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Infinite-Dimensional Hamiltonian Systems with Continuous Spectra: Perturbation Theory, Normal Forms, and Landau Damping

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Infinite-Dimensional Hamiltonian Systems with Continuous Spectra: Perturbation Theory, Normal Forms, and Landau Damping

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Various properties of linear infinite-dimensional Hamiltonian systems are studied. The structural stability of the Vlasov-Poisson equation linearized around a homogeneous stable equilibrium f_0 is investigated in a Banach space setting. It is found that when perturbations of $f_0'(v)$ are allowed to live in the space $W^{1,1}(\mathbb{R})$, every equilibrium is structurally unstable. When perturbations are restricted to area preserving rearrangements of f_0, structural stability exists if and only if there is negative signature in the continuous spectrum. This analogizes Krein's theorem for linear finite-dimensional Hamiltonian systems. The techniques used to prove this theorem are applied to other aspects of the linearized Vlasov-Poisson equation, in particular the energy of discrete modes which are embedded within the continuous spectrum.

In the second part, an integral transformation that exactly diagonalizes the Caldeira-Leggett model is presented. The resulting form of the Hamiltonian, derived using canonical transformations, is shown to be identical to that of the linearized Vlasov-Poisson equation. The damping mechanism in the Caldeira-Leggett model is identified with the Landau damping of a plasma. The correspondence between the two systems suggests the presence of an echo effect in the Caldeira-Leggett model. Generalizations of the Caldeira-Leggett model with negative energy are studied and interpreted in the context of Krein's theorem.

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Chapter 1

Introduction

Hamiltonian mechanics is a basis of modern physics. The Hamiltonian formulation can be used to describe both finite and infinite degree-of-freedom systems from classical physics. The purpose of this thesis is to describe new properties of Hamiltonian systems in the infinite-dimensional case. It is written in two sections, both of which describe results about linear infinite-dimensional Hamiltonian systems. It begins with an introduction that reviews those theorems from Hamiltonian systems theory that are essential for the results that follow in the main chapters. This includes theory of finite-dimensional symplectic vector spaces and a description of the formalism used to describe infinite-dimensional Hamiltonian systems.

The second chapter examines the stability of the spectrum of linear Hamiltonian time evolution operators under small perturbations. Much space will be devoted to introducing the literature on this subject, which was started by the classification of normal forms of linear finite-dimensional Hamiltonian systems by Weierstrass [1] and later Williamson [2], proceeding to the proof of the Krein-Moser theorem [3, 4] and attempts at its extension to infinite-dimensional systems [5]. This will be followed by new results proved by Morrison and me on the Vlasov-Poisson equation [6], which is noncanonical. There will be an extensive discussion of the implications of this type of work within both mathematics and physics, as well as a description of directions for future research.

The third chapter is about the diagonalization of linear infinite-dimensional Hamiltonian systems, a topic which is related in spirit to that of the second chapter. The primary result can be viewed as a continuation of work by Morrison, beginning with the derivation of an integral transformation from the Vlasov-Poisson equation to action-angle variables [7, 8]. We will present a similar integral transformation for the Caldeira-Leggett model [9], which is a model describing dissipation in quantum mechanics [10]. The novelty of this transformation will be that it implies an equivalence between the Vlasov-Poisson equation and the Caldeira-Leggett model. Plasma phenomenon predicted by the Vlasov equation will have analogs in the Caldeira-Leggett model and vice versa, which is particularly tantalizing if it could lead to new observations. This possibility will be explored for the particular case of the plasma echo [11]. The physics of dissipative systems will be discussed in a Hamiltonian context and some interesting results on heat bath models will be presented.

1.1 Notation conventions

Elements of phase space, whether finite or infinite dimensional, will be denoted with lowercase Latin letters a,b,c,..., matrices and operators will be denoted by uppercase Latin letters A,B,C,..., constants will be Greek letters

α,β,γ, etc. Integers may also be denoted by Latin letters, especially in indices. The real and complex numbers will be denoted by $\mathbb{R}$ and $\mathbb{C}$. Phase space will be represented by calligraphic symbols like $\mathcal{A}$,$\mathcal{B}$,$\mathcal{C}$,..., the inner product on these spaces will be $\langle\cdot,\cdot\rangle$, and general sesquilinear forms will be denoted by $(\cdot,\cdot)$ or $\omega(\cdot,\cdot)$, where the initial letter is allowed to vary. The canonical Poisson bracket on $\mathbb{R}^{2n}$ will be $[\cdot,\cdot]$ and a general Poisson bracket will be $\{\cdot,\cdot\}$. For operators and matrices M^* is the adjoint of M, and when it makes sense M^t will be the transpose of M.

1.2 Hamiltonian systems

All of the systems studied in this paper are linear, so the treatment here will be restricted to the Euclidean and Banach space cases rather than general Poisson manifolds. In this section we will outline the finite-dimensional theory and give references to more detailed treatments. We will present the most important results for understanding the development in the rest of thesis, along with some discussion of their meaning. We will also give a detailed introduction to infinite-dimensional Hamiltonian systems complete with remarks on the mathematical rigor which can be attached to the formalism.

1.2.1 Finite-dimensional Hamiltonian systems

This introduction will be brief and serves to remind the reader of the most relevant concepts to the rest of thesis. For a detailed introduction to linear symplectic geometry (and indeed also all of symplectic geometry), the book by McDuff and Salamon [12] is quite strong. An introduction to classical mechanics from the point of view of Hamiltonian systems with a large number of examples is the book by Arnold [13]. The review article by Morrison [14] contains a very detailed introduction to noncanonical Hamiltonian systems (among other things). The setting for finite-dimensional Hamiltonian systems considered here will be vector spaces with Poisson structures. The simplest example of such a space, and the general phase space for a canonical Hamiltonian theorem is a symplectic vector space.

Definition A symplectic vector space is a pairing $(\mathcal{V}, \omega)$, where $\mathcal{V}$ is a real vector space and $\omega(\cdot, \cdot)$ is an anti-symmetric, non-degenerate bilinear form referred to as the symplectic structure.

If $\mathcal{V}$ is finite-dimensional the dimension of $\mathcal{V}$ must be an even number $2n$, and $\mathcal{V}$ can be generically taken to be $\mathbb{R}^{2n}$. In this section the phase space $\mathcal{M}$ will be a symplectic vector space. Let the standard inner product on $\mathcal{M}$ be given by $\langle \cdot, \cdot \rangle$. Then the symplectic form ω can be written in terms of a linear operator J as $\omega(\cdot, \cdot) = \langle J \cdot, \cdot \rangle$. The matrix J is anti-symmetric and nonsingular. An important example of such a matrix is the canonical symplectic matrix:

$$J_0 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}. \tag{1.1}$$

Given a functional H on $\mathcal{M}$, it is possible to define a Hamiltonian system where the energy corresponds to H.

Definition Let $\mathcal{M}$ be a symplectic vector space with symplectic operator J and $z \in \mathcal{M}$ a point in this phase space. Then for each C_2 functional H on $\mathcal{M}$, the equations

$$\frac{dz}{dt} = -J\nabla H(z).$$

define a Hamiltonian system on $\mathcal{M}$. The continuity requirement on H ensures that $J\nabla H$ is Lipschitz, and that solutions to the ordinary differential equations exist.

Symplectic vector spaces always have a basis in which $J = J_0$.

Let $(\mathbb{R}^{2n}, J)$ be a symplectic vector space. There exists a vector space isomorphism K such that $J_0 = K^t J K$.

This isomorphism can be applied to Hamilton's equations, and the resulting system of equations is Hamilton's equations with a modified Hamiltonian and the canonical symplectic form. In the rest of this section it is assumed that $J = J_0$. If the point z is written as a pair (q, p) with $q, p \in \mathbb{R}^n$ then Hamilton's equations take the familiar form:

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}$$
$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}.$$

Transformations that leave the symplectic form invariant are known as canonical transformations, and make it possible to simplify the equations of motion without ruining their Hamiltonian form.

Definition Consider the symplectic operator J on $\mathbb{R}^{2n}$. A linear map K satisfying $K^TJK = J$ is called a symplectomorphism or canonical transformation.

One of the most important properties of canonical transformations are that they are always volume preserving maps, that is $\text{Det}K = 1$.

Symplectic manifolds and symplectic vector spaces are special cases of Poisson manifolds. Consider a finite dimensional vector space.

Definition A Poisson structure or Poisson bracket is a map $\{,\}$ from $C^\infty(\mathbb{R}^k)\times C^\infty(\mathbb{R}^k) \to C^\infty(\mathbb{R}^k)$ with a number of properties: bilinearity, antisymmetry, the Jacobi identity, which is

$$\{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} = 0. \tag{1.2}$$

and the Leibniz property:

$$\{fg, h\} = f\{g, h\} + g\{f, h\}. \tag{1.3}$$

To each symplectic operator J there corresponds a Poisson bracket defined by:

$$\{f, g\}_J = \frac{\partial f}{\partial z_i} J^{ij} \frac{\partial g}{\partial z_j}. \tag{1.4}$$

When $J = J_0$ the Poisson bracket is written $[\cdot, \cdot]$ and is of the form:

$$[f, g] = \sum_k \frac{\partial f}{\partial q_k}\frac{\partial g}{\partial p_k} - \frac{\partial f}{\partial p_k}\frac{\partial g}{\partial q_k}.$$

Hamilton's equations can be rewritten in terms of the Poisson bracket:

$$\frac{dz_j}{dt} = [z_j, H].$$

When the Poisson bracket of the system is not the canonical Poisson bracket the equations of motion can still be written in this form. The Poisson bracket is also a convenient tool to describe the time evolution of functions of the dynamical variables, i.e. observables. If $F(z)$ is an observable then $\frac{dF(z)}{dt} = [F, H]$. The Poisson bracket is necessary for the definition of noncanonical Hamiltonian systems, which we will describe later in this introduction.

1.2.2 Solving linear Hamiltonian ODEs

This section is a review of basic methods in linear constant coefficient ODEs, written from the point of view of Hamiltonian systems. The goal of this section is to review the types of behavior that can occur in these systems and to set the stage for the development and study of the infinite-dimensional theory.

Canonical Hamiltonian equations are linear when the Hamiltonian function $H(z)$ is quadratic in the components of z. This means that $H(z) = H(z, z)$, where $H(\cdot, \cdot)$ is a symmetric, bilinear form generated by H. There will exists a symmetric matrix H such that $H(z) = \langle Hz, z\rangle$. Assuming that the symplectic form is canonical, Hamilton's equations are:

$$\frac{dz}{dt} = -J_0 Hz.$$

This is a first order constant coefficient linear ODE, and it can be solved by computing the matrix exponential of $-tJ_0H$. The resulting time evolution depends on the modes of $-J_0H$ and their multiplicity. $-J_0H$ can be reduced into a finite number of Jordan blocks, each of which has a well-defined exponentiation. The most important factor in the time evolution of Hamilton's equations is the spectrum of the time evolution matrix $-J_0H$. The spectrum is symmetric with respect to complex conjugation and multiplication by -1.

Theorem 1.2.1. *Let H be a symmetric matrix. and let λ be in the spectrum of $-J_0H$. Then $-\lambda$,$\bar{\lambda}$, and $-\bar{\lambda}$ are also in the spectrum. Generically the spectrum comes in quartets when it is neither purely imaginary or purely real, and real or imaginary members come in pairs.*

The matrix $-J_0H$ can be simplified by similarity transformations to the direct product of Jordan blocks. The $k \times k$ Jordan block corresponding to the eigenvalue λ is:

$$(-J_0H)_{\lambda,k} = \begin{pmatrix} \lambda & 1 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \dots & \dots & 0 & \lambda \end{pmatrix}.$$

The matrix exponential of t times that matrix yields behavior of the form $e^{\lambda t}$ and secular growth.

$$e^{t(-J_0 H)_{\lambda,k}} = e^{t\lambda} \begin{pmatrix} 1 & t & t^2/2 & \dots & t^{k-1}/(k-1)! \\ 0 & \ddots & \ddots & \ddots & t^{k-2}/(k-2)! \\ \vdots & \ddots & \ddots & \ddots & t^{k-j}/(k-j)! \\ \vdots & \ddots & \ddots & 1 & t \\ 0 & \dots & \dots & 0 & 1 \end{pmatrix}.$$

It is possible to use canonical transformations to simplify the Hamiltonian of a linear Hamiltonian system. Solutions are stable when the spectrum is pure imaginary and there are no non-trivial Jordan blocks. In this case the Hamiltonian can be diagonalized by canonical transformations, and the Hamiltonian functional is a quadratic form that can be written (after splitting z into (q, p) as:

$$H(q, p) = \sum_k \sigma_k \frac{\omega_k}{2} (q_k^2 + p_k^2).$$

Here $\sigma_k = \pm 1$ is the signature of the pair of modes with frequency $\pm\omega_k$. There are analogous normal forms that are derived from the form of the Jordan blocks. A complete description of how to reduce a symplectic matrix with canonical transformations appears in Williamson [2], and an appendix of Arnold's book [13] contains a concise list of all the possible normal forms. For these systems the stability theory is easy to understand. If all the eigenvalues are imaginary and non-degenerate, then solutions are stable. If there are degeneracies, then there may be secular growth. If there are any eigenvalues with a non-zero real part, there is exponential growth of solutions.

1.2.3 Noncanonical Hamiltonian systems

Systems written in terms of a canonical Poisson bracket or symplectic form are referred to as canonical systems. The Poisson bracket of a canonical Hamiltonian system is non-degenerate, but this condition is not required for Hamilton's equations to be well-defined. Relaxing this condition and considering degenerate Poisson brackets leads to noncanonical Hamiltonian mechanics. Consider such a system, defined with a degenerate Poisson bracket:

$$\frac{dz}{dt} = \{z, H\}.$$

Let the functions C_j satisfy $\{C_j, \cdot\} = 0$. These phase space functions are then constants of motion for any Hamiltonian, as $\{C_j, H\} = 0$ for all H. The most common name for them in the literature is Casimir invariants, and they exist due to the structure of phase space rather than the symmetries of the Hamiltonian.

Consider a finite-dimensional Poisson bracket that can be written in the following form:

$$\{A, B\} = \frac{\partial A}{\partial z_i} J^{ij} \frac{\partial B}{\partial z_j}. \tag{1.5}$$

Here J is an anti-symmetric operator that may depend on z and that is degenerate. Then the Lie-Darboux theorem [15] states that there is a local diffeomorphism around each point z under which J realizes the following form:

$$J = \begin{pmatrix} 0 & I & 0 \\ -I & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{1.6}$$

The operator J locally becomes the canonical symplectic operator plus the zero matrix. The phase space is described by canonical coordinates, which are not in the kernel of J, and Casimir invariants, which are. The gradients of the Casimir invariants are a basis for the kernel of J.

This establishes the relationship between noncanonical and canonical systems. The phase space of noncanonical systems is composed of the level sets all of the Casimir invariants, which are referred to as symplectic leaves. The dynamics are restricted to take place within a single symplectic leaf, on which the equations of motion are the canonical Hamilton's equations.

This leads to a pleasing geometric picture that is made rigorous by theorems about the local structure of Poisson manifolds. Through each point of a Poisson manifold (or in our case a vector space with a Poisson structure), passes one symplectic leaf. Each symplectic leaf has a symplectic form that originates from the Poisson bracket, and therefore each symplectic leaf is a symplectic manifold. This was established by Lie for the case where the rank of the symplectic leaves are constant throughout the Poisson manifold, and by Kirillov, Kostant, and Souriau for the case where the rank of symplectic leaves are not constant throughout the Poisson manifold [16]. The evolution of a noncanonical Hamiltonian system is just like that of a canonical when the dynamics are considered only within a single symplectic leaf. The intimate connection between the geometry of the symplectic leaves and the perturbation theory of the spectra of linear Hamiltonian systems is one of the most important themes in this thesis, and we investigate it thoroughly in chapter 2.

1.3 Infinite-dimensional Hamiltonian systems

The systems of interest in this thesis are all infinite-dimensional. Infinite-dimensional Hamiltonian systems have a formalism that is very similar to that of the finite-dimensional theory. The main difference between the two theories is that the infinite-dimensional formalism is not automatically rigorous. There are also a number of important phenomenon that occur in the infinite-dimensional case without analogue in finite-dimensional systems. In the linear theory these are associated with the existence of the continuous spectrum.

Infinite-dimensional systems are field theories, and their dynamical variables are functions. The phase space for such systems are Banach spaces or Hilbert spaces. Hamilton's equations are derived using functional differentiation, and the equations of motion are partial differential equations. The obstruction to constructing a general theory of infinite-dimensional Hamiltonian systems, even in the linear case, is due to the lack of a general existence theory for partial differential equations. Without such existence results it is impossible to be certain whether or not Hamilton's equations are closed on a given phase space or whether the brackets and Hamiltonian functionals remain defined for all time.

This section will mirror the previous section and introduce the formalism for infinite-dimensional Hamiltonian systems, both canonical and non-canonical. Lie-Poisson systems, which are extremely important both in applications and in this thesis, will be discussed as well.

Like the finite-dimensional case, there will be two equivalent ways of defining an infinite-dimensional Hamiltonian system, using a symplectic operator and using a Poisson bracket. These structures require functional derivatives for their proper definition. In the canonical case, phase space is typically defined as the product $\mathcal{M} = \mathcal{B} \times \mathcal{B}$. The Hamiltonian is some functional from $\mathcal{M}$ to the real numbers, and we require that its derivative may be computed formally.

Represent a point in the phase space as two functions (q, p). Then Hamilton's equations are:

$$\frac{\partial q}{\partial t} = \frac{\delta H}{\delta p} \tag{1.7}$$

$$\frac{\partial p}{\partial t} = -\frac{\delta H}{\delta q}. \tag{1.8}$$

These equations can also be written in terms of the canonical symplectic operator, which is a map on $\mathcal{M}$ defined by 1.1. Then, letting $u = (q, p)$, Hamilton's equations can be written $\frac{\partial u}{\partial t} = -J_0 \nabla H$.

Likewise, in analogy with the finite-dimensional theory, Hamilton's equations can also be written using a Poisson bracket, which is a bilinear map on functionals. The canonical Poisson bracket is defined by:

$$\{A,B\}_c = \frac{\delta A}{\delta q}\frac{\delta B}{\delta p} - \frac{\delta A}{\delta p}\frac{\delta B}{\delta q}, \tag{1.9}$$

and Hamilton's equations are $\frac{\partial u}{\partial t} = \{u, H\}$. The formalism that comes from these definitions is used to derive partial differential equations, whose properties can be studied rigorously by mathematicians.

Canonical Hamiltonian systems lead to linear equations when the Hamiltonian is a quadratic form in the dynamical variables. When phase space is a Hilbert space, it is possible to use representation theorems to write the Hamiltonian using a self-adjoint operator and the inner product. This representation is always possible for a bounded, symmetric sesquilinear form, and is also possible in some unbounded cases, such as when the sesquilinear form is sectorial. This subject is discussed extensively in Kato's book on the perturbation theory of linear operators [17].

Because of this we will only consider canonical Hamiltonian systems in which the Hamiltonian functional can be written as a self-adjoint operator on a phase space which is assumed to be a Hilbert space.

Definition Let H be a (potentially unbounded) self-adjoint operator on $\mathcal{M} = \mathcal{B} \times \mathcal{B}$. These operators define a Hamiltonian system on the phase space $\mathcal{M} = \mathcal{B} \times \mathcal{B}$ with Hamilton's equations defined by $\frac{\partial u}{\partial t} = -J_0 H u$.

This will be the starting point for the investigation of canonical systems. Many properties of finite-dimensional systems generalize directly, perhaps most importantly the symmetry of the spectrum, which still occurs in quartets or doublets.

1.3.1 Noncanonical systems and Lie-Poisson form

Noncanonical formulations of Hamiltonian mechanics are extremely important for describing classical field theories. Eulerian descriptions of continuous media are generically derivable from noncanonical Hamiltonian structures. The most famous and important examples of these theories, the incompressible Euler equation of hydrodynamics, ideal magneto-hydrodynamics [18], the Vlasov equation with various types of interactions [19, 20], and the BBGKY hierarchy [21] are all of this type, see the review article [14] for a more extensive introduction to these systems and a list of references. In fact, Eulerian theories like the ones referenced above often have an additional property, having a Poisson bracket in Lie-Poisson form. Systems of this special type are closely related to Lie algebras, and in the special case where the structure constants are antisymmetric they also have an analog of the Liouville theorem, which is generally not apparent in noncanonical systems.

A general noncanonical infinite-dimensional Hamiltonian system is defined analogously to the finite-dimensional case: a noncanonical degenerate Poisson bracket is introduced.

Definition Suppose that $\mathcal{M}$ is a Banach space, Let $\{,\}$ be a bi-linear, anti-symmetric map from pairs of functionals on $\mathcal{M}$ to functionals on $\mathcal{M}$, that also satisfies the Jacobi identity. Then $\{,\}$ is called a Poisson bracket.

Given a Hamiltonian functional H, Hamilton's equations are $\frac{\partial f}{\partial t}(x) = \{f(x), H\}$, just as in the previous sections. As before Casimir invariants satisfy $\{C, H\} = 0.$, independent of the Hamiltonian. They frequently have natural physical interpretations in terms of the dynamical variables of the problem of interest. The existence of Casimir invariants is the factor that distinguishes noncanonical systems from canonical ones within the terminology of this paper. Often systems with non-degenerate, but also noncanonical symplectic forms are called noncanonical systems. All of these cases can of course be converted into the canonical case using a suitable transformation.

In the finite-dimensional case, this was made rigorous using the Lie-Darboux theorem. In the infinite-dimensional case the geometry of the symplectic leaves is much more complicated. The rank of the symplectic operator can change drastically near each point. Despite this, in cases of interest, it is often possible to transform the Poisson bracket to a form completely analogous to that of the finite-dimensional case. The ability to do this for certain linear systems, most importantly the linearized Vlasov-Poisson equation, is a critical aspect of the research within this thesis.

It was mentioned earlier that many Eulerian descriptions of continuous media could be described as noncanonical Hamiltonian systems using Lie-Poisson brackets. Suppose that the Poisson bracket can be written using a degenerate operator J:

$$\{A, B\}[u] = \left\langle \frac{dA}{du^i}, J^{ij} \frac{dB}{du^j} \right\rangle . \tag{1.10}$$

This bracket is said to be a Lie-Poisson bracket if the operator $J^{ij} = c^{ij}_k u^k$, and the constants c are the structure constants of a Lie algebra, satisfying $c^{ij}_k = -c^{ji}_k$ and $c^{ij}_m c^{mk}_l + c^{jk}_m c^{mi}_l + c^{ki}_m c^{mj}_l = 0$. In the infinite dimensional case, the analog is $\{A, B\}[u] = \langle \frac{\delta A}{\delta u^i}, J^{ij} \frac{\delta B}{\delta u^j} \rangle$, where $J^{ij} = C^{ij}_k u^k$, were the C^{ij}_k are the structure operators of an infinite-dimensional Lie algebra [14].

One of the reasons that Hamiltonian systems of this form appear commonly in Eulerian descriptions of continuous media is due to the fact that standard Hamiltonian reductions result in Hamiltonian systems with Lie-Poisson brackets [14]. Eulerian theories, such as the Euler equation and the MHD equations, can be naturally derived by reducing the corresponding Lagrangian theories. This guarantees that these theories will have Lie-Poisson brackets.

Chapter 2

Structural Stability of Infinite-Dimensional Hamiltonian Systems

2.1 Introduction

The perturbation of point spectra for classical vibration and quantum mechanical problems has a long history [22, 23]. The more difficult problem of assessing the structural stability of the continuous spectrum in e.g. scattering problems has also been widely investigated [24, 17]. Because general linear Hamiltonian systems are not governed by Hermitian or symmetric operators, the spectrum need not be stable and a transition to instability is possible. For finite degree-of-freedom Hamiltonian systems, the situation is described by Krein's theorem [3, 25, 4], which states that a necessary condition for a bifurcation to instability under perturbation is to have a collision between eigenvalues of opposite signature. The purpose of the present chapter is to investigate Krein-like phenomena in Hamiltonian systems with a continuous spectrum. Of interest are systems that describe continuous media which are Hamiltonian in terms of noncanonical Poisson brackets (see e.g. [14, 26]).

Our study differs from that of [5], which considered canonical Hamiltonian systems with continuous spectra in a Hilbert space where the time evolution operator is self-adjoint. The effects of relatively compact perturbations on such a system were studied and it was proved that the existence of a negative energy mode in the continuous spectrum caused the system to be structurally unstable. It was also proved that such systems are otherwise structurally stable. Also, our study differs from analyses of fluid theories concerning point spectra [27, 28] and point and continuous spectra [29], the latter using hyperfunction theory.

A representative example of the kind of Hamiltonian system of interest is the Vlasov-Poisson equation [19], which when linearized about stable homogeneous equilibrium gives rise to a linear Hamiltonian system with pure continuous spectra that can be brought into action-angle form [7, 30, 31, 8]. A definition of signature was given in these works for the continuous spectrum. In the present chapter we concentrate on the Vlasov-Poisson equation, but the same structure is possessed by Euler's equation for the two-dimensional fluid, where signature for shear flow continuous spectra was defined [32, 33] and, indeed, a large class of systems [34]. Thus, modulo technicalities, the behavior treated here is expected to cover a large class of systems.

In Sec. 2.4 we review on a formal level the noncanonical Hamiltonian structure for a class of systems that includes the Vlasov-Poisson equation as a special case. Linearization about equilibria is described, the concept of dynamical accessibility, and the linear Hamiltonian operator T, the main subject of the remainder of the chapter, are defined. In Sec. 2.2 we prove Krein's theorem in the time independent finite-dimensional case, and also discuss how the picture changes in the noncanonical case. Then in Sec. 2.3 we discuss the literature pertaining to the structural stability of infinite-dimensional Hamiltonian systems, to set the stage for the rest of the chapter. In the remainder of the chapter we sketch proofs, in varying levels of detail, pertaining to properties of this linear operator for various equilibria. In Sec. 2.5 we describe spectral stability in general terms and analyze the spectrum of T for the Vlasov case. The existence of a continuous component to the spectrum is demonstrated and Penrose plots are used to describe the point component. In Sec. 2.6 we describe structural stability and, in particular, consider the structural stability of T under perturbation of the equilibrium state. We show that any equilibrium is unstable under perturbation of an arbitrarily small function in $W^{1,1}$. In Sec. 2.7 we introduce the Krein-Moser theorem and restrict to dynamically accessible perturbations. We prove that equilibria without signature changes are structurally stable and those with changes are structurally unstable. In Sec. 2.8 we define critical states of the linearized Vlasov equation that are structurally unstable under perturbations that are further restricted. We prove that a mode with the opposite signature of the continuum is structurally unstable and that the opposite combination cannot exist unless the system is already unstable. Finally, in Sec. 2.9, we conclude.

2.2 Finite-dimensional perturbation theory: The Krein-Moser theorem

For linear finite-dimensional Hamiltonian systems, Hamilton's equations are a set of first order linear ODEs. If the Hamiltonian is time-independent and the Jordan forms of the time evolution operator is trivial, then the behavior of solutions is characterized by the eigenfrequencies. If all of the eigenfrequencies are on the real axis and non-degenerate, then the system will be stable. If there are degenerate eigenvalues the system will be stable as long as the time evolution operator does not have any nontrivial Jordan blocks, but there will be secular growth if it does. Any complex eigenfrequencies will lead to instability. The Hamiltonian of a linear finite-dimensional Hamiltonian system is a quadratic form in the canonical variables. If we consider perturbations of the coefficients of the quadratic form it is trivial to define a notion of small perturbations, as the resulting perturbation of the Hamiltonian will be a bounded operator. Krein [3] and Moser [4] independently proved a theorem characterizing the structural stability of these systems in terms of the Krein signature, a quantity that amounts to the sign of the energy evaluated on the eigenvector of a mode [35, 36, 37, 38]. It is of historical interest to note that the fact that bifurcations to instability occur through collisions of modes of opposite sign was observed by Sturrock [35, 36] in the plasma physics literature.

Theorem 2.2.1. (Krein-Moser) *Let H be a symmetric matrix that defines a stable linear finite-dimensional Hamiltonian system. Then H is structurally* *stable if all the eigenfrequencies are non-degenerate. If there are any degeneracies, H is structurally stable if the associated eigenmodes have energy of the same sign. Otherwise H is structurally unstable.*

This Krein-Moser theorem gives a clear picture of the behavior of these systems under small perturbations. The eigenfrequencies move around, but remain confined to the real line unless there is a collision between a positive energy and negative energy mode, in which case they may leave the axis.

This theorem was first proved by Krein in the early 1950s and later rediscovered by Moser in the late 1950s. They proved theorem under the assumption that the Hamiltonian was a periodic function of time, which is an extension of the result presented here. Proving theorem in the time independent case requires the following corollary of a theorem of Williamson:

Theorem 2.2.2. *Suppose J_0H has only purely imaginary eigenvalues, and that it has no non-trivial Jordan blocks. Then there exists a canonical transformation K such that K^tHK is diagonal. In its diagonal form, write the basis for phase space as $(q_1, ..., q_n, p_1, ...p_n)$. Then the quadratic form $\langle Hu, u\rangle$ can be written as a sum of terms of the form $\pm\omega_i(p_i^2 + q_i^2)$.*

This will be the starting point for the proof of the Krein-Moser theorem in the time-independent case.

Proof. Consider the n degree of freedom linear Hamiltonian system defined by $\frac{du}{dt} = -J_0Hu$. Assume that the solutions are stable. This means that

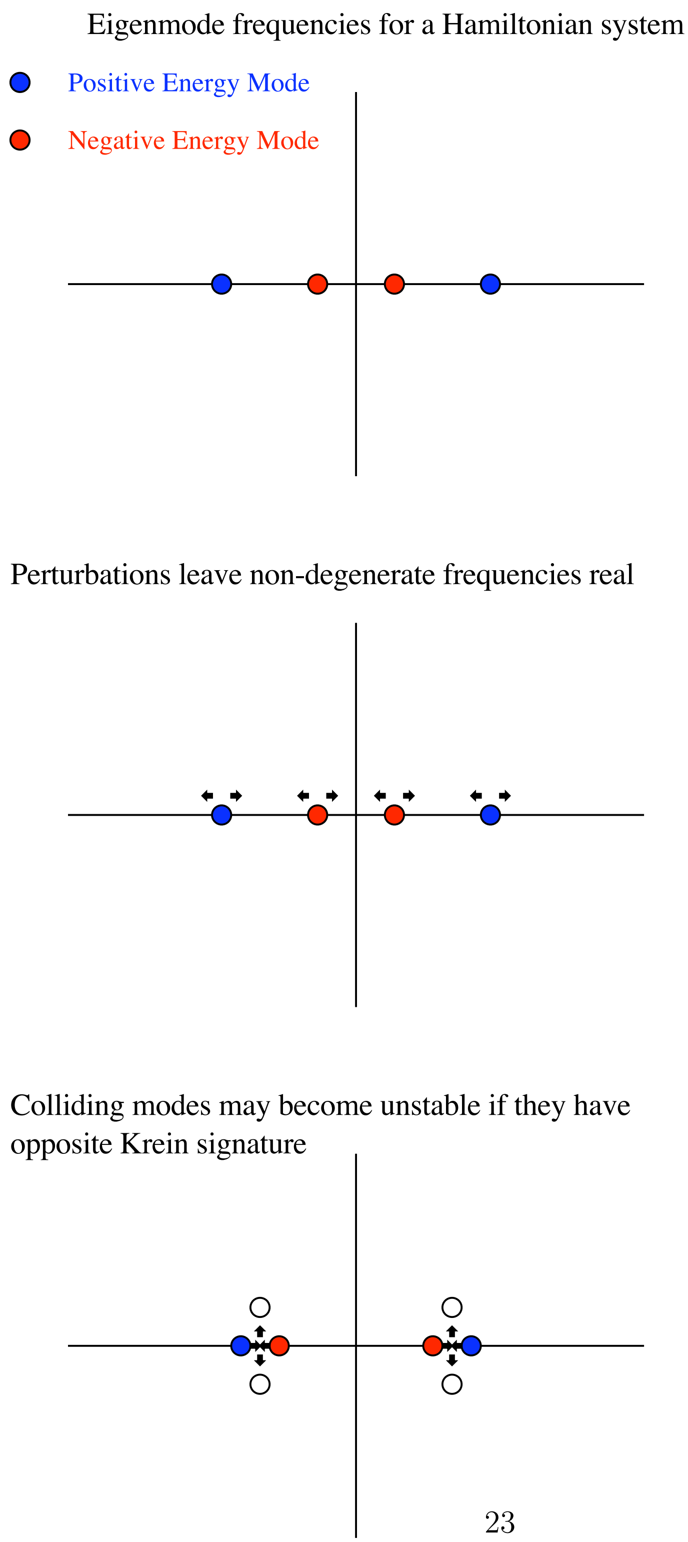

Figure 2.1: Krein Bifurcations

all of the characteristic frequencies are real (or that all the eigenvalues are purely imaginary), and that $J_0 H$ has no non-trivial Jordan blocks. Using Williamson's theorem there exists a canonical transformation which converts the system to one of the form $\frac{du_c}{dt} = -J_0 H_c u_c$, where H_c has been diagonalized in terms of the frequencies ω_i.

Suppose that H is definite on the eigenspace defined by each eigenvalue. Then for each $\epsilon > 0$, there is a $\delta > 0$ such that whenever H_1 is a symmetric matrix satisfying $\|H_1 - H\| < \delta$, there is a one to one map (labeling each multiple eigenvalue a number of times equal to its multiplicity) between eigenvalues of $J_0 H_1$ and $J_0 H$ such that each eigenvalue of $J_0 H$ is within a distance ϵ from the eigenvalue of $J_0 H_1$ to which it corresponds, and the inequality $\|P_{\omega_j} - P_{\omega_j,1}\| < \epsilon$ holds. Here P_{ω_j} is the projection operator onto the space spanned by the eigenvectors corresponding to the eigenvalue iω and $P_{\omega_j,1}$ is the projection onto the eigenspace corresponding to the eigenvalues of $J_0 H_1$ that correspond to the iω_j of $J_0 H$ (examine the first chapter of Kato [17] for theorems on finite-dimensional linear perturbation theory upon which this is based).

Using these inequalities:

$$|\left\langle H_1 P_{\omega_j,1} u, P_{\omega_j,1} u \right\rangle| \leq (\omega_j - 2\omega_j \epsilon - \omega_j \epsilon^2 - \delta - 2\delta\epsilon - \delta\epsilon^2) \left\langle P_{\omega_j,1} u, P_{\omega_j,1} u \right\rangle . \quad (2.1)$$

For small enough δ and ϵ, the right side is positive definite for every ω_j. In this case, H_1 is sign definite on each of the eigenspaces of $J_0 H_1$. Therefore each eigenvector of $J_0 H_1$ has a nonzero energy, which implies stability because exponentially unstable or secularly growing modes must have 0 Hamiltonian to maintain conservation of the Hamiltonian. Therefore the system is structurally stable in this case.

Suppose that for some $i\omega_j$ there is one Jordan block with a positive normal form and one with a negative normal form. Labeling this eigenspace with (q_1, q_2, p_1, p_2), the Hamiltonian can be written $H = \frac{\omega}{2}(q_1^2 + p_1^2 - q_2^2 - p_2^2)$. Perturb the Hamiltonian by the coupling q_1 to q_2, $H_1 = H + \delta q_1 q_2$. For all positive values of δ the resulting system has a quartet of unstable eigenmodes. Therefore the system is structurally unstable. □

This theorem is the rigorous statement of the general instability criteria described at the beginning of this section. This also illustrates the point that the existence of negative energy modes is essential for a Hamiltonian system to be structurally unstable. The Krein bifurcation into instability which is alluded to within this section and within theorem is readily observed in numerous physical examples.

In practical physical situations, the existence of negative energy modes introduces complications and potential instabilities that must be taken into consideration. Dissipation induced instability may occur when a system with negative energy modes is modified to include friction. If the friction strictly removes energy from the system, then it may be possible for it to remove energy from the negative energy mode, thereby increasing its amplitude and driving the system into instability. Indeed one of the earliest satellites, Explorer 1, failed catastrophically due to the dissipation induced instability. Explorer 1 was designed to maintain its orientation through spin stabilization. The configuration of Explorer 1 could be described using rigid body equations which predicted the existence of a pair of negative energy modes. Dissipation of energy through the vibrations of the antenna caused this mode to become unstable, which forced the satellite to tumble and lose its ability to communicate with Earth [39]. A similar mechanism has been proposed to explain the tumbling of comets and asteroids. Energy dissipation can occur in asteroids through inelastic dissipation driven by strains within the asteroids. This drives the growth of negative energy modes, and results in asteroids and comets that tend to spin about their axis of maximal inertia [39].

The purpose of this chapter is to study generalizations of the Krein-Moser theorem to infinite-dimensions. The most important case given here is the study of the Vlasov-Poisson equation, which is a noncanonical Hamiltonian system. To motivate the study of that system and to make clear the principles necessary to study noncanonical systems, we briefly discuss the finite-dimensional noncanonical case here.

2.2.1 Structural stability in the finite-dimensional noncanonical case

Considering the direct analog of the canonical theory leads one to consider a perturbation problem of the exact same form as in the canonical case, with the exception that J_0 is replaced with a degenerate a version of the form:

$$J = \begin{pmatrix} 0 & I & 0 \\ -I & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.2}$$

This case is identical to that of the canonical case because the system neatly splits into a piece that evolves on a symplectic leaf and a piece that spans the nullspace of J. Therefore, as long as the energy of eigenvalues is calculated using the normal form of the Hamiltonian on the symplectic leaf, the Krein-Moser theorem will remain intact.

Difficulty may arise when perturbations are viewed in a more general way. Consider a nonlinear Hamiltonian system. Many linear systems are derived by linearizing around equilibria of some original nonlinear system. This leads to the expansion of both the Hamiltonian functional and the Poisson bracket in a neighborhood of the equilibrium of choice. It is a physically important problem to study how the linear stability of these equilibria change under small changes to the Hamiltonian, perhaps by moving to neighboring equilibria, or by modifying the Hamiltonian of the original non-linear system so that its equilibria change slightly.

In these cases, both the Hamiltonian and the Poisson bracket of the resulting linear theory are perturbed. By simply allowing such perturbations within the previous formalism, by making it possible to change both J and H, it is possible to induce violations of Krein's theorem by changing the rank of J. The value of the Hamiltonian on the Casimir invariants does not effect the stability, but when the rank is increased suddenly it becomes relevant.

Rank changes do not need to be artificially introduced to witness this type of behavior, as many noncanonical Hamiltonian systems have points within their phase space where the rank of J changes. Equilibria located at these points can lead to rank changing perturbations of the linear theory.

The picture can be clarified by introducing the concept of dynamically accessible perturbations. In the infinite-dimensional case the rank of the symplectic leaf of a point in phase space may vary tremendously from point to point.

2.3 Other studies of the structural stability of infinite-dimensional Hamiltonian systems

There have been a number of previous [5] and contemporaneous [29] works studying the structural stability of infinite-dimensional Hamiltonian systems. The infinite-dimensional theory becomes different from the finite-dimensional theory when the spectrum of the infinite-dimensional time evolution operator becomes continuous.

The oldest such result is contained in a paper written by Grillakis [5]. This paper considered canonical Hamiltonian systems, and was motivated by the study of the non-linear Schrodinger equation and the Klein-Gordon equation. Grillakis formulated his Hamiltonian system to be in a phase space that could be decomposed into the Cartesian product of two Hilbert spaces. The Hamiltonian operator would be written as a matrix of operators on the phase space, in the form:

$$H = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}. \tag{2.3}$$

Each point in phase space is written $u = (q, p)$ and Hamilton's equations are $\frac{\partial u}{\partial t} = J_0 Hu$, as before. This leads to the eigenvalue problem:

$$\lambda q = Ap$$

$$\lambda p = -Bq.$$

Under the assumptions that $p \in X = \{kerA \cup kerB\}^{\perp}$, $R = A|_X$ and $S = B^{-1}|_X$, this eigenvalue problem can be reduced to:

$$(R - zS)p = 0. \tag{2.4}$$

Grillakis takes this as his starting point, assuming that R and S are self-adjoint operators with empty kernels. He considers the case where there is a continuous spectrum, which has positive signature, and a discrete eigenvalue embedded within the continuum. He studies perturbations of the Hamiltonian by relatively compact perturbations of R and S. He is able to show that if $R(\delta) = R + \delta W$ and $S(\delta) = (S^{-1} + \delta V)^{-1}$, and that the discrete eigenvalue is of positive energy, that there is some δ for which $(R(\delta_1), S(\delta_1))$ is spectrally stable for $\delta_1 < \delta$. Similarly, if the eigenvalue has negative energy, there exist W and V that make $(R(\delta), S(\delta))$ unstable for all values of δ.

The work of Grillakis also led to criteria for determining the number of negative eigenvalues of the systems that he considered. There has been a large amount of work in this direction since then, with the goal of finding the number and location of negative eigenvalues of linear wave equations arising from Hamiltonian systems [40, 41].

Hirota has studied bifurcations into instability in ideal MHD [29]. This is much closer to the work presented here, as MHD is a noncanonical Hamiltonian system with a Lie-Poisson bracket. Hirota has carried out a research program that mirrors the work of Morrison on Vlasov theory [8], with the first step being the derivation of an integral transformation that converts the linearized equations of MHD in a neighborhood of an equilibrium into action angle variables. This enabled him to attach a signature to the continuous spectrum, paving the way for studies of the structural stability in terms of signature.

Hirota used this signature to interpret hydrodynamic instabilities in terms of interactions between positive and negative energy modes and the continuous spectrum, in the linearization around a parallel shear flow.

While Grillakis achieved general results for canonical systems, he only considered the case where the continuous spectrum interacts with the discrete spectrum, and the case of continuous spectra interacting with each other remains an interesting problem. In the noncanonical case of the linearized Vlasov-Poisson equation, which is presented here (and in the paper [6], this is the cause of structural instability.

2.4 Noncanonical Hamiltonian form of the Vlasov-Poisson Equation

In this section we will introduce the Vlasov-Poisson equation and its Hamiltonian formulation. The Vlasov-Poisson equation has a single dependent variable $f(x, v, t)$, such that for each time t, $f\colon \mathcal{D} \to \mathbb{R}$, where the particle phase space $\mathcal{D}$ is some two-dimensional domain with coordinates (x, v). The variable f is a phase space density. The dynamics are Hamiltonian in with a noncanonical Poisson bracket of the form

$$\{F, G\} = \int_{\mathcal{D}} dxdv\, f\, \left[\frac{\delta F}{\delta f}, \frac{\delta G}{\delta f}\right]\, ,. \tag{2.5}$$

where $[f, g] := f_x g_v - f_v g_x$ is the usual Poisson bracket, where the subscripts denote partial differentiation.

For the Vlasov-Poisson equation we assume $\mathcal{D} = X \times \mathbb{R}$, where $X \subset \mathbb{R}$ or $X = S$, the circle, the distinction will not be important. The Hamiltonian is given by

$$H[f] = \frac{1}{2}\int_X dx \int_{\mathbb{R}} dv\, v^2 f + \frac{1}{2}\int_X dx\, |\phi_x|^2\, ,. \tag{2.6}$$

where ϕ is a shorthand for the functional dependence on f obtained through solution of Poisson's equation, $\phi_{xx} = 1 - \int_{\mathbb{R}} f dv$, for a positive charge species with a neutralizing background. Using $\delta H/\delta f = \mathcal{E} = v^2/2 + \phi$, we obtain

$$f_t = \{f, H\} = -[f, \mathcal{E}] = -v f_x + \phi_x f_v\, ,. \tag{2.7}$$

where, as usual, the plasma frequency and Debye length have been used to non-dimensionalize all variables.

This Hamiltonian form for the Vlasov-Poisson equation was first published in [19]. For a discussion of a general class of systems with this Hamiltonian form, to which the ideas of the present analysis can be applied, see [34]. In a sequence of papers [42, 7, 31, 43, 8, 44] various ramifications of the Hamiltonian form have been explored – notably, canonization and diagonalization of the linear dynamics to which we now turn.

Because of the noncanonical form, linearization requires expansion of the Poisson bracket as well as the Hamiltonian. Equilibria, f_0, are obtained by extremization of a free energy functional, $F = H + C$, as was first done for Vlasov-like equilibria in [45]. Writing $f = f_0 + f_1$ and expanding gives the Hamiltonian form for the linear dynamics

$$f_{1t} = \{f_1, H_L\}_L. \tag{2.8}$$

where the linear Hamiltonian, $H_L = \frac{1}{2}\int_{\mathcal{D}} dxdv\, f_1 \mathcal{O}\, f_1$, is the second variation of F, a quadratic from in f_1 defined by the symmetric operator $\mathcal{O}$, and $\{F, G\}_L = \int_{\mathcal{D}} dxdv\, f_0[F_1, G_1]$ with $F_1 := \delta F/\delta f_1$. Thus the linear dynamics is governed by the time evolution operator $T\cdot := -\{\,\cdot\,, H_L\}_L = [f_0, \mathcal{O}\,\cdot\,]$.

Linearizing the Vlasov-Poisson equation about an homogeneous equilibrium, $f_0(v)$, gives rise to the system,

$$\begin{aligned} f_{1t} &= -v f_{1x} + \phi_{1x} f_0' && (2.9)\\ \phi_{1xx} &= -\int_{\mathbb{R}} dv\, f_1\,, && (2.10) \end{aligned}$$

for the unknown $f_1(x, v, t)$. Here $f_0' := df_0/dv$. This is an infinite-dimensional linear Hamiltonian system generated by the Hamiltonian functional:

$$H_L[f_1] = -\frac{1}{2}\int_X dx \int_{\mathbb{R}} dv \, \frac{v}{f_0'}\,|f_1|^2 + \frac{1}{2}\int_X dx\,|\phi_{1x}|^2 \,.. \tag{2.11}$$

We concentrate on systems where x is an ignorable coordinate, and either Fourier expand or transform. For Vlasov-Poisson this gives the system

$$f_{kt} = -ikvf_k + \frac{if_0'}{k}\int_{\mathbb{R}} d\bar{v}\, f_k(\bar{v},t) =: -T_k f_k\,,. \tag{2.12}$$

where $f_k(v,t)$ is the Fourier dual to $f_1(x,v,t)$. Perturbation of the spectrum of the operator defined by Eq. (2.12) is the primary subject of this chapter. The operator T_k is a Hamiltonian operator generated by the Hamiltonian functional

$$H_L[f_k, f_{-k}] = \frac{1}{2}\sum_k \left(-\int_{\mathbb{R}} dv\, \frac{v}{f_0'}\,|f_k|^2 + |\phi_k|^2 \right)\,,. \tag{2.13}$$

with the Poisson bracket

$$\{F,G\}_L = \sum_{k=1}^{\infty} ik \int_{\mathbb{R}} dv\, f_0' \left(\frac{\delta F}{\delta f_k}\frac{\delta G}{\delta f_{-k}} - \frac{\delta F}{\delta f_{-k}}\frac{\delta G}{\delta f_k} \right) .. \tag{2.14}$$

Observe from (2.14) that $k \in \mathbb{N}$ and thus f_k and f_{-k} are independent variables that are almost canonically conjugate. Thus the complete system is

$$f_{kt} = -T_k f_k \qquad \text{and} \qquad f_{-kt} = -T_{-k} f_{-k}\,,. \tag{2.15}$$

from which we conclude the spectrum is Hamiltonian.

Lemma 2.4.1. *If λ is an eigenvalue of the Vlasov equation linearized about the equilibrium $f_0'(v)$, then so are $-\lambda$ and $\overline{\lambda}$ (complex conjugate). Thus if $\lambda = \gamma + i\omega$, then eigenvalues occur in the pairs, $\pm\gamma$ and $\pm i\omega$, for purely real and imaginary cases, respectively, or quartets, $\lambda = \pm\gamma \pm i\omega$, for complex eigenvalues.*

Proof. That $-\lambda$ is an eigenvalue follows immediately from the symmetry $T_{-k} = -T_k$, and that $\overline{\lambda}$ is an eigenvalue follows from $T_k f_k = -\overline{(T_k f_k)}$. □

In [7, 31, 8, 44] it was shown how to scale f_k and f_{-k} to make them canonically conjugate variables. In order to do this requires the following definition of dynamical accessibility, a terminology introduced in [46, 47].

Definition A particle phase space function k is *dynamically accessible* from a particle phase space function h, if k is an area-preserving rearrangement of h; i.e., in coordinates $k(x,v) = h(X(x,v), V(x,v))$, where $[X,V] = 1$. A perturbation δh is *linearly dynamically accessible* from h if $\delta h = [G,h]$, where G is the infinitesimal generator of the canonical transformation $(x,v) \leftrightarrow (X,V)$.

Dynamically accessible perturbations come about by perturbing the particle orbits under the action of some Hamiltonian. Since electrostatic charged particle dynamics is Hamiltonian, one can make the case that these are the only perturbations allowable within the confines of Vlasov-Poisson theory.

Given an equilibrium state f_0, linear dynamically accessible perturbations away from this equilibrium state satisfy $\delta f_0 = [G, f_0] = G_x f_0'$. Therefore assuming the initial condition for the linear dynamics is linearly dynamically accessible, we can define

$$q_k(v,t) = f_k \qquad \text{and} \qquad p_k(v,t) = -i f_{-k}/(k f_0'). \tag{2.16}$$

without worrying about a singularity at the zeros of f_0' and $k = 0$. With the definitions of (2.16), the Poisson bracket of (2.14) achieves canonical from

$$\{F, G\}_L = \sum_{k=1}^{\infty} \int_{\mathbb{R}} dv \left(\frac{\delta F}{\delta q_k} \frac{\delta G}{\delta p_k} - \frac{\delta F}{\delta p_k} \frac{\delta G}{\delta q_k} \right) .. \tag{2.17}$$

The full system has the new Hamiltonian $\bar{H} = H + UP$ in a frame moving with speed U, where $P = \int_{\mathcal{D}} dxdv\, v f$. Linearizing in this frame yields the linear Hamiltonian $\bar{H}_L = H_L + P_L$, from which we identify the linear momentum

$$P_L[f_k, f_{-k}] = \frac{1}{2} \sum_{k=1}^{\infty} \int_{\mathbb{R}} dv\, \frac{k}{f_0'} \, |f_k|^2 \, , . \tag{2.18}$$

which must be conserved by the linear dynamics. It is easy to show directly that this is the case.

Lemma 2.4.2. *The momentum P_L defined by (2.18) is a constant of motion, i.e., $\{P_L, H_L\} = 0$.*

Proof. This follows immediately from (2.15): $\int_{\mathbb{R}} dv\ (f_k T_{-k} + f_{-k} T_k) = 0$. □

Observe, that like the Hamiltonian, H_L, the momentum P_L is conserved for each k, which in all respects appears only as a parameter in our system. Assuming the system size to be L yields $k = 2\pi n/L$ with $n \in \mathbb{N}$, and, thus, this parameter can be taken to be in $\mathbb{R}^+/\{0\}$. Alternatively, we could suppose $X = \mathbb{R}$, Fourier transform, and split the Fourier integral to obtain an expression similar to (2.14) with the sum replaced by an integral over positive values of k. For the present analysis we will not be concerned with issues of convergence for reconstructing the spatial variation of $f_1(x, v, t)$, but only consider $k \in \mathbb{R}^+/\{0\}$

to be a parameter in our operator. We will see in Sec. 2.5 that the operator T_k possesses a continuous component to its spectrum. But, we emphasize that this continuous spectrum of interest arises from the multiplicative nature of the velocity operator, i.e. the term vf_k of T_k, not from having an infinite spatial domain, as is the case for free particle or scattering states in quantum mechanics. In the remainder of the chapter, f will refer to either f_1 or f_k, which will be clear from context, and the dependence on k will be suppressed, e.g. in T_k, unless k dependence is being specifically addressed.

2.5 Spectral stability

Now we consider properties of the evolution operator T defined by (2.12). We define spectral stability in general terms, record some properties of T, and describe the tools necessary to characterize the spectrum of T. We suppose f_k varies as $\exp(-i\omega t)$, where ω is the frequency and $i\omega$ is the eigenvalue. For convenience we also use $u := \omega/k$, where recall $k \in \mathbb{R}^+$. The system is spectrally stable if the spectrum of T is less than or equal to zero or the frequency is always in the closed lower half plane. Since the system is Hamiltonian, the question of stability reduces to deciding if the spectrum is confined to the imaginary axis.

Definition The linearized dynamics of a Hamiltonian system around some equilibrium solution, with the phase space of solutions in some Banach space $\mathcal{B}$, is *spectrally stable* if the spectrum $\sigma(T)$ of the time evolution operator T is purely imaginary.

Spectral stability does not guarantee that the system is stable, or that the equilibrium f_0 is linearly stable. (See e.g. [14] for general discussion). The solutions of a spectrally stable system are guaranteed to grow at most sub-exponentially and one can construct a spectrally stable system with polynomial temporal growth for certain initial conditions. (See e.g. [48] for analysis of the Vlasov system.)

Spectral stability relies on functional analysis for its definition, since the spectrum of the operator T may depend on the choice of function space $\mathcal{B}$. The time evolution operators arising from the types of noncanonical Hamiltonian systems that are of interest here generally contain a continuous spectrum [34] and the effects of perturbations that we study can be categorized by properties of the continuous spectrum of these operators. In general for the operators of [34], the operator T is the sum of a multiplication operator and an integral operator. In the Vlasov case, the multiplicative operator is $iv\cdot$ and the integral operator is $f_0' \int dv \,\cdot$. As we will see, the multiplication operator causes the continuous spectrum to be composed of the entire imaginary axis except possibly for some discrete points.

Instability comes from the point spectrum. In particular, the linearized Vlasov Poisson equation is not spectrally stable when the time evolution operator has a spectrum that includes a point away from the imaginary axis, with the necessary counterparts implied by Lemma 2.4.2.4. For the operator T this will always be a discrete mode; i.e. an eigenmode associated with an eigenvalue in the point spectrum.

Theorem 2.5.1. *The one-dimensional linearized Vlasov-Poisson system with homogeneous equilibrium f_0 is spectrally unstable if for some $k \in \mathbb{R}^+$ and u in the upper half plane, the plasma dispersion relation*

$$\varepsilon(k,u) := 1 - k^{-2}\int_{\mathbb{R}} dv \frac{f_0'}{v-u} = 0\,.$$

Otherwise it is spectrally stable.

Proof. The details of this proof are given in plasma textbooks. It follows directly from (2.9) and (2.10), and the assumption $f_1 \sim \exp(ikx - i\omega t)$. □

Using the Nyquist method that relies on the argument principle of complex analysis, Penrose [49] was able to relate the vanishing of $\varepsilon(k,u)$ to the winding number of the closed curve determined by the real and imaginary parts of ε as u runs along the real axis. Such closed curves are called Penrose plots. The crucial quantity is the integral part of ε as u approaches the real axis from above:

$$\lim_{u\to 0^+} \frac{1}{\pi}\int_{\mathbb{R}} dv\, \frac{f_0'}{v-u} = H[f_0'](u) - i f_0'(u)\,,$$

where $H[f_0']$ denotes the Hilbert transform, $H[f_0'] = \frac{1}{\pi}⨍ dv\, f_0'/(v-u)$, where $⨍ := PV\int_{\mathbb{R}}$ indicates the Cauchy principle value. (See [50] for an in depth treatment of Hilbert transforms.) The graph of the real line under this mapping is the essence of the Penrose plot, and so we will refer to these closed curves as Penrose plots as well. When necessary to avoid ambiguity we will refer to the former as ε-plots.

For example, Fig. 2.2 shows the derivative of the distribution function, f_0', for the case of a Maxwellian distribution and Fig. 2.3 shows the contour $H[f_0'] - if_0'(u)$ that emerges from the origin in the complex plane at $u = -\infty$, descends, and then wraps around to return to the origin at $u = \infty$. From this figure it is evident that the winding number of the $\varepsilon(k, u)$-plot is zero for any fixed $k \in \mathbb{R}$, and as a result there are no unstable modes.

Making use of the argument principle as described above, Penrose obtained the following criterion:

Theorem 2.5.2. *The linearized Vlasov-Poisson system with homogeneous equilibrium f_0 is spectrally unstable if there exists a point u such that*

$$f_0'(u) = 0 \qquad \text{and} \qquad \fint dv\, \frac{f_0'(v)}{v-u} > 0\,,$$

with f_0' traversing zero at u. Otherwise it is spectrally stable.

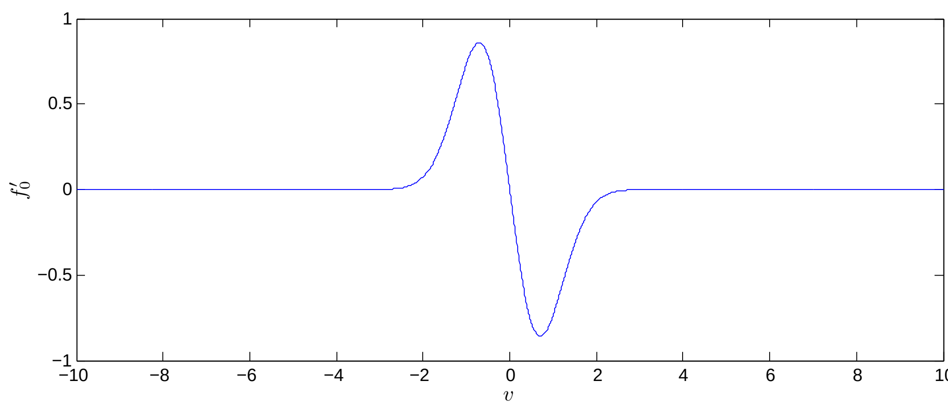

Figure 2.2: f_0' for a Maxwellian distribution.

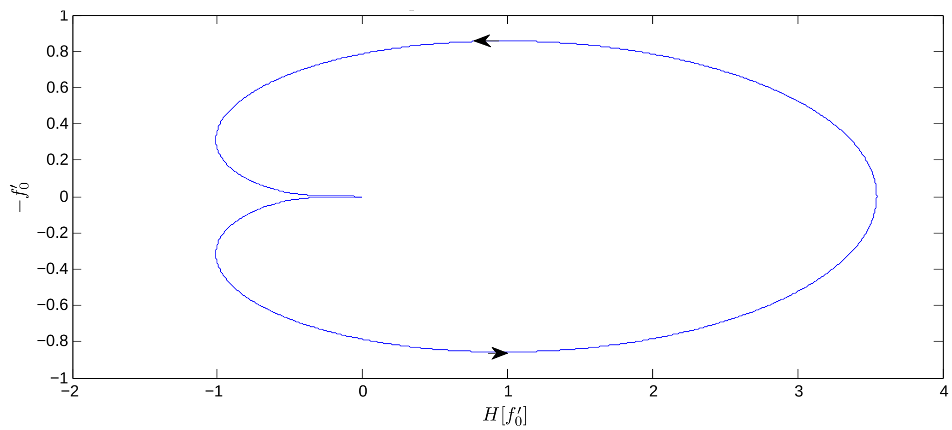

Figure 2.3: Stable Penrose plot for a Maxwellian distribution.

Penrose plots can be used to visually determine spectral stability. As described above, the Maxwellian distribution $f_0 = e^{-v^2}$ is stable, as the resulting ε-plot does not encircle the origin. However, it is not difficult to construct unstable distribution functions. The superposition of two displaced Maxwellian distributions, $f_0 = e^{-(v+c)^2} + e^{-(v-c)^2}$, is such a case. As c increases the distribution goes from stable to unstable. Figures 2.4 and 2.5 demonstrate how the transition from stability to instability is manifested in a Penrose plot. The two examples are $c = 3/4$ and $c = 1$. (Note, the normalization of f_0 only affects the overall scale of the Penrose plots and so is ignored for convenience.) It is evident from Fig. 2.5 that for some $k \in \mathbb{R}$ the ε-plot (which is a displacement of the curve shown by multiplying by $-k^{-2}$ and adding unity) will encircle the origin, and thus will be unstable for such k-values.

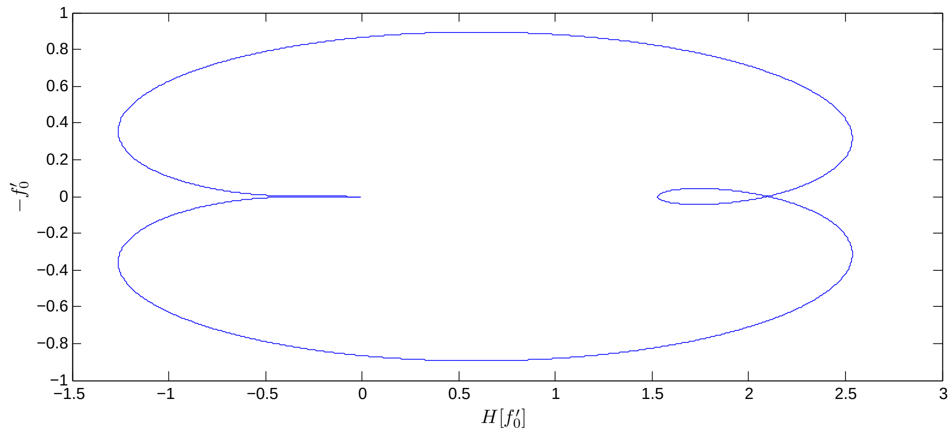

Figure 2.4: Penrose plot for a stable superposition of Maxwellian distributions

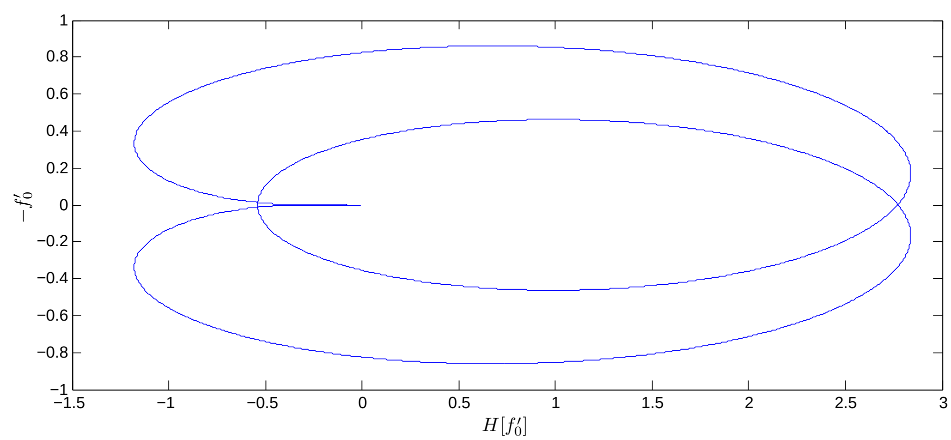

Figure 2.5: The unstable Penrose plot corresponding to two separated Maxwell distributions.

We now are positioned to completely determine the spectrum. For convenience we set $k = 1$ when it does not affect the essence of our arguments, and consider the operator $T\colon f \mapsto ivf - if_0' \int f$ in the space $W^{1,1}(\mathbb{R})$, but we also discuss the space $L^1(\mathbb{R})$. The space $W^{1,1}(\mathbb{R})$ is the Sobolev space containing the closure of functions under the norm $\|f\|_{1,1} = \|f\|_1 + \|f'\|_1$. Thus it contains all functions that are in $L^1(\mathbb{R})$ whose weak derivatives are also in $L^1(\mathbb{R})$. First we establish the expected facts that T is densely defined and closed.

In $W^{1,1}$ the operator T is the sum of the multiplication operator and a bounded operator – that it is densely defined and closed follows from the fact that the multiplication operator is densely defined and closed in these spaces, where

$$D_1(T) := \{f | vf \in W^{1,1}(\mathbb{R})\}.$$

Theorem 2.5.3. *The operator $T\colon W^{1,1}(\mathbb{R}) \to W^{1,1}(\mathbb{R})$ with domain $D_1(T)$ is both (i) densely defined and (ii) closable.*

Proof. (i) The set of all smooth functions with compact support, $C_c^\infty(\mathbb{R})$ is a subset of D_1. This set is dense in $W^{1,1}(\mathbb{R})$ so D_1 is dense and T is densely defined. (ii) The operator T is closable if the operator v is closable because T and v differ by a bounded operator. The multiplication operator v is closed if for each sequence $f_n \subset W^{1,1}(\mathbb{R})$ that converges to 0 either vf_n converges to 0 or vf_n does not converge. Suppose vf_n converges. At each point f_n converges to 0. Therefore vf_n converges to 0 at each point, so vf_n converges to 0 if it converges. $\square$

Therefore there exists some domain D such that the graph (D, TD) is closed.

In determining the spectrum of the operator T, denoted (T), we split the spectrum into point, residual, and continuous components as follows:

Definition For $\lambda \in \sigma(T)$ the resolvent of T is $R(T,\lambda) = (T - \lambda I)^{-1}$, where I is the identity operator. We say λ is (i) in the point spectrum, $\sigma_p(T)$, if $T - \lambda I$ fails to be injective, (ii) in the residual spectrum, $\sigma_r(T)$, if $R(T,\lambda)$ exists but is not densely defined, and (iii) in the continuous spectrum, $\sigma_c(T)$, if $R(T,\lambda)$ exists and is densely defined but unbounded.

Using this definition we characterize the spectrum of the operator T.

Theorem 2.5.4. *The component $\sigma_p(T)$ consists of all points $\lambda = iu \in \mathbb{C}$ where $1 - k^{-2} \int_{\mathbb{R}} dv\, f_0'/(v-u) = 0$, $\sigma_c(T)$ consists of all $\lambda = iu$ with $u \in \mathbb{R} \setminus (-i\sigma_p(T) \cap \mathbb{R})$, and $\sigma_r(T)$ contains all points $\lambda = iu$ in the complement of $\sigma_p(T) \cup \sigma_c(T)$ that satisfy $f_0'(u) = 0$.*

Proof. By the Penrose criterion we can identify all the points in the point spectrum. If $1 - k^{-2} \int_{\mathbb{R}} dv f_0'/(v-u) = 0$ then $iu = \lambda \in \sigma_p(T)$. Because the system is Hamiltonian these modes will occur for the linearized Vlasov-Poisson system in quartets (two for T_k and two for T_{-k}), as follows from Lemma 2.4.2.4.

It is possible for there to be discrete modes with real frequencies and these will occur in pairs. If for real u the map $u \mapsto \varepsilon$ passes through the origin then there will be such an embedded mode.

For convenience we drop the wavenumber subscript k on f_k and add the subscript n to identify f_n as an element of a sequence of functions that converges to zero with, for each n, support contained in an interval of length $2\epsilon(n)$ surrounding the point u and zero average value. Let $u \in \mathbb{R}$ and choose the sequence $\{f_n\}$ so that $\epsilon(n) \to 0$. Then for each n

$$\begin{aligned}\|R(T, iu)\| &\geq \frac{\|f_n\|_{1,1}}{\|(v-u)f_n\|_{1,1}} \\ &\geq \frac{\|f_n\|_{1,1}}{\|v-u\|_{W^{1,1}(u-\epsilon,u+\epsilon)}\|f_n\|_{1,1}} \\ &= \frac{1}{\|v-u\|_{W^{1,1}(u-\epsilon,u+\epsilon)}}.\end{aligned}$$

In the above expression, $W^{1,1}(u-\epsilon, u+\epsilon)$ refers to the integral of $|f|+|f'|$ over the interval $(u-\epsilon, u+\epsilon)$. Therefore the resolvent is an unbounded operator and $iu = \lambda$ is in the spectrum. If the frequency u has an imaginary component $i\gamma$ then $\|R(T, iu)\| < 1/\gamma$ so unless $iu = \lambda$ is part of the point spectrum it is part of the resolvent set.

The residual spectrum of T is contained in the point spectrum of T^*. The dual of $W^{1,1}$ is the space $W^{-1,1}$ defined by pairs $(g, h) \in W^{-1,1}$ with $\|(g, h)\|_{-1,1} < \infty$ (cf. [51]). The operator $T^*(g, h) = i(vg - h + \int(gf_0' - hf_0'')dv, -vh)$ is the adjoint of T. If we search for a member $iu = \lambda$ of the point spectrum we get two equations, one of which is $(v-u)h = 0$. This forces $h = 0$ because h cannot be a δ-function in $W^{-1,1}$. The other equation is then $(v-u)g + \int g f_0' dv = 0$ which can only be true if the integral is zero or if $(v-u)g$ is a constant. For this $g = \frac{1}{v-u}$ and the resulting equation for u is the same equation as that for the frequency of the point modes of T. If the integral is zero then $g = \delta(v-u)$ is a solution when $f_0'(u) = 0$. Therefore the residual spectrum contains the points $\lambda = iu$ satisfying $f_0'(u) = 0$. □

This characterization of the spectrum fails in Banach spaces with less regularity than $W^{-1,1}$, such as L^p spaces, because the Dirac δ is not contained in the dual space. In this case the residual spectrum vanishes because $\sigma_p(T^*) = \sigma_p(T)$. This calculation is nearly identical to that of Degond [48], who characterizes the residual spectrum slightly differently than we do. In any event, the result is that the Penrose criterion determines whether T is spectrally stable or not. If the winding number of the ε-plot is positive, then there is spectral instability and if it is zero there is spectral stability.

2.6 Structural stability

Spectral stability characterizes the linear dynamics of a nonlinear Hamiltonian system in a neighborhood of an equilibrium. The main question now is to determine when a spectrally stable system can be made spectrally unstable with a small perturbation. When this is impossible for our choice of allowed perturbations, we say the equilibrium is structurally stable, and when there is an infinitesimal perturbation that makes the system spectrally unstable we say that the equilibrium is spectrally unstable. We can make this more precise by stating it in terms of operators on a Banach space.

Definition Consider an equilibrium solution of a Hamiltonian system and the corresponding time evolution operator T for the linearized dynamics, with a phase space some Banach space $\mathcal{B}$. Suppose that T is spectrally stable. Consider perturbations δT of T and define a norm on the space of such perturbations. Then we say that the equilibrium is *structurally stable* under this norm if there is some $\delta > 0$ such that for every $\|\delta T\| < \delta$ the operator $T + \delta T$ is spectrally stable. Otherwise the system is *structurally unstable*.

Because we are dealing with physical systems it makes sense to have physical motivation for the choice of norm on the space of perturbations. In this chapter we are interested in perturbations of the Vlasov equation through changes in the equilibrium. This choice is motivated by the Hamiltonian structure of the equations and Krein's theorem for finite-dimensional systems. In general the space of possible perturbations is quite large, but perturbations of equilibria give rise to operators in certain Banach spaces and motivate the definition of norm. Even in the case of unbounded perturbations there may exist such a norm (see Kato [17], for instance).

Consider a stable equilibrium function f_0. We will consider perturbations of the equilibrium function and the resulting perturbation of the time evolution operator. Suppose that the time evolution operator of the perturbed system is $T + \delta T$. In the function space that we will consider these perturbations are bounded operators and their size can be measured by the norm $\|\delta T\|$. This norm will be proportional to the norm of $\|\delta f_0'\|$, where δf_0 is the perturbation of the equilibrium.

Definition Consider the formulation of the linearized Vlasov-Poisson equation in the Banach space $W^{1,1}(\mathbb{R})$ with a spectrally stable homogeneous equilibrium function f_0. Let $T_{f_0+\delta f_0}$ be the time evolution operator corresponding to the linearized dynamics around the distribution function $f_0 + \delta f_0$. If there exists some δ depending only on f_0 such that $T_{f_0+\delta f_0}$ is spectrally stable whenever $\|\delta T_{\delta f_0}\| = \|T_{f_0} - T_{f_0+\delta f_0}\| < \delta$, then the equilibrium f_0 is structurally stable under perturbations of f_0.

The aim of this work is to characterize the structural stability of the linearized Vlasov-Poisson equation. We will prove that if the perturbation function is some homogeneous δf_0 and the norm is $W^{1,1}$ (and L^1 as a consequence) every equilibrium distribution function is structurally unstable to an infinitesimal perturbation in this space. This fact will force us to consider more restricted sets of perturbations.

2.6.1 Winding number

We need to compute the winding number of Penrose plots and the change in winding number under a perturbation, both in this section and the rest of the chapter. We use the fact that one way to compute the winding number is to draw a ray from the origin to infinity and to count the number of intersections with the contour accounting for orientation.

Lemma 2.6.1. *Consider an equilibrium distribution function f_0'. The winding number of the Penrose ε-plot around the origin is equal to $\sum_u \operatorname{sgn}(f_0''(u))$ for all $u \in \mathbb{R}^-$, satisfying $f_0'(u) = 0$.*

To calculate the winding number of the Penrose ε-plot using this lemma one counts the number of zeros of f_0' on the negative real line and adds them with a positive sign if f_0'' is positive, a Penrose crossing from the upper half plane to the lower half plane, a negative sign if f_0'' is negative, a crossing from the lower half plane to the upper half plane, and zero if u is not a crossing of the x-axis, a tangency. This lemma comes from the following equivalent characterization of the winding number from differential topology [52].

Definition If X is a compact, oriented, l-dimensional manifold and $f\colon X \to \mathbb{R}^{l+1}$ is a smooth map, the winding number of f around any point $z \in \mathbb{R}^{l+1} - f(X)$ is the degree of the direction map $u\colon X \to S^l$ given by $u(x) = \frac{f(x)-z}{|f(x)-z|}$.

In our case the compact manifold is the real line plus the point at ∞ and $l = 1$. The degree of u is the intersection number of u with any point on the circle taken with a plus sign if the differential preserves orientation and a minus sign if it reverses it. The lemma is just a specialization of this definition to the negative x-direction on the circle. If more than one derivative of f_0 vanishes at a zero of f_0' there is a standard procedure for calculating the winding number by determining if there is a sign change in f_0' at the zero.

2.6.2 Structural instability of general f_0

In a large class of function spaces it is possible to create infinitesimal perturbations that make any equilibrium distribution function unstable. This can happen in any space where the Hilbert transform is an unbounded operator. In these spaces there will be an infinitesimal δf_0 such that $H[\delta f_0']$ is order one at a zero of f_0'. Such a perturbation can turn any point where $f_0' = 0$ into a point where $H[f_0' + \delta f_0'] > 0$ as well. Because $\delta f_0'$ is small and the region where $H[\delta f_0']$ is not small is also small, the only effect on the Penrose plot will be to move the location of the zero. Thus, such a perturbation will increase the winding number and cause instability.

We will explicitly demonstrate this for the Banach space $W^{1,1}(\mathbb{R})$ and, by extension, the Banach space $L^1 \cap C_0$. This will imply that any distribution function is infinitesimally close to instability when the problem is set in one of these spaces, implying the structural instability of every distribution function.

Suppose we perturb f_0 by a function δf_0. The resulting perturbation to the operator T is the operator mapping f to $\delta f_0' \int dv f$. In the space $W^{1,1}$ this is a bounded operator and thus we take the norm of the perturbing operator to be $\|\delta f_0'\|_{1,1}$. Now we introduce a class of perturbations that can be made infinitesimal, but have Hilbert transform of order unity.

Consider the function $\chi(v, h, d, \epsilon)$ defined by

$$\chi = \begin{cases} hv/\epsilon & |v| < \epsilon \\ h\,\mathrm{sgn}(v) & \epsilon < |v| < d + \epsilon \\ h + d/2 + \epsilon/2 - v/2 & 2h + d + \epsilon > v > d + \epsilon \\ -h - d/2 - \epsilon/2 - v/2 & 2h + d + \epsilon > -v > d + \epsilon \\ 0 & |v| > 2h + d + \epsilon \end{cases} .$$

Figures 2.6 and 2.7 show the graph of χ and its Hilbert transform, $H[\chi]$, respectively.

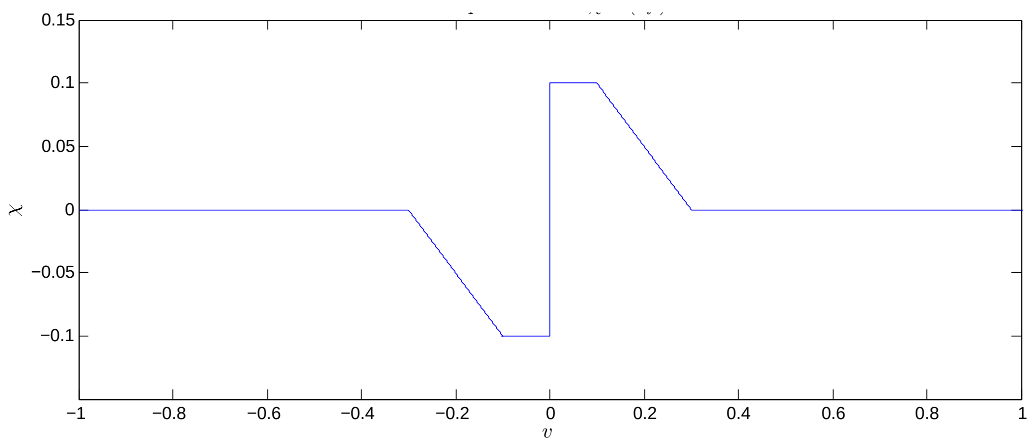

Figure 2.6: The perturbation χ for $\epsilon = e^{-10}$, $h = d = .1$.

Figure 2.7: The Hilbert transform of χ.

Lemma 2.6.2. *If we choose* $d = h$ *and* $\epsilon = e^{-(1/h)}$*, then for any* $\delta, \gamma > 0$ *we can choose an* h *such that* $\|\chi\|_{1,1} < \delta$ *and* $⨍ dv\, \chi/v > 1 - O(h)$*, and* $|⨍ dv\, \chi/(u - v)| < |\gamma/u|$ *for* $|u| > |2h + d + \epsilon|$.

Proof. In the space $W^{1,1}$ the function χ has norm $2h^2 + 2hd + h\epsilon + 4h$, which is less than any δ for small enough h. We can compute the value of the Hilbert transform of this function at a given point u by calculating the principal values:

$$\begin{aligned} ⨍ dv \, \frac{\chi}{v-u} &= \frac{hu}{\epsilon} \log\left(\frac{|u-\epsilon|}{|u+\epsilon|}\right) + h \log\left(\frac{|d+\epsilon-u||u+d+\epsilon|}{|\epsilon+u||\epsilon-u|}\right) \\ &+ \frac{1}{2}(d+\epsilon+2h-u)\log\left(\frac{|d+\epsilon+2h-u|}{|d+\epsilon-u|}\right) \\ &+ \frac{1}{2}(d+\epsilon+2h+u)\log\left(\frac{|d+\epsilon+2h+u|}{|d+\epsilon+u|}\right). \end{aligned} \quad (2.19)$$

We analyze the asymptotics of this function as h, d, and ϵ go to zero, with the desiderata that i) the norm of χ goes to zero, ii) the maximum of the Hilbert transform of χ is $O(1)$, and iii) there is a band of vanishing width around the origin outside of which the Hilbert transform can be made arbitrarily close to zero.

Note that (2.19) can be written as a linear combination of translates of the function $x \log x$:

$$\begin{aligned} ⨍ dv \, \frac{\chi}{v-u} &= \frac{h}{\epsilon}((u-\epsilon)\log(|u-\epsilon|) - (u+\epsilon)\log(|u+\epsilon|)) \\ &- \frac{1}{2}(d+u+\epsilon)\log(|d+u+\epsilon|) - \frac{1}{2}(d-u+\epsilon)\log(|d-u+\epsilon|) \\ &+ \frac{1}{2}(d+u+\epsilon+2h)\log(|d+u+\epsilon+2h|) \\ &+ \frac{1}{2}(d-u+\epsilon+2h)\log(|d-u+\epsilon+2h|). \end{aligned} \quad (2.20)$$

The function $x \log x$ has a local minimum for positive x at $x = 1/e$. This is the point at which the function is most negative. It has zeros at $x = 0$ and $x = 1$. For values of u, d, ϵ, h close to zero all of the arguments of the log functions are less than $1/e$. Therefore, for $|u| < d + \epsilon + 2h$ the $x \log x$ terms are all monotonically decreasing functions of the argument x. Of the terms of (2.20), $\frac{h}{\epsilon}((u-\epsilon)\log(|u-\epsilon|) - (u+\epsilon)\log(|u+\epsilon|))$ has by far the largest coefficient as long as ϵ is much smaller than h. We choose $h = d$ and $\epsilon = 0(e^{-1/h})$. Then the terms that do not involve ϵ are all smaller than $(6h+\epsilon)\log(6h+\epsilon)$. With these choices χ satisfies

$$\begin{aligned} \chi(0) &= 2 - (h + e^{-1/h})\log(|h + e^{-1/h}|) + (3h + e^{-1/h})\log(|3h + e^{-1/h}|) \\ &= 2 + O(h \log h)\,. \end{aligned}$$

Consider the pair of functions $-(u+c)\log(|u+c|) + (u-c)\log(|u-c|)$. The derivative with respect to u is $-\log(|u+c|) + \log(|u-c|)$. This is zero for $u = 0$ and for $u > 0$ it is always negative and the pair is always decreasing, and for small values of h the pair is guaranteed to be positive. Suppose that $u > \epsilon$. Then we can bound the term with the h/ϵ coefficient:

$$\begin{aligned} &\frac{h}{\epsilon}\,|(u-\epsilon)\log(|u-\epsilon|) - (u+\epsilon)\log(|u+\epsilon|)| \\ &\qquad = \left|\frac{h}{\epsilon}(u-\epsilon)\log\frac{|u-\epsilon|}{|u+\epsilon|} - 2\epsilon\log(|u+\epsilon|)\right| \\ &\qquad = \frac{h}{\epsilon}\left|(u-\epsilon)\log\frac{1-\frac{\epsilon}{u}}{1+\frac{\epsilon}{u}} - 2\epsilon\log(|u+\epsilon|)\right| \\ &\qquad < \frac{h}{\epsilon}\left|(u-\epsilon)\log(e^{-\epsilon/u})\right| + 2\left|h\log(|u+\epsilon|)\right| \\ &\qquad = \frac{h(u-\epsilon)}{u} + 2\left|h\log(|u+\epsilon|)\right|\,. \end{aligned}$$

For $u >> \epsilon$, for example if $u = O(h^2)$, this term is $O(h \log h)$. Therefore, for $|u| > h^2$ we have $\chi = O(h \log h)$ which can be made arbitrarily small. When

$|u| > 3h + \epsilon$ the function χ decreases at least as fast as $O(1/u)$. With these choices of h, d, and ϵ, the norm of χ is $O(h)$, which proves the Lemma. □

Now we state theorem that any equilibrium is structurally unstable in both the spaces $W^{1,1}$ and $L^1 \cap C_0$. In order to prove this theorem we will make use of a result from Morse theory [53]. A Morse function is a function that has no degenerate critical points.

Lemma 2.6.3. *Let M be a smooth manifold. The set of Morse functions is open and dense in the space $C^r(M, \mathbb{R})$.*

Therefore if f_0 is C^2 there is an infinitesimal perturbation f_1 such that $f_0 + f_1$ is a Morse function. Because the winding number is stable under homotopy there is an f_1 such that all the zeros of $f_0 + f_1$ are non-degenerate and the winding number of the Penrose plot is the same as that of f_0. Therefore we will assume that f_0 is a Morse function. A consequence of this assumption is that all of the zeros of f_0' are isolated.

Theorem 2.6.4. *A stable equilibrium distribution $f_0 \in C^2$ is structurally unstable under perturbations of the equilibrium in the Banach spaces $W^{1,1}$ and $L^1 \cap C_0$.*

Proof. If f_0 is stable then the Penrose ϵ-plot of f_0' has a winding number of zero. Because the point at ∞ corresponds to a crossing where f_0' goes from negative to positive there exists a point u_0 with $f_0'(u_0) = 0$ that is an isolated zero, $H[f_0'](u_0) < 0$, and $f_0''(u_0) < 0$. Let $F = \sup |f_0''|$. Choose h to always be smaller than the distance from u_0 to the nearest 0 of f_0'. Then if $\epsilon = O(e^{-1/h})$ and $d = h$ the support of $\chi(u - u_0)$ will contain only one zero of f_0'. For h small enough the slope of χ at u_0 will be greater than F so that the function $f_0' + \chi$ will be positive for u in the set (u_0, u^+) for some u^+ in the support of χ. Similarly $f_0' + \chi$ will be negative for u in the set (u^-, u_0) for some u^- in the support of χ. Because χ has compact support the function $f_0' + \chi$ is positive in a neighborhood outside of the support of χ so that the intermediate value theorem guarantees one additional zero of the function $f_0' + \chi$ for $u > u_0$ and also for $u < u_0$. Choose χ so that this Hilbert transform of $f_0' + \chi$ is positive at the point u_0 and h small enough that it is negative before the next zero of $f_0' + \chi$ on either side of u_0. Then the winding number of $f_0' + \chi$ is positive because an additional positive crossing has been added on the negative real line.

Because the norm of χ is $O(h)$ in both $W^{1,1}$ and L^1 the distribution f_0 is unstable to an arbitrarily small perturbation and is therefore structurally unstable. $\square$

Thus we emphasize that we can always construct a perturbation that makes our linearized Vlasov-Poisson system unstable. For the special case of the Maxwellian distribution, Fig. 2.8 shows the perturbed derivative of the distribution function and Fig. 2.9 shows the Penrose plot of the unstable perturbed system. Observe the two crossings created by the perturbation on the positive axis as well as the negative crossing arising from the unboundedness of the perturbation.

In some sense Theorem 2.6.4 represents a failure of our class of perturbations to produce any interesting structure for the Vlasov equation. Indeed signature appears to play no role in delineating bifurcation to instability. In order to derive a nontrivial result we develop a new theory analogous to the finite-dimensional Hamiltonian perturbation theory developed by Krein and Moser. This new theory involves a restriction to dynamically accessible perturbations of the equilibrium state. This is natural since the noncanonical Hamiltonian structure can be viewed as the union of canonical Hamiltonian motions (on symplectic leaves) labeled by the equilibrium state – to compare with traditional finite-dimensional theory requires restriction to the given canonical Hamiltonian motion under consideration.

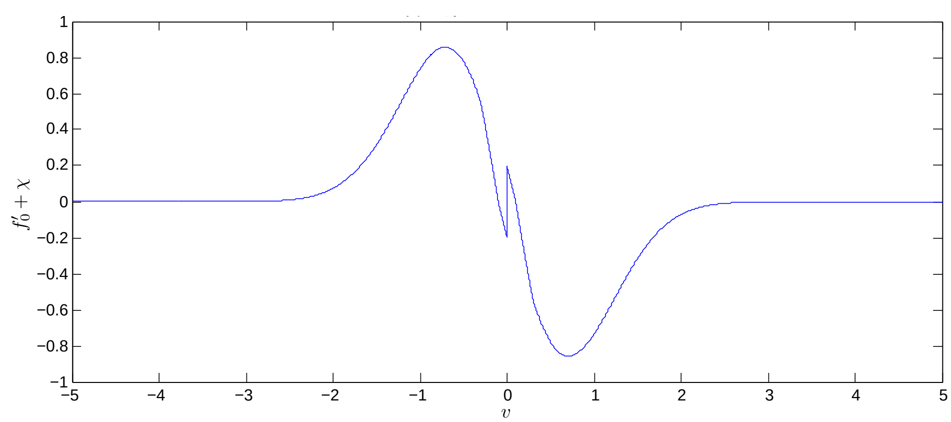

Figure 2.8: $f_0' + \chi$ for a Maxwellian distribution.

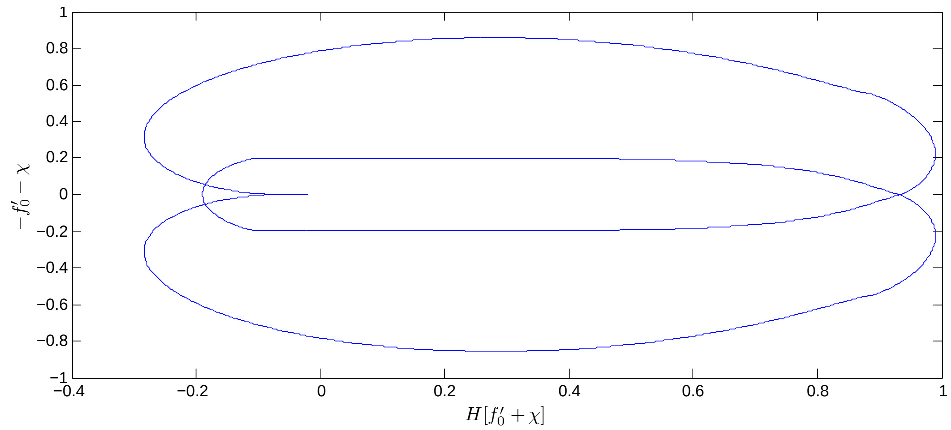

Figure 2.9: Penrose plot for perturbed Maxwellian.

2.7 The Krein-Moser theorem for Vlasov-Poisson

Our goal is to place the perturbation theory of infinite-dimensional Hamiltonian systems in the language of the finite-dimensional theory.

The appropriate definition of signature for the continuous spectrum of the Vlasov-Poisson equation was introduced in [7, 8] (see also [44]), where an integral transform was also introduced for constructing a canonical transformation to action-angle variables for the infinite-dimensional system. The transformation is a generalization of the Hilbert transform and it can be used to show that the linearized Vlasov-Poisson equation is equivalent to the system with the following Hamiltonian functional:

$$H_L = \sum_{k=1}^{\infty} \int_{\mathbb{R}} du\, \sigma_k(u) \omega_k(u) J_k(u,t)\,,. \tag{2.21}$$

where $\omega_k(u) = |ku|$ and $\sigma_k(u) = -\text{sgn}(kuf_0'(u))$ is the analog of the Krein signature corresponding to the mode labeled by $u \in \mathbb{R}$. (Note, the transformation can always be carried out in a frame where $f_0'(0) = 0$. Because the Hamiltonian does not transform as a scalar for frame shifts, which are time dependent transformations, signature is frame dependent. The Hamiltonian in a shifted frame is obtained by adding a constant times the momentum P_L of (2.18) to H_L. Later we will see that Hamiltonians that can be made sign definite is some frame are structurally stable in a sense to be defined.)

Definition 1. *Suppose $f_0'(0) = 0$. Then the signature of the point $u \in \mathbb{R}$ is $-\text{sgn}(uf_0'(u))$.*

Below is a graph that illustrates the signature for a bi-Maxwellian distribution function.

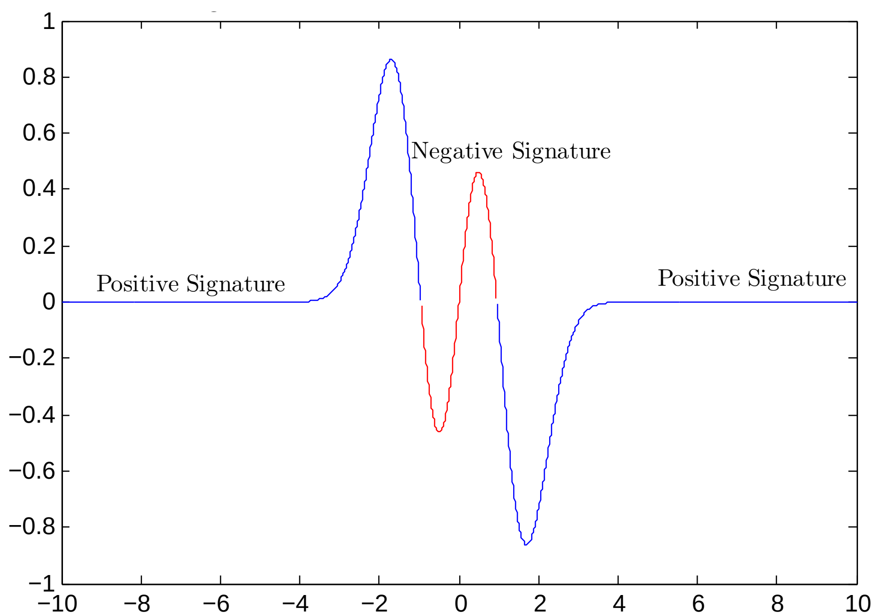

Figure 2.10: Signature for a bi-Maxwellian distribution function.

2.7.1 Dynamical accessibility and structural stability

Now we discuss the effect of restricting to dynamically accessible perturbations on the structural stability of f_0. In this work we only study perturbations of f_0 that preserve homogeneity. Because dynamically accessible perturbations are area preserving rearrangements of f_0, it is impossible to construct a dynamically accessible perturbation for the Vlasov equation in a finite spatial domain that preserves homogeneity.

To see this we write a rearrangement as $(x, v) \leftrightarrow (X, V)$, where V is a function of v alone. Because $[X, V] = 1$ and $V(v)$ is not a function of x, we have $V'\partial X/\partial x = 1$, or $X = x/V'$. If the spatial domain is finite, this map is not an diffeomorphism unless $V' = 1$. In the infinite spatial domain case, this is not a problem and these rearrangements exist. First we note that a rearrangement cannot change the critical points of f_0.

Lemma 2.7.1. *Let (X, V) be an area preserving diffeomorphism, and let V be homogeneous. Then the critical points of $f_0(V)$ are the points $V^{-1}(v_c)$, where v_c is a critical point of $f_0(v)$.*

Proof. By the chain rule $df_0(V(v))/dv = V(v)'f_0'(V(v))$. The function $V' \neq 0$ because (X, V) must be a diffeomorphism. Therefore the critical points occur when $f_0'(V) = 0$ or at points $v = V^{-1}(v_c)$. $\square$

Consider the perturbation χ that was constructed earlier. If v_c is a non-degenerate critical point of f_0 such that $f_0''(v_c) < 0$, then we want to prove that there is a rearrangement V such that $f_0(V) = f_0(v) + \int_{-\infty}^{v} \chi(v' - v_c)dv'$ or that $df_0(V)/dv = f_0'(v) + \chi(v - v_c)$. Such a rearrangement can be constructed as long as the parameters defining χ, the numbers h, d, ϵ, are chosen such that $f_0'(v) + \chi(v - v_c)$ has the same critical points as $f_0'(v)$. Using Morse theory it is possible to construct a V so that $f_0(V) = f_0(v) + \int \chi + O((v - v_c)^3)$, where $O((v - v_c)^3)$ has compact support and is smaller than $f_0(v) - f_0''(v_c)(v - v_c)^2/2$.

Theorem 2.7.2. *Let v_c be a non-degenerate critical point of f_0 with $f_0'(v_c) < 0$. Then there exists a rearrangement V such that $f_0(V) = f_0(v) + \int \chi + O((v - v_c)^3$, where O is defined as above.*

We omit the proof but it is a simple application of the Morse lemma. In order to apply the Morse lemma f_0 must be C^2. This is not restrictive for practical applications where typically f_0 is smooth. The rearrangement of f_0 can also be made to be smooth if desired.

Using this result we prove a Krein-like theorem for dynamically accessible perturbations in the $W^{1,1}$ norm.

Theorem 2.7.3. *Let f_0 be a stable equilibrium distribution function for the Vlasov equation on an infinite spatial domain. Then f_0 is structurally stable under dynamically accessible perturbations in $W^{1,1}$, if there is only one solution of $f_0'(v) = 0$. If there are multiple solutions, f_0 is structurally unstable and the unstable modes come from the zeros of f_0' that satisfy $f_0''(v) < 0$.*

Proof. Suppose that f_0' has only one zero on the real line. Because f_0 is an equilibrium this zero will have $f_0'' > 0$. Because a dynamically accessible perturbation can never increase the number of critical points, it will be impossible to change the winding number of the Penrose plot to a positive number. Therefore f_0 is structurally stable.

Suppose that $f_0' = 0$ has more than one solution on the real axis. Using the previous theorem perturb f_0' by $\chi(v - v_c)$ in a neighborhood of a critical point v_c with $f_0''(v_c) < 0$. This will increase the winding number to 1 since it will add a positively oriented crossing on the negative real axis for the correct choices of h, d, and ϵ in the definition of χ. The norm of χ can be made as small as necessary and therefore f_0 is structurally unstable. Since no other critical points with $f_0'' < 0$ can be created the only critical points that lead to instabilities are the ones that already exist having $f_0'' < 0$. $\square$

The implication of this result is that in a Banach space where the Hilbert transform is an unbounded operator the dynamical accessibility condition makes it so that a change in the Krein signature of the continuous spectrum is a necessary and sufficient condition for structural instability. The bifurcations do not occur at all points where the signature changes, however. Only those that represent valleys of the distribution can give birth to unstable modes.

2.8 Krein bifurcations in the Vlasov equation

We identify two critical states for the Penrose plots that correspond to the transition to instability. In these states the system may be structurally unstable under infinitesimal perturbations of f_0' in the C^n norm for all n. The first critical state corresponds to the existence of an embedded mode in the continuous spectrum. If the equilibrium is stable, then such an embedded mode corresponds to a tangency of the Penrose plot to the real axis at the origin. If the system is perturbed so that the tangency becomes a pair of transverse intersections, then the winding number of the Penrose plot would jump to 1 and the system would be unstable. Considering a parametrized small perturbation, we see that the value of k for the unstable mode, will correspond to some value of $k \neq 0$ for which the embedded mode exists. Figures 2.11 and 2.12 illustrate a critical Penrose plot for a bifurcation at $k \neq 0$. We explore this bifurcation in Sec. 2.8.1.

Another critical state occurs when $H[f_0'] = 0$ at a point where f_0' transversely intersects the real axis. If the Hilbert transform of f_0' is perturbed, there will be a crossing with a negative $H[f_0']$, and the winding number will be positive for some k. This mode enters through $k = 0$ because the smaller the perturbation of $H[f_0']$ the smaller k must be for T_k to be unstable. Figure 2.13 is a critical Penrose plot corresponding to the bi-Maxwellian distribution with the maximum stable separation. We explore this kind of bifurcation in Sec. 2.8.3.

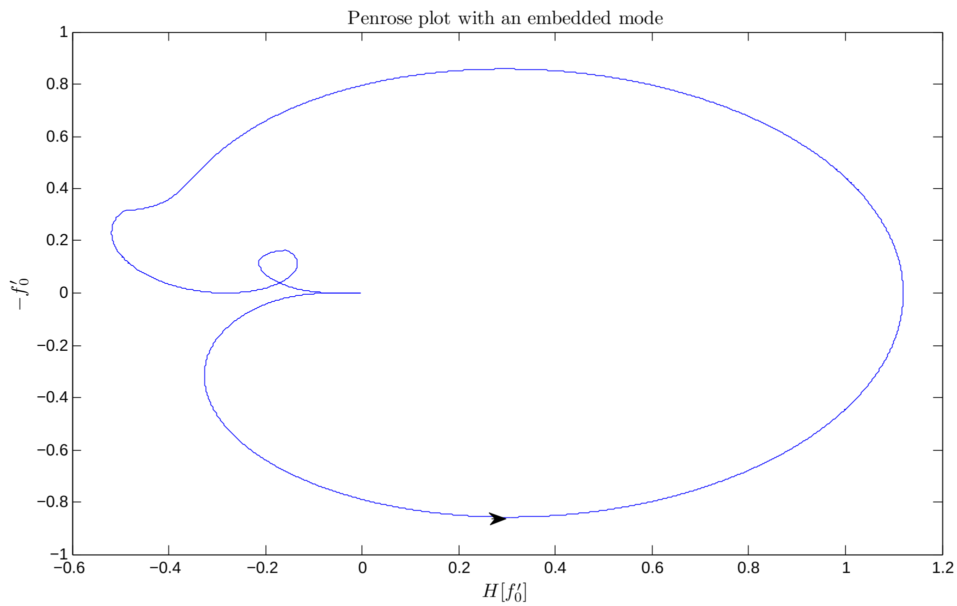

Figure 2.11: Critical Penrose plot for a $k \neq 0$ bifurcation.

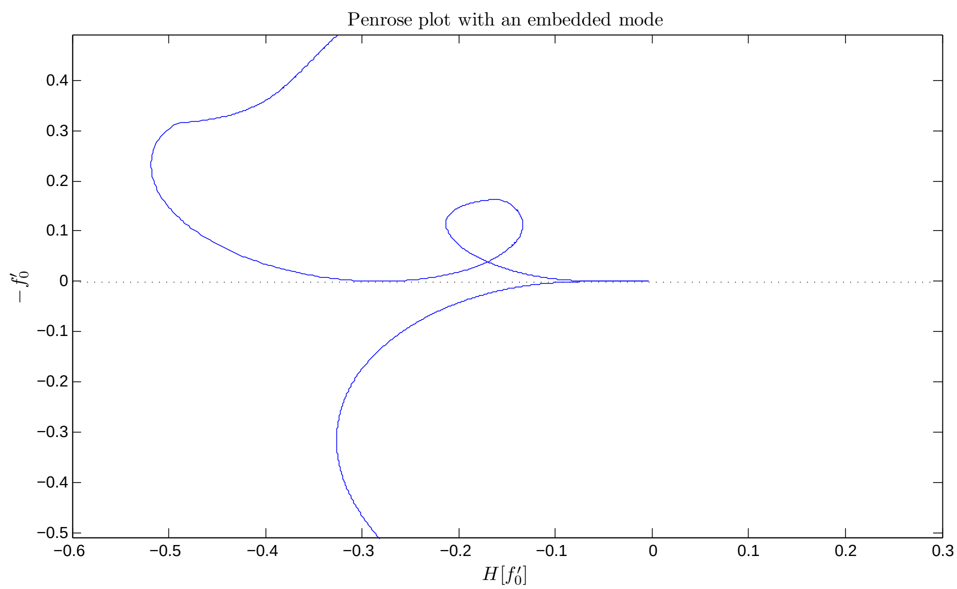

Figure 2.12: Close up of a critical Penrose plot for a $k \neq 0$ bifurcation.

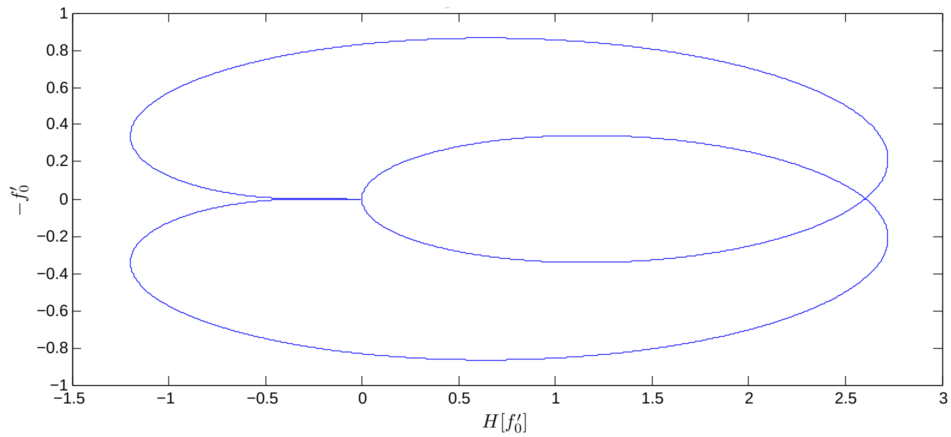

Figure 2.13: Critical Penrose plot for a bi-Maxwellian distribution function.

2.8.1 Bifurcation at $k \neq 0$

The linearized Vlasov equation can support neutral plasma modes embedded within the continuous spectrum. The condition for existence of a point mode is the vanishing of the plasma dispersion relation on the real axis,

$$\varepsilon(u) = 1 - H[f_0'] + if_0' = 0\,.. \tag{2.22}$$

If the spatial domain is unbounded the point modes will be analogues of the momentum eigenstate solutions of the Schrodinger equation and have infinite energy. Any violation of the Penrose criterion will guarantee the existence of zeros of the plasma dispersion function on the real axis because k can take any value in this case.

If the plasma dispersion vanishes at some u and $f_0''(u) = 0$, there is an embedded mode in the continuous spectrum. The signature of the continuous spectrum will not change signs at the frequency of the mode and we will extend the definition of signature to the point u even though $f_0'(u) = 0$. The signature of an embedded mode is given by $\text{sgn}(u\,\partial\epsilon_R/\partial u)$ (see [7, 43]). The signature of the continuous spectrum is $-\text{sgn}(uf_0')$. These signatures are the same if the value of f_0' in a neighborhood of its zero is the same sign as $H[f_0'']$.

We will prove that if f_0 is stable and mildly regular, it is impossible for there to be a discrete mode embedded in the continuous spectrum with signature that is the same as the signature of the continuous spectrum surrounding it. The proof has a simple conceptual outline. Suppose that there exists a discrete mode with the same signature as the continuum. Then there exists some point u satisfying $f_0'(u) = 0$, $-\text{sgn}(f_0') = \text{sgn}(\partial\epsilon_R/\partial u)$ in a neighborhood of u, and $H[f_0'](u) = 0$. Perturbations of f_0' centered around this point will give the Penrose plot a negative winding number, contradicting the analyticity of the plasma dispersion function in the upper half plane. We need f_0' to be Hölder continuous so that the Penrose plot is continuous and for the plasma dispersion function to converge uniformly to its values on the real line.

Lemma 2.8.1. *Let g be a function defined on the real line such that g is Hölder and let $h = H[f]$. Then the functions g_z, h_z that are the solutions of the Laplace equation in the upper half plane satisfying $f_z = f$ and $g_z = g$ on the real line converge uniformly to f and g.*

Proof. Because g can be defined as a bounded and continuous function on $\mathbb{R} \cup \{\infty\}$ and the g_z are analytic, the g_z must converge uniformly to g. The same properties hold for h and h_z must converge to $h = H[g]$. □

Lemma 2.8.2. *Let f_0' be the derivative of an equilibrium distribution function and let f_0' be sufficiently regular such that the assumptions of the previous lemma are true. Then the Penrose plot that is associated with f_0' cannot have a negative winding number.*

Proof. The Penrose plot associated with f_0' is the image of the real line under the map $\epsilon(u) = 1 - H[f_0'] + if_0'$. This is naturally defined as an analytic function if u is in the upper half plane. By the argument principle the image of $\mathbb{R} + it$ under this map has a non-negative winding number. Both the real and imaginary parts of this map converge uniformly to their values on the real line. Therefore the Penrose plot is a homotopy of these contours, making it possible to parametrize the contours by some t such that the distance from the Penrose plot to the contour produced by the image of $\mathbb{R} + i\delta$ is always less than some $\eta(t)$ that goes to 0. If the winding number of the Penrose plot were negative, there would be some t for which the winding number was negative because the winding is a stable property under homotopy, contradicting the analyticity of the map. □

Theorem 2.8.3. *Let f_0' and f_0'' be Hölder continuous. If f_0 is stable there are no discrete modes with signature the same as the signature of the continuum.*

Proof. Because f_0 is stable the winding number of the Penrose plot is equal to 0. Assume that there is a discrete mode with the same signature as the continuum surrounding it. Then there exists a point u with $f_0'(u) = 0$, $f_0''(u) =$

0, and $\text{sgn}(f_0'(u+\delta)dH[f_0'](u)/du) = 1$. Then we search for a function g such that the Penrose plot of $f_0' + g$ has a negative winding number. If such a function exists it will contradict Lemma 2.8.1. Because f_0'' is Hölder $\partial\epsilon_R/\partial u$ is bounded away from zero in a neighborhood of the point. Suppose that in this neighborhood there is only one zero of f_0'. Then define g such that g has one sign, is smooth and has compact support, and such that the $|\partial H[g]/\partial u| < |f_0''|$ in this neighborhood. Then for small enough g the function $f_0' + g$ will have two zeros in a neighborhood of the point. Then both of the crossings will correspond to crossings of negative orientation and the resulting winding number will be -1, a contradiction. □

Corollary 2.8.4. *If f_0 is stable it is impossible for there to be a point where $f_0' = 0$, $f_0'' < 0$, and $H[f_0'] > 0$.*

If f_0 is unstable the winding number is positive. In this case it may be possible for modes with the same signature as the continuum to exist. It is possible for a positive energy mode to be embedded in a section of negative signature and a negative energy mode to be embedded in a section of positive signature. This situation is structurally unstable under perturbations that are bounded by the C^n norm and remains so even when a linear dynamical accessibility constraint is enforced.

Theorem 2.8.5. *Let f_0' be the derivative of an equilibrium distribution function with a discrete mode embedded in the continuous spectrum. Then there exists an infinitesimal function with compact support in the C^n norm for each n such that $f_0' + \delta f'$ is unstable.*

Proof. Suppose that $H[f_0'']$ is non-zero in a neighborhood of the embedded mode. Define a dynamically accessible perturbation $\delta f = h f_0'$. Then assume that $f_0''' \neq 0$ at the mode. If we define h such that it does not vanish at the mode we find that $\delta f'' = h'' f_0' + h' f_0'' + h f_0'''$ and therefore we can choose h such that the discrete mode becomes a crossing. This can be done with h infinitesimal and smooth. The resulting perturbation will have an infinitesimal effect on f_0'. The new crossings will cause a violation of the Penrose criterion, and therefore the system with the embedded mode is structurally unstable. □

This is an analog of Krein's theorem for the Vlasov equation for the case where there is a discrete mode. As a result of this we see that all discrete modes are either unstable or structurally unstable.

2.8.2 Little-big man theorem

Consider a linearized equilibrium that supports three discrete modes. The signature of each mode depends on the reference frame. There is a result that applies to a number of Hamiltonian systems, the three-wave problem in particular [54, 28]), that gives a condition on the signature of the modes and their frequencies in some reference frame such that no frame shift can cause all the modes to have the same signature. In a shifted frame, the Hamiltonian changes so the frequencies in the action-angle form are Doppler shifted. Sometimes such shifts can render the Hamiltonian sign definite. A result for finite systems, which we call the little-big man theorem, indicates that this cannot happen when the mode of different signature has frequency with largest absolute value. A related result exists for the point spectrum of the Vlasov equation.

Theorem 2.8.6. *Let f_0' be the derivative of an equilibrium distribution function that has three discrete modes (elements of the point spectrum) with real frequencies. Consider a reference frame where all of the modes have positive frequency. Then represent the energies of the three modes as a triplet $(\pm\pm\pm)$ where the plus and minus signs correspond to the signature of each mode, with the first mode being the one with the lowest phase velocity (ω/k) and the last one with the highest phase velocity. Then, if the triplet is of the form $(+-+)$ or $(-+-)$ there is no reference frame in which all the modes have the same signature. If the triplet has any other form, then there is a reference frame in which all the modes have the same signature.*

Proof. The formula for the energy of an embedded mode is $\text{sgn}(\omega\, \partial\epsilon_R/\partial\omega)$ [43]. If there are three embedded modes in a frame where the frequencies are all positive, the triplet is

$$\left(\text{sgn} \frac{\partial \epsilon_R}{\partial \omega}\bigg|_{\omega_1}, \text{sgn}\ \frac{\partial \epsilon_R}{\partial \omega}\bigg|_{\omega_2}, \text{sgn}\ \frac{\partial \epsilon_R}{\partial \omega}\bigg|_{\omega_3} \right)$$

If this is $(+-+)$ then as we shift frames the possible triplets are $(0-+), (--+), (-0+), (-++), (-+0), (-+-)$. All of these are indefinite. The other possible initially indefinite triplet is $(--+)$. However if we shift the two $-$ modes to negative frequency the triplet becomes $(+++)$. All other examples are either definite or reduce to one of these two. □

A few observations are in order. First, frame shifts do not change the structure of Penrose plots, but only induce re-parameterizations. Next, Theorem 2.8.6 differs from its finite-dimensional counterpart in that no restriction on the wave numbers is involved, a necessary part of the three-wave problem. Lastly, we are not addressing nonlinear stability here, as in the finite dimensional case, but should a frame exist in which the energy is definite, this is an important first step in a rigorous proof of nonlinear stability.

2.8.3 Bifurcation at $k = 0$

Assume that there are no embedded modes and that f_0 is stable, but that there is a point that has $f_0' = 0$ and $H[f_0'] = 0$. This is the critical state for a bifurcation at $k = 0$. This can be destabilized in the same way as the critical state for $k \neq 0$. There will be a perturbation that makes $H[f_0'] < 0$ without changing f_0' at that point. Therefore the Penrose plot becomes unstable and the equilibrium is structurally unstable.

Theorem 2.8.7. *Suppose that f_0' is a stable equilibrium distribution function that has a zero at u of both f_0' and $H[f_0']$. Then f_0' is structurally unstable under perturbations bounded by the C^n norm for all n.*

Proof. Let δh be symmetric about the point u, be smooth with compact support and have its first n derivatives less than some ϵ. Then let $\delta f_0' = -H[\delta h]$. The resulting perturbation to $H[f_0']$ is h. If h is positive at u, then by the symmetry of h $f_0' + \delta f_0'$ has a zero at u and $H[f_0'] + h$ is positive there. Thus the Penrose plot has a positive winding number and is unstable. Therefore f_0' is structurally unstable. $\square$

The previous two sections demonstrated that when the Penrose plot is critical, no amount of regularity is sufficient to prevent f_0 from being structurally unstable. However, when the Penrose plot is not critical all that is required is that a small perturbation only change the Penrose plot by a small amount in addition to a condition to prevent perturbations near $v = \infty$. Suppose we arbitrarily restrict the support of the perturbations so that $|v| < v_{max}$. Then if we increase the required regularity such that $\sup(H[\delta f_0'])$ is bounded there will be some δ such that for all $\delta f_0'$ with $\|\delta f_0'\| < \delta$ the distribution $f_0 + \delta f_0$ is structurally stable. This restriction can be motivated physically by restricting the particles in the distribution function to be traveling slower than the speed of light.

2.9 Conclusion

We have considered perturbations of the linearized Vlasov-Poisson equation through changes in the equilibrium function. The effect of these perturbations on the spectral stability of the equations is determined by the class of allowable perturbations and the signature of the continuous and point spectra. Every equilibrium can be made unstable by adding an arbitrarily small function from the space $W^{1,1}$. If we rearrange f_0 then only when the signature of f_0 changes sign can an arbitrarily small perturbation destabilize it. When

f_0 is stable discrete modes always have the opposite signature of the spectrum surrounding them. This is the result of Theorem 2.8.3. The equilibria are structurally unstable under C^n small perturbations for all n. The signature of the spectrum and the signature of the discrete mode can never be the same.

This generalization of Krein's theorem is more complicated than the finite-dimensional original. However the basic ideas of Krein's theorem are still important in the infinite-dimensional case. When the perturbations are more restricted than just belonging to $W^{1,1}$ the structural stability is determined by the signature of the spectrum. Just as in Krein's theorem there must be a positive signature interacting with a negative signature to produce structural instability.

This chapter was devoted primarily to the Vlasov equation, but other noncanonical Hamiltonian systems admit to a similar treatment, e.g. the 2D Euler equation with shear flow equilibria, and we hope to chronicle such cases in future publications.

Chapter 3

Landau Damping and Hamiltonian Models of Friction

3.1 Introduction

In 1946 Landau [55] theoretically predicted the collisionless damping of the electric field in a plasma governed by the Vlasov-Poisson system. This result has been of great importance in the field of plasma physics, and indeed collisionless or continuum damping, as it is sometimes called, occurs in a wide variety of kinetic and fluid plasma models that possesses a continuous spectrum. For example, such damping occurs in the context of Alfven waves in magnetohydrodynamics (see e.g. Chap. 10 of [56]) and has been proposed as a mechanism for plasma heating in response to electromagnetic waves.

Many other systems also undergo Landau damping, both inside and outside of plasma physics. It is not surprising that Landau damping exists in stellar dynamics governed by the Jeans equation [57] because this equation is of Vlasov type but with an attractive interaction potential. In fact, Landau damping occurs in collisionless kinetic theories with a rather large class of potentials, and recently has been proven rigorously to exist in the nonlinear case [58, 59]. Landau damping exists in the context of the fluid mechanics of shear flow (see e.g. [60, 33] which contains a list of original sources over a period of more than 50 years) and the description of wind driven water waves. It also appears in multiphase media [61] and has been established for systems containing large numbers of coupled oscillators, most notably the Kuramoto model. This has implications for biological models describing the synchronization or decoherence of the flashing of fireflies and chirping of crickets as well as other phenomenon in mathematical biology [62].

Another class of continuum systems involves the interaction of a discrete oscillator with a continuous bath of oscillators. In these systems the oscillator can be a particle or one mode of some field, and the bath often represents thermal fluctuations or radiation. One of the first detailed treatments of such a system is due to Dirac [63], but early on Van Kampen also used such a model to describe the emission and absorption of light by an atom [64]. The single wave model of plasma physics, which describes both beam plasma and laser plasma interaction physics [65, 66, 67], is also an example. The example of interest in this chapter is the Caldeira-Leggett model [68].

The Caldeira-Leggett model was invented in order to study quantum tunneling in the presence of dissipation and the quantum limit of Brownian motion [10]. A model of this type was deemed necessary because quantum mechanics is incompatible with frictional forces. However, the Caldeira-Leggett model is a Hamiltonian system that exhibits dissipation by coupling to a continuum, i.e., it has Landau damping. The Caldeira-Leggett Hamiltonian is the sum of the Hamiltonian of a classical harmonic oscillator, the Hamiltonian of continuous bath of harmonic oscillators, and a linear coupling term between the discrete and continuous degrees of freedom. The discrete degree of freedom corresponds to a macroscopic system and the bath of oscillators represent the environment. The coupling causes the discrete oscillator to damp by transference of energy to the continuum. This system has become a standard model for studying the physics of low temperature quantum systems, and it has numerous applications ranging from the understanding of superconducting circuit elements to qubits in quantum computers [69].

We analyze the classical Caldeira-Leggett model using a procedure analogous to that used by Landau to analyze the Vlasov-Poisson system of plasma physics. Following Landau, the initial value problem can be solved using the Laplace transform and the rate of decay can be derived in the weak damping limit. This paralleling of Landau's original calculation suggests a connection between this system and the Vlasov-Poisson system. In fact, we will show that both systems can be mapped into a normal form that is common to a large class of infinite-dimensional Hamiltonian systems that have a continuous spectrum [7, 8, 33, 34].

The Caldeira-Leggett model, like all Hamiltonian systems in the class, has a continuous spectrum that is responsible for the damping through phase mixing (filamentation) and the Riemann-Lebesgue lemma. Because this structure is shared by a number of important physical systems, it is interesting to determine the nature of their similarities. It is well-known that the properties of linear ordinary differential equations are closely tied to the spectra of their time evolution operators. In fact, for given spectra there are a number of normal forms. Any linear finite-dimensional Hamiltonian system can be reduced to one of these normal forms (ODEs) through an appropriate transformation, and in this sense the behavior of such systems is completely understood. theory of normal forms for infinite-dimensional Hamiltonian systems is not nearly as well-developed as that for finite-dimensional systems, but for some systems much is known. For systems with continuous spectra, the analog of diagonalization is conversion into a multiplication operator. If the original system is $\dot{f} = \mathcal{L}f$, then a transformation T such that $T\mathcal{L}T^{-1}$ is a multiplication operator would diagonalize the system. Any two systems that have the same normal form would thus be equivalent through some linear transformation.

This procedure has been performed for the linearized Vlasov-Poisson equation [8], and when the spectrum is purely continuous the time evolution operator is equivalent to the multiplication operator x. This discovery led to the discovery of an entire class of transformations diagonalizing linear infinite-dimensional Hamiltonian systems of a certain form [34]. In fact, it is always possible to perform such a transform in the special case of a bounded, self-adjoint operator [70]. The operators dealt with here are usually unbounded and non-normal (even if they did exist in a Hilbert space), as is often the case when dealing with continuous Hamiltonian matter models. A precursor to the discovery of such transformations is existence of a complete basis of singular eigenfunctions of the original equation, a treatment that is common for systems with continuous spectra that dates back to Dirac [63]. In fact these methods have been developed in parallel within the field of plasma physics beginning with the work of Van Kampen [71] and within condensed matter physics through the work of Dirac and later Fano [72]. Caldeira and his collaborators developed a diagonalization method for the Caldeira-Leggett model [73], although they were primarily interested in the time evolution of the discrete degree of freedom and thus did not write down the full inverse of their transformation. In this chapter we complete the treatment of the Caldeira-Leggett system. Then, we note that the normal form is the same as that of the Vlasov-Poisson system and that the models are thus equivalent through the use of an integral transform.

Specifically, in Sec. 3.2 we review the Caldeira-Leggett model and then, in the spirit of Landau [55], present its Laplace transform solution in Sec. 3.3. This is followed by obtaining the singular eigenfunctions, in the spirit of Van Kampen [71] and Dirac [63], and the invertible integral transform akin to that of [8] for transforming to normal form. In Sec. 3.5 we show explicitly how the Caldeira-Leggett model is equivalent to a case of the linearized Vlasov-Poisson system. Then, in Sec. 3.6 we suggest that there may be an echo effect in the Caldeira-Leggett model, perhaps making the nature of dissipation directly testable. In Sec. 3.7 variants of the Caldeira-Leggett model which contain negative energy are introduced, and stability criteria are derived. The results are interpreted in terms of Krein's theorem. Finally, in Sec. 3.8 we conclude.

3.2 Caldeira-Leggett Hamiltonian

As noted above, the Caldeira-Leggett model is an infinite-dimensional Hamiltonian system describing the interaction of a discrete degree of freedom with an infinite continuum of modes [10]. The continuum is typically referred to as the environment. The Caldeira-Leggett model has the following Hamiltonian:

$$\begin{aligned} H_{CL}[q,p;Q,P] = \frac{\Omega}{2}P^2 + \frac{1}{2}\left(\Omega + \int_{\mathbb{R}_+} dx\, \frac{f(x)^2}{2x}\right) Q^2 \\ + \int_{\mathbb{R}_+} dx\, \left[\frac{x}{2}(p(x)^2 + q(x)^2) + Qq(x)f(x)\right]\,,. \end{aligned} \tag{3.1}$$

which together with the Poisson bracket

$$\{A,B\} = \left(\frac{\partial A}{\partial Q}\frac{\partial B}{\partial P} - \frac{\partial A}{\partial P}\frac{\partial B}{\partial Q}\right) + \int_{\mathbb{R}_+} dx \left(\frac{\delta A}{\delta q}\frac{\delta B}{\delta p} - \frac{\delta A}{\delta p}\frac{\delta B}{\delta q}\right). \tag{3.2}$$

produces the equation of motion for observables in the form $\dot{F} = \{F, \mathcal{H}\}$, where F is any functional of the discrete, (Q,P), and continuum, (q,p), coordinates and momenta. Note, it is assumed that $f(x)$ is chosen so that the integrals of (3.1) exist. The coefficient of Q^2 includes a frequency shift term that is used to make the Hamiltonian positive definite. We take p and q to be functions on the positive real line, $\mathbb{R}_+$, and P and Q to be real numbers. Hamilton's equations for the Caldeira-Leggett system are thus,

$$\dot{q}(x) = xp(x) \tag{3.3}$$

$$\dot{p}(x) = -xq(x) - Qf(x) \tag{3.4}$$

$$\dot{Q} = \Omega P \tag{3.5}$$

$$\dot{P} = -\left(\Omega + \int_{\mathbb{R}_+} dx\, \frac{f(x)^2}{2x}\right) Q - \int_{\mathbb{R}_+} dx\, q(x)f(x)\,.. \tag{3.6}$$

This system was originally introduced by Caldeira and Leggett in 1981 [68]. They initially considered a very massive harmonic oscillator coupled to a large number of light harmonic oscillators with varying frequencies, and then studied the limit of the light oscillators becoming a continuous spectrum. The coupling causes Q to decay to zero with time, and therefore the system can be used to model dissipation. This makes it an ideal system to model the effects of dissipation in quantum mechanics and especially quantum tunneling. It has been extensively studied and is frequently mentioned in the condensed matter literature. There have been some controversies about the physics of the damping and the physicality of the initial conditions [69]. Connecting this system with plasma physics, where much intuition has been developed over the years about wave-particle interaction, can help to improve the understanding of its behavior. For example, a clear picture of filamentation can be viewed in the numerical work of [74].

3.3 The Landau solution and Landau damping

One of the classical calculations in plasma physics is the solution of the linearized Vlasov-Poisson equation using the Laplace transform. This yields a formula for the solution of the initial value problem and also facilitates the derivation of the damping rate for the electric field. It is possible to do the same thing for the Caldeira-Leggett model. We begin with the set of Hamilton's equations that were written down in the previous section and eliminate the two momenta to derive a pair of second order equations for the coordinates,

$$\ddot{q}(x) = -x^2 q(x) - Qxf(x) \tag{3.7}$$

$$\ddot{Q} = -\Omega_c^2 Q - \Omega \int_{\mathbb{R}_+} dx\, f(x) q(x)\,,. \tag{3.8}$$

where for convenience we use the corrected frequency,

$$\Omega_c^2 := \Omega^2 + \Omega \int_{\mathbb{R}_+} dx\, f(x)^2/2x\,.. \tag{3.9}$$

Defining the Laplace transform of the coordinates by

$$\tilde{q}(x,s) = \int_{\mathbb{R}_+} dt\, q(x,t) e^{-st} \tag{3.10}$$

$$\tilde{Q}(s) = \int_{\mathbb{R}_+} dt\, Q(t) e^{-st}. \tag{3.11}$$

results in the following set of algebraic equations:

$$s^2 \tilde{q}(x,s) = -x^2 \tilde{q}(x,s) - \tilde{Q}(s) x f(x) + s q(0,x) + \dot{q}(0,x) \tag{3.12}$$

$$s^2 \tilde{Q}(s) = -\Omega_c^2 \tilde{Q}(s) - \Omega \int_{\mathbb{R}_+} dx\, \tilde{q}(x,s) f(x) + sQ(0) + \dot{Q}(0)\,,. \tag{3.13}$$

which can be easily solved for $\tilde{Q}(s)$,

$$\begin{aligned} \tilde{Q}(s) = \left[-\Omega \int_{\mathbb{R}_+} dx\, \frac{f(x)(sq(0,x) + \dot{q}(0,x))}{s^2 + x^2} + sQ(0) + \dot{Q}(0) \right] \\ \div \left[s^2 + \Omega_c^2 - \Omega \int_{\mathbb{R}_+} dx\, \frac{x f(x)^2}{s^2 + x^2} \right] .. \end{aligned} \tag{3.14}$$

The Laplace transform is inverted using the Mellin inversion formula,

$$Q(t) = \frac{1}{2\pi i} \int_{\beta - i\infty}^{\beta + i\infty} ds\, \tilde{Q}(s) e^{st}\,,. \tag{3.15}$$

where β is any real number that ensures $\tilde{Q}(s)$ is analytic for $Re(s) > \beta$. This integral is usually evaluated using Cauchy's integral formula, whence asymptotically in the long-time limit the behavior of the solution is given by the poles of $\tilde{Q}(s)$. Thus, the solution will be dominated by an exponentially decaying term arising from the pole of $\tilde{Q}(s)$ closest to the real axis. We assume the closest pole is indeed close to the real axis and that there are no poles with a positive real part, i.e., that the solutions are stable. This is the weak damping limit. As long as $f(x)$ is Hölder continuous, the poles of $\tilde{Q}(s)$ come from the zeros of the denominator, so we are interested in the roots of the equation

$$0 = s^2 + \Omega_c^2 - \Omega \int_{\mathbb{R}_+} dx \, \frac{x f(x)^2}{s^2 + x^2} = s^2 + \Omega_c^2 - \Omega \int_{\mathbb{R}} dx \, \frac{f(|x|)_-^2}{2(x - is)} \,.. \tag{3.16}$$

Here $f(|x|)_-^2$ is the antisymmetric extension of $f(x^2)$ defined by $f(|x|)_-^2 = sgn(x) f(|x|)^2$. Making the substitution $\omega = is$, yields the dispersion relation, which in the limit ω tends to the real axis becomes

$$\omega^2 - \Omega_c^2 + \frac{\Omega}{2} ⨍_{\mathbb{R}} dx \, \frac{f(|\omega|)_-^2}{x - \omega} + \frac{i\pi\Omega}{2} f(|x|)_-^2 = 0 \,,. \tag{3.17}$$

where $⨍$ denotes the Cauchy principal value integral. For quantities not yet integrated, we will denote this by **PV**. This equation can be viewed as the dispersion relation in the weak damping limit. Let ω_c be a real solution to the real part of the above equation. Then let ω be a root of the previous equation, assume $\gamma = Im(\omega)$ is small, and solve for γ to first order; i.e. $0 = 2i\omega_c\gamma + i\pi\Omega f(|\omega_c|)_-^2/2$ or

$$\gamma = -\frac{\pi\Omega}{4|\omega_c|} f(|\omega_c|)^2 \,.. \tag{3.18}$$

There are a large number of methods used to derive damping of Q for this model. The standard approach is to attempt to prove that after suitable approximations Q satisfies the equation of motion of a damped harmonic oscillator. The treatment here is almost identical to the method that was used to treat the Vlasov-Poisson equation by Landau, and agrees with other derivations of the damping rate in the weak damping limit [68].

3.4 Van Kampen modes: diagonalization of the Caldeira-Leggett model

The Laplace transform method is just one way to treat the Vlasov equation. Another way is to write the solution as a superposition of a continuous spectrum of normal modes, a method attributed to Van Kampen [71]. Such modes of the Vlasov equation are called the Van Kampen modes, and we will see that they exist for the Caldeira-Leggett model as well. We formally calculate the Van Kampen modes for this system and use them to motivate the definition of an invertible integral transform, akin to those of [7, 8, 33, 34], that maps the Caldeira-Leggett model to action-angle variables, the normal form for this Hamiltonian model. The nature of the transformation depends on the coupling function $f(x)$. In the present treatment we will assume that $f(0) = 0$, but that f does not vanish otherwise. We will also assume that the dispersion relation does not vanish anywhere. This excludes the possibility of discrete modes embedded in the continuous spectrum. The case where the Caldeira-Leggett model possesses such modes will be treated in future work.

As stated above, the normal form of the Caldeira-Leggett Hamiltonian will be seen to be equivalent to that for the Vlasov-Poisson system through the integral transformation introduced in [7, 8].

The first step is to obtain a solution with time dependence $\exp(-iut)$ and derive equations for the amplitudes of a single mode (q_u, p_u, Q_u, P_u). To this end consider

$$\begin{aligned} iuq_u(x) &= -xp_u(x) \\ iup_u(x) &= xq_u(x) + Q_u f(x) \\ iuQ_u &= -\Omega P_u \\ iuP_u &= \left(\Omega + \int_{\mathbb{R}_+} dx\, \frac{f(x)^2}{2x}\right) Q_u + \int_{\mathbb{R}_+} dx\, q_u(x) f(x) \,.. \end{aligned} \tag{3.19}$$

Note, although we use the subscript, $u \in \mathbb{R}$ is a continuum label. Eliminating the momenta from Eqs. (3.19) yields

$$(u^2 - x^2) q_u(x) = Q_u x f(x) \tag{3.20}$$

$$(u^2 - \Omega_c^2) Q_u = \Omega \int_{\mathbb{R}_+} dx\, q_u(x) f(x) \,,. \tag{3.21}$$

where recall Ω_c is defined by (3.9). Of these, (3.20) is solved following Van Kampen (a generalized function solution that dates to Dirac [63]) giving the general form for q_u

$$q_u(x) = \mathbf{PV}\, \frac{Q_u x f(x)}{u^2 - x^2} + C_u Q_u \delta(|u| - x) \,.. \tag{3.22}$$

Substitution of (3.22) into (3.21) determines C_u,

$$u^2 - \Omega_c^2 = \Omega \fint_{\mathbb{R}_+} dx \, \frac{x f(x)^2}{u^2 - x^2} + \Omega C_u f(|u|) \tag{3.23}$$

$$C_u = \frac{u^2 - \Omega_c^2}{\Omega f(|u|)} - \fint_{\mathbb{R}} dx \, \frac{f(|x|)_-^2}{2(u-x) f(|u|)} \,.. \tag{3.24}$$

Therefore we can specify an initial condition on the amplitudes Q_u and compute the corresponding coordinates and momenta by an integral over the real line. Each mode oscillates with a different real frequency, with the expression for the solution given by

$$q(x,t) = \fint_{\mathbb{R}} du \, \frac{Q_u x f(x)}{u^2 - x^2} e^{-iut} + \int_{\mathbb{R}} du \, C_u Q_u \delta(|u| - x) e^{-iut} \tag{3.25}$$

$$Q(t) = \int_{\mathbb{R}} du \, Q_u e^{-iut} \,,. \tag{3.26}$$

were Q_u acts an amplitude function that determines which Van Kampen modes are excited.

The Caldeira-Leggett model can be diagonalized and solved by making use of the integral transform alluded to above. Previously, Caldeira et al. [73] derived a transformation that diagonalizes the Caldeira-Leggett model. However, they were interested in solving for the evolution of the variable Q and therefore did not attempt to write down the full inverse of the operator (except in a special case where they made use of the evolution of the reservoir). We will extend their results by deriving the inverse map that we will use to establish the equivalence with the Vlasov-Poisson system.

In order to define the transform, we introduce a number of other important maps and introduce our notation. Extensive use will be made of the

Hilbert transform, which is defined for a function $g(x)$ on $\mathbb{R}$ by

$$H[g](v) = \frac{1}{\pi} \fint_{\mathbb{R}} dx \, \frac{g(x)}{x - v} \, .$$

We also need some Hilbert transform identities [50, 8]. Let g, g_1, and g_2 be functions of $x \in \mathbb{R}$ and suppose that all the expressions we write down are well defined, then the following hold:

$$H[H[g]] = -g \tag{3.27}$$

$$H[g_1 H[g_2] + g_2 H[g_1]] = H[g_1]H[g_2] - g_1 g_2 \tag{3.28}$$

$$H[vg] = vH[g] + \frac{1}{\pi} \int_{\mathbb{R}} dx \, g \, .. \tag{3.29}$$

Next we define two functions, ϵ_R and ϵ_I by

$$\epsilon_I = \pi f(x)^2 \qquad \text{and} \qquad \epsilon_R = 2\frac{x^2 - \Omega_c^2}{\Omega} + \pi H[f(|x|)_-^2] \, .. \tag{3.30}$$

These together with $|\epsilon|^2 := \epsilon_I^2 + \epsilon_R^2$ are used to define the following integral transforms:

Definition For functions $h(x)$ on $\mathbb{R}_+$, the transform

$$T_+[h](u) := \epsilon_R h(|u|) + \epsilon_I H[h(|x|)](u) \, ,$$

while

$$\widehat{T_+}[h](u) := \frac{\epsilon_R}{|\epsilon|^2} h(u) - \frac{\epsilon_I}{|\epsilon|^2} H[h(|x|)](u) \, ,$$

Related to the above transforms are two more transforms,

$$T_-[h](u) := \epsilon_R h(|u|) + \epsilon_I H[sgn(x) h(|x|)](u) \, ,$$

and

$$\widehat{T_-}[h](u) := \frac{\epsilon_R}{|\epsilon|^2} h(u) - \frac{\epsilon_I}{|\epsilon|^2} H[sgn(x)h(|x|)](u)\,.$$

Using the transform T_+ it is possible to write the map from the amplitudes of the Van Kampen modes Q_u to the functions $(q(x), Q)$. To see this consider the expression for $q(x)$ in terms of the amplitude function Q_u, and simplify it using the Hilbert transform as follows:

$$\begin{aligned} q(x) &= \fint_{\mathbb{R}} du\, \frac{Q_u x f(x)}{u^2 - x^2} + \int_{\mathbb{R}} du\, C_u Q_u \delta(|u| - x) \\ &= \fint_{\mathbb{R}} du\, \frac{x f(x) Q_u}{2u} \left(\frac{1}{u - x} + \frac{1}{u + x} \right) + C_x Q_x + C_{-x} Q_{-x} \\ &= \pi x f(x) \left(H\left[\frac{Q_u}{2u}\right](x) + H\left[\frac{Q_u}{2u}\right](-x) \right) + 2C_x(Q_x + Q_{-x})\,.. \end{aligned} \tag{3.31}$$

Next, decompose Q_u into its symmetric and antisymmetric parts: $Q_u = Q_{+u} + Q_{-u}$ and observe that the antisymmetric parts vanish from both sides of (3.31),

$$\begin{aligned} q(x) &= \pi x f(x) H[Q_{+u}/u](x) + 2C_x Q_{+x} \\ &= \pi f(x) H[Q_{+u}] + \left(2\,\frac{x^2 - \Omega_c^2}{\Omega f(x)} - \fint_{\mathbb{R}} dx'\, \frac{f(|x'|)_-^2}{(x - x') f(x)} \right) Q_{+x}\,,. \end{aligned} \tag{3.32}$$

where the second line follows from the third Hilbert transform identity combined with the fact that Q_{u+}/u is antisymmetric and thus has a vanishing integral. Now multiply both sides of (3.32) by $f(x)$ and find

$$\begin{aligned} f(x)q(x) &= \pi f(x)^2 H[Q_{+u}] + \left(2\,\frac{x^2 - \Omega_c^2}{\Omega} - \fint_{\mathbb{R}} dx'\, \frac{f(|x'|)_-^2}{(x - x')} \right) Q_{+x} \\ &= \epsilon_I H[Q_{+u}] + \epsilon_R Q_{+x} \\ &= T_+[Q_{+u}]\,.. \end{aligned} \tag{3.33}$$

Now we are set to define a transformation.

Definition Let Q_{+u} be a function on $\mathbb{R}_+$, then the map

$$I_c[Q_+] := \left(\frac{1}{f(x)} T_+[Q_{u+}], 2\int_{\mathbb{R}_+} du\, Q_{u+} \right) .$$

The map $I_c[Q_+]$, a map from the Van Kampen mode amplitudes to the original dynamical variables, has an inverse. To see this note that $\epsilon_R = S + H[\epsilon_I]$, where $S = 2(x^2 - \Omega_c^2)/\Omega$, and let g be a function on $\mathbb{R}_+$. Then,

$$H[Sg(|x|)] = SH[g(|x|)] + \frac{4u}{\pi\Omega} \int_{\mathbb{R}_+} dx\, g\, ,$$

where we have used our Hilbert transform identities to move the x^2 outside of the Hilbert transform of g. Using this, consider the following sequence of identities:

$$\begin{aligned}
\widehat{T_+}[T_+[g]] &= \frac{\epsilon_R}{|\epsilon|^2}(\epsilon_R g + \epsilon_I H[g]) - \frac{\epsilon_I}{|\epsilon|^2} H[\epsilon_R g + \epsilon_I H[g]] \\
&= \frac{\epsilon_R^2}{|\epsilon|^2} g + \frac{\epsilon_R \epsilon_I}{|\epsilon|^2} H[g] - \frac{\epsilon_I S}{|\epsilon|^2} H[g] - \frac{\epsilon_I}{|\epsilon|^2} H[H[\epsilon_I] g + \epsilon_I H[g]] - \frac{\epsilon_I}{|\epsilon|^2} \frac{4u}{\pi\Omega} \int_{\mathbb{R}_+} dx\, g \\
&= \frac{\epsilon_R^2}{|\epsilon|^2} g + \frac{\epsilon_R \epsilon_I}{|\epsilon|^2} H[g] - \frac{\epsilon_I S}{|\epsilon|^2} H[g] - \frac{\epsilon_I}{|\epsilon|^2} (H[\epsilon_I] H[g] - g\epsilon_I) - \frac{\epsilon_I}{|\epsilon|^2} \frac{4u}{\pi\Omega} \int_{\mathbb{R}_+} dx\, g \\
&= g + \frac{\epsilon_R \epsilon_I}{|\epsilon|^2} H[g] - \frac{\epsilon_I S}{|\epsilon|^2} H[g] - \frac{\epsilon_I}{|\epsilon|^2} H[\epsilon_I] H[g] - \frac{\epsilon_I}{|\epsilon|^2} \frac{4u}{\pi\Omega} \int_{\mathbb{R}_+} dx\, g \\
&= g + \frac{\epsilon_R \epsilon_I}{|\epsilon|^2} H[g] - \frac{\epsilon_I}{|\epsilon|^2} H[g](S + H[\epsilon_I]) - \frac{\epsilon_I}{|\epsilon|^2} \frac{4u}{\pi\Omega} \int_{\mathbb{R}_+} dx\, g \\
&= g + \frac{\epsilon_R \epsilon_I}{|\epsilon|^2} H[g] - \frac{\epsilon_R \epsilon_I}{|\epsilon|^2} H[g] - \frac{\epsilon_I}{|\epsilon|^2} \frac{4u}{\pi\Omega} \int_{\mathbb{R}_+} dx\, g \\
&= g - \frac{\epsilon_I}{|\epsilon|^2} \frac{4u}{\pi\Omega} \int_{\mathbb{R}_+} dx\, g\, ,.
\end{aligned} \tag{3.34}$$

where in each step use has been made of the various identities above. Because the integral of Q_+ is equal to $Q/2$, we can define the inverse of I as follows:

$$\widehat{I_c}[q(x), Q] = \widehat{T_+}[f(x)q(x)] + \frac{2u}{\pi\Omega}\frac{\epsilon_I}{|\epsilon|^2}Q\,.$$

The above transform ignores the (p, P) variables and only produces the symmetric part of the Van Kampen modes. We derive the other half of the transformation from a mixed variable generating functional. To this end, define $Q_+ = \bar{Q}$ and $Q_- = \bar{P}$ and rescale the coordinate part of the transformation by choosing $Q = 2\int_{\mathbb{R}_+} du\, \bar{Q}\sqrt{\epsilon_I/(\pi|\epsilon|^2)}$:

$$\bar{Q} = \sqrt{\frac{\pi|\epsilon|^2}{\epsilon_I}}\left(\hat{T_+}[f(x)q(x)] + \frac{2u}{\pi\Omega}\frac{\epsilon_I}{|\epsilon|^2}Q\right) = \widehat{I}[q(x), Q] \tag{3.35}$$

$$(Q, q(x)) = \left(2\int_{\mathbb{R}_+} du\, \bar{Q}\sqrt{\frac{\epsilon_I}{\pi|\epsilon|^2}}, \frac{1}{f(x)}T_+\left[\sqrt{\frac{\epsilon_I}{\pi|\epsilon|^2}}\bar{Q}\right]\right) = I[\bar{Q}]\,.. \tag{3.36}$$

Then we introduce the mixed variable type-2 generating functional

$$\mathcal{F}[q, Q, \bar{P}] = \int_{\mathbb{R}_+} du\, \bar{P}\,\widehat{I}[q(x), Q]\,,$$

which produces the transformations in the usual way:

$$p(x) = \frac{\delta\mathcal{F}}{\delta q} = f(x)\,\widehat{T_+}^{\dagger}\left[\sqrt{\frac{\pi|\epsilon|^2}{\epsilon_I}}\bar{P}\right] \tag{3.37}$$

$$P = \frac{\delta\mathcal{F}}{\delta Q} = \int_{\mathbb{R}_+} du\, \frac{2u\bar{P}}{\Omega}\sqrt{\frac{\epsilon_I}{\pi|\epsilon|^2}}\,.. \tag{3.38}$$

Calculating the adjoint of $\widehat{T}$ simplifies the resulting expression for $p(x)$, viz.,

$$p(x) = \frac{1}{f(x)}T_-\left[\sqrt{\frac{\epsilon_I}{\pi|\epsilon|^2}}\bar{P}\right]\,.. \tag{3.39}$$

Now, analogous to I we define the operator $J[\bar{P}] = (p(x), P)$ by

$$J[\bar{P}] = \left(\frac{1}{f(x)} T_- \left[\sqrt{\frac{\epsilon_I}{\pi|\epsilon|^2}} \bar{P} \right], \int_{\mathbb{R}_+} du \frac{2u\bar{P}}{\Omega} \sqrt{\frac{\epsilon_I}{\pi|\epsilon|^2}} \right), . \tag{3.40}$$

which can be inverted through the use of the Hilbert transform identities. Define $\bar{P}_c = \bar{P}\sqrt{\pi|\epsilon|^2/\epsilon_I}$, and consider the expression

$$\begin{aligned} &\frac{\epsilon_R}{|\epsilon|^2} p(x) - \frac{\epsilon_I}{|\epsilon|^2} H\left[sgn(x)p(|x|)\right] \\ &\quad = \frac{\epsilon_R}{|\epsilon|^2}(\epsilon_I H[sgn(u)\bar{P}_c] + \epsilon_R \bar{P}_c) - \frac{\epsilon_I}{|\epsilon|^2} H[sgn(x)\epsilon_I H[sgn(u)\bar{P}_c] + sgn(x)\epsilon_R \bar{P}_c] .. \end{aligned} \tag{3.41}$$

Paralleling the method used to invert the map from $\bar{Q}$ to (q, Q), we see a difference occurs when evaluating the term $\epsilon_I H[sgn(x)\epsilon_R \bar{P}]/|\epsilon|^2$, i.e.

$$\begin{aligned} \frac{\epsilon_I}{|\epsilon|^2} H[sgn(u)u^2 \bar{P}_c] &= x^2 \frac{\epsilon_I}{|\epsilon|^2} H[sgn(u)\bar{P}_c] + \frac{2}{\pi} \frac{\epsilon_I}{|\epsilon|^2} \int_{\mathbb{R}_+} du\, u\bar{P}_c \\ &= x^2 \frac{\epsilon_I}{|\epsilon|^2} H[sgn(x)\bar{P}_c] + \frac{\Omega}{\pi} \frac{\epsilon_I}{|\epsilon|^2} P .. \end{aligned} \tag{3.42}$$

With this expression we can directly use the inversion calculation for the (q, Q) case to obtain the following expression for the full transformation:

$$\bar{P} = \sqrt{\frac{\pi|\epsilon|^2}{\epsilon_I}} \left(\widehat{T_-}[f(x)p(x)] + \frac{2}{\pi} \frac{\epsilon_I}{|\epsilon|^2} P \right) \tag{3.43}$$

$$\bar{Q} = \sqrt{\frac{\pi|\epsilon|^2}{\epsilon_I}} \left(\widehat{T_+}[f(x)q(x)] + \frac{2u}{\pi\Omega} \frac{\epsilon_I}{|\epsilon|^2} Q \right) .. \tag{3.44}$$

Applying this transformation to Hamilton's equations yields the equations for a continuum of harmonic oscillators. This can be seen directly for both $\bar{P}$ and $\bar{Q}$.

$$
\begin{aligned}
\dot{\bar{Q}} &= \sqrt{\frac{\pi|\epsilon|^2}{\epsilon_I}}\left(\widehat{T_+}[f(x)\dot{q}(x)] + \frac{2u}{\pi\Omega}\frac{\epsilon_I}{|\epsilon|^2}\dot{Q}\right) \\
&= \sqrt{\frac{\pi|\epsilon|^2}{\epsilon_I}}\left(\widehat{T_+}[xf(x)p(x)] + \frac{2u}{\pi}\frac{\epsilon_I}{|\epsilon|^2}P\right) \\
&= \sqrt{\frac{\pi|\epsilon|^2}{\epsilon_I}}\left(\frac{\epsilon_R}{|\epsilon|^2}\,uf(u)p(u) - \frac{\epsilon_I}{|\epsilon|^2}\,H[|x|f(|x|)p(|x|)] + \frac{2u}{\pi}\frac{\epsilon_I}{|\epsilon|^2}P\right) \\
&= \sqrt{\frac{\pi|\epsilon|^2}{\epsilon_I}}\left(\frac{\epsilon_R}{|\epsilon|^2}\,uf(u)p(u) - \frac{\epsilon_I}{|\epsilon|^2}\,uH[sgn(x)f(|x|)p(|x|)] + \frac{2u}{\pi}\frac{\epsilon_I}{|\epsilon|^2}P\right) \\
&= u\bar{P}\,.. \qquad (3.45)
\end{aligned}
$$

Similarly, for $\bar{P}$,

$$
\begin{aligned}
\dot{\bar{P}} &= \sqrt{\frac{\pi|\epsilon|^2}{\epsilon_I}}\left(\widehat{T_-}[f(x)\dot{p}(x)] + \frac{2}{\pi}\frac{\epsilon_I}{|\epsilon|^2}\dot{P}\right) \\
&= \sqrt{\frac{\pi|\epsilon|^2}{\epsilon_I}}\left(\widehat{T_-}[-xf(x)q(x) - f(x)^2Q] - \frac{2\Omega_s}{\pi}\frac{\epsilon_I}{|\epsilon|^2}Q - \frac{2}{\pi}\frac{\epsilon_I}{|\epsilon|^2}\int_{\mathbb{R}_+} dx\, f(x)q(x)\right) \\
&= \sqrt{\frac{\pi|\epsilon|^2}{\epsilon_I}}\left(-u\widehat{T_+}[f(x)q(x)] + \frac{2}{\pi}\frac{\epsilon_I}{|\epsilon|^2}\int_{\mathbb{R}_+} dx\, f(x)q(x) - \widehat{T_-}[f(x)^2]\,Q\right) \\
&\qquad - \frac{2\Omega_s}{\pi}\frac{\epsilon_I}{|\epsilon^2|}\,Q - \frac{2}{\pi}\frac{\epsilon_I}{|\epsilon|^2}\int_{\mathbb{R}_+} dx\, f(x)q(x) \\
&= \sqrt{\frac{\pi|\epsilon|^2}{\epsilon_I}}\left(-u\widehat{T_+}[f(x)q(x)] - \frac{\epsilon_R}{|\epsilon|^2}\,f(x)^2\,Q + \frac{\epsilon_I}{|\epsilon|^2}\,H[sgn(x)f(|x|)^2]\,Q - \frac{2\Omega_s}{\pi}\frac{\epsilon_I}{|\epsilon|^2}\,Q\right) \\
&= \sqrt{\frac{\pi|\epsilon|^2}{\epsilon_I}}\left(-u\widehat{T_+}[f(x)q(x)] - \frac{2u^2}{\pi\Omega}\frac{\epsilon_I}{|\epsilon|^2}\,Q\right) \\
&= -u\,\bar{Q}\,.. \qquad (3.46)
\end{aligned}
$$

Let this full map be called I_f and consider I_f as a map from the Banach space $L_p \times L_p \times \mathbb{R}^2$, $p > 1$, to the Banach space $L_p \times L_p$. The operator I_f

is a bounded linear functional between these two spaces, because each term is either a multiplication operator that is bounded on L_p, an L_p function, or a bounded function multiplied by the Hilbert transform, which is another bounded operator. In order to establish the equivalence with the normal mode it is important to specify the phase space of the dynamical variables. Using this map we can simply choose each functional space to be L_p and have a well defined map in each case. This map demonstrates how the Caldeira-Leggett model can be written as a superposition of a continuous spectrum of singular eigenmodes.

Because the transformation to the normal form was a canonical one, the normal form Hamiltonian should be the original Hamiltonian of the Caldeira-Leggett model written in the new coordinates. We will verify this by direct substitution. For convenience we introduce the quantities

$$
\begin{aligned}
A &= \frac{\Omega}{2}P^2 + \frac{1}{2}\int_{\mathbb{R}_+} dx\, xp(x)^2 \\
B &= \frac{\Omega_s}{2}Q^2 + \int_{\mathbb{R}_+} \left(\frac{x}{2}q(x)^2 + f(x)q(x)Q\right) dx\,,.
\end{aligned}
$$

where $\Omega_s = \Omega_c^2/\Omega$. Evidently, $H_{CL} = A + B$. Then,

$$\begin{aligned}
A &= \frac{\Omega}{2} P \int_{\mathbb{R}_+} du \, \frac{2u\bar{P}}{\Omega} \sqrt{\frac{\epsilon_I}{\pi|\epsilon|^2}} + \frac{1}{2} \int_{\mathbb{R}_+} du \, xp(x) f(x) \widehat{T_+}^\dagger \left[\sqrt{\frac{\pi|\epsilon|^2}{\epsilon_I}} \bar{P} \right] \\
&= P \int_{\mathbb{R}_+} du \, u\bar{P} \sqrt{\frac{\epsilon_I}{\pi|\epsilon|^2}} + \frac{1}{2} \int_{\mathbb{R}_+} du \, \widehat{T_+} \left[xp(x) f(x) \right] \sqrt{\frac{\pi|\epsilon|^2}{\epsilon_I}} \bar{P} \\
&= P \int_{\mathbb{R}_+} du \, u\bar{P} \sqrt{\frac{\epsilon_I}{\pi|\epsilon|^2}} + \frac{1}{2} \int_{\mathbb{R}_+} du \, u\widehat{T_-} \left[f(x) p(x) \right] \sqrt{\frac{\pi|\epsilon|^2}{\epsilon_I}} \bar{P} \\
&= \frac{1}{2} \int_{\mathbb{R}_+} du \, u\bar{P} \left(\sqrt{\frac{\pi|\epsilon|^2}{\epsilon_I}} \left(\widehat{T_-}[f(x)p(x)] + \frac{2}{\pi} \frac{\epsilon_I}{|\epsilon|^2} P \right) \right) \\
&= \frac{1}{2} \int_{\mathbb{R}_+} du \, u\bar{P}^2 \, .. \qquad (3.47)
\end{aligned}$$

Similarly,

$$\begin{aligned}
\frac{1}{2} \int_{\mathbb{R}_+} dx \, xq(x)^2 &= \frac{1}{2} \int_{\mathbb{R}_+} du \sqrt{\frac{\epsilon_I}{\pi|\epsilon|^2}} \widehat{T_+}^\dagger \left[\frac{xq(x)}{f(x)} \right] \bar{Q} \\
&= \frac{1}{2} \int_{\mathbb{R}_+} du \, \bar{Q} \sqrt{\frac{\epsilon_I}{\pi|\epsilon|^2}} \left(u \frac{\epsilon_R q(u)}{f(u)} - uH \left[\frac{\epsilon_I(|x|) q(|x|)}{f(|x|)} \right] - 2 \int_{\mathbb{R}_+} dx \, f(x) q(x) \right) \\
&= -\frac{1}{2} \int_{\mathbb{R}_+} dx \, Q f(x) q(x) + \frac{1}{2} \int_{\mathbb{R}_+} du \, \bar{Q} u \sqrt{\frac{\pi|\epsilon|^2}{\epsilon_I}} \widehat{T_+} \left[f(x) q(x) \right] \, .. \qquad (3.48)
\end{aligned}$$

Now, analyzing the entire expression for B in a sequence of steps,

$$
\begin{aligned}
B &= \frac{1}{2}\int_{\mathbb{R}_+} dx\, Q f(x) q(x) + \frac{1}{2}\int_{\mathbb{R}_+} du\, \bar{Q} u \sqrt{\frac{\pi|\epsilon|^2}{\epsilon_I}} \widehat{T_+}\left[f(x)q(x)\right] + \frac{\Omega_s}{2} Q^2 \\
&= \frac{1}{2}\int_{\mathbb{R}_+} du\, \bar{Q} u \sqrt{\frac{\pi|\epsilon|^2}{\epsilon_I}} \widehat{T_+}\left[f(x)q(x)\right] + \Omega_s Q \int_{\mathbb{R}_+} du\, \bar{Q} \sqrt{\frac{\epsilon_I}{\pi|\epsilon|^2}} + \frac{Q}{2}\int_{\mathbb{R}_+} dx\, T_+\left[\sqrt{\frac{\epsilon_I}{\pi|\epsilon|^2}} \bar{Q}\right] \\
&= \frac{1}{2}\int_{\mathbb{R}_+} du\, \bar{Q} u \sqrt{\frac{\pi|\epsilon|^2}{\epsilon_I}} \widehat{T_+}\left[f(x)q(x)\right] \\
&\qquad + \frac{Q}{2}\int_{\mathbb{R}_+} dx\, \left(\left(\frac{2u^2}{\Omega}\bar{Q} + \pi H[sgn(x) f(|x|)^2]\bar{Q}\right)\sqrt{\frac{\epsilon_I}{\pi|\epsilon|^2}} + \epsilon_I H[\sqrt{\frac{\epsilon_I}{\pi|\epsilon|^2}}\bar{Q}]\right) \\
&= \frac{1}{2}\int_{\mathbb{R}_+} du\, \bar{Q} u \left(\sqrt{\frac{\pi|\epsilon|^2}{\epsilon_I}} \widehat{T_+}\left[f(x)q(x)\right] + \frac{2u}{\Omega}\sqrt{\frac{\epsilon_I}{\pi|\epsilon|^2}} Q\right) \\
&= \frac{1}{2}\int_{\mathbb{R}_+} du\, u \bar{Q}^2 \,..
\end{aligned}
\tag{3.49}
$$

With (3.47) and (3.49) we obtain $H_{CL} = \int_{\mathbb{R}_+} duu \left(\bar{Q}^2 + \bar{P}^2\right)/2$ – the Hamiltonian for a continuous spectrum of harmonic oscillators and the normal form for the Caldeira-Leggett model.

3.5 Equivalence to the Linearized Vlasov-Poisson equation

The treatment of the Caldeira-Leggett model of Sec. 3.4 is similar to an analysis of the linearized Vlasov-Poisson equation performed in [8, 44]. In those papers an integral transform was presented that transforms the Vlasov equation into a continuous spectrum of harmonic oscillators. The two systems are identical except the spectrum of the Caldeira-Leggett model only covers the positive real line. Now we explicitly produce a transformation that takes one system into the other.

The Vlasov equation describes the kinetic theory of a collisionless plasma. Spatially homogeneous distribution functions are equilibria, and linearization about such states are often studied in plasma physics. In the case of one spatial dimension and an equilibrium distribution function $f_0(v)$, the linearized Vlasov-Poisson equation around f_0 is given by

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} - \frac{e}{m}\frac{\partial \phi}{\partial x} f_0' = 0 \tag{3.50}$$

$$\frac{\partial^2 \phi}{\partial x^2} = -4\pi e \int_{\mathbb{R}} dv\, f. \tag{3.51}$$

where $f_0' = df_0/dv$. These equations inherit the noncanonical Hamiltonian structure of the full Vlasov-Poisson system [19] and have a Poisson bracket given by

$$\{F, G\}_L = \int\int dx dv\, f_0 \left[\frac{\delta F}{\delta f}, \frac{\delta G}{\delta f}\right] .. \tag{3.52}$$

This bracket is of a form that is typical for Hamiltonian systems describing continuous media (cf. e.g. [20, 14]). The Hamiltonian is given by

$$H_L = -\frac{m}{2}\int\int dv dx\, v\, \frac{f^2}{f_0'} + \frac{1}{8\pi}\int dx \left(\frac{\partial \phi}{\partial x}\right)^2 , . \tag{3.53}$$

and the Vlasov-Poisson equation can be written as $\dot{f} = \{f, H_L\}_L$.

The spatial dependence of the Vlasov-Poisson equation can be removed by performing a Fourier transform. This allows the potential to be explicitly eliminated from the equation

$$\frac{\partial f_k}{\partial t} - ikv f_k - \frac{4\pi i e^2}{mk} f_0'(v) \int_{\mathbb{R}} dv\, f_k = 0 .. \tag{3.54}$$

The Hamiltonian structure in terms of the Fourier modes has the new bracket

$$\{F,G\}_L = \sum_{k=1}^{\infty} \frac{ik}{m} \int_{\mathbb{R}} dv\, f_0' \left(\frac{\delta F}{\delta f_k} \frac{\delta G}{\delta f_{-k}} - \frac{\delta G}{\delta f_k} \frac{\delta F}{\delta f_{-k}} \right) .. \tag{3.55}$$

and the Hamiltonian functional is simply (3.53) written in terms of the Fourier modes.

One way to canonize this bracket is with the following scalings:

$$q_k(v,t) = f_k \qquad \text{and} \qquad p_k(v,t) = \frac{m f_{-k}}{ik f_0'}\,,. \tag{3.56}$$

where $k > 0$. In terms of these variables the Poisson bracket has canonical form, i.e.

$$\{F,G\}_L = \sum_{k=1}^{\infty} \int_{\mathbb{R}} dv \left(\frac{\delta F}{\delta q_k} \frac{\delta G}{\delta p_k} - \frac{\delta G}{\delta q_k} \frac{\delta F}{\delta p_k} \right) .. \tag{3.57}$$

From this point it is possible to derive a canonical transformation that diagonalizes the Hamiltonian. We make the following definitions:

$$\varepsilon_I(v) = -\frac{4\pi^2 e^2 f_0'}{mk^2 \int_{\mathbb{R}} dv\, f_0} \qquad \varepsilon_R(v) = 1 + H[\varepsilon_I]\,, \tag{3.58}$$

$$G_k[f] = \varepsilon_R f + \varepsilon_I H[f] \qquad \widehat{G_k}[f] = \frac{\varepsilon_R}{|\varepsilon|^2} f - \frac{\varepsilon_I}{|\varepsilon|^2} H[f]\,. \; . \tag{3.59}$$

It was proven in [8] that $G_k = \widehat{G_k}^{-1}$.

A transformation to the new set of variables $(\mathcal{Q}_k, \mathcal{P}_k)$ that diagonalizes the system will be given in terms of the variables (q_k, p_k). To this end we first introduce the intermediate variables $(\mathcal{Q}_k', \mathcal{P}_k')$ defined by

$$q_k = G_k[\mathcal{Q}_k'] \qquad \text{and} \qquad \mathcal{Q}_k' = \widehat{G_k}[q_k] .. \tag{3.60}$$

The corresponding momentum portion of the canonical transformation is induced by the following mixed variable generating functional:

$$\mathcal{F}[q_k, \mathcal{P}'_k] = \sum_{k=1}^{\infty} \int_{\mathbb{R}} du \, \mathcal{P}'_k \widehat{G_k}[q_k] \, , . \tag{3.61}$$

whence we obtain via $\mathcal{Q}'_k = \delta\mathcal{F}/\delta\mathcal{P}'_k$ and $p_k = \delta\mathcal{F}/\delta q_k$,

$$\mathcal{Q}'_k = \widehat{G_k}[q_k] \qquad \text{and} \qquad \mathcal{P}'_k = \widehat{G_k}^{\dagger}[p_k]. \tag{3.62}$$

Then, the variables $(\mathcal{Q}_k, \mathcal{P}_k)$ are defined as

$$\mathcal{Q}_k = \left(\mathcal{Q}'_k - i\mathcal{P}'_k\right)/\sqrt{2} \qquad \text{and} \qquad \mathcal{P}_k = \left(\mathcal{P}'_k - i\mathcal{Q}'_k\right)/\sqrt{2} \, , . \tag{3.63}$$

in terms of which the Vlasov-Poisson Hamiltonian has the form of a continuum of harmonic oscillators (see [44] for an explicit calculation),

$$H_L = \sum_{k=1}^{\infty} \int_{\mathbb{R}} du \left(\mathcal{Q}_k^2 + \mathcal{P}_k^2\right)/2 \, .. \tag{3.64}$$

Thus, for a single value of k, this is the same normal form as that of the Caldeira-Leggett model, with the exception that the integral here is over the entire real line instead of just the half line. If we consider two copies of the Caldeira-Leggett model the normal form would be the same as that for a single k value of the linearized Vlasov-Poisson system. By composing the transformation that diagonalizes the Caldeira-Leggett model with the inverse of the transformation that diagonalizes the Vlasov-Poisson system we obtain a map that converts solutions of one system into solutions of the other system. Explicitly suppose that we have two copies of the Caldeira-Leggett Hamiltonian, with the same coupling function $f(x)$. Then set the normal form of the second copy equal to the normal form of the Vlasov equation on the negative real line. Let $(q_1(x), p_1(x), Q_1, P_1)$ be one set of solutions to the Caldeira-Leggett model and let $(q_2(x), p_2(x), Q_2, P_2)$ be another and let $\Theta(x)$ be the Heaviside function. Then we can write a solution to the linearized Vlasov-Poisson equation using the following map:

$$f_k(v,t) = G_k\Bigg[\frac{1}{\sqrt{2}}\Bigg(\Theta(u)\sqrt{\frac{\pi|\epsilon|^2}{\epsilon_I}}\left(\widehat{T_+}[f(x)q_1(x)] + \frac{2u}{\pi\Omega}\frac{\epsilon_I}{|\epsilon|^2}Q_1\right) \tag{3.65}$$

$$+\Theta(-u)\sqrt{\frac{\pi|\epsilon|(-u)^2}{\epsilon_I(-u)}}\left(\widehat{T}_+[f(x)q_2(x)](-u) + \frac{-2u}{\pi\Omega}\frac{\epsilon_I(-u)}{|\epsilon(-u)|^2}Q_2\right) \tag{3.66}$$

$$+i\Theta(u)\sqrt{\frac{\pi|\epsilon^2|}{\epsilon_I}}\left(\widehat{T_-}[f(x)p_1(x)] + \frac{2}{\pi}\frac{\epsilon_I}{|\epsilon|^2}P_1\right) \tag{3.67}$$

$$+i\Theta(-u)\sqrt{\frac{\pi|\epsilon(-u)|^2}{\epsilon_I(-u)}}\left(\widehat{T_-}[f(x)p_2(x)] + \frac{2}{\pi}\frac{\epsilon_I(-u)}{|\epsilon(-u)|^2}P_2\right)\Bigg)\Bigg] \tag{3.68}$$

$$f_{-k}(v,t) = \frac{kf_0'}{m}G_k^\dagger\Bigg[\frac{1}{\sqrt{2}}\Bigg(\Theta(u)\sqrt{\frac{\pi|\epsilon|^2}{\epsilon_I}}\left(\widehat{T}_+[f(x)q_1(x)] + \frac{2u}{\pi\Omega}\frac{\epsilon_I}{|\epsilon|^2}Q_1\right) \tag{3.69}$$

$$+\Theta(-u)\sqrt{\frac{\pi|\epsilon|(-u)^2}{\epsilon_I(-u)}}\left(\widehat{T}_+[f(x)q_2(x)](-u) + \frac{-2u}{\pi\Omega}\frac{\epsilon_I(-u)}{|\epsilon(-u)|^2}Q_2\right) \tag{3.70}$$

$$-i\Theta(u)\sqrt{\frac{\pi|\epsilon^2|}{\epsilon_I}}\left(\widehat{T_-}[f(x)p_1(x)] + \frac{2}{\pi}\frac{\epsilon_I}{|\epsilon|^2}P_1\right) \tag{3.71}$$

$$-i\Theta(-u)\sqrt{\frac{\pi|\epsilon(-u)|^2}{\epsilon_I(-u)}}\left(\widehat{T_-}[f(x)p_2(x)] + \frac{2}{\pi}\frac{\epsilon_I(-u)}{|\epsilon(-u)|^2}P_2\right)\Bigg)\Bigg]. \tag{3.72}$$

This map is invertible using the formulas presented earlier in the chapter. Given a single mode of the linearized Vlasov-Poisson system, $f_k(v,t)$, we can write two solutions to the Caldeira-Leggett model as follows:

$$(q_1(x,t), Q_1(t)) = I[\frac{1}{2}\Re(\hat{G}[f_k](u,t) + \hat{G}[f_k](-u,t))] \tag{3.73}$$

$$(p_1(x,t), P_1(t)) = J[\frac{1}{2}\Im(\hat{G}[f_k](u,t) - \hat{G}[f_k](-u,t))] \tag{3.74}$$

$$(q_2(x,t), Q_2(t)) = I[\frac{1}{2}\Im(\hat{G}[f_k](u,t) + \hat{G}[f_k](-u,t))] \tag{3.75}$$

$$(p_2(x,t), P_2(t)) = J[\frac{1}{2}\Re(\hat{G}[f_k](-u,t) - \hat{G}[f_k](u,t))]. \tag{3.76}$$

Therefore one would expect the solutions of the Caldeira-Leggett model to share the same properties as the solutions of the Vlasov-Poisson system.

It was remarked earlier that both systems exhibit damping. In the Vlasov-Poisson case the electric field decays, and in the Caldeira-Leggett model it is the discrete coordinate Q. The existence of the transformation between the two systems gives us a way to understand what determines the damping rate in each case. In the standard calculation of the Landau damping rate for the Vlasov equation, it is clear that the rate depends only on the location of the closest zero in the lower half complex plane of the dispersion relation, which only depends on the equilibrium f_0. The same is true for the Caldeira-Leggett model, where the damping of Q depends on the coupling function f. It is clear that integral transformations change the rate of damping, as all the instances of the Vlasov equation and the Caldeira-Leggett model share the same normal form but generally have different damping rates.

It is possible to interpret Landau damping using the normal forms and canonical transformation. The dynamical variables of the normal form have a time evolution $\sim \exp(-iut)$. The observables can then be expressed as some operator on this oscillatory dynamical variable. The result will be an oscillatory integral over the real line, and by the Riemann-Lebesgue lemma we know that that such an integrated quantity will decay to zero in the long-time limit. For the systems at hand, this integral can be deformed into the lower half complex plane, and Cauchy's theorem can be used to see that the behavior is governed by the locations of the poles of the analytic continuation of the oscillatory integrand. These poles determine the exponential damping rate. In these systems the poles are clearly introduced by the continuation (following the Landau prescription) of the dispersion relation in the integral transformations, which is therefore the origin of Landau damping. We will demonstrate this explicitly for the damping of the coordinate Q in the Caldeira-Leggett model.

Starting from the solution,

$$\begin{aligned} Q(t) &= \int_{\mathbb{R}} du\,(\bar{Q}(|u|)\cos(ut) + sgn(u)\bar{P}(|u|)\sin(ut))\frac{f(|u|)}{|\epsilon|} \\ &= \int_{\mathbb{R}} du\,(\hat{I}[\mathring{q}(x), \mathring{Q}]\cos(ut) + \hat{J}[\mathring{p}(x), \mathring{P}]sgn(u)\cos(ut))\frac{f(|u|)}{|\epsilon|}\,,. \end{aligned} \quad (3.77)$$

we see that each term in the integrand of (3.77) has an oscillatory part and has poles at the zeros of $|\epsilon|^2$. The damping rate will be based on the closest zero of $|\epsilon|$, the dispersion relation for the Caldeira-Leggett model. Likewise, for the Vlasov-Poisson system we can write a similar expression for the density

$\rho_k(t)$,

$$\begin{aligned}
\rho_k(t) &= \int_{\mathbb{R}} dv\, G_k[\hat{G}_k[\mathring{f}]e^{-iut}] \\
&= \int_{\mathbb{R}} du\, \left(\varepsilon_R(\hat{G}_k[\mathring{f}]e^{-iut}) - H[\varepsilon_I]\hat{G}_k[\mathring{f}]e^{-iut}\right) \\
&= \int_{\mathbb{R}} du\, \hat{G}_k[\mathring{f}]e^{-iut} \,..
\end{aligned} \tag{3.78}$$

The damping rates are given by the poles of $\hat{G}_k$, and the observed rate will be due to the closest zero of $|\varepsilon|^2$ to the real axis. Therefore, mathematically the source of the damping in the Vlasov-Poisson and Caldeira-Leggett models are identical, it being the nearest pole introduced by the integral transformation that diagonalizes the system.

3.6 Application: echo effects in the Caldeira-Leggett model

The analysis of the past section established the equivalence between the Caldeira-Leggett model and the evolution equation of a single mode of the linearized Vlasov-Poisson equation. The establishment of mathematical connections between theories of very different physical systems is potentially extremely useful if the phenomenon of one of theories can be discovered in the other theory, or if the correspondence can hint at the existence of new effects. One promising scenario would be the potential existence of an echo in the Caldeira-Leggett model.

The plasma echo [75] is one of the hallmarks of plasma systems described by kinetic theory. The basis for the effect is that Landau damping does not destroy the structure of the distribution function in phase space, so that as long as collisions are weak, highly oscillatory distributions functions can persist long after the electric field has damped away. These structures can be observed by perturbing the plasma. An antenna can be used to disturb the electric field, causing a fluctuation of the distribution function. The same antenna can be used again, after the original fluctuation has Landau damped, to excite another fluctuation with a different wave number [76]. These fluctuations interact to the lowest order ignored by the linear theory, and the resulting non-linear interaction produces a disturbance in the electric field from seemingly out of nowhere. This is the plasma echo effect. The observation of the plasma echo [76] was an important experiment. It confirmed the nature of collisionless damping in plasmas and allowed for measurement of the collisional damping rate through observation of the decay of the echoes in time.

It is unlikely that a completely analogous effect could be observed within a system described by the Caldeira-Leggett model. The correspondence with the Vlasov equation only extends to the linearized version. On the other hand, the situation in both cases remains the same, there is a highly oscillatory dynamical variable. In order to verify the nature of damping in these models, the oscillatory structure must be uncovered after the damping has already caused moments of the dynamical variables to vanish.

The simplest way (from the prospective of a theorist) to cause such an effect is to directly drive the bath. Incidentally this method of exciting an echo in the Caldeira-Leggett model has been proposed before, in an unpublished paper on the arxiv dating back 10 years [11].

Suppose that the Hamiltonian of the bath can be controlled externally. Let $H[q(x), p(x)] = \int \frac{dx}{2} \left(c(t)q(x)^2 + p(x)^2\right)$ be the Hamiltonian of the bath. Then consider an initial condition where the bath is undisturbed, $P(0) = 0$, and $Q(0) = Q_0$. As time advances the discrete degrees of freedom damp through Landau damping and the energy is transferred to the continuum. Suppose the externally controlled parameter $c(t)$ can be chosen so that $c(t) = 1$ except for an extremely small interval of time centered around $t = \tau$. In the limit $c(t) = 1 + \epsilon\delta(t - \tau)$. The result of such a force would be to transform $p(x, \tau)$ to $p(x, \tau) + \epsilon q(x, \tau)$. Then consider the resulting initial value problem for the further evolution of the Caldeira-Leggett model.

The initial condition is a superposition of the solution to the Caldeira-Leggett model at time $t = \tau$ and an additional term from the impulse. This additional term in the initial condition has the form $q_a(x) = 0$ and $p_a(x) = \delta q(x, \tau)$. The first term can be rewritten as $q_a(x) = \frac{\delta}{2}(p(x, \tau) - p(x, \tau)$. There the additional initial condition has one term that is proportional to the time reversal of the solution to the original equations of motion, except with the roles of q and p reversed. The other term continues to Landau damp, just like the initial condition from the original evolution. The time reversed piece is equal to δ times the initial solution running backwards in time and with q and p reversed. Therefore at time $t = 2\tau$, the Landau damping is undone and there is an echo of the original initial condition in the discrete variables, with amplitude proportional to δ. An experimental study of this type of echo would reveal information about the nature of damping in system/bath models as well as the nature of actual dissipation within the bath.

3.7 Negative energy variants of the Caldeira-Leggett model

The original motivation for the present study of the Caldeira-Leggett model was a general desire to understand the interaction of the continuous spectrum with embedded eigenvalues of both positive and negative energy. This was done with an eye towards proving a generalization of the Krein-Moser theorem for infinite-dimensional Hamiltonian systems. The Caldeira-Leggett model turned out to be the simplest system imaginable with discrete modes embedded in the continuous spectrum. In fact we were completely unaware that the Caldeira-Leggett model was an important system in physics used to describe real phenomenon for more than two years after we had proposed the model and and finished initially studying it. At that time it was simply referred to as the toy model, and we used it because we believed that it was likely to be a normal form for the linearized Vlasov-Poisson equation in the case where there were discrete modes embedded in the continuous spectrum (in the case where the coupling vanishes).

Because the motivation was to study the effect of signature on the structural stability of linear Hamiltonian systems, variants of the Caldeira-Leggett model were introduced with interesting features, most notably different types of negative energy. Both the discrete modes and parts of the continuous spectrum were allowed to possess negative energy in the systems that were examined. The result of these studies was the production of interesting archetypal systems for exhibiting bifurcations into instability or structural stability in infinite-dimensional Hamiltonian systems. Furthermore, the stability analyses resulted in the development of geometric criteria in the spirit of the Penrose criterion for the Vlasov-Poisson equation.

Gyroscopically or magnetically stabilized systems generically have negative energy modes. Various authors have investigated the effects of coupling such systems to both discrete and continuous heat baths [77]. In a sense, their work quite similar to the work presented here, however the interpretation here will be in terms of Krein's theorem, the geometric stability criteria introduced here is intriguing, and some of the systems and perturbations are slightly more exotic.

Consider the Hamiltonian of a discrete oscillator with negative energy and a continuous bath of oscillators with positive energy with some coupling:

$$H = -\frac{\Omega}{2}\left(P^2 + Q^2\right) + \int_{\mathbb{R}_+} dx \frac{x}{2}\left(p(x)^2 + q(x)^2\right) + H_c. \tag{3.79}$$

When the coupling term H_c is equal to 0 this system is stable. It was mentioned earlier that if the discrete oscillator has positive energy, and care is taken to prevent the occurrence of zero frequency modes, general couplings between the continuum and the discrete oscillator lead to stability. This case will prove to be very different. Let $H_c = Q \int_{\mathbb{R}_+} dx f(x) q(x)$. Hamilton's equations are:

$$\frac{dQ}{dt} = -\Omega P$$
$$\frac{dP}{dt} = \Omega Q - \int_{\mathbb{R}_+} dx f(x) q(x)$$
$$\frac{\partial q}{\partial t} = xp$$
$$\frac{\partial p}{\partial t} = -xq - Qf.$$

Search for discrete modes by assuming that the time evolution is $\sim e^{-iut}$:

$$iuQ = \Omega P$$
$$iuP = -\Omega Q + \int_{\mathbb{R}_+} dx f(x) q(x)$$
$$iuq = -xp$$
$$iup = xq + Qf(x).$$

These can be simplified along the lines of the previous sections to yield the dispersion relation:

$$0 = u^2 - \Omega^2 + \Omega \int_{\mathbb{R}} dx \frac{sgn(x) f(|x|)^2}{2(x-u)}. \quad (3.80)$$

If this expression has any solutions for $Im(u) > 0$ the system is spectrally unstable. In order to determine whether or not there are zeros in the upper half plane, consider the fact that the expression on the right hand side is an analytic function for u in the upper half plane. Therefore the argument principle may be used to calculate the number of zeros that occur in the upper half plane in a manner analogous to the derivation of the Penrose criterion for the Vlasov-Poisson equation. Consider the limiting value of the dispersion function, which is now denoted as $D(u)$, as u approaches the real line from the top:

$$D(u) = u^2 - \Omega^2 + \frac{\Omega}{2} H[f(|x|)^2_-] + \frac{i\pi\Omega}{2} f(|x|)^2_-. \qquad (3.81)$$

The winding number of the image of the real line around the origin is equal to the number of zeros minus the number of poles in the upper half plane, assuming that the contour is traversed positively. Dividing both sides of the equation by $u^2+\Omega^2$ introduces a pole in the upper half plane, but makes it easier to visualize the contours. Therefore, to determine the stability, the winding number of $\frac{D(u)}{u^2+\Omega^2}$ is calculated.

This is simple to compute, as the imaginary part is negative for $u < 0$ and positive for $u > 0$. The behavior of the real part depends on the Hilbert transform of $f(|x|)^2_-$ and the location of Ω. One can assume that for a small perturbation, the coupling function $f(x)$ will also be small. In this case it is obvious that generic choices of $f(x)$ all lead to instability. When the discrete oscillator has positive sign, the orientation around which the contour is circled changes and the stability of the original Caldeira-Leggett model is recovered. Likewise, the behavior when $f(x)$ vanishes in a region near $x = \Omega$ indicates that the coupling did not produce unstable modes (although it does produce zero frequency modes). This allows a very simple interpretation in terms of Krein's theorem. The negative energy discrete mode is structurally unstable when placed in a continuous spectrum with positive energy, just as is the case in Chapter 2.

This should not be surprising and is not even new. Indeed this system satisfies all the conditions needed for the paper of Grillakis to apply, and the structural instability could be inferred from the results contained in that paper. A similar type of analysis can be used to study the behavior of interacting continua, which is not directly covered by theorems contained there.

3.8 Conclusion

To summarize, we have shown how the Caldeira-Leggett model can be analyzed the same way as the Vlasov-Poisson system. We wrote down the solution using the Laplace transform, an expression for the time evolution as an integral in the complex plane over the initial conditions. It was then indicated how Cauchy's theorem can be used to derive the time asymptotic behavior of the solution, and it was described how the long-time damping rate is equal to the distance from the real axis of the closest zero of the dispersion relation (when analytically continued into the lower half complex plane). Thus, the damping of the Caldeira-Leggett model can be seen to be a rediscovery of Landau (or continuum) damping. Caldeira and Leggett introduced their system to study damping in quantum mechanical systems, and it is now seen to be one of many interesting physical examples of Hamiltonian systems that exhibit such behavior.

Next we described how to analyze the Caldeira-Leggett model by means of singular eigenmodes, paralleling Van Kampen's well-known treatment of the Vlasov-Poisson system. Here the solution was written as an integral over a distribution of such modes, each of which is itself a solution that oscillates with some real frequency. We described how Hamiltonian systems with continuous spectra generally have a solution formula in terms of such an integral over singular eigenmodes. This type of formal expansion led to a an explicit integral transformation that transforms the original Caldeira-Leggett system into a pure advection problem, just as is the case for the Vlasov-Poisson system. It was noted that a general class of such transformations was written down in [8] and was subsequently extended to a larger class of Hamiltonian systems [34]. The existence of these transformations amounts to a theory of normal forms for systems with a continuous spectrum, analogous to theory of normal forms for finite degree-of-freedom Hamiltonian systems. This enabled us to write down an explicit transformation that converts the time evolution operator for the Caldeira-Leggett Hamiltonian into a multiplication operator and we found the inverse of this map. In this way we showed that the Caldeira-Leggett model shares the same normal form as the Vlasov-Poisson system, along with a number of other Hamiltonian systems that occur in different physical contexts.

One reason for investigating Hamiltonian structure is the existence of universal behavior shared by such systems. For example, linear Hamiltonian system with the same normal form are equivalent. This suggests some further avenues for research. Here we only treated the case where the dispersion relation of the Caldeira-Leggett model has no roots with real frequency; i.e.

spectrum was purely continuous. When there are roots, the spectrum is no longer purely continuous and there are embedded eigenvalues, as is known to be the case for the Vlasov-Poisson system [78]. Consequently, one obtains a different normal form, one with a discrete component, and this and more complicated normal forms could be explicated. We expect that there is a transformation that takes Vlasov-Poisson system with embedded modes into the Caldeira-Leggett model with embedded modes. Also, finite degree-of-freedom Hamiltonian systems are known to have only certain bifurcations of spectra, for example, as governed by Krein's theorem. Since there is a generalization of this theorem for Vlasov-like systems [6], one could investigate bifurcations in the context of the Caldeira-Leggett model. Another possibility would be to use the tools developed in [44] to do statistical mechanics over the continuum bath.

The integral transform we presented is intimately related to the Hilbert transform, which is known to be an important tool in signal processing. In the same vein the integral transform for the Vlasov-Poisson system of Ref. [8] has been shown to be a useful experimental tool [79, 80] and one could explore experimental ramifications in the context of the Caldeira-Leggett model. Indeed it would be interesting to search for echoes in the Caldeira-Leggett model, and we feel that it would be quite worthwhile if it was feasible.

Chapter 4

Conclusions and Directions for Future Research

The theory of infinite-dimensional Hamiltonian systems is relatively young from the point of view of mathematical physics. Although infinite-dimensional systems are typically much more difficult to study rigorously than finite-dimensional ones, in many important cases rigorous results from finite-dimensional theories can be extended to the infinite-dimensional case. The generalization of Krein's theorem to the Vlasov-Poisson equation, which was proved in Chapter 2, is an example of one such case. This work illustrated both the technical difficulties that arise in attempts to correctly define Hamiltonian systems on infinite-dimensional spaces and the power of the intuition that comes from applying the theorems of finite-dimensional Hamiltonian systems theory. There are a number of interesting problems that were left unsolved here, although in some cases there has been substantial effort already expended. These ideas are classified into two categories, canonical systems and noncanonical systems. In the canonical case much work has been done by Grillakis, but the case with a purely continuous spectrum remains and it would be interesting to solve that case to close that problem. In the noncanonical case, there are variety of additional systems to which Krein's theorem could be generalized. Since many of the most important classical field theories are of Lie-Poisson form, this problem is at least of equal interest to the canonical case. As was seen here, it appears as if Krein's theorem holds only within a single symplectic leaf, and it would be interesting to see how this is manifested in systems like Maxwell-Vlasov or incompressible Euler. Ultimately, it would also be interesting if it would be possible to make the Hamiltonian formulation completely rigorous for some of these infinite-dimensional noncanonical systems. The finite-dimensional picture has such a formulation and the resulting intuitive picture would be worth extending to infinite-dimensions.

The third chapter was about a specific system, the Caldeira-Leggett model, but ultimately it touches on a deep question: is all damping continuum damping? The Caldeira-Leggett model is quite believable as a general model of dissipation, and in many simple dissipative systems it seems clear that a system bath decomposition is valid description of the physics of damping. If we believe that all systems are Hamiltonian, then we must believe that all damping must arise as some sort of irreversible (or nearly so) transfer of energy from different parts of some Hamiltonian system. Continuum damping is a clear realization of this phenomenon, but it is not obvious how to formulate some physical systems, such as a highly collissional gas, in this way. It would be interesting to continue working to understand the nature of dissipation. In the case of Caldeira-Leggett model, some ways forward are clear as it may be possible to realize an echo in an experiment. Additionally, there are many other possible problems, some of which relate to nonlinear systems and quantum mechanics.

Progress in these areas may lead to a much improved understanding of the physics of real matter and to the realization of new technologies.

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