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Regularization in phase transitions with Gibbs-Thomson law

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Regularization in phase transitions with Gibbs-Thomson law

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Regularization in phase transitions with Gibbs-Thomson law

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We study the regularity of weak solutions for the Stefan and Hele-Shaw problems with Gibbs-Thomson law under special conditions. The main result says that whenever the free boundary is Lipschitz in space and time it becomes (instantaneously) $C^{2,\alpha}$ in space and its mean curvature is Hölder continuous. Additionally, a similar model related to the Signorini problem is introduced, in this case it is shown that for large times weak solutions converge to a stationary configuration.

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Chapter 1

Introduction

The Stefan problem is a well known model for phase transitions of materials whose temperature is undergoing diffusion. It says that if $u(x,t)$ is the temperature of a material with two different phases (say liquid and solid) in some container Ω, then

$$(u+\chi)_t = \Delta u \quad \text{in } \Omega$$

$$\text{where } \chi = \text{ characteristic function of the solid phase}$$

Usually, one assumes that $u \equiv 0$ along the solid-liquid interface. The Gibbs-Thomson law is a correction to this model which makes it more accurate at smaller scales. It says that the the temperature of the interface is not constant but proportional to the mean curvature of the interface. There is a vast literature considering the heuristics and rigorous justification of this law [10].

In this work we study the smoothness of u and of the solid-liquid interface for this model, we require the interface to be a Lipschitz hypersurface in space and time. Additionally, we review the existence theory for weak solutions developed by Luckhaus [14] and apply his method to a new modification of the Stefan problem.

The main results can be summarized informally as follows. See section 2 for details.

Any weak solution of the Stefan or Hele-Shaw problems with Gibbs-Thomson law is automatically $C^{2,\alpha}$ in space whenever its interface is Lipschitz in space and time. In the case of the Signorini-Gibbs-Thomson law one gets $C^{1,1}$ in space.

The Gibbs-Thomson law is actually used indirectly in this result. What will actually be shown is that in general any weak solution to the "Stefan condition" is Hölder continuous, as long as the free boundary ($\partial\{\chi = 1\}$) is given locally by a Lipschitz graph in space and time. This is true independently of what other condition might be imposed on u (in particular, it gives a new Hölder continuity estimate for u for the classical Stefan problem). In the case of the Gibbs-Thomson law we get that the mean curvature of the free boundary is Hölder continuous, thus one has a Lipschitz surface with a continuous curvature to which the well known elliptic regulariy estimates can be applied.

For the classical case where $u \equiv 0$ on the interface, one can use comparison principles and viscosity solutions and a greater deal is known. In terms of regularity of weak (viscosity) solutions a lot of progress has taken place since the the work of Athanasopoulos, Caffarelli and Salsa [3–5] for the parabolic case and Caffarelli [6–8] for the elliptic case. Drawing inspiration from the theory of minimal surfaces, these works have brought forward a paradigm for the study of regularity of (parabolic/elliptic) free boundary problems: free boundaries which are either *Lipschitz* or very *flat*[1] ought to be smooth. As can be expected, proving the smoothness result under the Lipschitz assumption tends to be easier, and is often a first step in developing the machinery to address the more general and harder case of free boundaries that are a priori only *flat*.

Instead, when one includes the effects of the Gibbs-Thomson law the comparison principle and viscosity solution approach no longer works. Heuristically, this is because the free boundary velocity is of the same order as $(-\Delta_s)^{\frac{1}{2}}\kappa$, where Δ_s denotes the Laplace-Beltrami operator on the interface. This is a non-local, third order operator (as the mean curvature is already of order 2) acting on the free boundary. In particular, as it has greater than 2 one cannot expect anything like a comparison principle. More concretely, most of the arguments in the works of Athanasopoulos et al cited above break down when the temperature is not constant along the interface, such as Harnack-like principles or the Alt-Caffarelli-Friedman monotonicity formula.

However, the free boundary regularity is now more directly connected to the function u: if the temperature were bounded or have enough integrability in space then the interface would be[2] $C^{1,\alpha}$ in space. As mentioned in a previous paragraph, under the Lipschitz assumption it will be shown that a solution to the Stefan condition (regardless of the values of u along the interface) becomes

[1]this means the free boundary is trapped between two parallel planes that are close together.

[2]that this is so follows from the regularity of almost minimal boundaries (cf. Section 4)

Hölder continuous for all positive times, thus proving for the Gibbs-Thomson law that *Lipschitz free boundaries become* $C^{2,\alpha}$ *in space instantaneously.*

The Hölder continuity of u (in space and time) will be proved pushing the De Giorgi-Nash-Moser regularity theory for linear parabolic equations so that it can handle singular right hand sides, namely the distribution χ_t, which under the Lipschitz assumption lives in H^{-1}. This will be proven in two ways: first by a modification of the usual iterations that will lead to a *non linear and homogeneous* estimate and secondly by a maximum principle related to that proven by Stampacchia, which will give a linear but non-homogeneous estimate. These estimates are proven for weak solutions in the sense of Luckhaus, but they can also be seen as a priori estimates for classical solutions and from that perspective a corollary of these results is that *whenever singularities form, they must be felt at least at the level of Lipschitz regularity*, one could hope that similar estimates might help understand the formation of singularities, as it has been done for geometric flows.

Besides reviewing Luckhaus' method, we also modify it to treat a new toy problem motivated by porous flow through semipermeable walls and the Signorini problem, the original model is discussed in [11]. The problem is similar to Hele-Shaw or Stefan with Gibbs-Thomson law, except that instead of the asking u = mean curvature on the interface, we ask only that $u \leq$ mean curvature and that it be the largest subharmonic (resp. subcaloric) function satisfying that property, which gives a time-dependent Signorini problem.

The organization of the paper is as follows: in Section 2 we state the main results in detail; in Section 3 we review Luckhaus' construction of weak solutions, almost minimal surfaces and adapt these ideas to the Stefan-Signorini problem; sections 4 and 5 deal with the regularity of Lipschitz free boundaries. Finally, the appendix contains a review of the linear the parabolic De Giorgi - Nash - Moser theory, where we prove the oscillation lemma adapting an estimate from the work of Caffarelli-Vasseur [9] on the quasigeostrophic equation, with this lemma in hand one can prove continuity without using a covering argument as it is usually done [15].

Chapter 2

Main results

2.1 Definitions and notation

To state the main results it will be helpful to fix some notation.

We will denote by Ω a generic bounded domain of $\mathbb{R}^n$ with a Lipschitz boundary. If $T > 0$ we shall also write Ω_T for the product $\Omega \times (0, T)$.

The functional spaces we will work with are: the Sobolev space $H_0^1(\Omega)$ of functions with square-summable gradients and vanishing on the boundary and the space $BV(\Omega)$ of functions with finite perimeter (see [12] for properties of BV functions). We are restricting ourselves to the case of zero Dirichlet boundary conditions for simplicity, although our methods allow to handle generic prescribed boundary values.

Definition 2.1.1. *A pair* (u, χ) *of functions*

$$u \in L^2(0, T; H_0^1(\Omega)) \cap L^\infty(0, T; L^2(\Omega))$$

$$\chi \in L^\infty(0, T; BV(\Omega)), \quad \chi \in \{0, 1\} \text{ a.e.}$$

are called a weak solution to the Stefan problem with Gibbs-Thomson law in Ω_T *if they satisfy 1) The weak Stefan condition*

$$\int_0^T \int_\Omega (u + \chi)\phi_t dxdt + \int_0^T \int_\Omega \nabla u \cdot \nabla \phi dxdt = 0 \quad \forall \phi \in C_c^\infty(\Omega_T)$$

and 2) The Gibbs-Thomson law in the BV sense: for a.e. $t \in (0,T)$ and every $Y \in C_c^\infty(\Omega;\mathbb{R}^n)$ we have

$$\int_\Omega (div(Y) - \nu \cdot DY(\nu))|\nabla\chi(t)| = \int_\Omega u(t) Y \cdot \nu |\nabla \chi(t)|,\ \nu = \frac{\nabla\chi(t)}{|\nabla\chi(t)|}$$

Definition 2.1.2. *A pair (u,χ) of functions*

$$u \in L^2(0,T;H_0^1(\Omega))$$

$$\chi \in L^\infty(0,T;BV(\Omega)), \quad \chi \in \{0,1\}\ a.e.$$

are called a weak solution to Hele-Shaw with Gibbs-Thomson law in Ω_T if they satisfy the same condition 2) above and instead of the weak Stefan condition we have

$$\int_0^T \int_\Omega \chi\phi_t dxdt + \int_0^T \int_\Omega \nabla u \cdot \nabla\phi dxdt = 0 \quad \forall \phi \in C_c^\infty(\Omega_T)$$

2.1.1 Additional conventions

Throughout this work we will refer to the Stefan problem with Gibbs-Thomson law simply as (SGT) and to the Hele-Shaw problem with Gibbs-Thomson law as (HS). Whenever we talk about a solution to (SGT) or (HS) we will mean it in the sense of Definitions 2.1.1 and 2.1.2. When we say that they have an initial condition (u_0,χ_0) we will mean it in the usual sense obtained by integrating by parts and allowing test functions to be non-zero at $t=0$.

As it is standard we will work with the parabolic cylinders

$$Q_r(x,t) = \left\{(y,s)\ : |x-y| \leq r,\ t-r^2 < s < t\right\}$$

By Q_r we will mean simply $Q_r(0,0)$. All of our estimates are *interior estimates* so we may assume we are always working at (say) Q_2.

2.2 Main results

Now we can state the two main results concerning Lipschitz free boundaries:

Theorem 2.2.1. *Let (u,χ) solve (SGT) in Q_2 and such that its free boundary is a special Lipschitz domain of the form:*

$$\{(x',x_n,t)\in Q_2 : x_n = f(x',t)\}, \quad f \text{ Lipschitz in both } x' \text{ and } t$$

If L and V denote respectively the Lipschitz constants of f in x' and t, we have for every $\alpha\in(0,1)$ that

$$\|u\|_{L^\infty(Q_1)} \le g\left(\|u\|_{L^2(Q_2)}\right) \qquad \|u\|_{C^\alpha(Q_1)} \le C_{L,n,\alpha}\left(\|u\|_{L^2(Q_2)} + V\right)$$

where $g(t)$ is the inverse to the function

$$t \to C_n \frac{t^{2+\delta}}{(t^2 + C_L V^2)^{\frac{1+\delta}{2\delta}}} \qquad \delta = \frac{2}{n}$$

The result for Hele-Shaw is very similar, except we get no further regularity in time.

Theorem 2.2.2. *Let (u,χ) solve (HS) in Q_2 and such that its free boundary is a special Lipschitz domain as above. If L and V denote respectively the Lipschitz constants of f in x' and t, we have for each $\alpha\in(0,1)$ and each $t\in(-2,0)$ that*

$$\|u(t)\|_{L^\infty(B_1)} \le g\left(\|u(t)\|_{L^2(B_2)}\right) \qquad \|u(t)\|_{C^\alpha(B_1)} \le C_{L,n,\alpha}\left(\|u(t)\|_{L^2(B_2)} + V\right)$$

where $g(t)$ is of the same form as in the previous theorem.

The third result deals with the existence of weak solutions for the Stefan-Signorini problem (explained in the introduction), the Stefan condition is to be understood in the same sense as in Definition 2.1.1, and the Signorini condition is also understood in the BV sense.

Theorem 2.2.3. *Let $\Omega \subset \mathbb{R}^n$ $(n \leq 3)$ be bounded with Lipschitz boundary. Given $u_0 \in H_0^1(\Omega)$ and $\chi_0 = \chi_{E_0} \in BV(\Omega)$ there exits a weak solution to the Stefan-Signorini problem defined for all positive times. M oreover, as $t \to +\infty$ the free boundary converges uniformly to the boundary of the smallest domain with positive mean curvature containing E_0.*

Chapter 3

Luckhaus Theorem revisited and the mixed Stefan-Signorini problem

In this chapter we shall review the Luckhaus existence theorem for (SGT) and apply the same ideas to the Stefan-Signorini problem. We start by introducing discrete solutions and reviewing their basic properties, that is done in the next section. We shall make a parenthesis to talk about almost minimal surfaces, which have an important role in Luckhaus' proof, once is done we will continue to prove the existence theorems. The result on long time behavior is proved at the end.

3.1 Luckhaus discrete solutions

As discussed in the introduction, the nature of the Gibbs-Thomson law is such that one cannot exploit the known methods for building weak solutions (as one can in the classical Stefan problem, Porous medium equation, etc). On the other hand, as was first pointed out by Visintin and Gurtin (cf. Section 2 of [14]), for smooth solutions one has the inequality

$$\frac{d}{dt}\left\{\mathrm{Per}(\Gamma(t)) + \frac{1}{2}\int u(t)^2 dx + \frac{1}{2}\int_0^t \int |\nabla u(t)|^2 dx dt\right\} \leq 0 \tag{3.1}$$

This Lyapunov functional points to an intrinsic gradient flow structure. Inspired by this fact Luckhaus [14] developed a scheme to built weak solutions starting from arbitrary initial data and defined for all times. The main idea was to discretize time and solve an elliptic variational problem at each discrete time step, the functional being determined by the Lyapunov functional above. Given the Gibbs-Thomson law relating u and the mean curvature of the interface, is not surprising that this minimization problem falls under the scope of the regularity theory of almost minimal surfaces. Thanks to this, and estimates for the velocity obtained by Luckhaus one has enough compactness to guarantee the existence of a limit as the time step goes to zero. This limit is then shown easily to be a solution in a weak sense that will be explained below.

A closely related result is that of Almgren and Wang [2], where time is also discretized. Their approximations are built in a somewhat different manner, in particular their idea involves the use of the Wasserstein distance. Both of these works just predate the emergence of gradient flows in Wasserstein space as a robust approach to many non-linear evolution problems. An entirely different approach we won't discuss here is that of phase fields, with it, Soner [16] managed to prove existence of weak solutions for large times.

Definition 3.1.1. *Let Ω be a domain with Lipschitz boundary and $T > 0$. Given $N > 0$, we fix a* time step *$h = 2^{-N}T$. By a* ***discrete solution*** *to*

(SGT) with time step $h > 0$ we will mean a pair of functions

$$u : \Omega_T \to \mathbb{R}$$
$$\chi : \Omega_T \to \{0, 1\}$$

Which are piece-wise constant in time

$$\left.\begin{array}{r} u(x,t) = u_k(x) \\ \chi(x,t) = \chi_k(x) \end{array}\right\} \textit{ if } t \in [(k-1)h, kh)$$

where the sequence $\{u_k, \chi_k\}_{k \geq 0}$ satisfies the following

- u_0, χ_0 *are given* initial conditions *with*

$$u_0 \in H_0^1(\Omega), \quad \chi = \chi_{E_0} \in BV(\Omega)$$

- *For any $k \geq 0$ the pair (u_{k+1}, χ_{k+1}) achieves the minimum of the functional*

$$F_{k,h}(u, \chi) = \int_\Omega |\nabla \chi| + \frac{h}{2} \int_\Omega |\nabla u|^2 dx + \frac{1}{2} \int_\Omega u(u - u_k) dx \tag{3.2}$$

  among all pairs (u, χ) with $u \in H_0^1(\Omega)$ and $\chi : \Omega \to \{0, 1\} \in BV(\Omega)$ that satisfy the constraint

$$u - u_k + \chi - \chi_k = h\Delta u \quad \textit{in } H^{-1} \tag{3.3}$$

Remark. It will be convenient to take the following convention: we will denote with the latin letter t a generic time in $(0, T)$, we will use the greek letter τ to refer to a time $\tau \in (0, T)$ which happens to be a multiple of the time step h. Moreover, we will denote by E or F the solid phase, i.e. $E = \{\chi = 1\}$.

When shall also write sometimes χ_E or χ_F for the characteristic function of the solid phase E or F.

Remark. By standard methods from calculus of variations one can show that for each $h > 0, T > 0$ and any initial data (u_0, χ_0) one can build a discrete weak solution with time step h in $(0, T)$. The challenge is to get an actual weak solution when $h \to 0$.

Remark. The minimization condition on $F_{k,h}$ is a way to force the Lyapunov condition (3.1) on the weak solutions. This will be seen in Proposition 3.3.3.

3.2 Almost minimal surfaces

Heuristically speaking, an almost minimal boundary E is a set whose perimeter cannot decrease too much by perturbations at a small scale, so in some sense it is close to a minimal surface in a neighborhood of each point. One might expect that if this closeness happens in a strong enough sense then such a set must be smooth, this is the content of the Almgren-Tamanini theory. Let us make some concrete definitions.

Definition 3.2.1. *Fix a modulus of continuity $\rho(r)$. A set E of finite perimeter is said to be* ***almost minimal*** *in Ω with respect to $\rho(r)$ if $\exists d > 0$ such that*

$$\int_\Omega |\nabla \chi_E| \leq \int_\Omega |\nabla \chi_F| + \rho(r) r^{n-1}$$

for any F such that $F \Delta E \subset \Omega$ has diamater smaller than $2r$, $r < d$.

It turns out that the solid phases $E_k = \{\chi_k = 1\}$ from the discrete solutions have such a property. In fact, pick $k \in \mathbb{N}$ and let be $(\tilde{u}_, \tilde{\chi})$ a competing function for the variational problem (3.2) solved by $(u, \chi) = (u_{k+1}, \chi_{k+1})$, then

$$F_{k,h}(u, \chi) \leq F_{k,h}(\tilde{u}, \tilde{\chi})$$

$$\Rightarrow \int_\Omega |\nabla \chi| \leq \int_\Omega |\nabla \tilde{\chi}| + \frac{h}{2} \int_\Omega |\nabla \tilde{u}|^2 - |\nabla u|^2 dx + \frac{1}{2} \int_\Omega \tilde{u}(\tilde{u} - u_k) - u(u - u_k) dx$$
$$= \int_\Omega |\nabla \tilde{\chi}| - \frac{1}{2} \int_\Omega \tilde{u} \left(\tilde{\chi} - \chi_k\right) - u \left(\chi - \chi_k\right) dx$$

We rewrite the above inequality:

$$\int_\Omega |\nabla \chi| \leq \int_\Omega |\nabla \tilde{\chi}| + \frac{1}{2} \int_\Omega u \left(\chi - \tilde{\chi}\right) - (\tilde{u} - u)(\tilde{\chi} - \chi_k) dx$$

Observe that $w = \tilde{u} - u$ satisfies $w - h\Delta w = \tilde{\chi} - \chi$, by performing a few integrations it can be shown that

$$\int_\Omega |w| dx \leq 2 \int_\Omega |\tilde{\chi} - \chi| dx$$

This allows to bound the second term and conclude that

$$\int_\Omega |\nabla \chi| \leq \int_\Omega |\nabla \tilde{\chi}| + \frac{1}{2} \int_\Omega u \left(\chi - \tilde{\chi}\right) dx + \int_\Omega |\tilde{\chi} - \chi| dx$$

What does this say?, applying Hölder inequality with $p > n$ to the term containing u we get further

$$\int_\Omega |\nabla \chi| \leq \int_\Omega |\nabla \tilde{\chi}| + C_p \|u\|_{L^p(\Omega)} \left(\int_\Omega |\chi - \tilde{\chi}| dx \right)^{\frac{p-1}{p}} + \int_\Omega |\tilde{\chi} - \chi| dx$$

Since $\int_\Omega |\chi - \tilde{\chi}| dx = |E \Delta F|$ and $|E \Delta F| \leq c_n r^n$ this implies ($\alpha = 1 - \frac{n}{p}$) that

$$\int_\Omega |\nabla \chi| \leq \int_\Omega |\nabla \tilde{\chi}| + C_{n,p} \left(\|u\|_{L^p(\Omega)} r^\alpha + r \right) r^{n-1}$$

If $n \leq 3$ then $2^* > n$, so plugging above the Sobolev embedding $\|u\|_{L^{2^*}} \leq C\|\nabla u\|_{H^1}$. Since we can pick *any* set of finite perimeter to play the role of the solid phase, we above inequality holds with $\tilde{\chi} = \chi_F$ for any F. We have proven the following proposition:

Proposition 3.2.2. *Suppose the space dimension is $n \leq 3$, and let (u, χ) be a discrete solution to the Stefan problem. Then each set $E(t) = \{\chi = 1\}$ is an almost minimal set with respect to $\rho_t(r)$, where*

$$\rho_t(r) = C_{n,p}\left(\|\nabla u(t)\|_{H^1} r^\alpha + r\right),\ \alpha = 1 - \frac{n}{2^*}$$

Remark. Note that the estimate above is independent of the time-step h. If anything, as $h \to 0$ the only thing that deteriorates in the estimate is the supremum in time of H^1 norm of the discretized temperature.

This fact is key in the existence result of Luckhaus (discussed in the next section), thanks to the regularity theory of F. Almgren and I. Tamanini, which extends the regularity theory of minimal surfaces of E. De Giorgi. We summarize the facts we need from this theory as a single result

Theorem 3.2.3 (Almgren-Tamanini)**.** *Let E be almost minimal in some domain Ω with respect to $\rho(r) = Ar^\alpha$ and let $n \leq 7$. Then there exists $r_0 = r_0(A, \alpha)$ such that if $x_0 \in \partial E \cap \Omega$ and $d(x_0, \Omega^c) > r_0$ we have*

$$E \cap B_{r_0}(x_0) = \{(x', x_n)\ :\ x_n < f(x')\} \cap B_r(x_0)$$

(after possibly rotating the coordinate system)

Here $f(x')$ is a function defined in $B'_r(x'_0)$ such that

$$\|f\|_{C^{1,\frac{\alpha}{2}}} \leq C(A,\alpha)$$

The theory behind the above result can be found for example in [1]. Before we go back to the Stefan problem we will prove a stability property of almost minimal surfaces which will be useful in the future.

Lemma 3.2.4 (Stability of almost minimal surfaces)**.** *Assume $n \leq 7$. Let $\{E_k\}_{k\in\mathbb{N}}$ be a sequence of sets each of which are almost minimal in Ω with respect to some $\rho_k(r)$, s.t. $\rho_k(r) \leq \rho_0(r)$ and $\rho_0(r) = Cr^{\alpha}$. If $\rho_k \to \rho$ uniformly and $E_k \to E$ uniformly (i.e. in the Haussdorff metric) then E is also almost minimal in Ω with respect to $\rho(r)$.*

Proof. Let F be such that $E\Delta F \subset B_r(x) \subset\subset \Omega$ and let us write $E_k = \{\chi_k = 1\}$, $E = \{\chi = 1\}$ and $F = \{\tilde{\chi} = 1\}$. We may assume without loss of generality that F has a smooth boundary, thus we may pick another sequence F_k with smooth boundary and such that $F_k\Delta E_k$ has radius less than $r+\epsilon_k$, $\epsilon_k \to 0$.

Using a covering argument and the Almgren-Tamanini theorem to control the oscillation of the normals we may also assume that

$$\int_\Omega |\nabla\chi_k| \to \int_\Omega |\nabla\chi|, \quad \int_\Omega |\nabla\tilde{\chi}_k| \to \int_\Omega |\nabla\tilde{\chi}|$$

For each k, we have by assumption

$$\int_\Omega |\nabla\chi_k| \leq \int_\Omega |\nabla\tilde{\chi}_k| + \rho_k(r+\epsilon_k)(r+\epsilon_k)^{n-1}$$

Taking $k \to +\infty$ we obtain

$$\int_\Omega |\nabla \chi| \le \int_\Omega |\nabla \tilde{\chi}| + \rho(r) r^{n-1}$$

which finishes the proof. $\square$

3.3 Existence of weak solutions

The goal of this section is to review the following theorem of Luckhaus (see [14]) which is the base for Theorem 2.2.3.

Theorem 3.3.1. *[14] Given u_0 and χ_0 there is a sequence of discrete solutions $(u^{(N)}, \chi^{(N)})$ with time step $h_N \to 0$ that converges in $L^1(\Omega_T)$ to a pair (u, χ) with the following properties:*

$$\begin{array}{l} u \in L^2\left(0, T; H^1(\Omega)\right) \cap L^\infty\left(0, T; L^2(\Omega)\right) \\ \chi \in L^\infty\left(0, T; BV(\Omega)\right) \\ (u + \chi)_t \in L^2\left(0, T; H^{-1}(\Omega)\right) \\ u(0) = u_0, \;\; \chi(0) = \chi_0 \end{array}$$

*Moreover, for **almost every time** $t \in (0, T)$ we have*

i) $(u + \chi)_t = \Delta u$ in the H^{-1} sense.

ii) The set $E(t)$ has a $C^{1,\alpha}$ boundary and its mean curvature in the BV sense agrees with the trace of u on $\partial E(t)$.

To start the proof we will collect some basic facts about discrete solutions that follow easily from their definition.

Proposition 3.3.2. *Let (u, χ) be a discrete solution with time step h. Then:*

(Discrete Stefan condition) For any $\phi \in C_c^\infty(\Omega_T)$ *we have*

$$\int_0^T \int_\Omega (u+\chi)(\partial_t^h \phi) dxdt + \int_\Omega (u_0+\chi_0)\phi(x,0)dx = \int_{\Omega_T} \nabla u \cdot \nabla \phi \, dxdt \quad (3.4)$$

(Discrete Gibbs-Thomson Law) For any $t \in (0,T)$, $Y \in C_c^\infty(\Omega;\mathbb{R}^n)$ *we have*

$$\int_\Omega (div(Y) - \nu \cdot DY(\nu))|\nabla \chi(t)| = \int_\Omega u(t) Y \cdot \nu |\nabla \chi(t)|, \ \nu = \frac{\nabla \chi(t)}{|\nabla \chi(t)|} \quad (3.5)$$

Proof. We omit the details as the proof is standard. For condition (3.4) one only needs to test against the constraint (3.3) which is satisfied by (u_k, χ_k). So testing (in space) against an arbitrary test function $\phi(x,t)$ for each k and adding up the resulting integral equation over k we get (3.4). The (discrete) Gibbs-Thomson condition (3.5) is nothing but the Euler-Lagrange equation associated to the functional $F_{k,h}$ defined in (3.2). □

Proposition 3.3.3. *Let* (u,χ) *be again a discrete solution with time step* $h>0$*. For any pair* $\tau_1 < \tau_2$ *we have the estimates:*

$$\int_\Omega |\nabla \chi(\tau_2)| \le \int_\Omega |\nabla \chi(\tau_1)| \quad (3.6)$$

$$\begin{aligned} \sup_{\tau \in (\tau_1,\tau_2)} \left\{ \frac{1}{2} \int_\Omega u(\tau)^2 dx \right\} + \frac{1}{2} \int_{\tau_1}^{\tau_2} \int_\Omega |\nabla u|^2 dxdt \quad (3.7) \\ \le \operatorname{Per}(E(\tau_1)) - \operatorname{Per}(E(\tau_2)) \le \operatorname{Per}(E(0)) \end{aligned}$$

$$\|e(\tau_2) - e(\tau_1)\|_{H_0^{-1}(\Omega)} \le (\tau_2 - \tau_1)^{1/2} (2 \operatorname{Per}(E_0))^{1/2}, \ \ e(t) = \chi(t) + u(t) \quad (3.8)$$

Proof. Inequalities (3.6) and (3.7) wil follow from the fact that $(0, \chi_k)$ is itself an admissible pair for the variational problem solved by (u_{k+1}, χ_{k+1}). In other words, we have

$$\int_\Omega |\nabla \chi_{k+1}| + \frac{h}{2}\int_\Omega |\nabla u_{k+1}|^2 dx + \frac{1}{2}\int_\Omega u_{k+1}(u_{k+1}-u_k)dx$$

$$= F_h(u_{k+1},\chi_{k+1}) \leq F_h(0,\chi_k) = \int_\Omega |\nabla \chi_k|$$

Adding these inequalities for each k with τ_k in (τ_1,τ_2) one gets the first estimate. For the second inequality, let $v = \chi_{m+k} - \chi_m + u_{m+k} - u_m$ and let ϕ be an arbitrary function in $H_0^1(\Omega)$, then

$$\int_\Omega v\phi dx = \int_\Omega \sum_{i=1}^{k} (\chi_{m+i} - \chi_{m+i-1} + u_{m+i} - u_{m+i-1})\phi dx = \int_\Omega \sum_{i=1}^{k} (h\Delta u_{m+i})\, w\; dx$$

$$= -\int_{mh}^{(m+k)h} \int_\Omega \nabla u \cdot \nabla w \; dxdt$$

then by Hölder inequality

$$\left|\int_\Omega vwdx\right| \leq (hk)^{1/2}\left(\int_{\Omega_T} |\nabla u|^2 dxdt\right)^{1/2} ||w||_{H_0^1}$$

From the first estimate, the right hand side is bounded by $(kh)^{1/2}(2\,\mathrm{Per}(E_0))^{1/2}\|w\|$, and since w was arbitrary this gives the estimate for $e(\tau)$. $\square$

Observe that equation (3.8) gives a (discrete) Hölder estimate on $u+\chi$ over time. Since one would not think that the discontinuities of u and χ cancel each other, we may expect to derive continuity for u and χ individually from the estimate for $u+\chi$. This is done in the following lemma, particularly, in the first step of the proof.

Lemma 3.3.4 (Luckhaus time estimates)**.** *Given a discrete solution with time step h, the following integral time-continuity estimates hold (recall τ is a multiple of h):*

$$\int_{\tau}^{T-\tau}\int_{\Omega}|\chi(x,t\pm\tau)-\chi(x,t)|dxdt \leq C\tau^{\gamma}$$

$$\int_{\tau}^{T-\tau}\int_{\Omega}|u(x,t\pm\tau)-u(x,t)|dxdt \leq C\tau^{\gamma}$$

Where C depends on the initial data (u_0,χ_0) and γ is a small dimensional constant.

Proof. Step 1. For any given $f \in H^1$ and $g \in BV$ satisfying $g(\Omega) \subset \{-2,0,2\}$, it can be shown that

$$\int_{\Omega}|g|dx \leq 4\int_{\Omega}|f+g|dx + C_n\left(\int_{\Omega}|f+g|dx\right)^{\frac{n}{2n-2}}\|\nabla f\|_{L^2}^{\frac{n}{n-1}} \tag{3.9}$$

One only needs to apply the Sobolev inequality to $h=\min\{(f-\frac{1}{2})_+,1\} \in H^1$ and use the fact that since g can only take the values 0 and ± 2 then

$$\left\{g\neq 0,\ |f|<\frac{3}{2}\right\} \subset \left\{|f+g|>\frac{1}{2}\right\}$$

Therefore

$$\frac{1}{2}\int|g|dx \leq \left|\left\{|f|>\frac{3}{2}\right\}\right| + 2\int_{\Omega}|f+g|dx$$

and then the estimate follows.

Step 2. Next we show that for a discrete solution u,χ we have with $\tau=|\tau_1-\tau_2|$

$$\int_{\Omega}|e(\tau_1)-e(\tau_2)|dx \leq C_{\Omega}\left(1+A+\tau^{-\frac{1}{2}}\|e(\tau_1)-e(\tau_2)\|_{H^{-1}}\right)B|\tau_1-\tau_2|^{\frac{1}{4}} \tag{3.10}$$

Where $A = \int_\Omega |\nabla(\chi(\tau_1) - \chi(\tau_2))|$ and $B^2 = \|\nabla\left(u(\tau_1) - u(\tau_2)\right)\|_{L^2(\Omega)}$. This is a standard interpolation estimate. To obtain it let $\phi_\epsilon = \epsilon^{-n}\phi(\epsilon^{-1}x)$ be an approximation to the identity, then

$$\int_\Omega |e(\tau_1) - e(\tau_2)|dx \leq \int_\Omega |(e(\tau_1) - e(\tau_2)) * \phi_\epsilon - (e(\tau_1) - e(\tau_2))|\, dx$$

$$+ \int_\Omega |(e(\tau_1) - e(\tau_2)) * \phi_\epsilon|\, dx$$

Thinking of ϕ_ϵ as a function in H^1 (and assuming ϕ_1 is supported in B_1) we see that the second integral is bounded by

$$\frac{C_\Omega}{\epsilon}\|e(\tau_1) - e(\tau_2)\|_{H^{-1}(\Omega)} + \int_{\{d(x,\partial\Omega)<\epsilon\}\cap\Omega} |e(\tau_1) - e(\tau_2)|dx$$

For the first integral, we obtain via the triangle inequality

$$\int_\Omega |(e(\tau_1) - e(\tau_2)) * \phi_\epsilon - (e(\tau_1) - e(\tau_2))|\, dx \leq \epsilon C_\phi \left(1 + \int_\Omega |\nabla(\chi(\tau_1) - \chi(\tau_2))|\right)$$

$$+\epsilon C_\phi \int_\Omega |\nabla(u(\tau_1) - u(\tau_2))|dx$$

We bound the L^1 norm of the gradient of $u(\tau_1) - u(\tau_2)$ in terms of its L^2 norm, and take $\epsilon = \frac{|\tau_1 - \tau_2|}{B}$ to get the estimate, after using inequality (3.8) from Proposition 3.3.3.

Step 3. We derive first the estimate for χ:

$$\int_\tau^{T-\tau} \int_\Omega |\chi(t \pm \tau) - \chi(t)|dxdt \leq \int_{\{t:\|\nabla u(t)\|_{L^2}^2 > K\}} \int_\Omega |\chi(t \pm \tau) - \chi(t)|dxdt$$

$$+ \int_{\{\tau<t<T-\tau:\|\nabla u(t)\|_{L^2}^2 \leq K\}} \int_\Omega |\chi(t \pm \tau) - \chi(t)|dxdt = I_1 + I_2$$

We proceed to bound each integral, the first can be controlled via Tchebyschev's inequality

$$I_1 \leq \frac{2\Omega}{K} \int_{\Omega_T} |\nabla u|^2 dxdt$$

To bound I_2, we apply for each t inequality (3.9) from step 1 with

$$g = \chi(t \pm \tau) - \chi(t), f = u(t \pm \tau) - u(t)$$

Then

$$I_2 \leq 4 \int_{\tau}^{T-\tau} |e(t \pm \tau) - e(t)| dx + C_n \int_{\tau}^{T-\tau} \left(K \int_{\Omega} |e(t \pm \tau) - e(t)| dx \right)^{\frac{n}{2n-2}} dt$$

We now want to apply inequality (3.10) from step 2. First, we use the basic estimates from the previous lemma to see that for some $C_0 = C_0(u_0, \chi_0)$ we have

$$1 + A + \tau^{-\frac{1}{2}} \|e(t \pm \tau) - e(t)\|_{H^{-1}} \leq C_0$$

Then plugging in the inequality we arrive at

$$I_2 \leq C_\Omega C_0 4TK|\tau|^{\frac{1}{4}} + C_\Omega T K^{\frac{n}{n-1}} C_0^{\frac{n}{2n-2}} \tau^{\frac{n}{8n-8}}$$

We still have the freedom to chose K, if we take $K = \tau^\gamma$, with γ small enough (depending only on the n), we get

$$I_1 + I_2 \leq C(u_0, \chi_0, \Omega) T \tau^\gamma$$

Which is the desired estimate for $\chi(t)$. For $u(t)$, now we only need to use the triangle inequality:

$$\int_{\tau}^{T-\tau} \int_{\Omega} |u(x, t \pm \tau) - u(x, t)| dxdt \leq \int_{\tau}^{T-\tau} \int_{\Omega} |\chi(t \pm \tau) - \chi(t)| dxdt$$

$$+\int_{\tau}^{T-\tau}\int_{\Omega}|e(t\pm\tau)-e(t)|dxdt \leq C(u_0,\chi_0,\Omega)T\tau^{\gamma}+C(u_0,\chi_0,\Omega)T\tau^{\frac{1}{4}}$$

This finishes the proof. □

Now we are ready to prove existence of solutions in the sense of Definition 2.1.1.

Proof of Theorem 3.3.1. The proof will consist in taking a converging sequence of discrete solutions as $h \to 0$. After that, one must show that the limiting solutions satisfy both the Stefan condition (weakly) and the Gibbs-Thomson law (in the sense of sets of finite perimeter). This we do step by step.

Convergence: The velocity estimates of Luckhaus and a compactness theorem of Kolmogorov tells us that the sequence $\{\chi^h\}$ and $\{u^h\}$ has a converging subsequence in $L^1(\Omega_T)$. Thus there is a pair of functions χ and u that are both the $L^1(\Omega_T)$ and pointwise a.e. limit of a subsequence of $\{\chi^h\}_h$ and $\{u^h\}_h$, respectively.

Stefan condition: By the basic estimates for discrete solutions we also conclude that $\chi \in L^\infty(0,T;BV(\Omega))$ and that $\chi = 0, 1$ almost everywhere. By the same reasoning we conclude that $u \in L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$. Moreover, using test functions we can use the "Discrete Stefan condition" along the subsequence to get (in the limit) for any $\phi \in C_c^\infty(\Omega_T)$ the weak Stefan condition

$$\int_{\Omega_T}(u+\chi)\phi_t dxdt+\int_{\Omega}\phi(x,0)(u_0+\chi_0)dx=\int_{\Omega_T}\nabla\phi\cdot\nabla u\,dxdt$$

This, and the fact that $u \in L^2((0,T);H^1(\Omega))$ imply that $(u+\chi)_t \in L^2((0,T);H^{-1})$.

Gibbs-Thomson Law: Observe that $\{u^{h_k}\}_{k\in\mathbb{N}}$ lies in a bounded set of $L^2\left(0,T;H^1(\Omega)\right)$ and additionally $\int_0^T\int_\Omega |\nabla u|^2 dxdt < \infty$. Therefore, for some $M \subset (0,T)$ of measure zero we know that if $t \notin M$ then there is some positive number $C(t)$ such that

$$\int_\Omega |\nabla u(t)|^2 dx \ , \int_\Omega |\nabla u^{h_k}(t)|^2 dx < C(t) \ \ \forall k \in \mathbb{N} \tag{3.11}$$

Additionally, we can prove the time estimate (see first part of Proposition 3.3.3)

$$\int_\Omega |\nabla \chi^{h_k}(t)| \leq C(\chi_0) \ \ \forall\, t \in (0,T)$$

Therefore, for any $t \notin M$ we have

$$\{u^{(h_k)}(t)\}_k \text{ is bounded in } H^1(\Omega),\ \chi^{(h_k)}(t) \to \chi(t) \ \text{ in } L^1(\Omega)$$

Now, recall[1] that $n \leq 3$. Then inequalities in (3.11) together with 3.2.3 (Almgren-Tamanini) guarantee that whenever $t \notin M$ then along another subsequence (that now may depend on t) we have $\partial E_k(t) \to \partial E(t)$ in the C^1 topology. Where $E(t)$ is some set with $C^{1,\frac{\alpha}{2}}$ boundary. This convergence allows us to (fixing a text function $\xi \in C_c^\infty(\Omega,\mathbb{R}^n)$ to the pass discrete Gibbs-Thomson Law (3.5) to the limit and conclude that the mean curvature ∂E (in the BV sense, cf Definition 2.1.1) is given by $u(t)$, with this we have finished the proof of Theorem 3.3.1. □

[1]This is the only step in the proof where the dimensional restriction $n \leq 3$ is used

3.4 Handling the Stefan-Signorini problem

The goal of this section is to adapt the Luckhaus argument to the case of the Stefan-Signorini problem.

Following Luckhaus, we consider discrete approximations to our potential solutions. How do we do that? The Stefan condition should be obtained in the same way, that is by an implicit discretization in time. The Signorini condition for the mean curvature will need some modifications.

Definition 3.4.1. *Let Ω be a domain with Lipschitz boundary and $T > 0$. Given $N > 0$, we fix a* time step $h = 2^{-N}T$. *By a **discrete solution** to the Stefan-Signorini problem with time step h we will mean a pair of functions*

$$\begin{aligned} u &: \Omega_T \to \mathbb{R} \\ \chi &: \Omega_T \to \{0,1\} \end{aligned}$$

Which are piece-wise constant in time

$$\left.\begin{aligned} u(x,t) &= u_k(x) \\ \chi(x,t) &= \chi_k(x) \end{aligned}\right\} \textit{ if } t \in [(k-1)h, kh)$$

where the sequence $\{u_k, \chi_k\}_{k\geq 0}$ satisfies the following

- *u_0, χ_0 are given* initial conditions *with*

$$u_0 \in H_0^1(\Omega), \;\; \chi = \chi_{E_0} \in BV(\Omega)$$

- *For any $k \geq 0$ the pair (u_{k+1}, χ_{k+1}) solves the following **obstacle problem**, that is, it minimizes the functional*

$$F_{k,h}(u,\chi) = \int_\Omega |\nabla \chi| + \frac{h}{2}\int_\Omega |\nabla u|^2 dx + \frac{1}{2}\int_\Omega u(u-u_k)dx$$

among all pairs (u, χ) with $u \in H_0^1(\Omega)$ and $\chi : \Omega \to \{0, 1\} \in BV(\Omega)$ that satisfy the constraint

$$u - u_k + \chi - \chi_k = h\Delta u \quad \textit{in } H^{-1}$$

and such that the new solid phase ***contains*** *the previous one, that is*

$$\{\chi_k = 1\} \subset \{\chi = 1\}$$

Remark. The added constraint makes the variational problem considered at each time step a *parametric obstacle problem.* As before, usual calculus of variations methods guarantee existence of discrete solutions for all times and all time steps $h > 0$.

Remark. The obstacle constraint forces the inclusion $E(\tau_1) \subset E(\tau_2)$ whenever $\tau_1 < \tau_2$, so that the free boundary is always *expanding.* It also will guarantee that the free boundary does not move at those points where its mean curvature is positive.

For this notion of weak solution, one can prove easily corresponding estimates as for the Stefan problem:

Proposition 3.4.2. *Let (u, χ) be a discrete solution with time step h.Then:*

(Discrete Stefan condition) For any $\phi \in C_c^\infty(\Omega_T)$ we have

$$\int_0^T \int_\Omega (u + \chi)(\partial_t^h \phi) dx dt + \int_\Omega (u_0 + \chi_0)\phi(x, 0) dx = \int_{\Omega_T} \nabla u \cdot \nabla \phi \, dx dt$$

(Discrete Signorini condition) For any $t \in (0,T)$, $Y \in C_c^\infty(\Omega;\mathbb{R}^n)$ *we have the following two conditions: If* $Y \cdot \nabla\chi(t) \geq 0$ *almost everywhere with respect to* $|\nabla\chi(t)|$ *then*

$$\int_\Omega (div(Y) - \nu \cdot DY(\nu))|\nabla\chi(t)| \geq \int_\Omega u(t) Y \cdot \nu |\nabla\chi(t)|,\ \nu = \frac{\nabla\chi(t)}{|\nabla\chi(t)|}$$

Plus, if $\operatorname{supp} Y$ *is a positive distance away from* $E(t-h)$ *then*

$$\int_\Omega (div(Y) - \nu \cdot DY(\nu))|\nabla\chi(t)| = \int_\Omega u(t) Y \cdot \nu |\nabla\chi(t)|,\ \nu = \frac{\nabla\chi(t)}{|\nabla\chi(t)|}$$

Proof. Again we omit the proof since it is standard, the conditions on the Signorini condition represent nothing but the standard variational inequality satisfied by the solution of a parametric obstacle problem. □

Remark. Heuristically, the Signorini condition is saying nothing else but the fact that $u|_{\partial E}$ is not larger than the mean curvature of ∂E, and that it agrees with it at those points where ∂E is moving, which is exactly what we want to have in the limit.

The basic energy and time estimates and the more delicate Luckhaus velocity estimates carry through to the Stefan-Signorini case. This we state without proof as the details are similar.

Claim. *For discrete solutions to the Stefan-Signorini problem there are analogous estimates corresponding to the basic estimates and the Luckhaus time estimates from the previous section.*

We are now ready to prove the existence result for the Stefan-Signorini problem.

Proof of Theorem 2.2.3. As for the Stefan case, proving the existence of a solution requires three steps: showing there is convergence, proving the Stefan condition holds and proving that the Signorini condition holds. We focus only on the last one.

Signorini condition. The main obstacle is getting in a situation where one can use the Almgren-Tamanini theorem. We overcome it by making the following observation

Claim . Suppose the space dimension is $n \leq 3$, and let (u, χ) be a discrete solution. Then each set $E(t) = \{\chi = 1\}$ is an almost minimal set with respect to $\rho_t(r)$, where

$$\rho_t(r) = C_{n,p} \left(\max\{\|\nabla u(t)\|_{H^1}, \|\nabla u_0\|_{H^1}\} r^\alpha + r\right), \ \alpha = 1 - \frac{n}{2^*}$$

Moreover: If F is another set containing $E(t)$ and $\tilde{\chi}$ denotes the characteristic function of F we have

$$\int_\Omega |\nabla \chi(t)| \leq \int_\Omega |\nabla \tilde{\chi}| + \int_\Omega u(t)(\tilde{\chi} - \chi) dx$$

Let us take the claim granted for a second and prove the statement of the theorem. Just as for the Gibbs-Thomson case we can now prove that there exists a set of measure zero $M \subset (0, T)$ such that: $t \notin M$ implies there exists some subsequence $(u^{h_k}(t), \chi^{h_k}(t))$ such that $\|u^{h_k}(t)\|_{H^1} \leq C(t)\ \forall k$, thus the Almgren-Tamanini theorem guarantees that the boundaries of the sets $\{E_k(t)\}_k$ are uniformly bounded in the $C^{1,\alpha}$ norm. This allows to pass the discrete Signorini condition to the limit.

It only remains to prove the claim. Let $F \subset \Omega$ such that $E(t)\Delta F \subset B_r(x_0) \subset\subset \Omega$, for some x_0 and $r > 0$ small enough. It is enough to consider the special cases $E(t) \subset F$ and $F \subset E$ (for a general F we can decompose it in two pieces with the corresponding properties).

If $E(t) \subset F$ then $F \supset E(t-h)$, so F itself is an candidate admissible candidate for the variational problem solved by $(u(t), \chi(t))$. In this case we can use the same inequality $F(u, \chi) \leq (u_F, \chi_F)$ as for the Gibbs-Thomson law to get

$$\int_\Omega |\nabla \chi| \leq \int_\Omega |\nabla \tilde{\chi}| + C_{n,p} \left(\|\nabla u(t)\|_{H^1} r^\alpha + r \right)$$

The difference arises when $F \subset E(t)$. In this case, let $F = F_1 \cup F_2$, where $F_2 \subset E(t-h)^c$. Then one can show easily by induction that with $\tilde{\chi}_i := \chi_{F_i}$ we have

$$\int_\Omega |\nabla \chi| \leq \int_\Omega |\nabla \tilde{\chi}_1| + C_{n,p} \left(\|\nabla u(0)\|_{H^1} r^\alpha + r \right)$$
$$\int_\Omega |\nabla \chi| \leq \int_\Omega |\nabla \tilde{\chi}_2| + C_{n,p} \left(\|\nabla u(t)\|_{H^1} r^\alpha + r \right)$$

Which proves the claim.

Long time behavior. The remaining issue is the behavior of $\chi(t)$ as $t \to +\infty$. We also divide this proof in a series of observations.

Step 1. Existence of a limit E_∞. From the previous theorem we have that any global weak solution (u, χ) satisfies the estimate

$$\int_0^{+\infty} \|\nabla u(t)\|^2_{H^1(\Omega)} dt < +\infty$$

In particular, we may pick a sequence $\{t_k\}_k$, $t_k \to +\infty$ along which the Gibbs-Thomson condition holds and such that $u(t_k) \to 0$ in $H^1(\Omega)$. Since $n \leq 3$ the Almgren-Tamanini theorem tells us that $\{E(t_k)\}_k$ has a boundary which is uniformly $C^{1,\alpha}$ in k and thus along some subsequence $E(t_k)$ converges (in the C^1 topology) to a set E_∞ with a $C^{1,\alpha}$ boundary.

Step 2. E_∞ has positive mean curvature in the weak sense. We can now apply the stability lemma to conclude that E_∞ is almost minimal with respect to

$$\begin{aligned} \rho(r) &= \lim_{k \to 0} \{C_{n,p} (\max\{\|\nabla u(t_k)\|_{H^1}, \|\nabla u_0\|_{H^1}\} r^\alpha + r)\} \\ &= C_{n,p} (\|\nabla u_0\|_{H^1} r^\alpha + r), \ \alpha = 1 - \frac{n}{2^*} \end{aligned}$$

so again by the Almgren-Tamanini E_∞ has a $C^{1,\frac{\alpha}{2}}$ boundary. Moreover, if we restrict to those sets such that $E_\infty \subset F$ we can remove the $\|\nabla u_0\|$ and r terms above (due to the second half of the Signorini condition) . In that case we may conclude, with an argument similar to Proposition 3.2.2 that

$$\int_\Omega |\nabla \chi_\infty| \leq \int_\Omega |\nabla \tilde{\chi}_F|$$

since F can be *any* set containing E_∞ we have proven that ∂E_∞ has positive mean curvature in the weak sense.

Step 3. E_∞ lies inside any positive mean curvature domain containing $E(0)$. This can be seen even at the level of the discrete solutions. Let $h > 0$, if F is a set with positive mean curvature containing $E(0)$ and E intersects F^c in a set of positive volume one readily sees that 1) the pair $E \cap F$ has perimeter no larger than E and 2) $u_{E \setminus F}$ has a strictly smaller H^1 norm in comparison to that of u_E. Thus the pair (χ_E, u_E) cannot be minimal, this means for each time step $h > 0$ the solid phases corresponding to the minimizers $\{\chi_k\}_k$ must lie inside F. We conclude that $E(t)$ lies inside F for every $t > 0$ and the assertion for E_∞ follows.

With steps 2 and 3 we have proved that E_∞ is the smallest domain with positive mean curvature containing the initial data $E(0)$.

Step 4. Uniform convergence: since $E(t)$ is a domain increasing with t, we conclude from the previous 3 steps that as $t \to +\infty$ the set $E(t)$ converges *uniformly* to the smallest domain with positive mean curvature containing $E(0)$, and that finishes the proof. $\square$

Chapter 4

Stefan: Lipschitz free boundaries

In this chapter we observe how the De Giorgi - Nash - Moser theory allows us to prove continuity of the temperature in the Stefan and Hele-Shaw problems whenever the free boundary is Lipschitz in space and time. Moreover, we prove an estimate that doesn't require Lipschitz in time but only some integrability of the free boundary velocity. In the first section we prove an interpolation lemma at the trace that will lead to an energy inequality with a non-linearity which will help redo de L^∞ bound.

From now on, whenever we speak of a solution we will assume it has a Lipschitz free boundary in space and time. The constants L and V will always denote the Lispchitz norm of the hypersurface with respect to space and time (cf statement of Theorem 2.2.1)

4.1 L^∞ bound

The first lemma uses the Lipschitz assumption on the free boundary to show how the Stefan condition holds in a stronger sense.

Lemma 4.1.1. *Let (u, χ) be a weak solution to (SGT) in Q_2 such that $\Gamma \cap Q_2$*

is a special Lipschitz hypersurface of the form

$$\{(x', x_n, t) \in Q_2 : x_n = f(x', t)\}, \quad f \text{ Lipschitz in both } x' \text{ and } t$$

Then:

χ_t *is a measure and it equals* $v|\nabla\chi(t)|$, *for some bounded function* $v : \Gamma \to \mathbb{R}$.

In particular, $\chi_t \in L^\infty H^{-1}$ *and* $u_t \in L^2 H^{-1}$.

Proof. The assumption says $E(t) = \{x : \chi(x, t) > 0\}$ is a Lipschitz domain changing in a Lipschitz manner over time, thus for every $\phi \in C_c^\infty(Q)$ and a.e. $t \in (-2, 0)$ we have

$$\frac{d}{dt} \int \phi \chi dx = \int v\phi|\nabla\chi(t)| + \int \chi\phi_t dx$$

where v is the normal speed of Γ which is a bounded function defined on Γ, a direct consequence of Rademacher's theorem. Integrating the above identity with respect to $t \in (-2, 0)$ we get

$$0 = \int_B \phi(0)\chi(0)dx - \int_B \phi(-2)\chi(-2) = \int_{-2}^0 \frac{d}{dt}\left(\int_B \phi\chi dx\right) dt$$

$$\Rightarrow \int_Q v\phi|\nabla\chi|dt = -\int_Q \chi\phi_t dx dt$$

Given that ϕ was an arbitrary test function, we conclude that for almost every time we have $\chi_t = v|\nabla\chi(t)|$. The Sobolev trace theorem for Lipschitz domains (see lemma below) then says the measures $v|\nabla\chi(t)|$ lie in a bounded set of H^{-1}, so $\chi_t \in L^\infty H^{-1}$. By definition $(\chi + u)_t \in L^2 H^{-1}$, thus $u_t \in L^2 H^{-1}$. $\square$

The next tool we need uses the arithmetic-geometric mean inequality to bound traces of u. This we will need in order to control terms involving integrals of the temperature along the free boundary in terms of the L^2 norm of u and a small enough multiple of the norm of its gradient. The Lipschitz assumption will be key as we will use the Sobolev trace theorem for boundaries of (Lipschitz) domains.

Lemma 4.1.2 (Trace Lemma). *Let Ω be an open domain in $\mathbb{R}^n$ and Σ a hypersurface such that $\Sigma \cap \Omega$ is given by the graph of a Lipschitz function with Lipschitz constant L. Then there exists $C = C(L) > 0$ such that for every $\epsilon > 0$ and every $\phi \in H_0^1(\Omega)$ we have*

$$\int_\Sigma \phi^2 d\sigma \leq C \left(\frac{1}{4\epsilon} \int_\Omega \phi^2 dx + \epsilon \int_\Omega |\nabla \phi|^2 dx \right) \tag{4.1}$$

Proof. By a density argument we may assume $\phi \in H_0^1(\Omega) \cap C_c^\infty(\Omega)$ without losing generality. If $\Sigma \cap \Omega$ is given by the graph of a Lipschitz function then we can find a bi-Lipschitz diffeomorphism T that flattens Σ into (say) the hyperplane $\Pi = \{x_n = 0\}$. Thus if $\psi = \phi \circ T$ we know that

$$C^{-1} \|\psi\|_{L^2} \leq \|\psi\|_{L^2} \leq C \|\psi\|_{L^2}$$

$$C^{-1} \|\nabla \psi\|_{L^2} \leq \|\nabla \phi\|_{L^2} \leq C \|\nabla \psi\|_{L^2}$$

For $C = C(L)$. Because of this we only need to prove the estimate for ϕ^*, namely

$$\int_\Pi \psi(x', 0)^2 dx' \leq C \left(\frac{1}{4\epsilon} \int_{\Omega^*} \psi^2 dx + \epsilon \int_{\Omega^*} |\nabla \psi|^2 dx \right)$$

If we denote by y the coordinate corresponding to the axis orthogonal to Π, we have

$$(\psi^2)_y = 2\psi\psi_y$$

so that

$$\psi(x',y)^2 - \psi(x',0)^2 = \int_0^y \psi(x',s)\psi_s(x',s)ds$$

if for each x' we take $y = y(x')$ so that (x,y') lies in $\Omega^* \setminus \operatorname{supp}\psi$ we get $\psi(x',y) = 0$. Therefore

$$\psi(x',0)^2 = -\int_0^{y(x')} \psi(x',s)\psi_s(x',s)ds$$

$$\Rightarrow \int_\Pi \psi(x',0)^2 dx' = -\int_{\Omega^* \cap \{y>0\}} \psi\psi_s dx \leq \int_{\Omega^*} |\psi\psi_s| dx$$

Now we finish via Cauchy-Schwartz, for each $\epsilon > 0$ we have

$$\int_\Pi \psi(x',0)^2 dx' \leq \int_{\Omega^*} \frac{1}{4\epsilon}\psi^2 + \epsilon\psi_s^2 dx$$

$$\Rightarrow \int_\Pi \psi(x',0)^2 dx' \leq \frac{1}{4\epsilon}\int_{\Omega^*} \psi^2 dx + \epsilon \int_{\Omega^*} |\nabla\psi|^2 dx$$

$\square$

Now we are ready to prove the energy inequality for Lipschitz solutions of (SGT).

Lemma 4.1.3 (Energy inequality)**.** *There exists a constant $C_L = C(L)$ such that for a.e. $t \in (-2,0)$, any $m > 0$ and $\eta \in C_c^\infty(B_2)$ we have*

$$(u_t, \eta^2 u_m) + \frac{1}{2}\int |\nabla(\eta u_m)|^2 dx \leq \int |\nabla\eta|^2 u_m^2 dx + C_L \frac{V^2}{m^2}\int \eta^2 (u_m)^2 dx$$

where $u_m(x,t) = \max\{u(x,t), m\}$.

Proof. By the previous proposition, for a.e. $t \in (-2, 0)$ and any $\phi \in H^1_0(B_2)$ we have (omitting t to simplify notation)

$$(u_t, \phi) + \int \nabla u \cdot \nabla \phi dx = -\int \phi v |\nabla \chi|$$

Fix an arbitrary $\eta \in C^\infty_c(B_2)$ and take $\phi = \eta^2 u_m \in H^1_0(\Omega)$. Using this test function in the equation above we get

$$(u_t, \eta^2 u_m) + \int \eta^2 \nabla u \cdot \nabla u_m dx + \int 2\eta u_m \nabla u \cdot \nabla \eta dx = -\int \eta^2 u_m v |\nabla \chi|$$

$$\Rightarrow (u_t, \eta^2 u_m) + \int |\nabla(\eta u_m)|^2 dx = \int u_m^2 |\nabla \eta|^2 dx - \int \eta^2 u_m v |\nabla \chi|$$

Since $u_m \geq m$ have $1 \leq \frac{u_m}{m}$ a.e. with respect to $|\nabla \chi|$, so that

$$\int |\eta^2 u_m V||\nabla \chi(t)| \leq m^{-1} \int (\eta u_m)^2 |v||\nabla \chi|$$

Applying the previous lemma with $\phi = \eta^2 u_m$ and $\Sigma = \Gamma(t)$ we get

$$\int \eta^2 |u_m||V||\nabla \chi| \leq \frac{C_L V}{m} \left(\epsilon^{-1} \int (\eta u_m)^2 dx + \epsilon \int |\nabla(\eta u_m)|^2 dx \right)$$

Taking $\epsilon = \frac{m}{2C_L V}$ so that $\frac{C_L V \epsilon}{m} \leq \frac{1}{2}$ we obtain

$$(u_t, \eta^2 u_m) + \frac{1}{2} \int |\nabla(\eta u_m)|^2 dx \leq \int |\nabla \eta|^2 u_m^2 dx + C_L \frac{V^2}{m^2} \int \eta^2 u_m^2 dx$$

$\square$

We remark further: by integrating this inequality in time, we may rewrite the energy inequality as a time average:

$$\int (\eta u_m(T_2))^2 dx - \int (\eta u_m(T_1))^2 dx + \frac{1}{2} \int_{T_1}^{T_2} \int |\nabla(\eta u_m)|^2 dx dt$$

$$\leq \int_{T_1}^{T_2} \int |\nabla \eta|^2 u_m^2 dxdt + C_L \frac{V^2}{m^2} \int_{T_1}^{T_2} \int \eta^2 u_m^2 dxdt$$
$$\forall \ T_1, \ T_2 : \ -2 < T_1 < T_2 < 0$$

The energy inequality together with the Sobolev embedding theorem allows us to show via modified De Giorgi-Nash-Moser iterations that the temperature becomes bounded in the interior (and by the scale invariance of the estimate, it does so *instantaneously*), similar to Evans and Caffarelli's work on the standard Stefan problem. Here we will omit the details of this proof as the L^∞ bound will already be implied by the estimates of the next section, which follow a classical extension of Stampacchia of work De Giorgi.

4.2 Hölder continuity

Next we prove that the temperature is even Hölder continuous (and thus the free boundary is $C^{2,\alpha}$ for some α). For this, we prove a second L^∞ bound for our solutions using ideas of Stampacchia ([13]).

Lemma 4.2.1 (Non-homogenous bound)**.** *Let w solve (with χ_t as in the previous lemmas)*

$$\begin{aligned} w_t - \Delta w &= -\chi_t \quad \text{in } Q_2 \\ w &= 0 \quad\quad \text{on } \partial_p Q_2 \end{aligned}$$

Then for any $p > n-1$ we have

$$\|w\|_{L^\infty(Q_2)} \leq C(L,n) \|v\|_{L^p(\Gamma \cap Q_2)} \left(\int_{Q_2} |\nabla \chi| \right)^{\frac{1}{n-1} - \frac{1}{p}}$$

Proof. Step 1. (Energy inequality) Let $w_\lambda = (w-\lambda)_+ \in H_0^1(B_2)$, then for almost every time we have

$$(w_t, w_\lambda) + \int \nabla w \cdot \nabla w_\lambda dx = \int_\Gamma w_\lambda v |\nabla \chi|$$

We now bound the right hand side. For every $\epsilon > 0$ we have

$$\int_\Gamma w_\lambda v |\nabla \chi| \le \epsilon \int_\Gamma w_\lambda^2 |\nabla \chi| + \epsilon^{-1} \int_{\Gamma \cap \{w > \lambda\}} v^2 |\nabla \chi|$$

$$\le C_L \epsilon \int |\nabla w_\lambda|^2 dx + \epsilon^{-1} \int_{\Gamma \cap \{w>\lambda\}} v^2 |\nabla \chi|$$

Where in the second inequality we used the Sobolev trace theorem and Poincare's inequality. Taking $\epsilon = \frac{1}{2C_L}$ we have the energy inequality

$$(w_t, w_\lambda) + \frac{1}{2} \int |\nabla w_\lambda|^2 dx \le 2C_L \int_{\Gamma \cap \{w>\lambda\}} v^2 |\nabla \chi|$$

Step 2. (Iteration) Fix $M > 0$ and for each $k \in \mathbb{N}$ let

$$\lambda_k = M(1 - \frac{1}{2^k}), \quad w_k := (w - \lambda_k)_+$$

Integrating the previous energy inequality from -2 to 0 (recall that $w(-2) = 0$) we have

$$\frac{1}{2} \int w_k(0)^2 dx + \frac{1}{2} \int_{-2}^0 \int |\nabla w_k|^2 dx dt \le 2C_L \int_{-2}^T \int_{\Gamma \cap \{w > \lambda_k\}} v^2 |\nabla \chi(t)| dt$$

$$\Rightarrow \int_{Q_2} |\nabla w_k|^2 dx dt \le 4C_L \int_{-2}^0 \int_{\Gamma \cap \{w>\lambda_k\}} v^2 |\nabla \chi(t)| dt$$

We now apply Hölder's inequality on the right hand side with $\frac{p}{2}$, we get

$$\int_{-2}^{0}\int_{\Gamma\cap\{w>\lambda_k\}} v^2|\nabla\chi(t)|dt \leq \left(\int_{Q_2}|v|^p|\nabla\chi|dt\right)^{\frac{2}{p}}\left(\int_{-2}^{0}|\nabla\chi(t)(\Gamma\cap\{w>\lambda_k\})|dt\right)^{1-\frac{2}{p}}$$

Since $w_{k-1}>0 \Rightarrow w_k > 2^{-k}M$, we have the relation for any $q>1$

$$|\nabla\chi(t)(\Gamma\cap\{w>\lambda_k\})|^{\frac{2}{q}} \leq \frac{2^{2k}}{M^2}\left(\int w_{k-1}^q|\nabla\chi(t)|\right)^{\frac{2}{q}}$$

Moreover, taking[1] $q = 1+\frac{n}{n-2} > 2$ one can apply the Sobolev trace inequality and get the bound

$$|\nabla\chi(t)(\Gamma\cap\{w>\lambda_k\})|^{\frac{2}{q}} \leq C_L\frac{2^{2k}}{M^2}\int|\nabla w_{k-1}|^2dx$$

Integrating this relation and applying the energy inequaliy as above, we reach the relation

$$\int_{-2}^{0}|\nabla\chi(t)(\Gamma\cap\{w>\lambda_k\})|^{\frac{2}{q}}dt \leq 4C_L^2\left(\frac{2^k}{M}\right)^2\|v\|_p^2\left(\int_{-2}^{0}|\nabla\chi(t)(\Gamma\cap\{w>\lambda_{k-1}\})|dt\right)^{1-\frac{2}{p}}$$

Finally, since $2<q$, we can apply Jensen's inequality to the left side of the inequality and get

$$\left(\int_{-2}^{0}|\nabla\chi(t)(\Gamma\cap\{w>\lambda_k\})|dt\right)^{\frac{2}{q}}$$
$$\leq 4C_L^2\left(\frac{2^k}{M}\right)^2\|v\|_p^2\left(\int_{-2}^{0}|\nabla\chi(t)(\Gamma\cap\{w>\lambda_{k-1}\})|dt\right)^{1-\frac{2}{p}}$$

[1] if $n=2$ we may pick any $q>2$

Define $A_k = \int_{-2}^{0} |\nabla\chi(t)(\Gamma\cap\{w>\lambda_k\})|dt$, the last inequality can be rewritten as

$$A_{k-1} \leq 2^q C_L^q \left(\frac{2^k}{M}\right)^q \|v\|_p^q A_k^{\left(1-\frac{2}{p}\right)\frac{q}{2}}$$

$$\int_{-2}^{0} |\nabla\chi(t)(\Gamma\cap\{w>\lambda_{k-1}\})|dt \leq 2^q C_L^q \left(\frac{2^k}{M}\right)^q \|v\|_p^q \left(\int_{-2}^{0} |\nabla\chi(t)(\Gamma\cap\{w>\lambda_k\})|dt\right)^{\left(1-\frac{2}{p}\right)\frac{q}{2}}$$

Note that $\left(1-\frac{2}{p}\right)\frac{q}{2} > 1$ iff $p > n-1$, then we can apply to our sequence A_k the same argument we used to get the first L^∞ bound. This way we may check that $A_k \to \infty$ and thus $u \leq M$ a.e. in Q_2 as long as

$$M = C(L,n)\|v\|_{L^p(\Gamma\cap Q_2)}\left(\int_{Q_2} |\nabla\chi|\right)^{\frac{1}{n-1}-\frac{1}{p}}$$

similarly we may prove the lower bound for u and we are done.

$\square$

The above estimate together with the regularity for the homogenous linear case (see Appendix) grant us the second L^∞ bound for the temperature.

Proposition 4.2.2. *(Second L^∞ bound) For a solution u in Q_3 and $p > n-1$ we have*

$$\|u\|_{L^\infty(Q_1)} \leq C(L,n)\left(\|u\|_{L^2(Q_3)} + C(P,p,n)\|v\|_{L^p(\Gamma\cap Q_3)}\right)$$

where $P = \int_{Q_3} |\nabla\chi|$ and $C(P,p,n) = P^{1-\frac{2(n-1)}{np}} + P^{\frac{1}{n-1}-\frac{1}{p}}$.

Proof. Decompose u in Q_2 as $w_1 + w_2$, where

$$\begin{cases} (w_1)_t - \Delta w_1 = 0 & \text{in } Q_2 \\ w_1 = u & \text{on } \partial_p Q_2 \end{cases}$$

$$\begin{cases} (w_2)_t - \Delta w_2 = -\chi_t & \text{in } Q_3 \\ w_2 = 0 & \text{on } \partial_p Q_3 \end{cases}$$

Therefore

$$\sup_{Q_1} |u| \ \leq \sup_{Q_1} |w_1| + \sup_{Q_1} |w_2|$$

From the theory for linear parabolic equations (see appendix) we know that

$$\sup_{Q_1} |w_1| \leq C \|w_1\|_{L^2(Q_2)}, \quad \text{and}$$

$$\|w_1\|_{L^2(Q_1)} \leq \|u\|_{L^2(Q_2)} + C_n \|\nabla u\|_{L^2(Q_2)}$$

Moreover, by the energy inequality we know that for any $p > n-1$ ($n \geq 2$) we have

$$\|\nabla u\|^2_{L^2(Q_2)} \leq C\|u\|^2_{L^2(Q_3)} + C_L \|v\|^2_{L^p(\Gamma\cap Q_3)} \left(\int_{\Gamma\cap Q_3} |\nabla \chi| \right)^{1-\frac{2(n-1)}{np}}$$

Finally, from the non-homogeneous bound we know that for $p > n-1$

$$\sup_{Q_3} |w_2| \leq C_L \|v\|_{L^p(\Gamma\cap Q_3)} \left(\int_{Q_3} |\nabla \chi| \right)^{\frac{1}{n-1}-\frac{1}{p}}$$

Putting all the estimates together we get the bound

$$\sup_{Q_1} |u| \leq C(L,n) \left(\|u\|_{L^2(Q_2)} + (P^\beta + P^\gamma) \|v\|_{L^p(\Gamma \cap Q_3)} \right)$$

$$P = \int_{\Gamma \cap Q_3} |\nabla \chi|, \quad \beta = 1 - \frac{2(n-1)}{np}, \;\; \gamma = \frac{1}{n-1} - \frac{1}{p}$$

□

To finish this chapter (and thus the proof of Theorem 2.2.1) we use again the estimates for the linear case and the non-homogenous bound to show u is continuous.

Lemma 4.2.3 (Continuity of the temperature)**.** *Let* (u, χ) *be a weak solution with a Lipschitz free boundary in* Q_2*, there exists a universal* $\alpha = \alpha(L, V, n) \in (0,1)$ *such that*

$$[u]_{C^\alpha(Q_{1/4})} \leq C$$

Proof. Fix $\rho \in (0,1)$, let us decompose u as $w_1 + w_2$, each given by

$$\begin{cases} (w_1)_t - \Delta w_1 = 0 & \text{in } Q_\rho \\ w_1 = u & \text{on } \partial_p Q_\rho \end{cases}$$

$$\begin{cases} (w_2)_t - \Delta w_2 = -\chi_t & \text{in } Q_\rho \\ w_2 = 0 & \text{on } \partial_p Q_\rho \end{cases}$$

Interior regularity for caloric functions tells us that

$$\operatorname{osc}_{Q_{\frac{\rho}{4}}} w_1 \leq \mu \operatorname{osc}_{Q_{\frac{\rho}{2}}} w_1$$

and the non-homogeneous bound says that (with $p = \infty$)

$$\operatorname{osc}_{Q_{\frac{\rho}{4}}} w_2 \leq 2 \sup_{Q_{\frac{\rho}{4}}} |w_2| \leq C(L,n) V \left(\int_{Q_{\frac{\rho}{4}}} |\nabla \chi| \right)^{\frac{1}{n-1}}$$

Since now we may assume Γ is at least C^1, we have

$$\int_{Q_{\frac{\rho}{4}}} |\nabla \chi| \leq C_\Gamma \rho^n$$

We have obtained

$$\operatorname{osc}_{Q_{\frac{\rho}{4}}} u \leq \mu \operatorname{osc}_{Q_{\frac{\rho}{2}}} w_1 + C(L,n) V \rho^{1+\frac{1}{n-1}}$$

Given the interior estimates for w_1 in terms of $\|u\|_{L^2}$, we can conclude via a standard argument that there exists a universal $\alpha = \alpha(V, L, n)$ such that

$$[u]_{C^\alpha(Q_{\frac{1}{4}})} < \infty$$

with a corresponding estimate in terms of the L^2 norm of u. $\square$

Chapter 5

Hele-Shaw: Lipschitz free boundaries

This chapter is very similar to the previous one, except that the estimates are more "elliptic". The organization is the same and in many points where the arguments are similar we refer to the previous chapter.

5.1 L^∞ bound

Lemma 5.1.1. *Let (u, χ) be a weak solution to (HS) in Q_2 such that $\Gamma \cap Q_2$ is a Lipschitz hypersurface of the form*

$$\{(x', x_n, t) \in Q_2 : x_n = f(x', t)\}, \quad f \text{ Lipschitz in both } x' \text{ and } t$$

Then χ_t is a measure of the form $v|\nabla\chi|$ for some bounded function $v : \Gamma \to \mathbb{R}$. Moreover, we have $\chi_t \in L^\infty H^{-1}$

The proof is exactly the same as for the Stefan problem.

Lemma 5.1.2 (Energy inequality)**.** *There exists a constant $C_L = C(L)$ such that for a.e. $t \in (-2, 0)$, any $m > 0$ and $\eta \in C_c^\infty(B_2)$ we have*

$$\frac{1}{2}\int |\nabla(\eta u_m)|^2 dx \leq \int |\nabla\eta|^2 u_m^2 dx + C_L \frac{V^2}{m^2}\int \eta^2 (u_m)^2 dx$$

where $u_m(x, t) = \max\{u(x, t), m\}$.

Proof. By the previous proposition, for a.e. $t \in (-2, 0)$ and any $\phi \in H_0^1(B_2)$ we have (again we omit t to simplify the formulas)

$$\int \nabla u \cdot \nabla \phi \, dx = -\int \phi v |\nabla \chi|$$

Pick any $\eta \in C_c^\infty(B_2)$ and take $\phi = \eta^2 u_m \in H_0^1(\Omega)$, plugging this test function in the equation above we get

$$\int \eta^2 \nabla u \cdot \nabla u_m dx + \int 2\eta u_m \nabla u \cdot \nabla \eta dx = -\int \eta^2 u_m v |\nabla \chi|$$

$$\Rightarrow \int |\nabla(\eta u_m)|^2 dx = \int u_m^2 |\nabla \eta|^2 dx - \int \eta^2 u_m v |\nabla \chi|$$

Since $u_m \geq m$ have $1 \leq \frac{u_m}{m}$ a.e. with respect to $|\nabla \chi|$, so that

$$\int |\eta^2 u_m V| |\nabla \chi| \leq m^{-1} \int (\eta u_m)^2 |v| |\nabla \chi|$$

Applying the previous lemma with $\phi = \eta^2 u_m$ and $\Sigma = \Gamma(t)$ we get

$$\frac{1}{m} \int (\eta u_m)^2 |v| |\nabla \chi| \leq \frac{C_L V}{m} \left(\epsilon^{-1} \int (\eta u_m)^2 dx + \epsilon \int |\nabla(\eta u_m)|^2 dx \right)$$

Taking $\epsilon = \frac{m}{2C_L V}$ so that $\frac{C_L V \epsilon}{m} \leq \frac{1}{2}$ we obtain

$$\frac{1}{2} \int |\nabla(\eta u_m)|^2 dx \leq \int |\nabla \eta|^2 u_m^2 dx + C_L \frac{V^2}{m^2} \int (\eta u_m)^2 dx$$

□

As we said for the Stefan problem, this energy inequality allows one to get L^∞ bounds in the interior, since there is "diffusion" in u for Hele-Shaw it should not be surprising that the space L^2 norm of the temperature for a given time controls the pointwise values of u (for that time). Also as done for the Stefan problem, we give a detailed proof of the L^∞ bound (and continuity in space) following ideas of Stampacchia.

5.2 Hölder continuity in space

The continuity proof follows the same approach as the Stefan case: we prove an Stampacchia-like maximum principle and apply the estimates for the linear theory. This will be done in a few lemmas.

Lemma 5.2.1 (Non-homogenous bound)**.** *Let $w \in H^1(B_2)$ solve for some $t \in (-2, 0)$*

$$\begin{aligned} -\Delta w &= -\chi_t \quad \textit{in } B_2 \\ w &= 0 \qquad \textit{on } \partial B_2 \end{aligned}$$

Then for any $p > n-1$ we have

$$\|w\|_{L^\infty(B_2)} \leq C(L,n)\|v\|_{L^p(\Gamma(t)\cap B_2)} \left(\int_{B_2} |\nabla\chi(t)|\right)^{\frac{1}{n-1}-\frac{1}{p}}$$

Proposition 5.2.2. *(Second L^∞ bound) Let $p > n-1$, and u, χ a solution with Lipschitz free boundary in Q_3. Then for a.e. t we have*

$$\|u(t)\|_{L^\infty(B_1)} \leq C(L,n)\left(\|u(t)\|_{L^2(B_3)} + C(P,p,n)\|v\|_{L^p(\Gamma(t)\cap B_3)}\right)$$

where $P = \int_{B_3} |\nabla\chi(t)|$ and $C(P,p,n) = P^{1-\frac{2(n-1)}{np}} + P^{\frac{1}{n-1}-\frac{1}{p}}$.

Proof. Fix t, decompose $u = u(t)$ in B_2 as $w_1 + w_2$, where

$$\begin{cases} -\Delta w_1 = 0 & \text{in } B_2 \\ w_1 = u & \text{on } \partial B_2 \end{cases}$$

$$\begin{cases} \Delta w_2 = \chi_t & \text{in } B_3 \\ w_2 = 0 & \text{on } \partial B_3 \end{cases}$$

Therefore

$$\sup_{B_1} |u| \;\; \leq \sup_{B_1} |w_1| + \sup_{B_1} |w_2|$$

From the theory for linear parabolic equations (see appendix) we know that

$$\sup_{B_1} |w_1| \leq C\|w_1\|_{L^2(B_2)}, \quad \text{and}$$

$$\|w_1\|_{L^2(B_1)} \leq \|u\|_{L^2(B_2)} + C_n\|\nabla u\|_{L^2(B_2)}$$

Moreover, by the energy inequality we know that for any $p > n-1$ $(n \geq 2)$ we have

$$\|\nabla u\|^2_{L^2(B_2)} \leq C\|u\|^2_{L^2(B_3)} + C_L\|v\|^2_{L^p(\Gamma(t)\cap B_3)} \left(\int_{\Gamma(t)\cap B_3} |\nabla \chi| \right)^{1-\frac{2(n-1)}{np}}$$

Finally, from the non-homogeneous bound we know that for $p > n-1$

$$\sup_{B_3} |w_2| \leq C_L\|v\|_{L^p(\Gamma(t)\cap B_3)} \left(\int_{B_3} |\nabla \chi(t)| \right)^{\frac{1}{n-1}-\frac{1}{p}}$$

Putting all the estimates together we get the bound

$$\sup_{B_1} |u| \leq C(L,n) \left(\|u\|_{L^2(B_2)} + (P^\beta + P^\gamma)\|v\|_{L^p(\Gamma(t)\cap B_3)} \right)$$

with the desired β and γ. $\square$

We now state without proof the lemma for the continuity of the temperature, the proof is done *mutatis mutandis* the proof of Lemma 4.2.3 at the end of the previous chapter. The same observation made before the proof of Lemma **??** explains why we should not be surprised that we have an estimate for each fixed time.

Lemma 5.2.3 (Continuity of the temperature in space)**.** *Let* (u, χ) *be a weak solution with a Lipschitz free boundary in* Q_2*. Then for almost every* t *we have with a universal* $\alpha \in (0, 1)$

$$[u(t)]_{C^\alpha(B_{1/4})} \leq C$$

Appendices

Appendix A

Reviewing De Giorgi-Nash-Moser for parabolic equations

For the sake of completeness, in this appendix we shall review some aspects of the regularity theory of linear parabolic equations in divergence form. Namely, we consider functions such that

$$\begin{cases} u_t - Lu = 0 \quad \text{in } Q_2, \quad Lu = \operatorname{div}(A(x,t)\nabla u) \\ u \in L^2H^1 \qquad u_t \in L^2H^{-1} \;\; \lambda I \leq A(x) \leq \Lambda I \;\; a.e.\ (x,t) \in Q_2 \end{cases}$$

In the sense that for almost every $t \in (-2,0)$ and every $\phi \in H_0^1(B_2)$ we have

$$(u_t, \phi) + \int (A(x,t)\nabla u \cdot \nabla \phi)\ dx = 0$$

As it is now well known, to prove continuity of the solution one proceeds in two stages: first one shows solutions are bounded pointwise (in the interior) by their L^2 norms, the second stage consists in showing that the oscillation of these (bounded) solutions decays geometrically as we look at a shrinking sequence of dyadic parabolic cylinders.

The first part uses the Energy and Sobolev inequalities, we state the two key lemmas used in this part. They are a special case of those proved in the section dealing with the Stefan problem. We only need to take in that case $v \equiv 0$, thus we will not write the proofs for the linear case.

Lemma A.0.4. *There exists $C = C(\lambda, \Lambda)$ such that*

$$(u_t, u_m) + \int |\nabla(\eta u_m)|^2 dx \leq C \int |\nabla \eta|^2 u_m^2 dx$$

Here $u_m = (u - m)_+$, $m \in \mathbb{R}$ and $\eta \in C_c^1(B_2)$ are arbitrary.

Lemma A.0.5. *There exists $C = C(\lambda, \Lambda, n)$ such that if u is a solution then*

$$\|u_m\|_{L^\infty(Q_1)} \leq C \|u_m\|_{L^2(Q_2)}$$

Where again we have $u_m = (u - m)_+$.

Corollary A.0.6. *Solutions are bounded in the interior. Moreover, we have the following scale invariant estimate*

$$\|u\|_{L^\infty(Q_r)} \leq \frac{C}{r^{n+2}} \|u\|_{L^2(Q_{2r})}$$

For the second part of the regularity theorem, we adapt a lemma from Caffarelli & Vasseur [9] used in the study of the quasi geostrophic equation. It allows one to do a complete analogue of De Giorgi's elliptic proof in the parabolic case. As opposed to Moser's [15] original parabolic theory this does not rely on a covering lemma.

Lemma A.0.7. *There exists $\delta = \delta(\lambda, \Lambda, n)$ such that if u is a subsolution such that*

$$\begin{gathered} a \leq u \leq b \text{ in } Q_{2r} \\ |\{(x,t) : u \leq \tfrac{a+b}{2}\}| \geq \tfrac{1}{2}|Q_{2r}| \\ |\{(x,t) : \tfrac{a+b}{2} < u < \tfrac{1}{4}a + \tfrac{3}{4}b\}| \leq \delta |Q_{2r}| \end{gathered}$$

Then

$$u \leq \frac{1}{8}a + \frac{7}{8}b \ \ a.e.\ in\ Q_r$$

Proof. Consider the sets (for each $t \in (-4, 0)$)

$$\begin{aligned} A(t) = &\quad \{x \in B_2 \ : u(x,t) \leq \underline{m}\} \\ B(t) = &\quad \{x \in B_2 \ : u(x,t) \geq \overline{m}\} \\ C(t) = &\ \{x \in B_2 \ : \underline{m} < u(x,t) < \overline{m}\} \end{aligned}$$

$$\text{and } K = \frac{4}{\epsilon_0} \int_{-4}^{0} \int_{C(t)} |\nabla u|^2 dxdt, \ \ \epsilon_0 > 0 \text{ to be chosen}$$

It is in our interest to bound $|B(t)|$ from above for $t \in (-1, 0)$. Namely,

$$\int_{Q_1} (u - \overline{m})_+^2 dxdt \leq \int_{-1}^{0} \int_{B_1 \cap B(t)} (b-a)^2 dxdt \leq (b-a)^2 \int_{-1}^{0} |B(t)| dt$$

Suppose that we had

$$\int_{-1}^{0} |B(t)| dt \leq C^{-1} 8^{-2}$$

$$\Rightarrow \int_{Q_1} (u - \overline{m})_+^2 dxdt \leq C^{-1} \left(\frac{1}{8}(b-a)\right)^2$$

by taking C as in the previous Lemma, we get

$$u - \overline{m} \leq \frac{1}{8}(b-a) \text{ in } \ Q_{\frac{1}{2}} \Rightarrow u \leq m^* \ \text{ in } \ Q_{\frac{1}{2}}$$

Note further that since $a \leq u \leq b$ and u is a solution in Q_{3r} that we can argue as in the proof of the previous lemma to get the bound

$$K \leq \frac{4C(\Gamma, n)}{\epsilon_0}(b-a)^2$$

The key tool to estimate $|B(t)|$ is De Giorgi's H^1-isoperimetric inequality (see Appendix) which guarantees that

$$|A(t)||B(t)| \leq K^{1/2}|C(t)|^{1/2} \quad \text{whenever} \int_{C(t)} |\nabla u(x,t)|^2 dx \leq K$$

In other words, for such times t we have the bound $|B(t)| \leq |A(t)|^{-1}K^{1/2}|C(t)|^{1/2}$. The times for which this estimate holds turn out to cover most of $(-4,0)$, for if we define

$$I = \{t \in (-4,0) : |C(t)|^{1/2} \leq \epsilon_1, \int_{B_2} |\nabla u(x,t)|^2 dx \leq K\}$$

then $(-4,0) \setminus I \subset \{t : |C(t)|^{1/2} \geq \epsilon_1\} \cup \{t : \int_{B_2} |\nabla u(x,t)|^2 dx \geq K\}$, so that by Tchebyschev's inequality

$$|(-4,0) \setminus I| \leq \epsilon_1^{-2} \int_{-2}^{0} |C(t)| dt + K^{-1} \int_{Q_2} |\nabla u|^2 dx dt \leq \epsilon_1^{-2}\delta|Q_2| + \frac{\epsilon_0}{4}$$

that is, by picking ϵ_0,ϵ_1 and δ accordingly we can make I cover most of the time interval. The last thing we need before we effectively use estimate (bla) is the lower bound for $|A(t)|$, we **claim** that for any $t \in I \cap (-1,0)$ we have $|A(t)| \geq \frac{1}{4}|B_2|$.

Indeed, since $|\{(x,t) \in Q_2 : u \leq \underline{m}\} \geq \frac{1}{2}|Q_2|$ there is at least one $t_0 \in (-4,-1) \cap I$ such that

$$|\{x \in B_2 : u(x,t_0) \leq \underline{m}\}| \geq \frac{1}{4}|B_2|$$

So that the H^1 isoperimetric inequality gives us

$$|B(t_0)| \leq 4|B_2|^{-1}K^{1/2}|C(t_0)|^{1/2} \leq 4|B_2|^{-1}\epsilon_0^{-1/2}C(\Gamma,n)^{1/2}(b-a)\epsilon_1$$

On the other hand, since $t_0 \in I$

$$\int_{B_2} (u(x,t_0) - \underline{m})_+^2 dx \le \int_{B(t_0)} (b - \underline{m})^2 dx + \int_{C(t_0)} (\overline{m} - \underline{m})^2 dx$$

$$\le \left(|B(t_0)| + \frac{1}{4}|C(t_0)| \right) (b-a)^2 \le \left(|B(t_0)| + \frac{1}{4}\epsilon_1^2 \right) (b-a)^2$$

so (by the energy inequality) we have for any $t > t_0$

$$\int_{B_r} (u(x,t) - \underline{m})_+^2 dx \le \int_{B_2} (u(x,t_0) - \underline{m})_+^2 dx + C(\Gamma, n)(b-a)^2(t-t_0)$$

$$\le \left(\left(|B(t_0)| + \frac{1}{4}\epsilon_1^2 \right) + C(\Gamma,n)(t-t_0) \right) (b-a)^2$$

Taking ϵ_0 small enough and using Tchebyschev's inequality, we see that for any $\delta_0 \le \frac{|B_2|}{4C(\Gamma,n)}$ and any $t \in (t_0, t_0 + \delta_0)$ the following inequality holds:

$$|\{x \in B_r : u(x,t) \ge \overline{m}\}| \le (b-a)^{-2} \int_{B_2} \eta^2 (u(x,t) - m_k)_+^2 dx$$

$$\le \left(\left(|B(t_0)| + \frac{1}{4}\epsilon_1^2 \right) + C(\Gamma,n)(t-t_0) \right) \le \frac{1}{2}|B_2|$$

and since $|C(t)| \le \epsilon_1^2 \le \frac{1}{4}|B_2|$ we have come to

$$|\{x \in B_1 : u(x,t) \ge \underline{m}\}| \le \frac{3}{4}|B_2| \Rightarrow |A(t)| \ge \frac{1}{4}|B_2| \;\; \forall t \in (t_0, t_0 + \delta_0) \cap I$$

Note that δ_0 is independent of δ and ϵ_0, so we can choose them all so that $\epsilon_1^{-2}\delta + 4^{-1}\epsilon_0 \le 2^{-1}\delta_0$, in which case any interval of lenght δ_0 must contain at least one $t \in I$. Since $t_0 < -r^2$, we conclude that the last inequality holds for *any* $t \in (-r^2, 0) \cap I$ and we get the desired lower bound on $|A(t)|$.

We use the H^1 isoperimetric inequality one last time to get that

$$\int_{-1}^{0} |B(t)|dt = \int_{(-1,0)\setminus I} |B(t)|dt + \int_{I\cap(-1,0)} |B(t)|dt$$
$$\leq |(-1,0)\setminus I||B_2| + 4^{-1}|B_2|K^{1/2} \int_{I\cap(-1,0)} |C(t)|^{1/2}dt$$
$$\leq |B_1|2^{n-2}\left[\left(\epsilon_1^{-2}\delta|B_1| + \frac{\epsilon_0}{4}\right) + C\epsilon_1\epsilon_0^{-\frac{1}{2}}(b-a)\right]$$

which can be made $\leq C$ by taking first ϵ_0 and then δ universally small. □

Lemma A.0.8. *There exists $\mu = \mu(\lambda, \Lambda, n)$ with $0 < \mu < 1$ such that if u is a solution then*

$$\operatorname{osc}_{Q_r} u \leq \mu \operatorname{osc}_{Q_{2r}} u$$

Proof. We may assume without loss of generality that

$$\sup_{Q_{2r}} u = 1, \ \inf_{Q_{2r}} u = 0 \Rightarrow \operatorname{osc}_{Q_{2r}} u = 1$$

Moreover, we may also assume that

$$|\{(x,t) : u \leq \frac{1}{2}\}| \geq \frac{1}{2}|Q_{2r}|$$

Otherwise, we apply the argument below to $v = 1 - u$ and reach a similar conclusion. Consider then the sequence $\lambda_k = 1 - \frac{1}{2^k}$. Suppose k_0 is such that

$$|\{(x,t) : \lambda_{k-1} < u < \lambda_k\}| > \delta|Q_{2r}|, \ \ \forall k \leq k_0$$

Since these k_0 sets are disjoint all contained in Q_{2r}, it must be that

$$\delta|Q_{2r}|k_0 < |Q_{2r}| \Rightarrow k_0 < \delta^{-1}$$

In other words, there is always some $k_0 < \delta^{-1}$ for which we have the inequality

$$|\{(x,t) : \lambda_{k_0-1} < u < \lambda_{k_0}\}| \leq \delta |Q_{2r}|$$

Picking such a $k_0 = [\delta] + 1$, consider $w = \max\{u, \lambda_{k_0-2}\}$, it is a subsolution to which we can apply the previous lemma with $a = \lambda_{k_0-2}$, $b = 1$, the lemma tells us that

$$u \leq w \leq \frac{1}{8}\lambda_{k_0-2} + \frac{7}{8} = \lambda_{k_0+1} \quad \text{in } Q_r$$

Let $\mu_0 = \lambda_{k_0}$. Note that $\mu_0 < 1$ is completely determined by δ and thus it is a constant depend only on λ, Λ and n, moreover we have showed that

$$\operatorname{osc}_{Q_r} u = \mu_0 - 0 \leq \mu_0 \operatorname{osc}_{Q_{2r}} u$$

and that proves the lemma. $\square$

Corollary A.0.9. *There exists $C = C(\lambda, \Lambda, n)$ and $\alpha = \alpha(\lambda, \Lambda, n)$ such that any solution u is C^α in the interior. Specifically, we have the estimate*

$$\|u\|_{C^\alpha(Q_1)} \leq C\|u\|_{L^2(Q_2)}$$

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Vita

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