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Three Essays on Valuation and Investment in Incomplete Markets

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Three Essays on Valuation and Investment in Incomplete Markets

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Dissertation

Presented to the Faculty of the Graduate School of

The University of Texas at Austin

in Partial Fulfillment

of the Requirements

for the Degree of

Doctor of Philosophy

The University of Texas at Austin

May 2011

Acknowledgments

I wish to thank, first and foremost, my advisor, Professor Thaleia Zariphopoulou. Without her years of guidance, insight and support, this would not have been possible. I also owe thanks to my committee members, Professors Todd Arbogast, Irene Gamba, Stathis Tompaidis and Gordan Zitkovic, for their time and their expertise. Lastly, I need to thank my collaborators. I worked extensively with Professors Mike Tehranchi, Cambridge University and Ronnie Sircar, Princeton University. Thank you.

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The University of Texas at Austin
May 2011

Three Essays on Valuation and Investment in Incomplete Markets

Publication No. ____________________

Nathanael David Ringer, Ph.D.
The University of Texas at Austin, 2011

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Incomplete markets provide many challenges for both investment decisions and valuation problems. While both problems have received extensive attention in complete markets, there remain many open areas in the theory of incomplete markets. We present the results in three parts. In the first essay we consider the Merton investment problem of optimal portfolio choice when the traded instruments are the set of zero-coupon bonds. Working within a Markovian Heath-Jarrow-Morton framework of the interest rate term structure driven by an infinite dimensional Wiener process, we give sufficient conditions for the existence and uniqueness of an optimal investment strategy. When there is uniqueness, we provide a characterization of the optimal portfolio. Furthermore, we show that a specific Gauss-Markov

random field model can be treated within this framework, and explicitly calculate the optimal portfolio. We show that the optimal portfolio in this case can be identified with the discontinuities of a certain function of the market parameters. In the second essay we price a claim, using the indifference valuation methodology, in the model presented in the first section. We appeal to the indifference pricing framework instead of the classic Black-Scholes method due to the natural incompleteness in such a market model. Because we price time-sensitive interest rate claims, the units in which we price are very important. This will require us to take care in formulating the investor's utility function in terms of the units in which we express the wealth function. This leads to new results, namely a general change-of-numeraire theorem in incomplete markets via indifference pricing. Lastly, in the third essay, we propose a method to price credit derivatives, namely collateralized debt obligations (CDOs) using indifference. We develop a numerical algorithm for pricing such CDOs. The high illiquidity of the CDO market coupled with the allowance of default in the underlying traded assets creates a very incomplete market. We explain the market-observed prices of such credit derivatives via the risk aversion of investors. In addition to a general algorithm, several approximation schemes are proposed.

Contents

Chapter 1

Optimal Portfolio Choice in the Bond Market

1.1 Introduction

We consider the problem of optimal portfolio choice when the traded instruments are the set of zero-coupon bonds. In particular, we fix a utility function U and a planning horizon $[0, T]$, $T > 0$, and consider the functional $J(\phi) = \mathbb{E}^{\mathbb{P}}\ U(X_T^\phi)$ where X_T^ϕ is the accumulated wealth at time T generated by the self-financing trading strategy ϕ. Our goal is to characterize the strategy that maximizes J.

This type of utility maximization problem has a long history in financial economics. A seminal paper of Merton [31] from 1969 provides a solution in the case where the investor can trade continuously in a finite set of stocks and the bank account. In the bond market setting, however, the problem of optimal portfolio choice presents new challenges. For example, there are bonds of so many maturities available to trade that many models assume that there exists a continuum of bonds indexed by their maturity date. So, for our model, let $P(t, T)$ denote the price at time t of a zero-coupon bond which is worth one unit of money at the maturity T, where $T \geq t$. In the Heath-Jarrow-Morton (HJM) modeling framework proposed in [23], the price process $(P(t,T))_{t\in[0,T]}$ is an Itô process for each $T \geq 0$. We will study the utility maximization problem within the HJM framework.

In the orginal HJM framework, each of the price processes $(P(t,T))_{t\in[0,T]}$ is driven by the same finite-dimensional Wiener process. This modeling assumption has some shortcomings. For example, in the context of such models, there are typically many strategies which hedge the same claim, but most of these hedging strategies are rather unnatural and probably would never be implemented by a bond trader. For example, consider a finite model driven by three independent Wiener processes and a trader wishing to hedge a option written on a ten year bond. In this finite model, one could show that the trader could possibly hedge this option with a portfolio of bonds maturities 15, 20 and 30 years. Whereas, in practice, the actual trader would prefer to hedge this option with bonds of maturity of ten years or less. In other words, in finite models, there is a lack of incorporating a sense of "maturity-specific" risk. See [7] for a further discussion of this point. Citing such concerns, as well as the need for models with greater flexibility which can be parsimoniously parametrized, the original HJM framework has been generalized by several authors. For instance, Goldstein [19], Kennedy [29], and Santa-Clara and Sornette [37] have proposed various HJM-type random field models. In such models, the bond price processes typically satisfy stochastic differential equations driven by an infinite-dimensional Wiener process. HJM type models with discountinuous bond price sample paths have also been proposed, but we do not address this generalization here.

It is important to note that models driven by an infinite-dimensional Wiener process may not be complete in the usual sense: there typically exist contingent claims that cannot be exactly replicated by a self-financing strategy, even if the martingale measure is unique, and even if the notion of a strategy is generalized to allow portfolios of bonds with an infinite number of maturities. The cause of this new type of incompleteness is that when the prices are driven by an infinite-dimensional Wiener process, the volatility cannot be bounded away from zero. This lack of completeness in the presence of a unique martingale measure, though in a slightly different context, has lead to the introduction by Björk et al [6] of the notion of *approximate completeness*. This notion, the approximate completeness of bond market models driven by an infinite-dimensional Wiener process, has been studied by De Donno and Pratelli [12] and Taflin [41] where the portfolios that investors are allowed to buy correspond to points in the dual of a Banach space of bond price curves.

The problem of optimal portfolio choice in the bond market has been studied by Ekeland and Taflin [15]. They work in an HJM framework and prove the existence of an optimal portfolio in two cases: when the driving Wiener process is finite-dimensional and when the Wiener process is infinite-dimensional but the market price of risk is deterministic. Furthermore, they give a representation of the optimal portfolio as the sum of two mutual funds. We will build upon the work of Ekeland and Taflin by studying the Merton problem in the case when the driving Wiener process is infinite-dimensional and the bond prices are Markovian. Using the Clark-Ocone formula and convex duality, we give sufficient conditions for the existence of an optimal trading strategy. Furthermore, we prove that the optimal porfolio naturally decomposes as a sum of three mutual fund trading strategies. Each mutual fund is an investment strategy which serves as a hedge against a certain type of market risk. The first fund is universal in the sense that each investor in the bond market invests a portion of her wealth in this portfolio, independently of the details of her utility function and planning horizon. This can be thought of as a market portfolio. The second fund consists of the investor's hedge against fluctations in the market price of risk. This second fund does not appear in Ekeland and Taflin's decomposition since the market price of risk is assumed deterministic, in which case this fund is simply zero. Lastly, the third fund comes from the self-financing constraint and hedges against the stochastic discount factor. Because we are maximizing expected utility of wealth, rather than of discounted wealth, this third fund is slightly different than the fund that appears in [15]. Under natural conditions on the bond price volatilities and market price of risk, we show that at time $t \in [0, T]$, this third portfolio consists of bonds with maturities in the interval $[t, T]$.

Finally, we examine in detail the optimal portfolio for a class of Gaussian random field models proposed by Kennedy [29] and studied by Goldstein [19] and by Santa-Clara and Sornette [37]. We assume that the market price of risk is deterministic so that we may concentrate our attention on the first fund of the decomposition. We show that the optimal portfolio in this case can be identified with the discontinuities of a certain function of the market parameters.

The outline of the chapter is as follows: In Section 1.2 we recall the various notions that arise in the study of the bond markets, in particular, the HJM framework. In Section 1.3 we specify a general Markovian HJM model of the infinite-dimensional dynamics of the bond prices. In Section 1.4 we introduce the various notions of strategies needed in context of this model. In Section 1.5 we present our main results: we solve a Merton utility maximization problem and analyze the optimal strategy. In Section 1.6, we exhibit a nontrivial example of an HJM model which satisfies the conditions of Theorem 1.5.2 and explicitly construct the optimal portfolio. In Section 1.7, we state some results from Malliavin Calculus, including the Clark-Ocone formula, which are used in the proofs of the main theorems. In Section 1.8 we present the proofs of the main results.

1.2 The HJM framework

In this section we recall the HJM framework, proposed by Heath, Jarrow, and Morton [23], for modeling the bond market. We include this section for motivation and context; our precise modeling assumptions are spelled out in Section 1.3. We also use the notation introduced here to describe an example HJM model in Section 1.6.

Using the parametrization popularized by Musiela [32], let $f_t(x)$ denote the forward rate at time t for time to maturity x. The forward rates are related to the price $P(t,T)$ at time t of a zero-coupon bond with maturity date $T = t + x$ by the formula

$$f_t(x) = -\frac{\partial}{\partial x} \log\left(P(t, t+x)\right),$$

whenever the derivative exists. In this framework, the risk-neutral dynamics formally satisfy the following stochastic partial differential equation

$$df_t(x) = \left(\frac{\partial}{\partial x} f_t(x) + a_t(x) \right) dt + b_t(x) dW_t^{\mathbb{P}},$$

where the process $(W_t^{\mathbb{P}})_{t \geq 0}$ is a Wiener process under the historical measure $\mathbb{P}$. The drift is given by the famous HJM no-arbitrage condition

$$a_t(x) = b_t(x)\lambda_t + b_t(x) \int_0^x b_t(s) ds,$$

where λ_t is the market price of risk. The random variables $b_t(x)$, λ_t, and $W_t^{\mathbb{P}}$ are allowed to be vector-valued, in which case products are interpreted as standard Euclidean inner products.

Let $B_t = \exp\left(\int_0^t r_s \, ds\right)$ denote the value at time t of the bank account with initial deposit one dollar, where the short rate is given by $r_t = f_t(0)$. The discounted bond price,

$$\tilde{P}_t(x) = B_t^{-1} P(t, t+x), \tag{1.1}$$

in Musiela notation, formally satisfies the stochastic partial differential equation

$$d\tilde{P}_t(x) = \left(\frac{\partial}{\partial x} \tilde{P}_t(x) - \tilde{P}_t(x) \int_0^x b_t(s) ds \ \lambda_t \right) dt - \tilde{P}_t(x) \int_0^x b_t(s) ds \ dW_t^{\mathbb{P}}.$$

Letting $\sigma_t(x) = -\tilde{P}_t(x) \int_0^x b_t(s) ds$ and rewriting the above SPDE in integral form, we see that the discounted bond prices satisfy

$$\tilde{P}_t(x) = \tilde{P}_0(t+x) + \int_0^t \sigma_s(x+t-s)\lambda_s ds + \int_0^t \sigma_s(x+t-s) \ dW^{\mathbb{P}}, \tag{1.2}$$

with initial data $\tilde{P}_0(\cdot) = P(0, \cdot)$. It is in this form that we specify the HJM model in the next section.

1.3 The model specification

In this section, we specify a general Markovian HJM model of the discounted bond prices. Following [15], we take the discounted bond price curve $\tilde{P}_t = \tilde{P}_t(\cdot)$ to be the state variable. We will interpret equation (1.2) as an evolution equation in the space F of real-valued functions on $\mathbb{R}_+$. We now list the relevant assumptions on F.

Assumption 1.3.1. *1. The real, infinite-dimensional vector space F is equipped with a norm $\|\cdot\|_F$ for which it becomes a separable Banach space. The topological dual space of F is denoted F^*. Furthermore, the norm obeys the parallelogram law so that F is in fact a Hilbert space.*

2. The elements of F are continuous, real-valued functions on $\mathbb{R}_+$. In particular, for every $x \in \mathbb{R}_+$, the functional

$$\delta_x(f) = f(x)$$

is well-defined as a continuous functional on F; that is, δ_x is an element of F^* for all $x \geq 0$.

3. The semigroup $(S_t)_{t\geq 0}$ is strongly continuous in F, where the left shift operators S_t is defined by

$$(S_t f)(x) = f(t + x).$$

Typical examples of spaces which satisfy Assumption 1.3.1 are Sobolev spaces. See [8] or [15] for a detailed discussion. Also, see Section 1.6 for an example.

Remark 1. We assume that there exists exactly one martingale measure $\mathbb{Q}$ equivalent to $\mathbb{P}$; that is, the market is approximately complete. Therefore, we specify the model directly under this measure. Also, we will reserve the expectation notation $\mathbb{E}$ for expectation under $\mathbb{Q}$. Expectation under $\mathbb{P}$ will be denoted $\mathbb{E}^{\mathbb{P}}$.

We then fix a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ and a real, separable, infinite-dimensional Hilbert space G with inner product $\langle , \rangle_G$, and we let $(W_t)_{t\geq 0}$ be a Wiener process defined cylindrically on G, and that the filtration $(\mathcal{F}_t)_{t\geq 0}$ is given by the augmentation of the filtration it generates. Without loss of generality, one could take $G = \ell^2$ as in [15] as this amounts to choosing an orthonormal basis of G and working with the coordinates in this basis. We, however, prefer to keep G unspecified.

We now formulate a model of the risk-neutral discounted price dynamics. In what follows, we let

$$F_+ = \{f \in F : f(x) > 0\}$$

be the positive cone of F. Also, the notation $\mathcal{L}_{\mathrm{HS}}(G, F)$ denotes the space of Hilbert-Schmidt operators from G into F with norm

$$\|A\|_{\mathcal{L}_{\mathrm{HS}}(G,F)} = \left(\sum_n \|Ag_n\|_F^2\right)^{1/2},$$

for any orthonormal basis $(g_n)_n$ of G.

Assumption 1.3.2. *Let $\sigma(\cdot,\cdot) : \mathbb{R}_+ \times F_+ \to \mathcal{L}_{\mathrm{HS}}(G, F)$ be such that $\sigma(\cdot, f)$ is continuous for all $f \in F_+$ and such that $\sigma(t, \cdot)$ is globally Lipschitz, uniformly in $t \geq 0$.*

Additionally, we assume that, for all $f \in F_+$ and $t \geq 0$, we have

$$\overline{\operatorname{ran}(\sigma(t, f))} = \{g \in F : g(0) = 0\}, \tag{1.3}$$

or, equivalently,

$$\ker(\sigma(t, f)^*) = \operatorname{span}\{\delta_0\} \subset F^*$$

and

$$\|\sigma(t, f)^* \delta_x\|_G \leq C f(x), \tag{1.4}$$

for some $C > 0$.

Definition 1.3.3. *We fix an initial discounted bond price curve $\tilde{P}_0 \in F_+$ and define the* *discounted bond price process* $(\tilde{P}_t)_{t\geq 0}$ *as the continuous solution to the evolution equation*

$$\tilde{P}_t = S_t\tilde{P}_0 + \int_0^t S_{t-s}\sigma(s, \tilde{P}_s)dW_s. \tag{1.5}$$

We use the abbreviation $\sigma_t = \sigma(t, \tilde{P}_t)$.

That the discounted bond price process is well-defined follows from the standard existence and uniqueness theorem for mild solutions of the evolution equation (1.5). See for instance the book of Da Prato and Zabczyk [10] for a proof.

Remark 2. The dynamics of the discounted bond price in this model are genuinely infinite-dimensional in the following sense: For every finite set of maturity dates $T_1, \ldots, T_d$, the submarket consisting of the bank account and those bonds with discounted prices $(\tilde{P}_t(T_1 - t), \ldots, \tilde{P}_t(T_d - t))$ is incomplete. Note that this property crucially depends on the infinite dimensionality of the state space F, as well as the infinite dimensionality of the driving Wiener process. Indeed, the rank of the martingale operator σ_t is infinite.

Notice that our model is not a finite-factor model, where a bond price model is said to be a finite-factor model if there exists a deterministic function $g : \mathbb{R}^n \to F$ and an n-dimensional diffusion $(Z_t)_{t\geq 0}$ such that $\tilde{P}_t = g(Z_t)$ almost surely for all $t \geq 0$. The condition that a model be genuinely infinite-dimensional in the sense described above is much stronger than not being finite-factor. Indeed, there exist infinite-factor bond price models driven by a one-dimensional Wiener process. A discussion of this phenomenon of hypoellepticity in HJM models can be found in [2].

We will make use of the bounds contained in the following Proposition, stated without proof:

Proposition 1.3.4. *For every* $t \geq 0$ *and* $x \geq 0$*, the random variable* $\tilde{P}_t(x)$ *is strictly positive and*

$$\mathbb{E}\tilde{P}_t(x)^p < +\infty,$$

for all $p \in \mathbb{R}$.

The above Proposition allows us to *define* the bank account.

Definition 1.3.5. *For $t \geq 0$, the bank account process is given by $B_t = \tilde{P}_t(0)^{-1}$.*

1.3.1 The market price of risk

We have specified the dynamics of the discounted bond prices under the risk neutral measure $\mathbb{Q}$. For the utility maximization problem considered here, we need to know the dynamics of the bond prices under the equivalent historical measure $\mathbb{P}$. However, since we are working on the finite time horizon $[0, T]$, we only need to consider the restriction of $\mathbb{P}$ to the sub-sigma-algebra $\mathcal{F}_T$. We have by Girsanov's theorem that the Radon-Nikodym derivative is of the form

$$\frac{d\mathbb{P}}{d\mathbb{Q}} = \exp\left(-\frac{1}{2}\int_0^T \|\lambda_s\|_G^2 ds - \int_0^T \lambda_s dW_s\right),$$

where $(\lambda_t)_{t\in[0,T]}$ is the G-valued market price of risk process. The process $(W^{\mathbb{P}})_{t\in[0,T]}$, defined by

$$W_t^{\mathbb{P}} = W_t + \int_0^t \lambda_s ds,$$

is a cylindrical Wiener process on G under the measure $\mathbb{P}$.

We make the following assumption about the market price of risk:

Assumption 1.3.6. *Let $\lambda(\cdot,\cdot) : \mathbb{R}_+ \times F_+ \to G$ be such that $\lambda(\cdot, f)$ is continuous for all $f \in F_+$ and such that $\lambda(t,\cdot)$ is bounded and globally Lipschitz uniformly in $t \geq 0$. We denote $\lambda_t = \lambda(t, \tilde{P}_t)$.*

We assume that there exists a subset $F_+^0 \subset F_+$ and a measurable function $\Gamma(\cdot,\cdot)$: $\mathbb{R}_+ \times F_+^0 \to F^*$ such that $\tilde{P}_t \in F_+^0$ for all $t \geq 0$, almost surely, and such that

$$\lambda(t, f) = \sigma(t, f)^*\Gamma(t, f), \tag{1.6}$$

for all $t \geq 0$ and $f \in F_+^0$.

The appearance of the subset F_+^0 will be clarified in the example of Section 1.6. We will make use of the following bound: For every $p \in \mathbb{R}$, we have

$$\mathbb{E}\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)^p < +\infty \tag{1.7}$$

which follows from the assumption that $(\lambda_t)_{t\in[0,T]}$ is bounded.

1.4 Bond portfolios and trading strategies

Consider an investor holding c_0 units of cash (that is, $B_t^{-1}c_0$ units of the bank account) and c_i units of the bond with maturity T_i for $i = 1, \ldots, N$. Her wealth at time t is given by

$$\begin{aligned} c_0 + \sum_{i=1}^{N} c_i P(t, T_i) &= B_t \left(c_0\delta_0 + \sum_{i=1}^{N} c_i \delta_{T_i - t} \right)(\tilde{P}_t) \\ &= B_t \; \phi_t(\tilde{P}_t), \end{aligned}$$

where we have used the fact that $\delta_0(\tilde{P}_t) = \tilde{P}_t(0) = B_t^{-1}$. That is, the vector of portfolio weights $(c_0, \ldots, c_N)$ corresponds to the functional $\phi_t \in F^*$.

It is interesting to note that the evaluation functionals span a dense subspace of F^*. Indeed, let $\mathcal{S}$ be the closure of $\text{span}\{\delta_x, x \geq 0\}$ in the F^* norm and let

$$\mathcal{S}^\perp = \{f \in F : \mu(f) = 0 \text{ for all } \mu \in \mathcal{S}\}.$$

If $f \in \mathcal{S}$ then $f(x) = 0$ for all x; that is, $\mathcal{S}^\perp = \{0\}$ and $\mathcal{S} = \mathcal{S}^{\perp\perp} = F^*$ as claimed. We will call the elements of F^* portfolios, and processes taking values in F^* strategies. To be precise, we make the following definition:

Definition 1.4.1. *An admissible investment strategy is a progressively measurable F^*-valued process $(\phi_t)_{t\geq 0}$ such that*

$$\mathbb{E}\int_0^t \|\sigma_s^* \phi_s\|_G^2 ds < +\infty,$$

for all $t \geq 0$.

Remark 3. The definition of admissibility given here is well-suited for the Malliavin Calculus techniques we will use. There are, however, other ways to define admissible strategies. A popular alternative is to consider strategies such that the process $(\int_0^t \sigma_s^* \phi_s dW_s)_{t\geq 0}$ is uniformly bounded from below. As we will see, in the important case where the utility function is finite only on a half-line, the solution of the utility maximization problem is the same with either definition of admissibility. However, if the utility function is finite everywhere, the two definitions of admissibilities may give rise to different solutions.

We now formulate a definition of the self-financing condition. It is equivalent to that found in [7] and [15].

Definition 1.4.2. *An admissible strategy $(\phi_t)_{t\geq 0}$ is self-financing if there exists a constant $X_0 \in \mathbb{R}$ such that*

$$\phi_t(\tilde{P}_t) - \int_0^t \sigma_s^* \phi_s dW_s = X_0$$

for almost all $(t, \omega) \in \mathbb{R}_+ \times \Omega$. The set of admissible self-financing strategies is denoted by $\mathcal{A}$.

The integrability condition in Definition 1.4.1 is sufficient for the stochastic integral in Definition 1.4.2 to be well-defined.

Note that to each strategy $(\phi_t)_{t\geq 0}$ and initial wealth X_0, we can associate a self-financing strategy $(\psi_t)_{t\geq 0}$ by the rule

$$\psi_t = \phi_t + B_t \left(X_0 + \int_0^t \sigma_s^* \phi_s dW_s - \phi_t(\tilde{P}_t) \right) \delta_0.$$

The term $\psi_t - \phi_t$ corresponds to the amount of money held in or borrowed from the bank account.

Definition 1.4.3. *For an initial wealth $X_0 \in \mathbb{R}$ and a self-financing strategy $(\phi_t)_{t\geq 0}$, the* *wealth process* $(X_t^\phi)_{t\geq 0}$ *is given by*

$$
\begin{aligned}
X_t^\phi &= B_t\ \phi_t(\tilde{P}_t) \\
&= B_t \left(X_0 + \int_0^t \sigma_s^* \phi_s dW_s \right).
\end{aligned}
$$

Note that for every self-financing strategy ϕ, the discounted wealth process $(B_t^{-1} X_t^\phi)_{t\geq 0}$ is a martingale for the equivalent martingale measure $\mathbb{Q}$.

Proposition 1.4.4. *There is no arbitrage in this market.*

Proof. We claim that since for every self-financing strategy ϕ, the discounted wealth process $(B_t^{-1} X_t^\phi)_{t\geq 0}$ is a martingale under the equivalent martingale measure $\mathbb{Q}$, this suffices to preclude arbitrage in our market. Suppose instead that an arbitrage X_t, $t \in [0, T]$, exists. Since $\mathbb{P}$ and $\mathbb{Q}$ are equivalent measures, we then have that $\mathbb{Q}(X_T \geq 0) = 1$ and $\mathbb{Q}(X_T > 0) > 0$. But, this is a contradiction to the fact that X_t is a martingale under $\mathbb{Q}$, and, in particular, that $\mathbb{E}^{\mathbb{Q}}[X_T] = 0$. Hence, the arbitrage cannot exist. $\square$

1.5 The utility maximization problem

We a fix terminal date $T > 0$, an initial wealth X_0, and a utility function $U : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ and let

$$
J(\phi) = \mathbb{E}^{\mathbb{P}}\ U(X_T^\phi)
$$

be the expected terminal utility of implementing the strategy ϕ. The investor's goal is then to find the admissible strategy $\phi \in \mathcal{A}$ which maximizes the functional J.

Following [15] we list our assumptions on the utility function U and the inverse marginal utility $I(y) = (U')^{-1}(y)$.

Assumption 1.5.1. *The utility function* $U : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ *is strictly concave, finite and twice continuously differentiable on an open interval* $(\underline{x}, \infty)$ *for some* $\underline{x} \leq 0$, *with the value* $\underline{x} = -\infty$ *allowed. Moreover, we assume* $U'(x) \to \infty$ *as* $x \searrow \underline{x}$. *Letting* $\underline{y} = \inf_{x > \underline{x}} U'(x) =$

$\lim_{x\to\infty} U'(x)$, *we assume that either* $\underline{y} = 0$ *or* $\underline{y} = -\infty$. *Define the decreasing function* $I : (\underline{y}, \infty) \to (\underline{x}, \infty)$ *by* $I(y) = (U')^{-1}(y)$.

It is also assumed that there exists some $q > 0$ *such that, for all* y, *the following bounds*

$$|I(y)| \leq C(|y|^q + |y|^{-q})$$

and

$$|I'(y)| \leq C(|y|^{q+1} + |y|^{-q-1})$$

hold.

Remark 4. In [15], the authors list bounds on the utility function from which they derive the above bounds on the inverse marginal utility. Also, the above bounds are designed to handle both cases when $\underline{y} = 0$ and $\underline{y} = -\infty$. Moreover, the notation $\underline{x}$ and $\underline{y}$ is introduced to treat several interesting cases in a systematic way. For instance, for any *increasing* utility function we would have $\underline{y} = 0$. Thus, for the CARA utility $U(x) = -e^{-\gamma x}$, we have $\underline{x} = -\infty$ and $\underline{y} = 0$. Whereas for the CRRA utility $U(x) = x^\gamma/\gamma$, we have $\underline{x} = 0$ and $\underline{y} = 0$. Although, for the quadratic "utility" function $U(x) = cx - x^2$, we would have $\underline{x} = -\infty$ and $\underline{y} = -\infty$.

Theorem 1.5.2. *Under Assumptions 1.3.2 and 1.5.1, there exists a unique admissible strategy* $\bar{\phi} \in \mathcal{A}$ *which maximizes* J.

Furthermore, the optimal portfolio decomposes into a sum of three mutual funds

$$\bar{\phi} = \Phi^1 + \Phi^2 + \Phi^3$$

with the following properties

1. *For every* $t \in [0, T]$, *the normalized random vector* $\Phi^1_t / \|\Phi^1_t\|_{F^*} \in F^*$ *is a deterministic function of the market parameters* σ_t *and* λ_t, *and is independent of the investor's initial weath* X_0, *utility function* U, *and planning horizon* T.

2. If the function $\lambda(t, \cdot) : F_+ \to G$ is constant for all $t \geq 0$, then $\Phi_t^2 = 0$.

3. If for every $x \geq 0$, the volatility is such that

$$\sigma(t, f)^* \delta_x = \sigma(t, g)^* \delta_x,$$

whenever $f(s) = g(s)$ for all $0 \leq s \leq x$, then

$$\text{supp}\{\Phi_t^3\} \subset [0, T - t],$$

for all $t \in [0, T]$.

Remark 5. The decompositon $\bar{\phi} = \Phi^1 + \Phi^2 + \Phi^3$ can be given a financial interpretation. The first fund Φ^1 is universal in the sense that each investor in this market invests a portion of his wealth in Φ^1. We shall see that it is a multiple of the familiar Merton ratio $\sigma^{*-1}\lambda$. The second fund Φ^2 can be interpreted as the investor's hedge against fluctations in the market price of risk. This portfolio is generally non-zero unless λ_t is deterministic. The mutual fund Φ^3 is unique to the bond market setting. It arises because the risk free asset, the bank account, can be replicated by a portfolio of just-maturing bonds. If the volatility satisfies a certain maturity-mixing condition, satisfied by the Gauss-Markov HJM model for instance, then the portfolio Φ^3 consists of bonds with maturities less than the terminal date T.

We defer the proof to Section 1.8. In the next section, we explicitly compute the optimal portfolio for a specific HJM model.

1.6 A Gauss-Markov example

To fix ideas and to demonstrate that there exists nontrivial models which can be expressed in the above framework, we offer an example in this section corresponding to a Gauss-Markov HJM random field model proposed by Kennedy [29] and further studied by Goldstein [19] and by Santa-Clara and Sornette [37].

The model we analyze is determined by the following: two twice-differentiable functions $m : \mathbb{R}_+ \to \mathbb{R}$ and $n : \mathbb{R}_+ \to \mathbb{R}_+$, and a postive constant $\alpha > 0$.

We consider the Gauss-Markov HJM model given by

$$df_t(x) = \left(\frac{\partial}{\partial x} f_t(x) + a(x) \right) dt + b(x) dW_t^{\mathbb{P}}.$$

The function $b : \mathbb{R}_+ \to G$ describes the instantaneous covariance of the forward rates and is related to the model parameters by

$$\langle b(x), b(y) \rangle_G = n(x)n(y)e^{-\alpha|x-y|}.$$

Indeed, let $G = L^2(\mathbb{R}_+)$ and, for each $x \geq 0$, let $b(x)$ be the element of G given by

$$b(x, s) = \sqrt{2\alpha} n(x) \mathbf{1}_{\{s \geq x\}} e^{-\alpha(s-x)}.$$

The instantaneous drift $a : \mathbb{R}_+ \to \mathbb{R}$ is related to the model parameters by

$$a(x) = m(x) + n(x) \int_0^x n(s) e^{-\alpha(s-x)} ds.$$

This forward rate model can be put in the framework of the discounted bond price models introduced in section 1.3. We will first identify conditions on the functions m and n so that there exist functions σ, λ, and Γ satisfying Assumptions 1.3.2 and 1.3.6 for a suitable choice of state space F. Since the model under consideration is time homogeneous, and, moreover, the market price of risk is constant, in this section we will abuse notation and let $\sigma(f) = \sigma(t, f)$, $\lambda = \lambda(t, f)$ and $\Gamma(f) = \Gamma(t, f)$. These functions are related to this forward rate model by the equations

$$\sigma(f)^*\delta_x = -f(x)\int_0^x b(s)ds,$$
$$(\sigma(f)\lambda)(x) = -f(x)\int_0^x m(s)ds,$$
$$\sigma(f)^*\Gamma(f) = \lambda,$$

for all $t \geq 0$ and $f \in F_+^0$.

We list here the assumptions on our model.

Assumption 1.6.1. *The functions m, m', m'', n' and n'' decay exponentially at infinity. Moreover, we have that the function n is bounded from below.*

We fix $\beta > 0$ sufficiently small and let the state space be

$$F = \left\{ f : \mathbb{R}_+ \to \mathbb{R}; \lim_{x\to\infty} f(x) = 0 \text{ and } \int_0^\infty f'(x)^2 e^{\beta x} dx < +\infty \right\},$$

where f' denotes the weak derivative of the absolutely continuous function f. The space F is a separable Hilbert space for the norm $\|f\|_F = (\int_0^\infty f'(x)^2 e^{\beta x} dx)^{1/2}$ and satisfies Assumption 1.3.1. The dual space F^* is the completion of the space of finite signed measures on $\mathbb{R}_+$ for the norm $\|\mu\|_{F^*} = (\int_0^\infty \mu[0,x)^2 e^{-\beta x} dx)^{1/2}$.

Proposition 1.6.2. *For every $f \in F$, the linear operator K_f on G, defined by*

$$(K_f g)(x) = -\sqrt{2\alpha}\int_0^\infty \left(f(x)\int_0^{s\wedge x} n(t)e^{-\alpha(s-t)}dt \right) g(s)ds,$$

is Hilbert-Schmidt from G into F. Moreover, the function $\sigma : F \to \mathcal{L}_{\mathrm{HS}}(G,F)$, defined by

$$\sigma(f) = K_f, \tag{1.8}$$

satisfies Assumption 1.3.2.

Proof. That K_f is Hilbert-Schmidt follows from the fact that

$$\begin{aligned}
\|K_f\|^2_{\mathcal{L}_{\mathrm{HS}}(G,F)} &= 2\alpha \int_0^\infty \int_0^\infty \left[\frac{\partial}{\partial x} \left(f(x) \int_0^{s\wedge x} n(t) e^{-\alpha(s-t)} dt \right) \right]^2 e^{\beta x} ds\ dx \\
&\leq 2 \int_0^\infty f'(x)^2 e^{\beta x} \int_0^x \int_0^x n(s) n(t) e^{-\alpha|t-s|} ds\ dt\ dx + 2 \int_0^\infty f(x)^2 e^{\beta x} n(x)^2 dx \\
&\leq (4\alpha^{-1} + 2\beta^{-1}) \|f\|_F^2 \int_0^\infty n(x)^2 dx.
\end{aligned}$$

Here, we have used the Sobolev-type inequality

$$\begin{aligned}
f(x)^2 &= \left(\int_x^\infty f'(s) ds \right)^2 \\
&\leq \left(\int_x^\infty e^{-\beta s} ds \right) \left(\int_x^\infty f'(s)^2 e^{\beta s} ds \right) \\
&\leq \beta^{-1} e^{-\beta x} \|f\|_F^2.
\end{aligned}$$

We have also used the following estimate on the norm of the integral operator

$$\begin{aligned}
\int_0^\infty \int_0^\infty n(t) n(u) e^{-\alpha|u-t|} dt\ du &\leq \frac{1}{2} \int_0^\infty \int_0^\infty (n(t)^2 + n(u)^2) e^{-\alpha|u-t|} dt\ du \\
&= 2 \int_0^\infty \int_t^\infty n(t)^2 e^{-\alpha(u-t)} du\ dt \\
&= 2\alpha^{-1} \int_0^\infty n(t)^2 dt.
\end{aligned}$$

Furthermore, since $\sigma : F \to \mathcal{L}_{\mathrm{HS}}(G,F)$ is linear, the above bounds show that σ is Lipschitz.

We omit the verification of equations (1.3) and (1.4). $\square$

Proposition 1.6.3. *Suppose the initial bond price curve $\tilde{P}_0 \in F_+$ satisfies*

$$\inf_{x\geq 0} e^{3\beta x/4} \tilde{P}_0(x) > 0.$$

Define the subset $F_+^0 \subset F_+$ by

$$F_+^0 = \left\{ f \in F_+ ; \inf_{x \geq 0} e^{\beta x} f(x) > 0 \right\}.$$

If $(\tilde{P}_t)_{t\geq 0}$ is the solution to equation (1.5) with σ given by equation (1.8), then $\tilde{P}_t \in F_+^0$ for all $t \in [0,T]$, almost surely.

Proof. By the above Sobolev inequality we have $\tilde{P}_t(x) \leq \beta^{-1/2} e^{-\beta x/2} \|\tilde{P}_t\|_F$. Since

$$\tilde{P}_t(x) = \tilde{P}_0(t+x) \exp\left(-\frac{1}{2} \int_0^t \| \int_0^{t-s+x} b(u)du \|^2 ds + \int_0^t \int_0^{t-s+x} b(u)du\ dW_s \right) \tag{1.9}$$

we have

$$\exp\left(\int_0^t \int_0^{t-s+x} b(u)dudW_s \right) \leq \beta^{-1/2} e^{-\beta x/2} \|\tilde{P}_t\|_F \tilde{P}_0(t+x)^{-1} \exp\left(t \int_0^\infty n(s)^2 ds \right).$$

Now, we may take the probability space Ω as the canonical space of continuous functions taking values in a suitable Banach space E. Also, we may choose that the Wiener process is the coordinate map $W_t(\omega) = \omega(t)$, such that the space $G \subset E$ is the reproducing kernel Hilbert space for the law of W_1. For a further discussion of the reproducing kernel Hilbert space and infinite dimensional integration theory, see [8]. With this choice of Ω, let $\tilde{P}_t^-$ be defined as the F-valued random variable given by $\tilde{P}_t^-(\omega) = \tilde{P}_t(-\omega)$.

Then,

$$\tilde{P}_t(x) \geq \beta^{1/2} \tilde{P}_0(t+x)^2 e^{\beta x/2} \|\tilde{P}_t^-\|^{-1} \exp\left(-t \int_0^\infty n(s)^2 ds \right),$$

and the assertion follows.

□

Proposition 1.6.4. *Let $\lambda \in G$ be defined via*

$$\lambda(s) = \frac{1}{\sqrt{2\alpha}} \left[\alpha \frac{m(s)}{n(s)} - \left(\frac{m(s)}{n(s)} \right)' \right].$$

Then

$$(\sigma(f)\lambda)(x) = -f(x)\int_0^x m(s)ds.$$

Moreover, there exists a function $\Gamma : F_+^0 \to F^*$*, such that* $\sigma(f)^*\Gamma(f) = \lambda(s)$*.*

If the function

$$R(s) = \frac{\left(\frac{m(s)}{n(s)}\right)'' - \alpha^2 \frac{m(s)}{n(s)}}{n(s)}$$

is locally bounded, then $\Gamma(f) \in F^*$ *can be realized as a signed measure* μ *on* $\mathbb{R}_+$*, which solves the equation*

$$\int_s^\infty f(x)\mu(dx) = \frac{1}{2\alpha}R(s). \tag{1.10}$$

Proof. With the formulas in hand, the verification is a tedious but straightforward integration by parts. Assumption 1.6.1 and Proposition 1.6.3 guarantee that the norm $\|\Gamma(f)\|_{F^*}$ can be controlled. □

Remark 6. Equation (1.10) can be given an financial interpretation: The points of discontinuity of R correspond to the atoms of the measure μ. Moreover, these atoms signify investment decisions for our investor. In particular, an optimal investor will hold the bonds of relative maturities given by the locations of the upward jumps of this function. And conversely, the investor will sell short the bonds given by the downward jumps.

1.7 Some results from Malliavin calculus

There has been much recent academic interest in the financial applications of Malliavin calculus. In this section we will present several results without proof. One may find a more detailed treatment of the following results in Carmona and Tehranchi [7] and Nualart [34] among others.

The Malliavian derivative is a linear mapping from a space of random variables to a space of processes. We are concerned with the case where the random variables are elements of $L^p(\Omega;H)$, where H is one of the spaces $\mathbb{R}$, F, G, or the Hilbert-Schmidt operators $\mathcal{L}_{\mathrm{HS}}(F,G)$. The Malliavin derivative of a random variable $\xi \in L^p(\Omega;H)$ is a process $D\xi \in L^p(\Omega;L^2([0,T];\mathcal{L}_{\mathrm{HS}}(G,H)))$.

The Malliavin derivative operator is unbounded on $L^p(\Omega;G)$, so we then define it first on a core and then extending its definition to the closure of this set in the graph norm topology.

Definition 1.7.1. *Smooth random variables $\xi \in L^p(\Omega;H)$ are of the form*

$$\xi = \kappa\left(\int_0^T h^1(s)dW_s, \dots, \int_0^T h^n(s)dW_s\right), \tag{1.11}$$

where $h^1 \dots h^n \in L^2([0,T];G)$ are deterministic, and where the infinitely differentiable function $\kappa : \mathbb{R}^n \to H$ is, along with all of its derivatives, polynomially bounded. The Malliavin derivative of a smooth random variable is defined to be

$$D_t\xi = \sum_{i=1}^n \frac{\partial \kappa}{\partial x_i}\left(\int_0^T h^1(s)dW_s, \dots, \int_0^T h^n(s)dW_s\right) \otimes h^i(t).$$

Definition 1.7.2. *If ξ is the $L^p(\Omega,H)$ limit of a sequence $(\xi_n)_{n\geq 1}$ of smooth random variables such that $(D\xi_n)_{n\geq 1}$ converges in $L^p(\Omega;L^2([0,T];\mathcal{L}_{\mathrm{HS}}(G,H)))$, then we define*

$$D\xi = \lim_{n\to\infty} D\xi_n.$$

We use the notation $\mathbb{D}^{1,p}(H)$ to represent the subspace of $L^p(\Omega;H)$ where the derivative can be defined by Definition 1.7.1. This subspace is a Banach space for the graph norm

$$\|\xi\|_{\mathbb{D}^{1,p}(H)} = \left(\mathbb{E}\|\xi\|_H^p + \mathbb{E}\left(\int_0^T \|D_t\xi\|^2_{\mathcal{L}_{\mathrm{HS}}(G,H)}dt\right)^{p/2}\right)^{1/p}.$$

Now, recall the Clark-Ocone formula. This is the crucial result we need which provides an explicit martingale representation for random variables in $\mathbb{D}^{1,2}(\mathbb{R})$. The martingal representation is given in terms of the Malliavin derivative. A proof of the infinite-dimensional version of this result can be found in [7]. Also, an early application of the Clark-Ocone formula to utility maximization problems can be found in Karatzas and Ocone [27].

Theorem 1.7.3 (Clark-Ocone formula)**.** *For every $\mathcal{F}_T$ measurable random variable $\xi \in \mathbb{D}^{1,2}(\mathbb{R})$, we have the representation*

$$\xi = \mathbb{E}\ \xi + \int_0^T \mathbb{E}\{D_t \xi | \mathcal{F}_t\} dW_t. \tag{1.12}$$

We close this section with two results which allow us to calculate explicit formulas in what follows. The first result is a generalization of the chain rule in the spirit of Proposition 1.2.3 of Nualart [34]:

Proposition 1.7.4. *Let H_1 and H_2 be real separable Hilbert spaces and let $\mathcal{L}(H_1, H_2)$ denote the Banach space of bounded linear operators from H_1 into H_2. Then, for a random variable $\xi \in \mathbb{D}^{1,p}(H_1)$ and a globally Lipschitz function $\kappa : H_1 \to H_2$ with Lipschitz constant C, the random variable $\kappa(\xi)$ is in $\mathbb{D}^{1,p}(H_2)$. Moreover, there exists a random variable Z satisfying the bound $\|Z\|_{\mathcal{L}(H_1,H_2)} \leq C$, almost surely, and such that*

$$D\kappa(\xi) = Z D\xi. \tag{1.13}$$

Remark 7. Although the function κ may not be differentiable, there still exists the random variable Z which plays the role of a derivative in the sense of the chain rule. Of course, if κ is Fréchet differentiable, then $Z = \nabla\kappa(\xi)$ is its Fréchet derivative evaluated at ξ. In Section 1.8 we use this result in the cases $\kappa = \lambda(t, \cdot) : F \to G$ and $\kappa = \sigma(t, \cdot) : F \to \mathcal{L}_{\mathrm{HS}}(G, F)$.

The second result which we state without proof is the infinite dimensional analog of (1.46) of Nualart [34].

Proposition 1.7.5. *If the adapted continuous square integrable process $(\alpha_t)_{t\in[0,T]}$ is such that, for all $t \in [0, T]$, the random variable $\alpha_t \in \mathbb{D}^{1,p}(\mathcal{L}_{\mathrm{HS}}(G, H))$ is differentiable, for $p \geq 2$, then*

$$D_t \int_0^T \alpha_s dW_s = \alpha_t + \int_t^T D_t \alpha_s dW_s.$$

1.8 Proof of the main results

In this section we give the proof of the main results. Recall that the inverse marginal utility function $I = (U')^{-1}$ plays a crucial role in the study of the optimal investment problem. We begin with a duality lemma.

Assumption 1.8.1. *The utility function $U : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ and inverse marginal utility $I : (\underline{y}, \infty) \to (\underline{x}, \infty)$ satisfy Assumption 1.5.1.*

Lemma 1.8.2. *Fix the investor's initial wealth $X_0 > 0$. There exists a unique real number z_0 such that*

$$\mathbb{E}^{\mathbb{Q}} B_T^{-1} I\left(z_0 B_T^{-1} \frac{d\mathbb{Q}}{d\mathbb{P}}\right) = X_0.$$

Furthermore, for each random variable X with $\mathbb{E}^{\mathbb{Q}}\ B_T^{-1} X = X_0$ we have

$$\mathbb{E}^{\mathbb{P}}\ U(X) \leq \mathbb{E}^{\mathbb{P}}\ U \circ I(\eta),$$

where $\eta = z_0 B_T^{-1} \frac{d\mathbb{Q}}{d\mathbb{P}}$.

Proof. By Assumption 1.5.1, there is a constant $C > 0$ such that $|I(y)| < C(|y|^{-q} + |y|^{-q})$, where again, the bound is designed to handle both cases for $\underline{y} = 0$ and $\underline{y} = -\infty$. The density $\frac{d\mathbb{Q}}{d\mathbb{P}}$ and the discounted bond price $\tilde{P}_t(0)$ have moments of all negative orders by Proposition 1.3.4, and, in particular,

$$\mathbb{E}^{\mathbb{Q}} B_T^{-1} I\left(z B_T^{-1} \frac{d\mathbb{Q}}{d\mathbb{P}}\right) < +\infty$$

for all $z > 0$. Since the function $I : (\underline{y}, \infty) \to (\underline{x}, \infty)$ is continuous and decreasing, the function $z \mapsto \mathbb{E}^{\mathbb{Q}} B_T^{-1} I\left(z B_T^{-1} \frac{d\mathbb{Q}}{d\mathbb{P}}\right)$ is continuous and decreasing by the Monotone Convergence Theorem, and hence invertible on its range. The number $X_0 > 0$ is, in fact, contained in the range since $\lim_{y \to \underline{y}} I(y) = +\infty$ and $\lim_{y \to \infty} I(y) = \underline{x} \leq 0$.

We expand U into a Taylor series about the point $I(\eta)$. Since the utility function is concave, we have

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}} U(X) &\leq \mathbb{E}^{\mathbb{P}}\, U \circ I(\eta) + \mathbb{E}^{\mathbb{P}}\, \eta(X - I(\eta)) \\
&= \mathbb{E}^{\mathbb{P}}\, U \circ I(\eta) + z_0 \mathbb{E}^{\mathbb{Q}}\, (B_T^{-1} X - B_T^{-1} I(\eta)) \\
&\leq \mathbb{E}^{\mathbb{P}}\, U \circ I(\eta),
\end{aligned}$$

completing the proof. $\square$

The main theorem is then proven if we can show that there exists an admissible strategy $\bar{\phi}$ such that $X_T^{\bar{\phi}} = I(\eta)$. We approach this question via the Clark-Ocone formula. To this end, first recall the following lemma from Carmona and Tehranchi [8].

Lemma 1.8.3. *For every $p \geq 2$ and all $t \geq 0$, we have $\tilde{P}_t \in \mathbb{D}^{1,p}(F)$.*

We now prove a representation formula for the derivative $D\tilde{P}_t$. For this, we appeal to Skorohod's theory of strong random operators as developed in Skorohod [40].

Definition 1.8.4. *A strong random operator from F into a separable Hilbert space H is a H-valued stochastic process $(Z(f))_{f \in F}$ indexed by F which is linear in $f \in F$.*

A strong operator process $(Z_t(f))_{t \geq 0, f \in F}$ is similarly defined.

Definition 1.8.5. *Let a strong operator process $(Z_t(f))_{t \geq 0, f \in F}$ be adapted and let $H = \mathcal{L}_{\mathrm{HS}}(G, F)$. Then, by setting*

$$\left(\int_0^t Z_s \cdot dW_s \right)(f) = \int_0^t Z_s(f)\, dW_s,$$

we define a strong random operator $\int_0^t Z_s \cdot dW_s$ from F into F.

The following Proposition gives a useful representation formula for the Malliavin derivative of the discounted bond price. See [8] for a proof.

Proposition 1.8.6. *The Malliavin derivative $D\tilde{P}_t$ is given by*

$$D_s\tilde{P}_t = Y_{s,t}\sigma(s, \tilde{P}_s), \tag{1.14}$$

for $s \in [0, t]$. Here, the strong operator process $(Y_{s,t})_{0\leq s\leq t}$, is the solution to the integral equation

$$Y_{s,t} = S_{t-s} + \int_s^t S_{t-u}\nabla\sigma_u Y_{s,u} \cdot dW_u,$$

for $t \geq s$. Also, $\nabla\sigma_t$ is the $\mathcal{L}(F, \mathcal{L}_{\mathrm{HS}}(G, F))$-valued random variable such that $D\sigma(t, \tilde{P}_t) = \nabla\sigma_t D\tilde{P}$.

Thus now that we have equation (1.14) as a representation of the Malliavin derivative of the discounted bond price. It then follows that is Malliavin differentiable.

Corollary 1.8.7. *For every $p \geq 2$, we have $B_T^{-1} = \tilde{P}_T(0) \in \mathbb{D}^{1,p}(\mathbb{R})$.*

Lemma 1.8.8. *For every $p \geq 2$, we have $\frac{d\mathbb{Q}}{d\mathbb{P}} \in \mathbb{D}^{1,p}(\mathbb{R})$.*

Proof. Recall that the density is given by the exponential

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(\frac{1}{2}\int_0^T \|\lambda(s, \tilde{P}_s)\|_G^2 ds + \int_0^T \lambda(s, \tilde{P}_s) dW_s\right).$$

Recall Assumption 1.3.6, namely that the function $\lambda(s)$ is bounded. We then have

$$D_t\frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{d\mathbb{Q}}{d\mathbb{P}}\left(\lambda(t, \tilde{P}_t) + \int_t^T \nabla\lambda_s D_t\tilde{P}_s \cdot (dW_s + \lambda(s, \tilde{P}_s)ds)\right),$$

where $\nabla\lambda_s$ is the bounded $\mathcal{L}(F, G)$-valued random variable such that $D\lambda(s, \tilde{P}_s) = \nabla\lambda_s D\tilde{P}_s$.

$\square$

Corollary 1.8.9. *For every $p \geq 2$, we have $\eta = z_0 B_T^{-1}\frac{d\mathbb{Q}}{d\mathbb{P}} \in \mathbb{D}^{1,p}(\mathbb{R})$.*

Lemma 1.8.10. *For every $p \geq 2$, we have $I(\eta) \in \mathbb{D}^{1,p}(\mathbb{R})$ and $DI(\eta) = I'(\eta)D\eta$.*

Proof. The chain rule is not directly applicable because of the singularities $\lim_{y\to\underline{y}} I(y) = \infty$ and $\lim_{y\to\underline{y}} I'(y) = -\infty$. We first find a sequence of Malliavin-differentiable random variables, $I_n(\eta)$, which converge to $I(\eta)$ in $L^p(\Omega;\mathbb{R})$, such that $\mathbb{E}\left(\int_0^T \|D_s I_n(\eta)\|_G^2 ds\right)^{p/2}$ is uniformly bounded. Recall Assumption 1.5.1, in particular, the growth constraints on the utility function. Also, recall the moment bounds in Proposition 1.3.4. We then have

$$\begin{aligned}
\mathbb{E}\,|I(\eta)|^p &\leq C\mathbb{E}\,(|\eta|^{-pq} + |\eta|^{pq}) \\
&= C(|z_0|^{-pq}\mathbb{E}\,\tilde{P}_T(0)^{-pq}\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)^{-pq} + |z_0|^{pq}\mathbb{E}\,\tilde{P}_T(0)^{pq}\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)^{pq}) < +\infty.
\end{aligned}$$

Note that $z_0 = 0$ only if $\underline{y} = -\infty$, in which case $|I(\eta)| < K$, for some constant K.

There are then two cases to consider for the limiting values as $y \to \underline{y}$. Recall that $\underline{y} = 0$ and $\underline{y} = -\infty$.

If $\underline{y} = 0$, we let

$$I_n(y) = \begin{cases} I(y) & \text{if } y > \frac{1}{n} \\ I(\frac{1}{n}) & \text{if } y \leq \frac{1}{n}, \end{cases}$$

whereas if $\underline{y} = -\infty$, we set

$$I_n(y) = \begin{cases} I(y) & \text{if } y > -n \\ I(-n) & \text{if } y \leq -n. \end{cases}$$

Note now that $(|I_n(y) - I(y)|)^p < 2^p I(y)^p$, since I is decreasing. We then have $\mathbb{E}|I_n(\eta) - I(\eta)|^p \to 0$ by the Dominated Convergence Theorem.

Now, we show that $\mathbb{E}\left(\int_0^T \|D_s I_n(\eta)\|_G^2 ds\right)^{p/2}$ is uniformly bounded. Since I_n is Lipschitz we can apply Proposition 1.7.4. This, in turn, yields

$$\begin{aligned}
\mathbb{E}\left(\int_0^T \|D_s I_n(\eta)\|_G^2\right)^{p/2} ds &= \mathbb{E}|I_n'(\eta)|^p \left(\int_0^T \|D_s\eta\|_G^2 ds\right)^{p/2} \\
&\leq \left(\mathbb{E}|I_n'(\eta)|^{2p}\right)^{1/2}\left(\mathbb{E}\left(\int_0^T \|D_s\eta\|_G^2 ds\right)^p\right)^{1/2},
\end{aligned}$$

where

$$I'_n(y) = \begin{cases} I'(y) & \text{if } y > \frac{1}{n} \\ 0 & \text{if } y \le \frac{1}{n}, \end{cases}$$

or

$$I'_n(y) = \begin{cases} I'(y) & \text{if } y > -n \\ 0 & \text{if } y \le -n, \end{cases}$$

depending on the value of $\underline{y}$.

The uniform bound follows from the estimate

$$\mathbb{E}|I'_n(\eta)|^{2p} \le C^{2p}\mathbb{E}(|\eta|^{-2p(q+1)} + |\eta|^{-2p(-q-1)}) < +\infty.$$

In fact, for $\underline{y} = 0$, for every $p \ge 2$, we have

$$DI_n(\eta) \to I'(\eta)D\eta \tag{1.15}$$

in $L^p(\Omega, L^2([0,T];G))$. This follows since

$$\begin{aligned}\mathbb{E}\left(\int_0^T \|D_t I_n(\eta) - I'(\eta)D_t\eta\|_G^2 dt\right)^{p/2} &= \mathbb{E}\left(\int_0^T \|(I'_n(\eta) - I'(\eta))D_t\eta\|_G^2 dt\right)^{p/2} \\ &= \mathbb{E}\ \mathbf{1}_{\{\eta \le \frac{1}{n}\}}|I'(\eta)|^p \left(\int_0^T \|D_t\eta\|_G^2 dt\right)^{p/2}\end{aligned}$$

converges to zero by the Dominated Convergence Theorem.

Moreover, the same result clearly holds for $\underline{y} = -\infty$, since

$$\begin{aligned}\mathbb{E}\left(\int_0^T \|D_t I_n(\eta) - I'(\eta)D_t\eta\|_G^2 dt\right)^{p/2} &= \mathbb{E}\left(\int_0^T \|(I'_n(\eta) - I'(\eta))D_t\eta\|_G^2 dt\right)^{p/2} \\ &= \mathbb{E}\ \mathbf{1}_{\{\eta \le -n\}}|I'(\eta)|^p \left(\int_0^T \|D_t\eta\|_G^2 dt\right)^{p/2}\end{aligned}$$

also converges to zero by the Dominated Convergence Theorem. □

Corollary 1.8.11. *We have* $B_T^{-1}I(\eta) \in \mathbb{D}^{1,2}(\mathbb{R})$ *and*

$$\begin{aligned} D_t B_T^{-1} I(\eta) &= B_T^{-1}\eta I'(\eta)\lambda_t - B_T^{-1}\eta I'(\eta)\int_t^T \nabla\lambda_s D_t\tilde{P}_s(dW_s + \lambda_s ds) \\ &+ (I(\eta) + \eta I'(\eta))D_t\tilde{P}_T(0). \end{aligned}$$

Combining the previous representation result with the Clark-Ocone formula yields

$$\begin{aligned} B_T^{-1}I(\eta) &= X_0 + \int_0^T \mathbb{E}^{\mathbb{Q}}\{D_t B_T^{-1}I(\eta)|\mathcal{F}_t\}dW_t \\ &= X_0 + \int_0^T \sigma_t^*(\Phi_t^1 + \Phi_t^2 + \Phi_t^3)dW_t, \end{aligned}$$

where

$$\begin{aligned} \Phi_t^1 &= \mathbb{E}^{\mathbb{Q}}\{B_T^{-1}\eta I'(\eta)|\mathcal{F}_t\}\Gamma(t,\tilde{P}_t) \\ \Phi_t^2 &= -\mathbb{E}^{\mathbb{Q}}\left\{ B_T^{-1}\eta I'(\eta)\int_t^T \nabla\lambda_s Y_{t,s}\cdot(dW_s + \lambda_s ds)\middle| \mathcal{F}_t\right\} \\ \Phi_t^3 &= \mathbb{E}^{\mathbb{Q}}\{\big(I(\eta) + \eta I'(\eta)\big)\, Y_{t,T}^*\delta_0|\mathcal{F}_t\}. \end{aligned}$$

Finally, from Theorem 5.7 of [7] we have that if, for every $x \geq 0$, the volatility is such that

$$\sigma(t,f)^*\delta_x = \sigma(t,g)^*\delta_x,$$

whenever $f(s) = g(s)$ for all $0 \leq s \leq x$, then

$$\mathrm{supp}\{\Phi_t^3\} \subset [0, T-t],$$

for all $t \in [0,T]$.

1.9 Conclusion

Here we have studied the Merton problem of optimal portfolio choice and investment when the underlying traded instruments are the zero-coupon bonds. This problem of utility maximization has a long academic history in financial mathematics and economics, and, here, in our particular study, we analyzed the problem while modeling the bond dynamics in an infinite-dimensional Heath-Jarrow-Morton framework. Moreover, we find conditions for the existence and uniqueness of the optimal portfolio. And in the case of a unique trading portfolio, we provide its representation.

The infinite-dimensional setting in which we work poses many mathematical challenges. To meet these demands we appeal to the Malliavin calculus. Most importantly, the Malliavin calculus provides a martingale representation formula, via the Clark-Ocone formula, which allows us to characterize the hedging portfolio which solves the Merton problem of optimal investment and allocation. Most interestingly, from this representation of the optimal strategy, we find a natural financial interpretation of the portfolio. In particular, this portfolio agrees with the trading strategies and intuition often used by Wall St fixed income traders.

Chapter 2

Indifference Pricing with Term Structure and Change-of-Numeraire

2.1 Introduction

We consider the problem of pricing a claim for an investor when the underlying security is a derivative priced on the set of zero-coupon bonds. Some commonly traded derivatives include interest rate claims such as fixed and floating rate swaps and swaptions. Our investor has a specific utility function, $U(x) = -e^{\gamma x}$ and fixed planning horizon $T > 0$. We model the dynamics of the underlying in an infinite-dimensional setting, in particular, letting the bond price process evolve in a Hilbert space. There has been academic interest in infinite-dimensional pricing theory, in particular, Goldys and Musiela, [20], in which the authors derive infinite-dimensional Black-Scholes partial differential equations for the pricing of interest rate claims in a complete market. Our goal here is to price such a derivative security in an incomplete market.

The classic options pricing methodology is the celebrated Black-Scholes framework.

In this framework, the price of a derivative in a complete market is its discounted expected payoff under a uniquely determined measure, the unique equivalent martingale pricing measure. This Black-Scholes framework is able to uniquely price derivatives because claims are replicable and risks are hedgeable. Completeness in an infinite factor model is, unfortunately, a difficult point. Even in a market with a unique equivalent martingale pricing measure, there may exist claims not entirely replicable by a self-financing strategy. This, of course, makes a Black-Scholes pricing theory difficult to achieve in these models. Björk et al. [6] have developed the notion of approximate completeness to deal with this problem. Here we choose rather to embrace the natural incompleteness of the market. This motivates a need to develop a pricing mechanism in such an incomplete market with unhedgeable risks. A leading method in incomplete market pricing is that of indifference pricing. Musiela and Zariphopoulou [33] and Rouge and El Karoui [36] are a couple of the leading treatments of indifference pricing problems. The first results concerning term structure and indifference pricing come from Young [42]. In [42], she models the short rate via a multi-factor diffusion and then solves the primal problem by deriving the associated Hamilton-Jacobi-Bellman equations for pricing a claim in an incomplete market.

An investor's indifference price comes from the study of the maximization of his or her expected utility of investment opportunities. In particular, it can be understood in terms of solving the investor's Merton problem. In our infinite-dimensional framework, the utility maximization problem has been well-studied, for example Carmona and Tehranchi [7]. There are multiple approaches one may take in the study of the investor's Merton problem. Here we will take the dual approach. There has been much research into classifying the dual solutions to the Merton investment problem. Notably, Delbaen et al [13] and Frittelli [17] have given nice characterizations of the dual problem, i.e., optimizing over a set of measures instead of the primal problem of optimizing over a set of trading strategies. This provides a helpful approach in computing the indifference price in our incomplete market.

In pricing interest rate claims we need to take explicit care in formulating the pricing units, i.e., whether we are talking about dollars in today's dollars or in future dollars. In this setting we introduce the notion of spot and forward units. Since we are pricing using the indifference pricing methodology, we have an important need to carefully formulate our notion of investment preferences through the investor's utility function. The investor's utility becomes dynamic; a function of the time value of money. We introduce a notion of pricing consistency across units in an incomplete market. This leads not only to pricing consistency across units in both the complete and incomplete markets, but yields a change-of-numeraire technique for pricing in our infinite factor incomplete market framework. Characterizing this change-of-numeraire in our framework, namely, this idea of numeraire consistency and indifference pricing, is the second aim of this study. Ultimately, whether we price a claim in today's units or tomorrow's, we can yield consistent prices for our claims in our framework. This prevents so-called numeraire arbitrage, where one could theoretically generate riskless cash from an investment simply by choosing one's units in a specified way.

There has been academic interest in the applications of the change-of-numeraire techniques developed by Geman et al. [18] and Jamshidian [24]. The change-of-numeraire tool finds a natural home in interest-rate pricing. In the complete market finite-dimensional world of Black-Scholes it is often advantageous to perform a change-of-numeraire and work under the so-called forward measure when pricing a stock option with a stochastic interest rate. The technique often allows for easier calculations. Benninga et al. [3] and Björk [6] illustrate several examples (not just of interest rate options) of option pricing problems which simplify via a change of measure.

Ultimately, we achieve, via a proper specification of an investor's risk preferences (defined implicitly in the utility function), a general change-of-numeraire theorem for incomplete markets. We illustrate this with the pricing of a general interest-rate option. We supply a calculation of the indifference price of a claim both in forward and spot units when the model is infinite-dimensional. Moreover, an incomplete market pricing consistency theorem, i.e., the change-of-numeraire theorem, is given for the interest rate case. We find that we can preclude numeraire arbitrage in an incomplete market if and only if the risk preferences are specified appropriately in terms of the chosen numeraire.

2.2 Market Model

We first fix a probabilty space for our model, $(\Omega, \mathcal{F}, \mathbb{Q})$ and a real, separable, infinite-dimensional Hilbert space G with inner product $\langle , \rangle_G$, and we let $(W_t)_{t\geq 0}$ be a Wiener process defined cylindrically on G, and the filtration $(\mathcal{F}_t)_{t\geq 0}$ is given by the augmentation of the filtration it generates. For example, one could take $G = \ell^2$ as in [15] as this amounts to choosing an orthonormal basis of G and working with the coordinates in this basis. We, however, will keep G unspecified.

We will work within a Heath-Jarrow-Morton framework, proposed in [23], where $P(t,T)$ denotes the price, at time t, of a zero-coupon bond which is worth one unit at the maturity date $T \geq t$.

Also, we will use the parametrization proposed by Musiela [32], where $f_t(x)$ denotes the forward rate at time t with time to maturity x. The forward rates are related to the price $P(t,T)$ at time t of a zero-coupon bond with maturity date $T = t + x$ by

$$f_t(x) = -\frac{\partial}{\partial x} \log \left(P(t, t+x) \right),$$

whenever the derivative exists. In this framework, the risk-neutral dynamics satisfy the following stochastic partial differential equation

$$df_t(x) = \left(\frac{\partial}{\partial x} f_t(x) + a_t(x) \right) dt + b_t(x) dW_t^{\mathbb{P}},$$

where the process $(W_t^{\mathbb{P}})_{t\geq 0}$ is a Wiener process under the historical measure $\mathbb{P}$. The drift is given by the HJM no-arbitrage condition

$$a_t(x) = b_t(x)\lambda_t + b_t(x) \int_0^x b_t(s) ds,$$

where λ_t is the market price of risk. The random variables $b_t(x)$, λ_t, and $W_t^{\mathbb{P}}$ are allowed to be vector-valued, in which case products are interpreted as standard Euclidean inner products.

Let $B_t = \exp\left(\int_0^t r_s \, ds \right)$ denote the value at time t of the bank account with initial deposit one dollar, where the short rate is given by $r_t = f_t(0)$. The discounted bond price,

$$\tilde{P}_t(x) = B_t^{-1} P(t, t+x),$$

written here in Musiela notation, formally satisfies the stochastic partial differential equation

$$d\tilde{P}_t(x) = \left(\frac{\partial}{\partial x} \tilde{P}_t(x) - \tilde{P}_t(x) \int_0^x b_t(s) ds \ \lambda_t \right) dt - \tilde{P}_t(x) \int_0^x b_t(s) ds \ dW_t^{\mathbb{P}}. \tag{2.1}$$

Letting $\sigma_t(x) = -\tilde{P}_t(x) \int_0^x b_t(s) ds$ and rewriting the above SPDE in integral form, we see that the discounted bond prices satisfy

$$\tilde{P}_t(x) = \tilde{P}_0(t+x) + \int_0^t \sigma_s(x+t-s) \lambda_s ds + \int_0^t \sigma_s(x+t-s) \ dW^{\mathbb{P}}, \tag{2.2}$$

with initial data $\tilde{P}_0(\cdot) = P(0, \cdot)$. We will interpret equation 2.2 as an evolution in the space F of real-valued functions on $\mathbb{R}_+$. We will denote F^* as the topological dual space of F.

Also, we will make use of the following assumption on the volatility.

Assumption 2.2.1. *Let $\sigma(\cdot, \cdot) : \mathbb{R}_+ \times F_+ \to \mathcal{L}_{\text{HS}}(G, F)$ be such that $\sigma(\cdot, f)$ is continuous for all $f \in F_+$ and such that $\sigma(t, \cdot)$ is globally Lipschitz, uniformly in $t \geq 0$.*

Additionally, we assume that for all $f \in F_+$ and $t \geq 0$ we have

$$\overline{\text{ran}(\sigma(t, f))} = \{g \in F : g(0) = 0\}, \tag{2.3}$$

or, equivalently,

$$\ker(\sigma(t, f)^*) = \text{span}\{\delta_0\} \subset F^*$$

and

$$\|\sigma(t, f)^* \delta_x\|_G \leq C f(x), \tag{2.4}$$

for some $C > 0$.

Here we will take the discounted bond price curve $\tilde{P}_t = \tilde{P}_t(\cdot)$ to be our state variable.

We can now define the investor's bank account.

Definition 2.2.2. *For $t \geq 0$, the bank account proces is given by $B_t = \tilde{P}_t(0)^{-1}$.*

Our investor will be trading in both units of cash (the bank account) and in the underlying bonds. He or she begins with x units of cash (that is, $B_t^{-1}x$ units of the bank account) and c_i units of the bond with maturity T_i for $i = 1, \ldots, N$. His or her wealth at time t is given by

$$\begin{aligned} x + \sum_{i=1}^{N} c_i P(t, T_i) &= B_t \left(x\delta_0 + \sum_{i=1}^{N} c_i \delta_{T_i - t} \right) (\tilde{P}_t) \\ &= B_t \ \phi_t(\tilde{P}_t). \end{aligned}$$

We have denoted $\delta_0(\tilde{P}_t) = \tilde{P}_t(0) = B_t^{-1}$. Note then that the vector of portfolio weights $(c_0, \ldots, c_N)$ corresponds to the functional $\phi_t \in F^*$.

We may now define and admissible trading strategy for our investor:

Definition 2.2.3. *An admissible investment strategy is a progressively measurable F^*-valued process $(\phi_t)_{t\geq 0}$ such that*

$$\mathbb{E} \int_0^t \|\sigma_s^* \phi_s\|_G^2 ds < +\infty,$$

for all $t \geq 0$.

Definition 2.2.4. *An admissible strategy $(\phi_t)_{t\geq 0}$ is self-financing if there exists a constant $X_0 \in \mathbb{R}$ such that*

$$\phi_t(\tilde{P}_t) - \int_0^t \sigma_s^* \phi_s dW_s = X_0$$

for almost all $(t, \omega) \in \mathbb{R}_+ \times \Omega$. The set of admissible self-financing strategies is denoted $\mathcal{A}$.

Definition 2.2.5. *Fix an initial wealth $X_0 \in \mathbb{R}$ and a self-financing strategy $(\phi_t)_{t\geq 0}$, the wealth process $(X_t^\phi)_{t\geq 0}$ is given by*

$$\begin{aligned} X_t^\phi &= B_t \ \phi_t(\tilde{P}_t) \\ &= B_t \left(X_0 + \int_0^t \sigma_s^* \phi_s dW_s \right). \end{aligned}$$

Note that for every self-financing strategy ϕ, the discounted wealth process $(B_t^{-1}X_t^{\phi})_{t\geq 0}$ is a martingale for the equivalent martingale measure $\mathbb{Q}$. It then follows that this market is arbitrage-free.

2.3 The Merton Problem

The Merton problem of optimal investment has a long academic history, beginning with Merton [31]. We begin with assigning our investor a utility function $U(x)$, $x \in \mathbb{R}$. We further fix a terminal date $T > 0$ and initial wealth $X_0 \in \mathbb{R}_+$. For a trading strategy ϕ, we consider an investor who looks to optimize the following functional,

$$J(\phi) = \mathbb{E}^{\mathbb{P}} U(X_T^{\phi}), \tag{2.5}$$

that is, his or her expected utility of wealth over the set of admissible trading strategies.

Following Ekeland and Taflin, [15], we assume the following on the investor's utility function U and the respective inverse marginal utility $I(y) = (U')^{-1}(y)$.

Assumption 2.3.1. *The utility function $U : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ is strictly concave, finite and twice continuously differentiable on an open interval $(\underline{x}, \infty)$ for some $\underline{x} \leq 0$, with the value $\underline{x} = -\infty$ allowed. Moreover, we assume $U'(x) \to \infty$ as $x \searrow \underline{x}$. Letting $\underline{y} = \inf_{x > \underline{x}} U'(x) = \lim_{x\to\infty} U'(x)$, we assume that either $\underline{y} = 0$ or $\underline{y} = -\infty$. Define the decreasing function $I : (\underline{y}, \infty) \to (\underline{x}, \infty)$ by $I(y) = (U')^{-1}(y)$.*

It is also assumed that there exists some $q > 0$ such that, for every y, the following bounds

$$|I(y)| \leq C(|y|^q + |y|^{-q})$$

and

$$|I'(y)| \leq C(|y|^{q+1} + |y|^{-q-1})$$

hold.

Note that there are several utility functions we can select for our study. We choose to focus on the exponential utility form. Our investor is given a utility $U(x) = -e^{-\gamma x}$, $x \in \mathbb{R}$, $\gamma > 0$. Here x denotes the investor's wealth, which we allow to be negative. Also, γ is a constant which represents the investor's risk aversion. Also note that if wealth is measured in dollars, then the risk aversion coefficient therefore has units in $\frac{1}{dollars}$. The inverse of the risk aversion, $\frac{1}{\gamma}$ is the investor's risk tolerance. And similarly note then that the risk tolerance has units in dollars. As we will see, there are several financial and mathematical benefits in choosing such a utility function, specifically that the exponential form yields a high degree of tractability due to its scaling properties. Moreover, it often leads to pleasing solution forms for value functions and prices; see, for example, Musiela and Zariphopoulou [33].

Theorem 2.3.2. *The exponential utility $U(x) = -e^{-\gamma x}$ satisfies assumption 2.3.1.*

Proof. $U(x) = -e^{-\gamma x}$ is clearly strictly concave, finite and twice-differentiable. Note that the inverse marginal utility of the exponential is given by

$$I(y) = -\frac{1}{\gamma} log(\frac{1}{\gamma} y).$$

The logarithmic growth of $I(y)$ also clearly satisfies the bounds of 2.3.1. One may further note that $\lim_{y \to 0} I(y) = +\infty$ and $\lim_{y \to \infty} I(y) = -\infty$. □

In pricing interest-rate sensitive claims on Wall St., pricing is normally done in terms of future value of wealth. In other words, for a future date $T > 0$, prices are in units not of today's dollars, but rather of time T dollars. We now introduce our notation for the forward units.

Definition 2.3.3. *Given a risk-neutral pricing measure $\mathbb{Q}$ and bank account process $(B_t)_{t \geq 0}$, the forward risk-neutral measure (or simply forward measure) is a measure $\mathbb{Q}^f$ such that $\frac{d\mathbb{Q}^f}{d\mathbb{Q}} = \frac{B_T^{-1}}{\mathbb{E}^{\mathbb{Q}}[B_T^{-1}]}$.*

Then, for our forward measure $\mathbb{Q}^f$, we can state the following Lemma.

Lemma 2.3.4. *Fix $x_0 > 0$. There exists a unique number $z_0 > 0$ such that*

$$\mathbb{E}^{\mathbb{Q}^f}[I\left(z_0 \frac{d\mathbb{Q}^f}{d\mathbb{P}}\right)] = x_0. \tag{2.6}$$

Proof. Note that we have logarithmic growth-bounds of $I(y)$ and that the density $\frac{d\mathbb{Q}^f}{d\mathbb{P}}$ has moments of all negative orders by equation 1.7. Thus we have that

$$\mathbb{E}^{\mathbb{Q}^f}[I\left(z \frac{d\mathbb{Q}^f}{d\mathbb{P}}\right)] < +\infty,$$

for all $z > 0$. Also note that the function $I : \mathbb{R}_+ \to \mathbb{R}$ is continuous and decreasing. The function $z \mapsto \mathbb{E}^{\mathbb{Q}^f}[I\left(z\frac{d\mathbb{Q}^f}{d\mathbb{P}}\right)]$ is then continuous and decreasing by the Monotone Convergence Theorem, and hence invertible on its range. As noted about the bounds of $I(y)$ in the above remark, clearly the initial wealth, $x_0 \in (0, \infty)$, is in this range. $\square$

Now recall, from the first essay, Theorem 1.5.2, which we re-state here:

Theorem 2.3.5. *Under Assumptions 2.2.1 and 2.3.1, there exists a unique admissible strategy $\bar{\phi} \in \mathcal{A}$ which maximizes J.*

Note then that for our investor there exists a trading strategy strategy $\bar{\phi}$, such that $X_T^{\bar{\phi}} = I(z_0 \frac{d\mathbb{Q}^f}{d\mathbb{P}})$. We now have a useful expression for our optimized terminal wealth. We can now procede to solve the investor's Merton problem.

Definition 2.3.6. *The relative entropy between two probability measures $\mathbb{Q}$ and $\mathbb{P}$ is given by $\mathbb{E}^{\mathbb{Q}}[\log(\frac{d\mathbb{Q}}{d\mathbb{P}})]$, which is denoted as $H(\mathbb{Q}|\mathbb{P})$.*

Remark 8. The relative entropy between a martingale measure $\mathbb{Q}$ and the physical measure $\mathbb{P}$ can be given a financial meaning. Relative entropy is often thought of as a way to quantify information divergence or difference between two probability measures. As is discussed in [33], in an incomplete market, the martingale pricing measure which minimizes the entropy between it and the physical measure can be interpreted as a metric of the incompleteness of the market.

Theorem 2.3.7. *Given some initial time $t \in [0, T)$, initial capital $x_0 > 0$ and a risk-neutral measure $\mathbb{Q}$, the Merton problem admits as solution the value function*

$$v(x_0, t) = -e^{-\gamma x_0 - H(\mathbb{Q}^f|\mathbb{P})}, \tag{2.7}$$

where $\mathbb{Q}^f$ is the $forward$ measure and γ is the coefficient of risk aversion in forward units and $H(\mathbb{Q}^f|\mathbb{P})$ is the relative entropy of the forward measure with respect to the physical measure.

Proof. What remains is a careful computation of the value function. So, given that the investor has some initial wealth x_0, the goal is to find $v(x_0, t) = \sup_\phi \mathbb{E}^{\mathbb{P}}[-e^{-\gamma X_T}/X_0 = x_0]$. From the above Lemma, note that there exists some $z_0 > 0$ such that

$$x_0 = \mathbb{E}^{\mathbb{Q}^f}[I\left(z_0 \frac{d\mathbb{Q}^f}{d\mathbb{P}}\right)].$$

We first calculate z_0.

For the exponential utility, we have $I(y) = -\frac{1}{\gamma}\log(\frac{1}{\gamma}y)$. So we have:

$$x_0 = -\mathbb{E}^{\mathbb{Q}^f}[\frac{1}{\gamma}log(\frac{1}{\gamma}z_0\frac{d\mathbb{Q}^f}{d\mathbb{P}})],$$

which yields that

$$z_0 = \exp(-\gamma x_0 - \mathbb{E}^{\mathbb{Q}^f}[\log(\frac{1}{\gamma}\frac{d\mathbb{Q}^f}{d\mathbb{P}})]).$$

Under the optimal investment portfolio, we have $X_T^{\bar{\phi}} = I(z_0\frac{d\mathbb{Q}^f}{d\mathbb{P}})$. Substituting yields:

$$X_T^{\bar{\phi}} = -\frac{1}{\gamma}\log(\frac{1}{\gamma}\frac{d\mathbb{Q}^f}{d\mathbb{P}}\exp(-\gamma x_0 - \mathbb{E}^{\mathbb{Q}^f}[\log(\frac{1}{\gamma}\frac{d\mathbb{Q}^f}{d\mathbb{P}})])). \tag{2.8}$$

We know that $v(x_0, 0) = \mathbb{E}^{\mathbb{P}}[-e^{\gamma X_T^{\bar{\phi}}}]$, which implies

$$v(x_0, t) = e^{-\gamma x_0}\mathbb{E}^{\mathbb{Q}^f}[e^{-\mathbb{E}^{\mathbb{Q}^f}[\log(\frac{d\mathbb{Q}^f}{d\mathbb{P}})]}],$$

but the integrand is clearly $\mathbb{Q}^f$-measurable, which shows then that

$$v(x_0, t) = e^{-\gamma x_0} e^{-\mathbb{E}^{\mathbb{Q}^f}[\log(\frac{d\mathbb{Q}^f}{d\mathbb{P}})]}.$$

This can then be re-written:

$$v(x_0, t) = -e^{-\gamma x_0 - H(\mathbb{Q}^f|\mathbb{P})}.$$

$\square$

2.4 The Forward Indifference Price

In the classical Black-Scholes market model, payoffs of options contracts and other financial derivatives are replicable via trading portfolios which consist of combinations of cash and the underlying financial instrument(s). The Black-Scholes price of the financial derivative contract is naturally deduced from the cost of this replicating portfolio. But, in incomplete market models, such as the one considered here, there exist derivative payoffs which are not perfectly replicable via some trading strategy. The problem then arises that a rational option price is not uniquely defined. Indifference pricing aims to create a coherent pricing methodology in the absence of perfectly replicable risks.

Indifference pricing begins with an idea from investment management. Wealth managers look to construct optimal trading strategies subject to a risk profile of the investor, in other words, the solve the investor's Merton problem. The solution to the investor's Merton problem is a statement about how the investor trades and manages wealth in an uncertain risky world. The solution, of course, depends on the investor's unique attitude towards risks. And since there are unhedgeable risks in our incomplete market model, indifference pricing suggests that we can price financial derivatives based on the investor's specific attitudes toward risk. Whether an investor trades a stock, or simply cash or other derivative contracts, optimal portfolio theory assumes she behaves so that he or she is optimizing his or her expected utility of her wealth. In other words, our investor is solving his or her Merton problem of optimal investment. Recall the investor's original Merton problem of optimal investment:

$$J(\phi) = \sup_{\phi} \mathbb{E}^{\mathbb{P}}[-e^{-\gamma(X_T)}]. \tag{2.9}$$

And recall its solution, equation (2.7):

$$v(x_0, t) = -e^{-\gamma x_0 - H(\mathbb{Q}^f|\mathbb{P})}.$$

We will consider pricing a generic claim, C_T, which is assumed to be a financial derivative contract priced on some underlying in our market, some $P(t, T)$. We will denote C_T in forward, time T, units. Now consider a new Merton problem for our investor, one in which he or she has a time T liability C_T. In other words, we now consider optimizing the functional $J_{C_T}(\phi)$:

$$J_{C_T}(\phi) = \sup_{\phi} \mathbb{E}^{\mathbb{P}}[-e^{-\gamma(X_T - C_T)}]. \tag{2.10}$$

We will define the solution to equation (2.10) by the value function:

$$u(x_0, t) = \sup_{\phi} \mathbb{E}^{\mathbb{P}}[-e^{-\gamma(X_T - C_T)}]. \tag{2.11}$$

And now we can define the investor's indifference price for C_T.

Definition 2.4.1. *The indifference price of the claim C_T, denoted as $h(C_T)$, is the amount for which the two value functions v and u, defined in equations (2.7) and (2.11), coincide. In other words, for any initial wealth $x_0 > 0$ and $t \in [0, T)$, we have the following*

$$v(x_0, t) = u(x_0 + h, t). \tag{2.12}$$

Remark 9. This means our investor considers two investment problems. In the first problem, she has her regular portfolio she aims to invest optimally. In the second problem, she has sold this claim, C_T, but has received compensation in return for the assumed future liability of C_T. So, in the second problem, she invests optimally around her new liability. In other words, she solves two Merton problems. One with no liability and one with. The indifference price is precisely the amount of cash she would take such that the solutions to the two Merton problems are equal. The investor is then said to be indifferent.

In order to compute the indifference price, we must first calculate $u(x_0, t)$, the solution to equation (2.10).

Borrowing a technique from Delbaen et al [13], we first define a new measure.

Definition 2.4.2. *Let $\hat{\mathbb{P}}$ be a measure equivalent to $\mathbb{P}$ such that*

$$\frac{d\hat{\mathbb{P}}}{d\mathbb{P}} = \frac{e^{\gamma C_T}}{\mathbb{E}^{\mathbb{P}}[e^{\gamma C_T}]} \tag{2.13}$$

For ease of calculations, we denote $\hat{c} = \frac{1}{\mathbb{E}^{\hat{\mathbb{P}}}[e^{\gamma C_T}]}$.

Lemma 2.4.3. *For a risk-neutral measure $\mathbb{Q}^f$ equivalent to $\mathbb{P}$ we have the following:*

$$H(\mathbb{Q}^f|\mathbb{P}) = \mathbb{E}^{\mathbb{Q}^f}[log(\frac{d\mathbb{Q}^f}{d\hat{\mathbb{P}}}) + log\hat{c} + \gamma C_T] = H(\mathbb{Q}^f|\hat{\mathbb{P}}) + log\hat{c} + \mathbb{E}^{\mathbb{Q}^f}[\gamma C_T]. \tag{2.14}$$

As will be evident, this change of measure essentially allows us to ignore the claim in the computation of the value function.

Lemma 2.4.4. *Under a change-of-measure to $\hat{\mathbb{P}}$, the Merton problem with the claim becomes*

$$J_{C_T}(\phi) = \hat{c}^{-1} \sup_{\phi} \mathbb{E}^{\hat{\mathbb{P}}}[-e^{-\gamma X_T}] \tag{2.15}$$

Proof. From equation (2.10), recall that

$$J_{C_T}(\phi) = \sup_{\phi} \mathbb{E}^{\mathbb{P}}[-e^{-\gamma(X_T - C_T)}].$$

Which we can re-write as

$$J_{C_T}(\phi) = \sup_{\phi} \mathbb{E}^{\hat{\mathbb{P}}}[-e^{-\gamma X_T} e^{\gamma C_T} \frac{d\mathbb{P}}{d\hat{\mathbb{P}}}]. \tag{2.16}$$

Recall that $\hat{c} = \frac{1}{\mathbb{E}^{\hat{\mathbb{P}}}[e^{\gamma C_T}]}$, which simplifies equation (2.16) to

$$J_{C_T}(\phi) = \hat{c}^{-1} \sup_{\phi} \mathbb{E}^{\hat{\mathbb{P}}}[-e^{-\gamma X_T}],$$

completing the proof. □

Theorem 2.4.5. *The Merton problem with the claim, $J_{C_T}(\phi) = \sup_{\phi} \mathbb{E}^{\mathbb{P}}[-e^{-\gamma(X_T - C_T)}]$, has as solution the value function*

$$u(x_0, t) = e^{-\gamma x - H_t(\mathbb{Q}^f | \hat{\mathbb{P}})} \hat{c}^{-1}.$$

Proof. Using Lemma 2.4.4 we rewrite equation (2.10) as

$$J_{C_T}(\phi) = \hat{c}^{-1} \sup_{\phi} \mathbb{E}^{\hat{\mathbb{P}}}[-e^{-\gamma X_T}].$$

Recall that $\sup_{\phi} \mathbb{E}^{\hat{\mathbb{P}}}[-e^{-\gamma X_T}]$ was calculated in Theorem 2.3.7, which implies

$$\sup_{\phi} \mathbb{E}^{\hat{\mathbb{P}}}[-e^{-\gamma X_T}] = e^{-\gamma x - H_t(\mathbb{Q}^f | \hat{\mathbb{P}})}.$$

It then follows that we have the solution

$$u(x_0, t) = e^{-\gamma x - H_t(\mathbb{Q}^f | \hat{\mathbb{P}})} \hat{c}^{-1}.$$

□

We may now calculate the complete market forward indifference price of the claim C_T.

Theorem 2.4.6. *The complete market forward indifference price h^f is equal to the forward Black-Scholes price $\mathbb{E}^{\mathbb{Q}^f}[C_T]$.*

Proof. By definition of the indifference price, we look to find h^f so that for all initial $x > 0$, we have $u(x, t) = u(x + h^f, t)$. Substituting we find

$$e^{-\gamma x - H(\mathbb{Q}^f|\mathbb{P})} = e^{-\gamma(x+h^f) - H(\mathbb{Q}^f|\hat{\mathbb{P}})}\hat{c}^{-1}.$$

Recall equation (2.14), namely that

$$H(\mathbb{Q}^f|\mathbb{P}) = \mathbb{E}^{\mathbb{Q}^f}[log(\frac{d\mathbb{Q}^f}{d\hat{\mathbb{P}}}) + log\hat{c} + \gamma C_T] = H(\mathbb{Q}^f|\hat{\mathbb{P}}) + log\hat{c} + \mathbb{E}^{\mathbb{Q}^f}[\gamma C_T].$$

We can thus substitute for $H(\mathbb{Q}^f|\mathbb{P})$. Simplifying, the equation reduces to

$$h^f = \mathbb{E}^{\mathbb{Q}^f}[C_T]. \tag{2.17}$$

This, of course, is the famous forward Black-Scholes price. □

Having calculated the complete market forward indifference price, we will next calculate the forward indifference price in an incomplete market.

Remark 10. Naturally, working in a complete market framework is nicer than the incomplete world. When pricing claims sensitive to term-structure, however, it is becomes important to price without ignoring the incompleteness in the market. This is motivated in [6], [7], [16], [33] and several others. The HJM model with which we work is purely infinite dimensional, thus its incompleteness can be generated from any number of factors. We could, for example, consider that the volatility operator σ is onto a subspace H_σ of the space $\mathcal{L}_{HS}(G;F)$ such that H_σ has finite codomain. One may also consider the notion of approximate completeness introduced in [6], in which even with measure-valued portfolio processes, some contingent claims remain not-perfectly hedgeable. Regardless of the source of the incompleteness, the incompleteness can be characterized in the dual setting by the existence of multiple martingale pricing measures. Given this fact, we use a duality result proved in Delbaen et al. [13].

Lemma 2.4.7. *Given at least one forward martingale measure with $H(\mathbb{Q}^f|\mathbb{P}) < +\infty$ and the above assumptions on the claim and wealth process, we have the dual formulation of the*

Merton problem:

$$\sup_{\phi} \mathbb{E}^{\mathbb{P}}[-e^{-\gamma(X_T - C_T)}] = -exp(-\inf_{\mathbb{Q}^f}(H(\mathbb{Q}^f|\mathbb{P}) - \mathbb{E}^{\mathbb{Q}^f}[\gamma C_T]). \tag{2.18}$$

Theorem 2.4.8. *The incomplete market forward indifference price h^f for a claim C_T can be written as*

$$h^f = \sup_{\mathbb{Q}^f}(\mathbb{E}^{\mathbb{Q}^f}[C_T] - \frac{1}{\gamma}H(\mathbb{Q}^f|\mathbb{P})) - \sup_{\mathbb{Q}^f}(\frac{1}{\gamma}H(\mathbb{Q}^f|\mathbb{P})). \tag{2.19}$$

Proof. This follows from a direct calculation from Lemma 2.4.7 and Theorem 2.4.6. □

2.5 Change-of-Numeraire and the Spot Indifference Price

In the classic Black-Scholes pricing framework, that is, in a complete market framework, the change-of-numeraire pricing technique in complete markets is well-known from Geman et al. [18]. What we introduce in this section, is a notion of change-of-numeraire in an incomplete market. Motivated by this desire and since we have calculated the indifference price in forward units, we wish to now change numeraire and calculate the indifference price in spot terms. And, once we have priced the claim C_T in both forward and spot terms, we establish a pricing consistency across our choice of numeraires.

Because the pricing is done via utility, we will find that the pricing consistency occurs with a proper specification of the investor's utility function across numeraires. We do this via the definition of the risk tolerance of the investor. Recall that the risk tolerance coefficient has more natural units than the risk aversion coefficient. It is this reason why we choose to frame the change-of-numeraire around the risk tolerance. In the complete market case, we find the same standard pricing consistency as in Geman et al. [18], and in the incomplete market case we derive a generalized change-of-numeraire pricing consistency.

Our first result is the spot unit analogue of Lemma 1.8.2. It is important to note that here we now express our initial capital x_0 in spot units. In other words, our spot initial wealth, x_0, is the forward initial wealth discounted by the bank account $(B_t)_{t\geq 0}$. Also, since the wealth process $(X_t)_{t\geq 0}$ is defined in forward units, it too needs to be discounted by the bank account so we can express wealth in the proper spot units.

Definition 2.5.1. *The investor's spot wealth process, $(X_t^s)_{t\geq 0}$, is given by*

$$(X_t^s)_{t\geq 0} = \mathbb{E}^{\mathbb{Q}}[B_T^{-1}](X_t)_{t\geq 0}. \tag{2.20}$$

Lemma 2.5.2. *Given initial spot wealth $x_0 > 0$, there exists a unique number $z_0 > 0$ such that*

$$\mathbb{E}^{\mathbb{Q}}[B_T^{-1} I(B_T^{-1} z_0 \frac{d\mathbb{Q}}{d\mathbb{P}})] = x_0. \tag{2.21}$$

Proof. First recall Proposition 1.3.4 and the remark following it. Namely that the bank account process $(B_T)_{T\geq 0}$ has moments of all real orders. Knowing this, the proof is the same as in Lemma 2.3.4. $\square$

Classically, in the complere Black-Scholes framework the bank account was assumed to be a simple process, one with a fixed interest rate r, making discounting calculations much easier. In such a setting, for some fixed time $T > 0$, the time T forward Black-Scholes price h^f and the spot Black-Scholes price h^s share the simple relationship $e^{rt}h^s = h^f$. We refer to these prices as numeraire consistent. Since in our model the bank account is a process, we need to define a notion of numeraire consistency for our market framework.

Definition 2.5.3. *Given a martingale measure $\mathbb{Q}$ and using a bank account (B_T) as numeraire, we say that the time T forward price h^f and the spot price h^s are numeraire consistent if*

$$h^s = \mathbb{E}^{\mathbb{Q}}[B_T^{-1}]h^f. \tag{2.22}$$

With a definition of numeraire pricing consistency, we need to find necessary and sufficient conditions such that our spot and forward indifference prices are numeraire consistent.

Definition 2.5.4. *With an investment horizon $T > 0$ and an exponential utility function*

$U(x) = -e^{-\gamma x}$, *an investor's spot risk tolerance* $\frac{1}{\gamma}$ *is given by*

$$\frac{1}{\gamma^s} = \frac{1}{\gamma}\mathbb{E}^{\mathbb{Q}}[B_T^{-1}]. \tag{2.23}$$

Theorem 2.5.5. *Let* $(B_t)_{t\geq 0}$ *denote our pricing numeraire. The incomplete time* T *forward indifference price* h^f *and the spot indifference price* h^s *are numeraire consistent if and only if* $\frac{1}{\gamma^s} = \frac{1}{\gamma}\mathbb{E}^{\mathbb{Q}}[B_T^{-1}]$.

Proof. First assume that $\frac{1}{\gamma^s} = \frac{1}{\gamma}\mathbb{E}^{\mathbb{Q}}[B_T^{-1}]$.

We first calculate z_0 from Lemma 2.5.2:

$$z_0 = e^{-\gamma x_0 \mathbb{E}^{\mathbb{Q}}[B_T^{-1}]^{-1} - \mathbb{E}^{\mathbb{Q}}[B_T^{-1}]^{-1}\mathbb{E}^{\mathbb{Q}}[B_T^{-1}\log(\frac{1}{\gamma}B_T^{-1}\frac{d\mathbb{Q}}{d\mathbb{P}})]}.$$

Note that $\mathbb{E}^{\mathbb{Q}}[B_T^{-1}]^{-1}\cdot\gamma = \gamma^s$. We can now write the spot value funcion as

$$v^s(x_0, 0) = -e^{-\gamma^s x_0 - H(\mathbb{Q}^f|\mathbb{P})}.$$

Following the calculations in Theorem 2.4.6, we find the spot Black-Scholes price:

$$h^s = \mathbb{E}^{\mathbb{Q}}[B_T^{-1}]\mathbb{E}^{\mathbb{Q}^f}[C_T]. \tag{2.24}$$

Now recall the definition of the forward measure, Definition 2.3.3. We then find that

$$h^s = \mathbb{E}^{\mathbb{Q}}[B_T^{-1}C_T] = \mathbb{E}^{\mathbb{Q}}[B_T^{-1}]h^f. \tag{2.25}$$

Thus the spot price is the discounted forward price, where the discounting takes place under the unique risk-neutral pricing measure. This is the precise foward-spot pricing consistency in complete markets found in [18].

Now, working in the same incomplete market defined in section 2.4, we find the incomplete spot indifference price

$$h^s = \sup_{\mathbb{Q}}(\mathbb{E}^{\mathbb{Q}}[B_T^{-1}]\mathbb{E}^{\mathbb{Q}^f}[C_T] - \frac{1}{\gamma^s}H(\mathbb{Q}^f|\mathbb{P})) - \sup_{\mathbb{Q}}(-\frac{1}{\gamma^s}H(\mathbb{Q}^f|\mathbb{P})).$$

Recall that we have assumed that $\frac{1}{\gamma^s} = \frac{1}{\gamma}\mathbb{E}^{\mathbb{Q}}[B_T^{-1}]$. This yields

$$h^s = \sup_{\mathbb{Q}}(\mathbb{E}^{\mathbb{Q}}[B_T^{-1}](\mathbb{E}^{\mathbb{Q}^f}[C_T] - \frac{1}{\gamma}H(\mathbb{Q}^f|\mathbb{P}))) - \sup_{\mathbb{Q}}(-\mathbb{E}^{\mathbb{Q}}[B_T^{-1}]\frac{1}{\gamma}H(\mathbb{Q}^f|\mathbb{P})). \tag{2.26}$$

We now work in the other direction. First, we have already computed the forward indifference price in Theorem 2.4.8.

$$h^f = \sup_{\mathbb{Q}^f}(\mathbb{E}^{\mathbb{Q}^f}[C_T] - \frac{1}{\gamma}H(\mathbb{Q}^f|\mathbb{P})) - \sup_{\mathbb{Q}^f}(\frac{1}{\gamma}H(\mathbb{Q}^f|\mathbb{P})). \tag{2.27}$$

What is then left is to calculate the spot indifference price, for the claim C_T^s, which is the claim expressed in spot units. Following the same steps that led to the forward indifference price in Theorem 2.4.8 we first find our spot value function as

$$v^s(x_0, 0) = -e^{-\gamma^s x_0 - H(\mathbb{Q}^f|\mathbb{P})}.$$

It follows then that the spot indifference price for a claim C_T^s is

$$h^s = \sup_{\mathbb{Q}}(\mathbb{E}^{\mathbb{Q}}[C_T^s] - \frac{1}{\gamma^s}H(\mathbb{Q}^f|\mathbb{P})) - \sup_{\mathbb{Q}}(-\frac{1}{\gamma^s}H(\mathbb{Q}^f|\mathbb{P})). \tag{2.28}$$

Assuming the time T forward indifference price in equation (2.27) and the spot indifference price in equation (2.28) are numeraire consistent implies

$$\frac{1}{\gamma^s} = \frac{1}{\gamma}\mathbb{E}^{\mathbb{Q}}[B_T^{-1}], \tag{2.29}$$

completing the proof.

$\square$

2.6 Conclusion

In the classical pricing theory of Black-Scholes, there exist unique rational prices for financial derivatives. Once we allow, however, for market risks we cannot hedge, prices are no longer unique. One such methodology to cope with such risks is the method of indifference pricing, in which prices are determined via an investor's risk-valuation of available trading/hedging opportunities. Our infinite-dimensional market presented here has such risks.

Here we have priced, using indifference, a term-structure option. There are two natural sets of units with which to work in pricing such options, forward and spot units. We have shown a way to consistently price claims regardless of unit-choice. In addition, we see that the correct specification of an investor's risk parameters via his or her utility function leads to a general change-of-numeraire theorem for incomplete indifference pricing of options. Our change-of-numeraire technique is shown to have a natural fit with indifference pricing using exponential preferences, when the investor's risk-tolerance is defined in the proper units.

This approach precludes a possible arbitrage via numeraire choice. Moreover, this extends the notion of pricing consistency across numeraires from the complete market world of Black-Scholes to the incomplete market world of indifference pricing. When the market is complete, the indifference pricing reduces to the Black-Scholes price, and the standard numeraire consistency holds. The difficulty, however, arises when the market is incomplete. Here the indifference price, unlike the Black-Scholes price, is non-linear and furthermore depends on the pre-specified risk preferences of the investor. But, through the proper financially-intuitive specification of the investor's exponential utility function, we find the necessary and sufficient conditions for an incomplete market notion of numeraire consistency.

Chapter 3

Utility Pricing of CDOs: a Numerical Algorithm

We consider the problem of the utility-based pricing of credit derivatives. In particular, we study the pricing of collateralized debt obligations (CDOs). A credit derivative, or a defaultable instrument is a financial contract whose payout is contingent on certain default events. Default events can include the bankruptcy or a company, the failure of a loan repayment or the failure of a coupon payment of a bond. Also, default instruments might be written such that they pay the holder when such a default does not occur. The market for such credit derivatives has been in the news much since the financial crisis began in 2007. At the peak of the boom in credit derivative trading in 2006, such instruments accounted for over 10% of the entire OTC (over the counter) derivatives market.

As is often the case on Wall St., the implementation and popularity of an instrument often well precedes its actual quantitative understanding. The CDO market has been no exception. Naturally, there are several difficulties in the pricing of such claims. Perhaps the main issue is the high dimensionality of a CDO basket. Such a claim is often written on over 100 underlyings. Therefore, the computational complexities are enormous. The prices at which such instruments trade is difficult, if not impossible, to justify with a classical Black-

Scholes type pricing methodology. In the classical pricing schemes, to replicate the market prices, unrealistic assumptions are often required. Here we aim to show that such prices are possibly explained as a result of the risk aversion of investors. In other words, investors strongly fear tail events, such as the credit crisis of 2007-2008, which came as a result of widely spread defaults in the housing market, an event considered so unlikely that most models priced its probability at 0. Much like a far out-of-the-money put option in the equity options world, the classical pricing methodology of Black-Scholes has difficulty explaining such prices for a large scale default scenario CDO payouts. We believe that such prices are a result of investors risk profiles and we aim to price CDOs through the utility-based indifference method.

In the classical derivatives-pricing world of Black-Scholes, the underlying financial market is assumed to be complete. In other words, the payouts of financial derivatives are replicable via appropriate trading strategies in the correct underlying instruments. The Black-Scholes price is a result of the cost of such replicating trading portfolios. In the credit market, however, there exist risks and therefore derivative payouts which are not replicable via a trading portfolio in the underlying instruments. The obvious example here is a default event. The existence of default events creates an incompleteness in the financial market model. Such incompleteness suggests using a pricing methodology other than Black-Scholes. In particular, any pricing methodology used should obviously take into account such sources of incompleteness. There has been some academic treatment of pricing credit claims in an incomplete market. Examples of such studies include Bielecki and Jeanblanc [4] and [5], Collin-Dufresne and Hugonnier [9] and Sircar and Zariphopoulou [39].

Most of Wall St. prices such claims using available market data. Arbitrage pricing bounds can often be determined dependent on the availability of related traded financial instruments. This approach suffices for many trading applications, but it is not entirely quantitatively satisfying. A robust, market-consistent and tractable pricing methodology is clearly more desirable, not only for the academic but for a trader as well. Hence, we suggest the use of utility-based pricing. In other words, the incompleteness of our market, namely the default events, is taken into account via a given investor's risk attitudes towards such default events. As this point is often made, there are many fundamental differences between Black-Scholes arbitrage-free pricing and utility-based pricing. The most important differences include that while arbitrage-free methods price use a unique risk-neutral measure to yield a linear derivative price, the indifference price incorporates the historical probability measure and an investor's unique risk aversion/tolerance to create a nonlinear derivative price.

Perhaps the most simple credit derivatives are credit default swaps (CDS) whose payouts are contingent on the default event of a single underlying. These instruments work like an insurance contract, where the holder makes regularly scheduled payments until a specified default occurs. The instruments we investigate in our study, however, are much more complicated. Here we study collateralized debt obligations (CDOs) whose payouts depend on the default events of a specified (usually large) number of underlyings. Typically, a CDO basket will be written on anywhere from 50 to 300 stocks and whose duration is several years in time. A CDO is broken down into component *tranches*, where holders of CDOs purchase these various tranches. The tranches each correspond to a pre-specified percentage of the basket's stocks defaulting. The *equity* tranche pays off when between 0%-3% of the assets in the basket default. The *mezzanine* tranches pay off when between $3\% - 7\%$ and $7\% - 10\%$ of the assets in the basket default. The *senior* tranche pays off when between $10\% - 15\%$ of the assets in the basket default. The *super senior* tranche pays off when between $15\% - 30\%$ of the assets in the basket default.

So far, the main quantitative industry approach in pricing CDOs has focused on correlation between default times and the use of one-factor Gaussian copula models to analyze the tranche prices via implied correlation. These models then focus on having enough correlation to fit market data. For examples of arbitrage-free pricing of credit claims see Davis and Lo [11], Duffie and Garleanu [14] and Di Graziano and Rogers [22]. In fact, market prices throughout the ongoing credit crisis, beginning in late 2007 have often implied impossibly high correlations, i.e., where correlations are greater than 1. In fact, even before the credit crisis, the non-trivial prices of a super senior tranche (albeit a couple basis points), implies a rather high probability of this end-of-the-world type scenario, which seems to suggest that investors fear an out-of-control cascade of defaults. We feel that this implied investor fear is best modeled via the risk aversion of the traders. Therefore, we aim to show that indifference pricing can fit data via the investor's risk aversion. Also due to the current credit crisis, is a lack of quality data. CDOs are considered toxic throughout Wall Street, and hence their market prices are often not truly reflective of any necessary intrinsic nor investment value(s).

Here we look to extend the result of Sircar and Zariphopooulou [39]. Following [39], we model the default processes for a basket of assets in a complex multi-name CDO. We also compute a *diversity-coefficient* to characterize the effects that several of the investor's investment opportunities are defaultable, i.e., as the CDOs begin to force a payout by defaults occurring, the very instruments which our investor uses to hedge his or her risks disappear as they default. Here we extend [39] in looking at a general case, where the underlying basket stocks have unique default profiles. In this case, a system of ODEs is derived for pricing the CDO. Moreover, we develop and implement a numerical pricing algorithm for this. Furthermore, we propose several approximation schemes to lessen the computational complexity of the problem. Lastly, we propose a more general default profile for the underlying assets. In this case, a system of quasilinear pdes is derived and again a numerical algorithm is proposed.

3.1 Market Model for Credit Derivatives

We begin with considering our market model for an investor looking at a CDO basket of N stocks. Thus we fix a probability triple, $(\Omega, \mathcal{F}, \mathbb{P})$ where $\mathbb{P}$ is the physical measure. For a given firm i of the basket, we associate with it a filtration $(\mathcal{F}_t^i)$. Here we work within an hybrid default framework in which we consider both the intensity-based model for default as well as the firm's stock price. The intensity-based default models have been studied by Artzner and Delbaen [1] and Jarrow and Turnbull [25], for example. In such a framework, default occurs at some random stopping time τ which is driven by some stochastic intensity process $\lambda \geq 0$ which depends on the firm's traded stock price.

An investor looks at a CDO basket of N stocks, each of whose stock price processes $(S_t^{(i)})$ follow geometric Brownian motions:

$$dS_t^{(i)} = S_t^{(i)}\mu_i dt + S_t^{(i)}\sum_{j=1}^{N}\sigma_{ij}dW_t^{(j)}, \tag{3.1}$$

for $t \geq 0$ where $\mu_i \in \mathbb{R}$ are constants for each i, $\sigma_{ij} > 0$ are the volatilities and $(W_t^i)_{t\geq 0}$ are correlated Brownian motions defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We assume the constant correlations

$$\mathbb{E}[dW_t^{(i)}dW_t^{(j)}] = \rho_{ij}dt, i \neq j, \tag{3.2}$$

for $\rho_{ij} \in (-1, 1)$.

We exogenously model the defaults of the firms in our CDO basket. We begin with a general model of the default times.

Definition 3.1.1. *For each $i \in N$, let $\{\psi_i\}$ be independent standard exponential random variable on our probability triple $(\Omega, \mathcal{F}, \mathbb{P})$ independent of the intensities λ_i.*

Definition 3.1.2. *For each $i \in N$, the firm i defaults at time τ_i which is exponentially distributed with parameter λ_i, given as follows*

$$\tau_i = \inf\{t : \int_0^t \lambda_s^{(i)} ds = \psi_i\} \tag{3.3}$$

Remark 11. Note we can specify the λ_i as either a process or as a constant intesity factor for each i. When it is a constant, we dentote it as λ_i, and when it is a process we denote it as $\lambda_t^{(i)}$, for each $i \in N$.

3.2 The Merton Problem

We first equip our investor with an exponential utility function $U(x) = e^{-\gamma x}$ and an investment horizon $T > 0$. For the CDO in question, T will correspond with the furthest dated expiry of the contracts in the basket. We consider the investor's Merton problem of optimal investment. The assets available to the investor are the bank account and the set of traded non-defaulted stocks in the CDO basket. It is very important to note, however, that since we are trading defaultable assets, that once a given firm defaults, its stock is no longer available to be traded. This means that the key for our evaluation here is that the investor's set of investment opportunities is going to be dynamic, i.e., as firms default, their stocks are no longer viable investment/hedge instrument available to our investor. Due to the computational complexity of our pricing problem, we will introduce some new notation.

Definition 3.2.1. *For each $n, k \in \{1, 2, ...N\}$, let I_n^k denote the index set of the possible combinations of n non-defaulted assets.*

Remark 12. Note that for each n there will be $C(N, n)$ index sets. Moreover, since we have a heterogeneous set of tradable assets and our investment opportunities determine our price, it is very important to keep track of precisely which assets are alive at any given time. In other words, our investor can only trade in an underlying if it has yet to default. Hence our need for fastidious bookkeeping.

Definition 3.2.2. *The investor's control process is given by*

$$\pi_t = (\pi_t^{(1)}, ..., \pi_t^{(N)}), \tag{3.4}$$

where $\pi_t^{(i)}$ is the dollar amount held in stock i at time $t \in [0, \tau_i \wedge T)$.

Definition 3.2.3. *A control process $\pi_t^{(i)}$ is called admissible if $\mathbb{E}[\int_0^T \|\pi_t\| dt] < \infty$.*

Also, we define the investor's wealth process for an admissible control policy.

Definition 3.2.4. *The investor has a discounted wealth process given by*

$$dX_t = \sum_i \pi_i \hat{\mu}_i + \sum_{i,j} \pi_i \sigma_{ij} dW_t^{(j)}, \tag{3.5}$$

where $\hat{\mu} = \mu - r$.

We can now state the investor's Merton problem. A well-studied problem, beginning with Merton [31], we begin with assigning our investor a utility function $U(x)$, $x \in \mathbb{R}$. We further fix a terminal date $T > 0$ and initial wealth $X_0 \in \mathbb{R}_+$. In particular, we specify that our investor has exponential preferences, that is

$$U(x) = -e^{-\gamma x},$$

where $x \in \mathbb{R}$ and $\gamma > 0$, a constant, is the investor's coefficient of risk aversion.

For an admisible trading strategy π, we consider an investor who looks to optimize the following functional,

$$J(\pi) = \mathbb{E}^{\mathbb{P}} U(X_T^{\pi}). \tag{3.6}$$

In other words, our investor seeks to optimize his or her expected utility of wealth given over the set of admissible trading strategies.

We will solve equation (3.6) using dynamic programming, meaning we will work backwards. We first consider the case when all of the firms have defaulted. In this case the Merton problem has a trivial solution as there are no investment decisions to make besides leaving the remaining cash in the bank account. Once the problem is solved here, we can move to the next-to-last state, namely when all the firms in the basket have defaulted save precisely one. In this case, we solve the investor's Merton problem when his or her investment options include simply one stock and the bank account. We will continue working backwards in this fashion until we reach the initial state when none of the firms have defaulted.

Like was stated above, here, in our first case where all the firms have defaulted, the investment decision is already made for our investor. There are no hedge instruments left, therefore he or she must simply put his or her money into the money market account and wait.

Lemma 3.2.5. *At time $t \in (0, T)$, with wealth $x \in \mathbb{R}$, if all the available assets have defaulted,* *the investor's Merton problem, equation 3.6 has the solution*

$$M^0(t,x) = -e^{-\gamma x(T-t)}. \tag{3.7}$$

Again, since we require careful bookkeeping of the investor's trading portfolio across time an as various defaults occur, we introduce some further notation.

Definition 3.2.6. *At time $t \in (0,T)$, with wealth $x \in \mathbb{R}$, for a given state I_n^k, the Merton value function is*

$$M^{(I_N^k)}(t,x) = \sup_{\{\pi^i | i \in I_N^k\}} \mathbb{E}_t^{\mathbb{P}}[-e^{-\gamma X_T}]. \tag{3.8}$$

Definition 3.2.7. *For any given default state of our trading basket, I_n^k, let $\Sigma(I_n^k)$ denote the covariance matrix for the non-defaulted assets.*

Definition 3.2.8. *For any given default state of our trading basket, I_n^k, let $\mu(I_n^k)$ denote the vector of returns for the n non-defaulted assets.*

Definition 3.2.9. *For a given state I_n^k the diversity coefficient, $D(I_n^k)$, is*

$$D(I_n^k) = \mu(I_n^k)^T \Sigma(I_n^k)^{-1} \mu(I_n^k) \in \mathbb{R}_{++}. \tag{3.9}$$

Then we may move dynamically backwards (in terms of defaults) to the previous state, i.e., when there was precisely 1 firm of the original N in our basket which has not defaulted. A firm which is yet to default we will refer to as alive.

Lemma 3.2.10. *When precisely one firm in the basket is left alive, there are N distinct Merton value functions $M_i^1(t,x)$ which must solve*

$$M_t - \frac{1}{2}\frac{\mu_i^2}{\bar{\sigma}_i^2}\frac{M_x^2}{M_{xx}} + \lambda_i(-e^{-\gamma x} - M) = 0, \tag{3.10}$$

with terminal condition

$$M_i(T,x) = -e^{-\gamma x},$$

where $\bar{\sigma}_i^2 = \sum_{j=1}^N \sigma_{ij}^2$.

Proof. That there are N Merton equations is obvious. Now, following Proposition 1 from [39], note that since the λ_i are constants, the Merton value function solves the Hamilton-Jacobi-Bellman equation:

$$M_t - \frac{1}{2}\frac{\mu_i^2}{\bar{\sigma}_i^2}\frac{M_x^2}{M_{xx}} + \lambda_i(-e^{-\gamma x} - M) = 0.$$

The terminal condition is merely the utlity of her wealth at time T

$$M_i(T, x) = -e^{-\gamma x}.$$

□

Lemma 3.2.11. *At time $t \in (0, T)$, for a given $i \in \{1, ...N\}$, when there is precisely one firm left alive, the Merton value function, definition 3.2.6, has the solution, denoted by $v_i(t)$,*

$$v_i(t) = e^{-\alpha(T-t)} + \frac{\lambda_i}{\alpha_i}(1 - e^{\alpha_i(T-t)}), \tag{3.11}$$

where $\alpha_i = \frac{\mu_i^2}{2\bar{\sigma}_i^2} + \lambda_i$.

Proof. We begin by taking advantage of the scaling properties of the exponential utility function. Recall that the N Merton functions solve equation (3.10):

$$M_t - \frac{1}{2}\frac{\mu_i^2}{\bar{\sigma}_i^2}\frac{M_x^2}{M_{xx}} + \lambda_i(-e^{-\gamma x} - M) = 0, \tag{3.12}$$

with terminal condition

$$M_i(T, x) = -e^{-\gamma x}.$$

Consider the substitution $M_i^1(t, x) = -e^{-\gamma x} v_i(t)$. We may then re-write the solutions to the N Merton functions as

$$v_i' - \alpha_i v + \lambda_i = 0, \tag{3.13}$$

wth terminal condition

$$v_i(T) = 1,$$

where $\alpha_i = \frac{\mu_i^2}{2\sigma_i^2} + \lambda_i$. Note that this is an ODE. This linear ODE has solution

$$v_i(t) = e^{-\alpha(T-t)} + \frac{\lambda_i}{\alpha_i}(1 - e^{\alpha_i(T-t)}).$$

□

As we work backwards through our tree of defaults, in the cases where more than one firm is left alive, we need to introduce some more new notation to simplify this complicated process.

Definition 3.2.12. *For every $n \in \{2, ..., N\}$, denoting the number of firms left alive, we denote these n firms as $I_n = \{i_1, ..., i_n\}$. Moreover, let $I_{n-1}^{(k)}$ denote the possible remaining $n-1$ firms after firm k has defaulted.*

Definition 3.2.13. *For each $n \in \{2, ..., N\}$, at a given state I_n, let $M^{(I_n)}(t, x)$ be the Merton value function. Also, denote by $A_{(I_n)}$ the associated $\sigma\sigma^T$. And let $B_{(I_n)}$ be the associated $\mu_{(I_n)}^T A_{(I_n)}^{-1} \mu_{(I_n)}$.*

Theorem 3.2.14. *At time $t \in (0, T)$, when there are $n \geq 2$ firms left alive, the Merton problem at given state has the following solution*

$$v(t) = e^{-\alpha(T-t)} + \int_t^T e^{-\alpha(s-t)} G(s) ds, \tag{3.14}$$

where $\alpha = (\frac{1}{2}B_{(I_n)} + \sum_{k \in I_n} \lambda_k)$.

Proof. Using our new notation, and following the proofs of Lemmas 3.2.10 and 3.2.11, first note that the solution of the Merton problem solves the Hamilton-Jacobi-Bellman equation

$$M_t^{(I_n)} - \frac{1}{2}B_{(I_n)}\frac{(M_x^{(I_n)})^2}{M_{xx}^{(I_n)}} + \sum_{k \in I_n} \lambda_k (M^{(I_{n-1}^{(k)})} - M^{(I_n)}) = 0. \tag{3.15}$$

We then can make the same scaling substitution

$$M = -e^{-\gamma x} v_{(I_n)}(t).$$

Equation (3.15) can then be re-written as the following system of ODEs:

$$v'_{(I_n)} - (\frac{1}{2} B_{(I_n)} + \sum_{k \in I_n} \lambda_k) v_{(I_n)} + \sum_{k \in I_n} \lambda_k v^{(k)}_{(I_n)} = 0, \tag{3.16}$$

with terminal condition

$$v_{(I_n)}(T) = 1.$$

For the sake of simplicity, let $\alpha = (\frac{1}{2} B_{(I_n)} + \sum_{k \in I_n} \lambda_k)$. Moreover, note that in equation (3.16), that we can re-write the term $\sum_{k \in I_n} \lambda_k v^{(k)}_{(I_n)}$ as some $G(t) \in \mathbb{R}$. Then this system of ODEs, equation (3.16), can be re-written as follows:

$$v'_{(I_n)} - (\alpha) v_{(I_n)} + G(t) = 0, \tag{3.17}$$

with terminal condition

$$v_{(I_n)}(T) = 1.$$

It follows that the solution of the ODE

$$v' - \alpha v + G(t) = 0,$$

$$v(T) = 1,$$

is simply

$$v(t) = e^{-\alpha(T-t)} + \int_t^T e^{-\alpha(s-t)} G(s) ds. \tag{3.18}$$

$\square$

3.3 The Indifference Price

Now we have a methodology to solve for the Merton value functions at each node of our default tree. This led to our solution of the Merton problem of optimal investment for our investor facing his or her CDO basket. Ultimately, we wish to use this to price the tranches of the CDO. Before pricing the CDO tranches, we first add the simplest default claim possible for our starting point or pricing. We consider a claim c, which pays 1 unit if all firms survive up to time T. We now wish to investigate our investor's new Merton investment problem, given the addition of this claim c to his or her trading portfolio. So, now with the liability c, for an admissible trading strategy π, we consider an investor who looks to optimize the following functional,

$$J^c(\pi) = \mathbb{E}^{\mathbb{P}} U(X_T^{\pi} - c). \tag{3.19}$$

Lemma 3.3.1. *For a given state I_n^k, the Merton problem for an investor with a liability c, equation (3.19), which pays one unit if no firms default, solves the following ODE*

$$w'_{(N)} - (\frac{1}{2}(\mu^T A^{-1}\mu) - \sum_{k=1}^{N} \lambda_k) w_{(N)} + \sum_{k=1}^{N} \lambda_k e^{\gamma c} w^{I_{N-1}^{(k)}}, \tag{3.20}$$

with

$$w_{(N)}(T) = 1.$$

Proof. Note that when all the firms are still alive, from the proof of Theorem 3.2.14 it follows that the investor's Merton problem solves the following HJB

$$H_t^{(N)} - \frac{1}{2}(\mu^T A^{-1}\mu) \frac{(H_x^{(N)})^2}{H_{xx}^{(N)}} + \sum_{k=1}^{N} \lambda_k (M^{I_{N-1}^{(k)}} - H^{(N)}) = 0, \tag{3.21}$$

where

$$H^{(N)}(T, x) = -e^{-\gamma(x+c)}.$$

Following the proof of Theorem 3.2.14, we make a scaling substitution,

$$H^{(N)}(t,x) = -e^{-\gamma(x+c)} w_{(N)}(t). \tag{3.22}$$

And substituting $H^{(N)}(t,x) = -e^{-\gamma(x+c)} w_{(N)}(t)$ into equation (3.21) yields

$$w'_{(N)} - (\frac{1}{2}(\mu^T A^{-1}\mu) - \sum_{k=1}^{N} \lambda_k) w_{(N)} + \sum_{k=1}^{N} \lambda_k e^{\gamma c} w^{I^{(k)}_{N-1}},$$

with

$$w_{(N)}(T) = 1.$$

$\square$

In arbitrage-free pricing theory, payoffs of options contracts and other financial derivatives are replicable via trading portfolios which consist of combinations of cash and the underlying financial instrument(s). The arbitrage-free price, in other words, the Black-Scholes price, of the financial derivative is then deduced from the cost of this replicating portfolio. But in incomplete market models, such as the one considered here, there exist derivative payoffs which are not perfectly replicable via some trading strategy. Recall that incompleteness in our market model stems from the fact that we allow our investor to trade assets which may default at some random time. So, in our market, since defaults do occur, a rational option price is not uniquely defined. We then turn to indifference pricing, which aims to create a coherent pricing methodology in the absence of perfectly replicable risks.

Indifference pricing begins with an idea from investment management. Whether explicitly stated or not, portfolio managers look to solve an investor's Merton problem. They seek to construct optimal trading strategies subject to a risk profile of the investor. The aim, explicitly stated as such or not, is to maximize the investor's expected utility of his or her wealth. The solution to the investor's Merton problem is a statement about how the investor trades and manages wealth in an uncertain risky world. The solution, of course, depends on the investor's unique attitude towards risks. Part of the wealth manager's job is to quantify the investor's risk aversion/tolerance. And, since there exist the default risks in our incomplete market model, indifference pricing suggests that we can price our CDOs based on the investor's specific attitudes toward such unhedgeable risks. Whether an investor trades a stock, or simply cash or other derivative contracts, optimal portfolio theory assumes he or she behaves so that he or she is optimizing her expected utility of his or her wealth. In other words, our investor is solving his or her Merton problem of optimal investment. Recall that we considered two Merton problems of optimal investment for our investor. We studied one Merton problem of just straight investment with no extra liability:

$$J(\pi) = \sup_{\pi} \mathbb{E}^{\mathbb{P}}[-e^{-\gamma X_T^{\pi}}]. \tag{3.23}$$

And, we considered a second Merton problem for our investor in which he or she has some terminal liability c:

$$J^c(\pi) = \mathbb{E}^{\mathbb{P}}[-e^{-\gamma(X_T^{\pi}-c)}]. \tag{3.24}$$

Recall that we solved both equation (3.23) and equation (3.24). We denoted their solutions, respectively, $v(x,t)$ and $w(x,t)$. In particular, we will now consider the solutions to the respective Merton problems at the base nodes of each respective default tree, in other words, the node I_n^k, $n = N$, where all firms are still alive. Without loss of generality we will denote the solutions to the respective Merton problems at these two base nodes as $v(x,t)$ and $w(x,t)$. We may now define our investor's indifference price for the claim c.

Definition 3.3.2. *The time t indifference price for a claim c, denoted as R is the amount such that the two value functions v and w, coincide. In other words, for any initial wealth $x > 0$ and $t \in [0,T)$, we have the following:*

$$v(x,t) = w(x+R,t). \tag{3.25}$$

Remark 13. This means our investor considers two investment problems. In the first problem, he or she has his or her regular portfolio which he or she aims to invest optimally. In the second problem, he or she has sold this claim, c, but has received compensation for the assumed terminal liability of c. So, in the second problem he or she invests optimally around his or her new liability. In other words, he or she solves two Merton problems; one with no liability and one with liability. The indifference price is precisely the amount of cash he or she would take such that the solutions to the two Merton problems are equal. The investor is then said to be indifferent.

3.4 A Numerical Algorithm

We construct an algorithm for computing the indifference price of a CDO tranche. First we work through an example computation. So, for example, consider a mezzanine tranche. In particular, let this mezzanine tranche pay when the defaults are between 5% and 10% of the portfolio. Thus we denote the attachment points $K_u = 0.1$ and $K_l = 0.05$. Suppose our CDO is written on $N = 30$ firms. So, then our mezzanine tranche pays out when the number of assets still alive, n, is either 27 or 28. As is often the case in the credit market, a default is not a complete loss for an investor. Normally 40% of the firm's value is distributed to the bondholders in cases of default. So, we will specify some recovery coefficient, $\delta = 0.4$. Then a default is not an entire loss to the investor's portfolio in that particlar firm. This way, the tranche holder pays out if $\psi^n = n + 0.4 * (100 - n) \in \{27, 28\}$, where there are n assets alive.

Moreover, in this case, fix $\lambda = .01$ as some constant for each stock in the CDO. We also fix the maturity $T = 5$ years, a fixed interest rate of $r = 0.025$ and a constant correlation coefficient $\rho = 0.025$. We will show the indifference prices as a function of the investor's coefficient of risk aversion.

Continuing, let $f(\psi) = \mathbf{1}_{[27,28]}$. Define $F(\psi^n) = \min(28, \psi) - \min(27, \psi)$, the current spread on the tranche. We need to find the price R such that the investor is indifferent at time t between holding the tranche or not. We then solve the system of ODEs for $w(t)$, which yields the following system

$$w(t) = e^{-\beta(T-t)} + \int_t^T e^{-\beta(s-t)} H(s) ds, \tag{3.26}$$

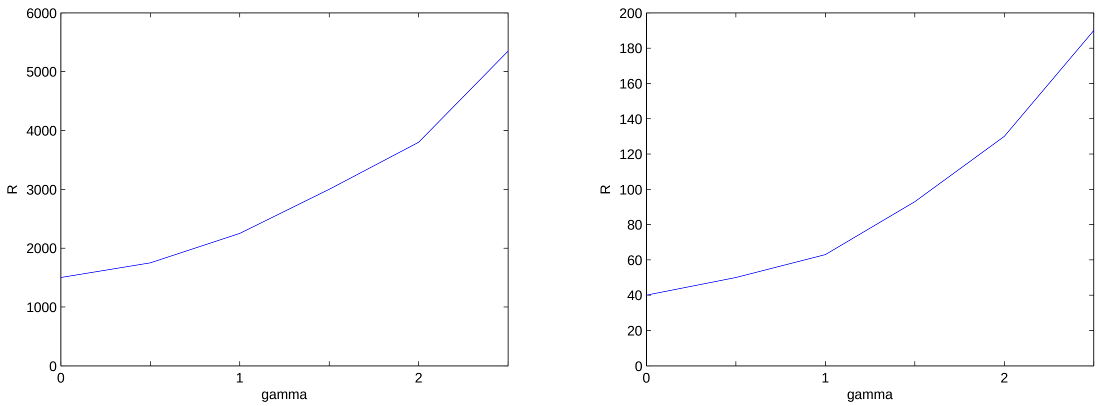

Figure 3.1: Mezzanine and Senior Tranche Prices

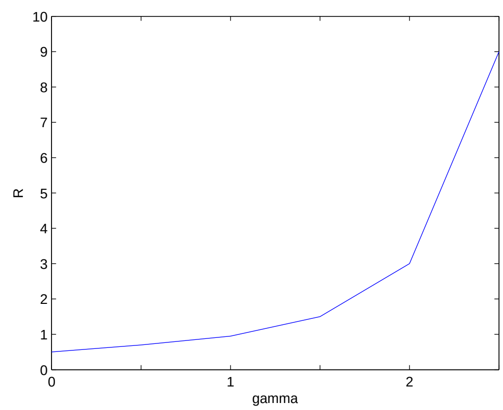

Figure 3.2: Super Senior Tranche Price

where $\beta(I_n^k) = \alpha(I_n^k) + \gamma * R * F(\psi^n)$ and $H^{(I_n^k)}(t) = \sum_k \lambda^k w^{(I_{n-1}^k)} e^{\gamma f(\psi^n)}$. Lastly, we find R such that $v(0) = w(0)$, satisfying the indifference equation.

These are the figures which show the investor's indifference price R for the CDO mezzanine, senior and super senior tranches as a function of γ, his or her coefficient of risk aversion. R is quoted in basis points.

Remark 14. These calculations are done numerically in C++. There are of course, several severe computational issues present. The first is that when N is large, there are huge combinatorial concerns. This requires capabilities for arbitrary length integer arithmetic. Moreover, it requires effective and efficient management of a very large multi-branching lattice. Next is the fact since there are so many nodes in our tree, and we have to solve an ODE at each step, this becomes time-expensive. For example, even in the small case of $N = 20$ we need to solve over $500,000$ ODEs. This also leads to unrealistic RAM requirements for a desktop PC if $N > 30$. The runtime for small N is reasonable. For $N = 10$ the runtime is under 5 minutes. These issues will be addressed again in our approximation section.

3.5 General Multi-Name Credit Derivatives

We wish to generalize the results of the previous section and propose an algorithm to price CDOs when there is a random default component. Here we include a default process $\lambda(\cdot)$ which itself follows some random driver. Other than λ, we will keep the same market model as in section 3.1 and section 3.2.

The ith firm has default time τ_i. The default intensity function of firm i is specified as exponential with parameter $\lambda_t^i(Y_t))$, where Y is an intensity driving factor, common for each $i \in \{1, 2, ..., N\}$. Y is given by the following diffusion equation

$$dY_t = \kappa(\theta - Y_t)dt + \beta\sqrt{Y_t}dW_t^{N+1}, \tag{3.27}$$

where (W_t^{N+1}) is correlated to (W_t^i) for $i \in \{1, 2, ..., N\}$.

We again consider the Merton problem for each node in our tree.

Theorem 3.5.1. *At each time $t \in (0, T)$, with wealth $x \in \mathbb{R}$, for a given state I_n^k, the Merton value function solves the following quasilinear partial differential equation*

$$v_t + \mathcal{L}_y v + \sum_k h^{i_k}(y)\left(v^{(I_n^k)-\{i_k\}} - v^{(I_n^k)}\right)$$

$$-\frac{1}{2}Dv - E\beta\sqrt{y}v_y - \frac{1}{2}F\beta^2 y\frac{v_y^2}{v} = 0,$$

with

$$v^{(I_n^k)}(T,y) = 1.$$

Here $\mathcal{L}_y$ is the generator of the CIR process

$$\mathcal{L}_y = \frac{1}{2}\beta^2 y \frac{\partial^2}{\partial y^2} + \kappa(\theta - y)\frac{\partial}{\partial y}, \tag{3.28}$$

$E(I_n^k) = c(I_n^k)^T \Sigma(I_n^k)^{-1}\mu(I_n^k)$, $F(I_n^k) = c(I_n^k)^T \Sigma(I_n^k)^{-1} c(I_n^k)$ *and* $c_i = \mu_i \sigma_i$.

Proof. The Merton problem for each node in our tree is

$$M^{(I_n^k)}(t,x,y) = \sup_{\{\pi^i | i \in I_N^k\}} \mathbb{E}_t^{\mathbb{P}}[-e^{-\gamma X_T}]. \tag{3.29}$$

The HJB pde now becomes

$$M_t + \sup_{\pi}\left(\frac{1}{2}(\pi^T \sigma \sigma^T \pi) M_{xx} + \beta\sqrt{y}(\nu\sigma^T\pi)M_{xy} + (\pi^T\mu)M_x\right)$$

$$+\frac{1}{2}\beta^2 y M_{yy} + \kappa(\theta - y)M_y + \sum_k h^i(y)\left(M^{(I_n^k)-\{i_k\}} - M^{(I_n^k)}\right) = 0,$$

where $\nu_j = \sigma_{N+1,j}$.

The next step is to simplify the HJB PDE. We begin with introducing some new notation.

First, let

$$E(I_n^k) = c(I_n^k)^T \Sigma(I_n^k)^{-1}\mu(I_n^k). \tag{3.30}$$

Second, let

$$F(I_n^k) = c(I_n^k)^T \Sigma(I_n^k)^{-1} c(I_n^k). \tag{3.31}$$

And third, let

$$c_i = \mu_i \sigma_i \tag{3.32}$$

We then may re-wrtite the HJB PDE as follows:

$$M_t - \frac{1}{2}\frac{DM_x^2 + E\beta\sqrt{y}M_xM_{xy} + \frac{1}{2}F\beta^2 yM_{xy}^2}{M_{xx}}$$

$$+\frac{1}{2}\beta^2 yM_{yy} + \kappa(\theta - y)M_y + \sum_k h^i(y)\left(M^{(I_n^k)-\{i_k\}} - M^{(I_n^k)}\right) = 0.$$

We will again to advantage of working with our investor's exponential utility function and make the following scaling substitution:

$$M^{(I_n^k)}(t,x,y) = -e^{-\gamma x}v^{(I_n^k)}(t,y). \tag{3.33}$$

Thus we then find the following quasilinear pde for $v^{(I_n^k)}(t,y)$:

$$v_t + \mathcal{L}_y v + \sum_k h^{i_k}(y)\left(v^{(I_n^k)-\{i_k\}} - v^{(I_n^k)}\right)$$

$$-\frac{1}{2}Dv - E\beta\sqrt{y}v_y - \frac{1}{2}F\beta^2 y\frac{v_y^2}{v} = 0,$$

with

$$v^{(I_n^k)}(T,y) = 1$$

where $\mathcal{L}_y$ is the generator of the CIR process

$$\mathcal{L}_y = \frac{1}{2}\beta^2 y\frac{\partial^2}{\partial y^2} + \kappa(\theta - y)\frac{\partial}{\partial y}.$$

□

While we may have an algorithm to solve for the value functions and ultimately the indifference price of such a CDO, it leaves impressive numerical challenges. First and foremost is the need to efficiently solve a large number of quasilinear PDEs. Here, dividing by the value function in the PDE provides a numerical difficulty which is not currently clear how to efficiently overcome. The RAM restraints are more severe for solving this system of PDEs than in the ODE case. Also, one would need to take great care in carefully specifying the $h^i(y)$ functions. For which we will suggest some approaches using various approximations to allow for numerical computations.

3.6 Approximations

Naturally, the problem with this approach is the numerical speed and efficiency of running through the trees, as the number of nodes can quickly grow out of control. We suggest some speed-up routines, which allow for computation of the large CDO baskets, for example when there are 120 or more stocks. Of course, these faster routines come at the expense of accuracy.

The first suggestion is to order the λ_i's in terms highest to lowest default intensities. In other words, this routine has the firms defaulting precisely in the order in which they are most likely to default. So, for the case of n firms, we would only need to solve n PDEs and we do not need to build a full tree, rather we just select one specific default path from the tree. This method has obvious limitations, as even though this path might be the most likely path, its individual path probability is dwarfed by the sheer number of possible default trajectories. Although given the small number of calculations needed, this could serve as a first sample approximation for a very large CDO basket.

The second possibility is, at each step going forward we order the $\alpha(I_n^k)$ terms. In other words, at a given default level n, we consider the next $n-1$ level and compute all the appropriate $\alpha(I_{n-1}^k)$ terms. Here we propose selecting the smallest α. This represents the node in our tree with highest default impact. So again, like with the previous speed-up routine, we are specifying, *a priori*, a specific default sample path. Whence we need only solve N PDEs. This is more accurate, in so far as it is more reflective of the true price, being that choosing the $\alpha(I_n^k)$'s takes into account the diversity coefficients and more market data than just the prior exogenously determined default intensities. Again, the actual probability of this sample path being taken is small, but would serve as a possible first approximation of the price for large CDOs.

3.7 Conclusion

Collateralized debt obligations (CDOs) have been a very important part of Wall St. for the last several years. The credit market grew to become a multi-trillion dollar market, of which CDOs played a major role. Moreover, unfortunately, CDOs have been at the center of the global financial credit-crisis since 2007. Using the classical Black-Scholes pricing framework, prices of CDOs are computed in terms of implied correlations. Given the high prices of some highly unlikely default scenarios, the Black-Scholes prices can often imply the impossible, such as implied correlations exceeding 1.

Here, instead, we describe the prices of CDO tranches as a result of the risk aversion of investors. Investors naturally fear bad events, observed for example, in the volatility smirk in equity options. In CDOs we believe the fear of a contagion of defaults leads to the high prices of the senior and super-senior CDO tranches, even for seemingly diversified portfolios. Given an investor with exponential preferences, we find an algorithm for numerically pricing CDO tranches for him or her. Since indifference pricing depends on an investor's unique attitudes towards his or her investment opportunities, the defaults of available assets can dramatically impact this price. In other words, as more firms in the CDO default, the investor is faced with fewer investment choices. This makes the bookkeeping in our algorithm vital. We show the numerical results in a few simple cases, but acknowledge that for the general cases, the numerical solutions remain a substantive challenge.

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