer w is defined as the inside diagonal of a Pythagorean box if and only if $w \neq 2^i$ or $w \neq 2^i \times 5$ (where i is a nonnegative integer) [**1**, p. 226]. Exploring only primitive boxes, w must be an odd integer such that $w \neq 1,5$. Every positive integer is one of the first components of a quadruple, so letting n be a positive, odd integer greater than 1 produces

$$(n, \frac{n^2 - 1}{2}, \frac{n^4 + 2n^2 - 3}{8}, \frac{n^4 + 2n^2 + 5}{8})$$

as a Pythagorean quadruple. For positive even integers, the box parametrization is

$$(h^2 + k^2 - 1, 2h, 2k, h^2 + k^2 + 1)$$

where $h, k \in \mathbb{Z}$. The special case where $h = k$ allows every positive even integer to be the dimensions of a square side of a Pythagorean box, since y and z are composed of $2h$ and $2k$, respectively. The quadruple can be stated as (x, y, y, w) for this case as $y = z$. The box is now a square prism with a $y \times y$ base and height x satisfying the equation

$$x^2 = w^2 - 2y^2 \ . \qquad (14)$$

Equation (14) along with a corresponding solution set can be used to generate an infinite amount of quadruples for square prisms.

Beauregard and Suryanarayan extend the discussion to the existence of perfect Pythagorean boxes: rectangular prisms where all the sides are *Pythagorean rectangles*—the side lengths and diagonal form a Pythagorean triple. The smallest known example is (153, 104, 672, 697), since (153, 104, 185) and (104, 672, 680) are both Pythagorean triples displaying that all dimensions of the box, including the diagonals, are integers [**1**, pp. 226-227].

Chapter 6: A Conclusion

The extensions of the Pythagorean Theorem and its triples continuously prove to be an open frontier for mathematicians. The Wade recursion is a grand accomplishment as it is capable of finding all primitive triples, something that was not able to be done previously with the Babylonian recursion method. The utilization of the distance and height of triples enabled the recursion to be discovered. Markov employed the parametrization of Pythagorean triples and applied it towards acute and obtuse triangles to find Heronian triangles where the area is an integer multiple of the perimeter. Using the Pythagorean Theorem along with acute and obtuse triangles displays the authority the theorem has in geometry. Pythagoras most likely did not contemplate how a square might be dissected into right triangles with integer sides, which are now named after the famous mathematician, but it seems to be an extended concept of his study of the square tiles in the palace hall long ago. One might suppose that Jepsen and Yang are intellectual descendants of Pythagoras himself. The three-dimensional version of the theorem involving Pythagorean boxes is truly an extension of the original. The similarities and differences between the two related topics determined by Beauregard and Suryanarayan provide perspective on the connections mathematical concepts hold. The current discoveries surrounding the Pythagorean Theorem exhibit how much content is still unknown today and waiting to be investigated and studied.

In secondary schools, teachers often do not have time to delve as deep into a topic as one would like. Most secondary students explore Pythagorean triples lightly in Geometry but do not extend past simple memorization of the first few primitive triples and how to find unreduced triples. The National Council of Teachers in Mathematics stresses that all students in grades nine through twelve should "establish the validity of geometric conjectures using deduction, prove theorems, and critique arguments made by others" [**4**]. Students may not be able to prove the theorems discussed but could come up with a conjecture on their own and defend it, given enough time. Students could also be given a list of Pythagorean triples and asked to explore the relationships found between them. Calculating distance D and height H could be added to the discovery process as well, further pushing them to see the various correlations available to be determined. In the secondary school level and beyond, the Pythagorean Theorem is an undeniably powerful truth in geometry and mathematics overall.

Bibliography

1. Beauregard, Raymond A., E.R. Suryanarayan, Pythagorean Boxes, *Mathematics Magazine* 74, No. 3 (2001) 222-227.

2. Jepsen, Charles, Roc Yang, Making Squares from Pythagorean Triangles, *The College Mathematics Journal* 29, No. 4 (1998) 284-288.

3. Markov, Lubomir P., Pythagorean Triples and the Problem $A = mP$ for Triangles, *Mathematics Magazine* 79, No. 2 (2006) 114-121.

4. Principles and Standards for School Mathematics. 2000. Retrieved November 24, 2010. http://standards.nctm.org/.

5. Veljan, Darko, The 25000-Year-Old Pythagorean Theorem, *Mathematics Magazine* 73, No. 4 (2000) 259-272.

6. Wade, Peter W., William R. Wade, Recursions That Produce Pythagorean Triples, *The College Mathematics Journal* 31, No. 2 (2000) 98-101.

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