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A Study of Pre-kindergarten Teachers' Mathematical Knowledge for Teaching

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A Study of Pre-kindergarten Teachers' Mathematical Knowledge for Teaching

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Dedication

To my parents

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A Study of Pre-kindergarten Teachers' Mathematical Knowledge for Teaching

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This dissertation investigates the ways in which pre-k teachers understood the math content that they were to teach and their math instruction. To investigate this, a qualitative case study examining five pre-k teachers was conducted. Data sources included observation field notes, teacher interviews, and documents such as state and district pre-k guidelines. The findings from this dissertation suggest that pre-k teachers' knowledge entails both knowledge of subject matter and pedagogical content knowledge. In addition, this study identified what these pre-k teachers knew about math and teaching/learning math as well as what they still needed to know to provide high quality and effective math instruction.

Chapter 1 introduces my research question and important terms, such as mathematical knowledge for teaching (MKT). Chapter 2 synthesizes relevant literature in the area of effective math instruction, theoretical framework of teachers' mathematical knowledge for teaching and early mathematics education. The literature review seeks to highlight the importance of early childhood teachers' deep understanding of

mathematical content and of their math instruction. Chapter 3 forwards the specific conceptual framework for this study while detailing the methodology that guided this investigation including data gathering and analysis. Chapter 4 presents the findings from this research. It examines pre-k teachers' understanding of mathematical content that they are to teach and their knowledge of how to teach mathematics. Chapter 5 addresses the significance of these two major findings. First, I discuss the four types of mathematical knowledge and skills that these pre-k teachers possess. I also compare and contrast them with the teacher knowledge examined in the literature. Then, by examining research literature on early math education, I suggest what mathematical knowledge and skills they still need to attain to offer high-quality and effective math instruction. This dissertation concludes with a discussion of implication for teachers, teacher educators, and suggestions for future research.

Table of Contents

List of Tables

INTRODUCTION

Chapter 1: Introduction

When I entered my first classroom as a teacher, I was full of passion and earnestness for the young children and for the job of teaching them. I also came filled with good faith efforts to provide them with best practices. Despite my passions, earnestness, and efforts, I always had to face my own questions and uncertainties about how to effectively teach young children the skills they would need to be successful. What I needed was encompassing knowledge, or what I call directionality of my practice. Directionality, like a compass, have helped me know where I was, where I was headed and would have confirmed whether that direction was the right direction. Possessing such knowledge, such directionality, not only clarifies teachers' vision of what they pursue but also presses them on to the goal of their practice. In this sense, teacher knowledge can serve as a compass, guiding teachers in the right direction as they educate our next generation of students.

Teacher knowledge consists of knowing the things that must be taught, the extent to which they should be taught, why they should be taught, and how they could be best be taught. A teacher's reservoir of this type of knowledge should be deep and applicable to specific content areas, such as literacy, math, and science. Teacher knowledge and teaching practice are like needle and thread. If a teacher lacks this knowledge, her teaching practice will have a limited impact on students' learning (a mental process) and their achievement (the outcome of learning). This work is a result of much time spent considering early childhood teachers' content knowledge for teaching. My dissertation focuses particularly on early childhood teachers' knowledge of teaching/learning mathematics. I have attempted here, as an early childhood researcher, to suggest two things – what kinds of mathematics should be taught and how they should be taught. I hope this dissertation will motivate prospective and practicing teachers to cultivate an in-depth mathematical knowledge for teaching young children.

THE CONTEXT

In the past ten years, mathematics teaching in the early childhood classroom has come to the forefront of research in both mathematics and early childhood education. A recent emphasis on early childhood mathematics has resulted in many researchers and policymakers attempting to identify what is effective in teaching math to young children. For instance, the National Association for the Education of Young Children (NAEYC) and the National Council of Teachers of Mathematics (NCTM), two leading professional organizations in their respective fields, assert that early mathematics education "should provide carefully planned experiences that focus children's attention on a particular mathematical idea or set of related ideas" (2002, p. 11). According to this view, teachers should "deliberately introduce basic mathematical concepts" (Ginsburg & Amit, 2008, p. 274) like number and operation, algebra, measurement, and so forth (NCTM, 2000). In addition, teachers should help children enhance their in-depth mathematical thinking. To do so, early childhood teachers themselves must possess a deep understanding of the mathematics involved (Lampert, 2001; Shulman, 1986).

Researchers have shown increased interest in the depth of prekindergarten (pre-k) teachers' understanding of the mathematics. Such interest is part of a "trickle down effect of the concern for academics in the early grades" (Saracho & Spodek, 2009, p. 305). Pre-k teachers feel mounting pressure to prepare students for success in elementary school mathematics. Rising with this pressure has been the importance of teachers' knowledge of and ability to teach mathematics. Pre-k is seen as an important step for students to "reach the standards they are expected to achieve in elementary school" (Stipek, 2006, p. 455). According to Neuman and Roskos (2005), educators are emphasizing the development of specific standards to ensure all pre-k children are academically successful in subject areas such as literacy and math. The demand for employing specific math standards and the pressure to prepare children for mathematical success in later grades requires pre-k teachers to possess a deep understanding of not only the content involved in their instruction but of the pedagogical strategies that best deliver such content to their students. Thus, what has become an important issue among researchers and educators is a pre-k teachers' knowledge of mathematics and their ability to teach it.

Although policymakers and professional organizations such as NCTM and NAEYC emphasize the need for pre-k teachers having in-depth knowledge for teaching mathematics, there is a lack of empirical work exploring this issue. Much of the research (e.g., Ball, 2000; Wilkins, 2008) is focused on elementary or secondary teachers' mathematical knowledge for teaching. Many early childhood educators believe that teaching math to 3- to 5-year-old children requires only a little knowledge of math (Ginsburg & Ertle, 2008). Ginsburg and Ertle (2008) suggest, however, that "early mathematics, even at the preschool level, entails broad and deep mathematical ideas that need to be understood by teacher" (p. 47). Ginsburg and Amit (2008), who conducted a case study of a preschool teacher's teaching of mathematics, also found that math at the preschool level entails all the processes involved in teaching mathematics to older children, including a deep knowledge of the subject matter and pedagogical content knowledge. Pre-k teachers need such knowledge in order to provide carefully planned and well-organized math instruction for prekindergarteners, and as such, there is a vital need to examine pre-k teachers' knowledge of what and how to teach this content area.

THIS STUDY

To address this need, this study investigated pre-k teachers' knowledge about mathematics and teaching mathematics in the research literature by examining three lines of inquiry: (a) teachers' understanding of the mathematical content that are taught in pre-k. This involves their understanding of broad strands of math's "big ideas" (e.g., number and operations, measurement, pattern, geometry, algebra), and the interconnections among such ideas. It also involves their understanding of how these ideas are the foundation for their students' later mathematical learning: (b) how teachers understand their teaching of this content, and (c) their reflections on their teaching of some of this mathematical content to their students, which involves their decision-making processes during the math lesson. Their reflections were prompted by questions from my observations of their teaching and by their own reflection of their teaching, which were done through having them observe a video of their teaching. Addressing these areas of study can help researchers begin to unpack and catalog the mathematical knowledge base of practicing pre-k teachers. Also, this dissertation will begin to provide researchers and teacher educators with insight into the knowledge and skills that pre-k teachers now have. They will also gain a sense of mathematical knowledge that pre-k teachers still need to offer high-quality and effective math instruction.

THIS DISSERTATION

In this dissertation, I present results from a study that investigated the ways in which pre-k teachers understood the math content that they were to teach and their math instruction. The research questions that guided my study were:

1. How do prekindergarten teachers understand the mathematical content they are to teach and its connection to children's further/later mathematical learning?
2. How do these pre-k teachers see themselves teaching this content?
3. Lastly, how do these teachers reflect on their teaching of some of this content during their math instruction?

In the following chapter, I present the relevant literature in the areas of early mathematics education and teachers' mathematical knowledge that guided my research and my analysis. I first describe how the literature defines effective early math instruction. I then introduce the theoretical framework of teachers' mathematical knowledge for teaching (MKT). MKT is defined as "the mathematical knowledge needed to carry out the work of teaching mathematics" (Ball, Thames, & Phelps, 2008). Ball et al.'s (2008) framework guided my data collection and analysis process. Next, I cover the recent literature on early mathematics education. I direct attention to its explanation of the importance of early childhood teachers' deep understanding of mathematical content and of their math instruction.

After examining the literature, I turn to the methodology that directed this investigation, which includes my conceptual understanding of the research process. Along with presenting my methods of investigation, I provide biographical information about the teachers I studied and demographic information on the city, school district, and elementary schools where the teachers worked is given.

I then present my findings by providing examples of pre-k teachers' MKT. In discussing these findings, I used Ball et al.'s (2008) framework of MKT including common content knowledge, specialized content knowledge, knowledge of content and students, and knowledge of content and teaching.

The final chapter presents four types of mathematical knowledge and skills that these pre-k teachers now have and how they are similar to or distinctive from teacher knowledge that other literature has examined. Then I suggest what mathematical knowledge and skills they still need to attain to offer high-quality and effective math instruction by examining research literatures on early math education. Finally, I offer suggestions for future research and the implications of this study.

CONCLUSION

Teachers' deep knowledge of subject matter and pedagogical content knowledge is the gateway of providing high-quality and effective instruction and of ensuring students' academic achievement. This dissertation looks at what five pre-k teachers' mathematical knowledge for teaching entails. Also, it suggests what knowledge and skills these pre-k teachers need. I believe this can help both practicing and prospective teachers see not only the directionality of their practice but also the ultimate goal they want to attain through their math instruction.

LITERATURE REVIEW

Chapter 2: Review of Relevant Literature

To understand pre-k teachers' MKT, the structures and domains of MKT must be examined. Also, the key role of MKT that impacts teaching/learning mathematics in early childhood classrooms should be addressed. Therefore, this review begins with an examination of the research on MKT and how it affects teaching and students' mathematical learning, and then, looks at the research that suggests the importance of early childhood teachers' deep knowledge of mathematical content and teaching mathematics.

First, I begin this chapter by demonstrating the importance of prekindergartners' math performance. Next, I examine how researchers and educators conceptualize effective math instruction that is appropriate for young children and enhances children's math performance. Then, I introduce Ball et al.'s theoretical framework that guides this study. This framework is rooted in Lee Shulman's ideas about the constructs of teacher knowledge, which includes content and pedagogical content knowledge. Their framework builds off of Shulman's constructs and was used in my data collection and analysis. Using this framework allowed me to better identify pre-k teachers' MKT and establish refinements to the categories of content and pedagogical content knowledge. I then cover the recent literature on early mathematics education, and focus on the importance of early childhood teachers' deep understanding of mathematical content and of their math instruction.

THE IMPORTANCE OF PREKINDERGARTNERS' MATH PERFORMANCE

Increasingly, researchers, educators, and policymakers have emphasized the significance of young children's math performance and of early math education. The

National Council of Teachers of Mathematics (NCTM), an organization that "provides vision, leadership, and professional development to support teachers" (NCTM, 2000, p. 1) in ensuring high-quality math education for all students, recently revised its standards to include prekindergarten for the first time. In 2000, early childhood and math educators held a conference to focus more explicitly on standards for preschool and kindergarten children (Cross et al., 2009; Clements, Sarama, & DiBiase, 2004). This conference was meant to facilitate communication among experts in the diverse fields relevant to the development of standards for young children's mathematics (Clements et al., 2004). Also, the NAEYC's and NCTM's joint position statement called, "Early Childhood Math: Promoting Good Beginnings" (2002) asserts that "high-quality, challenging, and accessible mathematics education for 3- to 6-year-old children is a vital foundation for future mathematics learning. In every early childhood settings, children should experience effective, research-based curriculum and teaching practices" (p. 1).

Across the U.S states and school districts continue to create and/or modify their own mathematics standards and curriculum guidelines for pre-k and kindergarten children (Clements, 2004). Much of this recent interest in early learning standards has grown out of the K-12 standards-based education reform movement. This process of education reform (e.g., the No Child Left Behind Act in 2002), positions pre-k mathematics as the starting point for preparing children to master mandated elementary school mathematics skills. Although pre-k was not included in NCLB's call for standards and accountability (Goldstein, 2007), the curricular and instructional requirements associated with this policy and the emphasis on academic achievement have begun to affect pre-k. In such a high-stakes standards-based education system, pre-k teachers are expected to prepare their students for schooling (Ginsburg & Ertle, 2008). They must begin preparing children to master mandated mathematics skills, skill that will extend through 12th grade (Hatch, 2002; Stipek, 2006).

The emphasis on early learning received even more attention with the Bush Administration's "Good Start, Grow Smart" (Office of the Whitehouse, 2002). This initiative set out to persuade states and local communities to strengthen early learning expectations of young children. The Bush Administration did this by linking federal funding to the requirement that all states develop voluntary early learning guidelines in language, literacy, and mathematics (Cross et al., 2009).

The standards-based accountability (SBA) reform that policymakers implemented called for "significantly raise[d] expectations for states, local educational agencies and schools in that all students are expected to meet or exceed State standards in reading and in math within 12 years" (Department of Education, 2002). This has impacted teachers (Goldstein, 2007), students, and families (Garcia, et al., 2006), changing the way that teachers both implement academic content and create learning environments. In Texas, for instance, the Texas Legislature enacted statutes that mandated the creation of the Texas public school accountability system to rate school districts and evaluate campuses. This rating system is based on the Texas Assessment of Knowledge and Skills (TAKS) indicator. The TAKS indicator is the percent of students who scored high enough to meet the standard to pass the test. There are three standards-"exemplary, recognized, and academically acceptable"- in rating school districts or campuses (Texas Education Agency, 2010). Exemplary indicates that, for every subject, at least 90% of the tested students passed the test. Recognized indicates that, for every subject, at least 80% of the tested students passed the test. Academically acceptable varies by the subjects. For instance, for reading, writing, and social studies, at least 70% of the tested students pass the test. For mathematics, at least 60% of the tested students passed the test. For science, at least 55% of the tested students passed the test (TEA, 2010).

Within the context of Texas public school accountability system, Texas pre-k programs were mandated by the legislature to serve high-risk 4-year-olds. The goal was to build a solid foundation of school successes among 4-year-olds and to break the debilitating cycle of costly remediation and school failure in later grades (TEA, 2008). School districts must offer pre-k if there are at least 15 eligible, at-risk 3- and 4-year-olds. Children must be at least 3 years old on or before September 1 of the program year, and either be unable to speak or comprehend English, educationally disadvantaged (i.e., eligible for free/reduced lunch program), or be homeless (TEA, 2008).

The education program of Texas pre-k is rooted in the National Association for the Education of Young Children's guidelines, developmentally appropriate practices (TEA, 2008). Also, Texas Pre-k Guidelines were developed at the state level and districts are encouraged to follow these guidelines. These voluntary guidelines describe the expected knowledge and skills across multiple subject domains that 4- to 5-year-old children are to exhibit and achieve. Also, it offers instructional strategies to help teachers provide "developmentally appropriate" and "effective" math instruction (TEA, 2008, p. 5). Texas pre-k teachers are encouraged to teach the expected mathematical content and skills and employ instructional strategies laid out in this guideline to help students successfully master state mandated math standards. In addition to state pre-k guidelines, pre-k teachers are expected to follow state approved curricula that districts have selected and implemented.

To promote students' mathematical learning and ensure students' academic gain, Texas pre-k teachers are required to implement TEA's guidelines or a district's math standards in a way that is effective at helping children's learning. The next section discusses what effective math instruction entails by examining the recent literature that identifies the characteristics of effective math instruction for young children. Many researchers and educators in early childhood (e.g., Epstein, 2007; Sarama & Clements, 2008; Ginsburg & Amit, 2008) suggested three aspects of effective early math instruction. The first is to build on children's informal mathematical knowledge (NCTM, 2000; Baroody & Wilkins, 1999): the second is to employ intentional teaching of mathematics (Epstein, 2007; Ginsburg & Amit, 2008), and the third is to make connections between concrete representations or instructional activities that are designed to actively engage students in learning and to support mastery of the objective and the mathematical ideas teachers represent (NCTM, 2000; Larson, 2002). Each of these, in the next part, will be discussed in detail.

EFFECTIVE EARLY MATH INSTRUCTION

As a result of recent efforts to strengthen the mathematics curricula in our nation's schools, many researchers in early math education have attempted to identify what is effective in teaching math to young children. Educators and the public also debate the most effective ways to teach mathematics. Some emphasize hands-on activities and students' active engagement and discovery: others advocate teacher-directed lessons and paper-and-pencil practice (Larson, 2002). Although effective math instruction takes many shapes and forms depending on the particular philosophical context (e.g., behaviorists advocate didactic instructional strategies), I conceptualize three aspects of effective math instruction that facilitates children's mathematical learning.

First, effective math instruction requires teachers to make use of whatever mathematical knowledge children bring to the classroom (NCTM, 2000). The NCTM (2000) in their Principles and Standards for School Mathematics point out that effective and high-quality math instruction must “build on and extend students’ intuitive and informal mathematical knowledge” (p. 1). This informal knowledge is gleaned from children’s everyday experiences such as at home, on the playground, in the grocery store, or at the shopping mall (Tudge, Li, & Stanley, 2008; Ginsburg, Pappas, & Seo, 2001).

Early childhood teachers should not underestimate this knowledge. Many cognitive research studies have consistently suggested that young children’s informal mathematical knowledge is relatively powerful (Ginsburg et al., 1998; Baroody & Wilkins, 1999; Gelman, 2000; Seo & Ginsburg, 2004). This research has shown that young children’s informal mathematical knowledge is a key foundation for their mathematics achievement in later schooling, for successfully mastering basic mathematics skills, and for devising and applying effective problem-solving (Ginsburg, Klein, & Starkey, 1998; Baroody, Lai, & Mix, 2006). Herron (2010) contended that mathematics in the early childhood classrooms build upon children’s informal mathematical knowledge in conjunction with “teacher guided experiences that allow children to use and develop this knowledge into a strong mathematical foundation” (p. 360). Also, effective math instruction should focus on connecting students’ ability to use informal math and their ability to understand the more formal math taught in elementary school (Epstein, 2007; NAEYC & NCTM, 2000).

Going hand in hand with this is my second aspect of effective mathematics instruction – the teacher’s “intentional teaching of mathematics” (Ginsburg & Amit, 2008; Epstein, 2007). Intentional teaching means that “teachers act with specific outcomes or goals in mind for children’s development and learning” (Epstein, 2007, p. 1). According to NAEYC’s and NCTM’s joint position statement, “Effective programs include intentionally organized learning experiences that build children’s understanding over time” (2002, p. 1). Epstein (2007) suggests the intentional teaching of mathematics entails integrating mathematics into children's everyday life and across other subject matters such as literacy or science in a coherent and planned manner. For many children, it has become increasingly evident that to promote solid mathematics learning it's not enough to use teachable moments that arise in children's play (e.g., building blocks) and daily routines (e.g., lining up or distributing snacks) (Ginsburg & Ertle, 2008). Instead, early math instruction should provide carefully planned math experiences (NAEYC & NCTM, 2002, Principle 9) that lead to learning the objectives described in the state or district math standards and employing the instructional strategies likely to help children achieve those objectives.

Finally, effective math teacher mediates students' understanding of the representations that are modeled by using concrete objects or hands-on activities and serves as a bridge between the concrete and the abstract mathematical ideas (Kilpatrick et al., 2001). As a recent report by the National Research Council points out, "Physical materials are not automatically meaningful to students and need to be connected to the situations being modeled" (Kilpatrick et al., 2001, p. 7). The use of manipulatives and other worthwhile hands-on activities alone does not ensure student understanding of mathematics. Cawelti (1999) also contended that the value of using concrete objects depends not on whether they are used, but on how they are used. Students have to do more than simply engage in planned math activities; the teacher must help students process activities in a meaningful way.

These three aspects of effective math instruction can satisfy, "in depth and in a logical sequence," children's intellectual needs and promote their mathematical thinking (Epstein, 2007, p. 46). Young children need many opportunities to relate their intuitive and experiential understanding of mathematics to the concepts or vocabulary that are used in school mathematics (Epstein, 2007). Also, they need to engage in deep mathematical thinking while exploring hands-on activities. To do this effectively, intentional teachers carefully plan for children's involvement with a number of key content mathematical ideas (Epstein, 2007) and create intentionally organized learning experiences that build on children's informal and intuitive understanding of mathematics (NAEYC & NCTM, 2002). In addition, they must assist students to make connection between the concrete and abstract (Larson, 2002; Kilpatrick et al., 2001).

In sum, effective pre-k math instruction entails three important components. First, it should provide instruction that reflects children's informal knowledge and experience of mathematics (Seo & Ginsburg, 2004). Effective early math instruction recognizes the importance of understanding children's capacities and their experiences and it works at connecting this informal mathematical knowledge to formal school instruction (Andrews & Trafton, 2002). Second, effective math instruction must be intentional (Ginsburg & Amit, 2008; Ginsburg & Ertle, 2008; Epstein, 2007). Early childhood educators must provide carefully planned math lessons that not only address national, state, or district math standards but employ appropriate instructional strategies that deliberately sequence the teaching of mathematical ideas in a coherent and developmentally appropriate manner. Third, to promote student understanding, teachers must help students cross the bridge the gap between concrete mathematical representations with abstract and symbolic ideas (Clements, 1999).

To provide effective math instruction, teachers need to have sound content knowledge and pedagogical content knowledge (Shulman, 1986; Lampert, 2001). Knowing both content and pedagogy is crucial to interpreting children's mathematical thinking and integrating with practice children's experience, interests, and needs (Ball & Bass, 2000). This study investigates pre-k teachers' understanding of mathematical content and their math instruction. To do this, what is needed is a theoretical framework that can catalog and make concrete the constructs of teacher's mathematical knowledge. Hence, in the next section, I address the theoretical framework that guides this study.

THEORETICAL FRAMEWORK

The theoretical framework of this study stems from the work of Ball and her colleagues (2008), using the Shulman's construct of teacher knowledge (e.g., content, pedagogical content knowledge). First, I review Ball et al.'s critiques of Shulman's work and then show the shortcomings they highlighted. Then I show how Ball et al. reconceptualized and refined Shulman's content and pedagogical content knowledge, specifically in regards to math education. Their refinements of teacher knowledge structure the theoretical framework I used to investigate pre-k teachers' understanding of mathematical content that they are to teach and of pedagogical strategies that they implement.

Lee Shulman's work

Shulman (1986) suggested the categories of teacher knowledge needed for effective teaching (e.g., subject matter knowledge, pedagogical content knowledge, general pedagogical knowledge). A central contribution of Shulman's work was to shift scholars' attention from general teaching behaviors to the nature and role of content knowledge for teaching (Ball et al., 2008). Another contribution was to provide a conceptual framework of content knowledge that is unique to teaching and essential to its profession of teaching (Grossman, 1990; Ball et al., 2008). Shulman's work explains what is needed for effective teaching and how teachers should be certified as professionals.

There was widespread interest in the ideas of Shulman and his colleagues (Calderhead, 1996). Although the definitions from other research studies capture the general idea of pedagogical content knowledge, Ball and her colleagues (2008) contend that they are too broad and inclusive to specify professional knowledge for teaching, particularly in terms of math instruction. For example, they ask how does a researcher operationalize the term pedagogical content knowledge (PCK)? How can he or she distinguish the PCK from other forms of knowledge such as content knowledge or pedagogical knowledge, teacher beliefs or attitude? Ball et al. (2008) suggest that what is necessary is a greater precision in defining the concepts involved. This would offer researchers a chance to better understand and specify the nature of the content knowledge for teaching. Thus, the following section introduces how Ball and his colleagues have approached this issue from mathematical perspective.

Ball et al.'s constructs of mathematical knowledge for teaching

In conceptualizing mathematical knowledge through the work teachers do, it is important to understand what teachers need to know about mathematics and how they put such knowledge into practice. Ball et al. (2008) approached this by focusing on what teachers actually do during the math lesson. Their premise, that teachers' knowledge is situated in their practice, compelled them to uncover how math was treated, taught, and involved in the regular day-to-day demands of teaching. Their analyses of the mathematical demands of teaching showed that special and unique mathematical knowledge was required in many everyday tasks of teaching, including such tasks as commenting on students' work, listening to students talk, and assigning homework (Ball et al., 2008). Take, for example, teaching the simple subtraction problem 37-18=19. The teacher must be able to understand the notion of place values and of borrowing a one from the tens place (Ball et al., 2008). Being able to do this simple procedure is necessary, but that ability is not the same as the ability to teach it. In addition, teachers need detailed mathematical understanding to handle the problems or errors that students make (Ball et al., 2008; Hill et al., 2008). Teachers must consider a number of things: Did the students think correctly? What would be a good task for my students that would help me teach this content? What are the important ideas or processes involved in the problem? Will students find this interesting? What might make the problem hard or easy? This distinct work that teachers do requires a kind of mathematical knowledge that few adults need on a regular basis and it involves sophisticated mathematical reasoning (Ball, 2000; Ball et al., 2008).

Relying on practice-based analyses of mathematical knowledge for teaching, Ball et al. (2008) suggest four subdomains of Shulman's content knowledge and pedagogical content knowledge. First, content knowledge can be divided into common content knowledge (CCK) and specialized content knowledge (SCK). The first subdomain, common content knowledge, refers to "the mathematical knowledge and skill used in settings other than teaching" (Ball et al., 2008, p. 399). Teachers need to be able to do the following: identify right or wrong answers, use correct terms or notation, and do the work or simple computation they assign their students. Such abilities require mathematical knowledge and skill that others have as well, which means it is not special to the work of teaching.

The second subdomain, specialized content knowledge, is "the mathematical knowledge and skill unique to teaching" (Ball et al., 2008, p. 400). This is the domain that is needed solely for teaching. For example, "finding and selecting appropriate examples or resources to present mathematical ideas, connecting a topic being taught to topics from prior or future years, modifying tasks to be either easier or harder, and asking productive mathematical questions etc." (Ball et al., 2008, p. 390) are all distinctive mathematical tasks related to SCK.

Each of these tasks requires unique mathematical understanding and reasoning. Teachers must be able to understand different interpretations of the problem or concept to make the particular content visible to and learnable by students. For instance, from my pilot study of effective kindergarten math instruction that I conducted with a practicing kindergarten teacher, I found that to teach their children about the number 84 required her to possess a variety of understandings. The teacher who was participating in my pilot study knew what each digit represents and how each digit corresponded with the tens' and ones' places. She was able to use multiple representations or examples to convey the idea that the number 84 is composed of 8 sets of ten plus 4 ones. This example shows that teaching place value to young children is a complex and challenging process. This type of mathematical teaching represents specialized mathematical knowledge that goes beyond the kind of tacit understanding of place value needed by most people.

Next, Ball et al. (2008) divide pedagogical content knowledge into two subdomains: knowledge of content and students and knowledge of content and teaching. Knowledge of content and students (KCS) refers to "knowledge that combines knowing about students and knowing about mathematics" (Ball et al., 2008, p. 401). KCS is crucial to "being inventive in creating worthwhile opportunities for learning that take learner's experiences, interests, and needs into account" (Ball & Bass, 2000, p. 86). KCS enables teachers to anticipate what students are likely to think, what they will be confused about, and what students will find interesting and motivating (Hill, Ball, & Schilling, 2008). Teachers, when assigning a task or planning an activity, must interpret students' mathematical thinking and build on it in their instruction. This task requires an interaction between specific mathematical knowledge and an understanding of students' thinking.

The other domain, knowledge of content and teaching (KCT), is a combination of "knowing about teaching and knowing about mathematics" (Ball et al., 2008, p. 401). This type of knowledge is related to how teachers design the instruction. Falling within this domain are such tasks as sequencing particular content for instruction, choosing appropriate examples to start with or to use to take students deeper into the content, deciding when to initiate whole group discussions or when to pause for more clarification, and when to ask a new question, and so forth (Ball et al., 2008, p. 402). All of these tasks require coordination between mathematics and instructional purposes or designs. In other words, KCT involves an understanding of pedagogical issues that affect student learning and of specific mathematical content or ideas.

Thus far, I have introduced how Ball and her colleagues investigated the nature of professionally oriented subject matter knowledge in mathematics. They provided four empirically discernable constructs within content knowledge and pedagogical content knowledge and examined how these four constructs are situated in teachers' practices. These four constructs are: recognizing a right or wrong answer and performing the simple computation (common content knowledge); making the mathematical concepts or ideas visible and learnable (specialized content knowledge); understanding students' mathematical thinking, interests, and needs (knowledge of content and students); and designing an instruction or making instructional decisions in a way that is essential to student learning (knowledge of content and teaching). These four categories of subject matter knowledge for teaching do not replace the construct of content and pedagogical content knowledge identified by Shulman and his colleagues; rather, these categories expand on it. Ball et al. (2008) add detail to it by carefully mapping and conceptualizing Shulman's constructs. These constructs also illuminate how subject matter knowledge for teaching can be used in teaching and how content and pedagogical content knowledge are connected to instruction and student learning. Moreover, these constructs helped me categorize and catalog pre-k teachers' MKT found in both their practice and their interviews.

EARLY CHILDHOOD TEACHERS' MATHEMATICAL KNOWLEDGE FOR TEACHING

Many researchers in early childhood (e.g., Ginsburg & Ertle, 2008) suggest that early childhood teachers need deep knowledge of subject matter. To understand what mathematical knowledge early childhood teachers need for teaching, I apply Ball et al.'s constructs of MKT (2008) to teaching mathematics to young children. This allows me to develop and specify the coherent constructs of pre-k teachers' MKT that affect their teaching practices and student's mathematical learning. I first introduce the research studies that emphasize the importance of teachers' content knowledge for teaching. Then, I focus on the significance of early childhood teachers' subject matter knowledge in mathematics and describe the rationales of why this is important and must be examined.

The major theme emerging from the research on teaching mathematics is that both the quality of instruction and students' learning are significantly correlated with teachers' deep understanding of the subject matter (Lampert, 2001; Ma, 1999; Hill, Rowan, & Ball, 2005). Several studies have explored the nature of teachers' subject knowledge in mathematics. These studies have suggested that teachers' mathematical content knowledge plays an important role in teachers' effectiveness and has a direct relationship with teachers' instructional practices (Ernest, 1989). Fennema and Franke (1992), in their review of research on mathematics teaching and learning, concluded that "when a teacher has a conceptual understanding of mathematics, it influences classroom instruction in a positive way" (p.115). Also, teachers' mathematical knowledge for teaching was significantly related to student achievement gains in both first and third grades (Hill et al., 2005).

For early childhood educators, the results of these studies offer many suggestions. Many researchers in early childhood math education argued that early childhood teachers also require a profound understanding of mathematics to provide high quality math instruction for all children. There are four important rationales for why knowing what to teach and how to teach is important. They are as follows: (a) early childhood mathematics entails broad and deep mathematical ideas, (b) children bring different types of informal mathematical knowledge to the classroom, (c) teachers should be able to implement math curriculum in an effective and developmentally appropriate manner, and (d) early mathematics ought to be integrated into children's daily routines. Each of these are connected to the characteristics of effective math instruction that I discussed in the previous section. Early childhood teachers need to know these four rationales to provide effective math instruction that reflects children's informal knowledge and experience of mathematics (Seo & Ginsburg, 2004), addresses national, state, or district math standards and employs appropriate instructional strategies in a coherent and developmentally appropriate manner. In the following section, I describe each of these in detail.

Early childhood mathematics is broad and deep

Early childhood mathematics entails broad and deep mathematical ideas (Ginsburg & Ertle, 2008). Most adults assume that math for young children is simple and easy (Baroody, 2004). However, Ginsburg and Ertle (2008) say that early mathematics consists of far more than counting or identifying numbers. In their framework, breadth implies that early mathematics involves broad strands of "big ideas," such as number and operations, measurement, pattern, geometry, and algebra (NAEYC & NCTM, 2002;

NCTM, 2000). Ginsburg and Ertle (2008) articulate that each of these entails several subtopics that are highly interrelated. For example, the big topical idea "Number and Operations" can be broken down into several subtopics (p. 48). These include counting (finding out how many in a collection), comparing and ordering (quantities can be compared and ordered, and numbers are one useful tool for doing so), grouping and place value (multi-digit numbers and the values of digits), composing and decomposing (a quantity can consist of parts and parts can be combined to form the whole), and adding and subtracting (NCTM, 2000; Clements, 2004). Another broad topic, geometry, includes the subtopics such as shape and spatial reasoning. The subtopic, shape, involves not only plane figures like circles and triangles but also hexagons, octagons, cubes, cylinders, and symmetries in two and three dimensions (NCTM, 2000). Spatial reasoning entails the concept of position (e.g., in front of, behind) and navigation (e.g., direction such as right or left) (NCTM, 2000; Clements, 2004). These examples suggest the breadth of early mathematics.

Just as evident is the depth of early mathematics. Pattern, for instance, which virtually all preschool educators would consider appropriate for preschoolers to learn, refers to "an underlying rule or concept" and describes "a regularity that determines, explains or predicts observed phenomena" (Ginsburg & Ertle, 2008, p. 51). The ability to recognize and analyze the pattern provides a foundation for the development of young children's algebraic thinking (Clements, 2004). Algebra, the study of structure, relation, and quantity, "begins with a search for patterns" (Clements, 2004, p. 52). In the pre-k years, children can learn to model simple concrete patterns. In kindergarten, they can learn to extend and create patterns (Klein & Starkey, 2004). For example, through pre-k and the primary grades, children learn to identify "the core unit (e.g., AB) that either repeats (ABABAB) or grows (ABAABAAAB)", and then they are able to predict what comes next and generalize the pattern (Clements, 2004, p.52). This is a crucial step in promoting children's algebraic thinking that allows children to recognize the order of data and to make predictions and generalizations beyond the available information (Clements, 2004). Also, understanding pattern is a foundational step in learning an important mathematical idea, linear function later on (Ginsburg & Ertle, 2008). To predict what comes next and generalize how things work is a basic ability to grasp the idea of linear function. This reveals how a mathematical concept often taught to preschoolers is actually quite deep.

These examples reviewed so far show the breadth and depth of early mathematics. The topics that 3- to 5-year-old children study may be basic, but they include many subtopics that entail deep mathematical ideas. The ability to understand basic math concepts, the interconnections among such concepts, and their relation to children's further mathematical learning is related to Ball et al.'s construct, specialized content knowledge. To teach such concepts and understand their underlying meanings requires early childhood teachers' rich understanding of specialized content knowledge.

Children's informal ways in understanding mathematical content

Combining children's mathematical thinking and mathematical content, a key aspect of knowledge of content and students (Ball et al., 2008), is an essential element of mathematical knowledge needed for teaching. Researchers (e.g., Empson & Junk, 2004; Carpenter et al., 1993) have proposed that teachers must integrate their knowledge of math content and practices with their knowledge of children's thinking. As children show different ways of solving problems, it is vital that teachers demonstrate a broad understanding of children's informal ways of understanding mathematical content. Consider division as an example of how math knowledge can be integrated with an understanding of children's thinking. According to *Webster's*, to "divide" is "to subject (a number or quantity) to the operation of finding how many times it contains another number or quantity." Division is the arithmetic inverse of multiplication. That is to say, the division of a by b is to multiply a by $1/b$ (Empson & Junk, 2004). This is a formal understanding of division. Informally, however, division can be defined in several different ways (Empson, 1999; Empson & Junk, 2004). Based on my teaching experience, 3-to 5-year-old children tend to see it as simply separating a quantity into two or more parts, areas, or groups and giving out in shares. For example, this includes the situation where a total number of cookies split into a given number of equally-sized groups or into groups of a given size. This situation, not part of formal mathematics (e.g., $a \div b = a \times 1/b$), is not seen by children as division. They learn only later on in their schooling that it is a manifestation of formal math (Empson & Junk, 2004).

Many scholars contend that teachers must understand how these informal models of mathematical concept develop in children's problem-solving processes. This means teachers must carefully observe and recognize children's informal ways of solving the math problems and build on and extend those ways when teaching mathematical content. This is an important task in terms of connecting children's informal math and their capability to grasp more formal school mathematics. The ability to combine children's informal mathematical knowledge and mathematical content is associated with knowledge of content and students (Ball et al., 2008). Early childhood teachers need KCS to integrate children's informal mathematical knowledge with more formal mathematics.

Implementing the math standards and curriculum

The math curriculum that is based on standards provides a plan of instruction and specifies the details of student learning, instructional strategies, and the teachers' roles. Most states and school districts across the U.S have developed their own mathematics standards and curriculum for pre-k and kindergarten children (Clements, 2004). To implement these curricula effectively, early childhood teachers need to understand the mathematical ideas involved.

In Texas, there are pre-k guidelines for teaching pre-kindergartners, which offer "detailed descriptions of expected behaviors across multiple skill domains that should be observed in 4- to 5-year-old children by the end of their pre-k experience" (TEA, 2008, p. 4). The guidelines lay out the behaviors and skills that children are to exhibit and achieve, as well as instructional strategies for the teachers. The purpose of this document is twofold. First, it helps early childhood educators make "informed decisions about curriculum content for prekindergarten children" (p. 4). Second, it helps align pre-k programs with the state adopted curricula and Texas Essential Knowledge and Skills (TEKS) for grades K-12. TEKS is a mandated learning standard that defines content and performance expectations for students (TEA, 2008). The TEA, throughout its guidelines, wants to ensure that all 4-year-old children develop the mathematics skills and knowledge needed for success in elementary school mathematics. This requires pre-k teachers to possess sound specialized content knowledge (Ball et al., 2008) that enables them to grasp the basic mathematical content and its connection to upper grade level.

Teaching mathematics according to the pre-k guidelines is not as easy as it might appear. Unlike many textbooks, instructional strategies or curriculum materials do not take the form of explicit mathematical statements. Instead, the guidelines typically involve various activities and games in which the math is embedded, sometimes in not so obvious ways. Also, the guidelines do not insist that teachers follow a rigid script in doing an activity. Although the guidelines suggest wording that the teacher may use, the basic assumption is that implementing a mathematics curriculum requires flexibility. They are also flexible so that they can be easily aligned with any of the approved state curricula for pre-k.

For instance, the first section of the guideline, which helps children learn to count from 1 to 30 encourages teachers to teach this by introducing the number words along with various songs, chants, or body movements. It states that the teacher "incorporates counting into everyday activities, such as counting songs and physical activities" (TEA, 2008, p. 84). In other words, teachers do not directly tell the children this content; they model the ideas through planned math activities. Through this process, children gradually elevate the upper limit to which they can count and learn something about the rules underlying the counting sequence. To be effective, teachers should have sound knowledge of content and teaching (Ball et al., 2008). Having such knowledge enables teachers to understand why the curriculum materials or instructional strategies are constructed as they are and to appreciate the importance of various elements of the activities. To flexibly implement and adapt the activity, teachers also need knowledge of content and students (Ball et al., 2008). With this knowledge, they can recognize the diverse and changing needs of her children. Without it, they might not implement the activity as intended and thus fail to promote key aspects of mathematical learning. In brief, implementing the curriculum and simple math activity to teach the standards entails many complex decisions that are based on teachers' KCS and KCT.

Summarized below in Table 1 is other broad and deep mathematical content that the Texas pre-k guidelines address. Pre-k mathematics content is divided into four big ideas – number and operation, geometry, measurement, and pattern/algebraic thinking.

Each of these, in turn, entails several subtopics. For example, number and operation covers such matter as counting, comparing/ordering, and adding to/taking away. The topic of geometry includes ideas like shape, locations or directions, and spatial reasoning. Measurement covers the subtopic of attributes. Pattern/algebraic thinking includes ideas such as classification/grouping, pattern, and data analysis. Again, teaching this content requires pre-k teachers' deep understanding of the mathematics involved.

Table 1-Mathematical content in Texas pre-k guidelines (TEA, 2008)

Mathematical ideas from Texas pre-k guidelines	
Number and Operation (p. 84-88)	Counting: count the items from 1-30 and find out how many in a collection
	Comparing/Ordering: compare or order collections, and know numbers are one useful tool for doing so.
	Adding To/Taking Away: make collection larger by adding items to it and make smaller by taking some away from it.
Geometry (p. 89-90)	Shape: recognize, identify, and name geometry figures
	Locations, Directions, & Coordinates: specify precisely directions, routes, and locations.
	Spatial Reasoning: know the shape of one's environment and create a mental image of geometric objects
Measurement (p. 91-92)	Attributes: recognize, compare, and order attributes such as length, volume, weight, area, and time
Pattern/Algebraic Thinking (p. 93-95)	Classification/Grouping: sort and classify objects using one or more attributes
	Pattern: recognize and create pattern
	Data Analysis: collect data and organize it in a graphic representation

Incorporating mathematics into pre-k daily routines

Finally, incorporating mathematics into pre-k daily routines requires proficient mathematical knowledge (Ginsburg & Ertle, 2008). One of the central features of early childhood math education is that teachers should provide rich opportunity for children to engage in math activities and then use these activities as a basis for teaching specific math content. The important idea is "to seize on the teachable moment" (Ginsburg & Ertle, 2008, p. 59). The teachable moment involves "the teacher's careful observation of children's play and other activities in order to identify the spontaneously emerging situation that can be exploited to promote learning" (Ginsburg, Lee, & Boyd, 2008, p. 7). In places like the block area, teachers must look for, particularly during free play, teachable moments. For instance, when a small group of children build a tower by placing rectangular blocks vertically, a teacher must be able to recognize that these children are involved in a mathematics activity. Further, the teacher could provide appropriate intervention such as asking them if they could build a structure the same size with different sizes or shapes of blocks. These are examples of how teachers can seize teachable moments and help children extend and elaborate their mathematical thinking. Doing so requires teachers to understand the mathematics underlying children's behavior or activities (Ginsburg, Inoue, & Seo, 1999). Based on their observations and analyses of such behavior and activities, teachers should design carefully planned and organized instruction to foster children's development. To provide coherent math instruction by seizing the teachable moments, teachers need knowledge of content and students that is based on teacher's deep understanding of both subject matter in mathematics and children's informal mathematics.

By highlighting these four rationales for knowing what to teach and how to teach it, I have shown that a deep understanding of mathematics is essential for providing children with high quality math experiences. Such experiences help ensure an effective and efficient pre-k year. Pre-k teachers must have SCK to understand the content of early mathematics, an area, as we have seen, with broad and deep mathematical concepts (Ginsburg & Ertle, 2008). Second, they must have a grasp of informal mathematics and of young children's mathematical thinking (Baroody & Wilkins, 1999; Empson & Junk, 2004), which is related to KCS. Third, they need both KCS and KCT to know the underlying ideas involved in math standards and curriculum described in the pre-k guideline and adapt the instructional strategies flexibly. Finally, they need to be able to recognize "teachable moments" and use them as a basis for planning and developing further math activities (Ginsburg & Ertle, 2008; Ginsburg et al., 1999). This also requires teachers to have thorough understanding of KCT. Meeting all such requirements is as complex and challenging when teaching young children as it is when teaching older children. What all of this underscores is the importance of pre-k teachers' understanding what and how to teach for their math instruction. Being cognizant of what they understand is an important objective and requires examination.

CONCLUSION

There have been attempts to make early childhood math education more academically rigorous and to take a big step forward in improving young children's math performance. As a result of this emphasis on increasing children's mathematical knowledge and academic performance, researchers and educators in early math education have become interested in implementing effective math instruction. Effective math instruction satisfies, "in depth and in a logical sequence," children's intellectual needs and promote their mathematical thinking (Epstein, 2007, p. 46). Effective early childhood teachers carefully plan for children's involvement with a number of key content mathematical ideas (Epstein, 2007) and create intentionally organized learning experiences that build on children's informal and intuitive understanding of mathematics (NAEYC & NCTM, 2002). In addition, they assist students to make connection between the concrete and abstract (Larson, 2002; Kilpatrick et al., 2001).

To provide effective math instruction, teachers need to have sound content and pedagogical content knowledge (Shulman, 1986; Lampert, 2001). Knowing both content and pedagogy is crucial to interpreting children's mathematical thinking and integrating into practice children's experience, interests, and needs (Ball & Bass, 2000). This dissertation unpacks and catalogs pre-k teachers' MKT by investigating pre-k teachers' understanding of mathematical content and their math instruction. In the next chapter, I explain how I conducted this study.

RESEARCH METHOD

Chapter 3: Methods

RESEARCH PARADIGM

Because teacher knowledge is socially constructed in and out of interaction between teachers and their world (Calderhead, 1996), constructivist paradigm framed this study. The constructivist approach focuses on "multiple realities constructed by human beings who experience a phenomenon of interest" (Krauss, 2005, p. 760). It regards realities as apprehendable in the form of mental constructions, socially based, local and specific in nature (Crotty, 2003). Using this approach, research is conducted by those who want to attempt to understand the world from the point of view of those who live it. According to Hatch (2002), constructivist researchers reject the notion that there is one objective reality that can be known and instead take the stance that a constructivist researcher's goal is to, "construct [with participants] the subjective reality that is under investigation" (p. 15). Thus, a constructivist perspective views knowledge as a social construction of reality and a social dimension of meaning. Teacher knowledge is contingent upon their teaching practices, and serves to generate ways of viewing situations and solving problems that teachers might be able to interpret within the context of their own classrooms (Calderhead, 1996).

The focus of this study was to examine pre-k teachers' MKT. Theoretically framing teacher knowledge through a constructivist paradigm provided a useful heuristic for knowing how they understand the mathematical content and their teaching, and how their MKT is being constructed. Within the constructivist perspective, a case study methodology (Yin, 2003) was used to gather and understand pre-k teachers' knowledge of math and teaching math. By drawing on multiple sources of information, a case study allows for in-depth investigation of what pre-k teachers know about teaching and learning math. This study encompassed the following case study characteristics, as described by Mertens (1998): it relied on the interaction between the context and the individual as well as looking deeply at the nature of the case and contexts as units of analysis and understanding. In addition, this study was inherently bounded and specific, the rationale for a case study approach was paramount (Stake, 2005).

RESEARCH METHODOLOGY

While the importance of early childhood teachers' MKT has been emphasized by researchers and educators to offer high-quality and effective math instruction (Ginsburg & Ertle, 2008), there is a lack of empirical work that examines their understanding of what and how to teach mathematics. This study used case study methodology to examine how pre-k teachers understand mathematics content and teaching mathematics. Case study methodology is well suited when the focus is on "a contemporary phenomenon within its real-life context, especially when the boundaries between phenomenon and context are not clearly evident" (Yin, 2003, p. 13). Thus, it is an effective method of investigation to examine pre-k teachers' understanding of mathematics instruction. In addition, case studies are intended to provide detailed and in-depth understanding of particular circumstances rather than broad and generalizable results (Stake, 1995). Employing this method provided me the chance to conduct an in-depth investigation into how pre-k teachers understand mathematics content and teaching mathematics. Five pre-k teachers' knowledge of math standards described in the state pre-k guidelines and their teaching practice was the cases investigated in this study. An investigation of this kind ought to yield a thick description of what is entailed in these teachers' mathematical knowledge for teaching. The questions that guide this qualitative case study were:

1. How do prekindergarten teachers understand the mathematical content they are to teach and its connection to children's further/later mathematical learning?

2. How do these pre-k teachers see themselves teaching this content?

3. Lastly, how do these teachers reflect on their decision-making practice that occurs during their instruction of planned content?

In this chapter, I discuss the research design, and methods of data collection that framed this dissertation. First, I discuss the rationale for the constructs that framed teachers' understanding of math content and instruction. I then discuss the rationale for the site selection, field entry, and participant involvement. The phases of the study are outlined and an overview of the project given, followed by a description of the data collection, documentation, and record keeping. I discuss sampling decisions and methods of data analysis and conclude with issues of quality and rigor and ethical considerations.

THE CONSTRUCTS THAT FRAME TEACHERS' UNDERSTANDING OF MATH CONTENT AND INSTRUCTION

In the literature review of this dissertation, the theoretical framework of teachers' MKT that I employed in this study was described. Ball et al. (2008) explained Shulman's constructs-content knowledge and pedagogical content knowledge, to indicate two subdomains within pedagogical content knowledge (knowledge of content and students and knowledge of content and teaching). Ball also designated an important subdomain of content knowledge unique to the work of teaching, specialized content knowledge. This type of knowledge is distinct from the common content knowledge that both teachers and nonteachers have. These four constructs were used in this study for data collection and analysis to identify teachers' understanding in regards to mathematics content, curriculum, and instruction. Table 2 addresses and operationalizes these constructs that frame the interview questions and observational data. These interview questions and categories of observed classroom behaviors reflect an adaptation of Ball et al.'s work (2008) that address teaching pre-k math and the teacher knowledge constructs found on the pre-k guidelines. To be clear, these questions were not presented in the order they were asked. See the appendices for this information.

Table 2-Constructs of Teachers' Mathematical Knowledge

Constructs	Observed classroom behaviors	Interview questions
Common content knowledge	• Teachers calculate or correctly solve the problem. • Teachers recognize when their students give a wrong answer. • Teachers use terms or notations correctly (Ball et al., 2008).	• What is a number that lies between ½ and ⅓? • What are the differences between a triangle, square, and rectangle? • What are non-standards or standard-units to measure height, weight, length, and volume? (TEA, 2008)
Specialized content knowledge	• Teachers respond to students' why questions. • Teachers explain mathematical goals or purposes to students. • Teachers modify the activities to be either easier or harder. • Teachers provide an appropriate example to make a specific mathematical idea meaningful to students. • Teachers choose, make, and use curriculum materials effectively. • Teachers connect a topic being taught to topics from future years. • Teachers ask productive mathematical questions (Ball et al, 2008).	• What is the best way to teach the concept of addition? • Please describe how you teach the number from 10-20. • What would you do when some students feel the planned activity is difficult or too easy? • How do you explain the concept of place value to little children?(TEA, 2008) • How do you think the big mathematical concept such as number and operation, pattern, or algebra is connected to children's further mathematical learning? (Ginsburg & Ertle, 2008)

Table 2, cont.

Knowledge of content and students	• Teachers anticipate what students are likely to think and what they will find confusing. • Teachers provide activities that students find interesting and motivating. • Teachers respond to and interpret students' emerging and incomplete thinking as expressed in the ways that students use languages. • Teachers figure out common errors that students are most likely to make (Ball et al., 2008).	• What are the kinds of shapes students are likely to identify as triangles?(TEA, 2008) • What is the likelihood that students will write 6 for 9? • How would you correct errors or misunderstandings that most students make?(Ball et al., 2008) • How do you respond to the students' answers when they solve the problems in a non-standard ways? (Empson & Junk, 2004) • Why do you think students make specific errors when teaching particular mathematics content? • What are the informal ways to solve adding and subtracting problems?
Knowledge of content and teaching	• Teachers sequence particular content for instruction. • Teachers balance teacher-directed and child-oriented instruction (Epstein, 2007). • Teachers employ whole-group, small group, and individualized instruction. • Teachers are able to allocate time appropriately. • Teachers choose which examples to start with and which examples to use to take students deeper into the content (Ball et al., 2008). • Teachers make instructional decisions about which student contributions to pursue and which to ignore or save for a later time. • During a classroom discussion, teachers decide when to pause for more clarification, when to use a student's remark, when to ask a new question, or pose a new task (Ball et al., 2008).	• Why do you choose these blocks as a major learning material? • Which do you think is a good way for teaching place value: a tape measure or unit blocks? (TEA, 2008) • Why do you choose the book for the first introduction to addition? • When is the best time to move on to the next stage? • What kinds of curriculum materials are good at taking students deeper into the content? (Ball et al., 2008) • How do you initiate the whole group discussion when you start to teach new mathematical concepts? • How do you organize the classroom or design the mathematical learning environment?

*Adapted from Ball, Thames, & Phelps, 2008

THE SCHOOL DISTRICT

This study was conducted during the spring of 2010 in Wood Trail and Dominion Independent School Districts (ISD). Wood Trail ISD is an urban district, located in central Texas. It has an enrollment of close to 45,000 students attending the district's 5 high schools, 10 middle schools, 31 elementary schools. The district has a diverse ethnic base with a student population that is approximately 9% African American, 11.2% Asian, 30.1% Hispanic, 0.5% Native American, 0.1% Pacific Islander, and 45.1% White with more than 77 languages spoken throughout the district. Wood Trail ISD offers half-day pre-k classes for children who are four years old on or before September 1, live in the district, and meet following criteria: 1) have a limited ability to speak or comprehend English language, 2) are homeless, 3) whose family income allows the child to qualify for free or reduced lunch, and 4) whose parents are military. All of these requirements except the fourth criteria are same as state's requirements.

Dominion ISD is a suburban district, located in East Texas. It has an enrollment of close to 3,100 students attending the district's 2 high schools, 1 middle school, and 3 elementary schools. The district has a diverse ethnic base with a student population that is approximately 9.8% African American, 62.5% Hispanic, and 25.5% White. Dominion ISD offers full-day pre-k classes for children who are four years old, live in Dominion ISD and meet at least one of the following: 1) are economically disadvantaged, 2) are limited English speaker, 3) are homeless, 4) have been in the foster care system, and 5) whose parents are military. All of these requirements except the fourth and fifth criteria are same as state's requirements.

PARTICIPANTS

In-depth case studies were developed with five pre-k teachers. To understand general pre-k teachers' understanding of my research questions, I worked with a variety of participants. I intended to select participants who have different years of teaching experience and who work in different school or school districts. I have chosen to study this type of participants because teachers' mathematical knowledge for teaching and their math instruction can be constructed and affected by teachers' background characteristics (e.g., years teaching, courses taken, or degree earned). According to Wilkins (2008), "years of teaching experience appeared to have a significant negative effect on content knowledge" (p.156). It had an indirect effect on teachers' instructional beliefs and practices. This study suggests that the teachers' background variables are related to teachers' knowledge, instructional beliefs, and practice.

Also, these pre-k teachers work in different elementary schools in two different school districts— three teachers in an urban school district and two teachers in a suburban school district. According to Causey, Thomas, and Armento (2000), teachers who work in urban school settings were more likely to have confidence in their knowledge of or ability to work with diverse student populations. These teachers tended to have higher expectations of student achievement. On the other hand, teachers with little urban education experience tend to hold a deficit model of students who differ from them in race, ethnicity, SES, or family structure. This study indicates that the environment and climate of school district or the types of students with which a teacher works influence teachers' views of teaching and learning, greatly impacting how and what they teach.

Because these studies suggest that contextual factors are directly or indirectly related to teacher knowledge and practice, I intended to select participants who have different years of teaching experience and who work in diverse school districts. This helped me generate data on how general pre-k teachers understand mathematics and teaching mathematics. Also, the diversity of participants most likely provided widely varying cases of the phenomenon. This could deliver more conceptually dense and potentially useful data (Merriam, 1998).

The first step to recruit participants was to identify public pre-k teacher who was willing to participate in this research project. For this, I met with district representatives who helped me connect to public elementary school principals. Next, I arranged a meeting with principals and presented this research project in brief and requested their permission to contact pre-k teachers who worked in their school. Then, I sent the email to seven pre-k teachers and explained them what this research was about. Finally, I gained informed consent from five pre-k teachers who were willing to participate in this research. These participants exemplified the characteristics that include different years of teaching, different types of teaching certificate, and teaching experiences of various grade levels. These teachers agreed to allow me to access to their daily classroom teaching practices and were willing to engage reflectively about their practices. Below, I provide some brief details about each participant.

Anna Smith

Anna Smith, identified herself as an Asian-American woman, was in her early 30's. At the time of this study, she was in her 6th year of teaching. She had been teaching at Kathy Elementary for six years since she moved to central Texas several years ago. Kathy Elementary was in an urban part of the Wood Trail school district and received Exemplary TEA rating on TAKS test. Kathy Elementary was composed of 69% of White, 19% Hispanic, 5% African American, and 5% Asian students.

Anna had earned a certificate of Early Childhood Education Grades (PK-6) and English as a Second Language Supplemental Grades (PK-6). Anna had previously taught second grade for three years and she had been teaching pre-k for three years. She had a

Bachelor's degree in Business Administration and a Master's degree in Education. Anna's class had 13 students - 10 White, 2 African American, and 1 Asian.

Brooke Gonzalez

Brooke Gonzalez, identified herself as a Hispanic woman, was in her mid 30's. At the time of this study, she was in her 7th year of teaching, her 5th year teaching pre-k. She had been teaching at Johnson Elementary for six years. She had taught ESL students for one year in different elementary school before she moved to Johnson Elementary. Johnson Elementary was in an urban part of the Wood Trail school district and received Exemplary TEA rating on TAKS test. Johnson Elementary was composed of 30% of Hispanic, 40% of White, 20% of African American, and 7% of Asian students.

Brooke had earned a certificate of Early Childhood Education Grades (PK-6) and English as a Second Language Supplemental Grades (PK-6). Brooke had previously taught third grade for one year and ESL students for one year. She had a Bachelor's degree in Education. Brooke's class had 15 students-10 White, 3 Hispanic, 1 African American, and 1 Asian.

Claire Wilson

Claire Wilson identified herself as a White woman, was in her mid 40's. At the time of this study, she was in her 18th year of teaching, her 5th year teaching pre-k. She had been teaching at Hill Elementary for six years. Hill Elementary was in an urban part of the Wood Trail school district and received Exemplary TEA rating on TAKS test. Hill Elementary was composed of 53% of White, 19% of Hispanic, 18% of Asian, and 8% of African American students.

Claire had earned a certificate of Early Childhood Education Grades (PK-6), Elementary Self-Contained Grades (PK-6) and English as a Second Language

Supplemental Grades (PK-6). She had a bachelor's degree in English. Claire had previously taught first grade for 13 years and has taught pre-k for 5 years at the time of this study. She had participated 5 times in professional development in math area throughout her teaching career. Claire's class had 18 students-10 White, 3 Hispanic, 3 Asian, and 2 African American students.

Diane Green

Diane Green, who identified herself as a White woman, was in her mid 20's. At the time of this study, she was in her second year of teaching pre-k. She had been teaching at Southpark Elementary for two years. Southpark Elementary was in a suburban part of the Dominion school district and received Academically acceptable on TEA rating. Southpark Elementary has a total enrollment of 912 students with a student population consisting of: less than .3% of Asian, 11.40 % African-American, 61.73 % Hispanic and 26.54 % White. 10.31% of students qualify for Special Education services and 22 % of the student population is English Language Learners who receive instruction in a language other than English. 71.86 % of the student population are economically disadvantaged and qualify for free or reduced lunch.

Diane had earned a certificate of Early Childhood Education Grades (PK-6). Diane had one year of teaching experience in pre-k at Southpark Elementary and she was in second year of teaching pre-k at the time of this study. Diane's class had 18 students-10 White, 5 Hispanic, and 3 African American students.

Eva Martinez

Eva Martinez, identified herself as a Hispanic woman, was in her early 30's. At the time of this study, she was in her fifth year of teaching pre-k at Southpark Elementary, which is the same school as Diane's. She had earned Special Education

Grades (EC-12), Generalist Grades (EC-4), and English as a Second Language Supplemental Grades (EC-12). She had participated in two types of professional development in math area during her teaching career. Eva's class had 19 students-12 White, four Hispanic, and three African American.

The Researcher

Within the constructivist perspective understanding of teaching math, the researcher herself is a necessary part of understanding the research. I was drawn to this research question for two reasons. As a former preschool teacher, I felt that young children's mathematical learning is critical in terms of building basic knowledge and skills that are needed in elementary school mathematics. In addition, as a teacher educator, I believe that having deep knowledge in math and math instruction is significant for preschool teachers who must ensure their students' math development. While these experiences affect my lens for interpretation, every effort has been made to present these findings as accurately as possible.

FIELD ENTRY

Upon IRB and district approval, I officially contacted the teachers in the end of January 2010. I met with each teacher to establish a time to conduct interviews and begin my observations in her classroom and developed a time-line.

It was made clear to the teachers and other participants through oral and written communication that: (a) participation in the study was voluntary; (b) participants could withdraw at any time; (c) interviews, discussions, and audio/video tapes were kept confidential, and (d) participants were given pseudonyms to protect their confidentiality.

PHASES OF INQUIRY

The following sections outline a timeline for the study. There were three phases, with phase one beginning in the January of 2010, and phase three ending in the May of 2010.

Phase 1: Participant selection

Phase one began with the participant selection described above. Using the selection process, I identified five teachers who were willing to participate in the requirements of the research. Since the primary focus of this research was to examine general pre-k teachers' mathematical knowledge for teaching, teachers who have different years of teaching experience and work in different elementary school and school district were selected.

Phase 2: Field Entry and Engagement

Once all necessary consent was granted, two semi-structured interviews with each participant, classroom observations, and post-observation interviews were conducted in February of 2010 and ended in May of 2010. First, I began to interview each teacher twice. The first interview was to gather teachers' background information and to understand their general perspectives on effective math instruction (See Appendix A, The Initial Formal Interview Protocol). The second interview was to generate a rich, thick description of my research questions (See Appendix B, The Follow-up Interview Protocol) Then, I observed each teacher's math lessons and her center time 4-5 times. I also conducted post observation interviews after I observed the lesson with the teacher to discuss what I observed in their teaching (See Appendix C, Post Observation Interview Protocol). During this interview, I discussed how their math lesson went, what they did during their math lesson and why they did such things. These data acted as a part of the expansion of field notes.

All interviews were audio-taped and transcribed and most observations were recorded with field notes. During the last week of observation, videotaping was used in addition to field notes because it enabled a more complete record of math instruction. Taping was limited to two individual lessons. These videoed sessions focused on the practices of the pre-k teachers. Given this study's emphasis on the knowledge of math and math instruction, it was important to acknowledge teacher-student interactions, the way teachers present the math ideas, and the types of resources or math activities that teachers utilized. This required my field notes and videotape to capture the details of teaching practice and student responses. Thus, videotaping was used to analyze the data more completely.

I periodically shared my field notes and selected data with the teachers, for purposes of member checking (Lincoln & Guba, 1985). The teachers had access to all data as requested and discussions as to the accuracy of assertions and episodes were frequent. Each teacher's interpretations were considered as the data was collected.

Phase 3: Closure and Analysis

The final phase occurred at the end of spring semester of 2010 when the last post observation interviews were scheduled. In this interview, I discussed how they perceived this research project and what they learned through this research. Member checks were made with the teacher to evaluate the accuracy of my interpretations (Lincoln & Guba, 1985) and informal interviews conducted in attempts to identify any significant events that occurred in my absence. Each teacher's insights and interpretations were included in the data analysis and reported in the final product.

Data analysis procedures emerged from the data actually collected; however, some preliminary actions were presupposed regarding the data, as opposed to quantitative research with which the validity and reliability can be established by adherence to proper statistical procedures, qualitative research requires detail and depth such that the readers' trust is gained and the conclusions valid and reasonable. Because of this, some decisions about data collection and analysis were made on an ongoing basis.

DATA SOURCES: COLLECTION, DOCUMENTATION, AND RECORD-KEEPING

Although data collection methods have been mentioned briefly in the preceding section, in this section, I further elaborate on each of the methods used to gather data.

Interviews

Qualitative interviews provide a "way for [teachers] to explain their unique perspectives on the issues at hand" (Hatch, 2002, p. 23). For this reason, I conducted two formal interviews before my classroom observations and 4-5 post observation interviews with each participant. The first formal interview was designed for gathering background information about the teacher, general ideas of pre-k teachers' view on mathematics, their understanding of mathematical content they are to teach and their math instruction (see Appendix A, The Initial Formal Interview Protocol). Based on the first formal interview, I conducted the second formal interview with each participant to generate a rich, thick description of my research questions. The interview questions included the following elements: (a) teachers' understanding of math content, (b) teachers' understanding of how to teach specific math content (e.g., teaching addition concept), (c) teachers' understanding of the math curriculum and curriculum materials provided by state and school district, (d) teachers' understanding of "children's mathematics" (Empson & Junk, 2004)—how children understand mathematics content and solve the problems, and (e) teachers' understanding of developmentally appropriate and effective math instruction (see Appendix B, The Second Formal Interview Protocol). The interview questions were adapted from the four constructs articulated by Ball et al. (2008) (see Appendix D, Teacher Knowledge Constructs). I piloted these questions with my colleagues who have a teaching experience in pre-k before implementing the study in order to refine and adjust them to my goals with this research. These interviews were recorded and transcribed.

After each classroom observation, I conducted post observation interviews to discuss what I observed in their teaching (See Appendix C, Post Observation Interview Protocol). Each interview lasted for approximately 20-30 minutes. The questions included (a) their own reflections on the lesson, (b) the goal of the lesson and whether this goal was achieved or not, (c) their choice of math activities and materials, and (d) what students learned through the lesson. I also have them clarify behaviors that caused confusion. These interview questions, adapted from A Study Package for Examining and Tracking Changes in Teachers' Knowledge, published by NCRTL (1993), focused on participants' reflection on pedagogical strategies they implemented and the adjustments they made. Responses to such questions helped me better understand these teachers' mathematical knowledge for teaching such as specialized content knowledge, knowledge of content and students, and knowledge of content and teaching (Ball et al., 2008). In the last post observation interview, member checks were made to evaluate the accuracy of my interpretations and to identify any significant events that occurred in my absence. These interviews were also audio-taped and transcribed.

Classroom observation

To examine pre-k teachers' MKT that is played out in their actual math lessons and to understand their decision-making practice that occurred during their math lesson, I observed each teacher's math lessons 4 to 6 times for up to 3 hours at a time after I completed the two formal interviews. Direct observations recorded with field notes and videotape was an important data source for this dissertation. In each classroom, I observed, made field notes on the teachers' talk or behaviors that were relevant to enhancing students' mathematical thinking. I also closely watched small group problem-solving activities that occurred during the lesson to see how these teachers interacted with students and incorporated students' informal mathematical knowledge to more formal mathematics. In each classroom, I acted primarily as an observer rather than a participant during these focused times. I sat in an unobtrusive location in the classroom and took notes.

Observations were used to identify teachers' mathematical knowledge that was used in their actual teaching practice. This aided the developing understanding of teachers' mathematical knowledge. Expanding field notes allowed me to take extended time to provide detail and enhancement of notes taken during actual time spent in the classroom (Lincoln & Guba, 1985). Because the classroom environment is a place of intense movement and dynamic timing, it becomes important to get notes written quickly, often requiring shorthanded scripts of events which later need detail and depth to capture the meaning and details of the interaction and situation (Hatch, 2002). As such, when I left the classroom, I used my written notes and videotape to elaborate and expand on the situations and contexts and note any methodological, theoretical, or personal thoughts based on the observations that I had. As stated previously, I periodically shared these notes with the teachers for purposes of triangulation and member checking (Lincoln & Guba, 1985).

Related Documents

During the data collection process, there were documentation and physical artifacts available that added to the sources of evidence I could draw from for analysis and interpretation. The documents included such things as math lesson plans; state and district pre-k guidelines, math curriculum or materials (if any), and/or students' work.

DATA ANALYSIS

In constructivist research, "Data analysis is a systematic search for meaning" (Hatch, 2002, p.148). Meaning is not an absolute but context specific and constructed by those involved. Because meaning is so contextually bound, analysis was initiated shortly after data collection had begun and continued throughout the course of the study; this approach was beneficial because it allowed, "shaping the direction of future data collection based on what [I was] actual finding or not finding" (Hatch, 2002, p. 149). After each observation, videotapes were reviewed and field notes expanded. Throughout the analysis process, I recorded reflections and insights in analytic memos where I wrote down my thoughts and impressions and refocused my observations and analysis on my research question (Hatch, 2002). I generated memos as a means to provide "a running record of insights, hunches, hypotheses, discussions about the implications of codes, additional thoughts, whatnot" (Strauss, 1996, p.110). I used the collected data—including analytic memos—to develop descriptions, to engage in analysis, to create interpretations, identify patterns and themes, and discover relationships. Data was coded using both external and internal codes (Graue & Walsh, 1998; Hatch, 2002).

External codes came from my conceptual perspectives about this research project (Graue & Walsh, 1998, p. 163), namely Mathematical Knowledge for Teaching (MKT) (Ball et al., 2008). Initially, data were coded into four categories including 1) common content knowledge, which refers to "the mathematical knowledge and skill used in settings other than teaching" (Ball et al., 2008, p. 399), 2) specialized content knowledge, the mathematical knowledge and skill unique to teaching (Ball et al., 2008, p. 400), 3) knowledge of content and students, which refers to "knowledge that combines knowing about students and knowing about mathematics" (Ball et al., 2008, p. 401), and 4) knowledge of content and teaching, a combination of "knowing about teaching and knowing about mathematics" (Ball et al., 2008, p. 401). The following table (Table 3) addresses and operationalizes the constructs of pre-k teachers' MKT. The table includes each aspect of MKT (external codes), a sample list of teacher behaviors from my field note and teachers' responses to my interview questions.

Table 3-External Codes of Teachers' Mathematical Knowledge

Constructs	Observed classroom behaviors	Responses to interview questions
Common content knowledge	• Teachers recognize when their students give a wrong answer. • Teachers use terms or notations correctly or incorrectly.	• Teachers define the mathematical concepts described in the state pre-k guidelines.

Specialized content knowledge	• Presenting mathematical ideas
	    • Using real-world examples
	    • Making the lesson concrete
	    • Explaining mathematical goals
	• Connecting the content to students' later mathematical learning
	• Promoting students' mathematical thinking
	    • Asking productive mathematical questions
	    • Understanding the breadth and depth of early mathematics
	    • Integrating the big ideas in mathematics to teach a specific mathematical concept
	• Modifying the lesson
	    • Repeating or re-teaching
	    • Differentiating the resources
	    • Redesigning the activity
		• Teachers talk about how they present the mathematical ideas.
		    • Teachers talk about how they make their lesson concrete.
		    • Teachers talk about how they make their lesson fun and interesting.
		• Teachers talk about how pre-k math is connected to students' later learning
		• Teachers talk about how they promote students' mathematical thinking
		    • Teachers talk about the questions that they posed during the lesson
		    • Teachers talk about how pre-k math includes basic but deep mathematical ideas
		    • Teachers talk about how they integrate the big ideas in mathematics to teach a specific mathematical concept
		• Teachers talk about how they modify the lesson based on students' ability

Table 3, cont.

Knowledge of content and students	• Understanding students' common errors
	  • Teachers anticipate what students are likely to think and what they will find confusing.
	• Understanding students' mathematical ability
	  • Teachers provide mathematical activities that doable and challengeable to students.
	• Noticing/attending to students' mathematical thinking
	  • Teachers notice, respond to, and scaffold students' informal ways of solving the mathematical problems.
		• Teachers talk about students' common errors and how they correct those errors.
		• Teachers talk about what students can do and can't do.
		• Teachers talk about why their students have difficulty in completing the tasks or understanding a certain concepts.
		• Teachers talk about students' informal ways of counting or adding/subtracting.
		• Teachers talk about how they respond to students' informal ways of thinking

Knowledge of content and teaching	• Making instructional decisions
	  • Teachers sequence particular content for instruction.
	  • Teachers choose which examples to start with and which examples or materials to use to take students deeper into the content.
	• Differentiating instructional settings
	  • Teachers employ whole-group, small group, and individualized instruction.
		• Teachers talk about why they choose certain examples or materials to each a specific content.
		• Teachers talk about instructional advantages or disadvantages of the materials or planned math activities.
		• Teachers talk about the effective way to teach specific content.
		• Teachers talk about how they plan the math activities.
		• Teachers talk about how they differentiate instructional settings.
		• Teachers talk about how they organize the classroom in a way that can help students' mathematical learning.

Internal codes were developed through my reading of the data (Graue & Walsh, 1998, p. 163) and emerged as a common theme that was found in all of the participants. The themes, such as "concrete mathematics" (Clements, 1999) and "connecting the concrete and the abstract mathematics" were developed (Haylock & Cockburn, 1997). I used these themes to develop an analysis to shape and support the data.

Also, I analyzed the observation field notes and video transcripts based on these four external codes. I reviewed my field notes many times throughout the data analysis process. I expected to find suitable categories and themes that allowed me to establish a sense of the pre-k teachers' mathematical knowledge base. These were derived from the repeated description and reconsideration of emerging patterns from the observation field notes rather than from the imposition of predetermined categories (Aubrey, 1996). In addition, I sought multiple examples of a particular math practice, behavior, or belief. I also gave careful consideration to unique aspects of the teachers' work (Goldstein, 2007). Also, I compared the classroom field notes with tentative themes that emerged in interview data. I did this to look for any overlooked disconfirming evidence.

I also investigated teachers' lesson plans used when teaching math as well as the state's or district's pre-k mathematics guideline or other mathematics curriculum (if any). This enabled me to analyze two things: (a) how teachers' mathematical knowledge for teaching was tied to its use in practice and (b) how these teachers understood and incorporated the curriculum in a way that was learnable to 4-year-old children.

As analysis continued, I refined my initial categories to ensure that each one addressed the research questions and that all relevant data fit into one category only (Merriam, 1998). After a satisfactory list of categories was created, data was recoded. Each coded unit of data was organized by its code and stored in a unique file.

During the descriptive and analytic processes, I made interpretations about the data using interpretive techniques suggested by Wolcott (1994) including extending the analysis, turning to theory, and listening to gatekeepers and knowledgeable others. 'Extending the analysis' allowed me to avoid oversimplifying my findings and offered a comparative perspective that "raises doubts or questions not lightly dismissed" (Wolcott, 1994, p. 40). When 'turning to theory', I connected my interpretations to large issues, namely Mathematical knowledge for teaching (MKT). 'Listening to gatekeepers and more knowledgeable others', such as my advisors and other committee members, facilitated accurate and thorough interpretations based on the advice of more experienced researchers. Finally, I clearly explained connections between my interpretations and the collected data, descriptions, and analysis (Wolcott, 1994).

After an initial draft of my findings was developed—descriptions, analysis, and interpretations—I developed a document to share with the participants. As a way of member checking I asked the following questions: What stood out to you as you read the summary of findings? Did reading the summary raise any questions or concerns that you would like to share? Teachers' responses to these questions were used to evaluate and to make further revisions to the initial findings.

QUALITY AND RIGOR

In any research, it is important to ensure quality and to demonstrate its rigor. In quantitative studies, results are discussed in terms of reliability and validity, because of the qualitative nature of this study; I focused on strengthening its trustworthiness using measures of credibility, dependability, transferability, and confirmability (Hatch, 2002; Lincoln & Guba, 1985). In the following section, each test for quality and rigor will be discussed.

Trustworthiness/Credibility

According to Wolcott (2001) the notion of internal validity—that is, the extent to which researchers effectively measure their research variables (Merriam, 1998)—does not align with qualitative research in which the researcher focuses on understanding the issues—people, places, ideas—under study. Thus, qualitative research addresses instead credibility (Lincoln & Guba, 1985; Mertens, 1998). I used four strategies to increase the credibility of this study. First, I used data triangulation (Lincoln & Guba, 1985; Hatch, 2002), in which multiple sources of data, such as field notes, video tape and teacher interviews were used to create and to support descriptions, analyses, and interpretations;

Second, prolonged engagement was made (Hatch, 2002). Third, I conducted member checking, in which interview transcripts, expanded field notes and initial research findings were reviewed by participants for accuracy (Lincoln & Guba, 1985; Hatch, 2002). For instance, I clarified emerging themes with participants and each had the opportunity to read his/her individual case study and the findings (See Appendix E). Lastly, I engaged in peer debriefing in which descriptions, analyses, and interpretations were discussed with colleagues and advisors (Lincoln & Guba, 1985; Mertens, 1998). For example, I met regularly with a group of fellow students and our advisor and discussed my emerging analysis and several drafts of the dissertation in progress.

Transferability

Transferability refers to the extent that the results are applicable to individuals beyond those who were sampled (Lincoln & Guba, 2005). As described before, participants were chosen through a process of theoretical sampling to represent general pre-k teachers' mathematical knowledge for teaching. As discussed, the sample differs in teaching experience, background and situation. As a result, this study, although case study in nature, and thus more individual than large-scale studies, has potential for transferability to other teachers based on similar experiences and situations. Limitations presented in the sample chosen, and thus important to the applicability of the study's findings were explored in the limitations section.

Dependability

Dependability is concerned with the replicability of a study to the end of similar findings. Dependability was established by providing the reader with evidence of the findings such that if it were to be repeated, it would end in similar results. One way to enhance dependability is through creation of an audit trail (Wideen et al., 1998). This record of the process of data collection and analysis included raw data (interview transcripts, observation transcripts, field notes, and documents), data reduction and analysis products (such as coding pages and hypothesis generation notes), synthesis pages (analysis sheets, concept maps) and process notes (journals). A written history that tracked changes that occurred both in the setting as well as other important features enhances the dependability of this study.

Confirmability

Confirmability in a qualitative study involves seeking feedback from others about the hypotheses generated from the data. It was established by showing that the data, rather than the researcher, were confirmable representations of the participants (Denzin & Lincoln, 2005). This study safeguarded confirmability through an audit trail, triangulation of results, peer debriefing and member checking. In addition, case reporting provided excerpts of "raw data" to illustrate assertions. These safeguards increased the confirmability of this study by providing the reader with access to actual data such that they may draw their own conclusions or further align their thoughts with those of the researcher about the accuracy of the representation. Case reporting, which included such samples of raw data, further supported interpretations.

Ethics

Information gathered from this research was shared with all teachers for member checking and triangulation of data, members of my doctoral committee, my learning community, in professional meetings, and/or publications. The data may undergo further analysis by me in the future. Inconveniences of the study were mainly to the involved teachers. The only risk to participants was the loss of confidentiality, and I have made every effort to safeguard the anonymity of all participants in this study.

Pseudonyms were assigned to all participants, district and school in this study and used in all written products of the research, including all reports and transcriptions. When students' names were on documents that were collected, pseudonyms were substituted when they were photocopied. Identifying information about the schools and district was approximated to further protect confidentiality. Tapes will not be shared publicly.

In conclusion, this dissertation used a constructivist paradigm and case study methodology to further develop our understanding of pre-k teachers' mathematical knowledge for teaching. This research will broaden the discourse on how early childhood teachers understand both math content that they are to teach and math instruction they implemented. In addition, this research will highlight what mathematical knowledge that early childhood teachers now have and further need.

FINDINGS

Chapter 4: Findings

The purpose of this study was to examine pre-kindergarten (pre-k) teachers' knowledge of mathematics and how they teach math to young children. In this chapter, I present the findings drawn from three semi-structured interviews and the observations of five pre-k teachers' math lessons. This study examined three questions: (a) How do pre-k teachers understand the mathematical content they are to teach? (b) How do these teachers see themselves teaching this content? And, (c) How do they reflect on their teaching some of this mathematical content to their students? The first research question involves teachers' understanding of broad strands of math's "big ideas" (e.g., number and operations, measurement, pattern, geometry, algebra), which are major components of pre-k math, and the interconnections among such ideas. It also involves their understanding of how these ideas are the foundation for their students' later mathematical learning. The second question investigates teachers' understanding of their math instruction and of how to teach pre-k mathematics. Lastly, the third question involves these teachers' decision-making processes during the lesson. Their reflections were prompted by questions from my observations of their teaching and on their own reflection of their teaching, which were done through having them observe a video of their teaching.

The data of this study were analyzed using Ball and her colleague's (2008) theoretical framework of mathematical knowledge for teaching. Ball and her colleagues developed four subdomains of the teacher knowledge suggested by Shulman (1986). These subdomains involve: (a) common content knowledge (CCK)-the mathematical knowledge that all educated adults should have, (b) specialized content knowledge

(SCK)- the mathematical knowledge and skill that is required for teaching, (c) knowledge of content and student (KCS)-“knowledge that combines knowing about students and knowing about mathematics,” and (d) knowledge of content and teaching (KCT)-a combination of “knowing about teaching and knowing about mathematics” (Ball et al., 2008, p. 401). To analyze the findings of the first research question, CCK and SCK were used. To analyze the second research question, SCK and KCS were used. And to analyze the third research question, KCT was used.

This chapter is divided into two parts. First, I examine these pre-k teachers' understanding of mathematical content that they are to teach. This includes two sub-categories: (a) pre-k teachers' common content knowledge and (b) pre-k teachers' specialized content knowledge. These types of knowledge inform how they understand the mathematical content that they are to teach, which is associated with Shulman's (1986) concept of content knowledge as well as my first and second research question. Second, I investigate these pre-k teachers' knowledge of how to teach mathematics. This knowledge also includes two sub-categories: (a) pre-k teachers' knowledge of content and students, which is related to my second research question and (b) pre-k teachers' knowledge of content and teaching, which is associated with my third research question. These types of knowledge inform how they understand teaching math, which is connected with their pedagogical content knowledge (Shulman, 1986).

PRE-K TEACHERS' KNOWLEDGE OF MATHEMATICAL CONTENT

In the research literature on teaching and teacher education, there is a shared understanding that a teacher's knowledge of the subject taught is a core component of her competence and of her effective teaching of that content (Mewborn, 2003; Grossman, 1990). The major theme emerging from the research on teaching mathematics is that both the quality of instruction and of students' learning are significantly correlated with a teacher's deep understanding of the subject matter (Lampert, 2001; Ma, 1999; Hill et al., 2005). Given the widespread agreement that enhancement of mathematical content knowledge can lead to higher-quality instruction, this section uses Ball and her colleagues' (2008) framework of common content knowledge (CCK) and specialized content knowledge (SCK) to address how pre-k teachers understand mathematical content they are responsible for teaching.

Common content knowledge (CCK)

Common content knowledge refers to "the mathematical knowledge and skill used in settings other than teaching" (Ball et al., 2008, p. 399). This involves identifying a right or wrong answer, using correct terms or notation, and doing the work or simple computation they assign their students. To examine pre-k teachers' CCK, I posed questions such as, "Would you describe for me the differences between a circle, triangle, square, and rectangle?" Identifying the characteristics of these shapes is the basic mathematical content described in the geometry sections of both Texas pre-k guidelines and NCTM's Principles and Standards for School Mathematics (PSSM). I also asked, "Would you describe for me the ways to measure length, weight, and volume?" Such a task is also related to PSSM's statement that pre-k children are to be able to recognize and compare the attributes such as "length, volume, weight, area, and time" (Clements, 2004). These questions required that a teacher knows that a square is a rectangle, a circle is a round figure whose boundary consists of points equidistant from the center, and that length is the extent of something from end to end which can be measured by a ruler. These questions required no specialized understanding. They could typically be answered by others who know mathematics.

The findings showed that the participants differed in their understanding of the core mathematical concepts covered in pre-k. For example, I asked each participant to define circle and their definitions of circle varied. Mathematicians define a circle as "a shape consisting of a curved line completely surrounding an area. Every part of the line is the same distance" (Mathwords, 2011). With this definition in mind, consider the following description by Anna. "A circle has no sides. It has no angles. It's a closed curve and it has the same distance from every part" (Second interview, 2/25/2010). This response successfully captures the key components of circle. Specifically, the participant addressed the concept of center and diameter. When a teacher's response was insufficient, it omitted some essential elements of a circle. For example, Diane said a circle is "a round shape" (Second interview, 3/10/2010). This description is too general. It misses the core characteristics of a circle such as a center and a diameter, and it fails to recognize that many other shapes meet that definition (e.g., oval). This indicates that Diane's CCK about circle is limited to the general feature of circle.

I also asked these participants to describe length, weight, and volume and to describe what tools could be used to measure them. Pre-k teachers must have this knowledge to teach the measurement section described in both Texas pre-k guidelines and district guidelines. Anna, Claire, and Eva answered accurately. For example, Eva said that length is "how long something is," which could be measured by "a ruler or other non-standard tools such as unifix cubes or paper clips." She described weight as "how heavy or light something is," which could be measured by "a scale." Eva described volume as "the amount of space that something occupies," which could be measured by "using cups and fill in with sand or water" (Second interview, 3/10/2010). Other participants, such as Brooke and Dianne, were unable to provide a clear definition of volume. Brooke stated, "Oh, gosh, that one is hard. The volume of something. I'm thinking we don't do that one in pre-k; I mean, not really. Volume? Remind me of volume. Is that how deep something is? (Second interview, 3/8/2010). Diane noted, "In pre-k, we don't really measure volume at all. Gosh, I don't know if I could tell you how to measure volume in general (Second interview, 3/10/2010). Neither of these participants could give the meaning of volume because their district guidelines do not address the concept. Texas pre-k guidelines don't directly address the concept of volume either. They do, however, address the concept of capacity. For example, in the measurement section, there is a clear goal stating that a pre-k child "compares the amount of space occupied by objects; demonstrates capacity by using sand and water; demonstrates capacity of container by size" (p. 91). Measuring and comparing the capacity is a basic skill that helps pre-kindergarteners understand the concept of volume. Since I didn't question the participants about capacity in particular, it's uncertain whether they could have defined it or not. Texas pre-k guidelines encourage teachers to introduce the concept of volume by measuring and comparing the capacity of different kinds of containers. Nevertheless, teachers such as Diane and Brooke did not go into detail in their descriptions of volume.

Teacher's understanding of subject matter has been considered as an important element that determines the quality of instruction (Lampert, 2001; Ma, 1999). In this study, however, the relationship between these teachers' CCK and the quality of their teaching was uncertain. Based on my observational field notes, not all of these teachers' CCK came into play in their actual math lessons. For example, Anna was able to provide a clear definition of circle, but in her geometry lesson she never engaged her CCK of circle (Field note, 2/20/2010). Anna addressed the general feature of circle rather than the essential characteristics of circle such as the concept of diameter. She just introduced to her students notion that a "circle is a round shape" to her students. On the other hand, Brooke, having clearly defined length and its measurement, engaged her CCK of length when she taught it (Field note, 4/15/2010). She provided her students accurate descriptions of what length represents and of how it can be measured (Field note, 4/15/2010). Evidence of other teachers' CCK in their lessons was not found. None of their math lessons involving the concept (e.g., shape, measurement) I asked about displayed their CCK. Since only Brooke's case indicates a positive correlation between her CCK and her math instruction, it is unclear that the differences among these pre-k teachers' CCK had direct or indirect effects on their quality of teaching.

In sum, this finding suggests that, within a school district or across school districts, pre-k teachers' CCK might differ substantially. This supports many other studies that investigate teachers' subject-matter knowledge (Calderhead, 1996). Researchers have shown than teachers' background characteristics–their years of teaching experience, the number of math courses taken, the experience of teacher training programs related to math instruction—have either direct or indirect influence on teachers' content knowledge, instructional beliefs, and practice (e.g., Wilkins, 2008).

In this study, the difference among pre-k teachers' CCK was associated with years of teaching and whether or not they had experience teaching at other grade levels. Teachers who had taught at other grade levels and whose years of teaching pre-k were between three and five years tended to describe the specific mathematical concepts more accurately than teachers with teaching experience of less than three years in pre-k or who had, more than five years, taught pre-k solely. Such differences, however, did not come into play in their actual math lessons leaving uncertain the relationship between pre-k teachers' CCK and their math instruction. These findings provide teacher educators with insight into how pre-k teachers' background characteristics as well as their training experience are associated with the mathematical content knowledge that these teachers now have and still need. Also, it suggests a need of study that investigates whether pre-k teachers' CCK influences their quality of math instruction.

Specialized content knowledge (SCK)

Ball and colleagues (2008) define specialized content knowledge as, "The mathematical knowledge and skill unique to teaching" (p. 400). This domain focuses on the knowledge needed to teach mathematics. Tasks related to SCK, for example, include "finding and selecting appropriate examples or resources to present mathematical ideas, connecting a topic being taught to topics from prior or future years, modifying tasks to be either easier or harder, and asking productive mathematical questions etc" (Ball et al., 2008, p. 390). In pre-k classrooms, teachers' SCK is associated with such tasks as modeling the basic mathematical concepts by providing appropriate math activities, asking questions that promote students' mathematical thinking, and knowing how pre-k math is the foundation for students' later learning. To examine pre-k teachers' SCK, I asked them a number of questions, including: Describe how the big ideas of mathematics taught in pre-k such as pattern, geometry, and number are related to students' later mathematical learning? These concepts are some of the biggest components of pre-k math and they are also described in state pre-k guidelines. How do you model simple addition concepts such as 1 + 2 = 3? And, how do you modify the mathematical activity based on students' ability? These are mental tasks that teachers do everyday to make a particular mathematical concept visible to and learnable for students. How pre-k teachers perform these tasks inform how they understand and teach their students this mathematical content, which is related to my first and second research questions. In addition, by observing their math lessons I could identify such knowledge as it was presented in their actual teaching practice.

My interview and observational data indicated that these pre-k teachers' specialized content knowledge was exemplified in their following of four mathematical tasks. The tasks were: understanding how the specific math content is connected to students' later mathematical learning, promoting students' mathematical thinking, presenting mathematical ideas, and modifying the lesson based on students' ability. In the following sections, I will describe each of these in detail.

Connecting the content to later mathematical learning

It's important that teachers know how mathematical content is connected to students' later learning. It's important in terms of understanding the logical and consistent order for teaching mathematical content from one grade level to the next (citation). The findings showed that among these five pre-k teachers the extent of such knowledge varied. I asked each participant how the idea of patterns was connected to children's later mathematical learning. Patterns, which most preschool educators would consider appropriate for preschoolers to learn, refer to "an underlying rule or concept" and describes "a regularity that determines, explains or predicts observed phenomena" (Ginsburg & Ertle, 2008, p. 51). The ability to recognize and analyze a pattern provides a foundation for the development of young children's algebraic thinking. It allows children to recognize the order of data and to make predictions and generalizations beyond the available information (Clements, 2004). Keeping this definition in mind, consider the following descriptions given by Anna.

> Patterning is very important. One of the things that they do in second grade is skipping cardinal numbers. That's a pattern and it ties right into when they're doing their multiplication or when they're doing their simple addition like doubles. So then your pattern is five, five, ten. Ten, ten, twenty. . . . You can see many things that have patterns. It helps their minds categorize it, like, "Okay, I can see how these things are related" and then be able to adapt it to more difficult

> subjects. Also, they can see that there is stuff that repeats that they could be looking for in life (Second interview, 2/25/2010).

Anna says why a pattern is important and details how it is connected to children's later mathematical learning. She mentions, for instance, that a pattern, "ties right into when they're doing their multiplication or when they're doing their simple addition like doubles." She recognized that patterns can be a basic mathematical concept that can aid specifically in understanding the concept of addition and multiplication. Moreover, she noted that patterns help children understand how things are related, can be categorized, and are repeated, all of which are the basics of algebraic thinking.

Other responses were not as detailed as to why patterns are an important concept in children's later mathematical learning. Although other participants, such as Brooke and Diane, had a shared understanding of the importance of pattern, they could not detail why such a concept was important. Consider the following description of patterns by these two teachers.

> Obviously they're going to hear about [patterns] and do them throughout the rest of their school years, obviously throughout college. I think it helps them see how things go in order and how things are repeated over and over again. It's okay that it's repeated because when they're this young, they don't get that concept (Brooke, second interview, 3/8/2010).

Brooke noted that patterns indicate "how things go in order" and "how things are repeated over and over again." She was able to capture the core characteristic of pattern—"regularity." She failed to mention, however, details of why patterns are important and foundational for later mathematical learning.

> I think patterns go on with everything. I mean there's day and night patterns. There's blue girl patterns. There's color patterns. There's different patterns that you develop every single day. There's number patterns. The kids fill in the blank number patterns. So I think starting with them when they're this young—they're developing and understanding what a pattern is—so as they grow and it gets

> harder, they understand the strategies of how to make a pattern and do a pattern (Diane, second interview, 3/10/2010).

Diane stated the various types of patterns that can be seen in our daily lives. However, she did not provide any description of what pattern is and how it is related to students' later learning. These two responses addressed the general features of a pattern. Yet, they omitted details of why patterns are important and how they connect with children's further mathematical learning. These two participants' statements leave open the question of whether they fully understood the concept of patterns. They both emphasized the importance of pattern but went into no detail as to why such concept was important. Clements (2004) wrote that a pattern is a basic concept in promoting children's algebraic thinking. Understanding patterns is also a foundational step later on for learning an important mathematical idea—linear function (Ginsburg & Ertle, 2008). Anna's response was consistent with these scholars' statements. Brooke's and Dianne's responses encompassed only the general idea of pattern and made no specific connections to children's later mathematical learning.

Unlike the other participants, Eva connected the concept of pattern directly to what students would need to know on the TAKS tests.

> I know they ask about patterns in TAKS questions, which comes next. They'll show them a bunch of shapes and it might not be as simple as A-B patterns, but that's where we need to start here in Pre-K—the basics. And I know, they ask those kinds of questions because we've learned them. We've brought questions that they have on the TAKS about patterns. And there are word problems, problem-solving skills that they'll need to use their patterns and, well, just the wording, the vocabulary of patterns. I know it does help them later on the TAKS (Second interview, 3/10/2010).

Eva's description indicates that teaching patterns in pre-k is important because it can help students succeed on their TAKS tests in elementary school. Although she had never taught other grades, Eva was familiar with TAKS questions through her third grade teacher friends.

> They often say how stressed out their children are and my friends get sick because they're also stressed out with just the TAKS…So, I think I should put in more lessons on certain areas or make the majority of stuff on the TAKS grade-appropriate for pre-k. I mean pre-k is so far from third grade, but they will take that test someday (First interview, 3/3/2010).

Eva recognized that pre-k mathematics was the starting point for preparing children to master mandated elementary school mathematics skills. Her awareness might have been a product of the "trickle down effect of the concern for academics in the early grades" (Saracho & Spodek, 2009, p. 305), which has grown out of the K-12 standards-based accountability (SBA) reform movement (e.g., 2002's No Child Left Behind Act). Although pre-k was not included in NCLB's call for standards and accountability (Goldstein, 2007), Eva's response reveals the curricular and instructional requirements associated with this policy. In such a high-stakes, standards-based education system, pre-k teachers are expected to prepare their students for schooling (Ginsburg & Ertle, 2008). They must begin preparing children to master mandated mathematics skills, skills that will extend through 12th grade (Hatch, 2002; Stipek, 2006). Eva's understanding of patterns and her commitment to teach this content to her pre-kindergarteners had clearly been shaped by current SBA reforms.

In sum, the ability to understand the basic math concepts, the interconnections among such concepts, and their relation to children's further mathematical learning are important components of SCK. The above interview data suggests that differing substantially were these pre-k teachers' SCK, specifically in terms of their notions of patterns. Most participants recognized the importance of teaching patterns in pre-k. What varied was their descriptions of why it was important and how it was connected to students' later mathematical learning. Teachers such as Anna could associate the pattern with other mathematical concepts that were taught in other grades (e.g., addition and multiplication); she understood how this often-taught concept to prekindergarteners was basic but still quite deep. Other teachers' responses, such as Brooke's and Diane's, were vague, lacking the specificity to describe why the pattern was an essential concept in students' later mathematical learning. The lack of such knowledge could have direct and indirect effects on teaching practices (Brophy, 1992). As Brophy explained, when teachers' content knowledge is more explicit, they will tend to teach the subject more fully and respond to students' questions more sensitively. In addition, the findings showed that pre-k teachers' knowledge of big math ideas being taught in pre-k and the relation of those ideas to topics in future years could be influenced by current SBA reforms. This suggests that educational contexts (e.g., SBA reforms) can affect features of teacher knowledge on how math concepts being taught now relate to students' later learning.

Promoting students' mathematical thinking

Many researchers in math education have emphasized the importance of focusing on students' mathematical thinking and reasoning (e.g., Ginsburg & Amit, 2008; NCTM, 2000; Russell, 1999, NCTM, 2000). Most professional development programs in mathematics have also been made it a key goal (e.g., Cognitively Guided Instruction). According to Russell (1999), deep mathematical thinking helps students develop, justify, and use mathematical generalizations. This leads to interconnections between aspects of mathematical knowledge, which is the foundation that provides insight into mathematical problems. Before formal schooling, young children are capable of learning complex mathematics and deal spontaneously with mathematical ideas (Ginsburg & Amit, 2008).

Related to SCK is a teacher's ability to assist pre-k students in developing a deeper understanding of mathematics. These teachers' SCK allows them to be inventive in creating worthwhile opportunities for learning that take into account learner's mathematical thinking, experiences, and needs (Ball & Bass, 2000).

All of the five pre-k teachers in this study tried to promote their students' mathematical thinking by asking productive mathematical questions during their whole-group or small-group math lessons. The following vignette of Brooke's math lesson illustrates how her SCK was put into use in her practice. In this lesson, students were to learn the concept of length, which the pre-k guidelines define as recognizing and comparing heights or lengths of people or objects (TEA, 2008). In her district's math guidelines, this objective is also described in the measurement domain. This lesson was Brooke's first time introducing the concept of length and how to measure it.

> During the whole-group math lesson, Brooke brings the ruler and measuring tape to the carpet where all of the children are sitting. She addresses the group with a question.
>
> **Brooke**: Does anyone of you know what it means to measure something? What is measuring?
> **Children**: [Several students raise their hands and share their own experiences about measuring].
> A girl says that she has been measured at the amusement park (e.g., Six Flags) to see whether her height was enough to ride certain types of roller coasters. Since most students have had similar experiences, they were very active in sharing their experiences with one another. Then, Brooke called one girl up to be measured. Brooke used a yard stick, but the girl was more than three feet tall. Brooke continued to discuss.
> **Brooke**: So, if she is taller than this ruler, I think she is able to ride the roller coaster. What did I use to measure how tall she was? I remembered somebody said the …..
> **Children**: Ruler!
> **Brooke**: Yes, it's a ruler. Look. What do you see on this ruler?
> **Children**: Numbers.
> **Brooke**: What is a ruler for? When do we use it?

Children: To measure something! [The students talked a lot about their experiences at home of using a ruler].
Brooke: Then, what's this [showing the measuring tape]? It's a measuring tape. Who uses this?
Children: The construction workers.
Brooke: How do they use it?
Child 1: When they make something with rocks or blocks….
Child 2: My dad uses this too.
Brooke: For what?
Child 2: [He explains how his dad uses it].
Brooke: Now, I am going to measure how tall we are. Come Sara! (Pseudonym) Because she is the smallest in our class, I will measure her height first. How do I measure her?
Child: From the bottom ~
Brooke: So, I start at the bottom. Where should I stop?
Children: At the top of her head.
Brooke: Then, what do I do?
Children: That's it!
Brooke: That's it? Well, to know how tall she is, I need to see the number here. It's 42 inches.
Children: Wow~! 42!
Brooke: [Brooke writes 42 on the board] Sara is 42 inches. Who do you think is taller than Sara?
Children: Me, me!
Brooke: Let's try Jane. Sara was 42. What do you think Jane is?
Child 1: 47
Child 2: 49!
Brooke: Good guess. It's 49 inches. [She writes this on the board]. So, who's taller?
Child 3: Jane.
Brooke: Why do you think she is taller?
Child 3: Because her number is bigger than Sara's.

Then, Brooke calls another boy to measure. The students loved being measured and they were all engaged and involved in this lesson. The boy's height was 48.
Brooke: Who is the smallest?
Children: Sara.
Brooke: Why?
Child 4: Because she has a smaller number.
Brooke: Then, who is the tallest?
Child 5: Jane.
Brooke: Why?

Child 5: Because nine is bigger than eight. One, two, three [by using his hand, he indicates that 42 is less than and 49 more than 48].
Brooke: So Sara is the smallest, Jane is the tallest and Mike is…
Children: Medium.
Brooke: Very good!

Brooke then measures the students' feet by using unifix cubes. Brooke calls on a boy and measures his foot.
Brooke: This time, I am not going to use a ruler or measuring tape. I'm using unifix cubes. How many cubes are here? Let's count!
Children: One, two…. eight.
Brooke: Whose foot is bigger than Carter's?
Children: Me, me!
Brooke: Kathy, come here. How many cubes do you think we need to measure Kathy's foot?
Children: [Guessing] Nine. Five…
Brooke: [She measured her foot with unifix cubes] Let's count.
Children: One, two . . . nine.
Brooke: Here's Carter's cubes and here's Kathy's cubes. Who has a bigger foot?
Children: Kathy!
Brooke: Why do you think Kathy's foot is bigger?
Children: Because they [the cubes] are longer.
Brooke: How many more unifix cubes does Kathy' foot have?
Children: One more.

Brooke began her lesson by asking students what measurement means. The students actively shared their experience of measuring things in real life, allowing them to think about what they already knew about measurement and to get a general understanding of what measuring something means. Next, Brooke called on a girl to demonstrate how to measure height using a ruler. She asked the students what a ruler was and when it was used. Again, the students began talking of their own experiences using a ruler. Then, Brooke discussed how to use a ruler and how it tells height. She measured two other students' heights and asked questions about who was taller and who was shorter.

Brooke also showed how to measure students' feet by using a non-standard tool, unifix cubes. While measuring, she asked many "why" and "how" questions: "Why do you think she is taller than her?" "How many cubes do you think we need to measure

Kathy's foot?" Pondering such questions is where students' deep mathematical thinking can occur. First, they can learn what measurement means and how to measure something. Second, they have to think about what each number represents from the ruler and unifix cubes. Lastly, they can make perceptual judgments of relative quantities by comparing students' heights and feet. This would suggest that Brooke's questions promoted students' mathematical thinking in regards to measurement as well as counting and comparison. By engaging in this kind of thinking process, children can learn about the significance of the results of measuring and counting. They can mentally construct that the same number implies the same quantities and figure out how many more (fewer) there are in one collection in another. This is a fundamental process of learning abstract mathematics such as solving comparison problems or learning arithmetic skills (e.g., adding to/taking away skills; Sarama & Clements, 2008).

Brooke's lesson showed the important components of SCK that help students engage in mathematical thinking processes. This is consistent with what Ball and her colleagues have suggested. Finding an example to make a specific mathematical point, presenting mathematical ideas (e.g., measurement, comparison), and asking productive mathematical questions (Ball et al., 2008) were all part of Brooke's lesson.

In addition, Brooke also understood the breadth and depth of early mathematics (Ginsburg & Ertle, 2008), an important, must-have ability of pre-k teachers. Her goal was to introduce the general meaning of measurement but she was careful to integrate other basic mathematical skills such as counting, identifying numbers, and comparing. She recognized that to teach the measurement concept, she needed to address the broad strands of big mathematical ideas as well as to understand the depth and complexity that underlie these ideas. Her post observation interview illustrated this point.

> My goal was for them to get a general understanding of what measurement is. A very general concept for them is to understand why we measure things and how we measure things. That was my focus, very general. I threw in the inches and higher numbers, but that's just because what's on the ruler. But I know it's too hard for them. They still won't understand that but I think they might get just a general idea of why we measure, what we use to measure. Other than that, they were able to use counting skills because they have to know how to count the unifix cubes. It would be great to have number identification. Also, I wanted them to compare their height and feet. These skills seem to be very basic but it requires abstract thinking, which can be a hard and complex process for these kids. And, well, I mean it would be great if they would have some skills about knowing measurement and that's why I did the lesson-to teach it to them (Post observation interview, 4/15/2010).

This interview describes how Brooke's intent was to promote students' mathematical thinking. She tried to encourage her students to practice counting, number identification, and comparing the quantities while teaching measurement. This indicates that playing an important role in her developing this lesson was her SCK—knowing what Sarama and Clements (2008) have suggested.

> Numbers can be used to tell us how many, describe order, and measure and that comparing and measuring can be used to specify how much of an attribute (e.g., length) object possess, and that measures can be determined by repeating a unit or using a tool (p. 70).

Brooke's SCK includes such components as what numbers represent (e.g., the meaning and the role of number), how numbers can be used in measurement, and how the measures of objects can be determined. These components are all interconnected and such interconnected knowledge was put into use in her measurement lesson.

From Brooke's math lesson, I compiled a list of the features of SCK that are needed to promote students' mathematical thinking. This type of knowledge involves knowing the breadth and depth of basic mathematics (e.g., number, counting, comparison, and measurement; Ginsburg & Ertle, 2008), integrating the big ideas in mathematics to teach a specific mathematical concept, and asking productive mathematical questions (Ball et al., 2008). Brooke put this knowledge to use in her actual teaching practice, which promoted students' engagement with deep mathematical thinking. In addition, she successfully met the goals that are described in the state and district's guidelines. For instance, she was able to meet "recognizing and comparing heights or lengths of people or objects" and "recognizing how much can be placed within an object" (TEA, 2008).

As with Brooke's, the SCK of other teachers, such as Anna's, Claire's, and Eva's, emerged in their math lessons. For instance, Claire asked many "why" and "how" questions such as "Why do think this number is bigger than that?" or "How many blocks do you have and how many more blocks do you need to have ten?" These questions enabled her students to think about the number concepts more deeply (Field note, 3/24/2010). Anna integrated various types of mathematical skills to teach a specific concept. She taught counting skills by encouraging students to count and identify the number of different kinds of shapes (Field note, 3/30/2010). The ability to integrate various types of mathematical skills when teaching a basic concept (e.g., using different types of shapes to teach counting skills) requires knowing the interconnections among such concepts, which is related to SCK. Although the teachers differed in how they did such tasks, they all tried to promote students' mathematical thinking.

All of Diane's math lessons, however, differed from those of her counterparts. A second year pre-k teacher, she enjoyed teaching math and was very energetic in preparing her lessons. She prepared many attractive manipulatives such as farm animals and real-world objects; she developed fun math activities and interacted individually with her students, especially those who were struggling (Field note, 3/3/2010). However, in her lessons observed, I did not notice Diane extending, promoting, or challenging students' mathematical thinking. This point is illustrated in the following vignette of Diane's math lesson on patterns. In this lesson, students were to learn the concept of patterns, which the pre-k guidelines states as recognizing and creating patterns (TEA, 2008). This lesson was the second time Diane had done a pattern activity. She said her goal in this lesson was to have her students create various kinds of patterns (Post observation interview, 3/31/2010).

> Diane and a small group of children are sitting around the table. Other children are playing in the center.
>
> **Diane**: We are going to make a pattern with the farm animals first. What does a pattern do?
> **Children**: Repeat itself.
> **Diane**: That's right. We can make a pattern by the color, shape …
>
> She gives an example of a pattern with a color. She starts with a yellow cow and then a green pig and creates a yellow-green-yellow-green pattern. Then, she creates a pattern using types of animals. She puts, one after the other, a red horse, yellow cow, green horse, and red cow to make a horse-cow-horse-cow pattern.
>
> **Diane**: Today, we're going to make a pattern by these animals [she puts down a duck, a horse, a duck, a horse]. What animal should come next?
> **Children**: A duck.
> **Diane**: Now, it's your turn to make a pattern.
>
> The students then begin to create patterns by themselves. Most students created their patterns by color rather than by type of animal. One boy, however, just puts together several purple animals and one green animal.
>
> **Diane**: Jason, is this a pattern?
> **Jason**: Yes.
> **Diane**: Well, you put all the purples here. I can't see any pattern. We need to fix it. You can make a pattern by the color or the types of animals.
> **Jason**: [He chooses a purple horse first and a yellow horse next].
> **Diane**: Okay. So which one will be the next?
> **Jason**: [He grabs a purple horse and put it right next to the yellow one].
> **Diane**: So, you are making a pattern by the color. Purple, yellow, purple, yellow…Good job.

In this vignette, Diane began her lesson by asking what patterns meant. Then, she asked her students to create patterns using farm animals. Jason failed to make a pattern. Diane informed him of this and asked him to make it again according to either the color or type of animal. Then, Jason created a pattern by color. Rather than allowing Jason to express his own thoughts, Diane pointed out his mistake directly to him and gave him explicit guidance. She interacted individually with her students, but her interaction was limited to explaining, correcting, and praising (Field note, 3/31/2010). She neither posed meaningful questions nor extended and challenged students' mathematical thinking. Her focus seemed to be on successfully completing the planned patterns activity. She stated in her first interview, "I don't want anybody to fall behind. I want them to feel successful" (3/3/2010). She seemed preoccupied with the type of activities in which students were engaged rather than with students' opportunities for reasoning and thinking. This example shows how Diane led the lesson and interacted with the students. She was satisfied with this lesson because she thought she had met the goal by having most students successfully create patterns (Post observation interview, 3/31/2010). Yet she gave only limited space for her students to think deeply on the subject matter. In Diane's lesson, students created patterns without understanding what a pattern actually meant. Having students successfully create the pattern seems meaningless if they knew nothing about what they were doing.

Teachers' SCK helps them promote students' mathematical thinking and reasoning. Four of the five pre-k teachers in this study demonstrated such knowledge, which involves knowing the breadth and depth of basic mathematics (e.g., numbers, counting, comparison, and measurement; Ginsburg & Ertle, 2008), integrating the big ideas in mathematics to teach a specific mathematical concept, and asking productive mathematical questions (Ball et al., 2008).

Presenting mathematical ideas

One of the most important tasks when teaching math to young children is to make the mathematical concepts visible to and learnable by students (Kilpatrick et al., 2004). Having SCK helps pre-k teachers present the specific mathematical ideas in an appropriate way for pre-kindergartners. To investigate this aspect of SCK, I asked the teachers such questions as, "Suppose you are teaching a simple addition problem, say 1 + 2 = 3. Where do you begin? What do you think is the best way to teach it?" I asked these questions because teaching addition and subtraction skills to pre-kindergartners requires teachers to have a strong SCK base. To understand addition and subtraction, children must recognize that "a collection can be made larger by adding items to it and made smaller by taking some away from it" (Baroody, 2004, p. 193). Understanding this simple principle is one of the most abstract ways of representing number operations. To teach addition or subtraction in a way that is visible to and learnable by the students, pre-k teachers must have SCK of this principle.

Four teachers—Anna, Brooke, Claire, and Eva—provided a detailed description of their own strategies for teaching addition or subtraction. Their responses to my original questions indicated that their SCK of addition and subtraction was exemplified in the following three tasks that these teachers did routinely. First, SCK includes teachers' ability to provide simple word problems using real life and everyday examples.

> Let's say I have five children sitting on the carpet right now. "Oh, look! Two more just put their backpacks away. Now we have seven. We had five, two more, now we have seven. So five plus two is seven." And I might just say things like that throughout the day just using the words (Claire, Second interview, 3/5/2010).

According to Claire, she addressed 5 + 2 = 7 by suggesting a real-world problem that students might encounter every day. Eva noted:

> Well, I'll try to make [addition and subtraction] interesting to them by introducing something close to their life or something interesting. I'll say "Heather, you have two sisters and your mommy just had another baby girl. How many do you have now?" So we can think that out together and I'm like, "Oh, great, that's called addition. You did addition." Or, we'll use a snack, something they love, and say, "Here, Malakia, I gave you three fish and do you want another two? Yes? Okay, so here's another two. How many do you now have?" So I'll use things that they're interested in—food, their family, their friends—so that they're more interested in the concept and they understand it better through the things that are familiar to them (Second interview, 3/10/2010).

Eva also said she made addition interesting to her students by introducing something close to their lives such as snack time or their families. For instance, she asked one student how many family members she would have when her baby sister was born. Claire and Eva described in detail how they used everyday or real life examples to introduce addition and subtraction concepts to their students. They stated that this was an appropriate way to enhance students' algebraic thinking. Recent research indicates that preschoolers are able to use addition and subtraction skills to comprehend and to solve simple arithmetic tasks and word problems (Gelman, 2000; Baroody, 1999). Claire and Eva were also able to suggest simple word problems to encourage their pre-k students to build addition skills. Thus, what seems to be an important aspect of SCK is being able to find and bring up everyday examples that involve the concept of addition and subtraction. This can help children construct an informal conceptual basis for understanding addition as an incremental process and subtraction as a decremental process. Anna and Brooke also stated the importance of using everyday examples that are closely related to children's daily routine such as snack time when introducing the adding and subtracting skills.

The second component of SCK in teaching addition and subtraction skills is being able to present the mathematical ideas through concrete objects. This allows children to use their own informal counting strategies to solve the addition and subtraction problems. The following two responses illustrate this point.

> Usually the way to do addition is we're at the carpet and I'll say, "Okay, go find me five things of something." And this is, like, the perfect example. You'll always have those students who will bring too many or too few, right? We will go over it as a whole group and they'll have, like, four more or two less and I'll say, "Uh, oh. How many do we have to take away or add to make it five?" So I'm not calling it addition or subtraction yet but we're already doing it as a general and we do that a lot (Brooke, Second interview, 3/8/2010).

According to Brooke, she asked her students to find five things in the classroom. Then, based on the number of objects that students found, she posed such questions as, "How many more or less do you need to make five?" This task addressed the concept of addition and subtraction as well as showed how Brooke used concrete objects to teach it.

Claire also described how she taught addition and subtraction.

> And then I would want them to actually use manipulatives, such as unifix cubes, buttons, blocks or anything that they could count, and put those things together to make sets. Sometimes they needed to use their fingers to show that. So I think I just weave it in with my wording and then intentionally try to show them by moving them or moving things together, putting things together to show them addition and then having them do their own (Second interview, 3/5/2010).

Claire stated that she might use "manipulatives," concrete objects such as "unifix cubes, blocks, or buttons" and, when teaching addition, intentionally try to show them by moving such manipulatives. These interviews indicate that both of these teachers tried to model the addition and subtraction concepts by using concrete objects. They recognized that using concrete objects was important because, "it visualizes what is happening" (Brooke, second interview, 3/5/2010). For example, for the initial amount students count out three items and then for the added amount count out two more items. To determine the solution the students count all the items that have been put out. This process allows them to use or invent more advanced and sophisticated counting strategies to determine sums and differences (Baroody, 2004). To encourage children's understanding of addition and subtraction, pre-k teachers need SCK, that is, knowledge that allowed them to present various ways of modeling simple equations such as 2 + 3 = 5 or 5 – 3 = 2. This type of SCK was also evidenced in Anna's and Eva's responses. Both of them emphasized using objects or manipulatives as an important way to teach arithmetic skills so that students could count them by moving them one by one.

The last aspect of these teachers' SCK of addition and subtraction was the ability to connect children's informal ways of thinking to formal arithmetic knowledge. Consider how Anna did this.

> I would start by getting one object. "Okay, we're starting with one, so let's get one object right here and then let's get two that you have here." Then I'll say, "Hey, we want to find out how much we have all together." I show this on what I call an "addition box." They can just make a box right here so they know the first number goes in there and I draw a plus sign next to it and then in another box, the second number goes in there and then the third box which is the total after the equal sign. That way they can separate: "Okay, one here, two here, and now I can push it all together on to that." It's important to separate the box because they still have difficulties making sure they have one. I mean if it's just in front of them in a line they can't remember. "Oh my gosh! Did I put one and two and move it all together?" So they have that visual of just the addition there and that helps. This kind of modeling helps them see: "So, there's a one here; there's a two here, and now we're going to add them together, so let's put them all together." And then have them count by moving them all one by one. And this can give them a little hint to how the former equation looks like (Second interview, 2/25/2010).

Anna said that she would model 1 + 2 = 3 by using concrete objects and an addition box. She described in detail how the addition box works. She stated that students put one object in the right box, two objects in the left box and finally they add them together and put the sum in the third box. This allows the students to use their counting strategies as well as to learn about mathematical symbols (e.g., numerals and formal equation). Anna's way of modeling the simple addition or subtraction problems can connect children's informal arithmetic knowledge to written arithmetic representations—a school-taught, symbolic mathematical knowledge. Many researchers have emphasized the importance of teaching this ability in early math education. According to Baroody (2004), early math education should "build on and extend children's informal knowledge by helping them to connect formal expression (e.g., 5 + 3 or 5 + ? = 8) to both problem situations and their informal solutions" (p. 197). NCTM (2000) also underscored the importance of connecting to formal mathematics various representations of children's problem strategies and solutions.

In addition to these interview data, the following examples of Anna and Claire's math lessons show how their SCK was used in their actual teaching practice. In Anna's lesson, she used a story book to teach the concept of addition. Her goal was just to "introduce what adding means" (Post observation interview, 3/4/2010) rather than to connect students' informal knowledge to formal mathematics, but she introduced the formal equations to her students. Anna read a book, *Cookie Count a Tasty Pop Up,* about counting cookies and mice. While reading this book, she encouraged her students to count the mice and cookies that appeared on every page.

> **Anna**: Three mice and three cookies! Get your fingers!
> **Children**: [The students raise three fingers on their left hand and three fingers on their right].
> **Anna**: How many all together?
> **Children**: Six.
> **Anna**: Let's count.
> **Children**: One, two, three…six!
> **Anna**: [turns to the next page] Now, how many cookies and mice are here?
> **Children**: Four mice and four cookies!
> **Anna**: Get your fingers!
> **Children**: [They show four left fingers and four right fingers].
> **Anna**: How many all together?
> **Children**: Eight!
> **Anna**: Let's count!

Anna writes the equations on the white board. 4 + 4 = 8
Anna: Four cookies plus four mice becomes eight. Now, how many cookies are here? How many mice?
Children: Five
Anna: [She writes 5 + 5 =] How many all together?
Children: Ten!
Anna: Show me your fingers!

Anna did not tell her students what the plus or equal signs meant. They just came up very naturally and the students were able to follow this lesson without any difficulties. This vignette illustrates Anna's SCK, the ability to connect children's informal ways of thinking to formal arithmetic knowledge. By reading a storybook, the students practiced the counting skill and addition strategy. Anna encouraged students to use their own fingers to count the whole amount of mice and cookies. This is the way of supporting students' use of informal addition strategies. At the same time, she wrote the equations on the board, which is a way of introducing formal expressions that represent the concept of addition. This math lesson showed Anna's SCK in regard to counting and addition and how she used it in her actual teaching. She not only encouraged students' informal ways of counting and addition strategies but also addressed the formal ways of representing the addition problems. In her post-observation interview, she noted, "They are familiar with using fingers because we do it all the time... I didn't expect them to understand those symbols. It's hard and they will learn it in the kindergarten. But I introduced this so they can see it a lot and hear it all the time" (Post observation interview, 3/4/2010). In this way, students were able to learn various problem-solving strategies.

This type of SCK was also found in Claire's math lesson on addition, which follows. It was an unplanned lesson, but still, her goal was to teach the addition concept.

The students are sitting around on the carpet. Claire starts an unscheduled math lesson because one of her students asks, “How many girls are in our classroom?” She seizes this moment to develop a meaningful math lesson.

Claire: A few minutes ago, Jack asked me how many girls are in our class. I thought it would be fun to think about this. I just drew a picture of a girl and a boy here [whiteboard]. Let’s count the girls first. Ready? [She walks around and points to a girl].
Children: One, two…four
Claire: [She writes “4” next to the picture of the girls]. Now, let’s count the boys.
Children: One, two … six.
Claire: [She writes “6” next to the picture of the boys]. Then how many are there all together?
Children: One, two….ten. [Claire walks around and points to each child].
Claire: [She writes “4 + 6 = 10” on the board].

Next to the equation, she draws six boys and four girls so that the students can better see how many boys and girls there are. They can also recognize that the boys outnumber the girls.

Child 1: If two teachers [Claire and her helper] are here, we have more girls in our classroom.
Claire: Do you want to add Ms. B and Ms. D? [She draws two more girls]. Let’s count.
Children: One, two, three…six.
Claire: You know what? We now have six and it’s the same as the boys. So, we can know that four plus two equals six [writes “4 + 2 = 6”]. Then, how many people are in our classroom right now? We have six girls and six boys. Let’s count.
Children: One, two, three…twelve!
Claire: [writes “6 + 6 = 12” on the board].

This vignette illustrates how Claire was able to spontaneously develop a mini-math lesson on counting and addition by taking advantage of her student’s unprompted question. On display in this vignette are her three aspects of SCK. The first aspect is her ability to use everyday or real life examples to introduce the concept. Responding to the question, “How many girls are in our class,” Claire asked her students to count their classmates, a real life math-related problem for these students. She then extended her lesson, using the second and third aspects of SCK, to teach addition by asking mathematical questions, such as, "How many all together?" These aspects of SCK include presenting the mathematical ideas by using concrete objects or drawings (e.g., pictures) and connecting children's informal ways of counting and addition strategies to formal expressions of representing the addition problems.

While introducing equations (4 + 6 = 10, 4 + 2 = 6, and 6 + 6 = 12), Claire drew pictures of boys and girls to help children visualize the total number of boys and girls. In her post-observation interview, Claire stated:

> At this age, they have to see the visual of the girls because if I had just written "six" and the word "girls," it would have been like just writing a different language. But they were seeing the girls and when we counted them, they could visualize, "Oh, it's six of them." …I just start using the math words [e.g., plus, minus] and it's just for them to start hearing it and I hope some of them make the connection that I've been using "plus," "plus," "plus." So, that's how we do it. It's really important for them to at least start hearing and seeing the language for math (Post observation interview, 3/24/2010).

She recognized the importance of seeing the visuals of counting and adding. For example, she drew six girls to help students visualize the number 6. She also used math-related languages such as plus or minus often. Claire's SCK manifests itself in her effort to model the addition problem in this manner.

Unlike these four teachers, Diane provided no answer to my question, believing that addition and subtraction were not supposed to be taught in pre-k. "We won't teach addition or subtraction at this age. We focus on teaching counting skills, sorting skills, or making a patterns" (Second interview, 3/10/2010). Although Texas pre-k guidelines do encourage teachers to teach "adding to or taking away" skills (TEA, 2008), Diane commented that teaching such content was not included in pre-k guidelines. Her thinking might have been due to feeling overwhelmed by the sheer volume of what a pre-k teacher is expected to teach in one year, leading to a misunderstanding of what and what not to teach during the pre-k years. So to investigate another aspect of her SCK, I asked her another question about how she taught numbers one to ten. I investigated here her SCK of teaching these numbers by asking her "Suppose you are teaching numbers one to ten. How would you teach this and what materials would you use?"

> Basically what I do is I have my flashcards of the numbers that I have in the room and we focus on those numbers. We practice using our hands and counting the numbers on our hands. We practice counting objects with those numbers. We practice tracing the numbers and just saying them over and over. Showing different things that represent those numbers and getting the visualization of the number (Second interview, 3/10/2010).

By her own account, she taught counting skills from one to ten by using concrete objects and practiced one to one correspondence repeatedly. With concrete objects, she visualized what each number represented. This illustrates Diane's SCK on presenting the basic mathematical concept (e.g., counting and recognizing the number from one to ten) in a way that was appropriate to her prekindergartners.

The examples from above illustrate these pre-k teachers' SCK. Such knowledge is necessary when presenting mathematical ideas (e.g., addition and subtraction) in a way that is visible to and learnable by pre-kindergarteners. Such presentations involve providing simple word problems using everyday examples, presenting the mathematical ideas by using concrete objects, and connecting children's informal ways of thinking to formal arithmetic knowledge. A teacher must be able to do all of these things to convey the abstract mathematical content in an appropriate and effective way to young children. Also, it is crucial for pre-k teachers to have this type of knowledge to help children construct mental representation of numbers and to build on or extend their informal knowledge by helping them to connect formal expressions to various problem situations (Baroody, 2004).

These pre-k teachers' SCK of addition and counting reveals their understanding of the concept of number. The interviews and vignettes illustrated above indicate that they tended to address the cardinal aspect rather than ordinal aspect of number when they presented the mathematical ideas such as addition, number identification, and counting skills. Haylock and Cockburn (1997) articulated that when thinking about the image of three, most teachers or adults might come up with the idea that three is a set of three things. When pre-k teachers need to explain a problem involving the number three, what they will most likely use are sets of three counters, three blocks, or three fingers. The five pre-k teachers' SCK in this study exemplify this point. They tended to present mathematical ideas (e.g., counting, addition, and subtraction) by first introducing a set of objects or manipulatives that involved the concept of a certain number. For instance, when teaching $1 + 2 = 3$, Anna stated, "I would start with getting one object to show my students there is one object right here. Then I would bring two objects and asked them how many objects are all together" (Second interview, 2/25/2010). Claire also noted that when teaching addition or subtraction, "I want them to actually use manipulatives, such as unifix cubes, buttons, blocks or anything that they can count, and put those things together to make sets together" (Second interview, 3/5/2010). These teachers' ways of presenting the mathematical ideas (e.g., addition) involved the number one or two or three and so forth, represented by the sets of one object, two blocks, or three unifix cubes – a cardinal aspect of number. Another participant Diane stated that when teaching numbers, "We practice using our hands and counting the numbers on our hands. We practice counting objects with those numbers. Showing different things that represent those numbers and getting the visualization of the number" (Second interview, 3/10/2010). This indicates that Diane uses a set of objects to relate the concept of number. This view of number is also found in Brooke's following description of how she taught the number 10.

> So when we do one on one, we now have an actual visual of the number. If they see a one and a zero, it's clueless, right? But if I have ten rocks then they can actually see a visual of ten rocks. So that's kind of how we do a lot of the number recognition, whether it be lower or higher—counting things in the classroom. They'll count ten crayons. "Bring me ten crayons. Bring me ten teddy bears." And that's how we do a lot of numbers (Second interview, 3/8/2010).

This description indicates that Brooke taught the number 10 by showing her students a set of ten things such as ten rocks, ten crayons or ten teddy bears. Both Texas pre-k guidelines and district curriculums address the ordinal aspect of number, stating, "The child uses ordinal numbers (first, second, third, and fourth) to count objects" (TEA, 2008, p. 86). Nevertheless, they mainly describe the instructional strategies that include the cardinal aspect of number. This is why these teachers might focus on presenting the cardinal aspect of number when teaching the addition concept or counting skills.

Haylock and Cockburn (1997) argued that this view of the way in which number concepts develop is very simplistic. Indeed, they asserted, there are other aspects to number, such as nominal or ordinal aspect. Also, they contended that this view made little allowance for the complex network of connections between concrete experience and the symbol, which constitute the concept of a certain number. Thus, these teachers' SCK of number ought to be expanded from cardinal to ordinal aspects of number. This will help them present the abstract mathematical ideas by making coherent connections between the concept of number and the meaning of such number.

Modifying the lesson

Modifying the lesson to be either easier or harder for students is another important aspect of SCK (Ball et al., 2008). This is important in terms of addressing students' individual differences. These teachers adjusted or changed their teaching practices based on where their children were as learners. My interviews and observational field notes reveal three ways teachers modified the lesson to address children's individual differences and needs. First, all of the pre-k teachers mentioned that repeating the mathematical tasks or re-teaching the content was an effective way to help the students who tended to misunderstand the mathematical concepts. Anna describes how this works.

> I just go back and show them again. For example, when teaching pattern, I asked them, "Hey, let's look again. You can touch it and see what it is. You have a choice of a pencil or a crayon. Which one do you think comes next?" So just showing it again and repeating it and having them see it and then they have a fifty-fifty shot. Usually they can try to get the next one right. And then the next day, do it again so eventually it'll click much better. I mean if you saw them now versus at the beginning of the year, they're much better at it. They'll actually say what they're doing and understand it because they've done it so much. They've had more practice doing it every day (Anna, second interview, 2/25/2010).

Anna mentioned that she just repeated the tasks (e.g., pattern) so that her students could see again. She suggested that repeating the mathematical tasks helped students understand the concept and capture the meaning eventually.

Diane expressed a similar sentiment.

> I think at this age, [it's effective to have] repetition and showing them again and again. So like last time when I taught the numbers, I showed different things that represented those numbers. Just counting different objects with them and showing that that's what that number is and just getting the visualization of this is the number and this is why. Just repeating this over and over until they get those numbers (Diane, second interview, 3/10/2010).

Diane also said she repeated the same activity until her students understood the number. According to Anna and Diane, they usually repeated the lesson and went back to basics when children had a hard time understanding the key ideas of mathematics. They also strongly believed that students could finally learn the concepts when they did the same mathematical activity over and over. These teachers held untested assumptions about their students. For instance, Anna stated that, "just showing it again and repeating it and having them see it, then they have a fifty-fifty shot. Usually they can try to get the next one right. And then the next day, do it again so eventually it'll click much better." This kind of assumption influenced the types of teaching strategies they implemented and the way they interacted with their students (Calderhead, 1996).

Second, these teachers differentiated their lesson according to students' prior knowledge or abilities. Rather than re-doing what they had just done, teachers, such as Eva, tried to offer enrichment with different types of materials or activities.

> If we're learning about the number ten, I'll be working with the ten with my other students, not my students who need to learn the number ten but my more advanced students. I'll come to them quickly and give them another activity with ten. Like, "Let's separate the numbers one plus zero." Or we'll do some addition, like five plus five equals ten. So I'll just make that lesson a little bit different for them and differentiate more with them and give them a higher-level thinking activity with the number ten than I do with the other students (Second interview, 3/10/2010).

According to Eva, she differentiated the lesson based on students' ability when teaching the number ten. She gave her advanced students another activity with the number ten that included higher-level thinking. The other four participants also agreed with the idea that differentiating the lesson was an effective way to address students' developmental needs. It led individual students to engage in "deep and sustained interaction with key mathematical ideas" (NAEYC & NCTM, 2002, p. 9).

Lastly, redesigning the mathematical activity was another important aspect of modifying the lesson. This was well illustrated in one of Claire's math lessons. The goal of the lesson was to sort objects. The students were supposed to sort the objects by type (e.g., dinosaurs, bears, etc.) and then, on their own, draw on the paper how they sorted.

While most students successfully sorted the objects, they had a hard time representing how they sorted. They did not know what to draw or how to draw. Claire reflected on this lesson:

> I think some kids need a lot more time to do their representation, but I didn't foresee that and felt frustrated with myself for not thinking through what's going to happen if they don't know what to draw or how to draw it. I felt like I gave them more answers more quickly than I wanted to, but I felt if I didn't move them forward, we weren't going to go anywhere (Post observation interview, 4/5/2010).

She expected her students to succeed with the task at hand and did not foresee that they might struggle with drawing the objects. When she realized this situation, she tried to scaffold by directly modeling ways to complete the task, but none worked well (Field note, 4/5/10). I asked her what she would do in her next math lesson. The following interview showed how she tried to deal with this problem.

> I don't know. The reason why I tried to do a representing activity is that…I tried to give direct guidance to the students but I stopped because I didn't want to give them, like, my way of doing it or one way of doing it. So now I'm thinking I might go through, maybe shortly, two ways to do it, or maybe not. Maybe just show them one way to do it and then have them try that way and practice instead of giving them so much freedom. Or, I might do different activity other than drawing. I don't know. (Post observation interview, 4/5/2010)

Claire kept on thinking about how to improve her lesson how to make it more doable and meaningful than the previous one. According to her thinking, she should revisit her goal for this activity and think about alternative ways to meet that goal. Claire had to manage her time, materials, activities, and students as well as revisit, in modifying her agenda, the district or state pre-k guidelines. This process of redesigning the lesson requires SCK.

In her next lesson, Claire changed several things. First, she changed the sorting objects from real things to different types of die-cut shapes, such as circles, triangles, and squares. She then gave to the students a piece of paper that she called a "sorting sheet" and asked them to glue the shapes on the sorting sheet based on how they had sorted them. Her students sorted the shapes by color and type and glued them on the paper. This time they completed the task successfully (Field note, 4/9/2010). Claire reflected on this lesson thus:

> I felt like the lesson went pretty well. I think the sorting went better today than my sorting lesson went the last [time]. So I felt like every kid was able to show some kind of sorting when they got to the table. They did it by shape or they did it by color and then a couple did it by shape and color. I think that having the different paper, little shapes or cut-outs that they could put on with a glue stick was a lot more attainable for the children than drawing their own pictures because it took less time. They could pick the shapes out of the bowl. They could see some more choices. Whereas when I tried to have them draw, they had to come up with that whole idea on their own and then draw it, and that was really difficult. So I feel like, although they could have fewer choices than the last one, they did well in representing their sorting by gluing them on. That helped me a lot with the timing, too. I think it went better because they could touch the things; they could see their choices and didn't have to come up with everything in their mind and then draw it somehow (Post observation interview, 4/9/2010).

Claire modified the activity by revisiting what her students could and could not do, re-designing the tasks to be more doable and attainable, and finding the appropriate materials or resources to facilitate that. In these pre-k teachers, another predominant aspect of SCK that was in evidence was the ability to change and adjust the teaching practice based on students' individual differences. This enabled them to "adjust the pace and content of their instruction in a range of ways" (Goldstein, 2008, p. 466). It also enabled them to effectively address students' unique strengths and needs.

To scaffold their students' understanding of a range of mathematical concepts, these pre-k teachers modified their lessons to be either easier or harder. There were three ways of doing this: repeating or re-teaching the content, differentiating the lesson according to students' ability, and re-designing the mathematical activity. This required the pre-k teachers to know their students' prior knowledge and their individual differences and to find appropriate examples or activities, based on students' abilities and needs that modeled the key mathematical point.

Summary

These pre-k teachers' understanding of the mathematical content they were to teach was divided into two subcategories—common content knowledge and specialized content knowledge (Ball et al., 2008). First, these teachers' CCK differed substantially within a school district or across the school districts. This difference was associated with years of teaching and whether or not they had experience teaching other grade levels. However, the relation between such differences and their actual teaching did not come into play. Second, these teachers' SCK was exemplified in the following four mathematical tasks:

- Understanding how the specific math content is connected to students' later mathematical learning: The extent of such ability varies among these teachers.
- Promoting students' mathematical thinking: These teachers tried to promote students' mathematical thinking by asking productive mathematical questions and integrating the big ideas of mathematics to teach a specific mathematical concept. This required them to know the breadth and depth of basic mathematics (e.g., number, counting, comparison, and measurement; Ginsburg & Ertle, 2008).
- Presenting mathematical ideas: These teachers used everyday or real life examples, and concrete objects to present a specific mathematical idea (e.g., addition and subtraction). Also, they tried to connect students' informal mathematical knowledge to formal, school-taught, and symbolic mathematical knowledge.

- Modifying the lesson: These teachers modified their lessons by re-teaching the content, differentiating the activity according to students' ability, and re-modeling the mathematical activity.

All of these tasks were something that these teachers routinely did but demanded unique mathematical understanding and reasoning. Examining pre-k teachers' CCK and SCK provides an insight into what these teachers need to do when teaching mathematics and into how this work demands mathematical reasoning, understanding, and skills. This might inform the features of materials for teacher education or professional development. In addition, it might inform the design of curriculum for the preparation of teachers' mathematical content knowledge for teaching that is tied to the knowledge and skills demanded by their tasks of teaching.

PRE-K TEACHERS' KNOWLEDGE OF HOW TO TEACH MATHEMATICS

A strong knowledge of the subject taught is a core component of teacher competence and of high-quality instruction. Such knowledge, however, remains inert in the classroom unless accompanied by mathematical knowledge and skills relating directly to the instruction and student learning (Baumert et al., 2010). Mathematical content knowledge such as CCK and SCK can help pre-k teachers recognize how to provide specific mathematical ideas effectively and how to enhance children's deep mathematical thinking. It does not, however, guarantee powerful experiences for students. What is required for such experiences is pedagogical content knowledge (Shulman, 1986). This involves "bundles of understandings that combine knowledge of mathematics, of students, and of pedagogy" (Ball et al., 2001, p. 453). It is pedagogical content knowledge that underlies the interpretation of student responses and understanding, the correct analysis of student errors and misunderstandings, the development and selection of mathematical tasks, and the choice of representations and teaching sequence (Ball et al., 2008). Pedagogical content knowledge that is needed for teaching prekindergarters includes knowing students' prior mathematical ability, noticing students' informal mathematical knowledge, and knowing what examples or materials might be effective when teaching a specific mathematical content.

To investigate pre-k teachers' pedagogical content knowledge when teaching mathematics, which is related to my second and third research questions, I employed Ball and her colleague's framework of knowledge of content and students (KCS) and knowledge of content and teaching (KCT). These two domains coincide with the two central dimensions of Shulman's pedagogical content knowledge: "the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning of those most frequently taught topics and lessons" and "the ways of representing and formulating the subject that make it comprehensible to others" (p. 9).

Knowledge of content and students (KCS)

Knowledge of content and students refers to "knowledge that combines knowing about students and knowing about mathematics" (Ball et al., 2008, p. 401). This includes knowing the following things: 1) what students are likely to think about particular mathematical ideas and what they will find confusing, 2) what students will find interesting and motivating, 3) what students are likely to do and whether they will find the task easy or hard, and 4) students' emerging mathematical thinking (Ball et al., 2008, p. 401). Each of these tasks requires an interaction between specific mathematical content knowledge and students' mathematical thinking.

To investigate such knowledge, I asked the teachers questions about, for example, the kinds of errors or mistakes that students make when the teachers cover specific mathematical content and about the concepts in math that students might have the hardest time grasping. I asked, for instance, "What are the kinds of errors that children make when you teach patterns? Which concept in math do you find students have the hardest time grasping and why do you think students have a hard time?" I also asked the teachers questions that required them to interpret students' informal and non-standard ways of thinking. In asking such questions as, "What are informal or non-standards strategies that your students use when counting the numbers," I attended less to the variety of what teachers notice and more to how and the extent to which teachers noticed students' mathematical thinking (Jacobs, Lamb, & Philipp, 2010). These questions helped me identify how these pre-k teachers understood both the ways their students grasped mathematical concepts and the nature of their student's mathematical thinking.

Understanding students' common errors

A central feature of KCS is the knowledge of students' common conceptions and misconceptions about particular mathematical content. My findings showed that most of the pre-k teachers in this study were capable of describing the common errors that students generally made as well as why such errors were made. The following interviews illustrate these teachers' understanding of students' conceptions and misconceptions about patterns. A pattern is, mathematically, "an underlying rule or concept" and describes "a regularity that determines, explains or predicts observed phenomena" (Ginsburg & Ertle, 2008, p. 51).

Anna was asked, "What are the common errors or misunderstandings that children often made in regards to patterns?" Anna responded:

> Skipping numbers. Skipping the elements of patterns. Like when we're doing the heart-pencil-heart-pencil pattern, whatever the last one was, some of my kids will say "Oh, it's the heart because it's the heart." So they'll just say what the last one

> was and then they'll put it right there next to it again. So whatever the last one was, they just see what will pop in their head and they'll just say that that was heart. I think this is because they just see, "Oh, there's a heart." So they're, like, remembering because when we say, "Heart, pencil, heart, pencil," and they'll say "Heart," so what comes next? "Heart." Because heart was the last thing they just said (Second interview, 2/25/2010).

According to Anna, one of the most common errors that her students made was skipping the elements in the pattern that were supposed to come next. She appeared to believe that this might have been because students were likely to remember the last thing they saw, which could lead them to make a mistake in predicting and creating the patterns.

Another participant, Claire, also noted the following:

> When you say patterns, the first thing that comes to my mind are those color patterns, that red, blue, red, blue. So then a common problem would be, like, putting two blues together or two reds together. On the worksheets that we talked about with the different shapes, not knowing what comes next in that pattern whether it was the circle, the square, or the triangle. Why? Maybe pattern is just not naturally occurring in their lives so far like language [is]. They've been listening to it and you're bouncing back and forth all these years but it's probably not at home—"Oh, look, I have this spoon, fork, spoon, fork on the table right now." I don't know that they've had to categorize their life in any way like that yet (Second interview, 3/5/2010).

In her response, Claire described one common problem found among her students was the putting of same objects or same colors together. She explained that patterns are something that children experience little of at home, making the concept, for some of them, rather difficult. These two responses capture students' common errors and mistakes usually made specifically regarding patterns.

Both Anna and Claire were able to provide mathematically significant details of students' misconceptions about pattern. Also, their descriptions included their own analysis of why their students might misunderstand the concept when doing a pattern activity. Responses demonstrating students' common errors were phrased in different ways, but they all tracked the key characteristics of students' thinking, which included skipping or repeating the elements in the patterns. Other participants were also able to describe students' misconceptions about pattern or other specific mathematical content. For example, Brooke stated that some of her students do not know that "pattern is something that goes over and over again. They just don't understand that a pattern repeats itself" (Second interview, 3/8/2010). Eva also said that one of common errors that her students make in regards to pattern was "losing track of what the starting pattern was" (Second interview, 3/10/2010). Her students often forgot how they started the pattern and put odd objects within the pattern. Diane mentioned that her students were likely to "forget how the pattern repeats" and omit the elements of pattern or put in extra things (Second interview, 3/10/2010).

In addition to patterns, some teachers were able to recognize students' misconceptions about other mathematical content such as subtraction. For instance, Eva stated,

> I used to do addition or subtraction activities with manipulatives. Sometimes they would forget what the original number was. So if I started with five unifix cubes and asked them to take three away and how many cubes were left, they would forget the starting number was five. I think that's the most common mistake they made. Subtraction for them is a big word. That's something very different for them. Most of my students have used numbers to count and write but they haven't used numbers in an equation before so it's a very new concept for them (Second interview, 3/10/2010).

According to Eva, one of the common errors that her students were most likely to make was to forget "what the original number was." This might have been because students were not yet familiar with "taking away" skills. They mostly used numbers as counting, so subtraction was a very new skill to these students.

The ability to recognize students' common errors or misunderstandings is an important component of KCS. This study found that all five of the participants possessed it. Having such knowledge is important in terms of assigning a task or choosing appropriate examples because the teachers can anticipate what the students are likely to think and what they will find confusing.

Understanding students' mathematical ability

Another important feature of KCS is understanding and recognizing what students can and cannot do. For instance, pre-k teachers need to know their students' mathematical ability. In examining such knowledge, I asked questions about what mathematical content the students had a hard time grasping. All participants were able to describe their students' ability to understand specific mathematical content covered in Texas pre-k guidelines. Although each participant chose different mathematical content, they could provide a reasonable, detailed explanation of why this content was challenging for their students. The following two interviews illustrate how these teachers saw their students' capability of understanding some essential mathematical ideas. I asked each participant, "What mathematical concepts do your students have the hardest time grasping?" The first response, from Diane, concerns students' understanding of sorting. Sorting is a basic skill of algebraic thinking (NCTM, 2000) and, in the Texas pre-k guidelines, is described within the domain of classification and patterns skills (TEA, 2008).

> I think, probably sorting is sometimes hard. They can sort by color, no problem. But when you throw in "Okay, I need you to sort – whatever the objects are that I give. How do you want me to sort it?" They can't answer, I mean a lot of times my kids aren't going to be able to sort by an attribute unless I tell them what attribute to sort by. So if I just say, "Okay, sort it," they're automatically going to sort it by color and I find that at this age, that's the easiest skill. They're not thinking, "Oh, okay, this animal has two legs. This animal has four or this animal has feathers, this animal has fur, so let's sort by these." So I think sorting by different attributes is a pretty hard skill for them. I just don't think it's something that they think about everyday (Diane, second interview, 3/10/2010).

Diane observed that since sorting objects by one or more attributes was difficult for her students, sorting in general was one of the hardest concepts for them to grasp. She recognized that her students could sort by color but couldn't sort by other attributes unless she told them what attribute to sort by. She knew her students could not fully understand what sorting meant, how to sort, or even why they sorted in a particular way. She was able to point out her students' ability and their current skills at sorting. This knowledge led her to concentrate more on providing sorting activities that included multiple attributes. She remarked, "That's why I want to do more on sorting and try to encourage my little kids to sort by different attributes other than to sort by the color" (Second interview, 3/10/2010). This important component of KCS-knowing what her students could and couldn't do-informs Diane of what she must focus on in her sorting lesson. I, then, asked her why her students might have difficulty in understanding sorting. Diane responded, "I think it's because my kiddos are not ready to sort things by different types of attributes other than color. They just need more time to think about them [different types of attributes]" (Second interview, 3/10/2010). Diane suggested that when her students were ready, they could all sort objects by various types of attributes (Second interview, 3/10/2010). Diane thought students were not capable of doing sorting activities because they were not yet developmentally ready to learn.

Eva also clearly explained what her students could and couldn't do in relation to the skill of measurement.

> We haven't talked about [measurement] that much so it might be hard to know it unless they have some prior knowledge on that. I find that for them that's hard, like the vocabulary can be hard, using tools to measure can be hard, to line them up correctly can be hard. And then like the capacity and things, like if you change the container, it's the same amount, they don't get that and those are kinds of concepts I think that are difficult for this age…just almost everything with measurement (Second interview, 3/10/2010).

Eva expressed knowledge of what her students might find confusing and difficult in understanding measurement. She was able to provide detailed descriptions about the range of her students' ability as well as some common examples that her students might find challenging. This is an important component of KCS. Based on such knowledge, she noted that, "I think measurement is very important and we do a lot of measurement. It's a new concept and I tried to help them be familiar with all those words. We do it in our small group but we also have a measurement activity in the center as well." Knowing that measurement was a challenging concept for her students helped Eva determine how she would address the concept and what parts to focus on. I also asked Eva why her students might have a hard time grasping the measurement concept. Eva stated that it was because she hadn't "talked about such a concept so far" (Second interview, 3/10/2010). Eva said that her students were confused by the measurement concept because they had not been taught the concept often enough compared to other mathematical content such as number or counting skills. This indicates that her math lessons were mainly focused on number identification and counting so that her students had relatively a hard time grasping measurement concept.

In a variety of ways these teachers described how they understood their students' capability and current level of understanding specific mathematical content. All the participants, however, made sense of what their students could and couldn't do. Anna discussed her students' abilities to understand addition and subtraction; Brooke and Claire discussed their students' abilities to understand measurement skill. Having this awareness of students' abilities in grasping mathematical concepts is an important component of KCS. It is important because teachers can develop math lessons that meet students' needs and levels (Graeber, 1999). Also, they can gain insight into what mathematical content they should focus on and into whether or not they can move on to the next step.

All five of the teachers in this study appeared to believe that students found particular mathematical ideas confusing or difficult because either they had not been taught the concepts or because they were not developmentally ready to learn the concept (see Diane's and Eva's interviews above). For example, Anna, in her first interview, noted that:

> When teaching addition, I could see that some of my students were not ready for it. They even had a hard time counting like counting the same objects repeatedly. So, they can't handle addition because they have to handle counting first so once they master it then I can teach addition (2/16/2010).

Anna thought of her students' difficulties in adding as a developmental issue. Graeber (1999) contended, however, that students' difficulties with understanding mathematical ideas didn't necessarily indicate they were not developmentally ready or able to think of any effective ways of solving the problems. Young children often have their own ways of solving problems, using words to refer to objects and situations. This underscores the importance of understanding students' reasoning, an issue that will be discussed in the next section. When teachers understand and support students' reasoning and thinking, they can, rather than classify them – prematurely – by developmental stage designations or by their lack of mathematical knowledge, suggest alternative ways that help students handle the problems.

Noticing and interpreting students' mathematical thinking

One of the most important aspects of KCS is this ability to integrate knowledge of math content and practices with knowledge of children's mathematical thinking. As children show different ways of solving problems, it is vital that teachers understand the way children think and reason about a particular mathematical idea (Ginsburg & Ertle, 2008). This means teachers must carefully observe and recognize children's informal ways of solving the math problems and build on and extend those when teaching mathematical content. For instance, pre-k teachers are required to notice students' own strategies of solving the math problems, understand what they know and don't know, and suggest alternative ways to solve the problems. To investigate such knowledge, I observed these teachers' math lessons. I wanted to see how the teachers, on the basis of their mathematical reasoning, attended to and interpreted students' informal ways of understanding, as well as how they responded to each student (Jacobs et al., 2010). I also asked questions that required teachers to interpret students' informal and non-standard ways of thinking. For instance, I asked such question as "Could you describe how your students solve addition problem? Have you ever found any informal strategies they use?" The results showed two important patterns. The first was that teachers varied in their level of attending and scaffolding students' mathematical thinking. The second was that, although some teachers could successfully engage their students' mathematical thinking, they could not articulate how they interpreted that thinking.

First, the teachers differed in how, on the basis of their mathematical thinking, they attended to students' informal ways of mathematical thinking and in how they scaffolded students. One teacher, Diane, seemed less likely to notice and respond to her student's mathematical thinking than were more experienced teachers. For example, consider the following vignette of Diane's small group math lesson. The goal of this lesson was to teach "basic counting skills" and "one-to-one correspondence" and to make the connection between numerals and the numbers of stickers (Diane, post observation interview, 4/22/2010). This is also stated in state pre-k guidelines in the counting skills section as "child counts 1-10 items, with one count per item" (TEA, 2008).

Diane and a group of students are doing a math activity while other students are playing in the center, putting the amount of stickers that match a given number on a worksheet. As the students get to higher numbers, more than eight, one of her students, Jacob, tended to omit some stickers or recount the same stickers twice.

Diane: Jacob, I think you must recount it. What is this number?
Jacob: Nine.
Diane: Then, how many stickers do you need?
Jacob: Nine.
Diane: But you didn't put nine stickers. It's only seven. How many more stickers do you need?
Jacob: …
Diane: Let's count. One, two, three…seven, eight, and nine! So, how many more do you need?
Jacob: …
Diane: Well, you need two more. Here're the stickers. Take two more.

Asked to identify the numeral and enumerate the stickers, Jacob kept forgetting which stickers had been counted and which had not. Such a task is difficult to do, especially when the large number of objects to be counted are arranged haphazardly (Tang & Ginsburg, 1999). Diane asked Jacob to identify the number and recount the stickers. She asked a question about how many more stickers he needed. Yet, he still did not know what to do. Diane provided him the answer and told him what to do (put two more stickers on).

In this vignette, Diane neither asked any questions about Jacob's mathematical thinking (such as why had he put on only seven stickers instead of nine), nor did she try to show alternative ways of counting and enumerating. This example illustrates Diane not attending to her student's mathematical thinking. Rather than figuring out what common mistake this student had made or why he was struggling, Diane told him what was right and wrong. Such a solution reveals nothing about whether this student understood the concept of the number nine or was engaged in connecting the numeral 9 and a set of nine objects.

In Diane's math lessons, I observed repeatedly this type of instruction. Consider another case, her small group math lesson. This lesson was about patterns and its goal was "to identify and create the patterns" (Field note, 3/30/2010).

> **Diane**: We are going to make a pattern with the farm animals. What does a pattern do?
> **Children**: Repeat itself.
> **Diane**: That's right. We can make a pattern by the color, shape …
>
> She offers an example of a pattern by color. She puts out a blue horse, a yellow chicken, a blue horse, and another yellow chicken, producing a blue-yellow-blue-yellow pattern. Then, she makes a pattern by type of animal. She puts a red pig, a green duck, a yellow pig, a blue duck, and a red pig, producing a pig-duck-pig-duck pattern.
>
> **Diane**: What animal would come next?
> **Children**: Duck.
>
> Diane then asks her students to create a pattern by themselves. One of her students, David, puts together several purple animals and one green animal.
>
> **Diane**: Is this a pattern?
> **David**: Yes.
> **Diane**: Well, you put all the purples here. I can't see any pattern. We need to fix it. You can make a pattern by the color or animals.
> **David**: [He makes a pattern using color]
>
> After practicing the pattern with animals, Diane uses unifix cubes. One of her students, Ashley, is rather meaninglessly building a tower with the cubes. Diane says to him, "This is not a pattern. I will show you first. You can start with blue, blue, blue, yellow, blue….." Ashley watches what Diane is doing.

In this vignette, Diane interacted with two students who were unable to create a pattern. Neither student created order when putting together objects (e.g., farm animals and unifix cubes). When Diane noticed this, she told them directly what they had made were not patterns. She corrected their errors for them. She asked no questions about why they had created their non-patterns as they had, nor did she provide any opportunity for either student to solve the problem himself. This might be because Diane was focused on completing the activity itself rather than engaging her students in mathematical thinking. She might have believed that just by completing the activity successfully could lead to students' mathematical learning. Such behavior fails to attend to students' mathematical thinking. Rather than figuring out why these students struggled with making patterns and asking questions that encouraged them to re-think, Diane gave them right and wrong answers. Such a solution offers no indication of whether the student understood what a pattern meant or how it worked. In Diane's math lessons, there were many interesting and fun math activities that her students were able to enjoy. Her major focus, however, seemed to be planning, organizing, and completing such activities rather than knowing and understanding students' ways of thinking (Field note, 3/22/2010).

The other teachers—Anna, Brooke, Claire, and Eva—provided evidence of attending and responding to students' informal ways of mathematical thinking so that they could provide meaningful instruction to their students. Consider, for example, the following vignette of Claire's small-group math lesson. The goal and type of math activity that Claire developed was similar to that of Diane's lesson. Students were to match a set of objects with a number. Instead of applying stickers, Claire's students were to glue the number of ducks that would match a given number.

> One of her students, Derek, is sitting at a table with other groups of students. He chooses the number 11 and tries to gather 11 ducks but keeps counting the same duck twice. Noticing this, Claire begins a discussion with him.
>
> **Claire**: I can see that you counted the same duck twice. Could you try again?
> **Derek**: [Counts the ducks again one by one but makes the same mistake].
> **Claire**: What number did you pick?
> **Derek**: Eleven.
> **Claire**: How many ducks do you have now?
> **Derek**: One, two, three…[He still counts the same duck twice].
> **Claire**: How can you *not* count the same duck twice?

Derek: …
Claire: Why don't you count five ducks first and glue them and count the rest of them later?
Derek: [He counts the five ducks and glues them and then counts another five ducks and glues them].
Claire: So, how many all together? You have five ducks here and five ducks here.
Derek: Ten!
Claire: How did you know it was ten?
Derek: Five, five, so it's ten [showing his fingers].
Claire: Then, how many more ducks do you need to make 11?
Derek: One
Claire: Good job. So, glue one more.

Having noticed that Derek was continually making the same mistake, Claire suggested an alternative way to count to eleven. Claire asked him to count the ducks in sets of five. Derek arranged the ducks in lines of five and five and interpreted it as involving the combination of two smaller sets. He was able to determine the number of each of the two smaller sets and combined these two values by using some form of mental addition, arriving at the correct total. In other words, he interpreted the problem as addition, not enumeration, saw the number of each subset, and added them mentally.

Claire helped Derek use this strategy, called "subitizing," which is more sophisticated and convenient than touching each object and counting one by one (Tang & Ginsburg, 1999). Claire noticed that Derek's informal way of counting objects was, when counting a large number, unmanageable. She then challenged him to count in a different way. Asking him questions allowed her to figure out the difficulty that Derek was facing and to understand his own way of counting. Claire's guidance clearly helped Derek complete the task successfully.

Such guidance was also found in other teachers' small-group math lessons. For instance, in Eva's small group lesson – drawing the dots of ladybugs so that they equaled a given number – she kept asking such questions as, "Why did you think this was number six? How come you drew seven dots?" These questions helped her get at her students' thinking (Field note, 3/10/2010). Based on a student's response, she either encouraged his or her own way of counting or suggested an alternative way. Although the math activities they developed were different, all of these experienced teachers carefully observed students' behaviors, focusing on how they solved the problems and, when necessary, scaffolding their thinking. These examples show that how teachers' respond to students' mathematical thinking can tap into students' potential competence.

The examples above illustrate how Diane and Claire attended to students' informal ways of mathematical thinking and helped students on the basis of their mathematical thinking. What differed was the extent to which these teachers attended to students' mathematical thinking. Four of the five teachers demonstrated that they were able to attend to the details of students' counting and to develop instruction that built on students' ways of thinking. The other teacher, Diane, might have had less knowledge of children's learning, which would in turn have provided her less leeway in recognizing children's mathematical thinking. Again, this ability to notice students' informal mathematical knowledge is an important aspect of KCS. Jacobs et al. (2010) called such ability the "professional noticing of children's mathematical thinking." They emphasized the importance of developing professional-noticing expertise in professional development. The development of professional noticing expertise in pre-k level is also needed to offer math instruction that builds on children's mathematical thinking.

In asking teachers to interpret students' informal and non-standard ways of thinking two patterns emerged. The first was this varied level of attending to and scaffolding of students' mathematical thinking. The second is what follows. Although four of the five teachers successfully engaged their students' mathematical thinking, none could articulate in any detail how they interpreted students' thinking. For example, I asked Claire how she dealt with the student who was struggling while doing the duck activity. She related her understanding of her students' mathematical thinking as follows:

> Two types of kids come to mind. They were recounting over and over or they didn't know what number came next. Some [have] short attention spans and I struggled with keeping enough attention with them long enough to get something done. So I really have to focus on listening to them count, watching them count, stick to them and not get distracted myself and try to keep them attentive and not distracted. And then, for others, just like taking time and giving them an opportunity to keep counting and not being frustrated with like they wouldn't know what number came next (Post observation interview, 3/24/2010)

This response provides no evidence of Claire's interpreting students' understandings of counting and numbers, even though she had addressed students' informal mathematical knowledge during her small group math lesson. She neither described students' informal counting strategies in detail nor explained the alternative way of counting that she had suggested for her students who continued to struggle. According to her description, she knew what challenges her students might face and what other convenient ways she could use to help them. She mentioned that the reason why some students struggled with counting was due to their short attention spans.

None of this, however, interprets students' mathematical thinking, which implies Claire established no specific connection to the students' informal strategies. Although I brought up Derek's case right after I observed her lesson, she could not recall her alternative way of counting (counting 11 by dividing the amount into two sets of five and adding one to that). This finding indicates that Claire's descriptions are difficult to justify on the basis of students' thinking and their ability. Although she successfully noticed student's mathematical thinking and challenged these students by giving an alternative way in her actual math lesson, this was not linked to her ability to provide explicit descriptions of students' informal strategies.

Other participants showed similar responses to my question that asked about students' counting strategies. For instance, Brooke, after her lesson on addition, said,

> I know my higher ones got [the addition concept] and the lower ones still need a lot of practice. A lot of them already knew what the number represented before we even started, so that was good for those kids. And I guess with my lower ones, they still don't understand the representation of the number (Post observation interview, 5/6/2010).

Again, this description includes no detailed interpretation of students' thinking even though Brooke focused on what her students were able to do and what they needed further. She explained none of the informal strategies of her students, nor did she connect such strategies to what the students did understand or did not understand yet. Eva's response follows a similar tack. After Eva's lesson on counting, I asked her about students' counting strategies. She stated, "Most of my kids are good at counting. We've practiced putting stickers many times so they have a confidence in counting less than 10. Now, I think I can give them more than ten stickers" (Post observation interview, 4/12/2010). Again, this description fails to include any detailed explanation of students' counting strategies even though Eva praised her students' ways of counting and noticed that her students could count one by one successfully (Field note, 2/12/2010).

Also, in Anna's small group lesson, which was about matching concrete objects with the numbers, she was helping one of her students, Monica, count 30 dinosaurs. According to my field notes:

> Anna is looking at how Monica counts 30 dinosaurs. Once counted, Monica moves the dinosaurs to the other side and, counting one by one, successfully counts the 30 dinosaurs. After Monica finishes counting, Anna asks her how she was able to count 30 (Field note, 3/4/2010).

Anna noticed and attended her student's way of counting – to push aside the counted items and not to count them again. I asked her about Monica's way of counting; one such question was “Did you find some informal strategies in Monica’s counting?” Anna responded:

> Well, she is one of the brilliant students in my class. I knew she was able to count over ten numbers. She figured out like two here and then she stared here and “Oh, yeah, I have two here. I have one more here so I have like three.” Just randomly. And she really remembered what she had counted so well (post observation interview, 3/4/2010).

Anna described here how Monica counted and how she had a good memory, yet her description included no specific counting strategy—pushing aside the items already counted. This might be because Monica’s counting strategy appeared so naturally that Anna failed to notice what her strategy was. Most children quickly and easily develop their own informal strategies (Whitebread, 1995). Tang and Ginsburg (1999) suggested that a teacher’s noticing such informal counting strategies that seem to be typical and common was important in terms of recognizing, nourishing, and promoting students’ mathematical abilities and thinking. Also, this ability could help teachers connect students’ informal mathematical knowledge to school mathematics, which includes more abstract representations of mathematical ideas (Whitebread, 1995).

All of these examples appear to indicate that these teachers’ KCS, specifically in regards to interpreting students’ mathematical thinking and reasoning, was deficient or missing. Anna, Brooke, Claire, and Eva were able to build on and extend students’ mathematical thinking in their math lessons, which means they had a tacit knowledge (Greeno, 1987) of interpreting students’ thinking. They could not, however, explicitly describe how they interpreted students’ informal ways of mathematical thinking. This indicates that, in terms of understanding students informal mathematics, these teachers had only tacit knowledge – “knowledge needed for performing a task, but whose presence is unsuspected by the performer” (Greeno, 1987, p. 62). Another teacher, Diane, seemed to lack both tacit and explicit knowledge of how to attend to students' thinking.

Since none of them could explain or recall students' informal ways of counting, their knowledge about students' informal mathematics appeared to be tacit rather than explicit. Ball (1991) wrote that, "tacit knowledge, whatever its role in independent mathematical activity, is inadequate for teaching. In order to help someone else understand and do mathematics, being able to "do it" oneself is not sufficient" (p. 16). Having an explicit knowledge of students' informal mathematics is important for several reasons. Explicit knowledge "corresponds directly to things we can discuss and observe" (Greeno, 1987, p. 62). This means it can be generalized and shared by other teachers. If teachers have only tacit knowledge, such knowledge can be communicated implicitly only, as an unseen and unanalyzed component of performance. For knowledge to be public, general, and shareable by others, it must be explicit. Tacit knowledge expressed explicitly could demystify the task and enable teachers to reflect on and rethink about what they've done. This is important in terms of making their teaching practice more professional as well as in terms of communicating more completely with students who have similar types of problems in understanding what is to be learned. Also, explicit knowledge enables teachers to apply or transfer such knowledge to various situations. For example, having an explicit knowledge of students' counting strategies can help teachers apply it to teaching addition or subtraction. In order for teachers to consistently provide high quality math instruction that builds on the basis of students' thinking, their knowledge must be explicit.

Summary

In sum, this study found that by using Ball and her colleague's notion of KCS, pre-k teachers' KCS can be divided into three subcategories. These involve (a) understanding students' common errors, (b) understanding students' ability, and (c) addressing students' mathematical thinking. The findings showed that all the pre-k teachers in this study were capable of describing not just what common errors students generally made but why they were made. In addition, they were able to describe their students' ability to understand specific mathematical content covered in pre-k guidelines.

In addressing students' mathematical thinking, however, there was a clear difference among these teachers in terms of attending to students' informal ways of mathematical thinking and of scaffolding students on the basis of their mathematical thinking. One teacher in particular was found to be less likely to notice and respond to her student's mathematical thinking than were her more experienced counterparts. The teachers who were able to build on and extend students' mathematical thinking in their math lesson could not provide explicit descriptions of how they interpreted students' informal ways of mathematical thinking. These findings suggest that pre-k teachers need a stronger knowledge base in interpreting students' mathematical thinking and reasoning. This entails noticing students' informal knowledge, "the mathematical understandings they glean from everyday life and the strategies they devise to cope with everyday situations" (Baroody & Wilkins, 1999, p. 48). Having an explicit knowledge of students' informal mathematics can help teachers suggest appropriate solutions or guidance that meets students' needs and enables them to recognize what other convenient strategies they can provide (Kennedy, 1998).

Knowledge of content and teaching (KCT)

Knowledge of content and teaching refers to "knowing about teaching and knowing about mathematics" (Ball et al., 2008, p. 401). This involves mathematical knowledge of the design of instruction such as choosing the instructional materials or examples, evaluating instructional advantages and disadvantages, and deciding teaching sequence. Many of the instructional decisions that teachers make required an "interaction between specific mathematical understanding and an understanding of pedagogical issues that affect student learning" (Ball et al., 2008, p. 401). The tasks that are related to pre-k teachers' KCT include choosing appropriate examples or materials to teach specific content, designing the instructional settings, and deciding teaching sequence. In investigating such knowledge, I observed each teacher's math lessons and conducted post-observation interviews to ask them how they made instructional decisions. For example, I asked them why they chose to use specific examples or materials when teaching the planned content, why they decided to do small groups instead of a whole group, or how they arrived at their teaching sequence. Investigating such questions helped me understand how these teachers reflected on their teaching mathematical content to their students, a concern related to my third research question that asks how pre-k teachers reflect on their decision making process. Teachers' reflections on their lessons inform their KCT, "involving a particular mathematical idea or procedure, and familiarity with pedagogical principles for teaching that particular content" (Ball et al., 2008, p. 402).

Making Instructional Decisions

The five pre-k teachers' KCT was examined through their reflection on their decision-making practices that occurred while they taught planned content. This was exemplified in the following field note taken from Brooke's math lesson and her post-observation interview.

> Today is the second time Brooke teaches a lesson on the concept of addition. The students are sitting around on the carpet as Brooke reviews a book – Rooster's Off to See the World that introduces the concept of addition. In this book, different types of animals are added and the total amount of animals increases. While reading this book, Brooke discusses the concept of addition with the students. After finishing the book, Brooke introduces the next activity that students will do so that the students know what they are doing next. This activity is designed to help students learn about what adding means. Then, she takes out a set of paper plates that have different amounts of dots. Brooke asks two students to come up and gives the plates to them. A boy receives a plate that has one dot and a girl receives a plate that has three dots. They hold them so that everyone is able to see how many dots are on each of the plates. Brooke asks them how many dots there are altogether. She did this activity with two more pairs of students (Field note, 5/6/2010).

According to this field note, Brooke made two instructional decisions. First, she decided to start the lesson by reading a book in a whole group to review the concept of addition. Second, to get students engaged with the content, she implemented a math activity that used paper plates with different numbers of dots. To examine her KCT on teaching addition, I asked her why she chose to read that specific book and how she decided to use this book. She responded:

> Because the book is about adding more animals and then taking more animals away, so I thought that would be a good way to start getting them to think about how to add things together. And so that's why before I started the math lesson, I just reviewed the addition of the animals to see if I could get their mind on that track. And a lot of time we do try to do literature in math and I think that was a perfect book. Also, I checked this book out because we're studying farms this week. This is perfect to blend it in so that's how I ended up with the book (Post observation interview, 5/6/2010).

She chose this book because it not only addressed the concept of addition but was also tied to the theme (e.g., farm animals) that her class was learning that week. By using this book, she appeared to believe that students could engage in, explore, and elaborate on mathematical content as it arose during their in-depth investigation of a central theme or topic.

Integrating mathematics into classroom routines (e.g., theme) and learning experiences across subject matters was a common aspect of KCT among other pre-k teachers. For example, Claire mentioned that, "I usually start with the book because it integrates so well like with literature, rhyming words, different units, and of course there are math in it" (Post observation interview, 3/24/2010). Claire used to start her math lessons with a book because it fit into the learning theme well, in addition to being integrated with other subjects such as language art. Like Claire, when choosing the examples or materials to start with, these teachers used both their knowledge of specific mathematical content and their understanding of integrated approaches to teaching mathematics and general pedagogical issues such as organization of classroom routines (Ernest, 1987) that affect student learning.

The second instructional decision that Brooke made was doing a paper plate dot activity with her students. To understand why, after reading the book, Brooke implemented a math activity using paper plates that had different quantities of dots, I asked her about the purpose, advantages, and disadvantages of this planned math activity. She stated,

> I want them to see that there's different ways of adding other than with unifix cubes so my goal was that they would see that even though there wasn't a number. Even though it was just a dot representation, you could still add them. You can still make a different number. So that was the goal. In this age, they need visualization and this activity can help them show adding something is about making more. And I didn't put numbers because, like I said, the whole point of addition is to teach them that more of something makes more. And if I put numbers, I think they would just be telling me the numbers instead of them having to count each dot so there was more of a representation than anything else,

> yeah. The only drawback of this activity was like most students played with this plate so I had a hard time making them concentrate on this activity. Also, some students didn't know what this activity was about. I think they need manipulatives or unifix cubes again so that they can actually touch it and count it (Post observation interview, 5/6/2010).

According to Brooke, her choice of implementing the plate dot activity was influenced by her knowledge of mathematical content (e.g., goal of teaching addition) and of how students learn mathematics. She wanted to show her students what adding was without using concrete objects and without seeing the numerals. However, she still recognized that pre-kindergarteners learn math by interacting with something that is visual. She thus decided to show her students a set of dots to help them represent what adding was about. When planning this activity, she had to think about the mathematical content, about how she could take her students deeper into that content, and how best to foster her students' mathematical learning. In addition, she evaluated the instructional advantages and disadvantages of this activity and identified what different methods and learning materials afford her goal of instruction. She identified these methods based on an understanding of her students' ability, on her mathematical goals, and on her belief in how young children learn math. Such ability indicates her KCT, her mathematical knowledge of the design of instruction (such as choosing the instructional materials or examples), and her evaluation of instructional advantages and disadvantages.

Another participant, Diane, also decided which math activities and examples she would choose based on her developmental knowledge of her students and her knowledge of what worked best for pre-k students. For example she mentioned that:

> I really like to find all of the different activities and examples that I think would be good for this age group and then I just pull the appropriate manipulatives and things that can foster their learning. In this age, they need different kinds of objects so that they can actually see and touch them. It could be the same activity, but the more experiences as you give it to them in different ways, could catch somebody else's attention (Post observation interview, 4/7/2010).

Diane recognized that pre-kindergarteners learn math by interacting with various types of objects. With this idea, she planned math activities and prepared learning materials that could help student learning. For example, she used different colors of unifix cubes to show color patterns, and developed different kinds of counting math activities by using various types of manipulatives that students liked (Field note, 3/3/2010).

In addition to Diane, other teachers' KCT was also influenced by their own beliefs of what comprises effective math instruction. Consider the following descriptions given by Brooke:

> Well, especially at this age, I think if it's not hands-on, you lose them. . . . at this age you have to catch them with something they can touch, something they can hold, something they can see. It has to be visual. It has to be physical otherwise they don't grasp it very well. Using their own body, anything that is real is what helps at this age and that is the foundation. So there has to be a physical compartment to the lesson. It has to be hands-on (First interview, 2/22/2010).

Brook suggested that pre-k students learned math through multi-sensory experiences. She conceptualized effective math instruction as providing physical math activities so that students could touch and see. Such an understanding of effective math instruction influenced her choice of math activities, representation, examples, or materials. These examples indicated that pre-k teachers' KCT include considering multiple things such as mathematical content that they are to teach and the ways they can help student learn that content effectively.

Also, these teachers' KCT entailed how to develop fun and engaging math activities that are tied with learning themes or topics. This is related to the early childhood education field's endorsement of an integrated curriculum approach, which means mathematics must be integrated into other subjects, learning themes, and children's everyday routines (Lee & Ginsburg, 2009). Eva, for instance, chose farm animal stickers to use in her math activity because she thought it would be interrelated with “the farm animal theme’ that her class was learning at the time (Post observation interview, 3/10/2010). Another teacher, Anna, also developed a math activity that used eggs and chickens because she wanted to relate her math lesson to Easter (Field note, 4/13/2010). In all of this study’s pre-k teachers, this type of KCT was predominant. This indicates that these teachers’ KCT was aligned with the idea of developmentally appropriate practice that is grounded both in the research on how young children develop and learn and in what is known about education effectiveness (Bredekamp & Copple, 2009).

In sum, these teachers’ KCT involved the following three things: 1) knowing which examples or materials to use to teach a planned math content, 2) knowing what mathematical activity works best to teach specific content, and 3) evaluating the instructional advantages and disadvantages of those examples and activities to confirm whether they promote students’ learning (Ball et al., 2008). All of these tasks were also found in other teachers’ decision-making processes. These abilities require an interaction between a teacher’s understanding of specific mathematical content, an understanding of how students learn, and general pedagogical knowledge.

Differentiating Instructional Settings

In addition, the pre-k teachers’ KCT in this study also included the ability to provide different types of instructional settings based on students’ individual differences. Differentiation in instructional settings enabled these teachers “to adjust the pace and content of their instruction in a range of ways and to address students’ unique constellations of strengths and needs” (Goldstein, 2008, p. 466). Three instructional settings made frequent appearances in all the participants’ classrooms: whole-group open-ended discussions, small group cooperative learning activities, and one-on-one teaching.

All of the participants, to introduce or review basic mathematical concepts, typically began their lessons with a whole-group discussion. Students' individual experiences related to the specific math skills or knowledge was often addressed during this time. For example, in her sorting lesson, Claire read and discussed, in a whole group, a book about how things could be sorted. While reading and discussing this book, she took students' ideas about the ways they sort their items in their home such as clothes, toys, or shoes (Field note, 4/5/2010). Wallace, Abbott, and Blary (2007) suggested that guiding classroom discussions about children's' problem-solving experiences is an effective way to motivate children to recognize, verbalize, and explore the basic math concepts.

After the introduction, the participants implemented small group cooperative learning activities. For instance, Anna, after leading a whole-group discussion on counting, implemented a "shape-counting book activity." This activity was about making a booklet that illustrated how students represent given numbers by using different types of shapes. During this activity, her students glued the appropriate number of shapes (e.g., triangles, circles, rectangles, and so on) to boxes labeled 1-10. These became counting booklets and students exchanged booklets with one another to see how their peers made them. They discussed each other what and how many shapes they used (Field note, 3/11/2010). This type of small group math activity allowed students to work together to maximize, within a cooperative context, their own and each other's learning. By giving students mathematical tasks to discuss, problems to solve, and goals to accomplish, small group cooperative learning increased students' willingness to engage in problem solving and peer interactions (Tarim, 2009). Also, teachers were able to interact more intensively with students based on their interests, needs, and abilities.

In addition to small group instruction, participants worked with children one-on-one, when necessary, during center time or when everyone else was doing their individual seatwork. Such interactions could challenge students' thinking and help children discover the important math concepts that they hadn't caught onto before. These strategies can be seen in Anna's descriptions of how she differentiated her instructional settings to make her math lesson effective.

> I still do the whole group to teach or introduce the content that we are learning. But then I'll break them in, that's when they're at centers and I'll do the small groups and see, "Okay, you and you, still don't know how to count. You two, come work with me at the same time." So as a small group, I know we'll be practicing more than as the whole group. That's how I try to handle students' individual differences (Second interview, 2/25/2010).

While other students were in the center, Anna would interact individually with those students struggling with understanding particular math content. Also, she divided her students into three or four groups based on students' ability and worked with each group during the center time (Field note, 3/11/2010). This was how she dealt with students' individual differences.

Other participants, such as Eva also discussed the importance of one-on-one instruction in helping students who are a little behind. She explained:

> I usually work one-on-one with whomever I think has fallen behind and need additional lessons. Today, Emily was the one that I wanted to practice with to reinforce the counting skills, so as you might have noticed, I called her while other students were playing in the center and repeated the counting activity with her (Post observation interview, 3/22/2010).

While other students were playing in the center, Eva practiced the counting activity individually with those students struggling with it. These participants made use of center time to work with students individually who needed more math lessons. To differentiate instructional settings effectively, they considered many instructional or curricular issues such as students' individual differences and capabilities, time management, the types of math activities and mathematical content that they were to teach. Such consideration required a strong KCT base, "a coordination between the mathematics at stake and the instructional options and purpose at play" (Ball et al., 2008, p. 401).

Summary

In this section, I examined pre-k teachers' knowledge of how to teach mathematics. This knowledge was divided into two subcategories – KCS and KCT. These teachers' KCS involves:

- Understanding students' common errors
- Understanding students' abilities and prior knowledge
- Noticing and addressing students' mathematical thinking

These abilities are important in terms of connecting children's informal math with their capability to grasp more academic mathematics (Ginsburg & Ertle, 2008).

All the pre-k teachers in this study were able to describe their students' common misconceptions, and regarding specific math content, their abilities. On the other hand, there appeared to be a clear difference among these teachers in terms of attending to students' informal ways of mathematical thinking and of scaffolding students on the basis of their mathematical thinking. Diane, a novice teacher, seemed less likely to notice and respond to her student's mathematical thinking than were the experienced teachers. This might be because teachers with more experience have developed meaningful ways to discern patterns and put together information in complex situations, which helps them be better able to notice the details of children's thinking and reasoning (Bransford, Brown,

& Cocking, 2000). Although the experienced teachers were able to build on students' mathematical thinking in their math lessons, they could not provide explicit descriptions of how they interpreted students' informal ways of mathematical thinking, which means they likely had a tacit rather than explicit understanding of students' mathematical thinking.

These teachers' KCT was examined through how they reflected on their decision-making practices that occurred while they taught planned content. Their KCT included:

- Knowing what mathematical activity or examples worked best to teach specific content,
- Evaluating the instructional advantages and disadvantages of those examples and activities,
- Providing different types of instructional settings based on students' individual differences.

All of these tasks are important because the teaching of mathematics to a group of more than 20 students requires organizational and managerial knowledge and skills (Ernest, 1987). Such knowledge and skills are in addition to the types of knowledge and skills that come with SCK and KCS. These tasks involve "a particular mathematical idea and familiarity with pedagogical principles for teaching that particular content" (Ball et al., 2008, p. 402).

PRE-K TEACHERS' KNOWLEDGE OF CONCRETE AND ABSTRACT MATHEMATICS

Pre-k teachers' MKT in this study was examined by using four subcategories that include CCK, SCK, KCS, and KCT (Ball et al., 2008). In addition to these four subcategories, a new category of these pre-k teachers' mathematical knowledge emerged. While investigating their MKT, knowledge of what it means to be concrete and abstract in teaching math was found in this study. Although such knowledge was not included in Ball et al.'s framework (2008), it was constantly present throughout this study. The recent report by the National Research Council points out that an effective teacher mediates students' understanding of the representations and serves as a bridge between the concrete and the abstract mathematics. For instance, early childhood teachers provide concrete objects or activities to teach abstract mathematical ideas such as number, addition, or subtraction. At the same time, they need to help students grasp the abstract concepts by making a coherent connection with concrete activities. Thus, investigating these teachers' understanding of concrete and abstract mathematics is important and meaningful in order for teachers to provide effective and high-quality math instruction. Although such understanding is not directly related to my research questions, it informs the core characteristics of pre-k teachers' MKT. In addition, it helps researchers understand how such knowledge impacts pre-k teachers' math instruction.

Knowledge of concrete mathematics

MKT that the pre-k teachers in this study had was centered on how to make math lessons concrete, emphasizing "doing math." To implement effective math instruction, these teachers focused on developing fun and engaging math activities. In one form or another, all of these teachers noted that young children learned mathematics by touching and moving concrete objects. In much of their talk about effective and developmentally appropriate math instruction, these teachers saw as essential for young children's mathematical learning concrete objects, physical materials, or manipulatives. They stated that making their math lesson concrete could promote pre-k students' mathematical thinking and help them understand the content. Consider Brooke's descriptions of her math teaching philosophy.

> Yes, I still have to [use manipulatives] even though I know that they can do two plus one. I'll still have the two teddy bears plus the one teddy bear. I think they need to have the visual. If I take the teddy bears away, they might not have that visual to make the connection. So, we always have to use those concrete manipulatives (Second interview, 3/8/2010).

To teach the addition concept of two plus one, Brooke stated that she must use concrete objects. She assumed that pre-k students needed to have the visual so that they could make connections with abstract mathematical ideas. The other teachers also emphasized the importance, when doing math, of touching and moving the concrete objects. Anna stated, "Children have to learn [mathematics] more on a real life level and tie it into concrete things" (First interview, 2/16/2010). Claire also noted that, "It's important to get a chance to work with the manipulatives and do things [when teaching math] with these [pre-k] children" (First interview, 2/22/2010). According to Diane, she stated that "being a hands-on learner" and doing fun activities are most effective way to teach math to four-year-old children (First interview, 3/3/2010). Eva mentioned that, "pre-k math must be fun and interesting. To do so, it must be based on children's everyday experiences, and something that is hands-on" (First interview, 3/3/2010). All of these teachers seemed to strongly believe in the effectiveness of using concrete objects when teaching math. Many early childhood teachers and educators have emphasized the importance of using concrete objects, physical materials or manipulatives.

Making math lessons concrete was clearly a major focus shared by these five pre-k teachers. Yet other scholars, such as Ginsburg, Lee, Baroody, and Clements, have expressed concern about early childhood teachers' overemphasis on implementing tangible and concrete math lessons. They contend that the use of materials is effective only when they are used to encourage children to think and make connections between the objects and the abstract mathematical idea. This suggests that early childhood teachers – including these five pre-k teachers – ought to re-consider how to promote students' thinking while implementing hands-on activities and how to make connections between concrete objects and abstract mathematical ideas.

Connecting the concrete and the abstract

As a recent report by the National Research Council points out, "physical materials are not automatically meaningful to students and need to be connected to the situations being modeled" (Kilpatrick et al., 2001, p. 7). An effective teacher mediates students' understanding of the representations and serves as a bridge between the concrete and the abstract. Investigating how these pre-k teachers make connections between the concrete and the abstract can help me understand their pedagogical content knowledge, which is related to my second and third research questions. To examine this, I asked such question as "How do you connect your planned hands-on math lessons with abstract mathematical ideas?" Although all the pre-k teachers in this study agreed that it was important to connect concrete math to abstract mathematical ideas, their knowledge of how to make such connections reflected a limited view. Consider the following descriptions of what each teacher mainly did to connect their planned math activity to abstract and symbolic mathematical ideas.

> Well, I write [the number] when they're doing it [solving the problems] but I don't expect them to have to write it. When we're doing them, I say, "So this is like three volcanoes and then two dinosaurs," – like I always have numbers written. So you're doing three plus two and then they know and we put it all together. After they put it all together, now I read the numbers so they can see it, but I don't expect them to have to write the equations (Anna, second interview, 2/25/2010).

Anna stated that when teaching 3+2, she wrote numbers or equations so that her students could see those numerals. Although she does not expect her students to have to write those numbers and equations, she thought it important to introduce them.

Brooke also noted that:

> Well, one of the ways I think that has helped is I use a lot of what they bring to school. If they bring a folder and a pencil and a snack, I use something that's theirs. So I'll say, "Let's see. How many folders did you bring to me today?" "Oh, Ms. Gonzalez. I brought one." "Oh, let me write that. One." And I'll make the number one. And I'll say, "How many did you bring me today? One." So you know I'll show her a line down. "This is a one." Obviously we do this many, many, many times (Brooke, second interview, 3/8/2010).

According to Brooke, she tied the number of real-life objects with the numerals. For instance, she asked her students how many folders they brought and, based on her students' responses, wrote the number that matched the number of folders.

Another participant, Claire, stated that:

> I think [making connections between the concrete and the abstract] would be like counting the number and writing it. Do a lot of that or matching a number with a set to figure out which ones are the same. But I do have some worksheets and stuff that have the plus or minus on there. The ones I can think of have, like, three birds, let's say, plus two birds equals blank. And you can see the three birds but you can also see the numerals so there are things like that to present together (Claire, Second interview, 3/5/2010).

Claire thought that making connections between the concrete and the abstract would be like "counting the number and writing it." In addition, she used some worksheets that mathematical symbols, such as plus, minus, or numbers as a means of connecting concrete with the abstract mathematical ideas. She seemed to believe that doing worksheets and writing numbers can mediate her students' hands-on experiences and serve as a bridge between the her planned concrete math activities and the abstract mathematical ideas.

Diane said:

> Well, I mean, I think the paperwork that we do is not concrete. It's not something that they're touching or moving around or playing with. They're actually making that connection. This is the number one, whether they're just tracing it and it's already written there for them, but we do have the opportunity for them to trace it and then have a space for them to try it on their own. So I think that's the kind of

> connection they're trying to do and then we also have on that page if we're learning about the number five, there's going to be five butterflies on the page that they can sit there and they can count and they match it to the number five and they're able to trace the number five (Diane, second interview, 3/10/2010).

Diane also mentioned that the paperwork not being a concrete activity, that is, it is more abstract ways of teaching math. On paperwork, it has pictures of concrete objects (e.g., five butterflies) and asks students to match those objects with the number (e.g., number five). Diane appeared to believe that mediating students' concrete math experiences and the abstract mathematical ideas was to give students a chance to trace the numerals that matched with the given objects. Eva said that:

> Let's say we are learning about eight. "Here's the number eight." So I see that they write it and, I mean, we're learning to recognize the eight, and writing it is just another way for them to learn to recognize that number better. And they can have the connection between eight and number eight (Eva, second interview, 3/10/2010)

Eva thought that writing the number helped students connect the concrete with the abstract mathematical ideas. For instance, she stated that she asked her students to write number eight when they learned about eight.

According to each of these teacher's descriptions, their knowledge of connecting concrete to abstract mathematical ideas meant showing or writing the numerals. They either showed the numerals or equations or did the paperwork so that students could gain experience with writing the numerals. My observational data also illustrated that these teachers wrote the numbers, introduced the equations or did the individual paperwork to help students internalize abstract mathematical ideas (see the vignette of Brooke's lesson on p.). However, early math educators (e.g., Clements, 1999; Gisnburg & Lee, 2009; Haylock & Cockburn, 1997; Sophian, 2004) have shown concern about such limited views. They contended that the approaches used by these teachers (e.g., writing numbers or equations as a means of connecting concrete math experience to abstract mathematical ideas) were too simple. They only represented one way of understanding numbers and number operations. Such scholars as Haylock and Cockburn (1997), Clements (1999), and Larson (2002) suggested that early childhood teachers need to understand the complex network of number and number operations represented by the symbols. This issue will be discussed more in the next chapter.

Summary

Understanding pre-k teachers' knowledge of the concrete and the abstract mathematics provides an empirical foundation in unpacking pedagogical content knowledge. Such understanding is related to my second and third research question, which addressed knowledge of how pre-k teachers teach mathematics. Pre-k teachers in this study, in terms of teaching math to young children, share an understanding of the importance of using concrete objects or manipulatives. They also agreed that it was important to connect concrete math to abstract mathematical ideas. Their knowledge of how to make such connections, however, was a product of a limited view. They all described that they showed the numerals or equations to their students or did the paperwork so that students could have experience writing the numerals. This simplistic view could limit children's understanding of numbers and number operation (Larson, 2002). To be an effective math teacher is to serve as a bridge between the concrete and the abstract. To do so, early childhood teachers – including these five pre-k teachers – need to reconceptualize the meaning of "concrete" (Clements, 1999) and understand various aspects of numbers.

CONCLUSION

In this chapter, five pre-k teachers' MKT was examined by using Ball et al.'s framework (2008), which includes CCK, SCK, KCS, and KCT. First, within the school district and across the school districts, these teachers' CCK differed substantially. However, the relation between such differences and their actual teaching did not come into play. Second, these teachers' SCK was exemplified in the following four mathematical tasks:

- Understanding how the specific math content is connected to students' later mathematical learning
- Promoting students' mathematical thinking
- Presenting mathematical ideas
- Modifying the lesson

All of these tasks were something that these teachers routinely did but demanded unique mathematical understanding and reasoning. Third, these teachers' KCS involved:

- Understanding students' common errors
- Understanding students' ability and prior knowledge
- Noticing and interpreting students' mathematical thinking

These abilities are important in terms of connecting children's informal math with their capability to grasp more academic mathematics (Ginsburg & Ertle, 2008). Finally, these teachers' KCT included:

- Knowing what mathematical activity or examples worked best to teach specific content
- Evaluating the instructional advantages and disadvantages of those examples and activities
- Providing different types of instructional settings based on students' individual differences

In addition to these four subcategories, what emerged in this study was their knowledge of what it means to be concrete and abstract when teaching math. Although such knowledge was not included in Ball et al.'s framework (2008), it was constantly present throughout this study and helped me involve in more detailed investigations on my second and third research questions. All of the five pre-k teachers seemed to strongly believe in the effectiveness of using concrete objects or manipulatives. They recognized the importance of connecting concrete math experiences with abstract mathematical ideas. Also, their knowledge of how to connect the concrete with the abstract was similar. Giving their students chances to write the numerals was how they tried to make such connections. However, this simplistic view could limit children's understanding of numbers and number operation (Larson, 2002). Many researchers in early math education contend that early childhood teachers need to understand the meaning of concrete and abstract mathematics. In the next chapter, this issue will be discussed in more detail. Also, I will examine the significance of these findings by comparing and contrasting these pre-k teachers' MKT with the teacher knowledge examined in the literature. Finally, I will discuss the implications for teachers and teacher educators and questions for future research.

DISCUSSION

Chapter 5: Significance and Implications

Teachers' content knowledge for teaching can serve as a compass, guiding teachers in the right direction as they educate our next generation of students. This dissertation focuses particularly on early childhood teachers' knowledge of teaching/learning mathematics. I have attempted here, as an early childhood researcher, to suggest two things – what kinds of mathematics should be taught and how they should be taught. Using Ball and her colleagues' framework (2008), this research study examined five unique pre-k teachers' mathematical knowledge for teaching (MKT). This framework includes common content knowledge (CCK), specialized content knowledge (SCK), knowledge of content and students (KCS), and knowledge of content and teaching (KCT). The findings from this research suggest that pre-k teachers' knowledge entails both knowledge of subject matter and pedagogical content knowledge, which are also found in teachers who teach math to older children (e.g., elementary school students). In addition, this research identified what these pre-k teachers knew about math and teaching/learning math as well as what they still needed to know to provide high quality and effective math instruction. These findings inform researchers and teacher educators how professional development and teacher training can help early childhood teachers have deep mathematical knowledge for teaching.

This research began with an examination of how pre-k teachers understand the mathematical content they are to teach and its connection to children's further/later mathematical learning. It also examined how these teachers saw themselves teaching and learning math. Having examined these areas, I'm able to address their knowledge of what and how to teach math to young children. In presenting this investigation, I introduced

Ball and her colleagues' theoretical framework of mathematical knowledge for teaching (MKT) (2008). MKT is defined as "the mathematical knowledge needed to carry out the work of teaching mathematics" (Ball et al., 2008). I then covered the recent literature on early mathematics education and directed attention to its explanation of the importance of early childhood teachers' deep understanding of mathematical content and of their math instruction. After discussing my research question and the terms that informed my investigation, I outlined the specific methodology for this case study of pre-k teachers' mathematical knowledge for teaching. I then presented two major findings. The first was that each of the five teachers' MKT was consistent with what has been suggested in the literature on math education. However, in these pre-k teachers, there was a unique aspect of MKT. The second findings was that these teachers needed two types of knowledge to promote children's mathematical thinking and reasoning: knowledge of children's mathematical thinking and knowledge of how to connect concrete math experiences to abstract mathematical ideas.

In this chapter, I address the significance of these two major findings. First, I discuss the four types of mathematical knowledge and skills (CCK, SCK, KCS, and KCT) that these pre-k teachers possess. I also compare and contrast them with the teacher knowledge examined in the literature. Then, by examining research literature on early math education, I suggest what mathematical knowledge and skills they still need to attain to offer high-quality and effective math instruction.

WHAT PRE-K TEACHERS' MKT ENTAILS

Each of the five pre-k teachers' MKT was divided into four subcategories including CCK, SCK, KCS, and KCT (Ball et al., 2008). These teachers' MKT was consistent with mathematical knowledge and skills that were found in other scholars' studies (e.g., Ball et al., 2008; Ginburg & Amit, 2008; Hill et al., 2008). Their MKT entailed the tasks involved in teaching mathematics to older children. At the same time, this study found characteristics of pre-k teachers' MKT that differed from that of elementary or secondary school teachers. In the following sections, I will briefly describe what these pre-k teachers' MKT entailed and how it is similar to or different from that found in the literature.

CCK

The findings from this study suggest that pre-k teachers' MKT entails CCK, SCK, KCS, and KCT (Ball et al., 2008). First, their CCK involves knowing mathematical terms or ideas that are covered in their state or district pre-k guidelines. For example, to teach shapes, these teachers must know the characteristics and definitions of each shape. Also, to teach measurement skill, these teachers must know what length, weight, and volume indicates and how they can be measured. To examine such knowledge, I asked each participant to define assorted shapes, such as a circle, and to describe length, weight, and volume and to name what tools could be used to measure them. This type of knowledge allows teachers to know the material they teach, recognize students' right or wrong answers, use mathematical terms and notations correctly (Ball et al., 2008). Such ability requires mathematical knowledge and skill that others have as well, which means it is not special to the work of teaching. Their CCK is similar to Shulman's (1986) notion of content knowledge, which refers to "knowledge of the facts or concepts" (p. 9) of a subject matter. Kennedy (1998) also used the term "recitational subject-matter knowledge" to refer to the ability to "recite specific facts on demand, to recognize correct answers, and to define terminology correctly" (p. 253).

Many research studies demonstrated a substantial link between strong knowledge of subject matter and high mathematical quality instruction (Ball et al., 2008; Hill et al., 2008; Kennedy, 1998). For instance, after analyzing the videos of teaching, Ball et al., (2008) also suggested that an understanding of the mathematics in the curriculum or standards plays a critical role in planning and carrying out instruction. My research found that the five pre-k teachers possessed varying levels of CCK. When asked about to define the mathematical concepts and terms used in pre-k, three of the five pre-k teachers, Anna, Claire, and Eva, were able to provide explicit definitions. Brooke and Diane were unable to provide clear descriptions.

Such differences among these teachers' CCK might not come into play in their actual math lessons. While many researchers (e.g., Ma, 1999; Lampert, 2001) found that teacher's understanding of the subject matter determines the quality of instruction, the relationship between these teachers' CCK and their teaching practice was uncertain in this study. For example, Anna was able to provide a clear definition of circle, but in her geometry lesson her CCK of circle did not come into play (Field note, 2/20/2010). Brooke was able to provide a clear definition of length and how to measure it and when teaching it, such knowledge did come into play (Field note, 4/15/2010). She provided her students accurate descriptions about what length represents and how it could be measured (Field note, 4/15/2010). The CCK of other teachers was not found in their lessons. None of their math lessons included the concept (e.g., shape, measurement) of what I had asked for to identify their CCK. Since only Brooke's case indicates a positive correlation between her CCK and her math instruction, it is unclear that the differences among these pre-k teachers' CCK had direct or indirect effects on their quality of teaching.

This might be because the mathematical content in pre-k includes relatively small amounts (compared to elementary or secondary school mathematics) of mathematical terms or ideas that are required knowledge of teachers. Although these pre-k teachers' CCK did not appear in their math lessons consistently, such knowledge can have an indirect effect on teaching practices (Brophy, 1992). As Brophy explained, when teachers' content knowledge is more explicit, they will tend to teach the subject more fully and respond to students' questions more sensitively. Also, they will have an insight into what to teach and what not to teach their pre-k students. CCK should not be neglected; indeed, it is an important component that pre-k teachers must have, and thus, the inconsistency found among the teachers found in this study demonstrate that further research that examines the relationship between pre-k teachers' CCK and their quality of teaching is needed.

SCK

The findings from this research suggest that these pre-k teachers' SCK involves all the tasks that are found in what elementary and secondary schoolteachers routinely do in their math lessons. The following list illustrates what these pre-k teachers' SCK entails.

- Presenting mathematical ideas
    - Using real-world examples:
    - Making the lesson concrete
    - Connecting children's informal ways of thinking to formal arithmetic knowledge
- Connecting the content to students' later mathematical learning
- Promoting students' mathematical thinking
    - Asking productive mathematical questions
    - Understanding the breadth and depth of early mathematics

        - Integrating the big ideas in mathematics to teach a specific mathematical concept
    - Modifying the lesson based on students' ability
        - Repeating the mathematical tasks or re-teaching the content
        - Differentiating their lesson according to students' prior knowledge or ability
        - Redesigning the mathematical activity or remodeling the problems

In chapter four, I documented each of these pre-K teachers routinely did most of these tasks. For example, four of the five pre-k teachers (Anna, Brooke, Claire, and Eva) were able to describe how they presented the concept of addition to young children. My interviews with these teachers suggested that they used real-world examples or activities, and concrete objects to present how addition works and that they connected the students' concrete experience with abstract and formal arithmetic knowledge. To promote students' mathematical thinking, Brooke asked productive mathematical questions in her measurement lesson and integrated the big ideas of mathematics such as counting and number identification, and comparing the quantities while teaching measurement (Field note, 4/15/2010). Also, these teachers modified the lessons when students seemed to have difficulty in understanding the content or completing the tasks. To modify the lesson, these teachers repeated the lesson and went back to the basics (Anna, second interview, 2/25/2010), differentiated the lesson according to students' prior knowledge (Eva, second interview, 3/10/2010), and redesigned or changed the math activity to be more doable for the students (see Claire's interview on p. 89).

These lists of SCK were also found in other research studies. Ginsburg and Amit (2008), for example, conducted a case study of one preschool teacher's teaching math. They found that her teaching entailed "profound knowledge of the subject matter, pedagogical content knowledge, lecturing, introduction of symbolism, connecting everyday experience to abstract idea, encouraging students to engage in concrete activity, and responding to students' questions and interests" (p. 284). Ball and her colleagues contended that these kinds of tasks demanded unique mathematical understanding and reasoning. They also noted that such tasks involve "an uncanny kind of unpacking of mathematics" because teachers must make features of particular content visible to and learnable by students (p. 400).

This study found numerous examples of unpacked knowledge. Teaching about adding, for instance, required understanding the adding system in a self-conscious way that went beyond the kind of tacit understanding of adding needed by most people. These teachers were able to talk explicitly about how to make and use mathematical representations effectively (e.g., recognizing how to use unifix cubes to teach 2 + 2 = 4, finding real-life examples that helped students understand what adding means). For example, Claire stated that, "I would want them to actually use manipulatives, such as unifix cubes, buttons, blocks or anything that they can count, and put those things together to make sets together" (Second interview, 3/5/2010). Another participant, Anna, noted that, "I'll show this [addition] on what I call an 'addition box.' They can just make a box right here so they know the first number goes in there. I draw a plus sign next to it and then, in another box, the second number goes in there and then [there is] the third box which is the total after the equal sign" (Second interview, 2/25/2010). They were able to provide in-detail descriptions of how they taught addition effectively. In addition, the teachers were able to integrate other basic math ideas to teach a specific concept. In Brooke's measurement lesson, she made a connection between adding and counting or measurement skills (see vignette of Brooke's measurement lesson on p. 67). All of these are examples of these teachers working with mathematics in its unpacked form.

The SCK of these five pre-k teachers was centered on making their math lesson concrete for their students by offering hands-on activities. According to Ball (1992), it is a widespread belief that providing concrete objects, physical materials, or manipulatives have been seen as developmentally appropriate math lesson. This was the unique characteristic of pre-k teachers' SCK found in this study. For instance, these teachers provided concrete math activities to present the mathematical ideas such as addition or counting skills. Consider Claire's descriptions of how she taught addition or subtraction.

> I would want them to actually use manipulatives, such as unifix cubes, buttons, blocks or anything that they can count, and put those things together to make sets. Sometimes they needed to use their fingers to show that. So I think I just weave in with my wording and then intentionally try to show them by moving them or moving things together, putting things together to show them addition and then having them do their own. (Second interview, 3/5/2010)

Claire stated that she might use "manipulatives," concrete objects such as "unifix cubes, blocks, or buttons" and, when teaching addition, demonstrate by moving those manipulatives. Brooke also said that to teach addition, she might ask her students to find five things in the classroom. Then, based on the number of objects that students find, she would, to address the concept of addition or subtraction, pose such question as, "How many more or less do you need to make five" (Second interview, 3/8/2010). These interviews show how Claire's and Brooke's SCK is focused on providing concrete math experiences to teach addition concept.

Unlike elementary or secondary school mathematics, pre-k math includes basic skills such as counting, identifying number or simple shapes, comparing the quantity, adding/taking away, and creating pattern (TEA, 2008). Thus, their SCK might be put to use by providing concrete math experiences such as providing hands-on activities or using concrete objects rather than "giving mathematical explanations, choosing and developing useable definitions, and linking representations to underlying ideas and to other representations" (Ball et al., 2008, p. 400).

KCS

KCS helps teachers offer math instruction that builds on children's mathematical knowledge in conjunction with "teacher guided experiences that allow children to use and develop this knowledge into a strong mathematical foundation" (Herron, 2010, p. 360). The five pre-k teachers' KCS involved: (a) understanding students' common errors, (b) understanding students' mathematical ability, and (c) noticing/attending to students' mathematical thinking. Understanding students' common errors and their mathematical ability are the important components of Shulman's notion of pedagogical content knowledge. Shulman (1986) described pedagogical content knowledge as follows:

> Pedagogical content knowledge also includes an understanding of what makes the learning of specific topics easy or difficult: the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning of those most frequently taught topics and lessons (p. 9).

Shulman suggested methods to provide the most useful ways of representing and formulating the subject to make it comprehensible to students. To do so teachers needed to understand what made understanding a particular concept difficult or easy and the conceptions or preconceptions students commonly brought to the frequently taught topics in the discipline. Also, teachers must know what students will find interesting and motivating. All the pre-k teachers in this study had such understandings and were able to talk explicitly about their students' misconceptions, errors, prior knowledge, and abilities.

The last domain of KCS in this research is noticing/attending to students' mathematical thinking. Many mathematics educators have accumulated substantial knowledge about both children's mathematical thinking within specific content areas and the significance of teachers' eliciting and building on children's thinking (Jacobs et al., 2010). Instruction that builds on children's ways of thinking can elicit rich instructional environments for students. Gearheart and Saxe (2004), for instance, examined two groups of teachers' math instruction. One group consisted of teachers who had participated in the professional development program Integrating Mathematics Assessment (IMA). The other group consisted of teachers who had not. IMA was "a well-researched example of professional development that helps teachers interpret children's mathematical thinking and guide children toward deeper understanding" (p. 304). IMA was designed to help teachers develop KCS. The results demonstrated the effectiveness of IMA. IMA classrooms made significantly greater gains in conceptual understanding than traditional classrooms. These findings demonstrate that, when teachers build their capacities to assess children's mathematical understandings, their students will grow in their knowledge of mathematical concepts.

Four of the five teachers in this study were capable of noticing students' mathematical thinking during their small group math lessons. In my observations of Diane's teaching, she did not attend to students' mathematical thinking and reasoning. In the vignette of Diane's math lesson on patterns, she asked no questions about why her students created them as they did, nor did she provide any opportunity for each student to solve the problem himself (Field note, 3/30/2010). Claire, on the other hand, attended to the details of students' counting strategy (e.g., subitizing) when teaching counting skills and scaffolded the students on the basis of students' way of thinking (Field note, 3/24/2010).

While noticing their students' mathematical thinking, these teachers were unable to provide in detail clear descriptions of how they interpreted students' thinking. They explained neither the details of student strategies nor how these details might have reflected what the students did understand. For instance, consider how Claire interpreted her students' informal counting strategies.

> Two types of kids come to mind. They were recounting over and over or they didn't know what number came next. Some [have] short attention spans and I struggled with giving them enough attention long enough to get something done. So I really have to focus on listening to them count, watching them count, stick to them and not get distracted myself. [I have to] try to keep them attentive and them not get distracted. And then, for others, just like taking time and giving them an opportunity to keep counting and not being frustrated with their not knowing what number came next (Post observation interview, 3/24/2010).

This response appears to provide no evidence of Claire's interpreting of students' understandings of counting and numbers even though she had addressed students' informal mathematical knowledge during her small group math lesson. She neither described students' informal counting strategies with detail nor explained the alternative way of counting that she had suggested for her students who kept struggling. This indicates that these teachers' knowledge of children's mathematical thinking was tacit rather than explicit. Just as I may know how to solve certain math problems but not be able to explain it, these teachers might have understood children's mathematical thinking and not been able to explain that understanding. Nevertheless, having explicit knowledge is important. It not only enables better explanations but also enables teachers to decide three things: what is most important to teach, what should be taught now rather than later, and what kind of problems could be posed to students that would most likely facilitate their understanding of a particular ideas (Kennedy, 1998). When students work the problems in erroneous or meaningless ways, these teachers might, unless their knowledge is explicit, have difficulty in deciding when to help, what to do, or how to do it. For instance with Claire, having this explicit knowledge may have enabled her to demonstrate students' counting strategies and how she scaffolded her students based on such strategies.

According to Graeber (1999), carelessness, never having been taught, or lack of thought are rarely the causes of students' wrong answers. Wrong answers frequently have theoretical underpinnings that teachers fail to anticipate. The teachers in this research also showed a tendency to think that students gave wrong answers to particular mathematical ideas (e.g., counting, adding, and subtracting) because they had not been taught such concepts or that they were not developmentally ready to learn the concept. For example, Anna, in her first interview, said:

> When teaching addition, I could see that some of my students are not ready for addition. They even had a hard time counting like counting the same objects repeatedly. So, they can't handle addition because they have to handle counting first and once they master that then I can teach addition (2/16/2010).

Anna thought of her students' mistake in counting as a developmental issue rather than as an issue with their own way of thinking or informal counting strategies. Tang and Ginsburg (1999) contended that students' difficulties with problem solving didn't necessarily indicate they could not think of any effective ways of solving the problems. Young children often have idiosyncratic ways of using words to refer to objects and situations. Also, they may solve the problems in unconventional ways. Their incorrect language or funny answers may not indicate a lack of reasoning ability. This underscores the importance of understanding students' reasoning. When teachers understand and support students' reasoning and thinking, they can suggest alternative ways that help students handle the problems rather than classifying them prematurely by developmental stage designations or by lack of mathematical knowledge. Thus, Anna could have provided appropriate math instruction to teach addition if she had knowledge of her students' informal ways of understanding what adding is.

KCT

KCT helps teachers understand pedagogical issues that affect student learning and of specific mathematical content or ideas. In this research, the five pre-k teachers' KCT entails the following four things: (a) knowing which examples or materials to use, (b) knowing what mathematical activity works best to teach specific content, (c) evaluating the instructional advantages and disadvantages of those examples and activities, and (d) providing different types of instructional settings based on students' individual differences. For instance, Brooke used a storybook that contained the concept of addition in her math lesson on addition. Also, she developed a paper plate activity to present the addition concept (Field note, 5/6/2010). Brooke was able to discuss the instructional advantages and disadvantages of paper plate activity (Post observation interview, 5/6/2010). In addition, these teachers such as Anna and Eva provided different types of instructional setting that include whole group, small group, and one-on-one lesson. The abilities reflected by such choices are important components of pedagogical content knowledge – "the ways of representing and formulating the subject that makes it comprehensible to students" (Shulman, 1986). Leinhardt (1986) also suggested that being an effective mathematics teacher required mathematical content knowledge as well as an understanding of the instruction process needed to efficiently transfer this knowledge to students.

These teachers' KCT was examined when they reflected on their decision-making process after they employed math activities. First, the general feature of KCT in this study was associated with how pre-k teachers utilized concrete objects. For instance, Diane stated that, "I really like to find all of the different activities and examples that I think would be good for this age group and then I just pull through the appropriate manipulatives and things that can foster their learning" (Post observation interview, 4/7/2010). Diane emphasized the effectiveness of using concrete objects and manipulatives. This might be due to the differences in the knowledge bases of elementary/secondary and early childhood mathematics. For example, elementary-level mathematics uses written notation to support mathematical thinking. Preschool children, despite being active mathematics learners, are typically unprepared to understand, much less use, written arithmetic. Instead, three- and four-year-olds are primed to notice and explore the quantitative relationships in the world around them, and to begin using other forms of primary representation (e.g., movement, models, pictures) to crystallize information about patterns, shapes, space, size, and numbers (Clements et al., 2004; Copley, 2000)

Second, these teachers' KCT entailed how to develop fun and engaging math activities that were tied to learning themes or topics. This is related to the field's endorsement of an integrated curriculum approach, which means mathematics must be integrated into other subjects, learning themes, and children's everyday routines (Lee & Ginsburg, 2009). Eva, for instance, chose farm animal stickers to use in her math activity because she thought it would be interrelated with "the farm animal theme" that her class was learning at the time (Post observation interview, 3/10/2010). Another teacher, Anna, also developed a math activity that used eggs and chickens because she wanted to relate her math lesson to Easter (Field note, 4/13/2010). In all of this study's pre-k teachers, this type of KCT was predominant. The integration approach to teaching mathematics is a general feature of early childhood education. It allows children to "engage in, explore, and elaborate on mathematics as it arises in the course of their in-depth investigation of a central theme or topic" (Lee & Ginsburg, 2009, p. 41). Also, it takes advantage of the natural relationships between subjects such as literacy, music, and art.

These teachers' KCT was also influenced by their own beliefs about how young children learn and of what comprises effective math instruction. Consider the following descriptions given by Brooke:

> Well, especially at this age, I think it has to be a lot of hands-on because if it's not hands-on, you lose them. They don't . . . they're not interested. So at four years old, even five, I mean, I'm sure five-year-olds, six-year–olds, too, but at this age you have to catch them with something they can touch, something they can hold, something they can see (First interview, 2/22/2010).

This observation informs Brooke's understanding of effective math instruction for young children and of how young children learn math. Brooke appeared to believe that pre-k students learned math through multi-sensory experiences. She conceptualized effective math instruction as providing physical math activities so that students could touch and see. This belief influenced her overall instruction decision processes including her choice of math activities, representation, examples, or materials. The other four teachers also had their own strong beliefs about how children learn mathematics and of what, for young children, is effective math instruction. These teachers held untested assumptions about their students and how they learned. This influenced how they thought about effective math instruction and how they developed mathematical tasks.

Teachers' beliefs about learners and learning influence their teaching practice (Calderhead, 1996) and their KCT. Anning (1988), in a study of teachers of young children, found that the teachers held various commonsense theories about children's learning that influenced how they structured tasks and how they designed instructional settings. Numerous case studies also have reported the interconnection between teachers' beliefs about teaching and learning and the ways in which they plan their work and teach in the classroom (Calderhead, 1996). In this study, pre-k teachers' beliefs about teaching and learning mathematics influenced their KCT. They have untested assumptions about learners and learning, which might have influenced by their teacher training or college experiences. They made instructional decisions and designed plans based on their beliefs about how young children learned math and about how best to foster their mathematical learning.

Summary

The findings from this study suggest that pre-k teachers' MKT entails CCK, SCK, KCS, and KCT (Ball et al., 2008). First, their CCK involves knowing mathematical terms or ideas that are covered in their state or district pre-k guidelines. This is similar to Shulman's (1986) notion of content knowledge, which refers to "knowledge of the facts or concepts" (p. 9) of a subject matter. This research also found that the five pre-k teachers had CCK, though they differed in how much of such knowledge they possessed. Still, such differences among these teachers' CCK had no direct effect on the quality of their teaching. Second, these pre-k teachers' SCK involved all the tasks that are found in what elementary and secondary schoolteachers routinely do in their math lessons. However, these pre-k teachers' SCK was centered on providing concrete math experiences rather than "giving mathematical explanations, choosing and developing useable definitions, and linking representations to underlying ideas and to other representations" (Ball et al., 2008, p. 400). Third, the five pre-k teachers' knowledge of students' mathematical thinking and reasoning, one of the major component of KCS, was tacit rather than explicit. While noticing their students' mathematical thinking, these teachers were unable to provide in detail clear descriptions of how they interpreted students' thinking. Finally, the general feature of KCT in this study was associated with how pre-k teachers utilized concrete objects and how to develop fun and engaging math activities that were tied to learning themes or topics. Such KCT was influenced by their own beliefs about how young children learn math and of what comprises effective math instruction.

MKT THAT PRE-K TEACHERS ARE STILL IN NEED OF

MKT that the pre-k teachers in this study had was centered on how to make math lessons concrete, emphasizing "doing math." To implement effective math instruction, these teachers focused on developing fun and engaging math activities. All of these teachers assumed that young children learned mathematics by touching and moving concrete objects. In much of their talk about effective and developmentally appropriate math instruction, these teachers saw the following as being essential for young children's mathematical learning concrete objects, physical materials, or manipulatives. They appeared to believe that making their math lesson concrete could promote pre-k students' mathematical thinking and help them understand the content. Consider Brooke's descriptions of her math teaching philosophy.

> Yes, yes, I still have to [use manipulatives] even though I know that they can do two plus one. I'll still have the two teddy bears plus the one teddy bear. I just think they need to have the visual. If I take the teddy bears away, they might not have that visual to make the connection. So, I always, we always have to use those concrete manipulatives (Second interview, 3/8/2010).

Brooke stated that she must use concrete objects to teach the addition concept of two plus one. She assumed that pre-k students needed to have the visual so that they could make connections with abstract mathematical ideas. The other teachers also emphasized the importance of doing math, touching, and moving the concrete objects. Murray (2001) also stated that, "Children learn better when they're using their senses; therefore, they should complete math tasks using three-dimensional objects to represent the numbers under examination" (p. 28).

However, other scholars such as Ginsburg, Lee, Baroody, and Clements express concern about early childhood teachers' overemphasis on implementing tangible and concrete math lessons. Baroody (1989) suggested that no matter how well-designed a math lesson, these manipulatives, in and of themselves, guaranteed no meaningful learning. The use of materials is effective only when they are used to encourage children to think and make connections between the objects and the abstract mathematical idea. Ginsburg and Lee (2009) also stated that, "mathematics in the early years does not need to be limited to the concrete or tangible" (p. 42). The medium, such as concrete objects or pictures of objects, might be insignificant unless they can be used to help students think and reflect, and to construct mathematical meanings and ideas.

Although all pre-k teachers in this study agreed that it was important to connect concrete math to abstract mathematical ideas, their knowledge of how to make such connections reflected a limited view. They thought that connecting concrete to abstract mathematical ideas was to either show the numerals or equations or do the paperwork so that students could have the experience of writing the numerals. For instance, Anna stated that she writes the numbers or equations so that her students can see but she doesn't expect them to have to write them (Second interview, 2/25/2010). Brooke also said that she wrote the numbers (Second interview, 3/8/2010). Claire mentioned that she would integrate "counting the number and writing it." Also, Claire often used worksheets that had the various types of symbols such as plus or minus (Second interview, 3/5/2010). Another participant, Diane stated that she would do the paperwork as a means of making connections between concrete math activities and abstract mathematical ideas (Second interview, 3/10/2010). Lastly, Eva also noted that she writes the number to help her students make connections between the number and the numerals (Second interview, 3/10/2010). According to each of these teacher's descriptions, they have a shared understanding of how to connect concrete to abstract mathematics. They connect concrete to abstract mathematical ideas by either showing the numerals or equations or doing the paperwork so that students could have the experience of writing the numerals. My observational data also established that these teachers wrote the numbers, introduced the equations, and did the individual paperwork to help students internalize abstract mathematical ideas (see the vignette of Brooke's lesson in Chapter 4).

However, certain math educators (e.g., Clements, 1999; Ginsburg & Lee, 2009; Haylock & Cockburn, 1997; Sophian, 2004) have contended that these teachers' approaches (e.g., writing numbers or equations as a means of connecting concrete math experience to abstract mathematical ideas) are too simple. They only represented one way of understanding numbers and number operations. Haylock and Cockburn (1997) suggested that an understanding of number and number operations includes a complex network of connections represented by the symbols. They maintained that a "mathematical symbol is not an abbreviation for just one category of concrete experiences" (p. 7). According to these scholars, these teachers' limited views cannot assist children in connecting one symbol with a confusing variety of concrete situations. Moreover, it can fail to establish an adequate conceptual foundation for the kinds of mathematical concepts that children will encounter in middle childhood and beyond. It also makes little allowance for the complex network of connections between concrete experience and symbols. These scholars suggested that early childhood teachers must reconsider what it means to be "concrete" and what it means to be "building up of cognitive connections" among concrete experiences, symbols, and abstract mathematical concepts (p. 15).

Early childhood teachers, including the five pre-k teachers in this study, must have MKT that helps them establish meaningful pedagogical sequences – "concrete to abstract." To do so, ECE teachers first need to reconceptualize the meaning of "concrete." Most early childhood teachers, including those in this study, tend to think of concrete as objects that students can grasp with their hands (Clements, 1999). The literature on early math education, however, points out problems with this view. Even if children begin to make connections between manipulatives and mathematical ideas, physical actions with certain manipulatives can suggest different mental actions than those we wish students to learn (Ball, 1992). Consider the following example suggested by Clements (1999). He found a mismatch among students using the number line to perform addition. When adding 5 + 4 by using concrete objects, the students located 5 first, counted 'one, two, three, four,' and then read the answer. This did not match with their mental actions to solve this problem, because to do so they have to count 'six, seven, eight, nine' and at the same time count the counts-6 as 1, 7 as 2, and so on (Clements, 1999, p. 47). Other researchers (e.g., Gravemeijer, 1991) also found that students' external actions with manipulatives do not always match the mental activity intended by the teacher. These researchers' arguments imply that although concrete objects have an important place in learning math, their physicality does not necessarily carry the meaning of the mathematical idea. Teachers can use concrete materials to build meaning initially, but to do so they must help students reflect on their actions with concrete objects.

Clements (1999) suggested that it was important to help students develop "integrated-concrete knowledge," knowledge that is connected in various ways (p. 48). The strength of such knowledge is the combination of many separate ideas in an interconnected structure of knowledge. For example, when thinking about the number four, "four-ness" is no more in four blocks than it is in a picture of four blocks (Kamii, 1986). Students with integrated-concrete knowledge can create "four" by building a representation of the number and connecting it with either physical or pictured blocks. Thus, integrated-concrete knowledge is “how ‘meaning-full’—connected to other ideas and situations—they are” (Clements, 1999, p. 49). ECE teachers need to provide math activities that can enhance children’s integrated-concrete knowledge. This requires them to think about how their planned math activity can support such knowledge whenever they plan their math lesson, which can be the unique aspect of KCT needed by ECE teachers.

Manipulatives and concrete objects can be used effectively before formal symbolic instruction. They can support students’ construction of meaningful mathematical ideas. Teachers should avoid, however, using concrete materials “as an end without careful thought rather than as a means to that end” (Clements, 1999, p. 56). They should also reconsider what “concrete” describes. Good concrete experiences are to assist students to establish, strengthen, and connect various representations of mathematical ideas.

So the first thing ECE teachers should do to establish meaningful pedagogical sequences “concrete to abstract” is to reconceptualize “concrete.” The second thing for them to do is to rethink the various aspects of number. For example, when thinking about the image of three, most teachers or adults might come up to the idea that three is a set of three things (Haylock & Cockburn, 1997). When a pre-k teacher needs to explain a problem involving the number three, what will most likely appear are sets of three counters, three blocks, or three fingers. Consider Brooke’s descriptions of how she taught the number 10.

> So when we do one on one, we now have an actual visual of the number. If they see a one and a zero, it’s clueless, right? But if I have ten rocks then they can actually see a visual of ten rocks. So that’s kind of how we do a lot of the number recognition, whether it is lower or higher—counting things in the classroom.

> They'll count ten crayons. "Bring me ten crayons. Bring me ten teddy bears." And that's how we do a lot of numbers (Second interview, 3/8/2010).

Brooke taught the number 10 by showing her students a set of ten things. Other teachers also taught the number concept by connecting the symbol with concrete situations consisting of sets of things. Also, Texas pre-k guidelines mainly focus on the number as a set of something, although they address the use of ordinal numbers (TEA, 2008). However, Haylock & Cockburn (1997) suggested that this view of the way in which number concepts develop is simplistic making little allowance for the complex network of connections between concrete experience and the symbol, which constitute the concept of certain number. They contended that this is only one aspect of a number, referred to as the cardinal aspect, and there are, apart from sets of things, various ways to think about numbers. They introduced the nominal aspect and cardinal aspect of numbers, which is appropriate to teach to young children. The nominal aspect of number is used "to label items or to distinguish them from one another" (p. 19). This includes the 3 on a clock face, page 3 in a newspaper, or the number 3 bus and so on. When used like this the symbol is hardly operating as a number at all. The ordinal aspect of number is used "not just to label things but also to put them in order" (p. 19). For example, the timetable for the Number 3 bus follows that for the Number 2 bus and precedes that for the Number 4. 3 here is being used not just in a nominal sense but also in an ordinal sense. In these two views, there is no "three-ness," no cardinal sense of the number 3. Haylock & Cockburn (1999) emphasized the ordinal aspect of numbers, which must not be treated as an obscure or secondary aspect of the concept of number in spite of the fact that almost no adults bring to mind an ordinal image when asked to think of three. This ordinal aspect of the number is as central and important as the cardinal aspect.

When both the cardinal and ordinal aspect of a number are taken into account, counting—one of the biggest portion that constitutes pre-k mathematics—can be taught differently. Teaching counting skills in pre-k involves learning a pattern of sounds, one, two, three, four…and so on, memorized by repetition and learning to co-ordinate the utterance of these sounds with the physical movements of a finger or of objects, matching one sound to one object, which demonstrates the cardinal aspect of the number (TEA, 2008). When the ordinal aspect of number is added, students begin to learn to label objects and to order them. Then, they have somehow to discover that the ordinal number of the last object is the cardinal number of the set (Haylock & Cockburn, 1997). This allows students to realize that when they get to three, it means a set of three things. In this process, students can have a connection among three concrete objects, the word three, using the cardinal aspect, and pictures of the number involving the ordinal aspect.

Haylock & Cockburn (1997) demonstrated how one symbol is used to represent vastly different situations. They contended that it is important for teachers of young children to understand various aspects of the number, extending their understanding of numbers to ordinal aspect. This is important in terms of laying foundations of experience and networks of connections on to which future experiences of number can be built. For example, they suggested that if the ordinal aspect is included in the concept of number, it is not difficult to extend students' understanding of numbers to make connections with negative numbers (p. 21). ECE teachers, including the five pre-k teachers in this study, ought to give their students as much opportunity to do number-related work moving up and down number lines, or similar manifestations of the ordinal aspect, as they give to manipulating sets of blocks or unifix cubes. For example, Brooke taught number 10 by providing a set of ten objects (Second interview, 3/8/2010). She could have taught number in a various ways if she understood the multiple aspects of number. In this manner teachers can connect the symbol 3 and the word "three" just as much with the idea of a label for a point or a position as with sets of things.

SUMMARY

In sum, I investigated the MKT that the five pre-k teachers already had and were still in need of. These teachers' MKT was consistent with what the literature suggested. It involves the framework of MKT put forward by Ball and her colleagues (e.g., common content knowledge, specialized content knowledge, and knowledge of content and students, and knowledge of content and teaching). It also entails some unique characteristics that are distinctive from elementary or secondary teachers' MKT. Since early mathematics emphasizes "doing math," the concrete experience, to crystallize the mathematical concepts rather than using written notation, these pre-k teachers' MKT was centered on how to make the math lesson active and concrete.

This unique characteristic of MKT highlights what MKT these teachers need further. The research on early mathematics education emphasized the significance of connecting concrete experience with abstract mathematical ideas. However, these teachers' understanding of such issues was too simplistic and limited. They appeared to believe that connecting concrete to abstract indicates introducing the written numbers or equations or doing some paperwork. This limited view can be problematic because it neither helps students make connections between symbols, concrete objects, pictures, or written numerals in a coherent way nor reinforces the foundations of future experiences of mathematical ideas. What these pre-k teachers need is to reconceptualize the meaning of concrete (Ball, 1992; Clements, 1999) and to extend their understanding of number from its cardinal to its ordinal aspect (Haylock & Cockburn, 1997).

This research study unpacks and catalogs the mathematical knowledge base of practicing pre-k teachers. It also provides an insight into the knowledge and skills that pre-k teachers now have and still need to offer high-quality and effective math instruction. This finding adds to the current literature with several implications for teacher trainers and educators.

IMPLICATIONS

I now offer specific contributions made by this research examining five pre-k teachers' mathematical knowledge for teaching. Examining the knowledge of these teachers in the classroom context adds to the literature in several areas. First, it supports the evidence of previous findings that teaching mathematics to young children entails all the processes involved in teaching mathematics to elementary or secondary school students including profound knowledge of subject matter, pedagogical content knowledge, and knowledge of children's informal mathematics. Second, it provides insights into what effective mathematics teaching entails and into the needs of pre-k teachers in terms of their mathematical knowledge. By presenting these case studies, I hope that more early childhood teachers will reflect on the importance of knowing what and how to teach math so as to provide high-quality and effective math instruction. Also, this study informs the design or curriculum for the content preparation of teachers that is tied to professional practice and to the knowledge and skill demanded by the work. ECE teachers and teacher educators all have a stake in ensuring that those who come to the work of teaching children are able to implement teaching of the highest quality.

Teachers

The findings of this research have several implications for early childhood teachers who are responsible for providing high-quality and effective math instruction.

ECE teachers can gain specific points of insight from the five pre-k teachers in this study by looking at the way in which they understood math and teaching/learning math. According to NCTM (2000), "What students learn is greatly influenced by how they are taught." This statement indicates that how a subject is taught tells students whether the subject is interesting or boring, clear or fuzzy, relevant or irrelevant, and challenging or routine. If teachers are to engage students in reasoning and thinking about important ideas in these subjects, they must themselves have a grasp of these ideas and they must have respect for the importance of having knowledge and skills needed for good teaching (Kennedy, 1998).

The findings from this research suggest how these pre-k teachers understood the mathematical contents that they were to teach, how they saw their teaching and students' learning math, and how such understandings were used in their teaching practice. Although the extent of each teacher's knowledge varied, this study confirms the importance of teachers' mathematical knowledge for teaching in terms of promoting and extending students' thinking and reasoning.

First, the findings suggest that to instill a deeper understanding in students of the central ideas and concepts in math and to enable students to see how these ideas connect to and can be applied to real-world situations, ECE teachers are also expected to understand the central concepts of math, see their relationships, and know how these ideas are related to students' later learning. Many researchers in early childhood math education (e.g., Ma, 1999; Lampert, 2001; Clements, 2004) contend that ECE teachers must have profound understanding of the depth and breadth of early mathematics. Such an understanding involves "appreciation of the connectedness among mathematical ideas, understanding basic concepts, and taking a longitudinal perspective that illuminates how some concepts must be learned before others and how some concepts set the stage for learning new ones" (Ginsburg & Ertle, 2008, p. 46). This indicates that ECE teachers should have SCK that informs the breadth and depth of early mathematics.

Second, the findings suggest that ECE teachers need to carefully observe and attend to children's own ways of solving the problems. The five pre-k teachers in this study were able to notice their students' current understanding and their capability. However, they did not appear to make specific connections with the students' informal or intuitive mathematical knowledge. This finding indicates that ECE teachers need knowledge of children's informal mathematics. Jacobs et al (2010) named such tasks as "professional noticing of children's mathematical thinking." They defined this expertise as a set of interrelated skills including "attending to children's strategies, interpreting children's understandings, and deciding how to respond on the basis of children's understandings." Clements (2004) also contended that successful early childhood teachers build on children's informal knowledge by carefully considering their language, cultural backgrounds, informal strategies, and thinking. One way to address this issue would be for teacher education programs to provide an adequate opportunity for both preservice and in-service teachers to discuss, research, and observe children's thinking and learning of mathematics (Copley, 2004). This can help children reflect on and extend the mathematics that arises in their everyday activities. Also, this can help teachers provide meaningful and engaging math lessons based on their understanding of what children know and are able to learn.

Finally, this study emphasizes the importance of integrating real-world situations, concrete mathematical experiences, and abstract mathematical ideas. The pre-k teachers in this study tended to have limited views in understanding the meanings of concrete and abstract. They viewed concrete as objects that students could grasp with their hands. They viewed abstract as being about introducing and writing the numerals or mathematical symbols such as plus, minus, or equal signs. However, many scholars, such as Ball, Clements, Haylock & Cockburn in math education, problematized this view. It made, they argued, few connections between concrete experience and the symbol, which would make it difficult to lay foundations of experience and networks of connections on to future experiences of number and other abstract mathematical concepts. This suggests that ECE teachers need to re-think how they can make a connection between the concrete and abstract. Mathematical ideas are not in the manipulatives or concrete objects (Lee & Ginsburg, 2009). In this sense, ECE teachers should recognize that "doing math" can be meaningless unless it helps children think, reflect, and construct meanings and ideas. Also, ECE teachers need to re-conceptualize what abstract mathematics is and how it can be taught to young children. They can do this by expanding their concept of mathematical ideas and appreciating the depth and breadth of early mathematics that young children should learn.

To expand and elaborate ECE teachers' understanding of mathematical ideas and the depth and breadth of early mathematics, the content of the professional development experiences should be focused on the big ideas of mathematics such as NCTM's mathematics standards (e.g., number and operation, geometry, measurement, algebra and patterns) (Sarama et al., 2004). Cohen and Hill (2000) concluded that when professional development is focused on learning and teaching academic content, both teaching practice and student achievement can be improved. Professional development in early childhood mathematics should focus on developing knowledge regarding specific subject-matter content, including deep conceptual knowledge of the mathematics to be taught.

Teacher educators

Every child has a right to receive a high-quality mathematics education (NCTM, 2000). Professional development has widely been regarded as the most significant way of achieving this goal (Darling-Hammond, 1998). It is the job of teacher educators who prepare their prospective teachers to be knowledgeable and responsive professional expertise. In examining the significance of this research, it is important for those in teacher education to consider the implications regarding pre-k teachers' knowledge of math and their teaching/learning math.

This research examined the following questions: What do teachers who teach young children need to know and be able to do in order to teach math effectively? This question places the emphasis on the importance of early childhood teachers' content and pedagogical content knowledge in math and the use of such knowledge in and for teaching. The findings from this research suggest that the pre-k teachers were in need of knowledge of how to interpret children's mathematical thinking and reasoning and knowledge about how to connect concrete mathematical experiences to abstract mathematical ideas. These findings add to the NAEYC and NCTM's joint statement (2002) that the pressing need in early mathematics education is to help teachers learn how broad and deep the content is of the early mathematics that they are teaching (Ginsburg & Ertle, 2008). The topics that three- to five-year-old children study may be basic, but they include many subtopics (e.g., numbers cover such matters as counting words, comparing/ordering, composing/decomposing, and adding/subtracting; Clements, 2004) that entail deep mathematical ideas (e.g., see Haylock & Cockburn's ideas of what numbers represent). Teacher educators should help pre-service ECE teachers understand how such mathematical concepts are interrelated and how they can be the foundation for later mathematical learning. To do so, they should provide extensive opportunities for pre-service teachers to focus on mathematics, goals, and children's mathematics and support interactions among teachers to share what they've learned and experienced (Sarama et al., 2004). This will allow ECE teachers to develop their knowledge of the mathematical content they teach. Also, it will help them engage in a deeper understanding of communicating this content to students, which provides an opportunity to attend to and respect students' ways of reasoning (Philipp, 2008).

LIMITATIONS AND QUESTIONS FOR FURTHER INQUIRY

Even though this study contributes to the field of early mathematics education, the study itself contains limitations. The qualitative case study design and scope of this research represent one limitation (Yin, 2003). Nevertheless, they point to the need for further inquiry. First, the intentional focus on pre-k teachers' MKT led to a subset of teachers that did not necessarily represent all pre-k teachers. What is still needed is more research on teachers who work in different states or in different types of schools other than public school.

In addition, this research, while including an extended field observation period, was limited to half of a school year. Because of this, this research could not obtain additional data on how these teachers taught other mathematical content, such as geometry, data analysis, and subtraction concept. Therefore, research, which follows teachers into different contexts and through more than one school year, would provide significant insight into what pre-k teachers' MKT entails and how it is played out in their math lessons.

A third limitation that developed was the relatively small amount of data related to students' mathematical learning. I had planned to include data about teachers' understanding of mathematical content and teaching. As a result, students' questions, responses, and their informal talk related to math were not fully considered as a major data source. Examining teacher-student or student-student interactions during the math lesson is important, though, in terms of understanding how these teachers interpret and notice students' mathematical thinking and reasoning. Thus, more research that includes adequate data on students would further extend teachers' knowledge base on students' thinking and inform researchers of how teacher knowledge is related to students' learning.

Finally, it is unclear how my presence in the classroom altered the ways that these teachers taught math and interacted with students. While precautions were taken, including member checking, listening to gate-keepers and knowledgeable others, and looking for counter-examples, the case studies presented were no doubt influenced by my respect and admiration for these teachers and the work they did with students. Regardless, further inquiry must also include this balance between honestly portraying teachers' practices and presenting helpful insight for teachers and teacher educators.

CONCLUSION

On the opening page of this dissertation, I mentioned that ECE teachers need encompassing knowledge, what I call directionality of their teaching practice. Possessing such knowledge, such directionality, not only clarifies teachers' vision of what they pursue but also presses them on to the goal of their practice. My dissertation focuses particularly on early childhood teachers' knowledge of teaching/learning mathematics. By working with five pre-k teachers, I have attempted to examine their MKT that consists of knowing the things that must be taught, the extent to which they should be taught, why they should be taught, and how they could be best be taught.

Each teacher possesses a definite set of knowledge, or a set of knowledge elements that may differ in scope, content, depth and quality (Antonijevic, 2006). The five pre-k teacher's MKT also differed in depth and was described in various ways, but there was a shared and consistent element found in each teacher's MKT. This research study examined, cataloged, and unpacked such knowledge. This is an important process in establishing sound and interconnected knowledge structures that can be shared and generalized by other teachers.

> One of the crucial, internal characteristics of organized and logically consistent knowledge system is represented by knowledge interconnectedness, which refers to the existence of definite connections and relations between the elements of personal knowledge. Instead of a great quantity of single and non-connected knowledge, the system of interconnected knowledge should enable optimal everyday and work adaptation of individuals, in the sense of the existence of stable foundation for solving everyday immediate problems in situations where direct or indirect applying of knowledge is requested. In the same way, complete and stable knowledge system should enable further development, in the sense of thinking abilities and skills development, and intellectual development, in general. It should be also the means of extending and strengthening the present base of knowledge and concepts, which may be a constituent part of the present knowledge system (Antonijevic, 2006, p. 146).

As Antonijevic suggests, complete and stable knowledge system provides solid foundation for solving everyday problems effectively, extends and strengthen the current knowledge base, and enables further development. Knowledge interconnectedness in teaching mathematics is necessary, in order to make the teaching process more efficient in general, through enabling students to achieve better and deeper understanding of the mathematical content that are taught. In this sense, this dissertation took a small but meaningful step toward establishing interconnected knowledge in teaching mathematics especially for young children by examining five pre-k teachers' MKT. By examining and unpacking teachers' knowledge base in teaching mathematics, I hope this dissertation provides an important insight into what it means to be effective and professional in regards to teaching mathematics to young children.

Appendix A: Initial Interview Protocol

During the initial interview I will do the following with each participant:

- Discuss the study and the participant's role in the data collection process
- Gather demographics information (Yrs teaching, SES of class, schools taught in, grade levels taught, etc.)
- Ask the following questions:

1. My research is focusing specifically on pre-k teachers' understanding about mathematical content they are to teach and math instruction they implement. Please tell me how you see your teaching during the math lesson up to this point.

   -What is teaching and what does it take to teach mathematics?

   - What do you see being your strengths when teaching mathematics?

   -What do you feel you still need to work when teaching mathematics?

   -As you think about your math instruction, are there some things you would like to know more about or be able to do better?

2. I am also interested in your own past experience with math.

   -What do you remember learning math in preschool or elementary school?

   -What about the high school level? What do you remember about learning math in high school?

   -How does your past experience with math influence your current math instruction?

3. What are you trying to accomplish in math with your prekindergartners? Why are these the most important things to accomplish?

4. What is your personal thought about effective and developmentally appropriate math instruction?

5. How do you think young children learn mathematics?

6. How do you incorporate state or district pre-k guideline in mathematics into your math instruction?

Appendix B: Follow-up Interview Protocol

During the follow-up interview I will ask the following questions, but they will need to be adjusted after I pilot them. The questions are neatly arranged into four groups. My asking them, however, may not be. Circumstances, for example, what the teacher is covering that day, may determine the best order for the questions.

In the previous interview, I asked you general questions about your experience teaching and, specifically, teaching math. Today, I want to go into more specific questions about two things. First, I would like to ask you questions that tell me something about your understanding of the mathematical content that you are to teach. Second, I will be asking questions about how you implement your math instruction. In other words, I will first ask you some simple math questions. Then, I will ask how you convey that understanding to your students.

1. Common Content Knowledge

-Would you describe for me the differences between a triangle, square, and rectangle?

-Would you describe for me the ways to measure length, weight, and volume?

2. Specialized Content Knowledge

-For this question, I'd like you to pick a big idea of mathematics taught in pre-k (e.g., number, pattern, algebra, measurement, geometry). How do you think the big idea is related to students' later learning?

-Suppose you are teaching a simple addition problem, say 3 + 5 = 8.

-Where do you begin? What do you think is the best way to teach it? If a student doesn't understand your explanation, what is an alternative way to teach the problem?

-How do you respond to the students' answers when they solve the problem in a non-standard way?

-Would you encourage non-standard ways of problem solving? Why or why not?

-Do you have more than one way of explaining the concept of subtraction? If so, which way is your standard explanation?

-What challenge do you face most often when you start to teach subtraction?

-Have you found an effective way around this challenge?

3. Knowledge of Content and Student

-Suppose that you were teaching the concept of equal and non-equal parts. What are informal strategies that children use to share or divide up to 10 items equally?

-How do you integrate their informal strategies with formal strategies?

-What are the kinds of errors that children make when you teach patterns?

-Why do you think children make these errors?

-How do you correct them?

-How do you, in general, correct common errors or misunderstandings?

-Which concept in math do you find students have the hardest time grasping?

4. Knowledge of Content and Teaching

- What learning activities or materials do you choose when you first introduce the concept of addition and subtraction?

-What instructional strategies work best when teaching geometry?

-How much do you rely on the shapes of everyday objects to help with teaching geometry?

-How do you supply and arrange the classroom to help promote children's general sense of everyday mathematics?

Appendix C: Post Observation Interview Protocol

During the post observation interview I will ask the following questions, but they will need to be adjusted for each lesson depending on what happened during the lesson that was observed.

1. How do you feel the lesson went? (After the teacher has talked about how she feels about the lesson we will watch the videotape of her lesson)

   - *What were you trying to achieve through your math instruction while I was observing you?*

   -How did things compare with what you had expected? Did anything surprise you?

   -Was there anything you were particularly pleased about? Why?

   -Did anything disappoint you? Why?

2. (After the teacher watched the video of her lesson) How do you feel when you see the video of your own math lesson?

   - *Were the students involved and excited about what they were learning? What did you do to keep them involved and excited?*

   -One thing I am interested in is how teachers select the activities, tasks, explanations, examples that they use or how they decide to explain thing to children. I noticed that you said/did ~

   -Where did this (example, task, explanation, etc.) come from?

   -Why did you decide to do this?

   -Does it have any particular advantages or disadvantages (benefits or drawbacks)?

-How did you decide whom to call on?

-I noticed the children (were; were not) divided into groups. Why? (or why not?)

3. What were the roles of state pre-k guideline or other district curriculum when designing this math lesson?

4. How did you try to address students' informal mathematical knowledge, everyday lives, and individual differences in this lesson?

Appendix D: Four Teacher Knowledge Construct/External Code

Table 4-Constructs of Teachers' Mathematical Knowledge

Constructs	Observed classroom behaviors	Interview questions
Common content knowledge	• Teachers calculate or correctly solve the problem. • Teachers recognize when their students give a wrong answer. • Teachers use terms or notations correctly (Ball et al., 2008).	• What is a number that lies between ½ and ⅓? • What are the differences between a triangle, square, and rectangle? • What are non-standards or standard-units to measure height, weight, length, and volume? (TEA, 2008)
Specialized content knowledge	• Teachers respond to students' why questions. • Teachers explain mathematical goals or purposes to students. • Teachers modify the activities to be either easier or harder. • Teachers provide an appropriate example to make a specific mathematical idea meaningful to students. • Teachers choose, make, and use curriculum materials effectively. • Teachers connect a topic being taught to topics from future years. • Teachers ask productive mathematical questions (Ball et al, 2008).	• What is the best way to teach the concept of fraction? • How do you respond to the students' answers when they solve the problems in a non-standard ways? (Empson & Junk, 2004) • Why do you choose these blocks as a major learning material? • What would you do when some students feel the planned activity is difficult or too easy? • How do you explain the concept of place value to little children?(TEA, 2008) • How do you think the big mathematical concept such as number and operation, pattern, or algebra is connected to children's further mathematical learning? (Ginsburg & Ertle, 2008)
Knowledge of content and students	• Teachers anticipate what students are likely to think and what they will find confusing. • Teachers provide activities that students find interesting and motivating. • Teachers respond to and interpret students' emerging and incomplete thinking as expressed in the ways that students use languages. • Teachers figure out common errors that students are most likely to make (Ball et al., 2008).	• What are the kinds of shapes students are likely to identify as triangles?(TEA, 2008) • What is the likelihood that students will write 6 for 9? • How would you correct errors or misunderstandings that most students make?(Ball et al., 2008) • How would students compare 2 half-sized cookies to one whole cookie? • Why do you think students make specific errors when teaching particular mathematics content? • What are the informal ways to solve adding and subtracting problems?
Knowledge of content and	• Teachers sequence particular content for instruction. • Teachers balance teacher-directed and child-oriented instruction	• Which do you think is a good way for teaching place value: a tape measure or unit blocks? (TEA, 2008) • Why do you choose the book for the

teaching	(Epstein, 2007). • Teachers employ whole-group, small group, and individualized instruction. • Teachers are able to allocate time appropriately. • Teachers choose which examples to start with and which examples to use to take students deeper into the content (Ball et al., 2008). • Teachers make instructional decisions about which student contributions to pursue and which to ignore or save for a later time. • During a classroom discussion, teachers decide when to pause for more clarification, when to use a student's remark, when to ask a new question, or pose a new task (Ball et al., 2008).	first introduction to fractions? • When is the best time to move on to the next stage? • What kinds of curriculum materials are good at taking students deeper into the content? (Ball et al., 2008) • How do you initiate the whole group discussion when you start to teach new mathematical concepts? • How do you organize the classroom or design the mathematical learning environment?

*Adapted from Ball, Thames, & Phelps, 2008

Appendix E: An Example of Member Checking

To establish trustworthiness, I conducted member checking, clarifying emerging themes with each participant and each had the opportunity to read her individual case study and the findings. The following email indicates how I interacted with one of the participants.

> Dear Diane,
>
> I am sending my field note of your lesson on 4/3 and my reflection or analysis of this. If you have any things that are uncertain or need corrections, please let me know. Also, feel free to share your thoughts, reflections, or ideas that occur while checking these data.
>
> Thanks,
>
> Jenn

> Dear Jenn,
>
> I've looked over your analysis and I think most of it well written. One thing I want to add is that kids are familiar with pattern activity but the stuff that I used in this lesson were new. That's why many children might want to play with those new farm animal creatures rather than doing the pattern. That's it I guess.
>
> Thanks,
>
> Diane

Bibliography

Andrews, A. G., & Trafton, P. R. (2002). *Little kids-powerful problem solvers*. Portsmouth, NH: Heinemann.

Anning, A. (1988). Teachers' theories about children's learning. In J. Calderhead (Ed.), *Teachers' Professional Learning* (pp. 128-145). London: Falmer Press.

Antonijevic, R. (2006). Characteristics of knowledge interconnectedness in teaching. *Philosophy, Sociology and Psychology 5*(1), 145-152.

Aubrey, C. (1996). An investigation of teachers' mathematical subject knowledge and the processes of instruction in reception classes. *British Educational Research Journal, 22*(2), 181-197.

Ball, D. L. (1991). Research on teaching mathematics: making subject matter knowledge part of the equation. In J. Brophy (Ed.), *Advances in research on teaching* (Vol. 2, pp. 1-47). Greenwich, CT: JAI Press.

Ball, D. L. (1992). Magical Hopes: Manipulatives and the reform of math education. *American Educator: The Professional Journal of the American Federation of Teachers, 16*(2), 14-18,46-47.

Ball, D. L. (2000). Bridging practices: Interwining content and pedagogy in teaching and learning to teach. *Journal of Teacher Education, 51*(3), 241-247.

Ball, D., & Bass, H. (2000). Making believe: The collective construction of public mathematical knowledge in the elementary classroom. In D. C. Phillips (Ed.), *Constructivism in education: Opinions and second opinions on controversial issues. Yearbook of the National Society for the Study of Education* (pp. 193-224). Chicago: University of Chicago Press.

Ball, D. L., & National Center for Research on Teacher Education, E. L. M. I. (1990). *With an Eye on the Mathematical Horizon: Dilemmas of Teaching Elementary School Mathematics. Craft Paper 90-3.*

Ball, D. L., Sleep, L., Boerst, T. A., & Bass, H. (2009). Combining the development of practice and the practice of development in teacher education. *Elementary School Journal, 109*(5), 458-474.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59*(5), 389-407.

Baroody, A. J. (1989). Manipulatives don't come with guarantees. *Arithmetic Teacher, 37*(2), 4-5.

Baroody, A. J., & Benson, A. (2001). Early number instruction. *Teaching Children Mathematics, 8*(3), 154-158.

Baroody, A. J., Lai, M., & Mix, K. S. (2006). The development of young children's early number and operation sense and its implications for early childhood education. In B. Spodek & O. N. Saracho (Eds.), *Handbook of Research on the Education of Young Children* (pp. 187-221). Mahwah, NJ: Lawrence Erlbaum Associates.

Baroody, A. J., & Wilkins, J. L. M. (1999). The development of informal counting, number, and arithmetic skills and concepts. In J. Copley (Ed.), *Mathematics in the early years* (pp. 48-65). Reston, VA: National Council of Teachers of Mathematics.

Baumert, J., Kunter, M., Blum, W., Brunner, M., Voss, T., Jordan, A., et al. (2010). Teachers' Mathematical Knowledge, Cognitive Activation in the Classroom, and Student Progress. *American Educational Research Journal, 47*(1), 133-180.

Borko, H., & Putnam, R. (1996). Learning to teach. In D. Berliner & R. Calfee (Eds.), *Handbook of Educational Psychology* (pp. 673-708). New York: MacMillan.

Bransford, J., Darling-Hammond, L., & Lepage, P. (2005). Introduction. In L. Darling-Hammond & J. Bransford (Eds.), *Preparing teachers for a changing world: What teachers should learn and be able to do* (pp. 1-39). San Francisco: Jossey-Bass Publishers.

Bredekamp, S., & Copple, C. (Eds.). (2009). *Developmentally Appropriate Practice in Early Childhood.* Washington, D.C.: National Association for the Education of Young Children.

Brophy, J. (1992). Conclusion to advances in research on teaching. In J. Brophy (Ed.), *Advanced in research on teaching: Teachers' knowledge of subject matter as it relates to their teaching practices* (pp. 347-361). Greenwich, CT: JAI Press.

Calderhead, J. (1996). Teachers: Beliefs and knowledge. In D. Berliner & R. Calfee (Eds.), *Handbook of Educational Psychology* (pp. 709-725). New York: Macmillan.

Carpenter, T. P., Fennema, E., Franke, M. L., Levi, L., & Empson, S. B. (1999). *Children's mathematics*. Portsmouth, NH: Heinemann.

Causey, V. E., Thomas, C. D., & Armento, B. J. (2000). Cultural diversity is basically a foreign term to me: The challenges of diversity for preservice teacher education. *Teaching and*

Teacher Education, 16(1), 33-45.

Cawelti, G. (1999). *Handbook of research on improving student achievement*. VA: Educational Research Service.

Clements, D. H. (1999). 'Concrete' manipulatives, concrete ideas. *Contemporary Issues in Early Childhood*, *1*(1), 45-60.

Clements, D. H., & Sarama, J. (2007). Effects of a preschool mathematics curriculum: Summative research on the "Building Blocks" project. *Journal for Research in Mathematics Education, 38*(2), 136-163.

Clements, D. H., Sarama, J., & DiBiase, A.-M. (2004). *Engaging young children in mathematics: Standards for early childhood mathematics education*. Mahwah, NJ: Lawrence Erlbaum Associates.

Cohen, D. K., & Hill, H. C. (2000). Instructional policy and classroom performance: The mathematics reforms in California. *Teachers College Record Volume, 102*, 294-343.

Copley, J. (2000). *The young child and mathematics*. Washington, DC: National Association for the Education of Young Children.

Copley, J. (2004). The early childhood collaborative: a professional development model to communicate and implement the standards. In D. H. Clements, J. Sarama & A.-M. DiBiase (Eds.), *Engaging young children in mathematics: Standards for early childhood mathematics education* (pp. 401-414). Mahwah, NJ Lawrence Erlbaum Associates.

Cross, C. T., Woods, T. A., & Schweingruber, H. (2009). *Mathematics learning in early childhood: Paths toward excellence and equity*.

Crotty, M. (1999). *The foundations of social research: Meaning and perspective in the research process*. London: Sage.

Darling-Hammond, L. (1998). Teacher education and the American Future. *Journal of Teacher Education, 61*(1-2), 35-47.

Denzin, N. K., & Lincoln, Y. S. (2005). Introduction: The discipline and practice of qualitative research. In N. K. Denzin & Y. S. Lincoln (Eds.), *Handbook of qualitative research* (2nd ed., pp. 443-466). CA: Sage: Thousand Oaks.

Department of Education. (2005). No Child Left Behind is working. Retrieved on January 23, 2011 from http://www.ed.gov/nclb/overview/importance/nclbworking.html.

Empson, S. B. (1999). Equal sharing and shared meaning: The development of fraction concepts in a first-grade classroom. *Cognition and Instruction, 17*(3), 283-342.

Empson, S. B., & Junk, D. L. (2004). Teachers' knowledge of children's mathematics after implementing a student-centered curriculum. *Journal of Mathematics Teacher Education, 7*(2), 121-144.

Epstein, A. (2007). *The intentional teacher: Choosing the best strategies for young children's learning*. Washington, DC: National Association for the Education of Young Children.

Ernest, P. (1989). The knowledge, beliefs and attitudes of the mathematics teacher: a model. *Journal of Education for Teaching, 15*(1), 13-33.

Fennema, E., & Franke, M. L. (1992). Teachers' knowledge and its impact. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 147-164). New York: Macmillan Publishing Company.

Garcia, V., Agbemakplido, W., Abdella, H. Lopez, O. & Registe, R. T. (2006) High school students perspectives on the 2001 No Child Left Behind Act's definition of a highly qualified teacher. *Harvard Educational Review*, 76 (4), 698-724.

Gearhart, M. (2007). Mathematical knowledge for teaching: Questions about constructs. *Measurement: Interdisciplinary Research and Perspectives, 5*, 173-180.

Gearhart, M., & Saxe, G. B. (2004). When teachers know what students know: Integrating mathematics assessment. *Theory Into Practice, 43*(4), 304-313.

Gelman, R. (2000). The epigenesis of mathematical thinking. *Journal of Applied Developmental Psychology, 21*(1), 27-37.

Ginsburg, H. P., & Amit, M. (2008). What Is teaching mathematics to young children? A theoretical perspective and case study. *Journal of Applied Developmental Psychology, 29*(4), 274-285.

Ginsburg, H. P., Klein, A., & Starkey, P. (1998). The development of children's mathematical knowledge: Connecting research with practice. In I. E. Sigel & K. A. Renninger (Eds.), *Handbook of child psychology: Vol. 4. Child psychology in practice* (pp. 401-476). New York: Wiley & Sons.

Ginsburg, H. P., Inoue, N., & Seo, K.-H. (1999). Young children doing mathematics: Observations of everyday activities. In J. Copley (Ed.), *Mathematics in the early years* (pp. 88-99). Reston, VA: NCTM.

Ginsburg, H. P., Lee, J. S., & Boyd, J. S. (2008). *Mathematics education for young children: What it is and how to promote it*: Society for Research in Child Development.

Ginsburg, H. P., Pappas, S., & Seo, K.-H. (2001). Everyday mathematical knowledge: Asking young children what is developmentally appropriate. In S. L. Goldbeck (Ed.), *Psychological perspectives on early childhood education: Reframing dilemmas in research and practice* (pp. 181-219). Mahwah, NJ: Lawrence Erlbaum Associates.

Ginsburg, H. P., & Seo, K.-H. (1999). Mathematics in children's thinking. *Mathematical Thinking and Learning, 1*(2), 113-129.

Glense, C. (1998). *Becoming qualitative researchers*. White Plains, NY: Longman.

Goldstein, L. S. (2007). Beyond the DAP versus standards dilemma: Examining the unforgiving complexity of kindergarten teaching in the United States. *Early Childhood Research Quarterly, 22*(1), 39-54.

Goldstein, L. S. (2008). Kindergarten teachers making "street-level" education policy in the wake of No Child Left Behind. *Early Education and Development, 19*(3), 448-478.

Graeber, A. O. (1999). Forms of knowing mathematics: What preservice teachers should learn. *Educational Studies in Mathematics, 38*(1-3), 189-208.

Graue, M. E., & Walsh, D. J. (1998). *Studying children in context: Theories, methods and ethics*. Thousand Oaks: CA: Sage.

Gravemeijer, K. P. E. (1991). An instruction-theoretical reflection on the use of manipulatives. In L. Streefland (Ed.), *Realistic mathematics education in primary school* (pp. 57-76). Utrecht: Freudenthal Institute: Utrecht University.

Greenes, C. (1999). Ready to learn: Developing young children's mathematical powers. In J. Copley (Ed.), *Mathematics in the early years* (pp. 39-47). Reston, VA: National Council of Teachers of Mathematics.

Greeno, J. (1987). Instructional representations based on research about understanding. In A. Schoenfeld (Ed.), *Cognitive science and mathematics education* (pp. 61-88). Mahwah: NJ: Erlbaum.

Greer, B. (1992). Multiplication and division as models of situations. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning*. New York: Macmillan.

Grossman, P. L. (1990). *The making of a teacher: Teacher knowledge & teacher education*. New York, NY: Teachers College Press.

Hatch, J. A. (2002). Accountability shovedown: Resisting the standards movement in early childhood education. *Phi Delta Kappan, 83*(6), 457-462.

Hatch, J. A. (2002). *Doing qualitative research in education settings*. Albany, NY: SUNY press.

Haylock, D., & Cockburn, A. (1997). *Understanding mathematics in the lower primary years*. London: Paul Chapman Publishing Ltd.

Herron, J. (2010). An evolution of mathematical beliefs: A case study of three pre-k teachers. *Journal of Early Childhood Teacher Education, 31*(4), 360-372.

Hill, H. C., Ball, D. L., & Schilling, S. G. (2008). Unpacking pedagogical content knowledge: Conceptualizing and measuring teachers' topic-specific knowledge of students. *Journal for Research in Mathematics Education, 39*(4), 372-400.

Hill, H. C., Blunk, M. L., Charalambous, C. Y., Lewis, J. M., Phelps, G. C., Sleep, L., et al. (2008). Mathematical knowledge for teaching and the mathematical quality of instruction: An exploratory study. *Cognition and Instruction, 26*(4), 430-511.

Hill, H. C., Rowan, B., & Ball, D. L. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. *American Educational Research Journal, 42*(2), 371-406.

Jacobs, V. R., Lamb, L., & Philipp, R. A. (2010). Professional noticing of children's mathematical thinking. *Journal for Research in Mathematics Education, 41*(2), 169-202.

Kamii, C. (1986). Place value: an explanation of its difficulty and educational implications for the primary grades. *Journal for Research in Childhood Education, 1*, 75-86.

Kennedy, M. (2005). *Inside teaching*. Cambridge, Massachusetts: Harvard University Press.

Kennedy, M., Ball, D., & McDiarmid, W. (1993). *A study package for examining and tracking changes in teachers' knowledge*. East Lansing, MI: National Center for Research on Teacher Learning.

Kennedy, M. M. (1998). Education reform and subject matter knowledge. *Journal of Research in Science Teaching, 35*(3), 249-263.

Kilpatrick, J., Swafford, J., & Bradford, F. (2001). *Adding it up: Helping children learn mathematics*. Washington DC: National Academy Press.

Klein, A., & Starkey, P. (2004). Fostering preschool children's mathematical knowledge:

Findings from the Berkeley Math Readiness Project. In D. H. Clements, J. Sarama & A.-M. DiBiase (Eds.), *Engaging young children in mathematics: Standards for early childhood mathematics education* (pp. 343-360). Mahwah, NJ: Lawrence Erlbaum Associates.

Krauss, S. (2005). Research paradigms and meaning making: a primer. *The Qualitative Report, 10*(4), 758-770.

Krauss, S., Brunner, M., Kunter, M., Baumert, J., Neubrand, M., Blum, W., et al. (2008). Pedagogical content knowledge and content knowledge of secondary mathematics teachers. *Journal of Educational Psychology, 100*(3), 716-725.

Larson, M. (2002). Essential characteristics of effective math instruction: Houghton Mifflin Math.

Lee, J. Exploring kindergarten teachers' pedagogical content knowledge of mathematics. *International Journal of Early Childhood, 42*(1), 27-41.

Lee, J., Meadows, M., & Lee, J. O. (2003). *What Causes Teachers to Implement High-Quality Mathematics Education More Frequently: Focusing on Teachers' Pedagogical Content Knowledge*.

Lee, J. S., & Ginsburg, H. P. (2007). What Is appropriate mathematics education for four-year-olds?: Pre-kindergarten teachers' beliefs. *Journal of Early Childhood Research, 5*(1), 2-31.

Lee, J. S., & Ginsburg, H. P. (2007). Preschool teachers' beliefs about appropriate early literacy and mathematics education for low- and middle-socioeconomic status children. *Early Education and Development, 18*(1), 111-143.

Lee, J. S., & Ginsburg, H. P. (2009). Early childhood teachers' misconceptions about mathematics education for young children in the United States. *Australasian Journal of Early Childhood, 34*(4), 37-45.

Leinhardt, G. (1986). Expertise in mathematics teaching. *Educational Leadership, 43*(7), 28-33.

Lincoln, Y. S., & Guba, E. G. (1985). *Naturalistic inquiry*. Thousand Oaks, CA: Sage.

Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States*. Mahwah, NJ: Erlbaum.

Mathwords. Retrieved January 10, 2011 from http://www.mathwords.com/

Merriam, S. B. (1998). *Qualitative research and case study applications in education*. San Francisco, CA: Jossey-Bass.

Mertens, D. M. (2005). *Research and evaluation in education and psychology* (2nd ed.). Thousand Oaks, CA: Sage.

Mewborn, D. (2001). Teachers content knowledge, teacher education, and their effects on the preparation of elementary teachers in the United States. *Mathematics Teacher Education and Development, 3*, 28-36.

Murray, A. (2001). Ideas on manipulative math for young children. *Young Children, 56*(4), 28-29.

National Association for the Education of Young Children & National Council of Teachers of Mathematics. (2002). *Position statement. Early childhood mathematics: Promoting good beginnings*. Retrieved August 20, 2009, from http://www.naeyc.org/positionstatements/mathematics

NCTM. (2002). *Principles and standards for school mathematics*. Reston, VA: NCTM.

Neuman, S. B., & Roskos, K. (2005). The state of state pre-kindergarten standards. *Early Childhood Research Quarterly, 20*, 125-145.

Philipp, R. A. (2008). Motivating prospective elementary school teachers to learn mathematics by focusing upon children's mathematical thinking. *Issues in Teacher Education, 17*(2), 7-26.

Russell, S. J. (1999). Mathematical reasoning in the elementary grades. In S. V. Lee (Ed.), *Developing mathematical reasoning in grades K-12* (pp. 1-12). Reston, VA: The National Council of Teachers of Mathematics.

Sarama, J., & Clements, D. H. (2008). Mathematics in early childhood. In O. N. Saracho & B. Spodek (Eds.), *Contemporary perspectives on mathematics in early childhood education* (pp. 67-94). NC: Information Age Publishing.

Sarama, J., DiBiase, A.-M., Clements, D. H., & Spitler, M. E. (2004). The professional development challenge in preschool mathematics. In D. H. Clements, J. Sarama & A.-M. DiBiase (Eds.), *Engaging young children in mathematics: Standards for early childhood mathematics education* (pp. 415-448). Mahwah, NJ: Lawrence Erlbaum Associates.

Seo, K.-H., & Ginsburg, H. P. (2004). What is developmentally appropriate in early childhood mathematics education? Lessons from new research. In D. H. Clements, J. Sarama & A. M. DiBiase (Eds.), *Engaging young children in mathematics: Standards for early* *childhood mathematics education* (pp. 91-104). Hillsdale, NJ: Lawrence Erlbaum Associates.

Shulman, L. (1986). Paradigms and research programs in the study of teaching: A contemporary perspective. In M. Wittrock (Ed.), *Handbook of research on teaching* (pp. 3-36). New York: MacMillan.

Shulman, L. (1987). Knowledge and teaching: Foundations of a new reform. *Harvard Educational Review, 57*(1), 1-22.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*(2), 4-14.

Sophian, C. (2004). Mathematics for the future: Developing a Head Start curriculum to support mathematics learning. *Early Childhood Research Quarterly, 19*(1), 59-81.

Stake, R. E. (1995). *The art of case study research*. Thousand Oaks, CA: Sage.

Strauss, A. L., & Corbin, J. M. (1998). *Basics of qualitative research: Grounded theory procedures and techniques*. Thousand Oaks: CA: Sage.

Tang, E., & Ginsburg, H. P. (1999). Young children's mathematical reasoning: A psychological view. In L. V. Stiff & F. R. Curcio (Eds.), *Developing Mathematical Reasoning in Grades K-12* (pp. 45-61). Reston: VA: National Council of Teachers of Mathematics.

Tarim, K. (2009). The Effects of cooperative learning on preschoolers' mathematics problem-solving ability. *Educational Studies in Mathematics, 72*(3), 325-340.

Texas Education Agency. (2008). *Texas Prekindergarten Guidelines*. Retrieved September 2008, from http://ritter.tea.state.tx.us/ed_init/pkguidelines/PKG_Final_100808.pdf .

Tudge, J. R. H., Li, L., & Stanley, T. K. (2008). The impact of method on assessing young children's everyday mathematical experiences. In O. N. Saracho & B. Spodek (Eds.), *Contemporary perspectives on mathematics in early childhood education* (pp. 187-214). Charlotte: Information Age Publishing, INC.

Wallace, A. H., Abbott, D., Blary, R. M. (2007). The classroom that math built, Encouraging young mathematicians to pose problems. *Young Children, 62*(5), 42-48.

Whitebread, D. (1995). Emergent mathematics or how to help young children become confident mathematicians. In J. Anghileri (Ed.), *Children's mathematical thinking in the primary years* (pp. 11-40). London: Cassell.

Wideen, M., Mayer-Smith, J., & Moon, B. (1998). A critical analysis of research on learning to teach: Making the case for an ecological perspective on inquiry. *Review of Educational Research, 68*, 130-178.

Wilkins, J. L. M. (2008). The relationship among elementary teachers' content knowledge, attitudes, beliefs, and practices. *Journal of Mathematics Teacher Education, 11*(2), 139-164.

Wolcott, H. F. (1994). *Transforming qualitative data: Description, analysis, and interpretation*. CA: Sage: Thousand Oaks.

Wolcott, H. F. (2001). *Writing up qualitative research* (2 ed.). CA: Sage: Thousand Oaks.

Woolfolk-Hoy, A., Davis, H., & Pape, S. (2006). Teachers knowledge and beliefs. In P. Alexander & P. Winne (Eds.), *Handbook of educational psychology*. Mahwah, NJ: Lawrence Erlbaum Associates.

Yin, R. K. (2003). *Case study research; Design and methods* (3 ed.). London: Sage.