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State Sums in Two Dimensional Fully Extended Topological Field Theories

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State Sums in Two Dimensional Fully Extended Topological Field Theories

by

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State Sums in Two Dimensional Fully Extended Topological Field Theories

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A state sum is an expression approximating the partition function of a d-dimensional field theory on a closed d-manifold from a triangulation of that manifold. To consider state sums in completely local 2-dimensional topological field theories (TFT's), we introduce a mechanism for incorporating triangulations of surfaces into the cobordism $(\infty, 2)$–category. This serves to produce a state sum formula for any fully extended 2-dimensional TFT possibly with extra structure. We then follow the Cobordism Hypothesis in classifying fully extended 2-dimensional G-equivariant TFT's for a finite group G. These are oriented theories in which bordisms are equipped with principal G-bundles. Combining the mechanism mentioned above with our classification results, we derive Turaev's state sum formula for such theories.

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Chapter 1

Introduction

The goal of this work is to derive state sum formulas in 2-dimensional fully-extended topological field theories (TFT's). A State Sum is an expression which approximates the value $\mathcal{Z}(M)$ of a n-dimensional quantum field theory $\mathcal{Z}$ on a n-dimensional closed manifold M using a triangulation of M or more generally a polyhedral decomposition. When the quantum field theory is topological a state sum formula computes $\mathcal{Z}(M)$ on the nose. The value $\mathcal{Z}(M)$ is independent of the chosen triangulation even though the formula depends on its combinatorics.

An axiomatic definition of a topological quantum field theory (TQFT) was laid out by Atiyah [1] in a manner closely related to Segal's definition of conformal field theory [14]. In Atiyah's formulation, a n-dimensional TQFT is a rule which associates to a n-dimensional closed manifold a complex number, and to a $(n-1)$-dimensional closed manifold a complex vector space, in a manner that respects disjoint unions and cutting-and-pasting. In some cases one may further extend a TQFT to a rule associating a $\mathbb{C}$-linear category to a $(n-2)$-dimensional closed manifold. In a fully extended n-dimensional TQFT the rule goes all the way down to points. A present day definition of a fully extended n-dimensional topological field theory takes the following form:

Definition. A fully extended n-dimensional topological field theory with values in a symmetric monoidal (∞, n)-category $\mathcal{C}$ is a symmetric monoidal functor,

$$\mathcal{Z} : (\mathrm{Bord}_n, \coprod) \longrightarrow (\mathcal{C}, \otimes)$$

Loosely speaking, the bordism category Bord_n consists of:

Objects: 0-manifolds,

1-Morphisms: bordisms of 0-manifolds,

2-Morphisms: bordisms of bordisms of 0-manifolds,

$\vdots$

n-Morphisms: bordisms of ... bordisms of bordisms of 0-manifolds,

$(n+1)$-Morphisms: diffeomorphisms of n-morphisms,

$(n+2)$-Morphisms: isotopies of diffeomorphisms of n-morphisms,

$(n+3)$-Morphisms: isotopies of isotopies of diffeomorphisms of n-morphisms,

$\vdots$

All this structure is packaged into what is known as a (∞, n)-category indicating that it is an ∞−category in which all k-morphisms are invertible for $k > n$. An ordinary category, for example, may be considered an $(\infty, 1)$-category all of whose k-morphisms for $k > 1$ are identities.

Possibly by abuse of terminology, we will say that a TFT has extra structure when bordisms in Bord_n are equipped with extra structure such as a framing, in which case we will say that the TFT is framed, or an orientation, in which case we will say that the TFT is oriented.

The construction and classification of TFT's exhibits two opposing and complementary trends. We refer to the first as a *bottom-up* construction, and to the other as a *top-down* construction. The bottom-up approach may be considered a generators-and-relations construction. Possibly the first example of a bottom-up classification of TFT's to appear in the literature [3] was the claim that every 2-dimensional oriented TFT with values in the category of vector spaces is determined by a finite dimensional commutative Frobenius algebra and vice versa. A more general example of a bottom-up classification is the *Cobordism Hypothesis* due to Baez and Dolan [2]. It determines the classification of fully extended TFT's from the object associated to the point.

The top-down approach may be considered an a-priori construction. Heuristically speaking, the invariant associated to a top-dimensional closed manifold is given by the path integral, which is then suitably generalized to manifolds of higher co-dimension. An example here is 3-dimensional Chern-Simons gauge theory associated to a compact gauge group G and a level $\lambda \in H^4(BG, \mathbb{Z})$ [4]. This construction was extended to a 0-1-2-3 theory when G is finite [6], and when G is toral [7].

The motivating example we used to explore the relationship between the two approaches was that of 2-dimensional G-equivariant TFT's for a finite group G.

Definition. Let G be a topological group. A n-dimensional G-equivariant topological field theory is a n-dimensional oriented theory in which manifolds are also equipped with principal G-bundles.

From now on let us assume G is finite. There are two constructions of 2-dimensional G-equivariant theories to compare. On the one hand, the Cobordism Hypothesis provides a classification of fully-extended 2-dimensional G-equivariant TFT's. On the other hand, Turaev [16] provides a classification of non-extended 2-dimensional G-equivariant TFT's with values in the category of vector spaces in terms of crossed G-Frobenius algebras. Turaev also provides a top-down construction of such non-extended theories from what he terms *biangular G-algebras* based on a state-sum formula.

In view of these two approaches to the construction and classification of 2-dimensional G-equivariant TFT's we ask:

(a) Can Turaev's theories be extended down to points?

(b) How does the Cobordism Hypothesis produce Turaev's state-sum formula?

The Cobordism Hypothesis, whose current formulation and proof is due to Lurie [10] states the following.

Theorem (Cobordism Hypothesis)**.** *Let $\mathcal{C}$ be a symmetric monoidal (∞, n)-category. The space of fully extended n-dimensional framed topological field theories with values in $\mathcal{C}$ is canonically equivalent to the underlying space of fully-dualizable objects in $\mathcal{C}$ via the map $F \mapsto F(\bullet)$.*

A fully extended n-dimensional *framed* TFT is a theory in which manifolds in the bordism (∞, n)-category are equipped with an n-framing. An *n-framing* of a k-manifold is a trivialization $\underline{\mathbb{R}}^n \xrightarrow{\cong} TX \oplus \underline{\mathbb{R}}^{n-k}$ of the tangent bundle TX completed to a rank n real vector bundle (here $\underline{\mathbb{R}}^n$ denotes the trivial rank n vector bundle over X).

The notion of a *fully-dualizabile object* is a higher categorical analogue of a dualizable object in a symmetric monoidal category. When $n = 1$ an object in a symmetric monoidal $(\infty, 1)$-category is (fully) dualizable if there exists an object $X^\vee$ and 1-morphisms $\mathrm{ev} : X \otimes X^\vee \to \mathbf{1}$ and $\mathrm{coev} : \mathbf{1} \to X^\vee \otimes X$ such that

$$X \cong X \otimes \mathbf{1} \xrightarrow{1 \otimes \mathrm{coev}} X \otimes X^\vee \otimes X \xrightarrow{\mathrm{ev} \otimes 1} \mathbf{1} \otimes X \cong X$$

$$X^\vee \cong \mathbf{1} \otimes X^\vee \xrightarrow{\mathrm{coev} \otimes 1} X^\vee \otimes X \otimes X^\vee \xrightarrow{1 \otimes \mathrm{ev}} X^\vee \otimes \mathbf{1} \cong X^\vee$$

are both identities up to invertible 2-morphisms.

When $n = 2$ an object in a symmetric monoidal $(\infty, 2)$-category is fully dualizable if there exist $X^\vee$, ev and coev as before and ev, coev admit left and right adjoints up to invertible 3-morphisms.

The *underlying space* of a (∞, n)-category is defined to be the $(\infty, 0)$-category we get by getting rid of all non-invertible morphisms.

Let k denote an algebraically closed characteristic zero field. Throughout this paper, we discuss the case where $\mathcal{C}$ is the 2-category whose objects are unital associative algebras, whose 1-morphisms are bi-modules, and whose 2-morphisms are morphisms of bimodules also known as intertwiners. We denote it by Alg_k. It is a symmetric monoidal category with product given by tensor product over the ground field k.

In section 3.2.1 we show that fully dualizable objects in Alg_k are exactly finite dimensional semi-simple algebras. Moreover, the subcategory with duals $\mathrm{Alg}_k^{\mathrm{fd}} \subset \mathrm{Alg}_k$ is the 2-category whose objects are the finite-dimensional semi-simple algebras, whose 1-morphisms are their finite rank bimodules, and whose 2-morphisms are their intertwiners. Note this subcategory is not full. In section 3.2.2 we prove:

Theorem. *The underlying space X of fully dualizable objects in Alg_k is equivalent to*

$$X \simeq \coprod_{r=1}^{\infty} E\Sigma_r \times_{\Sigma_r} K(k^{\times}, 2)^{\times r},$$

where Σ_r denotes the symmetric group on r elements, acting on the r-fold Cartesian product $K(k^{\times}, 2)^{\times r}$ of the Eilenberg - Mac Lane space by permutations.

The space in the above theorem is equivalent to the space of fully extended 2-dimensional framed TFT's with values in Alg_k. The theorem implies that the only invariant of such a theory is the Morita equivalence class of the algebra associated to the point. The Morita class appears in the theorem above as the positive integer $1 \leq r < \infty$ labelling the connected components of X.

In [10], Lurie introduces a generalization of the Cobordism Hypothesis for theories with extra structure. In section 3.4 we discuss the following result:

Proposition. *The space of fully extended* 2*-dimensional* G*-equivariant topological field theories with values in* Alg_k *is equivalent to*

$$\mathrm{Map}(BG, \mathrm{Map}(BSO(2), X)),$$

where BG *and* $BSO(2)$ *denote the classifying spaces of* G *and* $SO(2)$ *respectively.*

The mapping space $\mathrm{Map}(BSO(2), X)$ is equivalent to the space of fully extended 2-dimensional oriented TFT's with values in Alg_k as discussed in section 3.3. The Cobordism Hypothesis implies that the underlying space of fully dualizable objects in a symmetric monoidal (∞, n)-category admits a homotopy $O(n)$-action. The space of fully extended n-dimensional oriented TFT's is canonically equivalent to the space of homotopy $SO(n)$-fixed points. The underlying statement in the above proposition is that the homotopy $SO(2)$-action on X is trivializable, which is proved in section 3.2.3.

In section 3.3 we prove that the path-components of the space of fully extended 2-dimensional oriented theories with values in Alg_k are in bijection with

$$\pi_0 \mathrm{Map}(BSO(2), X) \cong \coprod_{r=1}^{\infty} (k^\times)^r / \Sigma_r$$

In particular, oriented theories refine framed theories in the sense that they capture not only the Morita class of the algebra associated to the point but the dimensions of its simple subalgebras as well.

Remark. In his thesis [13], Schommer-Pries provided a generators & relations description of the oriented bordism 2-category. He used this description to prove that the 2-groupoid of fully extended 2-dimensional oriented TFT's with values in Alg_k is equivalent to the 2-groupoid Frob_k of separable symmetric Frobenius algebras.

Remark. In their paper [5], Etingof, Nikshych and Ostrik study G-extensions of unital associative k-algebras for a finite group G. The 2-groupoid of G-extensions of a fixed finite-dimensional semi-simple algebra A, described in their work, is equivalent to the fundamental 2-groupoid of the connected component of $\mathrm{Map}(BG, X)$ associated to A.

In section 3.4 we prove:

Theorem. *Every fully-extended 2-dimensional G-equivariant TFT with values in Alg_k gives rise to a finite-dimensional strongly G-graded algebra $B = \bigoplus_g B_g$ with semi-simple trivial component B_e together with a non-degenerate cyclically symmetric form $\tau_e : B_e \otimes B_e \to k$, and vice versa.*

A *strongly* G-graded algebra is a G-graded algebra $B = \bigoplus_g B_g$ for which the multiplication factors through an isomorphism,

$$\begin{array}{ccc} B_g \otimes B_{g'} & \longrightarrow & B_{gg'} \\ & \searrow \quad \nearrow_{\cong} & \\ & B_g \otimes_{B_e} B_{g'} & \end{array}$$

Let $B = \bigoplus_g B_g$ be a finite dimensional strongly G-graded algebra with a semi-simple trivial component B_e. The trivial component B_e is a finite dimensional algebra, and as such, admits a canonical symmetric trace form

$$\tau_{B_e}(a, a') = \mathrm{Tr}(\mu_{aa'} : B_e \to B_e)$$

where $\mu_{aa'}$ denotes the operator given by left multiplication in aa'. The canonical trace form is non-degenerate since B_e is semi-simple.

The pair (B, τ_{B_e}) represents an equivalence class of fully-extended 2-dimensional G-equivariant TFT's with values in Alg_k. Denote the corresponding theory by $\mathcal{Z}_{(B,\tau_{B_e})}$. Then,

$$\mathcal{Z}_{(B,\tau_{B_e})}(\overset{+}{\bullet}) \cong B_e \quad , \quad \mathcal{Z}_{(B,\tau_{B_e})}(\overset{-}{\bullet}) \cong B_e^{\mathrm{op}}$$

$$\mathcal{Z}_{(B,\tau_{B_e})}(\circlearrowleft^g) \cong HH_0(B_e; B_g)$$

where $\circlearrowleft^g$ denotes an oriented circle equipped with a principal G-bundle with holonomy g and $HH_0(B_e; B_g)$ denotes the Hochschild homology of B_e with coefficients in B_g.

We can now answer question (a). Let $B = \bigoplus_g B_g$ denote a biangular G-algebra in the sense of Turaev (see Appendix A.2). Let $\mathcal{Z}_B^{\mathrm{Tur}}$ denote the 2-tier, 2-dimensional, G-equivariant TFT with values in Vect_k associated with B. The algebra B is a finite dimensional strongly G-graded algebra with a semi-simple trivial component B_e. In Section 3.6 we prove the following.

Proposition. *Let $\mathcal{Z}^{(1,2)}_{(B,\tau_{B_e})}$ denote the 2-tier truncation of the fully extended, 2-dimensional, G-equivariant TFT with values in Alg_k corresponding to the G-equivariant algebra (B, τ_{B_e}). Then,*

$$\mathcal{Z}_B^{Tur} \simeq \mathcal{Z}^{(1,2)}_{(B,\tau_{B_e})}$$

In other words, Turaev's theories, $\mathcal{Z}_B^{\mathrm{Tur}}$, extend down to 0-manifolds, and their extension is given by $\mathcal{Z}_{(B,\tau_{B_e})}$.

Let us now turn to question (b), namely, how to derive a state sum formula from the Cobordism Hypothesis. Let $\mathcal{Z}$ denote a fully extended n-dimensional TFT, M a closed n-manifold and T a triangulation of M. A state sum formula computes $\mathcal{Z}(M)$ from:

(i) the combinatorics of T,

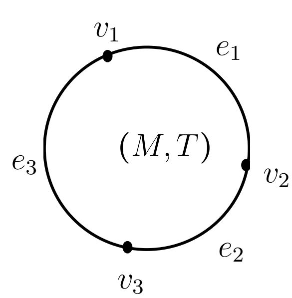
(ii) the invariant $\mathcal{Z}(\Delta_n)$ of the standard n-simplex.

Hence, to make sense of state sum formulas in the context of fully extended TFT's, we need to understand the invariant $\mathcal{Z}$ associates to Δ_n. To that end, we need to interpret Δ_n as a n-morphism in Bord_n.

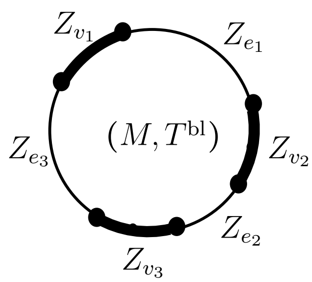
There are two issues to address here. The first is that already for $n = 1$ the 1-simplex Δ_1 can be viewed as a 1-morphism in three different ways:

$$\Delta_1 : \bullet \longrightarrow \bullet \quad , \quad \Delta_1 : \bullet \coprod \bullet \longrightarrow \varnothing \quad , \quad \Delta_1 : \varnothing \longrightarrow \bullet \coprod \bullet$$

In other words, we need to choose how to split $\partial\Delta_1 = (\partial\Delta_1)_{\mathrm{in}} \coprod (\partial\Delta_1)_{\mathrm{out}}$ into 'incoming' and 'outgoing' boundary. The second is that we need to make a compatible choice for all simplices of a given triangulation.



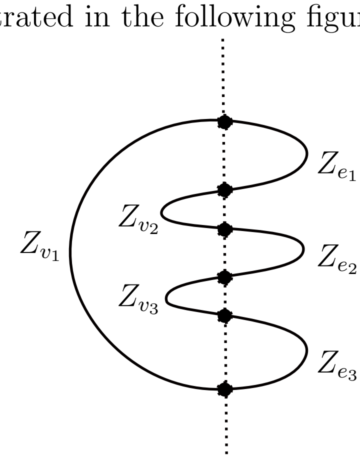
This work is dedicated in part to the resolution of both issues in dimension 2. To understand the ideas that go into resolving these problems let us go down in dimension. Consider the triangulation T of the 1-manifold M as in the above figure. We can blow-up the triangulation, so to speak, to get a new triangulation T^{bl} of M. Each vertex in T is blown-up to an edge of T^{bl} as depicted in the figure below. The edges of the blown-up triangulation T^{bl} are labeled either by a vertex Z_{v_i} or by an edge Z_{e_j} of the original triangulation.



The blown-up triangulation allows for a systematic choice of splitting $\partial Z = (\partial Z)_{\mathrm{in}} \coprod (\partial Z)_{\mathrm{out}}$ for every edge Z of T^{bl}. We may consider edges Z_e as 1-morphisms $Z_e : \bullet \coprod \bullet \to \varnothing$, and edges Z_v as 1-morphisms $Z_v : \varnothing \to \bullet \coprod \bullet$. Such a systematic choice gives rise to a factorization of M,

$$M = \left(\coprod_e Z_e \right) \circ \left(\coprod_v Z_v \right)$$

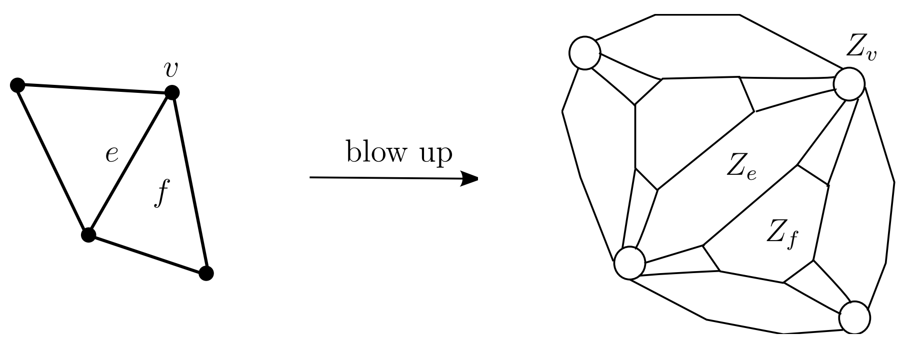
This factorization is illustrated in the following figure:



It has two important features. First, it demonstrably incorporates the combinatorics of T. Second, its building blocks, Z_v and Z_e, are part of the data making the point a fully-dualizable object in Bord_1.

One important difference in passing to dimension 2 is that the blow-up of a triangulation is no longer a triangulation but rather it admits faces with various number of sides. In section 2.1.1 we introduce the notion of a *polygonal complex* generalizing that of a simplicial complex. A polygonal decomposition of a closed surface is the data of a polygonal complex P together with a homeomorphism $|P| \to W$ from the geometric realization of P to W.

We start with a closed surface W endowed with a polygonal decomposition P. In section 2.2 we introduce a blow-up procedure of P resulting in a polygonal decomposition P^{bl} of W. Both vertices $v \in V(P)$ and edges $e \in E(P)$ of the original polygonal decomposition are blown-up to polygonal faces Z_v and Z_e. New polygonal faces are also introduced.



The figure above illustrates a local picture of the blow-up in dimension 2 consisting of two adjacent triangles. The blown-up polygonal decomposition P^{bl} of W has four types of polygonal faces. Three of them, labeled by the vertices Z_v, edges Z_e and faces Z_f of P, appear in the figure. The forth, $Z_{v,f}$, is labeled by a pair v, f, such that $v \in f$. It is adjacent to Z_v, Z_f and Z_e such that $v \subset e \subset f$. The blow-up gives rise to a factorization of W discussed in section 2.3:

Theorem. *Let P be a polygonal decomposition of a closed surface W. Then,*

$$|P^{\mathrm{bl}}| = \left(\coprod_f Z_f\right) \circ_v \left(\left(\coprod_e Z_e\right) \circ_h \left(\coprod_{v\in f} Z_{v,f}\right)\right) \circ_v \left(\coprod_v Z_v\right)$$

where v, e and f stand for the vertices, edges and faces of P.

Note that in the above factorization we distinguish between horizontal $\circ_h$ and vertical $\circ_v$ composition depending on which direction we choose to glue along. This factorization has the same couple of features we noted before: (i) it retains the combinatorics of P, (ii) its building blocks P_v, P_e and P_f are part of the data making the point a fully-dualizable object in Bord_2. In the 2-fold simplicial space model for Bord_2 introduced in [10] the right-hand-side denotes an element of $(\mathrm{Bord}_2)_{2,1}$. This factorization works for surfaces with extra structure since it does not interact with the blow-up.

Given a fully-extended 2-dimensional TFT $\mathcal{Z}$ with or without extra structure, a closed surface W and a polygonal decomposition P of W, we can compute $\mathcal{Z}(W)$ using the above factorization as long as we have the data making $\mathcal{Z}(\bullet)$ a fully-dualizable object in the target $(\infty, 2)$-category.

Consider the case where $\mathcal{Z}_{(B,\tau_{B_e})}$ is the fully extended 2-dimensional G-equivariant TFT with values in Alg_k, corresponding to a finite-dimensional, strongly G-graded algebra $B = \bigoplus_g B_g$, with semi-simple trivial component B_e equipped with its canonical trace form τ_{B_e}. We conclude section 4 with the following corollary:

Corollary. *Let W be a closed oriented surface equipped with a principal G-bundle $p : E \to W$. Let T be a triangulation of W whose faces, edges and vertices are labeled f, e and f respectively. Then,*

$$\mathcal{Z}_{(B,\tau_{B_e})}(W,p) = (\otimes_f \phi_f)\left((\otimes_e b_e) \otimes (\otimes_{v\in f} 1)\right)$$

where b_e is the canonical element in $B_g \otimes B_{g^{-1}}$ when the holonomy of p along the edge e is g.

The state sum formula on the right hand side of the above theorem is exactly Turaev's state sum formula used to define $\mathcal{Z}_B^{\mathrm{Tur}}$. Here we derived it from the Cobordism Hypothesis thereby answering question (b).

Chapter 2

Polygonal Decompositions.

In this chapter we define what we mean by a polygonal decomposition of a closed surface. Given a closed surface, together with a polygonal decomposition, we introduce a three-step procedure, which, at every step, replaces the old polygonal decomposition with a new one. We use this procedure to give a factorization of the closed surface into 2-manifolds with corners.

2.1 Polygonal Complexes.

2.1.1 Basic Concepts.

In this chapter S^1 denotes the unit circle $S^1 = \{ z \in \mathbb{C} \,|\, |z| = 1 \}$, and S_n its subset $S_n := \{1, e^{2\pi i/n}, \ldots, e^{2\pi i(n-1)/n}\}$ for $n \geq 1$. We let Π_n denote the convex hull of S_n inscribed inside S^1.

Definition 2.1.1. Let X be a finite set. A cyclic order on X is a path component in the space of injections $X \to S^1$.

Definition 2.1.2. Let X, X' be two cyclically ordered sets with representatives $\alpha, \alpha' : X, X' \to S^1$. A map $f : X \to S'$ is cyclic order preserving if for every section $f' : \mathrm{Im}(f) \to X$ we have $[\alpha \cdot f'] = [\alpha'|_{\mathrm{Im}(f)}]$.

Let X be cyclically ordered with $\alpha : X \to S^1$ a representative, and let $X' \subset S$ be a subset. Then the subset X' carries an induced cyclic order given by $[\alpha|_{X'}]$.

A cyclic order on X induces a ternary relation $R_X \subset X \times X \times X$. We have $R_X(x, x', x'')$ iff the cyclic order induced on $\{x, x', x''\} \subseteq X$ is the path component of the map $x, x', x'' \mapsto 1, i, -1$.

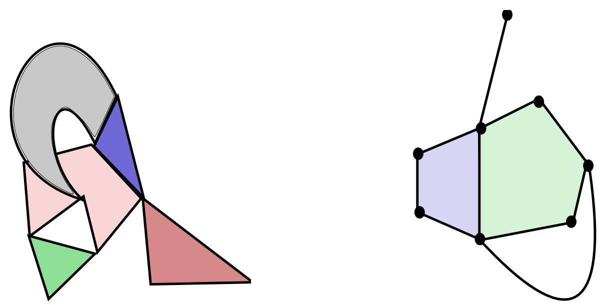
Let X be cyclically ordered. An element $x \in X$ is a *vertex of* X. A subset $\{x, x'\} \subseteq X$ is an *edge of* X if either $\{x, x'\} = X$ or having $y \in X \backslash \{x, x'\}$ such that $R_X(x, y, x')$ implies there is no $y' \in X \backslash \{x, x'\}$ for which $R_X(x', y', x)$.

Let F be a finite set and let $P \subset 2^F$. Assume each $X \in P$ is cyclically ordered. An element $x \in F$ is a *vertex of* P if there is $X \in P$ such that x is a vertex of X. A pair $\{x, x'\} \subset F$ is an *edge of* P if there is $X \in P$ such that $\{x, x'\}$ is an edge of X. An element $X \in P$ such that $|X| \geq 3$ is a *face of* P. We denote the vertices, edges and faces of P by the letters v, e and f respectively. We reserve the letter X for general elements of P. A priori vertices and edges of P are not necessarily in P.

Definition 2.1.3. Let F be a finite set and let $P \subset 2^F$. Assume each $X \in P$ is cyclically ordered. We say P is a *polygonal complex* if it satisfies the following conditions:

(i) if v is a vertex of P then $v \in P$,

(ii) if e is an edge of P then $e \in P$.



(a) (b)

Figure 2.1: Geometric models of polygonal complexes.

Definition 2.1.4. Let F be a finite set and let $P \subset 2^F$. Assume each $X \in P$ is cyclically ordered. The polygonal complex completion $\overline{P}$ of P is the smallest polygonal complex in 2^F containing P. It is given by adding to P all its vertices and edges.

Let P be a polygonal complex. We define a partial order on P,

(i) $v < e$ when $v \subset e$,

(ii) $v < f$ when $v \subset f$,

(iii) $e < f$ when e is an edge of f.

Alternately, we can view P as a category.

We define a functor $\mathcal{F}_\alpha : P \to \text{Top}$, which depends on a choice of a collection of cyclic order preserving bijection $\alpha_X : X \to S_{\#X}$ for every $X \in P$. On objects,

$$\mathcal{F}_\alpha : X \longmapsto \Pi_{\#X}$$

On arrows,

$$\mathcal{F}_\alpha : (X' \to X) \longmapsto (\Pi_{\#X'} \to \Pi_{\#X})$$

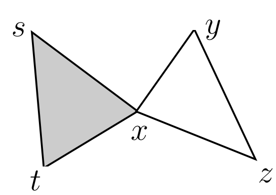
where $\Pi_{\#X'} \to \Pi_{\#X}$ is the unique affine linear map sending $\alpha_{X'}(s) \mapsto \alpha_X(s)$ for every $s \in X' \subseteq X$.

Definition 2.1.5. Let P be a polygonal complex. Choose a cyclic order preserving bijection $\alpha_X : X \to S_{\#X}$ for every $X \in P$. Define the *geometric realization* of P with respect to α to be the colimit,

$$|P|_\alpha := \varinjlim_P \mathcal{F}_\alpha(X)$$

Remark 2.1.6. For any $X \in P$ and any two choices of order preserving bijections $\alpha_X, \alpha'_X : X \to S_{\#X}$ there is a unique affine linear map $\Pi_{\#X} \to \Pi_{\#X}$ sending $\alpha_X(s)$ to $\alpha'_X(s)$ for each $s \in X$. This induces a natural homeomorphism $|P|_\alpha \to |P|_{\alpha'}$ by the universal property of colimits. Hence we are justified in dropping α from our notation.

Example 2.1.7. Let $F = \{s, t, x, y, z\}$ and let $X_1 = \{s, t, x\}$, $X_2 = \{x, y\}$, $X_3 = \{x, z\}$ and $X_4 = \{y, z\}$. Take $P = \{X_1, \ldots, X_4\}$. Choose the cyclic ordering on X_1 to be the component of the map $X_1 \to S^1$ sending $s, t, x \mapsto 1, i, -1$. Note that P is not a polygonal complex, for example, $\{s, t\}$ is an edge of P not in P. Its geometric realization appears bellow.



Let P be a polygonal complex and $Q \subset P$ a polygonal subcomplex. The complement Q^c of Q in P equals the completion $\overline{P \backslash Q}$. The intersection $Q \cap Q^c$ consists of edges and vertices only and is a polygonal complex without need for completion.

Definition 2.1.8. Let $P \subset 2^F$ be a polygonal complex and $A \subseteq F$. Then the star of A in P is the polygonal subcomplex

$$\mathrm{St}(A, P) = \overline{\{ X \in P \,|\, A \cap X \neq \emptyset \}}$$

Definition 2.1.9. Let $P \subset 2^F$ be a polygonal complex and $A \subseteq F$. Then P is *flat at* A if $|\mathrm{St}(A, P)|$ can be embedded in a disc.

It is straight forward to show $\mathrm{St}(A, P) \cup \mathrm{St}(A', P) = \mathrm{St}(A \cup A', P)$, in particular, P is flat at A and A' if and only if it is flat at $A \sqcup A'$.

Lemma 2.1.10. *Let $\mathcal{C}$ be a small category and $\mathcal{C}_1, \mathcal{C}_2 \subset \mathcal{C}$ full subcategories. Let $\mathcal{C}_3 = \mathcal{C}_1 \cap \mathcal{C}_2 \subset \mathcal{C}$ be the full subcategory with objects $\mathrm{Obj}\,\mathcal{C}_1 \cap \mathrm{Obj}\,\mathcal{C}_2$. Assume,*

$$\mathrm{Mor}\,\mathcal{C} = \mathrm{Mor}\,\mathcal{C}_1 \cup \mathrm{Mor}\,\mathcal{C}_2$$

Let $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ be a functor where $\mathcal{D}$ is a category admitting small colimits. Then there is a canonical isomorphism,

$$\underleftarrow{colim}_{\mathcal{C}} \mathcal{F}(c) \cong \underleftarrow{colim}_{\mathcal{C}_1} \mathcal{F}(c_1) \sqcup_{\underleftarrow{colim}_{\mathcal{C}_3} \mathcal{F}(c_3)} \underleftarrow{colim}_{\mathcal{C}_2} \mathcal{F}(c_2)$$

Proof. Pick $X \in \mathcal{D}$ for the colimit $\underleftarrow{\mathrm{colim}}_{\mathcal{C}} \mathcal{F}(c)$ and $X_i \in \mathcal{D}$ for the colimit $\underleftarrow{\mathrm{colim}}_{\mathcal{C}_i} \mathcal{F}(c_i)$. Pick $Y \in \mathcal{D}$ for the push out $X_1 \coprod_{X_3} X_2$.

Inclusion of categories implies existence of a canonical arrow, e.g. $\mathcal{C}_i \subset \mathcal{C}$ implies existence of canonical arrow $X_i \to X$. This gives a commutative square,

$$\begin{array}{ccc} X_3 & \longrightarrow & X_1 \\ \downarrow & & \downarrow \\ X_2 & \longrightarrow & X \end{array}$$

By universal property of push-out we get a canonical arrow $Y \to X$.

To get an arrow $X \to Y$ we need to show Y admits an arrow $\mathcal{F}(c) \to Y$ for each $c \in \mathcal{C}$ such that all triangles commute,

$$\begin{array}{ccc} \mathcal{F}(c) & \xrightarrow{\mathcal{F}(f)} & \mathcal{F}(c') \\ & \searrow \quad \swarrow & \\ & Y & \end{array} \tag{2.1.1}$$

Note the first assumption implies $\mathrm{Obj}\,\mathcal{C} = \mathrm{Obj}\,\mathcal{C}_1 \cup \mathcal{C}_2$. Let $c \in \mathcal{C}$, if $c \in \mathcal{C}_1$ take $\mathcal{F}(c) \to Y$ to be the composition $\mathcal{F}(c) \to X_1 \to Y$, and similarly if $c \in \mathcal{C}_2$. These compositions agree on $c \in \mathcal{C}_3$.

Let $f : c \to c' \in \mathrm{Mor}\,\mathcal{C}$, then $f \in \mathrm{Mor}\,\mathcal{C}_i$ for some $i = 1, 2$, and the following triangle commutes,

$$\begin{array}{ccc} \mathcal{F}(c) & \xrightarrow{\mathcal{F}(f)} & \mathcal{F}(c') \\ & \searrow \quad \swarrow & \\ & X_i & \end{array}$$

This implies all triangles in (2.1.1) commute. The compositions $X \to Y \to X$ and $Y \to X \to Y$ are both identities by the universal property. This proves the lemma. □

Lemma 2.1.11. *Let P be a polygonal complex and Q a subcomplex. Then there is a canonical homeomorphism,*

$$|P| \cong |Q| \coprod_{|Q\cap Q^c|} |Q^c|$$

Proof. Consider P a partially ordered set as defined above. Alternately, view it as a small category $\mathcal{C} \equiv P$. Take $\mathcal{C}_1 = Q$ and $\mathcal{C}_2 = Q^c$ as full subcategories. The condition on morphisms $\operatorname{Mor}\mathcal{C} = \operatorname{Mor}\mathcal{C}_1 \coprod \operatorname{Mor}\mathcal{C}_2$ in Lemma 2.1.10 is equivalent to having $\operatorname{Obj}\mathcal{C} = \operatorname{Obj}\mathcal{C}_1 \coprod \operatorname{Obj}\mathcal{C}_2$ and $\mathcal{C}(x,y) = \varnothing$ when either $x \in \mathcal{C}_1 \backslash \mathcal{C}_2$ and $y \in \mathcal{C}_2 \backslash \mathcal{C}_1$ or the other way around. The last two conditions are easily verifiable in our case. □

Following Section 3.3 in [15], we define a *subdivision* of a polygonal complex $P \subseteq 2^F$ to be a polygonal complex $P' \subseteq 2^{F'}$ such that,

(i) The set F' is a finite subset of points of $|P|$.

(ii) If $X' \in P'$ then there is $X \in P$ such that X' is contained in the image of $\Pi_{\#X}$ in $|P|$.

(iii) The linear map $|P'| \to |P|$ sending each vertex of P' to the corresponding point of $|P|$ is a homeomorphism onto its image.

Our definition differs from that of [15] in that the map $|P'| \to |P|$ is required to only be a homeomorphism onto its image rather than a homeomorphism.

Definition 2.1.12. Let P be a polygonal complex with partial order as defined above. The barycentric subdivision of P is the polygonal complex $\mathrm{sd}P \subseteq 2^P$ containing all totally ordered subsets.

Note that the barycentric subdivision of a polygonal complex is a simplicial complex. Therefore, the partial order defined on $\mathrm{sd}P$, considered as a polygonal complex, is simply inclusion. The barycentric subdivision is a subdivision P in the sense introduced above.

Assume a choice of order preserving bijections $\alpha_X : X \to S_{\#X}$ for every $X \in P$, and $\alpha'_Y : Y \to S_{\#Y}$ for every $Y \in \mathrm{sd}P$. The affine linear map $|\mathrm{sd}P|_{\alpha'} \to |P|_\alpha$, sending each vertex of $\mathrm{sd}P$ to the corresponding point of $|P|_\alpha$, is defined in the following way.

For every $Y \in \mathrm{sd}P$ we define $\Pi_{\#Y} \to |P|_\alpha$ such that for every pair $Y' \subseteq Y$ of elements of $\mathrm{sd}P$ the following triangle commutes,

$$\begin{array}{ccc} \Pi_{\#Y'} & \longrightarrow & \Pi_{\#Y} \\ & \searrow \quad \swarrow & \\ & |P|_\alpha & \end{array}$$

where the horizontal arrow is $F_{\alpha'}(Y' \to Y)$. Specifically, $\Pi_{\#Y'} \to \Pi_{\#Y}$ is the unique affine linear map sending $\alpha'_{Y'}(X) \mapsto \alpha'_Y(X)$ for every $X \in Y' \subseteq Y$.

Let $Y \in \mathrm{sd}P$ be a totally ordered subset of P. We define $\Pi_{\#Y} \to \Pi_{\#\max(Y)}$, so that for every $Y' \subseteq Y$ the following square commutes,

$$\begin{array}{ccc} \Pi_{\#Y'} & \longrightarrow & \Pi_{\#\max(Y')} \\ \downarrow & & \downarrow \\ \Pi_{\#Y} & \longrightarrow & \Pi_{\#\max(Y)} \end{array}$$

If $F : P \to \mathrm{Top}$ denotes the functor defining $|P|_\alpha \coloneqq \varinjlim_F$ then the right hand side vertical arrow is $F(\max(Y') \to \max(Y))$. More specifically, $\Pi_{\#\max(Y')} \to$

$\Pi_{\#\max(Y)}$ is the unique affine linear map such that $\alpha_{\max(Y')}(x) \mapsto \alpha_{\max(Y)}(x)$ for every $x \in \max(Y') \subseteq \max(Y)$.

The map $\Pi_{\#Y} \to \Pi_{\#\max(Y)}$ is defined to be the unique affine linear map sending,

$$\alpha'_Y(X) \longmapsto F(X \to \max(Y))(b(\Pi_{\#X}))$$

for every $X \in Y$, where $b(\Pi_{\#X})$ denotes the barycenter of the polygon $\Pi_{\#X}$. Now we can define $\Pi_{\#Y} \to |P|_\alpha$ as the composition $\Pi_{\#Y} \to \Pi_{\#\max(Y)} \to |P|_\alpha$, and all appropriate diagrams commute.

2.1.2 Truncation of a Polygonal Complex.

Let F be a finite set, and $P \subset 2^F$ a polygonal complex with each $X \in P$ cyclically ordered. From now on we will assume there are no two distinct $X, X' \in P$ with more than one common edge. For $A \subseteq F$ denote,

$$X_P(A) := \{\, y \in F \backslash A \,|\, \{x, y\} \text{ is an edge of } P \text{ for some } x \in A \,\}$$

We define a new set F_A by removing A from F and adding a second disjoint copy of $X_P(A)$. We denote the second disjoint copy by $X'_P(A)$. Elements of $X'_P(A)$ are denoted $v(x, y)$ where $x \in A$, $y \notin A$ and $\{x, y\}$ is an edge of P. Explicitly,

$$F_A := F \backslash A \coprod X'_P(A)$$

For every $X \in P$ we introduce an element of 2^{F_A},

$$S(A, X) := X \backslash A \coprod \{\, v(x, y) \in X'_P(A) \,|\, \{x, y\} \text{ is an edge of } X \,\}$$

If $A \cap X = \varnothing$ then $S(A,X) = X$, and if $X \subseteq A$ then $S(A,X) = \varnothing$.

Define a cyclic order on $S(A,X)$ in the following way. Choose a representative $\alpha : X \to S^1$ for the cyclic order on X. We can always find α such that the origin is contained in the convex hull of $\alpha(X)$ inscribed inside S^1, so we will assume so. Define a bijection $\beta : S(A,X) \to S^1$ in the following way,

$$\beta(s) = \begin{cases} \alpha(s) & s \in S(A,X) \backslash X'_P(A) = X \backslash A \\ \frac{\alpha(x)+\alpha(y)}{|\alpha(x)+\alpha(y)|} & s = v(x,y) \in S(A,X) \cap X'_P(A) \end{cases}$$

Take the cyclic order on $S(A,X)$ to be $[\beta]$. Note that $[\beta]$ is the unique cyclic order on $S(A,X)$ such that: (i) the map $S(A,X) \to X$, which is the identity on $X \backslash A$ and send $v(x,y) \in S(A,X) \cap X'_P(A)$ to x, is cyclic order preserving, (ii) if $v(x,y) \in S(A,X) \cap X'_P(A)$ then $\{v(x,y), y\}$ is an edge of $S(A,X)$. Clearly, if $X \cap A = \varnothing$ then $\alpha = \beta$ and the cyclic order defined on $S(A,X) = X$ agrees with that of X.

Definition 2.1.13. Let $P \subseteq 2^F$ be a polygonal complex with $A \subseteq F$. Assume no two distinct $X, X' \in P$ share more than one common edge. Using the above notation, define the *truncation* P^{tr}_A of P at A to be the minimal polygonal complex in 2^{F_A} containing $S(A,X)$ for each $X \in P$ with the unique cyclic order described above.

Consider the polygonal complex introduced in Example 2.1.7. The geometric realization of its truncation at $A = \{\}$ is depicted in Figure 2.3. Imagine the Cookie Monster biting off all $X \in P$ contained in A together with bits of their neighborhoods.

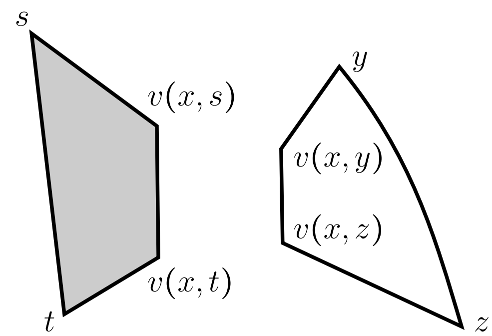


Figure 2.2: The geometric realization of the truncation of the polygonal complex in Example 2.1.7 at the vertex x.

Remark 2.1.14. A pair $\{v(x,y), v(x',y')\}$ is an edge of P_A^{tr} if and only if there exists a unique $S(A,X)$ with edge $\{v(x,y), v(x',y')\}$. The unique existence of such a $S(A,X)$ is equivalent to the unique existence of $X \in P$ with edges $\{x,y\}$, $\{x',y'\}$ and if $x \neq x'$ then also with edge $\{x,x'\}$. Uniqueness is guaranteed by our assumption that no two distinct $X, X' \in P$ share more than one common edge. A pair $\{v(x,y), v(x',y')\}$ is never a set of the form $S(A,X)$ for some $X \in P$. Hence if it is an edge of P_A^{tr} it must be an edge of some $S(A,X)$. The other direction is clear.

Lemma 2.1.15. *Let $P \subseteq 2^F$ be a polygonal complex with $A \subseteq F$. Then,*

(i) $St(X'_P(A), P_A^{\text{tr}})^c = St(A,P)^c$

(ii) $St(X'_P(A), P_A^{\text{tr}})^c \cap St(X'_P(A), P_A^{\text{tr}}) = St(A,P)^c \cap St(A,P)$

Proof. (i) Note that $\text{St}(A,P)^c = \{\, X \in P \mid X \cap A = \varnothing \,\}$ without need for completion. As a result,

$$\text{St}(X'_P(A), P_A^{\text{tr}})^c \;=\; \{\, X \in P_A^{\text{tr}} \mid X \cap X'_P(A) = \varnothing \,\}$$

$$
\begin{aligned}
&= \{\, X \in P_A^{\mathrm{tr}} \,|\, X \subseteq F \backslash A \,\} \\
&= \{\, X \in P \,|\, X \cap A = \varnothing \,\} \\
&= \mathrm{St}(A, P)^c
\end{aligned}
$$

(ii) If $X \in \mathrm{St}(X'_P(A), P_A^{\mathrm{tr}})^c \cap \mathrm{St}(X'_P(A), P_A^{\mathrm{tr}})$ then $X \cap X'_P(A) = \varnothing$ and X is a vertex or an edge of $S(A, X')$ for some $X' \in P$ such that $X' \cap A \neq \varnothing$. In particular $X \subseteq S(A, X')$. By definition of $S(A, X')$ necessarily $X \subseteq X' \backslash A$. This means X does not intersect A but is a vertex or edge of some $X' \in P$ which does. This means $X \in \mathrm{St}(x, P)^c \cap \mathrm{St}(x, P)$. This argument can be applied in the reverse direction, thus proving the second identity. □

Our goal is to define a map $f_A^{\mathrm{tr}} : |P_A^{\mathrm{tr}}| \to |P|$ which is a homeomorphism onto its image. Recall that we assume a choice of a collection of cyclic order preserving bijections $\alpha_X : X \to S_{\#X}$ for every $X \in P$ and $\alpha_Y^{\mathrm{tr}} : Y \to S_{\#Y}$ for every $Y \in P_A^{\mathrm{tr}}$. More accurately, we wish to define a map $f_A^{\mathrm{tr}} : |P_A^{\mathrm{tr}}|_{\alpha^{\mathrm{tr}}} \to |P|_{\alpha}$.

For brevity, let us denote,

$$
\begin{aligned}
Q &= \mathrm{St}(A, P) \\
Q_A^{\mathrm{tr}} &= \mathrm{St}(X'_P(A), P_A^{\mathrm{tr}})
\end{aligned}
$$

We wish to define f_A^{tr} via the commutative square,

$$
\begin{array}{ccc}
|P_A^{\mathrm{tr}}|_{\alpha^{\mathrm{tr}}} & \longrightarrow & |Q_A^{\mathrm{tr}}|_{\alpha^{\mathrm{tr}}} \coprod_{|Q_A^{\mathrm{tr}} \cap (Q_A^{\mathrm{tr}})^c|_{\alpha^{\mathrm{tr}}}} |(Q_A^{\mathrm{tr}})^c|_{\alpha^{\mathrm{tr}}} \\
{\scriptstyle f_A^{\mathrm{tr}}} \downarrow & & \downarrow \\
|P|_{\alpha} & \longleftarrow & |Q|_{\alpha} \coprod_{|Q \cap Q^c|_{\alpha}} |Q^c|_{\alpha}
\end{array}
$$

The horizontal arrows are given by the natural homeomorphisms introduced in Lemma 2.1.11. To complete the definition of f_A^{tr} we need to define the vertical arrow on the right hand side.

Using Lemma 2.1.15 and Remark 2.1.6, we have natural homeomorphisms as vertical arrows in a commutative square,

$$\begin{array}{ccc} |Q_A^{\mathrm{tr}} \cap (Q_A^{\mathrm{tr}})^c|_{\alpha^{\mathrm{tr}}} & \longrightarrow & |(Q_A^{\mathrm{tr}})^c|_{\alpha^{\mathrm{tr}}} \\ \downarrow & & \downarrow \\ |Q \cap Q^c|_\alpha & \longrightarrow & |Q^c|_\alpha \end{array}$$

To define f_A^{tr} what is left to do is define $|Q_A^{\mathrm{tr}}|_{\alpha^{\mathrm{tr}}} \to |Q|_\alpha$ which is a homeomorphism onto its image, and makes the following square commute,

$$\begin{array}{ccc} |Q_A^{\mathrm{tr}} \cap (Q_A^{\mathrm{tr}})^c|_{\alpha^{\mathrm{tr}}} & \longrightarrow & |Q_A^{\mathrm{tr}}|_{\alpha^{\mathrm{tr}}} \\ \downarrow & & \downarrow \\ |Q \cap Q^c|_\alpha & \longrightarrow & |Q|_\alpha \end{array}$$

To define $|Q_A^{\mathrm{tr}}|_{\alpha^{\mathrm{tr}}} \to |Q|_\alpha$, note,

$$\begin{aligned} Q_A^{\mathrm{tr}} &\overset{\mathrm{def}}{=} \overline{\{\, Y \in P_A^{\mathrm{tr}} \,|\, Y \cap X'_P(A) \neq \varnothing \,\}} \\ &= \overline{\{\, S(A,X) \in P_A^{\mathrm{tr}} \,|\, X \in P, X \cap A \neq \varnothing \,\}} \end{aligned}$$

Hence, it is enough to define $f^{\mathrm{tr}}(A,X) : \Pi_{\#S(A,X)} \to |Q|_\alpha$, which is a homeomorphism onto its image, for every $X \in P$ such that $X \cap A \neq \varnothing$. We must make sure that whenever $X' < X$, such that $X' \cap A \neq \varnothing$, the following triangle commutes,

$$\begin{array}{ccccc} \Pi_{\#S(A,X')} & & \longrightarrow & & \Pi_{\#S(A,X)} \\ & \searrow^{f^{\mathrm{tr}}(A,X')} & & \swarrow^{f^{\mathrm{tr}}(A,X)} & \\ & & |Q|_\alpha & & \end{array}$$

The horizontal arrow is given by the unique affine linear map which sends $\alpha^{\mathrm{tr}}_{S(A,X')}(s) \mapsto \alpha^{\mathrm{tr}}_{S(A,X)}(s)$ for every $s \in S(A,X') \subseteq S(A,X)$.

To define $f^{\mathrm{tr}}(A,X)$, for every $X \in P$ such that $X \cap A \neq \varnothing$, consider the unique affine linear map $\Pi_{\#S(A,X)} \to \Pi_{\#X}$ which sends,

$$\alpha^{\mathrm{tr}}_{S(A,X)}(s) \longmapsto \begin{cases} \alpha_X(s) & s \in S(A,X)\backslash X'_P(A) = X\backslash A \\ (\alpha_X(x) + \alpha_X(y))/2 & s = v(x,y) \in S(A,X) \cap X'_P(A) \end{cases}$$

Define $f^{\mathrm{tr}}(A,X)$ to be the composition $\Pi_{\#S(A,X)} \to \Pi_{\#X} \to |Q|_\alpha$.

The commutativity of triangles is easy to check since it only applies to edges of some face X of P which have one vertex in A and one in $X\backslash A$. It is important to note that f^{tr}_A can be extended to edges of Q^{tr}_A of the form $\{v(x,y), v(x',y')\}$ since by Remark 2.1.14 each is an edge of exactly one $S(A,X)$.

2.1.3 Blow Up of Vertices and Edges.

Let $P \subseteq 2^F$ be a polygonal complex flat at A such that $\#X_P(x) \geq 3$ for every vertex x of P in A. As always we assume no two distinct $X, X' \in P$ share more than one common edge.

Let P^{tr}_A be the truncation of P at A. The fact that P is flat at A implies that $X'_P(A) \subseteq F_A$ admits a cyclic order so that $\{v(x,y), v(x'y')\}$ is an edge of $X'_P(A)$ if it is an edge of P^{tr}_A. Let c denote a choice of a cyclic order on $X'_P(A)$ satisfying this condition.

Definition 2.1.16. Let P be a polygonal complex flat at A such that $\#X_P(x) \geq 3$ for every vertex x of P in A. Using the above notation, define the *blow up*

$P_A^{\mathrm{bl}}(c)$ of P at A, with respect to a choice of cyclic order c on $X'_P(A)$ described above, to be the minimal polygonal complex in 2^{F_A} containing: (i) P_A^{tr} (keeping the same cyclic order for every one of its elements), (ii) $X'_P(A)$ (with cyclic order c).

Remark 2.1.17. The blow up of P at A introduces a new face $X_A = X'_P(A)$. It also alters those faces of P that intersect A. When $A = \{x\}$ is a vertex of P, faces that intersect A change according to

$$(\dots, y, x, y', \dots) \longmapsto (\dots, y, v(x,y), v(x,y'), y', \dots)$$

thereby replacing two edges and introducing a new one. The overall number of edges is increased by 1. When $A = \{x, x'\}$ is an edge, faces that intersect A change according to

$$(\dots, y, x, y', \dots) \longmapsto (\dots, y, v(x,y), v(x,y'), y', \dots)$$
$$(\dots, y, x', y', \dots) \longmapsto (\dots, y, v(x',y), v(x',y'), y', \dots)$$
$$(\dots, y, x, x', y', \dots) \longmapsto (\dots, y, v(x,y), v(x',y'), y', \dots)$$

In the first couple of cases, two edges are replaced and a new edge is introduced. In the latter case three edges are replaced and there is no change to the overall number of edges. Note it is possible for x, x' to appear in a face of P without being an edge of that face. In that case the overall number of edges increases by two.

Our goal is to define $f_A^{\mathrm{bl}} : |P_A^{\mathrm{bl}}(c)| \to |P|$, when A is either a vertex or an edge of P, in such a way that the composition $|P_A^{\mathrm{tr}}| \to |P_A^{\mathrm{bl}}(c)| \to |P|$ coincides

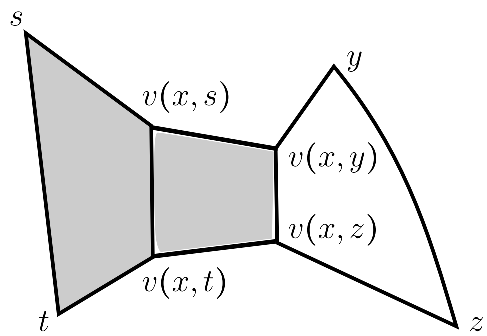


Figure 2.3: The geometric realization of the blow-up of the polygonal complex in Example 2.1.7 at the vertex x.

with f^{tr}_A. Note that $P^{\mathrm{bl}}_A(c)$ is not a subdivision of P in the sense introduced above. Hence the definition of f^{bl}_A will require a further refinement of $P^{\mathrm{bl}}_A(c)$.

Assume a choice of a collection of cyclic order preserving maps $\alpha^{\mathrm{bl}}_Y : Y \to S_{\#Y}$ for every $Y \in P^{\mathrm{bl}}_A(c)$. To be more specific, we wish to define f^{bl}_A such that the following triangle commutes,

$$\begin{array}{ccc} |P^{\mathrm{tr}}_A|_{\alpha^{\mathrm{tr}}} & \longrightarrow & |P^{\mathrm{bl}}_A(c)|_{\alpha^{\mathrm{bl}}} \\ & {\scriptstyle f^{\mathrm{tr}}_A}\searrow \quad \swarrow{\scriptstyle f^{\mathrm{bl}}_A} & \\ & |P|_\alpha & \end{array}$$

By definition, the truncation P^{tr}_A is a subcomplex of $P^{\mathrm{bl}}_A(c)$. Its complement $Q = (P^{\mathrm{tr}}_A)^c$ in $P^{\mathrm{bl}}_A(c)$ equals the completion of the single set $X'_P(A) \in 2^{F_A}$ to a polygonal subcomplex of $P^{\mathrm{bl}}_A(c)$. Applying Lemma 2.1.10 we get a natural homeomorphism,

$$|P^{\mathrm{bl}}_A(c)|_{\alpha^{\mathrm{bl}}} \cong |P^{\mathrm{tr}}_A|_{\alpha^{\mathrm{tr}}} \coprod_{|P^{\mathrm{tr}}_A \cap Q|_{\alpha^{\mathrm{tr}}}} |Q|_{\alpha^{\mathrm{bl}}}$$

Hence to define f^{bl}_A we need to define a map $|Q|_{\alpha^{\mathrm{bl}}} \to |P|_\alpha$ making the following square commute,

$$\begin{array}{ccc} |P_A^{\mathrm{tr}} \cap Q|_{\alpha^{\mathrm{tr}}} & \longrightarrow & |P_A^{\mathrm{tr}}|_{\alpha^{\mathrm{tr}}} \\ \downarrow & & \downarrow f_A^{\mathrm{tr}} \\ |Q|_{\alpha^{\mathrm{bl}}} & \longrightarrow & |P|_\alpha \end{array}$$

To define $|Q|_{\alpha^{\mathrm{bl}}} \to |P|_\alpha$ we consider case separately. When $A = \{x\}$ we use the barycentric subdivision of Q. As usual we need to pick a collection of order preserving bijections $\alpha'_Y : Y \to S_{\#Y}$ for every $Y \in \mathrm{sd}Q$.

We wish to define $|\mathrm{sd}Q|_{\alpha'} \to |P|_\alpha$. To do so, it is enough to define $\Pi_{\#Y} \to |P|_\alpha$ for $Y \in \mathrm{sd}Q$ of the form $Y = \{\{v(x,y)\}, \{v(x,y), v(x,y')\}, X'_P(x)\}$ where $\{v(x,y), v(x,y')\}$ is an edge of $X'_P(x)$ by the choice of cyclic order c.

If $\{v(x,y), v(x,y')\}$ is an edge of P_x^{tr}, then there exists a unique $X \in P$ such that $\{v(x,y), v(x,y')\}$ is an edge of $S(x,X)$. We then define $\Pi_{\#Y} \to |P|_\alpha$ to be the composition $\Pi_{\#Y} \to \Pi_{\#X} \to |P|_\alpha$, where $\Pi_{\#Y} \to \Pi_{\#X}$ is the unique affine linear map sending

$$\begin{aligned} \alpha'_Y(\{v(x,y)\}) &\longmapsto (\alpha_X(x) + \alpha_X(y))/2 \\ \alpha'_Y(\{v(x,y), v(x,y')\}) &\longmapsto \frac{1}{2}\left[(\alpha_X(x) + \alpha_X(y))/2 + (\alpha_X(x) + \alpha_X(y'))/2\right] \\ \alpha'_Y(X'_P(x)) &\longmapsto \alpha_X(x) \end{aligned}$$

If $\{v(x,y), v(x,y')\}$ is not an edge of P_x^{tr}, then we define the map $\Pi_{\#Y} \to |P|_\alpha$ to be the composition $\Pi_{\#Y} \to \Pi_{\#\{x,y\}} \to |P|_\alpha$, where $\Pi_{\#Y} \to \Pi_{\#\{x,y\}}$ is the unique affine linear map sending

$$\alpha'_Y(\{v(x,y)\}) \longmapsto (\alpha_X(x) + \alpha_X(y))/2$$

$$\alpha'_Y(\{v(x,y),v(x,y')\}) \longmapsto \alpha_X(x)$$
$$\alpha'_Y(X'_P(x)) \longmapsto \alpha_X(x)$$

We define $|Q|_{\alpha^{\text{bl}}} \to |P|_\alpha$ as the composition $|Q|_{\alpha^{\text{bl}}} \to |\text{sd}Q|_{\alpha'} \to |P|_\alpha$. This completes the definition of $f_x^{\text{bl}} : |P_x^{\text{bl}}(c)|_{\alpha^{\text{bl}}} \to |P|_\alpha$.

When $A = \{x,x'\}$ is an edge of P we use a different subdivision of Q. Let,

$$F' := \{x,x'\} \coprod X'_P(A)$$

Let $Q' \subseteq 2^{F'}$ be the minimal polygonal complex containing the following three types of faces:

(i) the cyclically ordered set $(v(x,y),v(x,y'),x)$ where $\{v(x,y),v(x,y')\}$ is an edge of $X'_P(A)$ and where there is $y'' \in X_P(A)\backslash\{y,y'\}$ such that $R_{X_P(A)}(y,y',y'')$.

(ii) same for x'.

(iii) the cyclically ordered set $(v(x,y),v(x',y'),x',x)$ where $\{v(x,y),v(x',y')\}$ is an edge of $X'_P(A)$ and where there is $y'' \in X_P(A)\backslash\{y,y'\}$ such that $R_{X_P(A)}(y,y',y'')$.

The resulting polygonal complex Q' is a subdivision of Q. We may view F' as a finite subset of $|Q|$ by choosing two appropriate points at the interior of $|Q| \equiv \Pi_{\#X'_P(A)}$.

As usual we pick a collection of order preserving bijections $\alpha'_Y : Y \to S_{\#Y}$ for every $Y \in Q'$. We wish to define $|Q'|_{\alpha'} \to |P|_\alpha$. To do so, it is enough to define $\Pi_{\#Y} \to |P|_\alpha$ for $Y \in Q'$ of the above three types in a way that agrees on their common edges.

Consider $Y = (v(x,y), v(x,y'), x)$ and assume there exists $X \in P$ such that $\{x,y,y'\} \subseteq X$ and $\{x,y\}$ and $\{x,y'\}$ are both edges of X. Consider $\Pi_{\#Y} \to \Pi_{\#X}$ the affine linear map sending: $\alpha'_Y(x) \mapsto \alpha_X(x)$, $\alpha'_Y((v(x,y)) \mapsto (\alpha_X(y) + \alpha_X(x))/2$ and same for y'. Define $\Pi_{\#Y} \to |P|_\alpha$ as the composition $\Pi_{\#Y} \to \Pi_{\#X} \to |P|_\alpha$.

If there is no such X then consider a map $\Pi_{\#Y} \to \Pi_{\#\{x,y\}} \coprod_{\Pi_{\#\{x\}}} \Pi_{\#\{x,y'\}}$ which restricts to the obvious,

$$\Pi_{\#\{x,v(x,y)\}} \coprod_{\Pi_{\#\{x\}}} \Pi_{\#\{x,v(x,y')\}} \longrightarrow \Pi_{\#\{x,y\}} \coprod_{\Pi_{\#\{x\}}} \Pi_{\#\{x,y'\}}$$

sending $\alpha'_Y((v(x,y))$ to $(\alpha_X(y) + \alpha_X(x))/2$ and $\alpha'_Y((v(x,y'))$ to $(\alpha_X(y') + \alpha_X(x))/2$. Define $\Pi_{\#Y} \to |P|_\alpha$ as the composition

$$\Pi_{\#Y} \to \Pi_{\#\{x,y\}} \coprod_{\Pi_{\#\{x\}}} \Pi_{\#\{x,y'\}} \to |P|_\alpha$$

For $Y = (v(x',y), v(x',y'), x')$ repeat the same definitions as in the previous case.

For $Y = (v(x,y), v(x',y'), x', x)$ assume there exists $X \in P$ such that $\{x,x',y,y'\} \subseteq X$ and $\{x,y\}$, $\{x,x'\}$ and $\{x,y'\}$ are edges of X. Consider $\Pi_{\#Y} \to \Pi_{\#X}$ the affine linear map sending: $\alpha'_Y(x) \mapsto \alpha_X(x)$ and the same for x', $\alpha'_Y((v(x,y)) \mapsto (\alpha_X(y) + \alpha_X(x))/2$ and same for x', y'.

If there is no such X then consider a map

$$\Pi_{\#Y} \to \Pi_{\#\{x,y\}} \coprod_{\Pi_{\#\{x\}}} \Pi_{\#\{x,x'\}} \coprod_{\Pi_{\#\{x'\}}} \Pi_{\#\{x',y'\}}$$

which restricts to the obvious,

$$\Pi_{\#\{x,v(x,y)\}} \coprod_{\Pi_{\#\{x\}}} \Pi_{\#\{x,x'\}} \coprod_{\Pi_{\#\{x'\}}} \Pi_{\#\{x',v(x',y')\}} \to \Pi_{\#\{x,y\}} \coprod_{\Pi_{\#\{x\}}} \Pi_{\#\{x,x'\}} \coprod_{\Pi_{\#\{x'\}}} \Pi_{\#\{x',y'\}}$$

sending $\alpha'_Y((v(x,y))$ to $(\alpha_X(y)+\alpha_X(x))/2$ and $\alpha'_Y((v(x',y'))$ to $(\alpha_X(y')+\alpha_X(x'))/2$. Define $\Pi_{\#Y} \to |P|_\alpha$ as the composition

$$\Pi_{\#Y} \to \Pi_{\#\{x,y\}} \coprod_{\Pi_{\#\{x\}}} \Pi_{\#\{x,x'\}} \coprod_{\Pi_{\#\{x'\}}} \Pi_{\#\{x',y'\}} \to |P|_\alpha$$

We define $|Q|_{\alpha^{\mathrm{bl}}} \to |P|_\alpha$ as the composition $|Q|_{\alpha^{\mathrm{bl}}} \to |Q'|_{\alpha'} \to |P|_\alpha$. This completes the definition of $f^{\mathrm{bl}}_{\{x,x'\}} : |P^{\mathrm{bl}}_{\{x,x'\}}(c)|_{\alpha^{\mathrm{bl}}} \to |P|_\alpha$.

Lemma 2.1.18. *Let P be a polygonal complex flat at A either a vertex or an edge such that $\#X_P(x) \geq 3$ for every $x \in A$. The map $f^{\mathrm{bl}}_A : |P^{\mathrm{bl}}_A(c)|_{\alpha^{\mathrm{bl}}} \to |P|_\alpha$ is a homeomorphism iff $|St(A,P)|$ is homeomorphic to a disc and the image of A is in its interior.*

Proof. Consider the case where $A = \{x\}$. Assume f^{bl}_x is a homeomorphism. By definition, this means every edge $\{y,y'\}$ of $X_P(x)$ is an edge of P^{tr}_x. This mean for every edge $\{y,y'\}$ of $X_P(x)$ there is $X \in P$ such that $\{x,y,y'\} \subseteq X$ and $\{x,y\}$, $\{x,y'\}$ are edges of X.

Let $\beta : X_P(x) \to S^1$ be a representative for c. For every edge $\{y,y'\}$ of $X_P(x)$ let $D_{y,y'}$ denote the sector of the unit disc D containing $\beta(y)$, $\beta(y')$ but no other element of $\mathrm{Im}(\beta)$. Since $\#X_P(x) \geq 3$ this is unambiguous.

Choose a collection of cyclic order preserving bijections $\alpha_X : X \to S_{\#X}$ for every $X \in \mathrm{St}(x,P)$. We wish to define a homeomorphism $\tilde{\beta} : |\mathrm{St}(x,P)|_\alpha \to D$.

Recall that $\mathrm{St}(x,P) = \overline{\{\, X \in P \,|\, x \in X \,\}}$. To define $\tilde{\beta}$ it is enough to define $\Pi_{\#X} \to D$, for every $X \in \mathrm{St}(x,P)$ such that $x \in X$. We need to make sure for every pair $X' < X$ the following triangle commutes,

$$\begin{array}{ccc} \Pi_{\#X'} & \longrightarrow & \Pi_{\#X} \\ & \searrow \quad \swarrow & \\ & D & \end{array}$$

If $X = \{x\}$ we take $\Pi_{\#X} \to D$ to be the map sending 1 to the origin. If $X = \{x,y\}$ we take $\Pi_{\#X} \to D$ to be the unique affine linear map taking $\alpha_X(x)$ to the origin and $\alpha_X(y)$ to $\beta(y)$.

If $\#X \geq 3$ then $\{x,y,y'\} \subseteq X$, where $\{x,y\}$ and $\{x,y'\}$ are edges of X. This means $\{v(x,y), v(x,y')\}$ is an edge of P_x^{tr}, and consequently an edge of $X_P(x)$. We take $\Pi_{\#X} \to D$ to be a homeomorphism from $\Pi_{\#X}$ to $D_{y,y'}$ which restricts to the appropriate affine linear maps on $\Pi_{\#\{x,y\}}$ and $\Pi_{\#\{x,y'\}}$.

By flatness of P at x the map $\tilde{\beta}$ is $1:1$. It is also onto by assumption. Because the spaces involved are compact and Hausdorff this is enough to guarantee $\tilde{\beta}$ is a homeomorphism. By definition of $\tilde{(}\beta)$, the image of x is in its interior.

Now assume $\tilde{\beta} : |\mathrm{St}(x,P)| \to D$ is some homeomorphism sending x to the interior of D. Note that the boundary of $|\mathrm{St}(x,P)|$ is $|\mathrm{St}(x,P) \cap \mathrm{St}(x,P)^c|$ which is mapped to S^1. The set $X_P(x)$ injects into $|\mathrm{St}(x,P) \cap \mathrm{St}(x,P)^c|$ which constitutes the geometric realization of all vertices and edges not containing x but contained in some $X \in P$ that does. This induces an injection $\beta : X_P(x) \to S^1$. Its class is necessarily c.

Let $\{y, y'\}$ be an edge of $X_P(x)$ with respect to c. Consider the image of $\Pi_{\#\{x,y\}} \coprod_{\Pi_{\#\{x\}}} \Pi_{\#\{x,y'\}} \hookrightarrow |\operatorname{St}(x,P)| \to D$ under $\tilde{\beta}$. It constitutes a non-self-intersecting path from $\beta(y) \in S^1$ to $\beta(y') \in S^1$ that intersects the interior of D. It therefore divides $\operatorname{Int}(D)$ into two connected components. One of them contains no elements of $\operatorname{Im}(\beta)$ in its boundary other than $\beta(y)$ and $\beta(y')$. Its pre-image must be contained in a single $\Pi_{\#X} \hookrightarrow |\operatorname{St}(x,P)|$. As a result there is a unique $X \in \operatorname{St}(x,P)$ such that $\{x,y,y'\} \subseteq X$ and $\{x,y\}$ and $\{x,y'\}$ are its edges. Thus every edge of $X'_P(x)$ is an edge of P_x^{tr}. This implies f_x^{bl} is a homeomorphism. □

2.1.4 Truncating and Blowing Up in Succession.

Lemma 2.1.19. *Let $P \subseteq 2^F$ be a polygonal complex with two* ***disjoint*** *$A, A' \subset F$. Then,*

(i) there is a natural identification $(F_A)_{A'} \leftrightarrow (F_{A'})_A$,

(ii) under the above identification, $(P_A^{\mathrm{tr}})_{A'}^{\mathrm{tr}} \leftrightarrow (P_{A'}^{\mathrm{tr}})_A^{\mathrm{tr}}$,

(iii) if P flat at A, A' which are either vertices or edges, then, under the above identification, $(P_A^{\mathrm{bl}})_{A'}^{\mathrm{bl}} \leftrightarrow (P_{A'}^{\mathrm{bl}})_A^{\mathrm{bl}}$,

(iv) the following diagrams commute,

$$\begin{array}{ccc} |(P_{A'}^{\mathrm{tr}})_A^{\mathrm{tr}}| & \longrightarrow & |P_{A'}^{\mathrm{tr}}| \\ \updownarrow & & \downarrow \\ |(P_A^{\mathrm{tr}})_{A'}^{\mathrm{tr}}| & & \\ \downarrow & & \downarrow \\ |P_A^{\mathrm{tr}}| & \longrightarrow & |P| \end{array} \qquad \begin{array}{ccc} |(P_{A'}^{\mathrm{bl}})_A^{\mathrm{bl}}| & \longrightarrow & |P_{A'}^{\mathrm{bl}}| \\ \updownarrow & & \downarrow \\ |(P_A^{\mathrm{bl}})_{A'}^{\mathrm{bl}}| & & \\ \downarrow & & \downarrow \\ |P_A^{\mathrm{bl}}| & \longrightarrow & |P| \end{array}$$

Lemma 2.1.19 guarantees that we can truncate and blow up a polygonal complex at disjoint subsets in succession and what we end up getting does not depend on the order at which we choose truncate or blow-up.

Proof. (i) By definition, $F_x = F\backslash x \coprod \{\, v(x,y) \,|\, y \in X_P(x) \,\}$. Hence,

$$(F_x)_{x'} = F\backslash\{x,x'\} \coprod \{\, v(x,y) \,|\, y \in X_P(x) \,\} \coprod \{\, v(x',y) \,|\, y \in X_{P_x^{\mathrm{tr}}}(x') \,\}$$

We have,

$$X_{P_x^{\mathrm{tr}}}(x') = \begin{cases} X_P(x') & x' \notin X_P(x) \\ X_P(x')\backslash x \coprod \{v(x,x')\} & x' \in X_P(x) \end{cases}$$

Therefore, if $x' \notin X_P(x)$, equivalently $x \notin X_P(x')$, we can clearly identify $(F_x)_{x'}$ and $(F_{x'})_x$. Otherwise,

$$\begin{aligned} (F_x)_{x'}\backslash(F\backslash\{x,x'\}) \;\; &= \;\; \{\, v(x,y) \,|\, y \in X_P(x)\backslash x' \,\} \coprod \\ & \{\, v(x',y) \,|\, y \in X_P(x')\backslash x \,\} \coprod \{v(x,x'), v(x', v(x,x'))\} \end{aligned}$$

And in this case the identification will take $v(x,x') \mapsto v(x, v(x',x))$ and $v(x', v(x,x')) \mapsto v(x',x)$. The rest is clear.

(ii) It is clear from the above considerations that the only case to check is when $x' \in X_P(x)$, or equivalently $x \in X_P(x')$.

As in Lemma 2.1.15, we have

$$\begin{aligned} \mathrm{St}(\{x,x'\},P)^c \;\; &= \;\; \mathrm{St}(X'_P(x) \sqcup X'_{P_x^{\mathrm{tr}}}(x'), (P_x^{\mathrm{tr}})_{x'}^{\mathrm{tr}})^c \\ &= \;\; \mathrm{St}(X'_P(x') \sqcup X'_{P_{x'}^{\mathrm{tr}}}(x), (P_{x'}^{\mathrm{tr}})_{x}^{\mathrm{tr}})^c \end{aligned}$$

$$\begin{aligned}\mathrm{St}(\{x,x'\},P)^c \cap \mathrm{St}(\{x,x'\},P) &= \\ &= \mathrm{St}(X'_P(x) \sqcup X'_{P^{\mathrm{tr}}_x}(x'), (P^{\mathrm{tr}}_x)^{\mathrm{tr}}_{x'})^c \cap \mathrm{St}(X'_P(x) \sqcup X'_{P^{\mathrm{tr}}_x}(x'), (P^{\mathrm{tr}}_x)^{\mathrm{tr}}_{x'}) \\ &= \mathrm{St}(X'_P(x') \sqcup X'_{P^{\mathrm{tr}}_{x'}}(x), (P^{\mathrm{tr}}_{x'})^{\mathrm{tr}}_x)^c \cap \mathrm{St}(X'_P(x') \sqcup X'_{P^{\mathrm{tr}}_{x'}}(x), (P^{\mathrm{tr}}_{x'})^{\mathrm{tr}}_x)\end{aligned}$$

Let us denote, $Q := \mathrm{St}(X'_P(x) \sqcup X'_{P^{\mathrm{tr}}_x}(x'), (P^{\mathrm{tr}}_x)^{\mathrm{tr}}_{x'})$ and $Q' := \mathrm{St}(X'_P(x') \sqcup X'_{P^{\mathrm{tr}}_{x'}}(x), (P^{\mathrm{tr}}_{x'})^{\mathrm{tr}}_x)$. We only need to show that, under the above identification $(F_{x'})_x \to (F_{x'})_x$, the subset $Q \subseteq 2^{(F_x)_{x'}}$ is mapped to the subset $Q' \subseteq 2^{(F_{x'})_x}$ and the mapping on each element $Z \in Q \to Z' \in Q'$ preserves the cyclic order. It is enough to show all maximal elements in Q are mapped to all maximal elements in Q' (recall that maximality is considered with respect to the partial order defined on a polygonal complex). Maximal elements in Q fall into exactly one of three categories:

(i) $S(x,X)$, $X \in P$, $x \in X$, $x' \notin X$.

(ii) $S(x',X)$, $X \in P$, $x' \in X$, $x \notin X$.

(iii) $S(x',S(x,X))$, $X \in P$, $x,x' \in X$.

And the same for Q'. The identification $(F_{x'})_x \to (F_{x'})_x$ switches between elements in the above two categories and sends each element in the third category to itself.

(iii) Given the work we did in (ii) this part becomes easier. In this case we take Q to be the minimal polygonal subcomplex of $(P^{\mathrm{tr}}_x)^{\mathrm{tr}}_{x'}$ containing $X'_P(x)$ and $X'_{P^{\mathrm{tr}}_x}(x')$, and Q' to be the minimal polygonal subcomplex of $(P^{\mathrm{tr}}_{x'})^{\mathrm{tr}}_x$ containing $X'_P(x')$ and $X'_{P^{\mathrm{tr}}_{x'}}(x)$. The map $(F_{x'})_x \to (F_{x'})_x$ sends the maximal element $X'_P(x) \in Q$ to the maximal element $X'_{P^{\mathrm{tr}}_{x'}}(x) \in Q'$, and the maximal element

$X'_{P_x^{\rm tr}}(x') \in Q$ to the maximal element $X'_P(x') \in Q'$. There is only one caveat, which is the cyclic order. When the cyclic order is unique, as in the case of polygonal decompositions, it is preserved. When the cyclic order is not unique we need the choices of cyclic orders to match. □

2.2 Three Step Blow-up.

All polygonal complexes P introduced in this section are assumed to satisfy the following two conditions:

(i) No two disjoint $X, X' \in P$ share more than one common edge.

(ii) For each vertex x of P we have $X_P(x) \geq 3$.

Let P be a polygonal complex. Recall that we denote the vertices, edges and faces of P by the letters v, e and f respectively. We reserve the letter X for a general element of P. We denote,

$$\deg_P(v) := \#X_P(x)$$

$$\deg_P(f) := \#f$$

Definition 2.2.1. A *polygonal decomposition* of a closed surface W is the data of a polygonal complex P together with a homeomorphism $|P| \to W$.

In particular, a triangulation of a surface is a special case of a polygonal decomposition. Let $|P| \to W$ be a polygonal decomposition of a closed surface W. Consequently, P is flat at $A = v$ or e, moreover, $|\operatorname{St}(A, P)|$ is homeomorphic to a disc with A in its interior.

We can blow up each vertex of P following some order chosen on its vertices. We get,

$$|(\cdots((P^{\rm bl}_{v_1})^{\rm bl}_{v_2})\cdots)^{\rm bl}_{v_n}| \to \ldots \to |(P^{\rm bl}_{v_1})^{\rm bl}_{v_2}| \to |P^{\rm bl}_{v_1}| \to |P| \to W$$

Each map in the composition is a homeomorphism by Lemma 2.1.18. By Lemma 2.1.19, the resulting polygonal decomposition is independent of the chosen order and we denote,

$$P^{\rm bl}_{\rm vert} \equiv |(\cdots((P^{\rm bl}_{v_1})^{\rm bl}_{v_2})\cdots)^{\rm bl}_{v_n}| \to W$$

Lemma 2.2.2. *Let $|P| \to W$ be a polygonal decomposition of a closed surface W. Then $P^{\rm bl}_{\rm vert}$ has two types of faces: X_v, for all $v \in P$, and X_f, for all $f \in P$, moreover,*

$$\deg_{P^{\rm bl}_{\rm vert}}(X_v) = \deg_P(v)\ ,\ \deg_{P^{\rm bl}_{\rm vert}}(X_f) = 2\deg_P(f)$$

Proof. By Remark 2.1.17, $P^{\rm bl}_{\rm vert}$ admits faces corresponding to those of P, namely X_f, and an additional face for each vertex blown, namely, X_v. Choosing an order on the vertices of P we can write down the faces of $P^{\rm bl}_{\rm vert}$ explicitly. For every vertex $v \in P$, $v = \{x\}$,

$$X_v = \{\, v(x,y) \,|\, y \in X_P(x)\,,\ x < y \,\} \coprod \{\, v(x, v(y,x)) \,|\, y \in X_P(x)\,,\ x > y \,\}$$

In particular, $\deg_{P^{\rm bl}_{\rm vert}}(X_v) = \deg_P(v)$. For every face $f \in P$,

$$X_f = \coprod_{\substack{e=\{y,y'\} \text{ is an edge} \\ \text{of } f \text{ and } y<y'}} \{v(y,y'), v(y', v(y,y'))\}$$

In particular, $\deg_{P^{\rm bl}_{\rm vert}}(X_f) = 2\deg_P(f)$. ◻

In the second step, we consider edges of $P^{\text{bl}}_{\text{vert}}$ of the form $X_f \cap X_{f'}$, for $f, f' \in P$ such that $f \cap f' = e$. Note that these edges are in correspondence with edges of P. Pick some order on the edges of P. Blow up each edge of $P^{\text{bl}}_{\text{vert}}$ of the above type following that order. We have,

$$|(\cdots(((P^{\text{bl}}_{\text{vert}})^{\text{bl}}_{e_1})^{\text{bl}}_{e_2})\cdots)^{\text{bl}}_{e_{n'}}| \to \ldots \to |((P^{\text{bl}}_{\text{vert}})^{\text{bl}}_{e_1})^{\text{bl}}_{e_2}| \to |(P^{\text{bl}}_{\text{vert}})^{\text{bl}}_{e_1}| \to |P^{\text{bl}}_{\text{vert}}| \to W$$

Each map in the composition is a homeomorphism as before. By Lemma 2.1.19, the resulting polygonal decomposition is independent of the chosen order on the edges and we denote,

$$P^{\text{bl}}_{\text{edge}} \equiv |(\cdots(((P^{\text{bl}}_{\text{vert}})^{\text{bl}}_{e_1})^{\text{bl}}_{e_2})\cdots)^{\text{bl}}_{e_{n'}}| \to W$$

To be more accurate, e_i is not an edge of $P^{\text{bl}}_{\text{vert}}$ but stands for the edge $X_f \cap X_{f'}$ where $f, f' \in P$ and $f \cap f' = e_i$.

Lemma 2.2.3. *Let $|P| \to W$ be a polygonal decomposition of a closed surface W. Then $P^{\text{bl}}_{\text{edge}}$ has three types of faces: Y_v, for every $v \in P$, Y_f, for every $f \in P$, and Y_e, for every $e \in P$, moreover,*

$$\deg_{P^{\text{bl}}_{\text{edge}}}(Y_v) = 2\deg_P(v)\ ,\ \deg_{P^{\text{bl}}_{\text{edge}}}(Y_f) = 2\deg_P(f)\ ,\ \deg_{P^{\text{bl}}_{\text{edge}}}(Y_e) = 4$$

Proof. By Remark 2.1.17, $P^{\text{bl}}_{\text{edge}}$ has faces corresponding to those of $P^{\text{bl}}_{\text{vert}}$, namely Y_v and Y_f, and an additional face for each edge blown, namely, Y_e. We will continue to be sloppy and let $e \in P$ also stand for the edge $X_f \cap X_{f'} \in P^{\text{bl}}_{\text{vert}}$ where $f \cap f' = e$. We choose an order both on the vertices and the edges of P.

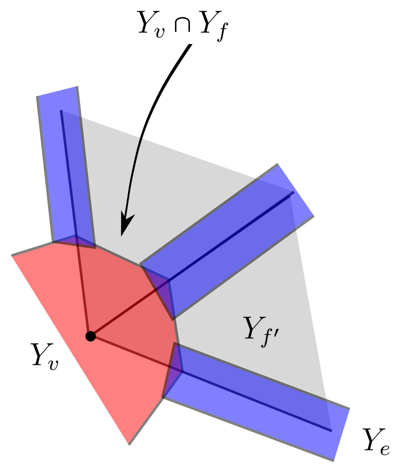


Figure 2.4: Faces and Edges of $P^{\mathrm{bl}}_{\mathrm{edge}}$.

Consider a configuration $v \subset e < e' \subset f$. Let $v = \{x\}$ and $e' = \{x, x'\}$. The face $Y_v \in P^{\mathrm{bl}}_{\mathrm{edge}}$ admits two vertices

$$\{v(e, v(x, x')), v(e', v(e, v(x, x')))\} \qquad x < x'$$
$$\{v(e, v(x, v(x', x))), v(e', v(e, v(x, v(x', x))))\} \qquad x > x'$$

For a fixed vertex v, the number of such configurations equals $\deg_P(v)$. As a result, $\deg_{P^{\mathrm{bl}}_{\mathrm{edge}}}(Y_v) = 2 \deg_P(v)$.

Blowing up the edge $X_f \cap X_{f'}$ effects those faces of $P^{\mathrm{bl}}_{\mathrm{vert}}$ adjacent to it, namely, X_f and $X_{f'}$, without changing their overall number of edges, meaning $\deg_{P^{\mathrm{bl}}_{\mathrm{edge}}}(Y_f) = \deg_{P^{\mathrm{bl}}_{\mathrm{vert}}}(X_f)$.

Fix an edge $e \in P$ and consider a configuration $x \in e \neq e' \subset f$. Let $e = \{x, y\}$ and $e' = \{x, y'\}$. The face Y_e admits a vertex

$$v(e, v(x, y')) \quad e < e', x < y' \qquad v(e, v(x, v(y', x))) \quad e < e', x > y'$$
$$v(e, v(e', v(x, y))) \quad e > e', x < y \qquad v(e, v(e', v(x, v(y, x)))) \quad e > e', x > y$$

There are 4 such configurations for a fixed e, since it is adjacent to two faces. Hence $\deg_{P^{\text{bl}}_{\text{edge}}}(Y_e) = 4$. □

In the third and last step, we consider edges of $P^{\text{bl}}_{\text{edge}}$ of the form $Y_f \cap Y_v$, for $f, v \in P$ such that $v \subset f$. Pick some order on the pairs v, f of P such that $v \subset f$. Blow up each edge of $P^{\text{bl}}_{\text{edge}}$ of the above type following that order. We have,

$$|(\cdots(((P^{\text{bl}}_{\text{edge}})^{\text{bl}}_{v_1,f_1})^{\text{bl}}_{v_2,f_2})\cdots)^{\text{bl}}_{v_{n''},f_{n''}}| \to \ldots \to |(P^{\text{bl}}_{\text{edge}})^{\text{bl}}_{v_1,f_1}| \to |P^{\text{bl}}_{\text{edge}}| \to W$$

Each map in the composition is a homeomorphism as before. By Lemma 2.1.19, the resulting polygonal decomposition is independent of the chosen order on the pairs and we denote,

$$P^{\text{bl}} \equiv |(\cdots(((P^{\text{bl}}_{\text{edge}})^{\text{bl}}_{v_1,f_1})^{\text{bl}}_{v_2,f_2})\cdots)^{\text{bl}}_{v_{n''},f_{n''}}| \to W$$

Lemma 2.2.4. *Let $|P| \to W$ be a polygonal decomposition of a closed surface W. Then P^{bl} has four types of faces: Z_v, for every $v \in P$, Z_f, for every $f \in P$, Z_e, for every $e \in P$ and $Z_{v,f}$ for every $v, f \in P$ such that $v \subset f$, moreover,*

$$\deg_{P^{\text{bl}}}(Z_v) = 2\deg_P(v)\ ,\ \deg_{P^{\text{bl}}}(Z_f) = 2\deg_P(f)$$

$$\deg_{P^{\text{bl}}}(Z_e) = 8\ ,\ \deg_{P^{\text{bl}}}(Z_{v,f}) = 4$$

Proof. By Remark 2.1.17, P^{bl} has faces corresponding to those of $P^{\text{bl}}_{\text{edge}}$, namely, Z_v, Z_f and Z_e, and an additional face for each edge blown, namely, $Z_{v,f}$. We

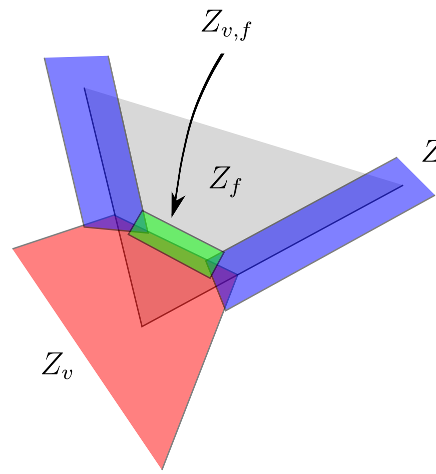


Figure 2.5: Faces of P^{bl}.

will continue to be sloppy and let $e \in P$ also stand for the edge $X_f \cap X_{f'} \in P^{\text{bl}}_{\text{vert}}$ where $f \cap f' = e$. We choose an order both on the vertices and the edges of P.

A face $Y_e \in P^{\text{bl}}_{\text{edge}}$ intersects an edge $Y_f \cap Y_v$, such that $v \subset e \subset f$, at a single vertex. By Remark 2.1.17, the blow of $Y_f \cap Y_v$ effects Y_e by introducing an additional edge. For a fixed edge e, there are 4 possible configurations $v \subset e \subset f$. As a result $\deg_{P^{\text{bl}}}(Z_e) = 8$.

Blowing up the edge $Y_f \cap Y_v$ effects those faces of $P^{\text{bl}}_{\text{edge}}$ adjacent to it, namely, Y_v and Y_f, without changing their overall number of edges, meaning, $\deg_{P^{\text{bl}}}(Z_{v/f}) = \deg_{P^{\text{bl}}_{\text{edge}}}(Z_{v/f})$. □

Remark 2.2.5. The procedure described above can be generalized to surfaces with boundary. This will require adapting the definition of $f^{\text{bl}}_{v/e}$ so that it becomes a homeomorphism. In its current definition, it is not (see Lemma 2.1.18).

2.3 Factorization of a Closed Surface in Bord_2.

Loosely speaking, the bordism category Bord_2 is a $(\infty, 2)$-category consisting of:

Objects: 0-manifolds,

1-Morphisms: bordisms of 0-manifolds,

2-Morphisms: bordisms of bordisms of 0-manifolds,

3-Morphisms: diffeomorphisms of 2-morphisms,

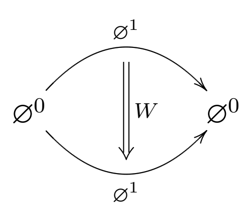
4-Morphisms: isotopies of diffeomorphisms of 2-morphisms,

$\vdots$

Composition of morphisms in Bord_2 is given by gluing along the boundary. Taking disjoint unions makes Bord_2 into a symmetric monoidal $(\infty, 2)$-category. A model for Bord_2 as a complete n-fold Segal space appears in [10]. A model for its $(2, 2)$-category truncation, Cob_2, appears in [13].

The empty set is either an object, the identity 1-morphism of the empty object, or the identity 2-morphism of the identity 1-morphism of the empty object. We designate these various interpretations by $\varnothing^0$, $\varnothing^1$ and $\varnothing^2$ respectively. Considered as an object, $\varnothing^0$ is the unit object for the tensor product.

A closed surface W is regarded as a 2-morphism in Bord_2 in the following way



Assume W is endowed with a polygonal decomposition $|P| \to W$. The three-step blow up procedure, described in Section 2.2, gives rise to a new polygonal decomposition $|P^{\mathrm{bl}}| \to W$ of W.

By abuse of notation, we denote the image of $\Pi_{\#Z}$ in $|P^{\mathrm{bl}}|$ by Z. Each edge of P^{bl}, or rather, the image of its geometric realization in $|P^{\mathrm{bl}}|$, is a 1-manifold with boundary. We consider it a 1-morphism in Bord_2 in the following way:

$$\begin{aligned} Z_v \cap Z_e &: \varnothing^0 \to \mathrm{pt} \sqcup \mathrm{pt} && v, e \in P,\ v \subset e \\ Z_f \cap Z_e &: \varnothing^0 \to \mathrm{pt} \sqcup \mathrm{pt} && e, f \in P,\ e \subset f \\ Z_v \cap Z_{v,f} &: \mathrm{pt} \sqcup \mathrm{pt} \to \varnothing^0 && v, f \in P,\ v \subset f \\ Z_f \cap Z_{v,f} &: \mathrm{pt} \sqcup \mathrm{pt} \to \varnothing^0 && v, f \in P,\ v \subset f \end{aligned}$$

Each face of P^{bl}, or rather, the image of its geometric realization in $|P^{\mathrm{bl}}|$, is a 2-manifold with boundary homeomorphic to S^1. We consider it a 2-morphism in Bord_2 in the following way:

$$\begin{aligned} &Z_v : \varnothing^1 \to (\sqcup_{e \in P,\, v \subset e} Z_v \cap Z_e) \sqcup (\sqcup_{f \in P,\, v \subset f} Z_v \cap Z_{v,f}) \\ &Z_f : (\sqcup_{e \in P,\, e \subset f} Z_f \cap Z_e) \sqcup (\sqcup_{v \in P,\, v \subset f} Z_f \cap Z_{v,f}) \to \varnothing^1 \\ &Z_{v,f} : Z_v \cap Z_{v,f} \to Z_v \cap Z_{v,f} \\ &Z_e : \sqcup_{v \in P,\, v \subset e} Z_e \cap Z_v \to \sqcup_{f \in P,\, e \subset f} Z_e \cap Z_f \end{aligned}$$

This interpretation culminates in a factorization of W into 2-morphisms in Bord_2 up to homeomorphism.

Theorem 2.3.1. *Let P be a polygonal decomposition of a closed surface W. Let v, e and f stand for the vertices, edges and faces of P respectively. Considered as a 2-morphism, $W \in \mathrm{Bord}_2(\varnothing^1, \varnothing^1)$ can be factored,*

$$\begin{array}{ccccc}
\varnothing^0 & & \xrightarrow{\quad\varnothing^1\quad} & & \varnothing^0 \\
& & \Uparrow \coprod_f Z_f & & \\
\varnothing^0 & \xrightarrow{\coprod_{f,e} Z_f \cap Z_e} & \coprod_{e,f}(\mathrm{pt} \sqcup \mathrm{pt}) = \coprod_{v,f}(\mathrm{pt} \sqcup \mathrm{pt}) & \xrightarrow{\coprod_{f,v} Z_f \cap Z_{v,f}} & \varnothing^0 \\
& \Uparrow \coprod_e Z_e & & \coprod_{v,f} Z_{v,f} \Uparrow & \\
\varnothing^0 & \xrightarrow[\coprod_{v,e} Z_v \cap Z_e]{} & \coprod_{v,e}(\mathrm{pt} \sqcup \mathrm{pt}) = \coprod_{v,f}(\mathrm{pt} \sqcup \mathrm{pt}) & \xrightarrow[\coprod_{v,f} Z_v \cap Z_{v,f}]{} & \varnothing^0 \\
& & \Uparrow \coprod_v Z_v & & \\
\varnothing^0 & & \xrightarrow[\quad\varnothing^1\quad]{} & & \varnothing^0
\end{array}$$

Chapter 3

Extended Topological Field Theories

3.1 The Cobordism Hypothesis.

We repeat essential definitions and results from [10]. Let Γ denote a topological group and fix a continuous homomorphism $\chi : \Gamma \to O(n)$. Let $\zeta_\chi = \mathbb{R}^n \times_\Gamma E\Gamma$ denote the associated rank n real vector bundle over the classifying space $B\Gamma$ determined by χ.

Definition 3.1.1. A Γ-structure on a manifold M of dimension $k \leq n$ consists of a continuous map $f : M \to B\Gamma$ and an isomorphism of vector bundles $TM^k \oplus \underline{\mathbb{R}}^{n-k} \xrightarrow{\simeq} f^*\zeta_\chi$.

Example 3.1.2.

(i) Γ is trivial, then a Γ-structure on a k-manifold is an n-framing.

(ii) $\Gamma = SO(n)$, χ is the embedding, then a Γ-structure is an orientation.

(iii) $\Gamma = G \times O(n)$, G a finite group, χ is trivial, then a Γ-structure is a principal G-bundle.

Let Bord_n denote the bordism (∞, n)-category. A model of Bord_n as a complete n-fold Segal space appears in [10]. Let Bord_n^Γ denote the bordism (∞, n)-category where all manifolds are equipped with Γ-structure. Another model for the bordism $(2,2)$-category appears in [13]. The latter suffices for the example that we will investigate in subsequent sections.

The data of a monoidal functor $F : (\mathcal{C}, \otimes) \to (\mathcal{C}', \otimes')$ between two monoidal categories includes morphisms $F(X) \otimes' F(Y) \to F(X \otimes Y)$ for every pair of objects $X, Y \in \mathcal{C}$ and a morphism $\mathbf{1}' \to F(\mathbf{1})$. When the monoidal functor F is *strong* these morphisms are assumed to be isomorphisms. Throughout this chapter all monoidal functors are assumed strong.

Theorem 3.1.3 (Cobordism Hypothesis for Manifolds with Γ-structure)**.** *Let $\mathcal{C}$ be a symmetric monoidal (∞, n)-category. Then there is a canonical equivalence of (∞, n)-categories,*

$$\mathrm{Fun}^{\otimes}(\mathrm{Bord}_n^{\Gamma}, \mathcal{C}) \xrightarrow{\simeq} ((\mathcal{C}^{\mathrm{fd}})^{\sim})^{h\Gamma}$$

In particular, the left hand side is a ∞-groupoid.

On the left hand side, $\mathrm{Fun}^{\otimes}(-,-)$ stands for the (∞, n)–category of symmetric monoidal functors between two symmetric monoidal (∞, n)– categories. On the right hand side, $\mathcal{C}^{\mathrm{fd}}$ denotes the (not necessarily full) subcategory of fully dualizable objects in $\mathcal{C}$ (see Definition B.0.11), and $(\mathcal{C}^{\mathrm{fd}})^{\sim}$ denotes its underlying ∞-groupoid gotten by discarding of non-invertible morphisms.

When Γ is trivial we get the Cobordism Hypothesis for framed manifolds. The space of n-dimensional fully extended framed TFT's with values in $\mathcal{C}$ is canonically equivalent to $(\mathcal{C}^{\mathrm{fd}})^{\sim}$. Since $O(n)$ acts on the space of n-framings of a given manifold, the space of framed theories carries a $O(n)$–action. As a consequence of the Cobordism Hypothesis for framed manifolds, the space $(\mathcal{C}^{\mathrm{fd}})^{\sim}$ carries a homotopy $O(n)$–action. The group Γ acts on $(\mathcal{C}^{\mathrm{fd}})^{\sim}$ via χ. This allows us to define the space of homotopy Γ-fixed points $((\mathcal{C}^{\mathrm{fd}})^{\sim})^{h\Gamma}$.

Let G denote a finite group. Let $\Gamma = G \times SO(2)$ and take $\chi : \Gamma \to O(2)$ to be trivial on G and the embedding on $SO(2)$. Let $\mathcal{C}$ denote a symmetric monoidal 2-category. By the Cobordism Hypothesis for manifolds with Γ-structure

$$\mathrm{Fun}^{\otimes}(\mathrm{Bord}_2^{\Gamma}, \mathcal{C}) \simeq ((\mathcal{C}^{\mathrm{fd}})^{\sim})^{h\Gamma}$$

By definition

$$\begin{aligned}
((\mathcal{C}^{\mathrm{fd}})^{\sim})^{h\Gamma} &= \mathrm{Hom}_{\Gamma}(E\Gamma, (\mathcal{C}^{\mathrm{fd}})^{\sim}) \\
&= \mathrm{Hom}_{G\times SO(2)}(EG \times ESO(2), (\mathcal{C}^{\mathrm{fd}})^{\sim}) \\
&= \mathrm{Hom}_{SO(2)}(BG \times ESO(2), (\mathcal{C}^{\mathrm{fd}})^{\sim}) \\
&= \mathrm{Hom}(BG, \mathrm{Hom}_{SO(2)}(ESO(2), (\mathcal{C}^{\mathrm{fd}})^{\sim}))
\end{aligned}$$

where Hom is understood in the category of ∞-groupoids. If the homotopy $SO(2)$-action on $(\mathcal{C}^{\mathrm{fd}})^{\sim}$ is trivializable then

$$\mathrm{Hom}_{SO(2)}(ESO(2), (\mathcal{C}^{\mathrm{fd}})^{\sim}) \simeq \mathrm{Hom}(BSO(2), (\mathcal{C}^{\mathrm{fd}})^{\sim})$$

We conclude that, under this hypothesis,

$$\mathrm{Fun}^{\otimes}(\mathrm{Bord}_2^{\Gamma}, \mathcal{C}) \simeq \mathrm{Hom}(BG, \mathrm{Hom}(BSO(2), (\mathcal{C}^{\mathrm{fd}})^{\sim}))$$

By the Homotopy Hypothesis, the 2-groupoid $(\mathcal{C}^{\mathrm{fd}})^{\sim}$ determines a 2-type X up to homotopy equivalence. The space of fully extended 2-dimensional Γ-

structured TFT's with values in $\mathcal{C}$ is then homotopy equivalent to:

$$\mathrm{Fun}^{\otimes}(\mathrm{Bord}_2^{\Gamma}, \mathcal{C}) \simeq \mathrm{Map}(BG, \mathrm{Map}(BSO(2), X))$$

where Map is understood in the category of topological spaces. Note that the space of fully extended 2-dimensional oriented TFT's with values in $\mathcal{C}$ is homotopy equivalent to

$$\mathrm{Fun}^{\otimes}(\mathrm{Bord}_2^{SO(2)}, \mathcal{C}) \simeq \mathrm{Map}(BSO(2), X)$$

3.2 The Category Alg_k.

In this section, all algebras are assumed associative, unital and defined over an algebraically closed field k of characteristic zero. Let Alg_k denote the 2-category with

$$\begin{aligned} \mathrm{Obj}(\mathrm{Alg}_k) &= \text{algebras over } k \\ 1-\mathrm{Mor}(\mathrm{Alg}_k) &= \text{bimodules} \\ 2-\mathrm{Mor}(\mathrm{Alg}_k) &= \text{intertwiners} \end{aligned}$$

We make the convention that a $(A,B)-$bimodule is an object in the category of 1-morphisms $\mathrm{Alg}_k(A,B)$ having A as a source and B as a target. Alg_k is a monoidal category with $\otimes$ given by tensoring over the ground field k. The tensor functor $\otimes$ should be distinguished from composition "$N \circ M$" $= M \otimes_B N$ which is given by tensoring over B, the target of M, and the source of N.

3.2.1 The Category $\mathrm{Alg}_k^{\mathrm{fd}}$.

Lemma 3.2.1. *The fully dualizable objects of* Alg_k *are the semi-simple finite-dimensional algebras.*

Proof. An object in a symmetric monoidal 2-category is fully dualizable if it admits a dual and the evaluation admits right and left adjoints (see Definition B.0.10). Every algebra $A \in \mathrm{Alg}_k$ admits its opposite algebra A^o as a dual. The opposite algebra A^o shares the same underlying vector space but multiplication is given in reverse order. As evaluation E we may take A considered as a $(A \otimes A^o, k)$–bimodule, and as co-evaluation C we may take A considered as a $(k, A^o \otimes A)$–bimodule. The isomorphisms of (A, A)–bimodules

$$\begin{array}{rcl} (A \otimes C) \otimes_{A \otimes A^o \otimes A} (E \otimes A) & \longrightarrow & A \\ (a_1 \otimes a_2) \otimes (a_3 \otimes a_4) & \longmapsto & a_1 a_3 a_2 a_4 \\ (C \otimes A^o) \otimes_{A^o \otimes A \otimes A^o} (A^o \otimes E) & \longrightarrow & A^o \\ (a_1 \otimes a_2) \otimes (a_3 \otimes a_4) & \longmapsto & a_3 a_1 a_4 a_2 \end{array}$$

imply that duality axioms are satisfied in the homotopy category $h\,\mathrm{Alg}_k$.

Let $A^e := A \otimes A^o$ denote the enveloping algebra of A, and let $\mu : A^e \to A$ denote the multiplication which is a left A^e-module morphism. By Definition, E admits a left adjoiont if there exist a (k, A^e)–bimodule E^L and unit and counit intertwiners

$$u^L : k \longrightarrow E^L \otimes_{A^e} E$$

$$v^L : E \otimes_k E^L \longrightarrow A^e$$

such that the compositions

$$E \cong E \otimes_k k \xrightarrow{\mathrm{id} \otimes u^L} E \otimes_k E^L \otimes_{A^e} E \xrightarrow{v^L \otimes \mathrm{id}} A^e \otimes_{A^e} E \cong E \tag{3.2.1}$$

$$E^L \cong k \otimes_k E^L \xrightarrow{u^L \otimes \mathrm{id}} E^L \otimes_{A^e} E \otimes_k E^L \xrightarrow{\mathrm{id} \otimes v^L} E^L \otimes_{A^e} A^e \cong E^L \tag{3.2.2}$$

are both identities. By Definition, E admits a right adjoint if there exist a (k, A^e)–bimodule E^R and unit and counit intertwiners

$$u^R : A^e \longrightarrow E \otimes_k E^R$$

$$v^R : E^R \otimes_{A^e} E \longrightarrow k$$

such that the compositions

$$E \cong A^e \otimes_{A^e} E \xrightarrow{u^R \otimes \mathrm{id}} E \otimes_k E^R \otimes_{A^e} E \xrightarrow{\mathrm{id} \otimes v^R} E \otimes_k k \cong E \tag{3.2.3}$$

$$E^R \cong E^R \otimes_{A^e} A^e \xrightarrow{\mathrm{id} \otimes u^R} E^R \otimes_{A^e} E \otimes_k E^R \xrightarrow{v^R \otimes \mathrm{id}} k \otimes_k E^R \cong E^R \tag{3.2.4}$$

are both identities.

Assume $A \in \mathrm{Alg}_k$ is fully dualizable. The unit u^L is determined by

$$u^L : 1 \in k \longmapsto y \otimes 1 \in E^L \otimes_{A^e} E$$

for some $y \in E^L$. The counit v^L maps

$$v^L : 1 \otimes y \in E \otimes_k E^L \longmapsto e \in A^e$$

for some $e \in A^e$. By left A^e–linearity of v^L,

$$\begin{aligned} v^L(x \otimes y) &= v^L(x \otimes 1 \cdot 1 \otimes y) &= x \otimes 1 \cdot e \\ &= v^L(1 \otimes x \cdot 1 \otimes y) &= 1 \otimes x \cdot e \end{aligned}$$

Hence

$$x \otimes 1 \cdot e = 1 \otimes x \cdot e \tag{3.2.5}$$

as elements in A^e. Since the composition in (3.2.1) is the identity,

$$1 = \mu(e) \tag{3.2.6}$$

An element $e \in A^e$ satisfying conditions (3.2.5) and (3.2.6) is called a *separating idempotent* for A. An algebra over a commutative ring has a separating idempotent if and only if it is separable (see chapter 10 in [12]). Since in our case A is defined over a perfect field, A is separable if and only if it is finite-dimensional and semi-simple (corollary 10.7.b in [12]). Hence a fully dualizable algebra in Alg_k is semi-simple and finite dimensional. The existence of a right adjoint to E_A implies that A is finite dimensional, which is already implied by the existence of a left adjoint.

Now assume A is finite-dimensional and semi-simple. We show that the evaluation E admits left and right adjoints. As a left adjoint E^L take A considered as a (k, A^e)–bimodule with $a \cdot x \otimes y = yax$. Choose a separating idempotent $e = \sum_i x_i \otimes y_i$ in A^e. Define the unit and co-unit intertwiners to be

$$u^L : 1 \longmapsto 1 \otimes 1$$
$$v^L : x \otimes y \longmapsto \sum_i x x_i \otimes y y_i$$

As a right adjoint E^R take $A^* := \mathrm{Hom}_k(A,k)$ considered as a (k, A^e)–bimodule with $(f \cdot x \otimes y)(a) = f(xay)$. Choose a basis $\{e_j\}$ for A and let $\{e^j\}$ denote its dual basis in A^*. Define the unit and co-unit intertwiners to be

$$u^R : 1 \otimes 1 \longmapsto \sum_i e_j \otimes e^j$$
$$v^R : y \otimes x \longmapsto y(x)$$

One can check that compositions (3.2.1), (3.2.2), (3.2.3) and (3.2.4) are all identities. Hence A is fully dualizable and the proposition is proved. □

Remark 3.2.2. By the Wedderburn theorem, a finite-dimensional semi-simple k-algebra, A, is isomorphic to a product of matrix algebras over k, that is,

$$A \cong \mathrm{End}_k(V_1) \times \cdots \times \mathrm{End}_k(V_r)$$

where V_i are finite dimensional k-vector spaces. The number of factors r is the *Morita class* of A, meaning that for two finite-dimensional semi-simple k-algebras A and A', we have

$$_A\mathrm{Mod} \simeq {_{A'}\mathrm{Mod}} \quad \text{if and only if} \quad r = r'$$

Lemma 3.2.3. *Let A, B be fully dualizable algebras and let $M \in \mathrm{Alg}_k(A,B)$. Then M admits right and left adjoints iff M is finite dimensional (equivalently finitely generated as a left A–module and a right B–module).*

Proof. In one direction assume M admits left and right adjoints. By Lemma A.1.5 in [13], the conclusion holds (the proof is a direct application of definitions). In the other direction, note the full subcategory of finite dimensional (A, B)-bimodules is semi-simple (see Lemma A.1.2). Hence, it is enough to show a simple (i.e. indecomposable) (A, B)-bimodule admits left and right adjoints. Note that when infinite sums of simples are involved, the units for right and left adjunction are not well-defined.

Let $A = \mathrm{End}(V_1) \times \cdots \times \mathrm{End}(V_r)$ and $B = \mathrm{End}(W_1) \times \cdots \times \mathrm{End}(W_s)$. Then every simple (A, B)-bimodule is isomorphic to $M = \mathrm{Hom}(W_i, V_j)$ for some $1 \leq i \leq s$ and $1 \leq j \leq r$. We show M admits left and right adjoints

$$M^R = \mathrm{Hom}(V_j, W_i) = M^L$$

Note that both compositions

$$\mathrm{Hom}(W_i, V_j) \otimes \mathrm{Hom}(V_j, W_i) \to \mathrm{Hom}(V_j, V_j) \xrightarrow{\mathrm{Tr}} k$$

$$\mathrm{Hom}(V_j, W_i) \otimes \mathrm{Hom}(W_i, V_j) \to \mathrm{Hom}(W_i, W_i) \xrightarrow{\mathrm{Tr}} k$$

are non-degenerate. Moreover, there exist

$$f_\alpha \in \mathrm{Hom}(V_j, W_i) \quad , \quad f^\alpha \in \mathrm{Hom}(W_i, V_j)$$

such that

$$\sum_\alpha \frac{1}{\dim(W_i)} \, f^\alpha \cdot f_\alpha = \mathrm{id}_{V_j}$$

$$\sum_\alpha \frac{1}{\dim(V_j)} \, f_\alpha \cdot f^\alpha = \mathrm{id}_{W_i}$$

We can define

$$\begin{aligned}
&u^R : 1 \in A \longmapsto \sum_\alpha \frac{1}{\dim(W_i)} f^\alpha \otimes f_\alpha \in M \otimes_B M^R \\
&v^R : f \otimes g \in M^R \otimes_A M \longmapsto f \cdot g \in B \\
&u^L : 1 \in B \longmapsto \sum_\alpha \frac{1}{\dim(V_j)}\, f_\alpha \otimes f^\alpha \in M^L \otimes_A M \\
&v^L : f \otimes g \in M \otimes_B M^L \longmapsto f \cdot g \in A
\end{aligned}$$

which determine the units and counits of right and left adjunctions. □

Corollary 3.2.4. *The subcategory* $\mathrm{Alg}_K^{\mathrm{fd}} \subseteq \mathrm{Alg}_k$ *consists of:*

$$\begin{aligned}
\mathrm{Obj}(\mathrm{Alg}_k^{\mathrm{fd}}) &= \textit{semi-simple finite dimensional algebras} \\
1-\mathrm{Mor}(\mathrm{Alg}_k^{\mathrm{fd}}) &= \textit{their finite dimensional bimodules} \\
2-\mathrm{Mor}(\mathrm{Alg}_k^{\mathrm{fd}}) &= \textit{their intertwiners}
\end{aligned}$$

Proof. $\mathrm{Alg}_k^{\mathrm{fd}}$ is the subcategory of Alg_k consisting of fully dualizable objects, their 1-morphisms admitting left and right adjoints, and their 2-morphisms (see Definition B.0.11). The rest follows from Lemmas 3.2.1 and 3.2.3. □

3.2.2 The Space $(\mathrm{Alg}_k^{\mathrm{fd}})^\sim$.

By the Cobordism Hypothesis for framed manifolds, the space of 2-dimensional fully extended framed TFT's with values in Alg_k is canonically equivalent to

$$\mathrm{Fun}^\otimes(\mathrm{Bord}_2^{\mathrm{fr}}, \mathrm{Alg}_k) \simeq (\mathrm{Alg}_k^{\mathrm{fd}})^\sim$$

The right hand side is a symmetric monoidal 2-groupoid that we get by discarding of non-invertible morphisms in $\mathrm{Alg}_k^{\mathrm{fd}}$. By Corollary 3.2.4 $(\mathrm{Alg}_k^{\mathrm{fd}})^\sim$

consists of:

$$\begin{aligned}
\mathrm{Obj}((\mathrm{Alg}^{\mathrm{fd}})^{\sim}) &= \text{semi-simple finite dimensional algebras} \\
1-\mathrm{Mor}((\mathrm{Alg}^{\mathrm{fd}})^{\sim}) &= \text{their } \textit{invertible} \text{ finite dimensional bimodules} \\
2-\mathrm{Mor}((\mathrm{Alg}^{\mathrm{fd}})^{\sim}) &= \text{their } \textit{invertible} \text{ intertwiners}
\end{aligned}$$

By the Homotopy Hypothesis, $(\mathrm{Alg}_k^{\mathrm{fd}})^{\sim}$ determines a 2-type X up to homotopy equivalence. The path components of X are labeled by equivalence classes of objects in $(\mathrm{Alg}_k^{\mathrm{fd}})^{\sim}$. These are the Morita equivalence classes of semi-simple finite-dimensional algebras, determined by the number r of factors in the decomposition of such an algebra into a product of matrix algebras. Hence,

$$X = \coprod_{r=1}^{\infty} X_r \tag{3.2.7}$$

where each X_r is a path-connected 2-type.

The homotopy type of a path-connected 2-type Y is completely determined by its fundamental group $\pi_1(Y)$, its second homotopy group $\pi_2(Y)$, the action of $\pi_1(Y)$ on $\pi_2(Y)$, and a class c_Y in the third cohomology group $H^3(\pi_1(Y);\pi_2(Y))$. This class is determined by the fibration

$$K(\pi_2(Y),2) \longrightarrow Y \longrightarrow K(\pi_1(Y),1)$$

which is the Postnikov tower associated to Y.

Proposition 3.2.5. *Let X_r be the path-connected 2-type as above. Then $\pi_1(X_r) \cong \Sigma_r$ is the symmetric group on r elements, and $\pi_2(X_r) \cong (k^{\times})^r$ is the r-fold product of $k^{\times} = k\backslash\{0\}$. The action of $\pi_1(X_r)$ on $\pi_2(X_r)$ is* *defined by permutation of entries. The associated cohomology class* $c_{X_r} \in H^3(\pi_1(X_r); \pi_2(X_r))$ *is trivial.*

Proof. Let A be a semi-simple finite dimensional k-algebra,

$$A \cong \mathrm{End}(V_1) \times \cdots \times \mathrm{End}(V_r)$$

The fundamental group $\pi_1(X_r)$ is isomorphic to the group of equivalence classes of (A, A)–bimodules. By Lemma A.1.4 in Appendix A.1, isomorphism classes of (A, A)–bimodules are in bijection with Σ_r.

The second homotopy group $\pi_2(X_r)$ is isomorphic to the group of invertible intertwiners $A \to A$ where A is considered a (A, A)–bimodule. These are given by the units $Z(A)^\times$ in the center of A. The center of A is the product of centers of its factors. Hence $\pi_2(X_r) \cong (k^\times)^r$.

Let M^σ be the (A, A)–bimodule corresponding to $\sigma \in \Sigma_r$, and let $M^{\sigma^{-1}}$ be its inverse,

$$M^\sigma = \mathrm{Hom}_k(V_{\sigma(1)}, V_1) \times \cdots \times \mathrm{Hom}_k(V_{\sigma(r)}, V_r)$$

$$M^{\sigma^{-1}} = \mathrm{Hom}_k(V_{\sigma^{-1}(1)}, V_1) \times \cdots \times \mathrm{Hom}_k(V_{\sigma^{-1}(r)}, V_r)$$

Let $\varphi : A \to A$ be an invertible intertwiner of A,

$$\varphi = (\lambda_1 \,\mathrm{id}_{V_1}, \ldots, \lambda_r \,\mathrm{id}_{V_r}) \quad , \quad \lambda_i \in k^\times$$

The action of $\pi_1(X_r)$ on $\pi_2(X_r)$ is given by,

$$\sigma \cdot \varphi := \mathrm{id}_{M^\sigma} \otimes \varphi \otimes \mathrm{id}_{M^{\sigma^{-1}}}$$

By direct calculation,

$$\begin{array}{ccc} (f_i)\otimes(f'_i)\otimes(f''_i)\in M^{\sigma}\otimes A\otimes M^{\sigma^{-1}} & \longmapsto & (f''_{\sigma(i)}\circ f'_{\sigma(i)}\circ f_i)\in A \\ \downarrow{\scriptstyle 1\otimes\varphi\otimes 1} & & \downarrow{\scriptstyle \sigma\cdot\varphi} \\ (f_i)\otimes(\lambda_i f'_i)\otimes(f''_i)\in M^{\sigma}\otimes A\otimes M^{\sigma^{-1}} & \longmapsto & (\lambda_{\sigma(i)} f''_{\sigma(i)}\circ f'_{\sigma(i)}\circ f_i)\in A \end{array}$$

therefore σ acts on φ by permuting its entries.

The class $c_{X_r}\in Z^3(\pi_1X_r;\pi_2X_r)$ is given by the associator of the 2-groupoid $(\mathrm{Alg}_k^{\mathrm{fd}})^{\sim}$. Given $\sigma,\sigma',\sigma''\in\Sigma_r$, the value $c_{X_r}(\sigma,\sigma',\sigma'')$ is the intertwiner $M^{\sigma''\sigma'\sigma}\to M^{\sigma''\sigma'\sigma}$ defined by the commutative diagram

$$\begin{array}{ccc} M^{\sigma}\otimes_A M^{\sigma'}\otimes_A M^{\sigma''} & \longrightarrow & M^{\sigma'\sigma}\otimes_A M^{\sigma''} \\ \downarrow & & \downarrow \\ M^{\sigma}\otimes_A M^{\sigma''\sigma'} & \longrightarrow & M^{\sigma''\sigma'\sigma} \end{array}$$

Each arrow is an isomorphism and $c_{X_r}(\sigma,\sigma',\sigma'')$ is gotten by looping around the commutative square starting at the bottom right corner. As a result $c_{X_r}(\sigma,\sigma',\sigma'')=\mathrm{id}$. Hence it is trivial. □

Corollary 3.2.6. *The space of 2–dimensional fully extended framed TFT's with values in* Alg_k *is homotopy equivalent to,*

$$\mathrm{Fun}^{\otimes}(\mathrm{Bord}_2^{\mathrm{fr}},\mathrm{Alg}_k)\simeq\coprod_{r=1}^{\infty}E\Sigma_r\times_{\Sigma_r}K(k^{\times},2)^{\times r}$$

3.2.3 The $SO(2)$–Action.

As a consequence of the Cobordism Hypothesis, the underlying space of fully dualizable objects $(\mathcal{C}^{\mathrm{fd}})^{\sim}$ of a symmetric monoidal (∞,n)–category $\mathcal{C}$

admits a homotopy $O(n)$–action. In this section we consider the special case of the homotopy $SO(2)$–action on $(\mathrm{Alg}_k^{\mathrm{fd}})^{\sim}$.

Let $\mathcal{G}$ be a 2-groupoid and let $\mathrm{Fun}^{\otimes}(\mathcal{G},\mathcal{G})$ denote its 2-groupoid of (strong) monoidal endo-functors. Note that $\mathrm{Fun}^{\otimes}(\mathcal{G},\mathcal{G})$ is a (generally not symmetric) monoidal 2-groupoid with tensor product given by composition. Let $\mathbf{Z}$ denote the 2-groupoid with one object, $\mathbb{Z}$ worth of 1-morphisms and only identity 2-morphisms. It is a (strict symmetric) monoidal category with monoidal product on 1-morphisms given by addition.

A homotopy $SO(2)$–action on $\mathcal{G}$ is a (strong) monoidal functor $F : \mathbf{Z} \to \mathrm{Fun}^{\otimes}(\mathcal{G},\mathcal{G})$. Because F is monoidal we may assume, up to equivalence, that F sends the unique object of $\mathbf{Z}$ to $\mathrm{id}_{\mathcal{G}}$. Then F is determined by a monoidal functor

$$F : \mathbf{Z}(\bullet,\bullet) \to \mathrm{Fun}^{\otimes}(\mathcal{G},\mathcal{G})(\mathrm{id}_{\mathcal{G}},\mathrm{id}_{\mathcal{G}})$$

The source of F is the symmetric monoidal 1-groupoid with $\mathbb{Z}$ worth of objects and only identity morphisms. The target of F is the symmetric monoidal category with objects monoidal natural isomorphisms of the identity functor. The following lemma was suggested by C. Schommer-Pries.

Lemma 3.2.7. *Let $(\mathcal{C},\otimes)$ be a monoidal category. There is a canonical equivalence of 1-groupoids,*

$$\begin{aligned} \mathrm{Fun}^{\otimes}(\mathbf{Z}(\bullet,\bullet),\mathcal{C}) &\xrightarrow{\simeq} (\mathcal{C}^{\mathrm{inv}})^{\sim} \\ F &\longmapsto F(1) \end{aligned}$$

where $\mathcal{C}^{\mathrm{inv}}$ denotes the full subcategory of invertible objects of $\mathcal{C}$.

Proof. We may assume $\mathcal{C}$ is strict monoidal. We show the functor $F \mapsto F(1)$ is essentially surjective and fully faithful.

To show it is essentially surjective, let $X \in \mathcal{C}^{\text{inv}}$. Pick $Y \in \mathcal{C}$ and isomorphisms $e : X \otimes Y \to \mathbf{1}$ and $c : \mathbf{1} \to Y \otimes X$ such that the compositions,

$$X = X \otimes \mathbf{1} \xrightarrow{\text{id} \otimes c} X \otimes Y \otimes X \xrightarrow{e \otimes \text{id}} \mathbf{1} \otimes X = X$$

$$Y = \mathbf{1} \otimes Y \xrightarrow{c \otimes \text{id}} Y \otimes X \otimes Y \xrightarrow{\text{id} \otimes e} Y \otimes \mathbf{1} = Y$$

are both identities. Such a choice is unique up to a unique isomorphism.

We define a monoidal functor $F_X : \mathbf{Z}(\bullet, \bullet) \to \mathcal{C}$ such that $F_X(1) = X$. On objects we let $F_X(n) = X^{\otimes n}$, where we understand $X^{\otimes n} \equiv \mathbf{1}$ when $n = 0$, and $X^{\otimes n} \equiv Y^{\otimes |n|}$ when $n < 0$.

In addition, we need to define natural isomorphisms,

$$F_2(m, n) : F_X(m) \otimes F_X(n) \to F_X(m + n)$$

$$F_0 : \mathbf{1} \to F_X(0)$$

such that the following diagrams commute,

$$\begin{array}{ccccc} & & F_X(l) \otimes F_X(m) \otimes F_X(n) & & \\ & \swarrow^{F_2(l,m)\otimes \text{id}} & & \searrow^{\text{id} \otimes F_2(m,n)} & \\ F_X(l+m) \otimes F_X(n) & & & & F_X(l) \otimes F_X(m+n) \\ & \searrow_{F_2(l+m,n)} & & \swarrow_{F_2(l,m+n)} & \\ & & F_X(l+m+n) & & \end{array} \tag{3.2.8}$$

$$\begin{array}{ccc} \mathbf{1} \otimes F_X(n) & \xrightarrow{F_0 \otimes \mathrm{id}} & F_X(0) \otimes F_X(n) \\ & \searrow\!\!\!\!= \quad \swarrow_{F_2(0,n)} & \\ & F_X(n) & \end{array} \qquad \begin{array}{ccc} F_X(n) \otimes \mathbf{1} & \xrightarrow{\mathrm{id} \otimes F_0} & F_X(n) \otimes F_X(0) \\ & \searrow\!\!\!\!= \quad \swarrow_{F_2(n,0)} & \\ & F_X(n) & \end{array}$$

We choose F_0 to be the identity. For $i = 1, 2, \ldots$ we denote

$$f^{(i)}(m,n) := \begin{cases} e : X \otimes Y \to \mathbf{1} & m > 0, n < 0 \\ c^{-1} : Y \otimes X \to \mathbf{1} & m < 0, n > 0 \\ \mathrm{id} : X^{\otimes i} \otimes X^{\otimes i} \to X^{\otimes i} \otimes X^{\otimes i} & m > 0, n > 0 \\ \mathrm{id} : Y^{\otimes i} \otimes Y^{\otimes i} \to Y^{\otimes i} \otimes Y^{\otimes i} & m < 0, n < 0 \end{cases}$$

For $1 \leq i \leq k(m,n) := \min\{|m|, |n|\}$ we denote,

$$F_2^{(i)}(m,n) = \underbrace{\mathrm{id} \otimes \cdots \otimes \mathrm{id}}_{|m|-i} \otimes f^{(i)}(m,n) \otimes \underbrace{\mathrm{id} \otimes \cdots \otimes \mathrm{id}}_{|n|-i}$$

We define $F_2(m,n) : X^{\otimes m} \otimes X^{\otimes n} \to X^{\otimes(m+n)}$ to be the composition,

$$F_2(m,n) = F_2^{(k(m,n))}(m,n) \circ \cdots \circ F_2^{(2)}(m,n) \circ F_2^{(1)}(m,n)$$

When $k(m,n) = 0$ we take $F_2(m,n) = \mathrm{id}$. This serves to guarantee the commutativity of the bottom couple of triangles. When m and n equal in sign $F_2(m,n)$ is just the identity. When they differ in sign and $m > 0$ we apply e in succession to inner most pairs $X \otimes Y$. When $m < 0$ we apply c^{-1} in succession to inner most pairs $Y \otimes X$.

The commutativity of the top diagram is non-trivial to check only when l, m and n alternate in sign. Consider the case $l > 0$, $m < 0$ and $n > 0$. Denote

$$d = l + m + n$$

$$s_l = \max\{l+m, 0\} \quad , \quad s_m = \max\{-d, 0\} \quad , \quad s_n = \max\{n+m, 0\}$$

Define a directed graph $\Gamma(l,m,n)$ whose set of nodes is the set of integral solutions

$$\{\,(x,y,z)\,|\, x+y+z = d, s_l \le x \le l, s_m \le y \le -m, s_n \le z \le n\,\}$$

and whose set of directed edges consists of

$$(x,y,z) \to (x-1, y-1, z) \quad \text{or} \quad (x,y,z) \to (x, y-1, z-1)$$

The graph $\Gamma(l,m,n)$ may alternately be considered a category by adding identity arrows. Consider the functor $\Gamma(l,m,n) \to \mathcal{C}$ defined by sending,

$$\begin{aligned} (x,y,z) &\longmapsto X^{\otimes x} \otimes Y^{\otimes y} \otimes X^{\otimes z} \\ (x,y,z) \to (x-1,y-1,z) &\longmapsto F_2^{(1)}(x,-y) \otimes \mathrm{id}_{X^{\otimes z}} \\ (x,y,z) \to (x,y-1,z-1) &\longmapsto \mathrm{id}_{X^{\otimes x}} \otimes F_2^{(1)}(-y,z) \end{aligned}$$

Every square in $\Gamma(l,m,n)$

$$\begin{array}{ccc} (x,y,z) & \longrightarrow & (x-1,y-1,z) \\ \downarrow & & \downarrow \\ (x,y-1,z-1) & \longrightarrow & (x-1,y-2,z-1) \end{array}$$

is mapped to a commutative diagram

$$\begin{array}{ccc} X^{\otimes x} \otimes Y^{\otimes y} \otimes X^{\otimes z} & \xrightarrow{F_2^{(1)}(x,-y)\otimes \mathrm{id}_{X^{\otimes z}}} & X^{\otimes (x-1)} \otimes Y^{\otimes (y-1)} \otimes X^{\otimes z} \\ \Big\downarrow {\scriptstyle \mathrm{id}_{X^{\otimes x}} \otimes F_2^{(1)}(-y,z)} & & \Big\downarrow {\scriptstyle \mathrm{id}_{X^{\otimes (x-1)}} \otimes F_2^{(1)}(-y+1,z)} \\ X^{\otimes x} \otimes Y^{\otimes (y-1)} \otimes X^{\otimes (z-1)} & \xrightarrow[F_2^{(1)}(x,-y+1)\otimes \mathrm{id}_{X^{\otimes (z-1)}}]{} & X^{\otimes (x-1)} \otimes Y^{\otimes (y-2)} \otimes X^{\otimes (z-1)} \end{array} \tag{3.2.9}$$

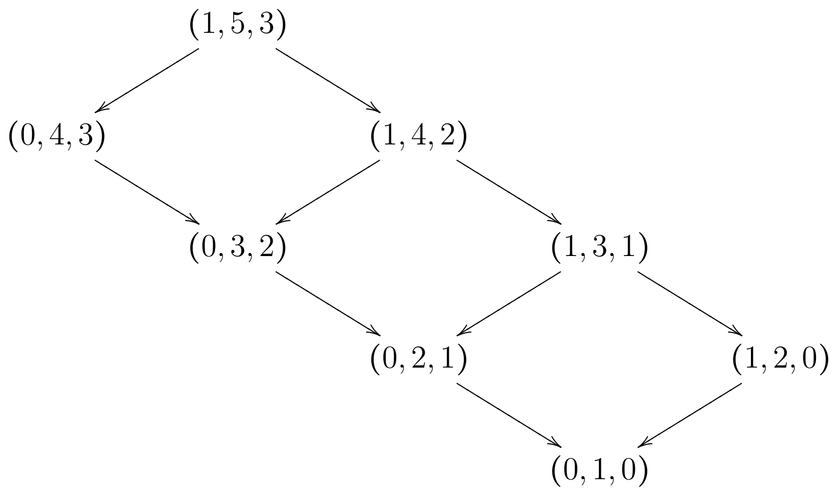


Figure 3.1: The directed graph $\Gamma(l, m, n)$.

Commutativity here is a consequence of the functoriality of the tensor product in $\mathcal{C}$.

Each composition of morphisms in diagram (3.2.8)

$$F_2(l, m+n) \circ \mathrm{id} \otimes F_2(m, n) \quad , \quad F_2(l+m, n) \circ F_2(l, m) \otimes \mathrm{id}$$

is the image of some directed path in $\Gamma(l, m, n)$ of maximum length. Each directed path in $\Gamma(l, m, n)$ of maximum length starts at the same node, namely, $(l, -m, n)$. The image of the terminal node of each maximum length directed path in $\Gamma(l, m, n)$ is identified in $\mathcal{C}$. By the commutativity of diagram (3.2.9), these two morphisms agree in $\mathcal{C}$.

For example, in the case of $\Gamma(1, 5, 3)$ depicted in Figure 3.1 the path $(1, 5, 3) \to (1, 4, 2) \to (1, 3, 1) \to (1, 2, 0) \to (0, 1, 0)$ is mapped to the composition $F_2(1,-2) \circ \mathrm{id}_X \otimes F_2(-5,3)$ and the path $(1,5,3) \to (0,4,3) \to (0,3,2) \to (0,2,1) \to (0,1,0)$ is mapped to the composition $F_2(-4,3) \circ F_2(1,-5) \otimes \mathrm{id}_{X^{\otimes 3}}$.

Similarly we get the commutativity of diagram (3.2.8) when $l < 0$, $m > 0$ and $n < 0$. We conclude that the functor $F \mapsto F(1)$ is essentially surjective.

A monoidal natural isomorphism $\eta : F \Rightarrow F'$, where $F, F' : \mathbf{Z}(\bullet,\bullet) \to \mathcal{C}$, is a collection of isomorphisms $\eta_n : F(n) \to F'(n)$ such that the following diagrams commute,

$$\begin{array}{ccc} F(m) \otimes F(n) & \xrightarrow{\eta_m \otimes \eta_n} & F'(m) \otimes F'(n) \\ {\scriptstyle F_2(m,n)} \downarrow & & \downarrow {\scriptstyle F'_2(m,n)} \\ F(m+n) & \xrightarrow[\eta_{m+n}]{} & F'(m+n) \end{array}$$

$$\begin{array}{ccc} F(0) & \xrightarrow{\eta_0} & F'(0) \\ {\scriptstyle F_0} \nwarrow & & \nearrow {\scriptstyle F'_0} \\ & \mathbf{1} & \end{array}$$

To show $F \mapsto F(1)$ is fully faithful we need to consider the map

$$\eta \in \mathrm{Hom}_{\mathrm{Fun}^{\otimes}(\mathbf{Z}(\bullet,\bullet),\mathcal{C})}(F, F') \longmapsto \eta_1 \in \mathrm{Hom}_{\mathcal{C}}(F(1), F'(1))$$

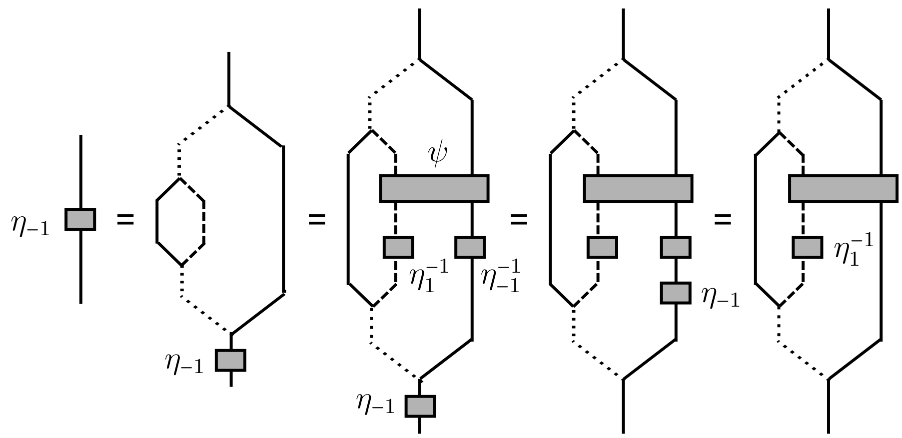
and show it is a bijection. By the above commutative diagrams it is clear that η_0 is determined by F_0 and F'_0 and that η_n is determined by η_1 and η_{-1} for $|n| > 1$. We need to show η_{-1} is determined by η_1 to establish the bijection. This dependency is demonstrated by the following identities,

$$\begin{aligned} \eta_{-1} &= \eta_{-1} \circ (F_2(-1,1)^{-1} \otimes \mathrm{id}) \circ (F_2(-1,1) \otimes \mathrm{id}) \\ &= \eta_{-1} \circ (F_2(-1,1)^{-1} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \eta_1^{-1} \otimes \eta_{-1}^{-1}) \circ \psi \circ (F_2(-1,1) \otimes \mathrm{id}) \end{aligned}$$

$$= \quad (F_2(-1,1)^{-1} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mathrm{id} \otimes \eta_{-1}) \circ (\mathrm{id} \otimes \eta_1^{-1} \otimes \eta_{-1}^{-1}) \circ \psi \circ (F_2(-1,1) \otimes \mathrm{id})$$

$$= \quad (F_2(-1,1)^{-1} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \eta_1^{-1} \otimes \mathrm{id}) \circ \psi \circ (F_2(-1,1) \otimes \mathrm{id})$$

where $\psi = F_2'(1,-1)^{-1} \circ \eta_0 \circ F_2(1,-1)$. These identities are represented by the following morphism diagrams:



They should be read top to bottom. Boxes represent morphisms, incoming lines represent sources while outgoing lines represent targets. Black lines denote $F(-1)$ or $F'(-1)$, dashed lines denote $F(1)$ or $F'(1)$ and dotted lines denote the unit object in $\mathcal{C}$. □

Proposition 3.2.8. *The homotopy $SO(2)$–action on* $(\mathrm{Alg}^{\mathrm{fd}})^{\sim}$ *is equivalent to the trivial action.*

Proof. We apply Lemma 3.2.7 to $\mathcal{C} = \mathrm{Fun}^{\otimes}(\mathcal{G},\mathcal{G})(\mathrm{id}_{\mathcal{G}},\mathrm{id}_{\mathcal{G}})$ for $\mathcal{G} = (\mathrm{Alg}_k^{\mathrm{fd}})^{\sim}$. The homotopy $SO(2)$–action is equivalent to the trivial action if and only if $F(1) \in \mathrm{Fun}^{\otimes}(\mathcal{G},\mathcal{G})(\mathrm{id}_{\mathcal{G}},\mathrm{id}_{\mathcal{G}})$ is isomorphic to the identity of $\mathrm{id}_{\mathcal{G}}$. The object

$F(1)$ is a monoidal natural isomorphism of the identity functor on $(\mathrm{Alg}_k^{\mathrm{fd}})^{\sim}$. As such, it consists of a collection of natural isomorphisms,

$$F(1)_A \in (\mathrm{Alg}_k^{\mathrm{fd}})^{\sim}(A, A) \quad , \quad \forall A \in (\mathrm{Alg}_k^{\mathrm{fd}})^{\sim}$$

This natural isomorphism $F(1)_A$ is the Serre automorphism S_A (see Definition B.0.12) as explained in §4.2 in [10]. It satisfies

$$ {}_k A_{A \otimes A^o} \cong {}_k A \otimes_{A^o \otimes A} (A^o \otimes S_A)_{A \otimes A^o} \tag{3.2.10}$$

as in equation (B.0.1). We have an isomorphism of (k, A^e)−bimodules

$$\begin{array}{rcl} {}_k A \otimes_{A^o \otimes A} (A^o \otimes S_A)_{A \otimes A^o} & \xrightarrow{\cong} & S_A \\ a \otimes (b \otimes x) & \longmapsto & (bc) \cdot x \end{array}$$

where S_A on the right hand side is considered a (k, A^e)−bimodule. By equation (3.2.10), $S_A \cong A$ as (A, A)−bimodules. As a result, $F(1)$ is equivalent to the identity natural isomorphism of the identity functor on $(\mathrm{Alg}_k^{\mathrm{fd}})^{\sim}$ and the proof is complete. □

3.3 Oriented Theories.

In this section we consider the space of fully extended 2-dimensional oriented TFT's with values in Alg_k. By Theorem 3.1.3 and Proposition 3.2.8 it is homotopy equivalent to $\mathrm{Map}(BSO(2), X)$. We can compute its homotopy groups as follows.

Recall X_r is the path-connected 2-type as in equation (3.2.7) with $\pi_1(X_r) \cong \Sigma_r$ and $\pi_2(X_r) \cong (k^\times)^r$. The action of $\pi_1(X_r)$ on $\pi_2(X_r)$ is given by permutation of entries (see Proposition 3.2.5). Let $f \in (k^\times)^r$ then

$$\mathrm{Stab}_{\Sigma_r}(f) = \{\, \sigma \in \Sigma_r \mid \sigma \cdot f = f \,\}$$

denotes the stabilizer of f in Σ_r

Lemma 3.3.1. *Let X be the space of fully extended 2–dimensional framed TFT's with values in Alg_k and X_r a path-connected component as in (3.2.7). Then,*

$$\pi_0 \,\mathrm{Map}(BSO(2), X) \cong \coprod_{r=1}^{\infty} (k^\times)^r/\Sigma_r$$

For every $f \in \mathrm{Map}(BSO(2), X_r)$,

$$\pi_i(\mathrm{Map}(BSO(2), X_r), f) \cong \begin{cases} \mathrm{Stab}_{\Sigma_r}(f) & i = 1 \\ (k^\times)^r & i = 2 \\ 0 & i \geq 3 \end{cases}$$

Proof. To compute the homotopy groups of $\mathrm{Map}(BSO(2), X)$ we first note that X is a 2-type. The CW complex $BSO(2) \simeq \mathbb{C}P^\infty$ has a cell structure with one cell in each even dimension. It may be considered the colimit $\mathbb{C}P^\infty = \varprojlim \mathbb{C}P^n$. Let us fix a map $f^{(1)} : \mathbb{C}P^1 \to X$ and consider the extension problem

$$\begin{array}{ccc} S^2 = \mathbb{C}P^1 & \xrightarrow{f^{(1)}} & X \\ \downarrow & \nearrow & \\ \mathbb{C}P^2 & & \end{array}$$

By obstruction theory, $f^{(1)}$ determines a class $c_{f^{(1)}} \in H^4(\mathbb{C}P^\infty; \pi_3(X))$, which vanishes if and only if f can be extended. Possible extensions of $f^{(1)}$ are classified by $H^4(\mathbb{C}P^\infty; \pi_4(X))$. But $\pi_3(X), \pi_4(X) = 0$, therefore $f^{(1)}$ can be uniquely extended up to homotopy. In the same way, extension of $f^{(2)}: \mathbb{C}P^2 \to X$ to $f^{(3)}: \mathbb{C}P^3 \to X$ exists and is unique up to homotopy. In this way we can show,

$$\mathrm{Map}(BSO(2), X) \simeq \mathrm{Map}(S^2, X)$$

Since X breaks into path-connected components $X = \coprod_r X_r$, the mapping space breaks into $\mathrm{Map}(S^2, X) = \coprod_r \mathrm{Map}(S^2, X_r)$. Our goal is to compute the homotopy groups of $\mathrm{Map}(S^2, X_r)$.

We have a product,

$$\mathrm{Map}_*(S^2, X_r) \times \mathrm{Map}_*(S^2, X_r) \to \mathrm{Map}_*(S^2, X_r)$$

which is associative and admits an identity only up to homotopy. This is enough to conclude that the homotopy groups of path-connected components of $\mathrm{Map}_*(S^2, X_r)$ are isomorphic.

Let us choose a base point $s \in S^2$. Consider the evaluation map,

$$\mathrm{ev}_s : \mathrm{Map}(S^2, X_r) \longrightarrow X_r$$
$$\mathrm{ev}_s : f \longmapsto f(s)$$

The evaluation map ev_s is a fibration (see Theorem 13.1 in [9]). Let us choose base points $x_r \in X_r$ and $f \in \mathrm{ev}_s^{-1}(x_r)$. The fibre $\mathrm{ev}_s^{-1}(x_r)$ is the space $\mathrm{Map}_*(S^2, X_r)$ of pointed maps $(S^2, s) \to (X_r, x_r)$.

There is a long exact sequence of homotopy groups,

$$\ldots \to \pi_n(\mathrm{Map}_*(S^2, X_r), f) \to \pi_n(\mathrm{Map}(S^2, X_r), f) \to \pi_n(X_r, x_r) \xrightarrow{\partial_n}$$
$$\xrightarrow{\partial_n} \pi_{n-1}(\mathrm{Map}_*(S^2, X_r), f) \to \ldots \to \pi_0(\mathrm{Map}(S^2, X_r)) \to 0$$

The path-connected component X_r is a 2-type, meaning, $\pi_i(X_r) = 0$ for $i \geq 3$. Hence $\pi_i(\mathrm{Map}_*(S^2, X_r), f) \cong \pi_i(\mathrm{Map}(S^2, X_r), f)$ for $i \geq 3$. On the other hand, $\pi_i(\mathrm{Map}_*(S^2, X_r), f) \cong \pi_{i+2}(X_r)$, therefore,

$$\pi_i(\mathrm{Map}(S^2, X_r), f) = 0 \quad , \quad i \geq 3$$

Consider the tail end of the long exact sequence,

$$\begin{aligned}0 \to \pi_2(\mathrm{Map}_*(S^2, X_r), f) \xrightarrow{\iota_*} \pi_2(\mathrm{Map}(S^2, X_r), f) \xrightarrow{\mathrm{ev}_*} \pi_2(X_r, x_r) \xrightarrow{\partial_2} \\ \xrightarrow{\partial_2} \pi_1(\mathrm{Map}_*(S^2, X_r), f) \xrightarrow{\iota_*} \pi_1(\mathrm{Map}(S^2, X_r), f) \xrightarrow{\mathrm{ev}_*} \pi_1(X_r, x_r) \xrightarrow{\partial_1} \\ \xrightarrow{\partial_1} \pi_0 \mathrm{Map}_*(S^2, X_r) \xrightarrow{\iota_*} \pi_0 \mathrm{Map}(S^2, X_r) \to 0\end{aligned} \tag{3.3.1}$$

Since $\pi_2(\mathrm{Map}_*(S^2, X_r), f) = \pi_4(X_r) = 0$ and $\pi_1(\mathrm{Map}_*(S^2, X_r), f) = \pi_3(X_r) = 0$, the induced homomorphism ev_* establishes an isomorphism

$$\pi_2(\mathrm{Map}(S^2, X_r), f) \cong \pi_2(X_r, x_r) \cong (k^\times)^r$$

by Proposition 3.2.5.

There is a right action of $\pi_1(X_r, x_r)$ on $\pi_0 \mathrm{Map}_*(S^2, X_r)$ given by the homotopy lifting property of ev (see Proposition 4A.1. in [8]). The differential ∂_1 in (3.3.1) is given by,

$$\partial_1 : [\gamma] \in \pi_1(X_r, x_r) \longmapsto [f]_* \cdot [\gamma] \in \pi_0 \mathrm{Map}_*(S^2, X_r)$$

where $[f]_*$ denotes a homotopy class of pointed maps. Saying the sequence (3.3.1) is exact at $\pi_0 \mathrm{Map}_*(S^2, X_r)$ means $\iota_*^{-1}([f]) = \mathrm{Im}\partial$. Saying it is exact at $\pi_1(X_r, x_r)$ means $\mathrm{Stab}_{\pi_1(X_r,x_r)}([f]_*) = \mathrm{Im}\,\mathrm{ev}_*$. Hence ev_* establishes an isomorphism,

$$\pi_1(\mathrm{Map}(S^2, X_r), f) \cong \mathrm{Stab}_{\pi_1(X_r,x_r)}([f]_*)$$

As mentioned above $\pi_1(X_r, x_r)$ acts on $\pi_0 \mathrm{Map}_*(S^2, X_r)$. The set of path-connected components $\pi_0 \mathrm{Map}(S^2, X_r)$ is the set of orbits for that action. When $\pi_0 \mathrm{Map}_*(S^2, X_r)$ is identified with $\pi_2(X_r, x_r)$, this action is identified with the natural action of π_1 on π_2. By Proposition 3.2.5, the action of Σ_r on $(k^\times)^r$ is defined by permutation, therefore,

$$\pi_0(\mathrm{Map}(S^2, X_r), f) \cong (k^\times)^r/\Sigma_r$$

This completes the proof. □

More than computing the fundamental groups of the mapping space $\mathrm{Map}(BSO(2), X)$, we can compute its fundamental 2-groupoid $\Pi_{\leq 2}\mathrm{Map}(BSO(2), X)$. The classifying space $BSO(2)$ is an Eilenberg-MacLane space $K(\mathbb{Z}, 2)$. Its fundamental 2-groupoid $\Pi_{\leq 2}BSO(2)$ is equivalent to the 2-groupoid $\mathcal{G}(\mathbb{Z}, 2)$ with one object, one identity 1-morphism and $\mathbb{Z}$ worth of 2-morphisms.

Proposition 3.3.2. *The 2-groupoid* $\mathrm{Grp}_2(\mathcal{G}(\mathbb{Z},2), (\mathrm{Alg}_k^{\mathrm{fd}})^\sim)$ *is equivalent to the 2-groupoid* $\mathcal{G}_{\mathrm{ori}}$ *consisting of:*

$$\begin{aligned}
\mathrm{Obj}(\mathcal{G}_{\mathrm{ori}}) &= \textit{pairs } (A, f) \textit{ with } A \in \mathrm{Alg}_k^{\mathrm{fd}} \textit{ and } f \in Z(A)^\times \\
\mathcal{G}_{\mathrm{ori}}((A,f),(A',f')) &= \textit{bimodules } M \in (\mathrm{Alg}_k^{\mathrm{fd}})^\sim(A, A') \textit{ such that} \\
&\quad\ f \cdot m = m \cdot f' \textit{ for every } m \in M \\
\mathcal{G}_{\mathrm{ori}}(M, M') &= (\mathrm{Alg}_k^{\mathrm{fd}})^\sim(M, M')
\end{aligned}$$

Remark 3.3.3. For $\mathcal{G}_{\text{ori}}((A,f),(A',f'))$ not to be empty means there is an invertible (A,A')–bimodule M such that $f \cdot m = m \cdot f'$ for all $m \in M$. This implies A and A' have the same Morita class r. In that case $M \simeq M^{\sigma}$ for some $\sigma \in \Sigma_r$. We have $Z(A)^{\times} = (k^{\times})^r = Z(A')^{\times}$. Denote $f = (\lambda_1, \dots, \lambda_r)$ and $f' = (\lambda'_1, \dots, \lambda'_r)$. The condition $f \cdot m = m \cdot f'$ implies $\lambda_i = \lambda'_{\sigma(i)}$. The symmetric group Σ_r acts on $(k^{\times})^r$ by permutation. This condition means that f and f' are in the same orbit. There are $(k^{\times})^r/\Sigma_r$ such orbits by definition. This agrees with our computations in Lemma 3.3.1 as indeed it should.

Proof. The objects of $\text{Grp}_2(\mathcal{G}(\mathbb{Z},2),(\text{Alg}_k^{\text{fd}})^{\sim})$ are functors. A functor $\mathcal{G}(\mathbb{Z},2) \to (\text{Alg}_k^{\text{fd}})^{\sim}$ consists of a choice of a semi-simple finite dimensional algebra $A \in (\text{Alg}_k^{\text{fd}})^{\sim}$ and a monoidal functor

$$\mathcal{G}(\mathbb{Z},2)(\bullet,\bullet) \longrightarrow (\text{Alg}_k^{\text{fd}})^{\sim}(A,A)$$

The category $\mathcal{G}(\mathbb{Z},2)(\bullet,\bullet)$ consists of one object $\text{id}_{\bullet}$ and $\mathbb{Z}$ worth of morphisms. The category $(\text{Alg}_k^{\text{fd}})^{\sim}(A,A)$ is the category of invertible (A,A)–bimodules. The monoidal structure on each is given by composition, namely, tensor product of morphisms in $\mathcal{G}(\mathbb{Z},2)(\bullet,\bullet)$ is given by addition, and tensor product of objects in $(\text{Alg}_k^{\text{fd}})^{\sim}(A,A)$ is given by tensoring over A.

A monoidal functor $\mathcal{G}(\mathbb{Z},2)(\bullet,\bullet) \to (\text{Alg}_k^{\text{fd}})^{\sim}(A,A)$ is a triple (ψ, F, ϕ) consisting of

(i) a functor $F : \mathcal{G}(\mathbb{Z},2)(\bullet,\bullet) \to (\text{Alg}_k^{\text{fd}})^{\sim}(A,A)$

(ii) an isomorphism of (A,A)–modules $\phi : F(\text{id}_{\bullet}) \otimes_A F(\text{id}_{\bullet}) \to F(\text{id}_{\bullet})$ which is bi-functorial

(iii) an isomorphism of (A,A)–modules $\psi : A \to F(\mathrm{id}_\bullet)$

satisfying the following commuting diagrams,

$$
\begin{array}{ccccc}
(F(\mathrm{id}_\bullet) \otimes_A F(\mathrm{id}_\bullet)) \otimes_A F(\mathrm{id}_\bullet) & \xrightarrow{\phi\otimes 1} & F(\mathrm{id}_\bullet) \otimes_A F(\mathrm{id}_\bullet) & \xrightarrow{\phi} & F(\mathrm{id}_\bullet) \\
\downarrow & & & & \| \\
F(\mathrm{id}_\bullet) \otimes_A (F(\mathrm{id}_\bullet) \otimes_A F(\mathrm{id}_\bullet)) & \xrightarrow[1\otimes\phi]{} & F(\mathrm{id}_\bullet) \otimes_A F(\mathrm{id}_\bullet) & \xrightarrow[\phi]{} & F(\mathrm{id}_\bullet)
\end{array}
$$

$$
\begin{array}{ccc}
F(\mathrm{id}_\bullet) \otimes_A A & \xrightarrow{1\otimes\psi} & F(\mathrm{id}_\bullet) \otimes_A F(\mathrm{id}_\bullet) \\
\downarrow & & \downarrow \phi \\
F(\mathrm{id}_\bullet) & = & F(\mathrm{id}_\bullet)
\end{array}
\qquad
\begin{array}{ccc}
A \otimes_A F(\mathrm{id}_\bullet) & \xrightarrow{\psi\otimes 1} & F(\mathrm{id}_\bullet) \otimes_A F(\mathrm{id}_\bullet) \\
\downarrow & & \downarrow \phi \\
F(\mathrm{id}_\bullet) & = & F(\mathrm{id}_\bullet)
\end{array}
$$

Bi-functoriality of ϕ reads

$$
\begin{array}{ccc}
F(\mathrm{id}_\bullet) \otimes_A F(\mathrm{id}_\bullet) & \xrightarrow{\phi} & F(\mathrm{id}_\bullet) \\
{\scriptstyle F(n)\otimes F(m)}\downarrow & & \downarrow{\scriptstyle F(n+m)} \\
F(\mathrm{id}_\bullet) \otimes_A F(\mathrm{id}_\bullet) & \xrightarrow[\phi]{} & F(\mathrm{id}_\bullet)
\end{array}
\tag{3.3.2}
$$

The 2-groupoid $\mathrm{Grp}_2(\mathcal{G}(\mathbb{Z},2),(\mathrm{Alg}_k^{\mathrm{fd}})^{\sim})$ is equivalent to the full sub-groupoid

$$\mathrm{Grp}_2^0(\mathcal{G}(\mathbb{Z},2),(\mathrm{Alg}_k^{\mathrm{fd}})^{\sim}) \subset \mathrm{Grp}_2(\mathcal{G}(\mathbb{Z},2),(\mathrm{Alg}_k^{\mathrm{fd}})^{\sim})$$

consisting of functors for which $F(\mathrm{id}_\bullet) \equiv A$ and ϕ is given by multiplication. Since F is a functor, $F(n+m) = F(n) \circ F(m)$, in particular, all $F(n)$'s are determined by $f := F(1)$. Since $\otimes$ is a functor $(F(n) \circ F(m)) \otimes 1 = (F(n) \otimes 1) \circ (F(m) \otimes 1)$. Since $F(\mathrm{id}_\bullet) \equiv A$ we have $(F(n) \otimes 1) \circ (F(m) \otimes 1) = (F(n) \otimes 1) \circ (1 \otimes F(m)) = F(n) \otimes F(m)$. Bi-functoriality is then established. The commuting diagram (3.3.2) is satisfied by associativity of multiplication in A. The bottom two commuting diagrams force ψ to be the identity. In short, objects of $\mathrm{Grp}_2^0(\mathcal{G}(\mathbb{Z},2),(\mathrm{Alg}_k^{\mathrm{fd}})^{\sim})$ are pairs (A,f), where A is a finite dimensional semi-simple algebra and f is an invertible intertwiner $F : A \to A$, i.e. an element of $Z(A)^{\times}$.

The 1-morphisms of $\mathrm{Grp}_2^0(\mathcal{G}(\mathbb{Z},2),(\mathrm{Alg}_k^{\mathrm{fd}})^{\sim})$ are natural transformations. A natural transformation $(A,f) \to (A',f')$ consists of a choice of an invertible (A,A')-bimodule M, and a choice of a natural isomorphism of (A,A')-bimodules,

$$\sigma : M \otimes_{A'} A' \longrightarrow A \otimes_A M$$

satisfying the following commuting diagrams,

$$\begin{array}{ccc}
(A \otimes_A M) \otimes_{A'} A' & \longrightarrow & A \otimes_A (M \otimes_{A'} A') \\
\downarrow{\scriptstyle \sigma^{-1}\otimes 1} & & \downarrow{\scriptstyle 1\otimes\sigma} \\
(M \otimes_{A'} A') \otimes_{A'} A' & & A \otimes_A (A \otimes_A M) \\
\downarrow & & \downarrow \\
M \otimes_{A'} (A' \otimes_{A'} A') & & (A \otimes_A A) \otimes_A M \\
\downarrow{\scriptstyle 1\otimes\phi'} & & \downarrow{\scriptstyle \phi\otimes 1} \\
M \otimes_{A'} A' & \xrightarrow[\sigma]{} & A \otimes_A M
\end{array}$$

$$\begin{array}{ccccc}
A \otimes_A M & \longleftarrow & M & \longrightarrow & M \otimes_{A'} A' \\
\downarrow{\scriptstyle \psi\otimes 1} & & & & \downarrow{\scriptstyle 1\otimes\psi'} \\
A \otimes_A M & & \xrightarrow[\sigma]{} & & M \otimes_{A'} A'
\end{array}$$

By abuse of notation $\sigma : m \otimes 1 \mapsto 1 \otimes \sigma(m)$ for every $m \in M$. The above conditions force $\sigma(m) = m$. Naturality of σ reads

$$\begin{array}{ccc} M \otimes_{A'} A' & \xrightarrow{\sigma} & A \otimes_A M \\ {\scriptstyle 1\otimes f'}\downarrow & & \downarrow{\scriptstyle f\otimes 1} \\ M \otimes_{A'} A' & \xrightarrow{\sigma} & A \otimes_A M \end{array}$$

By abuse of notation $f := f(1)$. Naturality is then equivalent to,

$$f \cdot m = m \cdot f' \quad , \quad \forall\, m \in N$$

In short, a 1-morphism $(A, f) \to (A', f')$ in $\mathrm{Grp}_2^0(\mathcal{G}(\mathbb{Z}, 2), (\mathrm{Alg}_k^{\mathrm{fd}})^{\sim})$ consists of a choice of an invertible (A, A')-bimodule M such that $f \cdot m = m \cdot f'$ for every $m \in M$.

The 2-morphisms of $\mathrm{Grp}_2^0(\mathcal{G}(\mathbb{Z}, 2), (\mathrm{Alg}_k^{\mathrm{fd}})^{\sim})$ are modifications. A modification $M \to M'$ consists of a choice of an isomorphism $\gamma : M \to M'$ of (A, A')–bimodules, satisfying the following commuting diagram,

$$\begin{array}{ccc} M \otimes_{A'} A' & \xrightarrow{\gamma\otimes 1} & M' \otimes_{A'} A' \\ {\scriptstyle \sigma}\downarrow & & \downarrow{\scriptstyle \sigma'} \\ A \otimes_A M & \xrightarrow[1\otimes\gamma]{} & A \otimes_A M' \end{array}$$

Since $\sigma(m) = m = \sigma'(m)$ the above is automatically commutative for any choice of γ. $\square$

3.4 G-equivarient Theories.

Let G denote a topological group. Let $\Gamma = G \times SO(n)$ and take $\chi : \Gamma \to O(n)$ to be trivial on G and the embedding on $SO(n)$. Let $\mathcal{C}$ denote a symmetric monoidal (∞, n)-category.

Definition 3.4.1. A *fully extended n−dimensional G-equivariant TFT* with values in $\mathcal{C}$ is a monoidal functor,

$$F : \mathrm{Bord}_n^{\Gamma} \longrightarrow \mathcal{C}$$

The bordism (∞, n)−category Bord_n^{Γ} consists of oriented manifolds equipped with principal G-bundles.

In this section we consider the case where $n = 2$ and G is a finite group. We classify fully extended 2−dimensional G-equivariant TFT's with values in Alg_k. Following our discussion in Section 3.1, the space of such theories is homotopy equivalent to $\mathrm{Map}(BG, \mathrm{Map}(BSO(2), X))$, where X is the underlying space of fully dualizable objects in Alg_k introduced in Section 3.2.1. Its fundamental 2-groupoid is equivalent to

$$\mathrm{Fun}^{\otimes}(\mathrm{Bord}_2^{\Gamma}, \mathrm{Alg}_k) \simeq \mathrm{Grp}_2(BG, \mathcal{G}_{\mathrm{ori}})$$

The 2-groupoid $\mathcal{G}_{\mathrm{ori}}$ was introduced in Proposition 3.3.2, and BG stands for the 2-groupoid with one object, G worth of 1-morphisms and only identity 2-morphisms.

Classification of G-equivariant theories in this case means identifying the set of path-connected components $\pi_0 \mathrm{Map}(BG, \mathrm{Map}(BSO(2), X))$ or the set of equivalence classes of functors $\pi_0 \mathrm{Grp}_2(BG, \mathcal{G}_{\mathrm{ori}})$.

Definition 3.4.2. A G-graded algebra $B = \bigoplus_g B_g$ is said to be *strongly graded* if the multiplication factors through an isomorphism

$$\begin{array}{ccc} B_g \otimes B_{g'} & \longrightarrow & B_{gg'} \\ & \searrow \qquad \nearrow_{\cong} & \\ & B_g \otimes_{B_e} B_{g'} & \end{array}$$

Definition 3.4.3. A G-*equivariant algebra* is a pair (B, τ) consisting of a finite dimensional strongly G-graded k-algebra $B = \bigoplus_g B_g$ whose trivial component B_e is semi-simple, and a non-degenerate cyclically symmetric form $\tau : B \otimes B \to k$ such that $\tau|_{B_g \otimes B_{g'}} \equiv 0$ whenever $gg' \neq e$.

The form $\tau : B \otimes B \to k$ is completely determined by its restriction $\tau_e := \tau|_{B_e \otimes B_e} : B_e \otimes B_e \to k$. On the other hand, any non-degenerate cyclically symmetric form $\tau_e : B_e \otimes B_e \to k$ can be uniquely extended to a non-degenerate cyclically symmetric form $\tau : B \otimes B \to k$ such that $\tau|_{B_g \otimes B_{g'}} \equiv 0$ whenever $gg' \neq e$.

Lemma 3.4.4. *Let $B = \bigoplus_g B_g$ be a finite dimensional strongly G-graded k-algebra whose trivial component B_e is semi-simple. There is a canonical bijection between $Z(B_e)^\times$ and the set of non-degenerate cyclically symmetric forms $\tau : B \otimes B \to k$ such that $\tau|_{B_g \otimes B_{g'}} \equiv 0$ whenever $gg' \neq e$. It is given by,*

$$f \in Z(B_e)^\times \longmapsto f \cdot \tau$$

where τ is the unique extension of the canonical trace form $\tau_{B_e} : B_e \otimes B_e \to k$ (see definition A.1.7).

Proof. Having an element in $Z(A)^\times$ is the same as having an isomorphism $A \to A$ of (A,A)–bimodules. Since A is semisimple, it is isomorphic, as a (A,A)–bimodule, to its dual $A^* = \mathrm{Hom}_k(A,k)$. The isomorphism is given by $a \in A \mapsto \tau_A(a,-) \in A^*$ where τ_A is the canonical trace form on A. Having an isomorphism $A \to A$ of (A,A)–bimodules we can compose it with $\tau(-,-)$ to get an isomorphism $A \to A^*$ of (A,A)–bimodules. Having the latter isomorphism is the same as having a bilinear form $\tau : A \otimes A \to k$. It is non-degenerate and cyclically symmetric. Non-degeneracy corresponds to $A \to A^*$ being an isomorphism. Cyclic symmetry of τ corresponds to $A \to A^*$ being a morphism of (A,A)–bimodules.

Having an element $f \in Z(B_e)^\times$ therefore gives rise to a non-degenerate cyclically symmetric form $\tau : B_e \otimes B_e \to k$. This form can be uniquely extended to a non-degenerate cyclically symmetric form on B such that $\tau|_{B_g \otimes B_{g'}} \equiv 0$ whenever $gg' \neq e$. □

Proposition 3.4.5. *A functor $F : BG \to \mathcal{G}_{\mathrm{ori}}$ gives rise to a G-equivariant algebra (B^F, τ^F), and vice versa, a G-equivariant algebra (B,τ) gives rise to a functor $F^{(B,\tau)} : BG \to \mathcal{G}_{\mathrm{ori}}$.*

Proof. A functor $F : BG \to \mathcal{G}_{\mathrm{ori}}$ consists of a choice of an object $(A,f) \in \mathcal{G}_{\mathrm{ori}}$ and a choice of a (strong) monoidal functor $BG(\bullet,\bullet) \xrightarrow{\otimes} \mathcal{G}_{\mathrm{ori}}((A,f),(A,f))$. This (strong) monoidal functor consists of:

(i) A finite dimensional invertible (A,A)-bimodule B_g for every $g \in G$,

(ii) An isomorphism of (A,A)-bimodules $\phi(g,g') : B_{g'} \otimes_A B_g \to B_{gg'}$ for every pair $g, g' \in G$,

(iii) An isomorphism of (A,A)-bimodules $\phi : {}_A A_A \to B_e$.

such that the following diagrams commute,

$$\begin{array}{ccccc} (B_h \otimes_A B_g) \otimes_A B_f & \xrightarrow{\phi(g,h)\otimes 1} & B_{gh} \otimes_A B_f & \xrightarrow{\phi(f,gh)} & B_{fgh} \\ {\scriptstyle\cong}\downarrow & & & & \| \\ B_h \otimes_A (B_g \otimes_A B_f) & \xrightarrow[1\otimes\phi(f,g)]{} & B_h \otimes_A B_{fg} & \xrightarrow[\phi(fg,h)]{} & B_{fgh} \end{array} \tag{3.4.1}$$

$$\begin{array}{ccccc} B_g \otimes_A A & \xrightarrow{1\otimes\phi} & B_g \otimes_A B_e & \xrightarrow{\phi(e,g)} & B_g \\ {\scriptstyle\cong}\downarrow & & & & \| \\ B_g & = & = & = & B_g \end{array} \qquad \begin{array}{ccccc} A \otimes_A B_g & \xrightarrow{\phi\otimes 1} & B_e \otimes_A B_g & \xrightarrow{\phi(g,e)} & B_g \\ {\scriptstyle\cong}\downarrow & & & & \| \\ B_g & = & = & = & B_g \end{array} \tag{3.4.2}$$

Up to equivalence we may assume $B_e \equiv A$ and $\phi = 1$. We will assume this from now on.

By the above, a functor $F : BG \to \mathcal{G}_{\mathrm{ori}}$ gives rise to a finite dimensional strongly G-graded algebra $B^F = \bigoplus_g B_g$ whose trivial component B_e is semi-simple. Associativity of multiplication and existence of unit are given in (3.4.1) and (3.4.2).

We have an additional piece of information, namely, $f \in Z(B_e)^\times$. By Lemma 3.4.4 it corresponds to a unique non-degenerate cyclically symmetric form τ^F making (B^F, τ^F) a G-equivariant algebra.

It is clear how to go the other way around. To construct the functor $F^{(B,\tau)}$ from a G-equivariant algebra (B,τ), we set $F^{(B,\tau)}(\bullet) = (B_e, f)$, where $f \in Z(B_e)^\times$ is such that $\tau_e = f\tau_{B_e}$ (see Lemma 3.4.4), and $F^{(B,\tau)}(\bullet \xrightarrow{g} \bullet) = B_g$.

Multiplication $B_{g'} \otimes_A B_g \xrightarrow{\cong} B_{gg'}$ provides the other piece of data making $F^{(B,\tau)}$ a strong monoidal functor. The commutativity of (3.4.1) and (3.4.2) is the consequence of associativity of multiplication and existence of unit in B. □

Let $(B, \tau) \equiv (B^F, \tau^F)$ be a G-equivariant algebra corresponding to a functor $F : BG \to \mathcal{G}_{\text{ori}}$, and let $f \in Z(B_e)^\times$ be such that $\tau_e = f\tau_{B_e}$ (see Lemma 3.4.4). Recall B_e is a finite dimensional semi-simple algebra

$$B_e \cong A \equiv \text{End}(V_1) \times \cdots \times \text{End}(V_r)$$

Each B_g corresponds to a unique (A, A)-bimodule of the form $M^{\sigma(g)}$ for some $\sigma(g) \in \Sigma_r$. Up to equivalence of functors, we may assume $\sigma : G \to \Sigma_r$ is a homomorphism of groups.

The invertible bimodules B_g must satisfy a compatibility condition with respect to $f \in Z(B_e)^\times$. This compatibility condition is equivalent to $\operatorname{Im} \sigma \subseteq \text{Stab}_{\Sigma_r}(f)$; the stabilizer of $f \in (k^\times)^r$ in Σ_r with respect to the action of Σ_r on $(k^\times)^r$ by permutation. We can view ϕ as a 2-cochain $\phi : G \times G \to (k^\times)^r$. The commutative diagrams in (3.4.1) imply ϕ is a 2-cocycle normalized so that $\phi(e, g) = 1 = \phi(g, e)$.

On the other hand, given a finite dimensional semi-simple algebra A with Morita class r, an element $f \in Z(A)^\times$, a homomorphism $\sigma : G \to \text{Stab}_{\Sigma_r}(f)$ and a normalized 2-cocycle ϕ, one can construct a G-equivariant algebra, equivalently a functor $F : BG \to \mathcal{G}_{\text{ori}}$.

Let $F_i : BG \to \mathcal{G}_{\text{ori}}$, $i = 1, 2$, be two functors with $F_i(\bullet) = (A_i, f_i)$. Let $\sigma_i(-) \in \text{Hom}(G, \text{St}_{\Sigma_r}(f_i))$ be the corresponding homomorphisms such that

$B_g^{F_i} \equiv M^{\sigma_i(g)}$. Let ϕ_i be the corresponding normalized 2-cocycles. A natural isomorphism

$$BG \underset{F_2}{\overset{F_1}{\rightrightarrows}} \mathcal{G}_{\mathrm{ori}}, \qquad \eta : F_1 \overset{\cong}{\Rightarrow} F_2$$

constitutes a choice of an invertible (A_1, A_2)-bimodule M, compatible with the left f_1 and right f_2 actions, plus a choice of a collection of isomorphisms,

$$\eta(g) : M \otimes_{A_2} B_g^{F_2} \to B_g^{F_1} \otimes_{A_1} M$$

such that the following diagrams commute,

$$\begin{array}{ccccc} M \otimes_{A_2} B_g^{F_2} \otimes_{A_2} B_f^{F_2} & \xrightarrow{\eta(g)\otimes 1} & B_g^{F_1} \otimes_{A_1} M \otimes_{A_2} B_f^{F_2} & \xrightarrow{1\otimes\eta(f)} & B_g^{F_1} \otimes_{A_1} B_f^{F_1} \otimes_{A_1} M \\ \downarrow{\scriptstyle 1\otimes\phi_2(f,g)} & & & & \downarrow{\scriptstyle \phi_1(f,g)\otimes 1} \\ M \otimes_{A_2} B_{fg}^{F_2} & & \xrightarrow{\eta(fg)} & & B_{fg}^{F_1} \otimes_{A_1} M \end{array} \tag{3.4.3}$$

$$\begin{array}{ccccc} M \otimes_{A_2} A_2 & \xrightarrow{\cong} & M & \xrightarrow{\cong} & A_1 \otimes_{A_1} M \\ \downarrow{\scriptstyle 1\otimes\phi_2} & & & & \downarrow{\scriptstyle \phi_1\otimes 1} \\ M \otimes_{A_2} B_e^{F_2} & & \xrightarrow{\eta(e)} & & B_e^{F_1} \otimes_{A_1} M \end{array}$$

Up to equivalence we may identify $M \equiv M^\sigma$ for some $\sigma \in \Sigma_r$ such that $\sigma \cdot f_1 = f_2$. We may also replace $\eta(g)$ by an isomorphism

$$\eta(g) : M^\sigma \otimes_{A_2} M^{\sigma_2(g)} \otimes_{A_2} M^{\sigma^{-1}} \to M^{\sigma_1(g)}$$

This implies $\sigma_1(-) \sim \sigma_2(-)$ are conjugate via σ. It also allows us to interpret η as a normalized 1-cochain $\eta : G \to (k^\times)^r$. Diagram (3.4.3) informs us that

ϕ_1 and ϕ_2 differ by a coboundary. This discussion gives rise to the following proposition,

Proposition 3.4.6. *The set of equivalence classes of fully extended* $2-$ *dimensional G-equivariant TFT's with values in* Alg_k *is in bijection with*

$$\pi_0 \,\mathrm{Fun}^{\otimes}(\mathrm{Bord}_2^{\Gamma}, \mathrm{Alg}_k) \cong \coprod_{r=1}^{\infty} \coprod_{[f]\in(k^\times)^r/\Sigma_r} H^2(G; k^\times)^r \times \mathrm{Hom}(G, \mathrm{Stab}_{\Sigma_r}(f))/\sim$$

where equivalence $\sim$ *is given by conjugation i elements of* $\mathrm{Stab}_{\Sigma_r}(f)$.

3.5 Duality Data.

Fix a G-equivariant algebra (B,τ). Let $\mathcal{Z}_{(B,\tau)}$ denote the fully extended 2-dimensional G-equivariant TFT with values in Alg_k corresponding to (B,τ). Denote $A \equiv B_e$ and $A^e = A \otimes A^o$.

Let $\mathrm{pt}_+ \xrightarrow{g} \mathrm{pt}_+$ denote the oriented interval, considered as a 1-bordism from pt_+ to itself, equipped with a principal G-bundle, which is trivialized along the boundary, and has holonomy g. Let $\mathrm{pt}_+ \sqcup \mathrm{pt}_- \xrightarrow{g} \varnothing^0$ denote the same oriented interval, considered as a 1-bordism from $\mathrm{pt}_+ \sqcup \mathrm{pt}_-$ to $\varnothing^0$, and equipped with the same principal G-bundle. Let $\varnothing^0 \xrightarrow{g} \mathrm{pt}_- \sqcup \mathrm{pt}_+$ denote the same oriented interval, considered as a 1-bordism from $\varnothing^0$ to $\mathrm{pt}_- \sqcup \mathrm{pt}_+$, and equipped with the same principal G-bundle. Duality data for pt_+ is given by,

$$(\mathrm{pt}_-, \underbrace{\mathrm{pt}_+ \sqcup \mathrm{pt}_- \xrightarrow{g} \varnothing^0}_{\mathrm{ev}_+(g)}, \underbrace{\varnothing^0 \xrightarrow{g^{-1}} \mathrm{pt}_- \sqcup \mathrm{pt}_+}_{\mathrm{coev}_+(g)})$$

We have,

$$\mathcal{Z}_{(B,\tau)}(\mathrm{pt}_+) = A \quad , \quad \mathcal{Z}_{(B,\tau)}(\mathrm{pt}_-) = A^o$$

$$\mathcal{Z}_{(B,\tau)}(\mathrm{pt}_+ \xrightarrow{g} \mathrm{pt}_+) = B_g \quad , \quad (A, A) - \text{bimodule}$$

$$\mathcal{Z}_{(B,\tau)}(\mathrm{pt}_+ \sqcup \mathrm{pt}_- \xrightarrow{g} \varnothing^0) = B_g \quad , \quad (A \otimes A^o, k) - \text{bimodule}$$

$$\mathcal{Z}_{(B,\tau)}(\varnothing^0 \xrightarrow{g} \mathrm{pt}_- \sqcup \mathrm{pt}_+) = B_g \quad , \quad (k, A^o \otimes A) - \text{bimodule}$$

Recall that the restriction $\tau|_{B_g \otimes B_{g^{-1}}}$ is non-degenerate. Let $\{x_i^g\}$ be a basis for B_g with dual basis $\{y_i^g\}$ in $B_{g^{-1}}$. The element $b_g = \sum_i x_i^g \otimes y_i^g \in B_g \otimes B_{g^{-1}}$ is independent of the choice of basis. We have $b_{g^{-1}} = \sum_i y_i^g \otimes x_i^g \in B_{g^{-1}} \otimes_k B_g$. By abuse of notation, we denote the image of b_g in $B_g \otimes_{A^e} B_{g^{-1}}$ by the same symbol.

Lemma 3.5.1. *Consider B_g a (A^e, k)-bimodule. Its left and right adjoints are given by*

$$B_g^L = B_{g^{-1}} = B_g^R$$

considered as (k, A^e)-bimodules. The units and counits for adjunctions are given by,

$$\begin{aligned} u_g^R :\ & A^e \longrightarrow B_g \otimes_k B_{g^{-1}} & v_g^R :\ & B_{g^{-1}} \otimes_{A^e} B_g \longrightarrow k \\ & 1 \otimes 1 \longmapsto b_g & & b' \otimes b \longmapsto \tau(b, b') \\ u_g^L :\ & k \longrightarrow B_{g^{-1}} \otimes_{A^e} B_g & v_g^L :\ & B_g \otimes_k B_{g^{-1}} \longrightarrow A^e \\ & 1 \longmapsto b_{g^{-1}} & & b \otimes b' \longmapsto bb' \otimes 1 \end{aligned}$$

Proof. We need to check the following compositions are identities.

$$B_g \cong A^e \otimes_{A^e} B_g \xrightarrow{u^R \otimes 1} B_g \otimes_k B_{g^{-1}} \otimes_{A^e} B_g \xrightarrow{1 \otimes v^R} B_g \otimes_k k \cong B_g$$

$$B_{g^{-1}} \cong B_{g^{-1}} \otimes_{A^e} A^e \xrightarrow{1 \otimes u^R} B_{g^{-1}} \otimes_{A^e} B_g \otimes_k B_{g^{-1}} \xrightarrow{v^R \otimes 1} k \otimes_k B_{g^{-1}} \cong B_{g^{-1}}$$

$$B_g \cong B_g \otimes_k k \xrightarrow{1\otimes u^L} B_g \otimes_k B_{g^{-1}} \otimes_{A^e} B_g \xrightarrow{v^L\otimes 1} A^e \otimes_{A^e} B_g \cong B_g$$

$$B_{g^{-1}} \cong k \otimes_k B_{g^{-1}} \xrightarrow{u^L\otimes 1} B_{g^{-1}} \otimes_{A^e} B_g \otimes_k B_{g^{-1}} \xrightarrow{1\otimes v^L} B_{g^{-1}} \otimes_{A^e} A^e \cong B_{g^{-1}}$$

By definition, $b = \sum_i x_i^g \tau(b, y_i^g)$ for every $b \in B_g$ and $b' = \sum_i \tau(x_i^g, b') y_i^g$, which ensures the first couple of compositions are identities. In addition, $\sum_i x_i^g y_i^g = 1 = \sum_i y_i^g x_i^g$ which ensures the second couple of compositions are identities. □

Let $\circlearrowleft_g$ denote the oriented circle equipped with a principal G-bundle with holonomy g. We can factor,

$$\begin{aligned} \circlearrowleft_g &= (\mathrm{pt}_+ \sqcup \mathrm{pt}_- \xrightarrow{g} \varnothing^0) \circ (\varnothing^0 \xrightarrow{e} \mathrm{pt}_+ \sqcup \mathrm{pt}_-) \\ &= (\mathrm{ev}_+(e))^L \circ (\mathrm{ev}_+(g)) \end{aligned}$$

We then have,

$$\mathcal{Z}_{(B,\tau)}(\circlearrowleft_g) \cong B_g \otimes_{A^e} A \cong HH_0(A; B_g)$$

3.6 Turaev's G-equivaraiant Theories.

In [17], Turaev introduces the notion of a bi-angular finite dimensional G-graded algebra B and uses it to construct a non-extended 2-dimensional G-equivariant TFT $\mathcal{Z}_B^{\mathrm{Tur}}$ with values in Vect_k.

Let us fix a G-equivariant algebra (B, τ) for which $\tau_e \equiv \tau|_{B_e \otimes B_e}$ is the canonical trace form on B_e. In Proposition A.2.4 we show this is equivalent to fixing a biangular finite dimensional G-graded algebra B (see Definition A.2.1).

Proposition 3.6.1. *Let $\mathcal{Z}^{(1,2)}_{(B,\tau)}$ denote the 2-tier truncation of the fully extended 2-dimensional G-equivariant TFT corresponding to (B,τ). Then,*

$$\mathcal{Z}^{\mathrm{Tur}}_B \simeq \mathcal{Z}^{(1,2)}_{(B,\tau)}$$

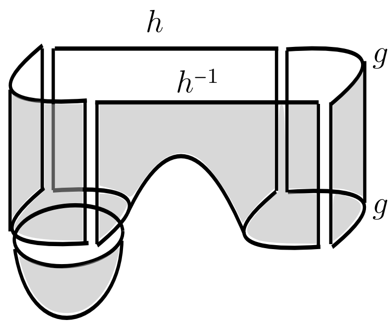
Proof. In [17], Turaev proves that to give a non-extended 2-dimensional G-equivariant TFT with values in Vect_k is to give a *crossed G-algebra*. The underlying crossed G-algebra of $\mathcal{Z}^{\mathrm{Tur}}_B$ is shown to be the G-center of B (see Definition A.2.5). To establish the proposition, we need to show the G-center L of B is equivalent to the underlying crossed G-algebra L' of $\mathcal{Z}^{(1,2)}_{(B,\tau)}$.

By definition, the underlying crossed G-algebra of a non-extended 2-dimensional G-equivariant TFT $\mathcal{Z}$ with values in Vect_k is the G-graded vector space $L' = \bigoplus_{g\in G} L'_g$ where $L'_g = \mathcal{Z}(\circlearrowleft_g)$ with multiplication given by the value of $\mathcal{Z}$ on the pair of pants. Hence, the underlying crossed G-algebra of $\mathcal{Z}_{(B,\tau)}$ is $L' \cong \bigoplus_{g\in G} HH_0(A;B_g)$. For any (A,A)-bimodule M we have,

$$HH_0(A;M) \cong HH^0(A;M^*)$$

Hence $L' \cong \bigoplus_{g\in G} HH^0(A;B_{g^{-1}})$. In Lemma A.2.6 we show $L_g = HH^0(B_e;B_g)$ hence $L'_g \cong L_{g^{-1}}$. The multiplication in L' defined via the pair of pants is induced by the multiplication in B. In other words, $L \cong L'$ as graded G-algebras.

An additional part of the data of a crossed G-algebra consists of an action $G \to \mathrm{Aut}(L')$. This action is given by the following 2-bordism,



It translates to the composition,

$$\begin{aligned} L'_g &\cong A \otimes_{A^e} B_g \\ &\cong k \otimes_k (A \otimes_{A^e} B_g) \\ &\to (A \otimes_{A^e} A) \otimes_k (A \otimes_{A^e} B_g) \\ &\cong A \otimes_{A^e} (A \otimes_k A) \otimes_{A^e} B_g \\ &\to A \otimes_{A^e} (B_h \otimes B_{h^{-1}}) \otimes_{A^e} B_g \\ &\cong A \otimes_{A^e} (B_h \otimes_A B_g \otimes_A B_{h^{-1}}) \\ &\to A \otimes_{A^e} B_{h^{-1}gh} \\ &\cong L'_{h^{-1}gh} \end{aligned}$$

$$\begin{aligned} 1 \otimes b &\to (1 \otimes 1) \otimes (1 \otimes b) \\ &\to \sum_i 1 \otimes x_i^h \otimes y_i^h \otimes b \\ &\to 1 \otimes \sum_i x_i^h b\, y_i^h \end{aligned}$$

As a consequence, the isomorphism $L \cong L'$ respects the G-action on $\mathrm{Aut}(L) \cong \mathrm{Aut}(L')$. □

Chapter 4

State Sums

A state sum is an expression computing the value of a fully extended n-dimensional TFT on a closed n-manifold from a triangulation of the manifold. In this section we derive a state sum formula for fully extended 2-dimensional G-equivariant TFT's with values in Alg_k. This formula coincides with the one introduced by Turaev in [17] for the purpose of defining X-HQFT's (Homotopy Quantum Field Theories) for $X = BG$.

Fix a G-equivariant algebra (B, τ). Let $\mathcal{Z}_{(B,\tau)}$ denote the fully extended 2-dimensional G-equivariant TFT with values in Alg_k corresponding to (B, τ). Let W be a closed oriented surface equipped with a principal G-bundle $p : E \to W$. Let T be a triangulation of W. We assume $E_v = p^{-1}(v)$ is trivialized for every $v \in V(T)$.

We wish to compute $\mathcal{Z}_{(B,\tau)}(W, p)$. For that purpose, we use the factorization of W derived from the blow up T^{bl} in Theorem 2.3.1,

$$W = \underbrace{(\coprod_f Z_f)}_{W_3} \circ_v \underbrace{((\coprod_e Z_e) \circ_h (\coprod_{v \in f} Z_{v,f}))}_{W_2} \circ_v \underbrace{(\coprod_v Z_v)}_{W_1}$$

We denote $\Psi_i = \mathcal{Z}_{(B,\tau)}(W_i, p|_{W_i})$, in other words, $\mathcal{Z}_{(B,\tau)}(W, p) = \Psi_3 \circ_h \Psi_2 \circ_h \Psi_1$.

We pick a convention for the orientation of edges. The edges $Z_v \cap Z_e$ and $Z_v \cap Z_{v,f}$ are oriented such that they lie to the right of the faces Z_v, and the edges $Z_f \cap Z_e$ and $Z_f \cap Z_{v,f}$ are oriented such that they lie to the right of the faces Z_f. Note that $Z_v \cap Z_f = \varnothing$, which makes this choice of orientation on edges well-defined. The trivialization of p on vertices allows us to trivialize $p|_{Z_v}$ and $p|_{Z_{v,f}}$.

We start with Ψ_1. It is given by,

$$\Psi_1 = \bigotimes_v \mathcal{Z}_{(B,\tau)}(Z_v, p|_{Z_v})$$

where,

$$Z_v : \varnothing^1 \to (\coprod_{v \in f} Z_v \cap Z_{v,f}) \circ_h (\coprod_e Z_v \cap Z_e)$$

therefore,

$$\mathcal{Z}_{(B,\tau)}(Z_v, p|_{Z_v}) : k \to \mathcal{Z}_{(B,\tau)}(\coprod_e Z_v \cap Z_e, p) \bigotimes_{\otimes_{i=1}^{d_v} A^e} \mathcal{Z}_{(B,\tau)}(\coprod_f Z_v \cap Z_{f,v}, p)$$

where $d_v = \deg_T(v)$, and

$$\mathcal{Z}_{(B,\tau)}(\coprod_e P_v \cap P_e) = \bigotimes_{e, v \subset e} A \quad , \quad A \text{ as a } (k, A^e)\text{-bimodule}$$
$$\mathcal{Z}_{(B,\tau)}(\coprod_f P_v \cap P_{v,f}) = \bigotimes_{f, v \subset f} A \quad , \quad A \text{ as a } (A^e, k)\text{-bimodule}$$

Lemma 4.0.2. *Fix a face Z_v of T^{bl}. Then,*

$$\mathcal{Z}_{(B,\tau)}(Z_v, p|_{Z_v}) : 1 \longmapsto (\otimes_{v \subset e} 1) \otimes (\otimes_{v \subset f} 1)$$

Proof. The edges of Z_v in the polygonal decomposition T^{bl} are cyclically ordered $(e_1', e_1'', \dots, e_d', e_d'')$ where $d = \deg_T(v)$, and $e_i' = Z_{e_i} \cap Z_v$, $e_i'' = Z_{v,f_i} \cap Z_v$.

If the cyclic order does not match the induced orientation on ∂Z_v flip it. We can factor,

$$\begin{array}{ccc} k & \xrightarrow{\mathcal{Z}_{(B,\tau)}(Z_v, p|_{Z_v})} & \\ {\scriptstyle u_e^L}\downarrow & \searrow & \\ A \otimes_{A^e} A & \xrightarrow[\cong]{} & \left(A_{e'_1} \otimes \ldots \otimes A_{e'_d}\right) \bigotimes_{\underbrace{A^e \otimes \ldots \otimes A^e}_{d}} \left(A_{e''_1} \otimes \ldots \otimes A_{e''_d}\right) \end{array}$$

The horizontal isomorphism is given by repeated applications of 'S-move' isomorphisms. As such, it maps,

$$1 \otimes 1 \longmapsto (\underbrace{1 \otimes \ldots \otimes 1}_{d}) \bigotimes (\underbrace{1 \otimes \ldots \otimes 1}_{d})$$

The intertwiner u_e^L is given by,

$$1 \longmapsto 1 \otimes 1$$

as in Lemma 3.5.1. □

The intertwiner Ψ_3 is given by,

$$\Psi_3 = \bigotimes_f \mathcal{Z}_{(B,\tau)}(Z_f, p|_{Z_f})$$

where,

$$Z_f : (\coprod_{v \in f} Z_f \cap Z_{v,f}) \circ_h (\coprod_{e \in f} Z_f \cap Z_e) \to \varnothing^1$$

therefore,

$$\mathcal{Z}_{(B,\tau)}(Z_f, p|_{Z_f}) : \mathcal{Z}_{(B,\tau)}(\textstyle\coprod_{e \in f} Z_f \cap Z_e, p) \displaystyle\bigotimes_{\otimes_{i=1}^{d_f} A^e} \mathcal{Z}_{(B,\tau)}(\textstyle\coprod_{v \in f} Z_f \cap Z_{v,f}, p) \to k$$

where $d_f = \deg_T(f)$. The holonomy along $Z_f \cap Z_e$ is possibly nontrivial. We denote it by $g(f,e)$.

$$\mathcal{Z}_{(B,\tau)}(\textstyle\coprod_{e\in f} Z_f \cap Z_e, p) = \otimes_{e\subset f} B_{g(f,e)} \quad , \quad B_{g(f,e)} \text{ as a } (k, A^e)\text{-bimodule}$$
$$\mathcal{Z}_{(B,\tau)}(\textstyle\coprod_{v\in f} Z_f \cap Z_{v,f}) = \otimes_{v\subset f} A \quad , \quad A \text{ as a } (A^e, k)\text{-bimodule}$$

Lemma 4.0.3. *Fix a face Z_f of T^{bl}. Its edges are endowed with cyclic order $(e_1', e_1'', \ldots, e_d', e_d'')$, where $d = \deg_T(f)$ and $e_i' = Z_{e_i} \cap Z_f$, $e_i'' = Z_{v_i,f} \cap Z_f$. If the cyclic order does not match the induced orientation on ∂Z_f flip it. Then,*

$$\mathcal{Z}_{(B,\tau)}(Z_f, p|_{Z_f}) : (x_1 \otimes \cdots \otimes x_d) \bigotimes (1 \otimes \cdots \otimes 1) \longmapsto \tau(x_1 \cdot \ldots \cdot x_d, 1)$$

where $x_i \in B_{g(f,e_i)}$.

Proof. Since $g(f,e_1) \cdot \ldots \cdot g(f,e_d) = e$, we can factor,

$$\begin{array}{ccc} \left(B_{g(f,e_1)} \otimes \ldots \otimes B_{g(f,e_d)}\right) \bigotimes_{A^e \otimes \ldots \otimes A^e} \left(A_{e_1''} \otimes \ldots \otimes A_{e_d''}\right) & & \\ \downarrow \cong & \searrow^{\mathcal{Z}_{(B,\tau)}(Z_f, p|_{Z_f})} & \\ A \otimes_{A^e} A & \xrightarrow[v_e^R]{} & k \end{array}$$

The vertical arrow is given by repeated application of 'S-move' isomorphisms:

$$(x_1 \otimes \ldots \otimes x_d) \otimes (a_1'' \otimes \ldots \otimes a_d'') \longmapsto x_1 a_1' \cdots x_d a_d' \otimes 1$$

The intertwiner v_e^R is given in Lemma 3.5.1. □

The intertwiner Ψ_2 is given by,

$$\Psi_2 = (\otimes_e \mathcal{Z}_{(B,\tau)}(Z_e, p|_{Z_e})) \bigotimes (\otimes_{v,f} \mathcal{Z}_{(B,\tau)}(Z_{v,f}, p|_{Z_{v,f}})$$

where,

$$Z_{v,f} : Z_v \cap Z_{v,f} \to Z_f \cap Z_{v,f}$$

therefore,

$$\mathcal{Z}_{(B,\tau)}(Z_{v,f}, p|_{Z_{v,f}}) : \mathcal{Z}_{(B,\tau)}(Z_v \cap Z_{v,f}, p|_{Z_v \cap Z_{v,f}}) \to \mathcal{Z}_{(B,\tau)}(Z_f \cap Z_{v,f}, p|_{Z_f \cap Z_{v,f}})$$

It is the identity on $\mathcal{Z}_{(B,\tau)}(Z_v \cap Z_{v,f}, p) = A = \mathcal{Z}_{(B,\tau)}(Z_f \cap Z_{v,f}, p)$ considered as a (A^e, k)-bimodule. Also,

$$Z_e : \coprod_{v \in e} Z_e \cap Z_v \to \coprod_{e \in f} Z_e \cap Z_f$$

therefore,

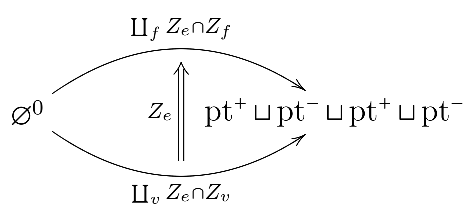
$$\mathcal{Z}_{(B,\tau)}(Z_e, p|_{Z_e}) : \mathcal{Z}_{(B,\tau)}(\textstyle\coprod_v Z_e \cap Z_v, p) \longrightarrow \mathcal{Z}_{(B,\tau)}(\textstyle\coprod_f Z_e \cap Z_f, p)$$

Lemma 4.0.4. *Fix a face Z_e of T^{bl}. Then,*

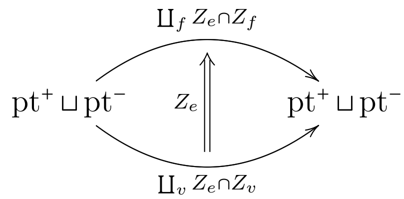
$$\mathcal{Z}_{(B,\tau)}(Z_e, p|_{Z_e}) : 1 \otimes 1 \longmapsto b_e$$

where b_e denotes the canonical element in $\otimes_f B_{g(f,e)}$.

Proof. Let us take a closer look at the 2-bordism Z_e. We understand it as a 2-morphism in the following way.



Let us change incoming and outgoing 0-boundary,



in such a way that $\coprod_v Z_e \cap Z_v$ is the identity on $\mathrm{pt}^+ \sqcup \mathrm{pt}^-$. This makes Z_e the unit of an adjunction between $Z_e \cap Z_f : \mathrm{pt}^+ \sqcup \mathrm{pt}^- \to \varnothing^0$ and $Z_e \cap Z_{f'} : \varnothing^0 \to \mathrm{pt}^+ \sqcup \mathrm{pt}^-$. The proof now follows from Lemma 3.5.1. □

Corollary 4.0.5. *Let W be a closed oriented surface equipped with a principal G-bundle $p : E \to W$, and a triangulation T, with faces, edges and vertices f, e and v respectively. Then,*

$$\mathcal{Z}_{(B,\tau)}(W,p) = (\otimes_f \psi_f)((\otimes_e b_e) \bigotimes (\otimes_{v \in f} 1)) \tag{4.0.1}$$

where $\psi_f = \mathcal{Z}_{(B,\tau)}(Z_f, p|_{Z_f})$, and $b_e \in \otimes_f B_{g(f,e)}$ is the canonical element, where $g(f,e)$ is the holonomy of p along e oriented so that it lies to the left of f.

In [17], Turaev defines a non-extended 2-dimensional G-equivarient TFT $\mathcal{Z}_B^{\mathrm{Tur}}$ from a biangular finite dimensional G-graded algebra. The invariant $\mathcal{Z}_B^{\mathrm{Tur}}(W,p)$ associated to a closed oriented surface W, equipped with a principal G-bundle $p : E \to W$, and a triangulation T, is defined in the following way.

Each pair (f,e), where e is a side of f, endows e with an orientation such that it lies to the right of f. Let $g(f,e) \in G$ denote the holonomy along the directed edge e and denote $B(f,e) = B_{g(f,e)}$. For any edge e we have two pairs (f_1, e),(f_2, e). There is a canonical element $b_e \in B(f_1, e) \otimes B(f_2, e)$ (see Appendix A.2).

Fix a face f and consider a cyclic ordering of its sides $e_1, e_2, \ldots, e_{d_f}$ which is compatible with the induced orientation. Clearly $g(f, e_1) \cdot \ldots \cdot g(f, e_{d_f}) = e$. Define a form,

$$\phi_f : B(f, e_1) \otimes \ldots \otimes B(f, e_n) \longrightarrow k$$

$$\phi_f : x_1 \otimes \ldots \otimes x_{d_f} \longmapsto \eta(x_1 \cdot \ldots \cdot x_{d_f}, 1)$$

Then,

$$\mathcal{Z}_B^{\mathrm{Tur}}(W, p) \stackrel{\mathrm{def}}{=} (\otimes_f \phi_f)(\otimes_e b_e)$$

By equation (4.0.1),

$$\mathcal{Z}_B^{\mathrm{Tur}}(W, p) = \mathcal{Z}_{(B,\tau)}(W, p)$$

where τ is the unique extension of the canonical trace form on B_e.

Appendices

Appendix A

Algebras.

All algebras in this appendix defined over a field k are assumed associative and unital. All k-vector spaces are finite dimensional. Let $\mathrm{End}_k(V)$ denote the endomorphism algebra of V.

A.1 Modules over Endomorphism Algebras.

Let V denote a finite dimensional k-vector space. For $A = \mathrm{End}_k(V)$ it is well-known that the category ${}_A\mathrm{Mod}$ of left A-modules is equivalent to Vect_k and every left A-modules is a multiple of $\mathrm{End}(k, V) \cong V$. A similar statement holds for the category Mod_A of right A-modules with $\mathrm{End}(V, k)$ as a generator. We let ${}_A\mathrm{Mod}_B$ denote the category of finite-dimensional (A, B)−bimodules.

Lemma A.1.1. *Let* $A_i = \mathrm{End}_k(V_i)$*,* $i = 1, 2$*. Then,*

$${}_{A_1}\mathrm{Mod}_{A_2} \simeq \mathrm{Vect}_k$$

as abelian categories. In particular, every (A_1, A_2)*-bimodule is isomorphic to a multiple of* $\mathrm{Hom}_k(V_1, V_2)$*.*

Proof. Define additive functors,

$$M \in_{A_1} \mathrm{Mod}_{A_2} \longmapsto \mathrm{Hom}_k(V_1, k) \otimes_{A_1} M \otimes_{A_2} \mathrm{Hom}_k(k, V_2) \in \mathrm{Vect}_k$$

$$V \in \mathrm{Vect}_k \longmapsto \mathrm{Hom}_k(k, V_1) \otimes_k V \otimes_k \mathrm{Hom}_k(V_2, k) \in {}_{A_1}\mathrm{Mod}_{A_2}$$

These define an equivalence between the two categories since,

$$\mathrm{Hom}_k(k, V_1) \otimes_k \mathrm{Hom}_k(V_1, k) \simeq A_1 \quad \text{as right } A_1\text{-modules}$$

$$\mathrm{Hom}_k(k, V_2) \otimes_k \mathrm{Hom}_k(V_2, k) \simeq A_2 \quad \text{as left } A_2\text{-modules}$$

$$\mathrm{Hom}_k(V_i, k) \otimes_{A_i} \mathrm{Hom}_k(k, V_i) \simeq k$$

The equivalence shows every bimodule is a multiple of

$$\mathrm{Hom}_k(k, V_1) \otimes_k \mathrm{Hom}_k(V_2, k) \simeq \mathrm{Hom}_k(V_1, V_2)$$

□

Lemma A.1.2. *Let $A = A_1 \times \cdots \times A_r$ and $B = B_1 \times \cdots \times B_s$ where A_i and B_j are matrix algebras. Then,*

$${}_A\mathrm{Mod}_B \simeq \prod_{i,j} {}_{A_i}\mathrm{Mod}_{B_j}$$

as abelian categories. In particular, the full subcategory of finitely generated (A, B)–bimodules is semi-simple.

Proof. Consider A_i as a (two-sided) ideal of A and B_j as a (two-sided) ideal of B. Define additive functors,

$$M \in {}_A\mathrm{Mod}_B \longmapsto (A_i \otimes_A M \otimes_B B_j)_{i,j} \in \prod_{i,j} {}_{A_i}\mathrm{Mod}_{B_j}$$

$$(M_{i,j}) \in \prod_{i,j} {}_{A_i}\mathrm{Mod}_{B_j} \longmapsto \bigoplus_{i,j} A_i \otimes_{A_i} M_{i,j} \otimes_{B_j} B_j$$

These define an equivalence of abelian categories. By Lemma A.1.1, ${}_A\mathrm{Mod}_B$ is a finite product of Vect_k's. Its subcategory of finitely generated bimodules is equivalent to $\prod_{i,j} \mathrm{Vect}_k^{\mathrm{fin}}$, hence semi-simple. □

Notation A.1.3. Let $A = A_1 \times \cdots \times A_r$ be a product of matrix algebras. In the above lemma we encountered a natural decomposition of (A, A)–bimodules. For a (A, A)–bimodule M we define,

$$M_{i,j} = A_i \otimes_A M \otimes_A A_j$$

We can regard $M_{i,j}$ either as a (A, A)–bimodule or as a (A_i, A_j)–bimodule. We have a decomposition of (A, A)–bimodules,

$$M = \bigoplus_{i,j} M_{i,j}$$

Let N be an additional (A, A)–bimodule. Then,

$$(M \otimes_A N)_{i,j} = \bigoplus_{k=1}^{r} M_{i,k} \otimes_{A_k} M_{k,j}$$

Lemma A.1.4. *Let $A = A_1 \times \cdots \times A_r$ where $A_i = \mathrm{End}_k(V_i)$. Let M be an invertible (A, A)–bimodule.*

(i) There exists a unique $\alpha \in \Sigma_r$ such that $M \cong \bigoplus_{i=1}^{r} \mathrm{Hom}_k(V_{\alpha(i)}, V_i)$,

(ii) Let $M' \cong \bigoplus_{i=1}^{r} \mathrm{Hom}_k(V_{\alpha'(i)}, V_i)$. Then, $M \otimes_A M' \cong \bigoplus_{i=1}^{r} \mathrm{Hom}_k(V_{\alpha'(\alpha(i))}, V_i)$,

(iii) Every inverse of M is isomorphic to $M^{-1} \cong \bigoplus_{i=1}^{r} \mathrm{Hom}_k(V_{\alpha^{-1}(i)}, V_i)$.

Proof. Since M is invertible there exists a (A, A)–bimodule N and invertible intertwiners $u : A \to M \otimes_A N$ and $v : N \otimes_A M \to A$ such that both compositions

$$M \cong A \otimes_A M \xrightarrow{u\otimes 1} M \otimes_A N \otimes_A M \xrightarrow{1\otimes v} M \otimes_A A \cong M$$

$$N \cong N \otimes_A A \xrightarrow{1\otimes u} N \otimes_A M \otimes_A N \xrightarrow{v\otimes 1} A \otimes_A N \cong N$$

are identities.

Since $M \otimes_A N \cong A$,

$$\bigoplus_k M_{i,k} \otimes_{A_k} N_{k,j} = (M \otimes_A N)_{i,j} = \begin{cases} A_i & i = j \\ 0 & i \neq j \end{cases}$$

As a result, for every i there exists a unique $\alpha(i) \in \{1, \dots, r\}$ such that $M_{i,\alpha(i)} \otimes_{A_{\alpha(i)}} N_{\alpha(i),i} \cong A_i$ and for all other $k \neq \alpha(i)$ we have $M_{i,k} \otimes_{A_k} N_{k,i} = 0$. Both u and v restrict to invertible intertwiners,

$$u_i : A_i \xrightarrow{\cong} M_{i,\alpha(i)} \otimes_{A_{\alpha(i)}} N_{\alpha(i),i}$$

$$v_{\alpha(i)} : N_{\alpha(i),i} \otimes_{A_i} M_{i,\alpha(i)} \xrightarrow{\cong} A_{\alpha(i)}$$

For every $k \in \{1, \dots, r\}$ we have a composition of isomorphisms,

$$M_{i,k} \cong A_i \otimes_{A_i} M_{i,k} \xrightarrow{u_i \otimes 1} M_{i,\alpha(i)} \otimes_{A_{\alpha(i)}} N_{\alpha(i),i} \otimes_{A_i} M_{i,k}$$

If $k \neq \alpha(i)$ then $N_{\alpha(i),i} \otimes_{A_i} M_{i,k} = 0$ and $M_{i,k} = 0$. We conclude that for every i there is a unique $\alpha(i)$ such that $M_{i,k} = 0$ for every $k \neq \alpha(i)$ and $M_{i,\alpha(i)} \otimes_{A_{\alpha(i)}} N_{\alpha(i),i} \cong A_i$. In particular, $M = \bigoplus_{i=1}^r M_{i,\alpha(i)}$.

Since A_i and $A_{\alpha(i)}$ are endomorphism algebras, every $(A_i, A_{\alpha(i)})$– bimodule is a multiple of $\mathrm{Hom}_k(V_{\alpha(i)}, V_i)$ and every $(A_{\alpha(i)}, A_i)$–bimodule is a multiple of $\mathrm{Hom}_k(V_i, V_{\alpha(i)})$:

$$\begin{aligned} A_i &\cong M_{i,\alpha(i)} \otimes_{A_{\alpha(i)}} N_{\alpha(i),i} \\ &\cong \mathrm{Hom}_k(V_{\alpha(i)}, V_i)^{\oplus N_i} \otimes_{A_{\alpha(i)}} \mathrm{Hom}_k(V_i, V_{\alpha(i)})^{\oplus N_i'} \\ &\cong (\mathrm{Hom}_k(V_{\alpha(i)}, V_i) \otimes_{A_{\alpha(i)}} \mathrm{Hom}_k(V_i, V_{\alpha(i)}))^{\oplus N_i N_i'} \end{aligned}$$

Therefore $N_i = N_i' = 1$ and our proof is complete. □

Notation A.1.5. Let $A = \mathrm{End}(V_1) \times \cdots \times \mathrm{End}(V_r)$ be a finite dimensional semi-simple k-algebra. For every $\sigma \in \Sigma_r$ we let M^σ denote the invertible (A, A)-bimodule,

$$M^\sigma = \mathrm{Hom}_k(V_{\sigma(1)}, V_1) \times \cdots \times \mathrm{Hom}_k(V_{\sigma(r)}, V_r)$$

with left action given by post-composition, and right action given by permuted pre-composition.

Corollary A.1.6. *Let $A = A_1 \times \cdots \times A_r$ where each A_i is an endomorphism algebra. Let M be an invertible (A, A)−bimodule. Then M is free of rank 1 both as a right and as a left A−module.*

Definition A.1.7. Let A be a finite dimensional k-algebra. The *canonical trace form* of A is the symmetric bi-linear form

$$\tau_A(a) = \mathrm{Tr}(\mu_a : A \to A)$$

where μ_a is the operator given by left multiplication in a.

The following proposition can be found in [12]. It serves as a characterization of semi-simple algebras in terms of their canonical trace forms.

Proposition A.1.8. *Let k be an algebraically closed field of characteristic 0. Let A be a finite dimensional k-algebra. Then A is semi-simple if and only if τ_A is non-degenerate.*

Lemma A.1.9. *Let A be a product of matrix algebras. Let M and N be both (A, A)-bimodules and let*

$$\phi : M \otimes_A N \longrightarrow A$$

be an isomorphism of (A, A)-bimodules. Define a bilinear form,

$$\tau : M \otimes N \longrightarrow k$$
$$\tau(m \otimes n) = \phi(\tau_A(m \otimes n))$$

Then τ is non-degenerate. Moreover, for every basis $\{x_i\}$ in M with dual basis $\{y_i\}$ in N with respect to τ,

$$\phi : \sum_i x_i \otimes y_i \longmapsto 1$$

Proof. By Corollary A.1.6, M and N are free rank of rank 1 both as right and as left A-modules. There exist a left cyclic generator $m \in M$ and a right cyclic generator $n \in N$ such that $\phi : m \otimes n \longmapsto 1$. We have

$$\tau(a \cdot m, n \cdot a') = \tau_A(aa')$$

which implies τ is non-degenerate.

The element $b = \sum_i x_i \otimes y_i$ is independent of a choice of basis, therefore it is enough to show this holds for one particular basis. There is a basis $\{e_i\}$ of A such that,

$$\sum_i e_i e^i = 1$$

where $\{e^i\}$ is the dual basis with respect to τ_A (when A is a matrix algebra this is given by the matrices $E_{i,j}$). The set $\{e_i \cdot m\}$ is a basis for M and $\{n \cdot e^i\}$ is a basis for N. Clearly they are dual with respect to τ. The canonical element b then maps to 1. □

Remark A.1.10. For a semi-simple finite dimensional algebra A we have identification,

$$A^* = A = A^o$$

as (A, A)-bimodules. The one on the left is given by the non-degenerate trace form, the one on the right is given by the anti-homomorphism given by the transpose.

A.2 Bi-Angular Algebras.

Let G denote a finite group. In [17], Turaev introduces the notion of a bi-angular finite dimensional G-graded algebra and uses it to construct a non-extended G-equivariant 2-dimensional TFT with values in Vect_k.

Let B denote a finite dimensional G-graded algebra $B = \oplus_{g\in G} B_g$. For any $h \in G$, there is a unique symmetric bi-linear form $\tau_h : B \otimes B \to k$, defined on a pair $a \in B_g$ and $a' \in B_{g'}$ to be

$$\tau_h(a, a') = \begin{cases} 0 & gg' \neq e \\ \mathrm{Tr}(\mu_{aa'} : B_h \to B_h) & gg' = e \end{cases}$$

where $\mu_{aa'}$ is the operator given by left multiplication in $aa' \in B_e$.

Definition A.2.1. A finite dimensional G-graded algebra $B = \oplus_{g\in G} B_g$ is *bi-angular* if τ_h is non-degenerate and independent of h.

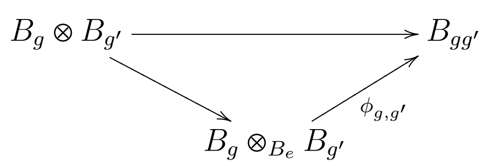
There are a number of immediate consequences to the above definition. If B is bi-angular then the subalgebra $B_e \subseteq B$ is semi-simple and each homogeneous piece B_g is a (B_e, B_e)-bimodule. Since τ_h is independent of h we drop the subscript. Non-degeneracy of τ provides, for every $g \in G$, a canonical element,

$$b_g := \sum_{i=1}^{n} x_i^g \otimes y_i^g \in B_g \otimes B_{g^{-1}} \tag{A.2.1}$$

where $\{x_i^g\}$ is a basis of B_g and $\{y_i^g\}$ is its dual basis in $B_{g^{-1}}$ with respect to τ. The image of b_g under the multiplication map $B_g \otimes B_{g^{-1}} \to B_e$ is the unit, namely,

$$\sum_{i=1}^{n} x_i^g \, y_i^g = 1$$

The multiplication factors through an intertwiner,



Lemma A.2.2. *Let B be bi-angular. Then $\phi_{g,g'}$ is an isomorphism, and each homogeneous piece B_g is an invertible (B_e, B_e)-bimodule. In particular, it is free of rank 1 as a left and a right B_e-module.*

Proof. The character of a finite dimensional left module of a semi-simple finite-dimensional k-algebra determines its isomorphism class. For every $g \in G$ and every $a \in B_e$,

$$\mathrm{Tr}(\mu_a : B_g \to B_g) = \tau_g(a, 1) = \tau_e(a, 1) = \mathrm{Tr}(\mu_a : B_e \to B_e)$$

Hence, $B_g \cong B_e$ as left B_e-modules. In particular, B_g is free of rank 1 as a left B_e-module. The intertwiner $\phi_{g,g^{-1}} : B_g \otimes_{B_e} B_{g^{-1}} \to B_e$ is surjective as $\phi_{g,g^{-1}}(b_g) = 1 \in B_e$ is in the image. Pick cyclic left generators $x \in B_g$ and $x' \in B_{g^{-1}}$. Every element of $B_g \otimes_{B_e} B_{g^{-1}}$ is of the form $a \cdot x \otimes x'$. Since $\phi_{g,g^{-1}}(x \otimes x') \in B_e$ must be invertible, $\phi_{g,g^{-1}}$ is necessarily injective. We conclude $\phi_{g,g^{-1}}$ is an isomorphism of (B_e, B_e)-bimodules. Therefore each B_g is an invertible bimodule which proves the lemma. □

Lemma A.2.3. *Let B be bi-angular. Then the intertwiner*

$$\phi_{g,g'} : B_g \otimes_{B_e} B_{g'} \to B_{gg'}$$

is an isomorphism, in other words, B is strongly graded.

Proof. In the proof of Lemma A.2.2 we showed $\phi_{g,g'}$ is an isomorphism for $gg' = e$. It is immediate from the definition that $\phi_{g,g'}$ is an isomorphism for either $g = e$ or $g' = e$. To prove the general case, consider the commutative diagram,

$$\begin{array}{ccc} B_{g^{-1}} \otimes_{B_e} B_g \otimes_{B_e} B_{g'} & \xrightarrow{1 \otimes \phi_{g,g'}} & B_{g^{-1}} \otimes_{B_e} B_{gg'} \\ \downarrow{\scriptstyle \phi_{g^{-1},g} \otimes 1} & & \downarrow{\scriptstyle \phi_{g^{-1},gg'}} \\ A \otimes_{B_e} B_{g'} & \xrightarrow{\phi_{e,g'}} & B_{g'} \end{array}$$

Since left vertical and bottom arrows are isomorphisms, $\phi_{g,g'}$ is injective. To show it is surjective, choose cyclic generators x,y and z of $B_{g^{-1}}$, B_g and $B_{g'}$ respectively, such that $\phi_{g^{-1},g}(x \otimes y) = 1 \in B_e$. Let $\tilde{y} = \phi_{g,g'}(y \otimes z)$. We are done if we can show $\tilde{y}$ is a cyclic generator. This is true iff the annihilator of $\tilde{y}$ in

B_e is zero. Let $a \in \mathrm{Ann}(\tilde{y})$, then

$$0 = x \otimes a \cdot \tilde{y} = x \cdot a \otimes \tilde{y} = a' \cdot x \otimes \tilde{y}$$

$$0 = \phi_{g^{-1},gg'}(a' \cdot x \otimes \tilde{y}) = a' \cdot \phi_{g^{-1},gg'}(x \otimes \tilde{y}) = a' \cdot z$$

Since z is a cyclic generator $a' = 0$ hence $0 = x \cdot a$ hence $a = 0$ and the proof is complete. □

Proposition A.2.4. *A bi-angular finite dimensional G-graded algebra B is a G-equivariant algebra (B, τ) for which τ is the unique extension of the trace form on B_e.*

Proof. Recall that a G-equivariant algebra is a pair (B, τ) consisting of a finite dimensional strongly G-graded algebra $B = \oplus_g B_g$, such that $B_e \subseteq B$ is semi-simple, and a symmetric non-degenerate form τ on B, with the property $\tau(ab, c) = \tau(a, bc)$, such that $\tau|_{B_g \otimes B_{g'}} \equiv 0$ whenever $gg' \neq e$.

If B is biangular then by Lemma A.2.3 it is strongly G-graded. The canonical trace form τ_{B_e} on B_e is symmetric non-degenerate with the property $\tau_{B_e}(ab, c) = \tau_{B_e}(a, bc)$. It extends uniquely to a form on B with the same properties if we require $\tau|_{B_g \otimes B_{g'}} \equiv 0$ whenever $gg' \neq e$.

Now assume (B, τ) is a G-equivariant algebra with the restriction $\tau|_{B_e \otimes B_e}$ being the canonical trace form on B_e. It is a finite dimensional G-graded algebra. As such, we have τ_h for every $h \in G$ as before. To show B is biangular we need to prove $\tau_h = \tau$ for every $h \in G$. To show τ_h is independent of h it is enough to show $\tau_h(c, 1) = \tau(c, 1)$ for any $c \in B_e$. By Lemma A.1.9, for every basis $\{x_i^h\}$ in B_h with dual basis $\{y_i^h\}$ in $B_{h^{-1}}$ with respect to τ, we have $\sum_i x_i^h\, y_i^h = 1$. Repeating an argument from [17],

$$\tau(c,1) = \tau(c, \sum_i x_i^h\, y_i^h) = \sum_i \tau(cx_i^h, y_i^h) = \mathrm{Tr}(\mu_c : B_h \to B_h) = \tau_h(c,1)$$

This completes the proof. □

As before, let $\{x_i^g\}$ be a basis for B_g with dual basis $\{y_i^g\}$ in $B_{g^{-1}}$ with respect to τ. We have map $\psi_g : B \to B$ given by,

$$\psi_g : b \longmapsto \sum_i x_i^g b\, y_i^g$$

Definition A.2.5. The *G-center* of a biangular G-graded algebra B is the G-graded subalgebra $L = \bigoplus_{g\in G} L_g$ given by,

$$L_g = \psi_e(B_g) \subseteq B_g$$

Lemma A.2.6. *$L_g = HH^0(B_e; B_g)$.*

Proof. By definition

$$HH^0(B_e; B_g) = \{\, b \in B_g \,|\, a \cdot b = b \cdot a,\ \ \forall a \in B_e \,\}$$

By Lemma A.1.9,

$$\sum_i x_i^e\, y_i^e = 1$$

For every $b \in HH^0(B_e; B_g)$,

$$b = b \cdot 1 = b \cdot \sum_i x_i^e\, y_i^e = \sum_i x_i^e \cdot b \cdot y_i^e$$

hence $b \in L_g$. For every $a \in B_e$, $a = \sum_i \tau(a, y_i^e)\, x_i^e$ and $a = \sum_i \tau(x_i^e, a)\, y_i^e$.

$$\begin{aligned}
a \cdot \left(\sum_i x_i^e \cdot b \cdot y_i^e \right) &= \sum_i (a x_i^e) \cdot b \cdot y_i^e \\
&= \sum_i \left(\sum_j \tau(a x_i^e, y_j^e)\, x_j^e \right) \cdot b \cdot y_i^e \\
&= \sum_j x_j^e \cdot b \cdot \left(\sum_i \tau(a x_i^e, y_j^e)\, y_i^e \right) \\
&= \sum_j x_j^e \cdot b \cdot \left(\sum_i \tau(x_i^e, y_j^e a)\, y_i^e \right) \\
&= \sum_j x_j^e \cdot b \cdot y_j^e a \\
&= \left(\sum_i x_i^e \cdot b \cdot y_i^e \right) \cdot a
\end{aligned}$$

hence $L_g \subseteq HH^0(B_e; B_g)$ and the proof is complete. □

Appendix B

Higher Categories.

What we refer to as 2-category stands for a $(2,2)$-category in [10] and is also known in the literature as a bicategory. The category of 2-categories is itself a 3-category. Its 1-morphisms are functors between 2-categories, its 2-morphisms are natural transformations between functors of 2-categories, and finally, its 3-morphisms are modifications between natural transformations between functors of 2-categories.

Definition B.0.7. Let $\mathcal{C}$ be a 2-category. Its *homotopy category* $h\mathcal{C}$ is the category whose objects are the objects of $\mathcal{C}$. Given $X, Y \in h\mathcal{C}$, $\mathrm{Hom}_{h\mathcal{C}}(X,Y)$ is the set of equivalence classes of objects in $\mathrm{Hom}_{\mathcal{C}}(X,Y)$.

When a 2-category $\mathcal{C}$ admits a symmetric monoidal structure, its homotopy category $h\mathcal{C}$ inherits one as well.

Definition B.0.8. An object in a symmetric monoidal 2-category $\mathcal{C}$ admits a dual if it admits a dual in $h\mathcal{C}$.

Definition B.0.9. A 1-morphism $f : X \to Y$ in a 2-category $\mathcal{C}$ *admits a right adjoint* if there exists a 1-morphism $f^R : Y \to X$ and 2-morphisms $u^R : \mathrm{id}_X \to f^R \circ f$ and $v^R : f \circ f^R \to \mathrm{id}_Y$ such that the compositions,

$$f \cong f \circ \mathrm{id}_X \xrightarrow{\mathrm{id}_f \times u^R} f \circ (f^R \circ f) \cong (f \circ f^R) \circ f \xrightarrow{v^R \times \mathrm{id}_f} \mathrm{id}_Y \circ f \cong f$$

$$f^R \cong \mathrm{id}_X \circ f^R \xrightarrow{u^R \times \mathrm{id}_{f^R}} (f^R \circ f) \circ f^R \cong f^R \circ (f \circ f^R) \xrightarrow{\mathrm{id}_{f^R} \times v^R} f^R \circ \mathrm{id}_Y \cong f^R$$

are both identities.

In the above definition, the 2-morphism u^R is called the *unit of a right adjunction* while v^R is called the *co-unit of a right adjunction.* Such an adjunction exhibits f as left adjoint to f^R and f^R as right adjoint to f. Right adjunction data (f^R, u^R, v^R), if it exists, is unique up to unique isomorphism. The definition of a 1-morphism admitting a left adjoint is the same with the roles of f and f^R reversed.

Definition B.0.10. Let $\mathcal{C}$ be a symmetric monoidal 2-category. An object in $\mathcal{C}$ is *fully dualizable* if it admits a dual such that the evaluation map admit both right and left adjoints.

Definition B.0.11. The collection of fully dualizable objects, 1-morphisms of fully-dualizable objects with left and right adjoint and their 2-morphisms, forms a monoidal subcategory of $\mathcal{C}$ which is denoted $\mathcal{C}^{\mathrm{fd}}$.

Definition B.0.12. The *Serre automorphism* of a fully dualizable object X in a symmetric monoidal 2-category is a 1-morphism $S_X : X \to X$ satisfying,

$$\mathrm{ev}_X^R = P_{X^\vee, X} \circ (\mathrm{id}_{X^\vee} \otimes S_X) \circ \mathrm{coev}_X \tag{B.0.1}$$

where $P_{X^\vee, X} : X^\vee \otimes X \to X \otimes X^\vee$ denotes the symmetric braiding (i.e. $P^2 = \mathrm{id}$).

Up to a choice of right adjoint ev_X^R for the evaluation map, one can define the Serre automorphism to be,

$$S_X = (\mathrm{id}_X \otimes \mathrm{ev}_X) \circ (P_{X,X} \otimes \mathrm{id}_{X^\vee}) \circ (\mathrm{id}_X \otimes \mathrm{ev}_X^R)$$

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Vita

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