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COEXISTENCE OF ATTRACTORS AND WADA BASIN BOUNDARIES IN DYNAMICAL SYSTEMS: A SURVEY OF RESULTS

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A SURVEY OF RESULTS

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This is a summary report on some existing results and methods regarding the problem of determining the basins of attraction of dynamical systems (in particular, two-dimensional diffeomorphisms) when there is a coexistence of attractors. Based on the work of Helena Nusse and James Yorke, it presents existence and characterization results for a certain kind of basin boundaries (namely, the Wada boundaries). The key feature of their approach is to redefine the idea of a basin boundary by introducing the notion of a `basin cell', which bypasses the problem of exactly locating the attractor of a system, which is often either not well-defined or hard to locate in practice. Moreover, the basin cells and their boundaries are characterized by utilizing the stable and unstable manifolds of the system, which are easier to locate by numerical methods, and thus their method provides both numerically verifiable characteristics and algorithms for computation.

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1 Introduction

This is a short report on some results and methods regarding the problem of determining the basins of attraction of dynamical systems when there is a coexistence of attractors. As it is often the case with dynamical systems, there are no really general results. Many authors have analyzed and characterized different types of dynamical systems and their attractors. This report is survey of some results contributed by James Yorke and his coauthors on a two-dimensional diffeomorphisms that possess more than one basins of attraction. In a series of papers (Nusse and Yorke [1996a,b, 1997]), they present existence and characterization results for a certain kind of basin boundaries (namely, the Wada boundaries) which have a complicated structure. The key results are presented in Chapter 2. One of the key features of their approach is to redefine the idea of a basin boundary by introducing the notion of a 'basin cell', which bypasses the problem of exactly locating the attractor of a system, which is often either not well-defined or hard to locate in practice. Moreover, the basin cells and their boundaries are characterized by utilizing the stable and unstable manifolds of the system, which are easier to locate by numerical methods, hence their theoretical definitions and existence results become operationalizable because they provide numerically verifiable characteristics. Chapter 3 summarizes some numerical methods proposed by the authors that can be used to verify the existence of and locate Wada boundaries.

We start with a section on the required mathematical preliminaries, mainly based on Chueshov [2002].

1.1 Preliminaries

Definition 1. A dynamical system is a pair of objects (X, S_t) where X is a complete metric space and S_t is a family of continuous mappings f of X into itself with the properties:

$$S_t : X \to X$$

$$S_{t+\tau} = S_t \circ S_\tau, \quad t, \tau \in \mathbb{T}_+, \quad S_0 = I$$

Here $\mathbb{T}_+$ is an index set. If $\mathbb{T}_+ = \mathbb{Z}_+$, then the dynamical system is discrete, and if $\mathbb{T}_+ = \mathbb{R}_+$, or more generally, $\mathbb{T}_+ = \mathbb{R}$, then the dynamical system is continuous. X is called the phase space and S_t is called the evolutionary operator. The dimension of X is called the dimension of the dynamical system.

Definition 2. A trajectory (or orbit) of a dynamical system is a set

$$\gamma = \{u(t) : t \in \mathbb{T}_+\}$$

where $u : \mathbb{T} \to X$ is continuous and

$$S_\tau u(t) = u(t + \tau) \quad \forall \tau \in \mathbb{T}_+ \quad \textit{and}\ \forall t \in \mathbb{T}$$

A trajectory $\gamma = \{u(t) : t \in \mathbb{T}_+\}$ is called periodic if there exists $T \in \mathbb{T}_+,\ T > 0$ such that $u(t + T) = u(t)$. The minimal such T is called the period of the trajectory.

Definition 3. An element $u_0 \in X$ is called a fixed point of a dynamical system (X, S_t) if

$$S_t u_0 = u_0 \quad \forall t \geq 0$$

The notion of invariant set plays an important role in the theory of dynamical systems.

Definition 4. A subset Y of the phase space X is said to be: a) positively invariant, if $S_t Y \subset Y$ for all $t > 0$; b) negatively invariant, if $S_t Y \supset Y$for all $t < 0$; c) invariant, if it is both positively and negatively invariant, i.e. if $S_t Y = Y$ for all $t \geq 0$.

The simplest examples of invariant sets are trajectories and semi-trajectories.

Other important example of invariant set is connected with the notions of ω -limit and α -limit sets that play an essential role in the study of the long-time behavior of dynamical systems.

Definition 5. Let $A \subset X$. Then the ω-limit set for A is defined by

$$\omega(A) = \bigcap_{s \geq 0} \left[\bigcup_{t \geq s} S_t(A) \right]_X$$

where $S_t(A) = \{v = S_t u : u \in A\}$.

Definition 6. The set

$$\alpha(A) = \bigcap_{s \geq 0} \left[\bigcup_{t \geq s} S_t^{-1}(A) \right]_X$$

where $S_t^{-1}(A) = \{v : S_t v \in A\}$ is called the $\alpha - limit$ set for A.

Here the $[\cdots]_X$ denotes closure w.r.t X.

1.2 Attractors

The concept of an attractor is fundamental to the analysis of limit behavior of a dynamical system. It is an important example of an invariant set. But there are multiple definitions of attractors in the literature. We start with presenting the definition of global attractor, which is unique. But many dynamical systems do not possess unique global attractors as defined below. Then the problem is to have an appropriate definition of an 'attractor' or 'attractors' and finding a way to locate them and their 'basins'. The difficulty of this problem varies from case to case. The main task of this report is to focus on a particular method of dealing with this problem for 2-dimensional diffeomorphisms that have co-existent 'attractors'.

Definition 7. A bounded closed set $A_1 \subset X$ is called a global attractor for a dynamical system (X, S_t) if

1. A_1 is an invariant set
2. the set A_1 uniformly attracts all trajectories starting in bounded sets, i.e. for any bounded $B \subset X$,

$$\lim_{t\to\infty} \sup\{dist(S_t y, A_1) : y \in B\} = 0$$

Here the distance between an element z and a set A is defined by

$$dist(z, A) = \inf\{d(z, y) : y \in A\}$$

where $d(z, y)$ is the distance between the elements z and y in X.

But global attractors do not always exist. So we get into the problem of possibly 'non-unique' attractors, for which there are many definitions. In the next chapter we will present a way to talk about asymptotic properties of dynamical systems without having to define an attractor, using a concept called 'trapping region'. Before that, we need some more definitions.

Let R be a subset of the state space for F. The stable set $S(R)$ of F is $\{x \in R : F^n(x) \in R for n = 0, 1, 2, ...\}$; the unstable set $U(R)$ of F is $\{x \in R : F^{-n}(x) \in R for n = 0, 1, 2, ...\}$. The set of points x for which $F^n(x)$ is in R for all integers n is called the invariant set $Inv(R)$ of Fin R, that is, $Inv(R) = S(R) \cap U(R)$ (recall definition in Chapter 1). A component of $S(R)$ (respectively, $U(R)$), which contains a point of $Inv(R)$ is called a stable (resp. unstable) segment. We call $Inv(R)$ a chaotic saddle when it includes a Cantor set.

Definition 8. A hyperbolic set A is called saddle-hyperbolic if $dim(E^s) \geq 1$ and $dim(E^u) \geq 1$. A region R is a saddle-hyperbolic region if R satisfies all the following conditions:

1. Hyperbolic property: the invariant set, $Inv(R)$ is a non-empty saddle-hyperbolic set;
2. Boundary property: $\overline{U(R)} \cap \partial R$ is mapped outside the closure $\overline{R}$ of R;
3. Intersection property: each non-trivial component y of $U(R)$ is an unstable segment, that is, y intersects $Inv(R)$;

2 Coexistence of Attractors and Wada Boundaries

The discussion in this chapter are based on Nusse and Yorke [1996a,b, 1997]. In a series of papers, the authors propose way to define boundaries that makes it more analytically tractable, prove the existence of Wada boundaries and propose numerical techniques to verify the existence of such boundaries. I present a summary of their work.

The starting point of this discussion is the consideration of the cases where at least three 'basins' of attraction coexist. As the authors point out, when a map appears to have a chaotic attractor, traditional approaches gave no way to determine if it is chaotic or to determine if there is more than one attractor. Hence, 'basin' is then ill-defined. So they redefine the concept of a 'basin' to get around the problem and propose a general way to describe the structure and properties of basins and their boundaries for two-dimensional diffeomorphisms. For one-dimensional maps, topological basin boundaries are well understood (see, for example, Nusse [1987]); in the example of the logistic map above the boundaries of the three basins (of the third iterate), coincide. In fact, many global structures for one-dimensional maps including attractors and basin boundaries are well known; see De Melo and Van Strien [1993]. The basin cells introduced in this paper allow us to discuss the global structure of basin boundaries for many choices of parameters in well-known two-dimensional maps including the Henon map and the time-2π map of the forced damped pendulum differential equation.

Let M denote a compact, smooth two-dimensional manifold without boundary; let $F : M \to M$ be a C^1-diffeomorphism. In the traditional approach, a basin (for F) is usually defined to be the set of points x for which $\omega(x)$ is contained in a specified attractor [1]. But, in case of a chaotic attractor, there is often no way to tell whether there is more than one attractor by this definition, so the 'basin' of a particular attarctor is not well defined. The authors redefine 'basin' to get around the problem.

Definition 9. A *compact region* is a simply connected, compact set with nonempty interior and consisting of finitely many components. A compact region Q is a *trapping region* if $F(Q) \subset Q$ and $F(Q) \neq Q$.

It is assumed that the *trapping regions* have piecewise smooth boundaries.

Definition 10. If Q is a trapping region, then the *basin* of Q is the set of points which eventually map into the interior of Q.

So, the above definition avoids the problem of defining the attractors of the system since it implies that a set B is a basin if there exists a trapping region Q such that B is the basin of Q. Note that the union of finitely many basins is a basin. [2]

Now, take a diffeomorphism F. Assume that F has at least two basins. Let B denote a basin.

[1] see definition of $\omega(x)$ in Ch. 1

[2] The authors emphasize the fact that a basin may include invariant Cantor sets, in that case the orbits of some points in the basin will not converge to an attractor.

Definition 11. A point $x \in M$ is a *boundary point* of B if $x \in \partial\overline{B}$. The boundary of B is the set $\partial\overline{B}$.

Here $\partial\overline{B}$ denotes the boundary of the closure of B. A basin boundary $\partial\overline{B}$ is fractal[3] if it contains a transversal homoclinic point[4].

An important concept is 'accessible' periodic points which play a crucial role in describing phenomena of basin boundaries.

Definition 12. A point $p \in \partial\overline{B}$ is accessible from basin B if a curve can be drawn, starting in B so that p is the first boundary point of B the curve hits.

When the basin boundary $\partial\overline{B}$ is fractal, only a relatively small subset of $\partial\overline{B}$ consists of accessible points, and generally no accessible points that are accessible from B will be accessible from another basin.

Definition 13. $p \in M$ is B-accessible if $p \in \partial\overline{B}$ and p is accessible from B.

If one point of a periodic orbit is B-accessible, then so are the other points, so we have B-accessible periodic orbits.

The paper introduces the notion of 'basin cell' which determines the structure of a basin.

Definition 14. A *basin cell* is a connected trapping region C whose boundary is piecewise smooth and $\partial C \subset W_s(P) \cup W_u(P)$ for some specified saddle-

[3]Fractal basin boundaries have been studied extensively, see, e.g. McDonald et al. [1985]; Grebogi et al. [1987]; Nusse and Yorke [1992], and references therein.

[4] A homoclinic point belongs to the intersection of the stable and unstable manifolds of an equilibrium

hyperbolic periodic orbit P.

It is possible to have three or more basins, such that for every basin boundary point x, each open neighborhood of x intersects each of the basins.

To describe this phenomenon, Nusse and Yorke [1996a,b, 1997] introduced the notion of 'Wada property' for dynamical systems. Basins of attraction have the Wada property if each point that is on the boundary of two basins is on the boundary of every basin. This concept is based on an earlier result (the proof of which follows from the λ-lemma) that goes like this:

Theorem 1. *Assume p is a periodic point on the basin boundary such that the unstable manifold of p intersects every basin and the stable manifold of p is dense in each of the basin boundaries. Then the basins have the Wada property.*

Assume that F has at least three basins. $x \in M$ is a Wada point if every open neighborhood of x has a nonempty intersection with at least three basins. B is a Wada basin if every $x \in \partial\overline{B}$ is a Wada point. Before presenting the formal results, we need to introduce basin cells.

2.1 Basin Cells

A cell is a connected compact region, which is homeomorphic to a disk. A manifold cell C is a cell whose boundary is piecewise smooth and for which there exists a saddle-hyperbolic periodic orbit P such that:

- The boundary of C consists alternately of pieces of the stable and unstable manifold of the periodic orbit P. We denote those by $W_s(P)$ and $W_u(P)$.
- Every point $x \in \partial C \cap W_s(P) \cap W_u(P)$ is a point of transverse intersection of $W_s(P)$ and $W_u(P)$.

In this case, we also say that the cell C is a manifold cell (for P), or, the cell C is generated by the orbit P, or also, the orbit P generates the manifold cell C.

Each of the components of $\partial C \cap W_s(P)$ is a stable edge of C and each of the components of $\partial C \cap W_u(P)$ is an unstable edge of the cell C. The common point of a stable and an unstable edge of a cell C is a corner point of the cell C. Note that each of the corner points of a cell C generated by a periodic orbit P is either a periodic point, a homoclinic point, or a heteroclinic point[5]. For two edges S and U of a cell C, S is adjacent to U if there is a corner point c of the cell C such that c is an endpoint of both S and U. A basin cell is a manifold cell C for which there exists an integer $k \geq 1$ such that C, $F(C)$,..., $F^{k-1}(C)$ are disjoint, $F^k(C) \subset C$, and $F^k(C) \neq C$. Hence, $C$is a trapping region for F_k. In general, the basin of a basin cell C is the trapping region $\bigcup_{i=0}^{k-1} F^i(C)$ so it consists of k components. In what follows, it is assumed that $k = 1$. So, for our purpose, a basin cell is a manifold cell C which is a trapping region, so $F(C) \subset C, F(C) \neq C$. Hence, the basin

[5] A heteroclinic point lies in the intersection of stable and unstable manifolds of different periodic equilibria

consists of one component are therefore assumed to be connected. The results generalizes to $k > 1$.

The following two propositions give the criteria that characterizes a basin cell.

Proposition 1. *Let $P = \{p_n\}_{1 \le n \le m}$ be a saddle-hyperbolic periodic orbit that generates a manifold cell C_p . Assume that C_p satisfies the following conditions:*

1. *P is contained in ∂C_p*
2. *For every integer $n(1 \le n \le m)$,p_n is not a corner point of C_p*
3. *F maps each of the unstable edges of C_p into C_p; and*
4. *C_p has $2m$ edges (that is, m stable and m unstable edges).*

Then C_p is a basin cell.

Proof. Let P and C_p be as in the proposition, and assume that the cell C_p satisfies the conditions specified. Let $P = \{p_n\}_{1 \le n \le m}$ be such that $p_{n+1} = F(p_n)(1 \le n \le m-1)$ and $p_1 = F(p_m)$.

Let $S_n(1 \le n \le m)$ denote the stable edges of C_p , and let $U_n(1 \le n \le m)$ denote the unstable edges of C_p . By the definition of cell, the union of the $2m$ arcs $\bigcup_{n=1}^{m}(S_n \cup U_n)$ is a simple closed piecewise smooth curve homeomorphic to the circle; call this curve J. The curve J separates the plane into two regions, and the closure of the bounded region is the manifold cell C_p. We assume that $p_n \in S_n(1 \le n \le m)$. The properties (1)–(4) imply that p_n is not a corner point of C_p and $F(U_n) \subset C_p$ for every integer $n(1 \le n \le m)$.

Note that $S_n \subset W_s(p_n)$ and $U_n \subset W_u(P)$, but $U_n \subset W_u(p_n)$ does not necessarily hold $(1 \leq n \leq m)$. It is even possible that all unstable edges are all compact arcs in the unstable manifold of one of the periodic points of P . For a compact arc A, we write A^0 for A minus its end points. We want to prove that $F(S_n) \subset C_p$ for every $n (1 \leq n \leq m)$. If $m = 1$, then $F(S_1) \subset S_1$, and we are done.

Assume from now on that $m \geq 2$. Suppose that there exists an integer $r (1 \leq n \leq m)$ such that $F(S_r)$ is not contained in C_p. In particular, we have that $F(S_r)$ is not contained in S_{r+1} if $1 \leq r \leq m-1$ or $F(S_r)$ is not contained in S_1 if $r = m$. But property (3) implies that F maps the end point of S_r into C_p . Hence, $F(S_r)$ must cross some arc $U_k (1 \leq k \leq m)$. Let q_1 be the point of intersection. The point q_1 is of course not a corner point of C_p . It is no restriction to assume that $r = m$, and so we have that $W_s(p_1)$ intersects U_k^0 transversally at q_1. We write K_1 for the compact arc in $W_s(p_1)$ with end points p_1 and q_1. We can assume that our q_1 and k have been chosen such that for every $x \in K_1^0$ either $x \in S_1$ or $x \notin C_p$. If it were not true we could have chosen K_1 shorter. Since p_1 is a period-m point, we have p_1 is a fixed point of F^m. Note that $F^m(K_1 \setminus \{p_1\}) \subset K_1^0$.

Let A be a compact arc in U_k^0 such that $q_1 \in A_0$. We now have $F^m(A)$ is a compact arc that intersects $W_s(p_1)$ transversally at $F^m(q_1)$. This fact together with the property that $F^m(q_1)$ is contained in K_1^0 imply that the compact arc $F_m(A)$ is not contained in C_p . On the other hand, we have $F^m(U_k) \subset C_p$, since $F^m(\partial U_k) \subset \partial C_p$ and $F^m(U_k)$ cannot intersect U_k. Since

$A \subset U_k^0$, we now have a contradiction. Therefore, $F(S_n) \subset S_{n+1}$, where $1 \leq n \leq m-1$, and $F(S_m) \subset S_1$. Hence, $F(\partial C_p) \subset C_p$, so C_p is a trapping region. (Notice that $F(CP) \neq C_p$.We conclude that C_p is a basin cell. □

Proposition 1 gives conditions under which a manifold cell is a basin cell. The next result deals with the converse situation and provides a characterization of basin cells.

Proposition 2. *Let $P = \{p_n\}_{1 \leq n \leq m}$ be a saddle-hyperbolic periodic orbit such that P generates a manifold cell C_p . Assume that C_p is a basin cell. Then C_p has the following properties:*

1. *For every integer $n (1 \leq n \leq m)$, p_1 is not a corner point of C_p ;*
2. *F maps each of the unstable edges of C_p into C_p ; and*
3. *There exists a basin cell C_p^* such that: (i) C_p^* has m stable edges and m unstable edges; (ii) every unstable edge of C_p^* is an unstable edge of C_p ; and (iii) every stable edge of C_p is contained in a stable edge of C_p^* .*

Proof. Let P and C_p be as in the proposition, and assume that the cell C_p is a basin cell. Let $P = \{p_n\}_{1 \leq n \leq m}$ be such that $p_{n+1} = F(p_n)(1 \leq n \leq m-1)$ and $p_1 = F(p_m)$. Start with following set of observations. **a.** Suppose there exists a point p of P that is not contained in ∂C_p . Then C_p is not forward invariant, so C_p is not a trapping region. SinceC_p is a basin cell, we now have a contradiction. Hence, P is contained in ∂C_p . **b.** Let p be one of the m periodic points of P. Suppose that p is a corner point of C_p. Let U be the unstable edge of C_p of which p is an end point, so $U \subset W_u(p)$.

Let q be the other end point of U. Consider a compact arc E in $W_u(p)$ such that E and C_p have only the point q in common. Such an arc exists since (by definition of a manifold cell) the point q is a point of transverse intersection of stable and unstable manifolds of P. Then there exist an arc $A \subset U$, an arc $B \subset E$ and a positive integer n such that $F^{2mn}(A) = B$. This implies that there exists an arc $I \subset U$ such that $F(I)$ is not contained in C_p. But then C_p is not a trapping region, since C_p is not forward invariant. This implies that C_p is not a basin cell for which $F(C_p) \subset C_p$, and we have a contradiction. Therefore, p is not a corner point of C_p . We conclude that p_n for all $n(1 \leq n \leq m)$ is not a corner point of C_p. **c.** Let U be an unstable edge of the cell C_p . Since C_p is a basin cell and $F(C_p) \subset C_p$, it is a trapping region. Hence, $F(U) \subset C_p$. **d.** By (a) and (b)we have that $p_n \in \partial C_p$ and it is not a corner point of $C_p(1 \leq n \leq m)$. Therefore C_p has at least m stable edges and at least m unstable edges. Write S_n for the stable edge of C_p that contains $p_n(1 \leq n \leq m)$. Assume that C_p has $m + r$ stable edges for some integer $r \geq 0$. If $r = 0$, then we are done. From now on, we assume that $r \geq 1$. Write $S_n(m + 1 \leq n \leq m + r)$ for the stable edges of C_p which do not contain a periodic point. Write $U_n(1 \leq n \leq m + r)$ for the unstable edges of C_p .

Now we set up some definitions to be used in the later part of the proof. We say that an unstable edge U of C_p is nonessential if the end points of U are both in the stable manifold $W_s(p_n)$ for some $n(1 \leq n \leq m)$; otherwise, we say that U is essential. For a nonessential unstable edge U of C_p , we say that the U-cell is the closed region bounded by the unstable edge U and the compact arc A in $W_s(p_n)$ such that the end points of A coincide with the end points of U. We denote the U -cell by C_u . Note that both $C_u$and the union $C_p \cup C_u$ are manifold cells. The idea of the proof is to construct a larger set C^* by replacing each nonessential edge U of C_p by the U-cell and then to show that the set C^* is a basin cell which satisfies (i), (ii) and (iii). We say that a manifold cell C is a C_p -admissible cell if: (i) every unstable edge of C is an unstable edge of C_p ; (ii) every stable edge of C that does not contain a periodic point is a stable edge of C_p ; and (iii) every stable edge of C_p is contained in a stable edge of C. The following property follows immediately from the definitions.

C_p **-admissible cell property.** Let C be a C_p -admissible cell which has $2N_c$ edges. Assume that U is a nonessential unstable edge of C. Then the union of the manifold cells C and C_u is a C_p -admissible cell which has $2(N_{c-1})$ edges. The next intermediate basic result for C_p -admissible cells is fundamental in getting the desired manifold cell.

We need the following lemma in order to complete the proof.

Lemma 1. *Let C be any C_p -admissible cell. Let n be any given integer* $(1 \leq n \leq m)$. *Write S_n^c for the stable edge of C that contains the periodic point p_n. Then, $\partial C \cap W_s(p_n)$ consists of one component (which is S_n^c) if and only if the two unstable edges which are adjacent to S_n^c are both essential.*

Proof. Let C, n, and S_n^c be as in the lemma.

Necessity Assume that $\partial C \subset W_s(p_n)$ consists of one component. This implies immediately that the two unstable edges which are adjacent to S_n^c are both essential.

SufficiencyAssume that the two unstable edges which are adjacent to S_n^c are both essential. Suppose that $\partial C \cap W_s(p_n)$ consists of two or more components. We will show that at least one of the unstable edges adjacent to S_n^c is nonessential. Let $E_n \subset W_s(p_n)$ be a stable edge of C that does not contain a point of P. (Hence, $E_n = S_{m+i}$ for some $i(1 \leq i \leq r)$.) Let a_n and b_n be the end points of E_n. Let I_n andJ_n be unstable segments of C which are adjacent to E_n. (Hence, $I_n = U_i$ and $J_n = U_j$ for some $i,\ j\ (1 \leq i,\ j \leq m+r)$.) Assume that a_n is an end point of I_n and that b_n is an end point of J_n. Let integers $s,\ t\ (1 \leq s,\ t \leq m)$ be such that $I_n \subset W_u(p_s)$ and $J_n \subset W_u(p_t)$. Then (by the definition of manifold cell) $W_s(p_n)$ intersects $W_u(p_s)$ transversally at a_n and intersects $W_u(p_t$ at b_n. We assume that the distance between a_n and p_n(measured along $W_s(p_n)$ is smaller than the distance between b_n and p_n . In other words, we assume that a_n is between p_n and b_n on $W_s(p_n)$. We write K_n for the compact arc in $W_s(p_n)$ with end points p_n and a_n. We assume that for every $x \in K_n^0$, either $x \in S_n^c$or $x \notin C_p$ .Note that $F_m(K_n \setminus \{p_n\}) \subset K_n^0$.If I_n is not adjacent to S_n^c, then at least two of the manifolds $W_s(p_n)(1 \leq n \leq m)$ must intersect, which is impossible. Hence, I_n is adjacent to S_n^c, so I_n is adjacent to both S_n^c and E_n. Consequently, $I_n$is a nonessential unstable edge of C . We now have a contradiction. We conclude that $\partial C \cap W_s(p_n)$ consists of one component. This completes the proof of the lemma. □

Now coming back to the proof of Proposition 2, let C be any C_p -admissible cell, and let S_n^c be the stable edge of C as in the lemma, where $1 \leq n \leq m$. Lemma 1 implies the following:

- If S is a stable edge of C such that $S \subset W_s(p_n) \setminus \{p_n\}$, then at least one of the unstable edges of C which is adjacent to $S_n^c$is nonessential; and
- The cell C_p has m essential unstable edges.

Write $U_n(m+1 \leq n \leq m+r)$ for the nonessential unstable edges of C_p . We denote by D_{n-m} the U -cell that corresponds to Un . We define $C_0 = C_p$. By induction, for every integer $k(1 \leq k \leq r)$ we define $C_k = C_{k-} \cup D_k$. It is no restriction to assume that the nonessential unstable edges are labeled such that for every $k(1 \leq k \leq r)$, the unstable edge U_{m+k} is adjacent to a stable edge of C_{k-1} that contains a periodic point of P . By the C_p -admissible cell property, we have that for every integer $k(1 \leq k \leq r)$ the cell C_k is a C_p -admissible cell and has $2(m+r-k)$ edges. In particular, the cell C_r satisfies (1), (2), and (3) of Proposition 2.

We now show that C_r is a trapping region. We show by induction that for every $k(1 \leq k \leq r)$ the cell D_k is mapped into C_r. For every $k(1 \leq k \leq r)$, write $S_{n;0}^c$ for the stable edge of C_k that contains the periodic point p_n. (Note that $S_{n;0}^c = S_n^c$ for every $n(1 \leq n \leq m)$, where S_n^c is as in the lemma.) We first show that for $k = 1$ the cell D_1 is mapped into C_r. By the definition of D_1. the unstable edge $U_{m+1}$is contained in the boundary of D_1 . Let $S_{n;0}^c$

and $E_n \subset W_s(p_n) \setminus S^c_{n;0}$ be the two stable edges of C_0 which are adjacent to U_{m+1} for some $n(1 \leq n \leq m)$. Since $U_{m+1} \subset C_p$ and C_p is a basin cell, we have that $F(U_{m+k}) \subset C_p \subset C_1$. We now write $S^*_{n;0}$ for the compact arc in $W_s(p_n)$ that contains both $S^c_{n;0}$ and E_n such that one end point of $S^*_{n;0}$ is an end point of $S^c_{n;0}$ and the other end point of $S^*_{n;0}$ is an end point of E_n. Since $S^c_{n;0} \subset C_p = C_0$ and $En \subset C_p$ we have that $F(S^c_{n;0}) \subset C_p \subset C_r$ and $F(E_n) \subset C_p \subset C^r$.

This together with the fact that C_r has m stable edges implies that $F(S^*_{n;0}) \subset C_r$. Since D_1 is bounded by U_{m+1} and $S^*_{n;0}$, we now have that $F(\partial D_1) \subset C_r$, and so $F(D_1) \subset C_r$. Note that $S^*_{n;0} = S^*_{n;1}$. If $r = 1$, then this implies that $F(\bigcup_{k=1}^{r} D_{k)} \subset C_r)$. Hence, the cell C_r is a trapping region, since for every $x \in C_r$ either $x \in C_0 = C_p$ or $x \in D_1$, and so $F(x) \in C_r$. By defining $C^*_p = C_r$ we conclude that the cell C^*_p is a basin cell and satisfies the properties (i), (ii), and (iii).

From now on, assume that $r > 1$. Let an integer $k(1 \leq k \leq r)$)be given. By the definition of D_k , the unstable edge U_{m+k} is contained in the boundary ofD_k . Let $S^c_{n;k-1}$ and $E_n \subset W^s(p^n) \setminus S^c_{n;k-1}$ be the two stable edges of C_{k-1} which are adjacent to U_{m+k} for some $n(1 \leq n \leq m)$, where E_n is also a stable edge of C_p. Assume that $F(S^c_{n;k-1}) \subset C_r$. Since $U_{m+k} \subset C_p$ and C_p is a basin cell, we have that $F(U_{m+k}) \subset C_p \subset C_r$. By construction, we have that for every $n(1 \leq n \leq m)$ the cell C_r has one stable arc in $W_s(p_n)$. We now write $S^*_{n;k-1}$ for the compact arc in $W_s(p_n)$ that contains both $S^c_{n;k-1}$ and E_n such that one end point of $S^*_{n;k-1}$ is an end point of $S^c_{n;k-1}$, and the other end point of $S^*_{n;k-1}$ is an end point of E_n. Since $E_n \subset C_p$ we have $F(E_n) \subset C_p \subset C_r$ and by assumption we have $F(S^c_{n;k-1}) \subset C_r$. This together with the fact that C_r has m stable edges implies that $F(S^*_{n;k-1}) \subset C_r$. Since D_k is bounded by U_{m+k} and $S^*_{n;k-1}$, we now have that $F(\partial D_k) \subset C_r$, and so $F(D_k) \subset C_r$. The fact that $S^*_{n;k-1} = S^c_{n;k-1}$ immediately implies that $F(S^c_{n;k-1}) \subset C_r$. By the induction hypothesis, we now have for every $k(1 \leq k \leq r)$that $F(D_k) \subset C_r$ This implies that $F(\bigcup_{k=1}^r D_k) \subset C_r$. Hence, the cell C_r is a trapping region, since for every $x \in C_r$ either $x \in C_0 = C_p$ or there exists an integer $k(1 \leq k \leq r)$ such that $x \in D_k$, and so $F(x) \in C_r$. By defining$C^*_p = C_r$ we conclude that the cell C^*_p is a basin cell and satisfies the properties (i), (ii), and (iii). □

Observation 1. Proposition 2 implies that a manifold cell that has a corner point which is a point of the saddle-hyperbolic periodic orbit that generates the cell, cannot be a basin cell. A special case of this situation is the fixed point p on a horseshoe that generates a manifold cell C and p is a corner point of C.

The following proposition shows that the accessible boundary points of the basin of a basin cell are all on the stable manifold of the periodic orbit that generates the cell.

Proposition 3. *Let P be a saddle-hyperbolic periodic orbit which generates a manifold cell C_p . Assume thatC_p is a basin cell. Let B_p be the basin of the cell C_p . Then every B_p -accessible point is on the stable manifold $W_s(P)$ of P.*

Proof. Let P , C_p and C_p be as in the proposition. Let C^0_p for the interior of C_p . Let $Q \subset W_u(P)$ be the union of the edge-generating arcs. Note that Q is compact. Let q be any arbitrarily given B_p-accessible point. There are three cases: (a)$q \notin Q$; (b) $q \in Q \setminus P$; and (c) $q \in P$.

Case (a). Assume that $q \notin Q$. This implies that $q \in P$. Let γ be an access arc to q, that is, γ is an arc, q is an endpoint to γ, and $\gamma \setminus q \subset B_p$. Since Q and γ are compact, there exists a subarc $\gamma^* \subset \gamma$ such that γ^* is an access arc to q and $\gamma^* \cap Q = \emptyset$. Now, assume that the access arc γ has endpoints q and r, and is chosen such that $\gamma \cap Q = \emptyset$. Write $\gamma_0 = \gamma \setminus q$. Let an integer $n \geq 0$ be given such that $F_n(r) \in C_p^0$. (Such an integer n exists since B_p is the basin of C_p and $r \in C_p$. The assumption that $\gamma \cap Q = \emptyset$ implies $F_n(\gamma) \cap Q = \emptyset$. Hence, $F_n(\gamma_0) \subset C_p^0$ and so $F_n(\gamma) \subset C_p$. Therefore, $F_n(q)$ is a B_p-accessible point which is in ∂C_p, so $F_n(q) \in W_s(P)$. The conclusion is that $qinW_s(P)$.

Case (b). Assume that $q \in Q \setminus P$. Then there exists an integer $k \geq 1$ such that $F_k(q) \in Q$.Now apply *Case(a)* to $F_k(q)$ and the conclusion is that $F_k(q) \in W_s(P)$ and so is q.

Case (c). Assume that $q \in P$. Then $q \in W_s(P)$. □

Observation 2. Let P be a hyperbolic periodic orbit which generates a basin cell C_p . Then there exists a minimal basin cell for P.

2.2 Wada Basins

Let $N \geq 3$ be any integer. let $F : M \to M$ be an area contracting C^1-diffeomorphism satisfying the following two conditions. (a) There exist at least three disjoint basins, and N of them will be denoted by $B_k (1 \leq k \leq N)$, where $N \geq 3$. (b) Lock-out property: there exists a compact set $K \subset M$ such that (i) K contains all points whose trajectories are bounded under forward and backward iteration of F, and (ii)if $x \in K$ and $F(x) \notin K$,then for every positive integer n, $F_n(x) \notin K$. Assumption (a) does not require that $B_k (1 \leq k \leq N)$ are the only basins, and assumption (b) implies that the compact set K contains all attractors and all periodic points. For every integer $k (1 \leq k \leq N)$, assume that P_k is a saddle-hyperbolic period- m_k orbit that generates a basin cell C_k, and assume that B_k is the basin of the cell C_k. Note that, by assumption, B_k is simply connected $(1 \leq k \leq N)$.

Proposition 4. *Let $p \in P_1$, and assume that $W_u(p)$ intersects B_k for some $k(1 \leq k \leq N)$. Then $W_s(p) \subset \partial \overline{B}_k$.*

Proof. This is actually a version of the λ-Lemma. □

Proposition 5. *Let $p \in P_1$, and assume that $W_u(p)$ intersects each $B_k (1 \leq k \leq N)$. Then, every $z \in W_s(p)$ is a Wada point w.r.t. $B_k (1 \leq k \leq N)$.*

Proof. Let p be as in Proposition 5. Then, by Proposition 1, $W_s(p) \subset \bigcup_{k=1}^{N} \partial B_k$. Hence, every $z \in W_s(p)$ is a Wada point w.r.t. $B_k (1 \leq k \leq N)$. □

The theorem below is the main result on the existence of Wada basins.

Theorem 2. *$W_u(P_1)$ intersects all $B_k (1 \leq k \leq N) \iff B_1$ is a Wada basin w.r.t. $B_k (1 \leq k \leq N)$.*

Proof. *Necessity* Assume that $W_u(P_1)$ intersects all $B_k (1 \leq k \leq N)$. For every

$p \in P_1$, Proposition 5 implies that every $z \in W_s(p)$ is a Wada point w.r.t. $B_k (1 \leq k \leq N)$, and Proposition 4 implies that $W_s(p) \subset \partial \overline{B}_k (1 \leq k \leq N)$. By Proposition 3 we have that $W_s(P_1)$ is the set of all B_1-accessible points. Let $z \in \partial \overline{B}_1$ and $\epsilon > 0$ be given. Since $z \in \partial \overline{B}_1$ and the fact that the B_1-accessible points are dense in $\partial \overline{B}_1$, there exists a B_1-accessible periodic point p_z such that $B_\epsilon(z) \cap W_s(p_z) \neq \emptyset$. Let $x \in B_\epsilon(z) \cap W_s(p_z)$, and select $\delta > 0$ such that $B_\delta(x) \subset B_\epsilon(z)$. Since x is a Wada point w.r.t. $B_k (1 \leq k \leq N)$ and $x \in \partial \overline{B}_1 (1 \leq k \leq N)$, the open ball $B_\delta(x)$ contains points of each $B_k (1 \leq k \leq N)$. Therefore $B_\epsilon(z)$ contains points of each $B_k (1 \leq k \leq N)$. Since $\epsilon > 0$ arbitrarily chosen, we have that z is a Wada point w.r.t. $B_k (1 \leq k \leq N)$. The conclusion is that B_1 is aWada basin w.r.t $B_k (1 \leq k \leq N)$.

Sufficiency Assume that B_1 is a Wada basin w.r.t. $B_k (1 \leq k \leq N)$. Notice that each $B_k (1 \leq k \leq N)$ is connected. Let p be a hyperbolic B_1 -accessible periodic point. By Proposition 3, the periodic orbit P_1 is the only periodic orbit on $\partial \overline{B}_1$ which is B_1- accessible, so $p \in P_1$. We write $W_{u+}(p)$ and $W_{u-}(p)$ for the two components of $W_u(p) \setminus p$. One of these two components, say $W_{u+}(p)$, is contained in B_1^-. Since each$x \in \partial \overline{B}_1$ is a Wada point, we now show that the other component $W_{u-}(p)$ intersects every $B_k (2 \leq k \leq N)$).

Suppose there exists an integer $m (2 \leq m \leq N)$ such that $W_{u+}(p)$ does not intersect basin B_m. Since P_1 generates a basin cell, we have by the definition of manifold cell together with the λ-Lemma that this immediately implies that the point p admits a transverse homoclinic point. Therefore, $W_s(p)$ piles up on itself, and intersects $W_{u-}(p)$ infinitely many times. This implies that there exist three disjoint, connected, bounded, open regions $D_k (1 \leq k \leq 3)$ such that: (a) D_1 is bounded by one segment of $W_s(p)$, say S_1, and one segment of $W_{u-}(p)$, say U_1; (b) D_3 is bounded by two segments of $W_s(p)$, say S^2 and S^3, and two segments of $W_{u-}(p))$, say U_3 and V_3; and (c) D_2 is bounded by the two segments S_1 and S_2 of $W_s(p)$, and two segments of $W_{u-}(p)$, say U_2 and V_2. Let three points $x_1 \in S_1 \setminus W_{u-}(p)$, $x_2 \in S_2 \setminus W_{u+}(p)$, and $x_3 \in S_3 \setminus W_{u-}(p)$ be given. Since $x_n \in W_s(p) \subset \partial\overline{B}_1$, we have that $x_n (1 \leq n \leq 3)$ is a Wada point. In particular, each of the points $x_n (1 \leq n \leq 3)$ is a boundary point of B_m. The assumption that $W_{u-}(p)$does not intersect B_m implies that B_m is not connected, and we have a contradiction. Therefore, there exists no integer $m (1 \leq m \leq N)$ such that $W_{u-}(p)$ does not intersect basin B_m. Hence, $W_u(p)$ intersects B_k, for every $k (1 \leq k \leq N)$. We conclude that $W_u(p_1)$ intersects all $B_k (1 \leq k \leq N)$. □

Observation 3. Let P_1 be as in the theorem, then $W_s(p_1)$ is dense in $\partial\overline{B}_1$.

The above result was for establishing the properties of a single basin B1. The next result is for showing those properties hold for a collection of basins. We present it without proof.

Corollary 2.1. *Assume that for every $m (1 \leq m \leq N)$, the set $W_u(p_m)$ intersects every $B_k (1 \leq k \leq N)$. Then $\partial\overline{B}_1 = \partial\overline{B}_k$ for every $1 \leq k \leq N$.*

3 Numerical Verification of Wada Boundaries

This section presents three numerical methods devised by Nusse and Yorke [1996b].

The first numerical method makes it easy to find accessible saddle periodic orbits. It is for finding accessible numerical trajectories on basin boundaries. Accessible trajectories often are quite simple. For many examples the accessible trajectories converge to a periodic trajectory which is accessible and is a saddle. For those periodic trajectories, one branch of its local unstable manifold is an access curve. This method the ABST method. The ABST method is valid assuming that the diffeomorphism is hyperbolic on the basin boundary. In practice, it works fine for many typical dynamical systems. It is used to find an accessible saddle periodic point, and it is easy to verify that it has found an accessible hyperbolic periodic point. The details of this procedure are presented in Zhiping et al. [1991].

Let the manifold M, the diffeomorphism F, the transient region R, and generalized attractors A and B be as defined. Assume that each point that leaves R under iteration of F is either in basin A or in basin B. The escape time $T_R(x)$ of a point x in R is defined by

$$T_R(x) = min\{n \geq 1 : F_n(x) \notin R)\}$$

, and $T_R(x) = \infty$ if $F_n(x) \in R$ for all $n \geq 1$. We say, $T_R(x) = 0$ if $x \notin R$.

Let J be an unstable segment in R. The notation $\{x, y\}$ for a pair means that

x and y lie on J. Since J is homeomorphic to an interval, we may assume it has the ordering of an interval. For $\{x, y\}$ we always assume for convenience that the ordering on J is such that we may write $x < y$, and denote $[x, y]$, for the segment on J joining x and y. Let $L \subset J$ be any connected subset of J. Assume L intersects the stable set $S(R)$ transversally, and let $\{a, b\}$ be a pair on L. For each $\epsilon > 0$, an $\epsilon - refinement$ of $\{a, b\}$ is a finite set of points $a = go < g_1 < \ldots < g_N = b$ in $[a, b]$, such that

$$(\epsilon/2) \cdot \rho([a, b]_J) \leq \rho([g_k, g_{k+l}]_J) \leq \epsilon \cdot \rho([a, b]_J)$$

for all k, $0 \leq k \leq N-1$

We say the pair $\{a, b\}$ is a straddle pair if $a \in \text{basin}A$ and $b \in \text{basin}B$. We call $\{a, b\}$ a proper straddle pair if $\{a, b\}$is a straddle pair, and at least one of the points a and b is in the interior of L. If $\{a, b\}$is a (proper) straddle pair, then we call the interval $[a, b]$ a (proper) straddle segment. The objective is to describe the 'accessible basin boundary refinement procedure' that selects in a unique way a proper straddle pair from any ϵ-refinement of a given straddle pair (on J). When we repeatedly apply the procedure to the end points of the ever decreasing straddle segments (with lengths converging to zero), the resulting nested sequence converges to an accessible point p in the basin boundary; ofcourse, this point p is in $J \cap S(R)$.

we now describe the accessible basin boundary refinement procedure which is the refinement procedure that generates a uniquely defined proper straddle pair from a given straddle pair. This procedure plays a dominant role in the method that generates a numerical trajectory on the basin boundary that is accessible from basin $\{A\}$.

Let $\{a, b\}$ be a straddle pair on a curve segment J such that a is contained in basin$\{A\}$, and b is contained in basin$\{B\}$. Let $P = \{x_i : 0 \leq i \leq N(\epsilon)\}$ any ϵ-refinement of $\{a, b\}$ we of course have $P \subset J$ and $a = x_0 < x_1 < \ldots < X_{N_\epsilon} = b$. We choose the proper straddle pair $\{a, b\}$ from P in the following way:

1. select b^* to be the leftmost point of P that is in basin$\{B\}$;

2. define m to be the minimum of the escape time of the points in P to the left of b^*, and write a^0 to denote the rightmost point to the left of b_* that has the minimum escape time m. Then (2a) If $m < T_R(a)$ then choose $a^* = a^0$; otherwise, (2b) if $m = T_R(a)$ then the choice of a^* depends on the grid P_* consisting of b^* and all the points in P to the left of b^* and all the points in P to the left of b^* (that is, $P* = \{x \in P : x \in [a, b^*]\}$). (i) If the grid P^* is not an ϵ-refinement of $\{a, b*\}$,then choose $a^* = a$; otherwise, (ii) if the grid P^* is an ϵ-refinementof $\{a, b\}$ then choose a^* to be the adjacent point in P^* to the right of a^*, unless b^* is that adjacent point, in which case choose $a^* = a^0$.

Observation 4. Assume that $\epsilon > 0$ is suitably chosen. In case of step (2b) the equality $a^* = a^0$does not occur and one has $a^* > a^0$.

(1) As the accessible basin boundary refinement procedure is applied repeatedly, step (2a) only occurs at most finitely many times, and the segment $[a, a^*]$ in (2a) may include points that are in basin $\{B\}$. However, once

step (2b) occurs, step (2) will never occur again. When step (2b) is applied, the entire segment $[a, a^*]$ (not just the grid points) is in basin $\{A\}$ but $[a, a^*]$ may include points that have escape time infinity. It should be emphasized that all the points between a and a^* in step (2b) whose escape time is finite, go to attractor A. This is why the method produces an accessible point as the refinement is repeated. The problem of course is to find ϵ small enough.

(2) When a^* and b^* have been chosen, if the grid consisting of a^*, a^* and all the points in P between a^* and b^* is still an ϵ-refinement of the pair $\{a, b\}$, then set $a^* = a$ and $b^* = b$ and apply step (2b). Repeat this until the grid $\{x \in P : x \in [a^*, b^*]\}$ fails to be an ϵ-refinement of $\{a^*, b^*\}$. Notice that in cases when only step (2b) is repeated, the point b does not move. (3) We can repeatedly apply the accessible basin boundary refinement procedure obtaining a sequence of straddle pairs that converges to an accessible point on the basin boundary.

The second numerical method is for plotting one dimensional stable and one dimensional unstable manifolds. One might erroneously expect that plots of unstable manifolds would be grossly inaccurate due to the sensitivity of initial data near a saddle periodic orbit, but plots are usually very accurate. The authors call the method for plotting unstable manifolds the UM method, and for plotting the stable manifolds the SM method. See Zhiping et al. [1991] for a detailed justification of the algorithm and more details on plotting of manifolds.

The third method is for finding periodic orbits of some specified period. The authors call this method the RP method. There may be several periodic orbits accessible from a basin B. All these periodic orbits have either the same period or the period differs by a factor of 2. If the map is dissipative, then the periods must be all the same [Alligood and Yorke, 1992]. Assume, for example, that the map is dissipative and assume that the ABST method yields a period-m orbit accessible from basin B. The question is to find all the periodic orbits which are accessible from basin B. The RP method attempts to plot Randomly Periodic points of a specified period. This routine has a basic step in which it chooses a point in the screen area at random and then applies a Quasi-Newton method up to 50 times. If the point is in the basin of infinity and is diverging or if the Quasi-Newton method applied to the initial point converges to a point which is not periodic, then nothing is plotted. The routine plots when it has found a point which is within a distance of 10^{-11} from a periodic point of a specified period. The routine RP keeps repeating this basic step, each time choosing a new random seed for starting the process.

Thereafter, one can use the UM method to verify which of these periodic orbits are indeed accessible from basin B. If a periodic point p is accessible, one of the two branches $W_{u+}(p)$ and $W_{u-}(p)$ of the unstable manifold of p does not intersect the stable manifold of p. Then one should attempt to find all the periodic orbits of period m, and test these to see if they are accessible. If a branch of the unstable manifold of some periodic point on the basin boundary, say P, goes to the regional attractor of its basin, then this branch of the unstable manifold of P is a curve in the basin that hits the basin boundary at the point P, so the point P is accessible. The RP method in [citation 18] finds the hyperbolic orbits of a specified period (assuming the period is not too high). It iteratively chooses an initial point at random and applies a Quasi-Newton method many times, plotting the periodic orbit it finds. If the period m is low, say m = 1or 2, the number of periodic orbits found will be small. Each can be tested to see if it is accessible. If p is accessible (from a basin B), then one of its branches enters the basin B, and that branch cannot leave the basin (assuming the boundary of the basin is everywhere fractal).

Whether a specific example has a Wada basin or Wada basin boundary, can usually be verified numerically using a program with the following steps

1. Determine the attractors and plot the basins.
2. Use the ABST method to find accessible periodic orbits on the basin boundary.
3. Use the UM method to plot the unstable manifolds of one point of each of the accessible periodic orbits. We need to plot stable and unstable manifolds to find a trapping region.
4. One may wish to use the RP method to numerically verify that all accessible periodic points have been detected.

4 Conclusion

We have presented a short summary of results regarding a certain class of basins of attraction, namely the Wada Basins. They have a complicated structure and a common boundary. In a series of papers, Yorke and coauthors proved existence and characterization results and numerical verification methods. In practice, these kind of basins seem to be important for many two-dimensional diffeomorphisms including the Henon map and the forced damp pendulum.

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