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A Summary of M396C: Analysis and the Real Line
UTeach Summers Master's course
Mathematics Department at the University of Texas at Austin

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by

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Report

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Dedication

This paper is dedicated to my wife, Denise Boyd, in recognition and appreciation for the help and unwavering support that she has provided to me in this endeavor.

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Abstract

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The University of Texas at Austin, 2011

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The purpose of this paper is to review and summarize the topics involved in the study of real analysis. Real analysis is a branch of mathematics that studies the field of real numbers including the calculus of real numbers, analytical properties of real functions and sequences. This includes limits of sequences of real numbers, continuity, completeness, and related properties of real functions. While all topics in the course were important and vital to understanding analysis, the goal of this paper is to review, research, and report on a few of the more interesting topics covered in the class.

Table of Contents

List of Tables

Chapter 1: Introduction

The number line has been a part of the study of mathematics since the beginning of the use of the term "number". "Number" originally meant something that could be counted, or referring to such things as horses, cows, or sons. These *counting numbers* are also called the *natural numbers.* Natural numbers can be defined by the set $\{1,2,3,...\}$. In a counting system some notation was needed to indicate nothing, and the idea of "zero" came into use. The set of natural numbers in union with 0 is called the *whole numbers*. The concept of negative numbers took longer to come into use, but the negative numbers combined with the whole numbers yields the set of *integers.* The concept of fractions was devised for use when one is not dealing with a whole (as a half-bucket of water). The set of fractions in union with the set of integers is known as the set of *rational numbers.* Rational numbers are defined as all numbers in the form $\frac{a}{b}$, where a and b are integers ($b \neq 0$). A number that cannot be written in this form (cannot be expressed a ratio of integers) is an *irrational number* [**1**].

The Real Numbers is the set of all the rational and irrational numbers. These numbers can also be defined as all points on the number line or all possible distances on the number line. The real numbers have the property of being *ordered,* which means that given any two different numbers, one number will always be greater than or less than the other. This means that the numbers can be arranged along a line (the line consisting of an infinite number of points so as to form a solid line). The points are ordered on the line so that points to the right are greater than points to the left.

Analysis can be defined as the study of functions, the real number line, and the ideas of continuity and limit. Many of these concepts and properties are learned at lower levels of mathematics but real analysis is more of a rigorous development of these properties. Analysis is a large area of study and is often subdivided into smaller areas of study that may be labeled Calculus and Real Analysis, Complex Analysis, Differential Equations, Theory of Functions, Numerical Analysis, and Optimization. The Analysis and the Real Line course explored many of these areas of mathematics and will be reviewed in this paper.

Chapter 2: The Isoperimetric Problem

Much has been written about the Isoperimetric Problem, the Isoperimetric Inequality, or the Isoperimetric Proof. Known since the time of ancient scholars, the most essential contribution towards a rigorous proof was given by Jacob Steiner (1796-1883) in 1841. While Steiner accepted the validity of the analytic method (Calculus), he only used synthetic methods (Geometry) [**2**].

The Isoperimetric Problem used for study in the Analysis and the Real Line class is presented by the instructor as two questions:

1) Consider all smooth closed curves in the plane having a given length. Is there a least one that encloses a greatest area?

2) Consider all smooth closed curves in the plane all of which enclose a given area. Is there one of shortest length? [**5**]

Two assumptions are made:

1) The curves can be considered to have no crossings.

2) The enclosed regions are convex.

To begin answering question 1 we will make use of parametric representations. A curve in the plane can be described as the set of all points (x, y) whose coordinates satisfy a relation $R(x, y) = 0$. A third variable t (a parameter) is typically employed and the coordinates of the points on the curve can now be described in terms of two functions $f(t)$, $g(t)$; $x = f(t)$ and $y = g(t)$ where $R\left(f(t), g(t)\right) = 0$.

If derivatives are to be considered to parametrize the curve, then the parametrizing functions should be differentiable.

The curves in question 1 are parametrized in terms of arc length, s. Assume that the given length is 2π. A point on the curve, call it $(0,0)$, is chosen and measured along the curve in one direction a distance s. Thus $0 \le s \le 2\pi$. For each s, $0 \le s \le 2\pi$, there is a point $(x(s), y(s))$ on the curve and this is the parametric representation. The coordinate axes is chosen so that $y(\pi) = 0$.

If A is the area enclosed by the curve, and the curve is convex, then

$$A = \int_0^\pi y\left(\frac{dx}{ds}\right)ds + \int_\pi^{2\pi} y\left(\frac{dy}{ds}\right)ds\,. \tag{1}$$

Let

$$A_1 = \int_0^\pi y\left(\frac{dx}{ds}\right)ds\,. \tag{2}$$

Using the inequality

$$ab \le \frac{a^2 + b^2}{2}\,, \tag{3}$$

and letting $y = a$ and $\dfrac{dx}{ds} = b$, substitute into (3) to yield

$$y\frac{dx}{ds} \le \frac{y^2 + \left(\frac{dx}{ds}\right)^2}{2}\,.$$

Integrating will produce

$$\int_0^\pi y\left(\frac{dx}{ds}\right)ds \le \int_0^\pi \frac{y^2 + \left(\frac{dx}{ds}\right)^2}{2}\,ds\,.$$

Substituting (2) yields

$$A_1 \le \frac{1}{2}\int_0^\pi \left[y^2 + \left(\frac{dx}{ds}\right)^2\right]ds\,. \tag{4}$$

To verify that $\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 = 1$ using s as arc length, start with the equation

$(\Delta x)^2 + (\Delta y)^2 = (\Delta s)^2$. This equation can be rewritten as

$$\frac{\Delta x}{\Delta s} = \sqrt{1 - \left(\frac{\Delta y}{\Delta s}\right)^2} .$$

If the limit of each side is taken as $\Delta s \to 0$ the result will be:

$$\lim_{\Delta s \to 0} \frac{\Delta x}{\Delta s} = \lim_{\Delta s \to 0} \sqrt{1 - \left(\frac{\Delta y}{\Delta s}\right)^2} ,$$

which becomes

$$\frac{dx}{ds} = \sqrt{1 - \left(\frac{dy}{ds}\right)^2} .$$

Squaring both sides results in:

$$\left(\frac{dx}{ds}\right)^2 = 1 - \left(\frac{dy}{ds}\right)^2 . \qquad (5)$$

By substituting (5) into (4):

$$A_1 \leq \frac{1}{2} \int_0^{\pi} \left(y^2 + 1 - \left(\frac{dy}{ds}\right)^2 \right) ds . \qquad (6)$$

Now consider

$$y(s) = u(s) \cdot \sin(s), 0 \leq s \leq \pi , \qquad (7)$$

to find $\frac{dy}{ds}$:

$$\frac{dy}{ds} = y'(s) = u'(s) \cdot \sin(s) + u(s) \cdot \cos(s)$$

$$\frac{dy}{ds} = \frac{du}{ds}\sin(s) + u(s) \cdot \cos(s) . \qquad (8)$$

Substitute (8) into (6) to yield:

$$A_1 \le \frac{1}{2}\int_0^{\pi}\left(y^2 + 1 - \left(\frac{du}{ds}\sin(s) + u(s) \cdot \cos(s)\right)^2 \right) ds . \qquad (9)$$

Then substitute $y(s) = u(s) \cdot \sin(s)$ into (8):

$$A_1 \le \frac{1}{2}\int_0^{\pi}\left(u^2 \sin^2(s) - u^2 \cos^2(s) + 1 - 2u\left(\frac{du}{ds}\right)\sin(s)\cos(s) - \left(\frac{du}{ds}\right)^2 \sin^2(s) \right) ds .$$

Simplification will result in:

$$A_1 \le \frac{1}{2}\int_0^{\pi}\left[u^2(\sin^2(s) - \cos^2(s)) - 2u\frac{du}{ds} \cdot \sin(s)\cos(s) - \left(\frac{du}{ds}\right)^2 \sin^2(s) + 1 \right] ds . \quad (10)$$

Using $\sin^2(s) - \cos^2(s) = -\cos(2s)$ and $2\sin(s)\cos(s) = \sin(2s)$, the inequality can now be written as:

$$A_1 \le \frac{1}{2}\int_0^{\pi}\left[u^2(-\cos 2s) - u\sin 2s\frac{du}{ds} - \left(\frac{du}{ds}\right)^2 \sin^2(s) + 1 \right] ds - \int_0^{\pi} u^2 \cos 2s(ds) . \quad (11)$$

Integrate using integration by parts.

Substituting into the integration by parts equation results in:

$$\int_0^{\pi} u^2 \cos 2s ds = \frac{1}{2}u^2 \sin 2s - \int_0^{\pi} u \sin 2s du$$

$$= \frac{1}{2}\left(-\frac{1}{2}u^2 \sin 2s + \int_0^{\pi} \sin 2s \frac{du}{ds}\right) - \frac{1}{2}\int_0^{\pi} u \sin 2s \frac{du}{ds}$$

$$= -\frac{1}{4}u^2 \sin 2s \int_0^{\pi} + \frac{1}{2}\int_0^{\pi}\left(1-\left(\frac{du}{ds}\right)^2 \sin^2 s\right).$$

The integration of $-\frac{1}{4}u^2 \sin 2s \int_0^{\pi}$ in the above equation will result in a value of 0

resulting in:

$$A_1 \le \frac{1}{2}\int_0^{\pi}\left[1-\left(\frac{du}{ds}\right)^2 \sin^2(s)\right]ds \le \frac{\pi}{2}. \qquad (12)$$

The maximization of area will only hold true when the inequality from (3) is an equation, meaning $a = b$ and $ab = \frac{a^2+b^2}{2}$.

Therefore:

$$A_1 = \frac{1}{2}\int_0^{\pi}\left[1-\left(\frac{du}{ds}\right)^2 \sin^2(s)\right]ds = \frac{\pi}{2}$$

only when $\frac{du}{ds} = 0$, which means $u = C$ for some constant *C*. So (7) yields

$y(s) = C\sin(s)$ and

$$y = \frac{dx}{ds} = \sqrt{1 - \left(\frac{dy}{ds}\right)^2} \ .$$

This means $y = \pm\sin(s)$ and thus $x(s) = \pm\cos(s) + C$.

This exercise has led to the conclusion that the maximum area will be obtained when the arc is always at an equal distance from a fixed center and is thus on the circumference of a circle.

Chapter 3: L'Hôpital's Rule

If continuous functions $f(x)$ and $g(x)$ are both zero at $x = a$, then $\lim_{x \to a} \frac{f(x)}{g(x)}$ cannot be determined by substituting $x = a$. The resulting quotient is $0/0$, an expression known as an *indeterminate form*. It is sometimes possible to find the limit of an indeterminate form by algebraic manipulation or construction. It is often more convenient, however, to apply "L'Hôpital's Rule." This rule, the work of John Bernoulli, appeared in the first calculus textbook ever published, by the Marquis de L'Hôpital, in 1696 [**3**, p.402].

L'Hôpital's Rule:

Suppose that the functions f and g are differentiable for $x \neq a$ in some open interval containing the point a and that $g'(x)$ is nonzero there. Suppose also that

$$\lim_{x \to a} f(x) = 0 = \lim_{x \to a} g(x).$$

Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

provided that the limit on the right either exists (as a finite real number) or is $+\infty$ or $-\infty$.

Further, suppose that the functions f and g are not only differentiable but have continuous derivatives near $x = a$ and that $g'(a) \neq 0$. Then

$$\lim_{x\to a}\frac{f'(x)}{g'(x)}=\frac{\lim\limits_{x\to a}f'(x)}{\lim\limits_{x\to a}g'(x)}=\frac{f'(a)}{g'(a)}$$

by the quotient rule for limits. In this case L'Hôpital's Rule reduces to the limit

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{f'(a)}{g'(x)}$$

Proof

(Working from right side)

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{f'(a)}{g'(a)}$$

$$\frac{f'(a)}{g'(a)}=\frac{\lim\limits_{x\to a}\dfrac{f(x)-f(a)}{x-a}}{\lim\limits_{x\to a}\dfrac{g(x)-g(a)}{x-a}} \qquad \text{(definition of derivative)}$$

$$\frac{f'(a)}{g'(a)}=\lim_{x\to a}\frac{\dfrac{f(x)-f(a)}{x-a}}{\dfrac{g(x)-g(a)}{x-a}} \qquad \text{(quotient law of limits)}$$

$$\frac{f'(a)}{g'(a)}=\frac{f(x)-f(a)}{g(x)-g(a)} \qquad \text{(simplification)}$$

$$\frac{f'(a)}{g'(a)}=\lim_{x\to a}\frac{f(x)}{g(x)}$$

This is often referred to as the "weak form". For analysis a more rigorous proof is used.

There are many examples of proofs for L'Hôpital's Rule in textbooks and mathematical journals. The following proof was an example used in the analysis class. While it is a proof of the $\frac{\infty}{\infty}$ case, a similar proof for the $\frac{0}{0}$ case can be shown.

The proof begins with definitions of limits.

(1) DEFINITION 1:

$$\lim_{x\to c} h(x) = L.$$

Let *c* and *L* be real numbers. The function *h* has limit *L* as *x* approaches *c* if, given any positive number *ε*, there is a positive number δ such that for all *x*,

$$0 < |x - c| < \delta \Rightarrow |h(x) - L| < \varepsilon.$$

(2) DEFINITION 2:

$$\lim_{x\to +\infty} h(x) = L.$$

Let *L* be a real number. The function *h* has a limit *L* as *x* approaches $+\infty$ if, given any positive number *ε*, there is a positive number *k* such that for all *x*,

$$|h(x) - L| < \varepsilon$$

for all $x \in$ *domain(h)* satisfying $x > k$.

(3) DEFINITION 3:

$$\lim_{x\to +\infty} h(x) = +\infty.$$

$\lim_{x\to +\infty} h(x) = +\infty$ if and only if for each positive number *ε* there is a positive number *k* such that $h(x) > \varepsilon$ for all $x \in domain(h)$ satisfying $x>k$.

(4) DEFINITION 4:

$$\lim_{x\to c} h(x) = +\infty.$$

Let c be a real number. $\lim_{x \to c} h(x) = +\infty$ if and only if for any positive number ε, there is a positive number δ such that $h(x) > \varepsilon$ for all $x \in domain(h)$ satisfying $|x - c| < \delta$.

L'Hôpital's Rule: $\frac{\infty}{\infty}$ case

Let f, g be functions with continuous derivatives with $g'(x) \neq 0$. Assume that $\lim_{x \to \infty} f(x) = \infty$, $\lim_{x \to \infty} g(x) = \infty$, and $\lim_{x \to \infty} \frac{f'(x)}{g'(x)} = L$, a real number. Then

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = L \tag{13}$$

Proof:

(1) Assume that $g(x) > 0$ for all $x > 0$, and $g'(x) > 0$ for all $x > 0$. Let $\varepsilon > 0$.

(2) There exists $m > 0$ such that

$$\left| \frac{f'(x)}{g'(x)} - L \right| < \frac{\varepsilon}{3} \tag{14}$$

whenever $x > m$, therefore by multiplying each side by $g'(x)$ and simply to produce:

$$|f'(x) - Lg'(x)| < \frac{\varepsilon}{3} g'(x) . \tag{15}$$

(3) Let a be a fixed real number, $a > m$. Then for all $t > a$

$$\left| \int_a^t [f'(x) - Lg'(x)] dx \right| < \frac{\varepsilon}{3} \int_a^t g'(x) dx . \tag{16}$$

Now integrate each side and simplify by multiplying each side by $\frac{1}{g(t)}$ that results in this inequality:

$$\left|\frac{f(t)-f(a)}{g(t)}-L\left(\frac{g(t)-g(a)}{g(t)}\right)\right|<\frac{\varepsilon}{3}\left(\frac{g(t)-g(a)}{g(t)}\right). \tag{17}$$

Simplify again to yield:

$$\left|\frac{f(t)}{g(t)}-\frac{f(a)}{g(t)}-L\left[1-\frac{g(a)}{g(t)}\right]\right|<\frac{\varepsilon}{3}\left[1-\frac{g(a)}{g(t)}\right]. \tag{18}$$

Since $g(t)>g(a)$ then $\frac{g(a)}{g(t)}<1$ which means $\frac{\varepsilon}{3}\left[1-\frac{g(a)}{g(t)}\right]<\frac{\varepsilon}{3}$.

Rearrange terms and simplify again

$$\left|\frac{f(t)}{g(t)}-L\right|-\left|\frac{f(a)}{g(t)}\right|+L\frac{g(a)}{g(t)}<\frac{\varepsilon}{3}\left[1-\frac{g(a)}{g(t)}\right]<\frac{\varepsilon}{3},$$

$$\left|\frac{f(t)}{g(t)}-L\right|-\left|\frac{f(a)}{g(t)}\right|+L\frac{g(a)}{g(t)}<\frac{\varepsilon}{3},$$

$$\left|\frac{f(t)}{g(t)}-L\right|<\frac{\varepsilon}{3}+\left|\frac{f(a)}{g(t)}-L\frac{g(a)}{g(t)}\right|.$$

(4) Therefore:

$$\left|\frac{f(t)}{g(t)}-L\right|<\frac{\varepsilon}{3}+\frac{|f(a)|}{g(t)}+|L|\frac{g(a)}{g(t)}.$$

(5) Since $\lim_{t\to\infty} g(t)=\infty$, there exists $n>m$ so that $\frac{|f(a)|}{g(t)}<\frac{\varepsilon}{3}$ for all $t>n$ and $\frac{|L|\,g(a)}{g(t)}<\frac{\varepsilon}{3}$ for all $t>n$.

(6) Then for all $t>n$,

$$\left|\frac{f(t)}{g(t)}-L\right|<\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}<\varepsilon$$

therefore,

$$\left|\frac{f(t)}{g(t)}-L\right|<\varepsilon\,.$$

Chapter 4: The Exponential Function

The **exponential function** is the function e^x, where e is the number such that the function e^x is its own derivative (e is approximately 2.718281828). The function is sometimes written as $\exp(x)$ especially when it is impractical to write the independent variable as a subscript. The exponential function occurs whenever a value or quantity grows or decays at a rate proportional to its current value. One of the most common uses is in the continuous compounding of interest. If an initial amount of $1 earns interest at an annual rate of 1% compounded monthly, then the interest earned each month is $1/12$ times the current value, so each month the total value is multiplied by $(1+1/12)$, and the annual value is $(1+1/12)^{12}$. If the interest is compounded daily the annual value becomes $(1+1/365)^{365}$. By allowing the number of time intervals per year to grow without limit (infinite intervals…continuous compounding) leads to the limit definition of the exponential function,

$$e = \lim_{n\to\infty}(1+\frac{1}{n})^n . \qquad (19)$$

The symbol e was first attributed to Leonhard Euler. [**4**, p.338]

In equation (19) the *n* represents the number of compoundings in a year. The table below demonstrates the influence of the number of compoundings. For the purpose of showing the influence of the frequency of compounding, all variables for principal, interest rate, and number of years are all set equal to 1.

Table 1. Compound Frequency Table

Compound frequency	Computation
Yearly	$\left(1+\frac{1}{1}\right)^{1} = 2$
Semi-annually	$\left(1+\frac{1}{2}\right)^{2} = 2.25$
Quarterly	$\left(1+\frac{1}{4}\right)^{4} = 2.44140625$
Monthly	$\left(1+\frac{1}{12}\right)^{12} \approx 2.61303529022$
Weekly	$\left(1+\frac{1}{52}\right)^{52} \approx 2.69259695444$
Daily	$\left(1+\frac{1}{365}\right)^{365} \approx 2.71456748202$
Hourly	$\left(1+\frac{1}{8780}\right)^{8780} \approx 2.71812669063$
Every minute	$\left(1+\frac{1}{525600}\right)^{525600} \approx 2.7182792154$
Every second	$\left(1+\frac{1}{31536000}\right)^{31536000} \approx 2.71828247254$

As is shown in the table, the more often compounding occurs the larger the computed value, but the value appears to be approaching a fixed value. This value is e and is referred to as the "natural" exponential, because it arises naturally in math and the physical sciences.

The exponential function e^x can be demonstrated or characterized in a number of ways. One characterization is its being defined by the power series:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots.$$

One of the most interesting and useful properties of the exponential function e^x is that it is equal to its derivative. It is known that:

$$e = \left(1+\frac{1}{n}\right)^n \tag{20}$$

for large values of n.

Let $h = \frac{1}{n}$, then equation (20) can be rewritten as:

$$e = \lim_{h \to 0} (1+h)^{\frac{1}{h}}$$

So for a small enough *h*, $e \approx (1+h)^{\frac{1}{h}}$ and $e^h \approx (1+h)$. Therefore, $e^h - 1 \approx h$ and taking the limit will result in:

$$\lim_{h \to 0} \left(\frac{e^h - 1}{h} \right) = 1.$$

$$\frac{d}{dx}(e^x) = e^x$$

The derivative of this function is itself.

The exponential function has many applications in the mathematics and science. As was shown, the function has a financial application in the compounding of interest. In the natural sciences the exponential function's applications include modeling growth (and decay) of things such as bacteria. In social sciences the exponential function is used to model population growth (or decline).

Chapter 5: Completeness of the Number Line

To begin to understand the completeness of the real number line, it is necessary to review the description of a line. A *point* is a defined term for a place or spot normally represented by a dot. The point, unlike the dot, has no size, no width, and can be thought of as infinitely small. A line consists of an infinite series of these points extending to infinity in both directions. There are no spaces or gaps on this line. Each point could be said to be represented by a real number and all real numbers are represented by a point. Thus the real number line is *complete*.

DEFINITION. Upper Bound and Lower Bound

The set *S* of real numbers is said to be *bounded above* if there is a number *b* such that $x \le b$ for every number *x* in *S*, and the number *b* is then called an *upper bound* for *S*. Similarly, if there is a number *a* such that $x \ge a$ for every number *x* in *S*, then *S* is said to be *bounded below***,** and *a* is called a *lower bound* for *S*.

DEFINITION. Least Upper Bound and Greatest Lower bound

The number λ is said to be a *least upper bound* for the set *S* of real numbers provide that: **1.** λ is an upper bound for *S*, and

2. If *b* is an upper bound for *S*, then $\lambda \le b$.

Similarly, the number γ is said to be a *greatest lower bound* for *S* if γ is a lower bound for *S* and $\gamma \ge a$ for every lower bound *a* of *S*.

In addition to these definitions there are some very important theorems that are used to help with the understanding of the idea of completeness.

The first of these is the Intermediate Value Theorem.

THEOREM: Intermediate Value Property of Continuous Functions

Suppose *f* is continuous on a closed interval $[a,b]$. Let *p* be any number between $f(a)$ and $f(b)$, so that $f(a) \le p \le f(b)$ or $f(b) \le p \le f(a)$. Then there exists a number *c* in $[a,b]$ such that $f(c) = p$.

Proof: If $f(a) = p$ or $f(b) = p$, then let $c = a$ or $c = b$. Otherwise, either $f(a) < p < f(b)$ or $f(b) < p < f(a)$. The proof is similar for either of the cases, so it is only necessary to prove the theorem under the assumption that $f(a) < p < f(b)$. Let *S* be all the xs in $[a,b]$ such that $f(x) < p$. Then *a* is in *S* and *b* is an upper bound of *S*. By the Least Upper Bound Axiom, *S* has a least upper bound, so let *c* be the least upper bound. Now $a \le c \le b$. Either $f(c) < p$, $f(c) > p$, or $f(c) = p$. If $f(c) < p$, it follows that $c < b$, since $p < f(b)$ by previous assumption, and that $f(c) + \varepsilon < p$ for sufficiently small ε. Since *f* is continuous at *c*, there is a $\delta > 0$ such that if $|x - c| < \delta$, then *x* is in $[a,b]$ and $|f(c) - f(x)| < \varepsilon$. In particular if $x = c + \frac{\delta}{2}$, then $\left|f(c) - f(c + \frac{\delta}{2})\right| < \varepsilon$, which implies that

$$f\left(c + \frac{\delta}{2}\right) < f(c) + \varepsilon < p.$$

But since $c + \frac{\delta}{2}$ is in *S*, *c* is not the least upper bound of *S*. Consequently the assumption that $f(c) < p$ is false. Analogously, the assumption that $f(c) > p$ is also false. Therefore $f(c) = p$.

Secondly, there is the Mean Value Theorem.

The Mean Value Theorem is one of the most frequently used topics in mathematics classes and texts. It is often used to prove other theorems in Differential and Integral Calculus, as well as in Analysis. The Mean Value Theorem is frequently derived by use of its own special case, Rolle's Theorem.

Rolle's Theorem

THEOREM: Rolle's Theorem

Let f be continuous on a closed interval $[a,b]$ and differentiable on the open interval (a,b). If $f(a) = f(b)$, then there is at least one point *c* in (*a,b*) where $f'(c) = 0$.

The theorem states that the tangent to a graph of *f* where the derivative is 0 (a minima or maxima) is parallel to the *x*-axis, and the line (the secant) joining the end points $(a, f(a))$ and $(b, f(b))$ is too. So Rolle's theorem claims that there is a point at which the tangent to the graph is parallel to the secant, assuming the secant is horizontal.

The Mean Value Theorem

THEOREM: The Mean Value Theorem

Let f be continuous on a closed interval $[a,b]$ and differentiable on the open interval (a,b). Then there is at least one point c in (a,b) where

$$f'(c) = \frac{(f(b) - f(a))}{(b-a)}.$$

The Mean Value Theorem claims that there is a point at which the tangent to the graph is parallel to the secant.

Proof: The equation of the secant line through the points $(a, f(a))$ and $(b, f(b))$, written in point-slope form, is:

$$y - f(a) = \frac{f(b) - f(a)}{b - a}(x - a)$$

which can be rewritten as

$$y = \frac{f(b) - f(a)}{b - a}(x - a) + f(a).$$

Now auxillary function g is introduced as

$$g(x) = f(x) - \left[\frac{f(b) - f(a)}{b - a}(x - a) + f(a)\right] \text{ for } a \le x \le b,$$

where g is continuous on $[a,b]$ and differentiable on (a,b) since f is. Therefore, $g(a) = g(b) = 0$. By Rolle's theorem there is a number c in (a,b) such that $g'(c) = 0$. But

$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a} \text{ for } a \le x \le b$$

and so

$$0 = g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}.$$

Therefore,

$$f'(c) = \frac{(f(b) - f(a))}{(b - a)}.$$

The Completeness Property of Real Numbers and the theorems studied in Analysis class are essential tools in the understanding and study of the real number system.

Chapter 6: Conclusion

For most students the subject of Analysis and the Real Line may not sound as exciting as other subjects in mathematics. Until a student has studied analysis there is probably no way to fully appreciate the extent of knowledge of the real numbers that comes from the study. The student is asked to recall properties and rules learned in Calculus, Discrete Mathematics, Number Theory, and Topology. Problems take on a "micro-surgery" atmosphere as the concepts studied are analyzed in greater and greater detail. In studying the different aspects of analysis a true sense of connectedness and continuity of the different branches of mathematics surfaces. It is for this reason that the author wanted to research and write about this subject.

As a teacher of High School Algebra and Geometry the level of intensity that is dealt with and learned in an analysis class is hard to relate to high school students. However, the study of analysis does relate in that the continuity and flow of the subject of mathematics has an influence on our high school students.

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