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Rational Embeddings of the Severi-Brauer Variety

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Rational Embeddings of the Severi-Brauer Variety

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DISSERTATION

Presented to the Faculty of the Graduate School of

The University of Texas at Austin

in Partial Fulfillment

of the Requirements

for the Degree of

DOCTOR OF PHILOSOPHY

THE UNIVERSITY OF TEXAS AT AUSTIN

August 2010

Dedicated to Susan Elizabeth Johnson.

Acknowledgments

I wish to thank Mark Meth and Kylee Meth, my parents, and Elizabeth Johnson for their support. I also wish to thank all the graduate students with whom I shared my experience at the University of Texas, especially Eric Katerman and Marlene Costa.

I owe a great deal to many of the professors, both here and at other universities, who supported me and spent so much time answering my questions. In particular, I wish to thank John Tate, David Helm, Kelly McKinnie, Daniel Krashen, and Darrell Haile. I would also like to thank my committee.

Rational Embeddings of the Severi-Brauer Variety

Publication No. __________________

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The University of Texas at Austin, 2010

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The aim of this thesis is to generalize results relating Severi-Brauer variety of a central, simple algebra, A, over an infinite field F with its norm hypersurface. We generalize this in two ways, first by enlarging the collection of rational embeddings of the Severi-Brauer into the norm hypersurface, and second by extending these ideas to rational embeddings of the generalized Severi-Brauer variety into related determinantal subvarieties of the algebra A.

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Chapter 1

Introduction

To each F-CSA, A, we can associate both a Severi-Brauer variety, $SB(A)$, and a norm hypersurface, $V(A)$, defined by a norm polynomial. In [1], Amitsur proposes that if two algebras A and B of degree n generate the same cyclic subgroup of the Brauer group, $Br(F)$, then their Severi-Brauer varieties are birational. The problem is known in many cases, but is still open. In [12], Saltman proves that the norm hypersurface variety satisfies a variant of this conjecture, that is, A and B generate the same cyclic subgroup of $Br(F)$ if and only if $V(A)$ and $V(B)$ are birational. In [13], Saltman constructs a collection of rational embeddings of $SB(A)$ into its norm hypersurface $V(A)$. An overall strategy for solving Amitsur's Conjecture is the following: for A and B of the same degree, which generate the same cyclic subgroup of the Brauer group, $V(A)$ and $V(B)$ are birational. If we analyze the rational embeddings of $SB(A)$ into $V(A)$ and $SB(B)$ into $V(B)$, it may be possible to compose and find a birational map from $SB(A)$ to $SB(B)$.

In this thesis, we focus on enlarging the collection of rational embeddings of $SB(A)$ into $V(A)$. In [13], Saltman created these rational embeddings by selecting a separable, splitting subring L in A, and α from a dense set of

A, and creating an intersection map taking a dense open subset of ideals in $SB(A)$ to their unique intersections with $L + \alpha$. We generalize this process in Chapter 4 to a dense set of $N \in Gr(n, A)$, not just a separable, splitting subring of A. For each of these N in a dense subset, there is a dense subset of $d \in A$ such that the intersection map, taking an ideal to its intersection with $N + d$, can be built, and yields a rational embedding of $SB(A)$ into $V(A)$ (Theorem 4.0.18).

The Severi-Brauer variety has a natural embedding $SB(A) \hookrightarrow Gr(n^2 - n, A)$. In Chapter 5 we use this embedding to define the tautological bundle on $SB(A)$ as the pullback of the tautological bundle on $Gr(n^2 - n, A)$, and show via Galois descent that this is a desingularization of the norm hypersurface. Each rational section of this bundle bijects with a rational point in the fiber over the generic point, which we label M, the generic, maximal right ideal of A. We show that any rational point of M which generates M will yield a rational embedding of $SB(A)$ into $V(A)$. We then analyze elements of M of lower rank in Chapter 5 and Chapter 6. We also apply this construction to the rth generalized Severi-Brauer variety, $SB_r(A)$, of right ideals of rank r. We show that for each r there exists a tautological bundle $T_r(A)$ on $SB_r(A)$ which desingularizes a twist of the rank r determinantal variety.

In Chapter 7 we show two results concerning the subspaces N from Chapter 4. First, we show that if $N \otimes_F F(A)$ intersects the generic, maximal right ideal $M \subseteq A \otimes_F F(A)$, then A is split. Second, we show that the collection of N and d we have constructed actually produce new examples, not of the form given in [13].

Chapter 2

Preliminary Results

The Brauer Group

The following results can be found in [9] and [5]. As a set, the Brauer group of a field F is given by equivalence classes of F-algebras called F-central simple algebras.

Definition 2.0.1. An algebra A is an F-central simple algebra (or F-CSA) if it is a simple ring with center $Z(A) = F$, and as a vector space over F it is finite dimensional.

Given a field F, we form an equivalence relation on the set of all F-CSA's, where two F-CSA's A and B are equivalent, $A \sim B$, if and only if there exist positive integers r and s such that $A \otimes_F M_r(F) = M_r(A)$ and $B \otimes_F M_s(F) = M_s(B)$ are isomorphic as F-algebras. As a set, the Brauer group of F, denoted $Br(F)$, is the quotient set of all F-CSA's under this equivalence relation. We will refrain from defining $Br(F)$ formally until we have the accompanying group structure.

By the Wedderburn-Artin Theorem, every F-CSA is isomorphic to a matrix ring over an F-CSA called a division algebra.

Definition 2.0.2. An F-CSA, A, is a division algebra over F if every element of A is invertible.

Given an F-CSA A, $A \cong M_s(D)$ for some positive integer s, and some division algebra D/F. Both s and D are unique up to isomorphism. Thus every equivalence class in $Br(F)$, $[A]$, contains a unique element of minimal dimension, which is a division algebra. In fact, $Br(F)$ may equivalently be defined as the set of isomorphism classes of division algebras over F. That is not the approach we will take here.

Given two F-CSA's A and B, their tensor product over F, $A \otimes_F B$, is again an F-CSA. This operation is well defined under the equivalence relation we defined above, so it is a binary operation on $Br(F)$, and creates a group structure on $Br(F)$.

Definition 2.0.3. Given a field F, the Brauer group of F, $Br(F)$, is the set of equivalence classes of F-CSA's. The operation is defined by $[A] * [B] = [A \otimes_F B]$.

The identity element is the equivalence class of the field F, $[F] = \{F, M_2(F), M_3(F), ...\}$. Given an algebra A, its inverse is an F-CSA known as the opposite algebra. This is denoted A^{op}. As an abelian group under addition, it is identical to A, but the multiplication in A^{op} is given by $a * b = ba$, where the right hand side is multiplication in the ring A. The product $A \otimes_F A^{op}$ is isomorphic via the obvious action to the endomorphism ring of A as an

F-vector space:

$$A \otimes_F A^{op} \cong End_F(A) \cong M_{dim_F A}(F),$$

thus $[A] * [A^{op}] = [F]$. The Brauer group is abelian and torsion in all cases ([5] Corollary 4.4.8, p. 99), and we often use additive notation for the operation.

Given a field extension L/F, and an F-CSA A, the tensor product $A \otimes_F L$ is an L-CSA ([5] Lemma 2.2.2, p. 20). In fact, there is a group homomorphism from $Br(F)$ to $Br(L)$ called "restriction of scalars":

$$\begin{aligned} r_{L/F} : Br(F) &\rightarrow Br(L) \\ [A] &\mapsto [A \otimes_F L]. \end{aligned}$$

We are particularly interested in the kernel of this map.

Definition 2.0.4. Given a field extension L/F, the relative Brauer group, $Br(L/F)$, is the kernel of the restriction of scalars homomorphism, $r_{L/F}$:

$$Br(L/F) = \{[A] \mid A \otimes_F L \cong M_s(L) \text{ for some } s\}.$$

We say that L is a *splitting field* for an F-CSA A if $[A] \in Br(L/F)$. Equivalently, we say that A is *split by* L. We also refer to $M_n(F)$ as the *split algebra*.

Theorem 2.0.1. *Given an algebraically closed field, $F = \overline{F}$, every F-CSA is split.*

Proof. The following proof is from [13] Lemma 1.2, p. 9. Given a division algebra D/F, let $\alpha \in D$. Then $F[\alpha]$ forms a commutative subring of D, which must be a domain since D has no zero divisors. It is also finite dimensional over F, since D is, thus it is a field ([8] Lemma 17.1, p. 255). But F is algebraically closed, so $F = F[\alpha]$, so $D = F$. If every division algebra over F is exactly F, then every F-CSA is matrices over F, and thus split. $\square$

We have shown that $Br(\overline{F}) = 0$. This also implies that $Br(\overline{F}/F) = Br(F)$, or that every F-CSA is split by the algebraic closure of F. Thus, given an F-CSA A, $A \otimes_F \overline{F} \cong M_n(\overline{F})$ for some n. Dimension does not change after extending scalars, so $dim_F A = dim_{\overline{F}} M_n(\overline{F}) = n^2$ for some n. So the dimension of every F-CSA is a square.

Definition 2.0.5. Let A be an F-CSA. Define the degree of A, $deg(A)$, to be the square root of the dimension: $deg(A) = \sqrt{dim_F A}$. Let $A \cong M_s(D)$, then the degree of D is called the index of A: $ind(A) = deg(D)$.

It is also the case that $Br(F) = 0$ for every finite field F ([11] Theorem 7.24, p. 104). Thus we may assume F is infinite at certain points.

In the proof of Theorem 2.0.1, we built a subfield of the division algebra D generated by F and an element α. This proof also shows that every element in a division algebra is contained in some subfield of D, and thus in a maximal subfield of D. These maximal subfields actually split D, and D is their union. Splitting fields are very important to the following results, and we gather several basic results from the theory in the next theorem.

Theorem 2.0.2. *Let D/F be a division algebra of degree n.*

1. *Given a subfield* $F \subseteq L \subseteq D$*,* L *is a maximal subfield if and only if* $dim_F L = n$*.*

2. *Each maximal subfield of* D *splits* D*.*

3. *Each splitting field* L/F *of* D *of minimal dimension is isomorphic to a maximal subfield of* D*.*

4. *Of the maximal subfields of* D*, there is always at least one which is separable over* F*.*

5. *Each division algebra* D *is split by a finite, Galois extension* L/F*.*

We would like to look at restricting scalars by one more type of field extension.

Definition 2.0.6. A field extension L/F is called totally transcendental if every element of L not in F is transcendental over F. It is called purely transcendental, or rational, if it is isomorphic to $F(Z_1, ..., Z_r)$, where the Z_i are algebraically independent.

Theorem 2.0.3. *Let* $L = F(Z_1, ..., Z_r)$ *be a rational extension of* F*. Then*

1. $r_{L/F} : Br(F) \to Br(L)$ *is injective, and*

2. *if* D/F *is a division algrebra, then* $D \otimes_F L/L$ *is a division algebra.*

We are interested in subrings of an arbitrary F-CSA that are similar to separable, splitting subfields of a division algebra D/F. These are called separable, splitting subrings of A. The terminology comes from the theory of Azumaya algebras, where the Brauer group $Br(\cdot)$ is extended to arbitrary commutative rings. These separable, splitting subrings are the analogue of separable, splitting subfields, although we will not use them in this capacity.

Definition 2.0.7. Let A be an F-CSA. A separable, splitting subring $L \subseteq A$ is a separable, commutative F-algebra whose dimension over F is equal to the degree of A.

This ring has the property that if we split the algebra A by $\overline{F}$, then we can choose a basis of the matrix algebra $A \otimes_F \overline{F} \cong M_n(\overline{F})$ such that the subring $L \otimes_F \overline{F}$ is the subalgebra of diagonal matrices ([5] Lemma 2.2.9, p. 23).

The Severi-Brauer Variety

To each F-CSA we may associate a collection of projective varieties which are closed subvarieties of Grassmannians.

Definition 2.0.8. Given a vector space V over F of dimension m, for any integer $k = 1, ..., m$, the Grassmannian of k-subspaces in V, $Gr(k, V)$, is the projective variety parameterizing all k-dimensional subspaces $W \subseteq V$.

The Grassmannian can be described as a closed subvariety of projective space via Plücker coordinates. One fact we will use quite frequently is that the Grassmannian is rational.

Definition 2.0.9. A variety X over F is rational if it is birational to affine space $\mathbb{A}_F^{dimX}$. In this case, its function field $F(X)$ is a rational extension of F (see definition 2.0.6). For any rational variety, its F points are dense.

The Grassmannian is also universal, or functorial. We are only interested in its universality with respect to field extensions L/F. Given a variety X over F, and an extension field L/F, the L-points of X, $X(Spec\, L)$, are the scheme morphisms from $Spec\, L \to X$ such that

$$\begin{array}{ccc} Spec\, L & \longrightarrow & X \\ & \searrow \quad \swarrow & \\ & Spec\, F & \end{array}$$

where $Spec\, L \to Spec\, F$ comes from the inclustion $F \subseteq L$. These points biject with the set of L-points of the variety $X \times_F L$, $X \times_F L(Spec\, L)$, where they can be thought of as the set of L-rational points of the variety $X \times_F L$. So we will also refer to these as the L-rational points of X. The Grassmannian is functorial in the sense that $Gr(k, V) \times_F L \cong Gr(k, V \otimes_F L)$. In other words, the L-rational points of $Gr(k, V)$ are the rational points of $Gr(k, V \otimes_F L)$.

Let A be an F-CSA. We will describe the generalized Severi-Brauer varieties first for the split algebra, $A = M_n(F)$. The ring A has no two-sided ideals, but it has a large collection of left and right ideals. Each of these ideals I is principal, generated by a single matrix $a \in I \subseteq A$. The dimension of I over F is $dim_F I = n \cdot rank(a)$, and we say I has rank $rank(a)$. Thus $I \in Gr(n \cdot rank(a), A)$. Given a fixed rank r, we can parameterize the full collection of rank r right ideals (respectively, left ideals) via the generalized Severi-Brauer variety. The conditions for being an ideal are polynomial, thus the collection of ideals forms a variety (see [3] pp. 98-100). This variety can be defined for an arbitrary F-CSA, split or not.

Definition 2.0.10. Let A be an F-CSA of degree n, r an integer between 1 and $n-1$. The generalized Severi-Brauer variety $SB_r(A)$ is the projective subvariety of $Gr(n \cdot r, A)$ parameterizing the collection of rank r, right ideals of A. Similary, ${}_rSB(A)$ is the projective variety of left ideals of rank r in $Gr(n \cdot r, A)$.

As a convention we will discuss the variety of right ideals $SB_r(A)$, although all statements will apply to ${}_rSB(A)$ also, unless otherwise noted. We are particularly interested in the variety of maximal right ideals, $SB_{n-1}(A)$, which we will denote $SB(A)$. We collect some results about these varieties.

Theorem 2.0.4. *Let A and B be F-CSA's.*

1. *$A \cong B$ if and only if $SB(A) \cong SB(B)$.*
2. *$SB(A) \times_F L \cong SB(A \otimes_F L)$.*
3. *If $A \cong M_n(F) \cong End_F(W)$, then $SB_r(A) \cong Gr(r, W)$.*
4. *$SB(A) \cong {}_1SB(A)$ via the map taking a maximal right ideal to its left annihilator.*

Proof. The proof of 1 can be found in [2] p. 196. The proof of 2 can be found in [3] Proposition 1, p. 100.

We will describe the isomorphism in 3. Suppose A is split, $A = End_F(W)$, where $dim_F W = n$. Then there is an isomorphism $SB_r(A) \to Gr(r, W)$ which can be described in one of two ways. Given an ideal $I \in SB_r(A)$, I is principal and so $I = a \cdot A$, where $rank(a) = r$. Given the rank of a, $a(W) \in Gr(r, W)$ is a subspace of dimension r. This map is well-defined and gives the isomorphism. Equivalently, given $I \in SB_r(A)$, let $W' = \cup_{a \in I} a(W)$. This union is a subspace of dimension r, $W' \in Gr(r, W)$. The full proof is given in [3] Corollary 1, p. 102.

We also note that the map $ann : SB(A) \to {}_1SB(A)$, taking a maximal right ideal to its left annihilator, has an analogue for ${}_{n-1}SB(A) \cong SB_1(A)$, although it is not true in general that ${}_1SB(A) \cong SB_1(A)$, or ${}_{n-1}SB(A) \cong SB(A)$. The full proof is given in [13] Lemma 13.6, p. 92. □

A division algebra D/F has no singular elements (except 0), so it can have no ideals, left or right. Thus $SB_r(D)$ has no F-rational points. Given a splitting field L/F, $SB_r(D) \times_F L \cong SB_r(D \otimes_F L) \cong SB_r(End_L(W)) \cong Gr(r, W)$, which is rational and thus has dense L-points. In particular, if L splits A, then $SB(A) \times_F L \cong Gr(n-1, W) \cong \mathbb{P}_L^{n-1}$. Thus Severi-Brauer varieties are equivalently defined as twists of projective space. We collect some results about the situation in the following theorems.

Theorem 2.0.5. *Let A be an F-CSA, then the following are equivalent:*

1. *A is split.*

2. $SB(A) \cong \mathbb{P}_F^{n-1}$.

3. $SB(A)$ *has an* F*-point.*

Proof. The point of interest is in 3 implies 1. Suppose $A \cong M_k(D)$ for some division algebra D with $deg(D) = l$, and $deg(A) = kl = n$. A minimal right ideal for A is a vector space over D of dimension k, thus it has dimension $kl^2 = nl$ over F. Any right ideal is isomorphic to a direct sum of m copies of a minimal right ideal, and must have dimension mnl over F. If A has a right ideal of dimension $(n-1)n$, then $(n-1)n = mnl$ for some m, and thus l divides $n-1$. But $lk = deg(A) = n$, so l divides n and $n-1$, thus $l = 1$ and the algebra is split. □

Theorem 2.0.6. *Let* A *be an* F*-CSA and* L/F *an extension field, then the following are equivalent:*

1. L *splits* A.

2. $SB(A)$ *has an* L*-point.*

3. $SB(A) \times_F L \cong \mathbb{P}_L^{n-1}$.

Theorem 2.0.4 states that two F-CSA's are isomorphic if and only if their Severi-Brauer varieties of maximal right ideals are isomorphic. We are now interested in the case where two Severi-Brauer varieties are birational, thus we are interested in their function fields.

Definition 2.0.11. Let A be an F-CSA, and $SB_r(A)$ a generalized Severi-Brauer variety. Define $F_r(A)$ to be the function field of $SB_r(A)$. In particular, define $F(A) = F_{n-1}(A)$, the function field of $SB(A)$.

The function field $F(A)$ has transcendence degree $n-1$. Since the generic point of $SB(A)$ is an $F(A)$-point, $F(A)$ splits the algebra A. This field is rational if and only if A is split, and something stronger is true.

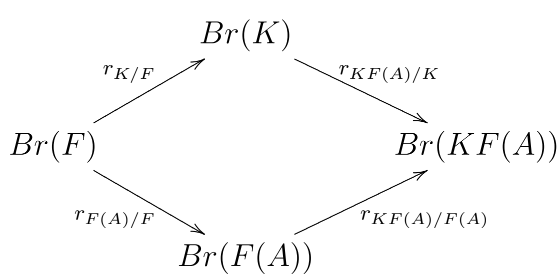
Definition 2.0.12. A field L/F is called a generic splitting field for an F-CSA, A, if

1. L splits A, and

2. K splits A if and only if the join KL is rational over K.

Theorem 2.0.7 (Amitsur)**.** *Let A be an F-CSA, then $F(A)$ is a generic splitting field for A.*

Proof. Since the generic point is an $F(A)$-point of $SB(A)$, $F(A)$ splits A. Let K/F split A. The join $KF(A)$ is the function field of $SB(A) \times_F K \cong \mathbb{P}^{n-1}_K$, which is rational over K. Conversely, if $KF(A)$ is rational over K, then we consider the following commutative diagram:



Since $F(A)$ splits A, $[A]$ is in the kernel of the composition $Br(F) \to Br(F(A)) \to Br(KF(A))$. Thus $[A]$ is in the kernel of the composition $Br(F) \to Br(K) \to Br(KF(A))$. But since $KF(A)$ is rational over K, the map $Br(K) \to Br(KF(A))$ is injective (Theorem 2.0.3). So $[A]$ is in the kernel of $Br(F) \to Br(K)$, thus K splits A. □

Since $F(A)$ splits A, and $Br(F) \to Br(F(A))$ is a homomorphism, $F(A)$ splits all powers of A. That is to say, the cyclic subgroup of the Brauer group generated by $[A]$ is in the kernel of $r_{F(A)/F}$, $\langle [A] \rangle \subseteq Br(F(A)/F)$. We have the following stronger fact.

Theorem 2.0.8 (Amitsur)**.** *For an F-CSA, A, $Br(F(A)/F) = \langle [A] \rangle$. That is, the following sequence is exact:*

$$0 \to \langle [A] \rangle \to Br(F) \to Br(F(A)).$$

The result was proved by Amitsur. The proof can be found in [5] Theorem 5.4.1, p. 125. This quickly implies the following theorem.

Theorem 2.0.9 (Amitsur)**.** *Let A and B be F-CSA's. If $SB(A)$ and $SB(B)$ are birational, then A and B generate the same cyclic subgroup of the Brauer group:*

$$F(A) \cong F(B) \Rightarrow \langle [A] \rangle = \langle [B] \rangle \leq Br(F).$$

An long standing problem in the theory of division algebras is the converse. This is known as Amitsur's conjecture.

Conjecture 2.0.10 (Amitsur's Conjecture). For two F-CSA's A and B, $\langle [A] \rangle = \langle [B] \rangle$ implies $F(A) \cong F(B)$.

We will now give similar statements about the generalized Severi-Brauer variety for arbitrary r, $SB_r(A)$. These results can be found in [3] Proposition 3, p. 103.

Theorem 2.0.11 (Blanchet)**.** *For an F-CSA, A, the following are equivalent:*

1. *$SB_r(A)$ has an L-rational point.*

2. *$ind(A \otimes_F L)$ divides r.*

3. *The join $LF_r(A)$ is rational over L.*

Definition 2.0.13. A field extension K/F is a $\frac{1}{r}$-splitting field for A if and only if $ind(A \otimes_F K)$ divides r.

Definition 2.0.14. A field extension K/F is a generic $\frac{1}{r}$-splitting field for A if

1. K is a $\frac{1}{r}$-splitting field for A, and

2. L is a $\frac{1}{r}$-splitting field for A if and only if the join LK is rational over L.

Theorem 2.0.12 (Blanchet)**.** *Let A be an F-CSA.*

1. *$F_r(A)$ is a generic $\frac{1}{r}$-splitting field for A.*

2. *$Br(F_r(A)/F) = \langle [A]^r \rangle$.*

3. $F_r(A) \cong F_r(B)$ *implies* $\langle [A]^r \rangle = \langle [B]^r \rangle \leq Br(F)$.

All three results can be found in [3]. The first is Theorem 2, p. 103. The second is Theorem 7, p. 115. The third follows easily from the second.

The Norm Hypersurface

Let A be an F-CSA, then there is a map $n_A : A \to F$ called the reduced norm. We will list some of its properties (see [5] Section 2.6).

Theorem 2.0.13. *Let* A *be an* F*-CSA, and* $n_A : A \to F$ *its reduced norm.*

1. n_A *is multiplicative:* $n_A(ab) = n_A(a)n_A(b)$*, and homogeneous of degree* n.
2. *If* A *is split, then* $n_A(a)$ *is the determinant of* a.
3. $a \in A$ *is invertible if and only if* $n_A(a)$ *is invertible.*
4. *If* $L \subseteq A$ *is a maximal subfield, then* $n_A|_L$ *is the usual norm of field extensions:* $n_A|_L = n_{L/K}$.

Given a commutative F-algebra R, the norm extends uniquely to the algebra $A \otimes_F R$, which may no longer be a central, simple algebra. Regardless, this function, $n_{A\otimes R} : A \otimes_F R \to R$, has many of the same properties that n_A does. We will use it briefly in order to define a norm polynomial. Let $m = n^2 = dim_F A$, and choose a basis $\{e_1, ..., e_m\}$ for A over F. Let $R = F[X_1, ..., X_m]$

and let $\gamma = X_1e_1 + \cdots + X_me_m \in A \otimes_F R$ (the tensor is represented by multiplication). Then $n_{A\otimes_F R}(\gamma) = f \in R$ is a polynomial, which we will call a *norm polynomial*. The polynomial f has the property that $n_A(a_1e_1 + \cdots a_me_m) = f(a_1, ..., a_m)$ for all $a_j \in F$. Thus if F is an infinite field, n_A determines the polynomial f uniquely.

Definition 2.0.15. Given an F-CSA, A, with norm polynomial f, define $V(F) \subseteq A$ to be the closed subvariety of A defined by f. This is called the norm hypersurface.

We will develop more results about the norm polynomial and the norm hypersurface in the next chapter.

Chapter 3

The Norm Polynomial

Our goal in this chapter is to review the results from David Saltman's 1978 paper *Norm Polynomials and Algebras*. In particular, we are concerned with those results and arguments which I generalize for this thesis.

The main results we want to prove are

1. The function field of the norm hypersurface, K, is a generic splitting field for the algebra, A.

2. K is purely transcendental over the function field of the Severi-Brauer variety, $F(A)$, of transcendence degree $n^2 - n$.

3. In the proof of 2 is the main idea we want to generalize, a realization of $F(A)$ as the function field of a restricted norm polynomial, via a rational embedding of the Severi-Brauer variety into the norm hypersurface.

4. For two algebras A and A', the function fields K and K' satisfy a variant of Amitsur's conjecture.

Let A be an F-CSA, and let f be a norm polynomial on A. Then $V(f)$ is the norm hypersurface. Let K be the function field of the norm hypersurface. Thus K is the quotient field of $F[X_1, ..., X_m]/(f)$.

We will start with a fact about the norm hypersurface, given in [12] Lemma 1.1, (a), and Lemma 1.2, (a), p. 335.

Lemma 3.0.14 (Saltman)**.** *Let L be a field extension of F.*

(a) $f(x_1, \ldots, x_m)$ is absolutely irreducible.

(b) L splits A if and only if $A \otimes_F L$ contains an element of rank r coprime to n.

(c) The L points of V are dense if and only if L splits A.

Proof.

(a) We see that a norm polynomial is well-defined up to a choice of basis. Thus if f and f' are two norm polynomials, a change of basis for the algebra A will give an isomorphism of $V(f)$ with $V(f')$. Over the algebraic closure $\bar{F}$, there is an isomorphism of V with the zero locus of the determinant polynomial, which is absolutely irreducible ([6] Proposition 12.2, p. 151). Hence f is absolutely irreducible.

(b) Assume $A \otimes_F L \cong M_t(D)$ where D is a division algebra of degree s. Every singular element of $A \otimes_F L$, that is, every L point of V, has rank a multiple of s. If there is an element of rank r coprime to n, then s is coprime to n. But s must divide n, thus $s = 1$ and the algebra is split. Conversely, if the algebra is split, then there are elements of any rank less than n, so in particular there are elements of rank 1, which is coprime to n.

(c) If the L points of V are dense, then they must intersect the dense, open subset V^+ of elements of rank $n-1$. Thus there is an element in $A \otimes_F L$ of rank $n-1$, which is coprime to n, and so L splits A by part (b).

Now suppose L splits A, $A \otimes_F L \cong M_n(L)$. Then the points of $V(L)$ can be identified with the singular matrices in $M_n(L)$. Consider $P = (x_{ij})$ an $n \times n$ matrix with indeterminate entries. Let h_i be the polynomial given by the $(1, i)$ minor of P. Then $det(P) = x_{11}h_1 - x_{12}h_2 + \cdots + (-1)^n x_{1n}h_n$. Let $U \subseteq V$ be the open subset defined by $h_1 \neq 0$. Then

$$L[V] = L[x_{i,j}]/(det(P)) \text{ and}$$

$$L[U] = \big(L[x_{i,j}]/(det(P))\big)[1/h_1] = L[x_{i,j}, 1/h_1]/(det(P)).$$

Now in $L[x_{i,j}, 1/h_1]$,

$$det(P) = 0 \Leftrightarrow x_{1,1} = \frac{x_{1,2}h_2 + \cdots + (-1)^{n+1}x_{1,n}h_n}{h_1}.$$

So

$$L[U] = L[\widehat{x_{1,1}}, x_{1,2}, ..., x_{i,j}, ...][1/h_1]$$

and U is a dense, open subset of $\mathbb{A}_L^{n^2-1}$. Thus the L points of U are dense. Since U is open and nonempty in the irreducible variety V, the L points of V are dense.

□

As noted above, one of our main goals is to prove that K is a generic splitting field of A. Previous definitions of a generic splitting field involved places, so the following theorem from [12] (Theorem 4.1, p. 341) has been modified slightly to emphasize the definition given here. Also, the proof that K splits A has been modified. In a remark during the proof, Saltman claims that the idea behind the proof is that the generic element of V has rank $n-1$, and thus K must split A. His proof, however, relies on a different argument. Here we formalize the idea in his remark and use it as the proof.

Theorem 3.0.15 (Saltman)**.** *The function field of the norm hypersurface, K, is a generic splitting field for A.*

Proof. First we show that K splits A. We will show that the generic element of V has rank $n-1$. To form the generic element, we consider the image of the generic point in $V \times_F K$, where it is a K rational point. Since the generic point is in V^+, the set of elements of rank $n-1$, defined over F, the generic element is in $V^+ \times_F K = (V \times_F K)^+$, and is a K rational element of rank $n-1$ in $V \times_F K$. By Lemma 3.0.14, part (b), K splits A.

Now let L split A. Then $F[V] \otimes_F L \cong L[x_1, ..., x_m]/(g)$, where g is any norm polynomial of $A \otimes_F L$. Since L splits A we may take g to be the determinant polynomial. By the argument in Lemma 3.0.14, part (c), we see that the variety defined by the determinant polynomial is rational, and so its function field is purely transcendental of transcendence degree $m-1$. Here the join LK is the field of fractions of $F[V] \otimes_F L \cong L[x_1, ..., x_m]/(g)$, which is exactly such a function field, so $LK \cong L(y_1, ..., y_{m-1})$.

Conversely, suppose $LK \cong L(y_1, ..., y_{m-1})$. Since K splits A, and LK

is an extension of K, we know that LK splits A. So $[A]$ is in the kernel of $r_{LK/F} : Br(F) \to Br(LK)$. This group homomorphism is also given by the composition

$$r_{LK/F} = r_{LK/L} \circ r_{L/F}.$$

The map $r_{LK/L}$ is injective, since LK is rational over L (Theorem 2.0.3). It follows that $[A]$ must be in the kernel of $r_{L/F}$, and thus L splits A. □

The next theorem shows the relationship between K and $F(A)$, and gives a realization of $F(A)$ as a function field defined by a single relation. We do this by showing the existence of an element $\phi(d)$ in $A \otimes_F K$, and a field of definition, $K_{\phi(d)} \subseteq K$, such that $K_{\phi(d)}$ is defined by a restricted norm polynomial, $K_{\phi(d)}$ is isomorphic to $F(A)$, and K is rational over $K_{\phi(d)}$ with transcendence degree $n^2 - n$.

Fix a maximal, separable subfield, $N \subseteq A$, and a basis of N given by $\{e_1, ..., e_n\}$. Extend this to a basis of A, $\{e_1, ..., e_n, e_{n+1}, ..., e_m\}$. Let W be the span of the elements $e_{n+1}, ..., e_m$, i.e. a complement of N in A. Extending scalars to $K = F(x_1, ..., x_m)$, we define $a = x_1e_1 + \cdots + x_me_m \in A \otimes_F K$, i.e. the generic element of V. We define $M = a \cdot A \otimes_F K$, the right ideal generated by a. As seen above, a has rank $n - 1$, and so M has dimension $n(n - 1)$ over K. In the proof we use the ideal $I = ann(M)$, the left annihilator of M. Equivalently, I can be defined as the left annihilator of a, and M as the right annihilator of I.

When we extend scalars to K, we see that $N \otimes_F K$ is still a field (since

N is algebraic and K is totally transcendental, they are linearly disjoint). Since the only singular element in $N \otimes_F K$ is 0, and M contains only singular elements, we have $(N \otimes_F K) \cap M = \{0\}$. Thus M is a complement to $N \otimes_F K$ in $A \otimes_F K$, and from above we have $W \otimes_F K$ is a complement to $N \otimes_F K$ in $A \otimes_F K$. Since $W \otimes_F K$ and M are both complements to $N \otimes_F K$, there is a natural isomorphism $W \otimes_F K \to M$, by embedding W into $A \otimes_F K$ and projecting onto M. We will call this map ϕ, but following [12], we will define it more explicitly via intersecting subspaces. What's important from the above remark about ϕ is that this intersection map is clearly a map of varieties, in fact it is linear.

To define ϕ explicitly, take $d \in W \otimes_F K$, nonzero, and consider the space $N \otimes_F K + K \cdot d$. This is a vector space of dimension $n+1$ over K, and M is a vector space of dimension $n^2 - n$. Since they live in $A \otimes_F K$ of dimension n^2, we know they must intersect nontrivially. Since $N \otimes_F K \cap M = (0)$, we see that their intersection must be a line. This one-dimensional subspace contains a unique element of the form $c_1 e_1 + \cdots + c_n e_n + d$. To see this, take any nonzero element in the line. The coefficient of d must be nonzero, otherwise this would be an element of $N \otimes_F K$. Then scaling by the coefficient of d we get the element we want. We call this element $\phi(d)$. Alternatively, consider the affine subspace $N \otimes_F K + d$, then its intersection with M is a singleton, and $(N \otimes_F K + d) \cap M = \{\phi(d)\}$. The ideal M, being comprised of singular elements, naturally sits in $V \times_F K$, and we think of ϕ as being a map of varieties $\phi : W \to V$ defined over K.

Consider $\phi^{-1}(V^+)$, the preimage of all singular elements of rank $n-1$. We want to show it is nonempty. Let $d = x_{n+1}e_{n+1} + \cdots + x_m e_m$. Note that $x_1e_1+\cdots+x_ne_n \in N\otimes_F K$, so $a = x_1e_1+\cdots+x_ne_n+d \in N\otimes_F K+d$. Of course $a \in M$, so $a \in (N \otimes_F K + d) \cap M$, thus $a = \phi(x_{n+1}e_{n+1}+\cdots+x_me_m)$. Also, a has rank $n-1$, so $a \in V^+$, thus $d \in \phi^{-1}(V^+)$. So $\phi^{-1}(V^+)$ is an open, nonempty subset of $W \otimes_F K$. Since W is rational, its F points are dense, and there must exist a $d \in W$ defined over F such that $\phi(d)$ has rank $n-1$. After I state the theorem, I will make a remark about another condition to put on d, but this will mostly be quoted and we can take this $\phi(d)$ to have the properties we need. The crucial point here is that $\phi(d)$ can be defined by only n transcendental elements in K. Namely, we again write $\phi(d) = c_1e_1 + \cdots c_ne_n + d$, then $\phi(d)$ will be defined over the field $K_{\phi(d)} = K(c_1, ..., c_n) \subseteq K$.

Finally, let's write $d = f_{n+1}e_{n+1} + \cdots + f_me_m$, $f_i \in F$, and define $g(X_1, ..., X_n) = f(X_1, ..., X_n, f_{n+1}, ..., f_m)$. Here f is the norm polynomial given by the basis $\{e_1, ..., e_m\}$, and g is the norm polynomial restricted to the affine variety $N + d$ defined over F.

Theorem 3.0.16 (Saltman)**.** *From the situation above*

(a) $K_{\phi(d)}$ *is isomorphic to* $F(A)$*.*

(b) K *is rational over* $K_{\phi(d)}$ *of transcendence degree* $n^2 - n$*.*

(c) $g(X_1, ..., X_n)$ *is absolutely irreducible and* $K_{\phi(d)}$ *is isomorphic to the field of fractions of the affine ring defined over* F *by* g*.*

Proof.

(a) Here Saltman quotes a proof by Amitsur, and gives a description of the Severi-Brauer variety with no reference. We will simply quote Saltman's result, [12] Theorem 4.2, part (c), p. 343.

(b) Consider $A \otimes_F K_{\phi(d)}$ as a subring of $A \otimes_F K$. Since $\phi(d) \in A \otimes_F K_{\phi(d)}$ and $\phi(d)$ has rank $n-1$ by construction, $K_{\phi(d)}$ splits A by Lemma 3.0.14 (b). So $A \otimes_F K_{\phi(d)}$ is a matrix ring, and we want to use this fact to show that K can be obtained from $K_{\phi(d)}$ by adjoining $n^2 - n$ elements. To do this we consider the ideal I in $A \otimes_F K$, defined to be the left annihilator of M.

Since M is a maximal right ideal, I will be a minimal left ideal. Since M is generated by a, I can also be defined as the left annihilator of a, $I = ann_{A \otimes_F K}(a)$. In fact, any rank $n-1$ element $m \in M$ will generate M as a maximal right ideal, and I can be defined as the left annihilator of m. In particular, $\phi(d) \in M$ has rank $n-1$, so I is also the left annihilator of $\phi(d)$, $I = ann_{A \otimes_F K}(\phi(d))$. Consider $I' = I \cap \left(A \otimes_F K_{\phi(d)}\right)$. Since $\phi(d) \in A \otimes_F K_{\phi(d)}$, $I' = ann_{A \otimes_F K_{\phi(d)}}(\phi(d))$, and I' is a minimal left ideal of $A \otimes_F K_{\phi(d)}$. As such, we can choose matrix units $e_{i,j}$ for $A \otimes_F K_{\phi(d)}$ such that I' is spanned by $e_{1,1}, ..., e_{n,1}$. Using these matrix units and extending scalars to K, we have $I \cdot a = 0$ implies a is a matrix with zero first row. In other words, $a = \sum d_{i,j} e_{i,j}$, where $d_{i,j} \in K$ and $d_{1,j} = 0$ for all j.

Let $K' = K_{\phi(d)}(d_{i.j}) \subseteq K$, generated over $K_{\phi(d)}$ by $n^2 - n$ elements. By construction of the $d_{i,j}$, we know $a \in AK'$. Since the basis e_i is defined over F, we can express $a = \sum_i k_i e_i$ with $k_i \in K'$, but over K we must have $k_i = x_i$, so $x_i \in K'$ for all i. Thus $K' = K$. Now we compare transcendence degrees. With $K_{\phi(d)} = F(c_1, ..., c_n)$, we have $t.d._F(K_{\phi(d)}) \leq n$. Since

$$\phi(d) = c_1e_1 + \cdots + c_ne_n + f_{n+1}e_{n+1} + \cdots + f_me_m$$

has rank $n - 1$, we know $f(\phi(d)) = 0$, thus

$$f(c_1, ..., c_n, f_{n+1}, ..., f_m) = g(c_1, ..., c_n) = 0$$

which implies $t.d._F(K_{\phi(d)}) \leq n - 1$. Since we have generated K over $K_{\phi(d)}$ by $n^2 - n$ elements $d_{i,j}$, we have $t.d._{K_{\phi(d)}}(K) \leq n^2 - n$. Finally,

$$n^2-1 = t.d._F(K) = t.d._F(K_{\phi(d)})+t.d._{K_{\phi(d)}}K \leq (n-1)+(n^2-n) = n^2-1$$

shows that the above inequalities are actually equalities. Thus $t.d._{K_{\phi(d)}}(K) = n^2 - n$, the $d_{i,j}$ must form a transcendence base, and K is rational over $K_{\phi(d)}$.

(c) Recall that f is a homogeneous polynomial of degree n. Write $f = f_1+f_2$, where f_1 is the sum of the monomials involving only the variables $X_1, ..., X_n$, and f_2 is the sum of the monomials involving at least one of $X_{n+1}, ..., X_m$. We also write $g = g_n + \cdots + g_0$ where g_i has degree i. The polynomial g is defined by evaluating f at $(X_1, ..., X_n, f_{n+1}, ..., f_m)$, which will cause the degree of every monomial in f_2 to decrease by at least one, and leave f_1 unperturbed. Thus $f_2(X_1, ..., X_n, f_{n+1}, ..., f_m) = g_{n-1} + \cdots + g_0$ and $f_1 = g_n$. Note that evaluating f at $(X_1, ..., X_n, 0, ..., 0)$ would kill f_2, thus $f(X_1, ..., X_n, 0, ..., 0) = f_1 = g_n$, and this is the norm polynomial restricted to the subfield $N = span_F\{e_1, ..., e_n\} \subseteq A$. We know that f restricted to a maximal subfield of the algebra is a norm polynomial for the field extension, thus g_n is a norm polynomial for $n_{N/F}$, which is irreducible by [4]. Thus g itself is irreducible.

Let $J \subseteq F[X_1, ..., X_n]$ be the kernel of $F[X_1, ..., X_n] \to F(c_1, ..., c_n) = K_{\phi(d)}$, $X_i \mapsto c_i$. Then J is prime, and $F(c_1, ..., c_n)$ is the function field of the variety defined by J. Since $F(c_1, ..., c_n)$ has transcendence degree $n - 1$, the variety has dimension $n - 1$, and J must be height 1. As $g(c_1, ..., c_n) = f(c_1, ..., c_n, f_{n+1}, ..., f_m) = f(\phi(d)) = 0$, we have $g \in J$ irreducible, thus $J = (g)$. So $K_{\phi(d)}$ is isomorphic to the field of fractions of the affine ring defined over F by g.

To show g is absolutely irreducible, we use the fact that the function field defined by g is isomorphic to $F(A)$ (by part (a)). Then we have an exact sequence

$$0 \to (g) \to F[X_1, ..., X_n] \to F(A).$$

We will extend scalars to $\bar{F}$, an algebraic closure of F. From [13] Lemma 13.3, p. 90, we have that $SB(A) \times_F \bar{F} \cong SB(A \otimes_F \bar{F})$, and thus $F(A) \otimes_F$

$\bar{F} \cong \bar{F}(A \otimes_F \bar{F})$, a field. This gives the exact sequence

$$0 \to \bar{F}[X_1, ..., X_n] \cdot g \to \bar{F}[X_1, ..., X_n] \to \bar{F}(A \otimes_F \bar{F})$$

and thus the ideal generated by g over the algebraic closure is prime, and g remains irreducible. Thus g is absolutely irreducible.

□

Using these results, we see that K satisfies a variant of Amitsur's conjecture ([12] Theorem 4.6, p. 344).

Theorem 3.0.17 (Saltman)**.** *Two F-CSA's, A and A', generate the same cyclic subgroup of the Brauer group, $Br(F)$, if and only if their norm hypersurfaces $V(A)$ and $V(A')$ are birational. That is,*

$$K \cong K' \Leftrightarrow \langle [A] \rangle = \langle [A'] \rangle \leq Br(F).$$

Proof. Since K is rational over $F(A)$, $Br(K/F) = Br(F(A)/F)$ (Theorem 2.0.3). Thus $Br(K/F) = \langle [A] \rangle$, and $Br(K'/F) = \langle [A'] \rangle$. Since $K \cong K'$, $Br(K/F) = Br(K'/F)$, and so $\langle [A] \rangle = \langle [A'] \rangle$.

If $\langle [A] \rangle = \langle [A'] \rangle$, then $[A']$ is in the kernel of $Br(F) \to Br(F(A))$, and so $F(A)$ splits $[A']$. Since $F(A')$ is a generic splitting field for A', and $F(A)$ splits A', we have that their join $F(A)F(A')$ is rational over $F(A)$ of transcendence degree $n-1$. Similarly, $F(A)F(A')$ is rational over $F(A')$ of the same transcendence degree. Since K is rational over $F(A)$ of transcendence degree $n^2 - n$, it is rational over $F(A)F(A')$ of transcendence degree $n^2 - n - (n-1) = (n-1)^2$. Similarly for K'. Thus K and K' are isomorphic. □

Chapter 4

Rational Embeddings

Let A be an F-CSA, with norm polynomial f and norm hypersurface $V = V(f) \hookrightarrow A$. Let K be the function field of V, and $V^+ \hookrightarrow V$ the dense, open subset of elements of rank $n-1$.

Although it is not obvious, there is a geometric interpretation of the result form Chapter 3 realizing K as a rational extension of $F(A)$, via a rational embedding of $SB(A)$ into K.

Definition 4.0.16. A rational embedding $X \dashrightarrow Y$ is a rational morphism such that the restriction to a dense, open subset $U \subseteq X$ yields an immersion $U \to Y$. A morphism $U \to Y$ is an immersion if it gives an isomorphism of U with an open subscheme of a closed subscheme of Y.

First, define $\Phi : V^+ \to SB(A)$ to be the morphism taking an element of rank $n-1$ to the maximal, right ideal it generates. This map is surjective and defined over F, and gives an embedding $F(A) \hookrightarrow K$ ([13] p. 100). Recall that the generic, maximal right ideal is the $F(A)$ rational point of $SB(A) \times_F F(A)$ corresponding to the generic point $Spec\, F(A) \to SB(A)$. This gives a unique maximal, right ideal $M \subseteq A \otimes_F F(A)$, called the generic maximal right ideal. As in Chapter 3, let $N \subseteq A$ be a maximal, separable subfield, and $d \in A$ such that $N+d$ intersects the maximal right ideal M in a unique point of rank $n-1$ (over $F(A)$). Then there is a map $\Psi : SB(A) \dashrightarrow V^+$ which takes almost every maximal, right ideal to its unique intersection with $N+d$, which is a point of rank $n-1$ in V^+. This rational section of Φ gives a rational embedding of $SB(A)$ into V, and expresses the function field $F(A)$ as the function field defined by $f|_{N+d} = p_{N,d}$. That is, $F(A)$ is isomorphic to the quotient field of $F[X_1, ..., X_n]/(p_{N,d})$. Our goal is to generalize this embedding to almost every n-dimensional subspace $N \subseteq A$, not just maximal, separable subfields, and to arbitrary F-CSA's. This gives a larger collection of rational embeddings $SB(A) \dashrightarrow V$. The following result was found simultaneously by Eli Matzri.

Theorem 4.0.18. *Let A be an F-CSA of degree n. For almost every subspace of A of dimension n (a. e. $N \in Gr(n, A)$), there exists a dense subset $U_N \subset A$ such that for all $d \in U_N$, the Severi-Brauer variety of A, $SB(A)$, is birational to a reduced, irreducible component of $V(p_{N,d}) \hookrightarrow N + d$.*

We will use several lemmas in proving this theorem.

Lemma 4.0.19. *For almost every N in $Gr(n, A)$, and for every $d \in A$, there is a rational map*

$$\psi_{N,d} : SB(A) \dashrightarrow (N + d) \cap V.$$

Proof. Let M be the generic, maximal right ideal. To be more explicit, consider the generic point of $SB(A)$. This is an $F(A)$ rational point. If I fiber A up to $F(A)$, this point picks out an explicit ideal M in $A \otimes_F F(A) = A^{F(A)}$. There is a dense open subset of $Gr(n, A^{F(A)}) = Gr(n, A) \times_F F(A)$ of n dimensional subspaces which intersect trivially with M.

To see that it is open, choose a basis of M, extend it to a basis of $A^{F(A)}$, and call this basis $\mathscr{B}$. Express an arbitrary basis of an n dimensional space in the basis $\mathscr{B}$, and write them as the rows of an $n \times n^2$ matrix. Trivial intersection with M is the non vanishing of the right most $n \times n$ minor of this matrix. To see that it is dense, we must show that it has a point. Choose a separable, splitting subring of A, L/F, which must have dimension n, and consider its image $L \otimes_F F(A)$ in $Gr(n, A^{F(A)})$. This subring has trivial intersection with M ([13] p. 100).

Now the F points of $Gr(n, A)$ are dense ($Gr(n, A)$ is rational), so there is a dense subset of n dimensional subspaces that intersect M trivially over $F(A)$, i.e. there is a dense subset of n dimensional subspaces of A, such that for each N in the subset, $N \otimes_F F(A) \cap M = (0)$. In fact, L above is one of these points.

From now on, the subspaces N come from this dense subset of $Gr(n, A)$. Given an N, choose $d \in A \backslash N$. Choose a basis of N, $\{\eta_1, ..., \eta_n\}$, add d, and extend this to a basis of A, $\mathscr{C} = \{\eta_1, ..., \eta_n, d, \eta_{n+2}, ..., \eta_{n^2}\}$. Counting dimensions, and using the fact that $N^{F(A)} \cap M = (0)$, we see that M is a complement of $N^{F(A)}$. The subspace generated over $F(A)$ by N and d will then intersect M in a line. For $d \notin N$, the coefficient of d must be nonzero, and by scaling we get a unique element $(N^{F(A)} + d) \cap M = \{m_{N,d}\}$. Note that for $d \in N$ we still get a unique element $(N^{F(A)} + d) \cap M = (0)$.

Fix such an N and d. We will first define the rational map

$$\begin{array}{rcl} Gr(n^2-n, A) & \dashrightarrow & N+d \\ H & \mapsto & H \cap (N+d) \end{array}$$

whose domain of definition as an F morphism is the open set $U = \{H | H \cap N = (0)\}$. To define the scheme theoretic domain of definition for this morphism, cover U by open affine subschemes U_i and identify U_i with some affine scheme $Spec\, A_i$. For $p \in U_i$, define $\kappa(p)$ to be its residue field, that is, $p \subseteq A_i$ is prime and $\kappa(p)$ is the field of fractions of A_i/p. This prime can be identified uniquely with a $\kappa(p)$-rational point of $Gr(n^2-n, A) \times_F \kappa(p) \cong Gr(n^2-n, A \otimes_F \kappa(p))$, that is to say, it is identified uniquely with a subspace H_p of $A \otimes_F \kappa(p)$ of dimension n^2-n. Now $N \otimes_F \kappa(p)$ is a subspace of $A \otimes_F \kappa(p)$, and the point p is in the domain of definition of the intersection map if and only if $H_p \cap N \otimes_F \kappa(p) = (0)$. Thus

$$U = \bigcup_i \{p \in U_i \,|\, H_p \cap N \otimes_F \kappa(p) = (0) \subseteq A \otimes_F \kappa(p)\}.$$

Notice that if η is the generic point of $SB(A) \hookrightarrow Gr(n^2-n, A)$, then $M = H_\eta$, $F(A) = \kappa(\eta)$, and $M \cap N \otimes_F F(A) = (0)$ implies that $\eta \in U$. So the morphism is defined on a dense open subset of $SB(A)$ and we call the restriction to this open subset the rational map $\Psi_{N,d}$.

The rational map is given by

$$\begin{array}{rcl} \Psi_{N,d} : SB(A) & \dashrightarrow & (N+d) \cap V \\ I & \mapsto & (N+d) \cap I. \end{array}$$

For $d \in N$, the maps can be defined to include $\Psi_{N,d} = 0$, the map that takes all ideals to $0 \in V$. If you like, you can define it only on the dense open subset of ideals that intersect N trivially. □

Lemma 4.0.20. *For almost every N in $Gr(n, A)$, there is a dense subset of $d \in A$ such that $(N^{F(A)} + d) \cap M$ is a singleton of rank $n - 1$.*

Proof. Working over $F(A)$ again, take $d \in A^{F(A)} \backslash N^{F(A)}$. As above, there will be a unique element $m_{N,d}$ which is the intersection of $N^{F(A)} + d$ with M. This gives a map $A^{F(A)} \backslash N^{F(A)} \to M$, defined over $F(A)$. In fact it is just the map

$$\Psi_{N,(\,\cdot\,)}(M) : A^{F(A)} \backslash N^{F(A)} \to M,$$

which we may also denote $m_{N,(\,\cdot\,)}$. We can easily extend it to include $d \in N^{F(A)}$ as above by defining $\Psi_{N,d}(M) = 0$, which is still the unique intersection of $N^{F(A)} + d$ with M. This map,

$$m_{N,(\,\cdot\,)} : A^{F(A)} \to M,$$

is a morphism, and in fact it is linear. If $d, c \in A^{F(A)}$, and $\alpha, \beta \in F(A)$, let $m_{N,d} = n_d + d$ and $m_{N,c} = n_c + c$. Then

$$\alpha m_{N,d} + \beta m_{N,c} = (\alpha n_d + \beta n_c) + (\alpha d + \beta c) \in (N^{F(A)} + (\alpha d + \beta c)) \cap M$$

which is a singleton, and so must be $m_{N,\alpha d + \beta c}$.

If I choose any element $d \in M$, then $m_{N,d} = d$. In particular we can see our map is onto. The generators of M form a dense open subset of M. In fact it is the intersection of $M \subset V$ with the dense open subset $V^+ \subset V$ of elements of rank $n-1$, which we will denote M^+. The preimage of this open subset is a dense open subset of $A^{F(A)}$. Since $A^{F(A)}$ is an affine space, its F points are dense, so there is a dense subset of $d \in A$ such that $(N^{F(A)}+d) \cap M$ is a point of rank $n-1$.

□

proof of Theorem 4.0.18. Let N and d be as in Lemma 4.0.20. We have $p_{N,d} = f|_{N+d}$ and $V(p_{N+d}) \hookrightarrow N+d$. Then we see that $V^+ \cap (N+d)$ is an open, nonempty subvariety of $V(p_{N,d})$. Suppose $p_{N,d}$ factors into a product $\prod_i q_i^{j_i}$, where the q_i are distinct, irreducible polynomials $(N+d \cong \mathbb{A}^n)$. Then $V(q_i) \hookrightarrow V(p_{N,d})$ are the reduced, irreducible components of $V(p_{N,d})$. Let $U \subseteq SB(A)$ be the domain of definition of $\Psi_{N,d}$. Then U is open, reduced and irreducible, so $\Psi_{N,d}$ will factor through $V(q_i)$ for some i ([7] p. 79, ex. 2.3 c). Since the image of M, $m_{N,d} \in V(q_i)$ has rank $n-1$, $V^+ \cap V(q_i)$ is a dense, open subvariety of $V(q_i)$. Where $\Psi_{N,d}$ is defined, it maps an ideal to an element in the ideal. Since $\Psi_{N,d}$ maps the generic point to an element of rank $n-1$, we know that for a dense, open subset of these ideals, $\Psi_{N,d}$ actually maps the ideal to a generator of the ideal, an element in the ideal of rank $n-1$. We restrict U further to be an open, affine subvariety of $SB(A)$ such that each ideal I in U is mapped to a generator of I. If we restrict Φ to $V^+ \cap V(q_i)$, and $\Psi_{N,d}$ to U, we see that $\Phi \circ \Psi_{N,d} = id_U$. This gives an immersion of U into $V^+ \cap V(q_i)$, which by dimension must be dominant. Thus U and $V(q_i)$ are birational.

$\square$

Theorem 4.0.21. *Let A be a division algebra. If $N \subseteq A$ is an n-dimensional subspace, $d \in A$, and $(N+d) \cap V^+ \neq \emptyset$, then $V \cap (N+d) = V(f|_{N+d}) \subseteq A$ is an irreducible variety.*

Proof. Let L_j be the degree 1 affine polynomials that define $N+d$ as a closed subvariety of A. Then

$$N + d = Spec\left(\frac{F[X_1, ..., X_{n^2}]}{(L_j)}\right) \cong \mathbb{A}^n_F = Spec\, F[Y_1, ..., Y_n]$$

gives a homomorphism $\mu : F[X_1, ..., X_{n^2}] \to F[Y_1, ..., Y_n]$ which does not increase the degree of polynomials: $deg(\mu(p)) \leq deg(p)$. The ring homomorphism takes f to $f|N + d = g \in F[Y_1, ..., Y_n]$, and thus $deg(g) \leq deg(f) = n$. We also have

$$V \cap (N + d) \cong Spec\left(\frac{F[Y_1, ..., Y_n]}{(g)}\right).$$

Assume g factors, $g = \prod_i g_i$ where the g_i are irreducible, but not necessarily distinct. Then $deg(g_i) = r_i < n$. Since $V^+ \cap V(g)$ is nonempty, $V^+ \cap V(g_i)$ is nonempty for some i. Since g_i is irreducible, $V(g_i)$ is irreducible, and $V^+ \cap V(g_i)$ must contain the generic point, the prime ideal (g_i). Thus this point represents an element of rank $n-1$, which is rational over the residue field $\kappa((g_i)) \cong F[Y_1, ..., Y_n]/(g_i)$. So (g_i) yields a unique element in $A \otimes_F \kappa((g_i))$ of rank $n-1$, and $\kappa((g_i))$ splits A.

Now

$$\kappa((g_1)) = frac\left(\frac{F[\mathbb{A}^n]}{(g_1)}\right) = F(x_1, ..., x_n).$$

The hypersurface has dimension $n-1$, so $F(x_1, ..., x_n)$ has transcendence degree $n-1$. There is a subset, S, of the generators $\{x_1, ..., x_n\}$ which forms a transcendence basis for $F(x_1, ..., x_n)$ over F. Without loss of generality, suppose $S = \{x_2, ..., x_n\}$, and let $K = F(x_2, ..., x_n)$. Then $g \in K[X_1]$ has degree less than or equal to r in X_1, $F(x_1, ..., x_n) \cong K[X_1]/(g)$, and so $[F(x_1, ..., x_n) : K] \leq r < n$. We have

$$r_{F(x_1,...,x_n)/F}(A) = r_{F(x_1,...,x_n)/K}(r_{K/F}(A))$$

but K being rational implies $A \otimes_F K$ is a division algebra of index n, which is split by $F(x_1, ..., x_n)$, a field extension of degree strictly less than n. This is a contradiction. □

Corollary 4.0.22. *Let A/F be a division algebra of index n. For almost every subspace of A of dimension n (a. e. $N \in Gr(n, A)$), there exists a dense subset $U_N \subset A$ such that for all $d \in U_N$, the Severi-Brauer variety of A, $SB(A)$, is birational to $V(p_{N,d}) \hookrightarrow N + d$.*

Proof. This is exactly the statement of Theorem 4.0.18, with the addition that $p_{N,d}$ is irreducible, which follows from Theorem 4.0.21. □

Chapter 5

Tautological Bundle

We will need to apply techniques from the theory of Galois descent for the main theorems in this chapter. Our main reference for this material is [10]. Fix L/F to be a finite, Galois extension with Galois group G. Galois descent gives an equivalence of objects, both algebraic and geometric, defined over F, and objects defined over K together with a twisted G action.

Definition 5.0.17. We say that G acts on a vector space V (respectively, algebra) over L via a twisted action if each $\sigma \in G$ acts σ-linearly. That is to say for $v \in V$, $\alpha \in K$, $\sigma(\alpha v) = \sigma(\alpha)\sigma(v)$.

There is an equivalence of categories of F vector spaces (respectively, algebras) and K vector spaces with a twisted G action via the map that takes an K vector space with twisted G action, V, to the F subspace of invariant, V^G, and the inverse map taking an F vector space, W, to the K vector space, $W \otimes_F L$, where G acts on the second factor of the tensor product. There is an equivalence of homorphisms of F vector spaces $W \to W'$ and G invariant homomorphisms of K vector space $V \to V'$ with twisted G actions, again by taking invariants and extending scalars.

In the case of F-CSA's, each F-CSA, A, which is split by the Galois extension L/F has a representative in its Brauer class, $[A]$, of degree $[L : F]$, and containing L as a maximal subfield. We will denote $Az_n^{L/F}$ to be the isomorphism classes of F-CSA's which have degree n and are split by L. There is a bijection of pointed sets between $Az_n^{L/F}$ and the first cohomology set $H^1(G, PGL_n(L))$ as follows: let $G \rightarrow PGL_n(L)$, $\sigma \rightarrow a_\sigma$ be a 1-cocycle. Since the automorphism group of the algebra $M_n(L)$ is $PGL_n(L)$, this cocycle induces a twisted G action. First, for $T \in M_n(L)$ a matrix, let $\sigma(T)$ denote the usual G action on the scalar entries. Then the twisted G action on $M_n(L)$ is given by $\sigma \cdot T = a_\sigma(\sigma(T))$. Occasionally we will denote $M_n(L)$ with the twisted G action coming from the cocycle a as ${}_aM_n(L)$.

Given such an action, the algebra of invariants, $A = ({}_aM_n(L))^G$ is an F-CSA, such that $A \otimes_F L$ is isomorphic to ${}_aM_n(L)$ via a G invariant isomorphism. Two F-CSA's arising from equivalent cocycles will be isomorphic (over F), and every F-CSA arises from such an action. Often we will want to view the action explicitly, by taking a lift of a_σ, $C_\sigma \in GL_n(L)$. Then the twisted G action is given by $\sigma \cdot T = C_\sigma(\sigma(T))C_\sigma^{-1}$.

A similar construction works for quasi-projective varieties. Given a quasi-projective variety Y over L, we say Y has a twisted G action if G acts on Y via automorphisms T_σ for $\sigma \in G$ such that

$$\begin{array}{ccc} Y & \xrightarrow{T_\sigma} & Y \\ \downarrow & & \downarrow \\ Spec\, L & \xrightarrow{S_\sigma} & Spec\, L \end{array}$$

is commutative, where S_σ is the automorphism induced by σ^{-1} on L. Then there exists a scheme X over F, unique up to isomorphism, such that $X \times_F L \cong Y$ via a G invariant isomorphism. Constructing X involves covering Y by affine, G invariant subvarieties, constructing the associated F varieties for each of these, and then gluing. For an affine variety, the twisted G action on Y yields a twisted G action on the F-algebra $\mathscr{O}_Y(Y)$. Then we apply the algebraic version of Galois descent given above to this ring, that is, we take the ring of invariants. This ring gives the F variety X. The construction extends to G invariant morphisms of quasi-projective L varieties.

If Y is $\mathbb{P}^{n-1}$, then the automorphism group of Y is $PGL_n(L)$, and we have that the isomorphism classes of varieties X over F given by twisted G actions on Y biject with the cohomology set $H^1(G, PGL_n(L))$. This was originally the connection between varieties X over F such that $X \times_F L \cong \mathbb{P}_L^{n-1}$ and F-CSA's of degree n split by L.

Definition 5.0.18. By a *twist* of a vector space over K (respectively, algebra, variety, morphism, or commutative diagram), we will mean a vector space over F (respectively, algebra, variety, morphism, or commutative diagram) given by Galois descent. Twists of vector spaces, V, (respectively, algebras and homomorphisms) will be denoted V^G. Twists of varieties, Y, (respectively, morphisms and commutative diagrams) will be denoted Y_G.

Lemma 5.0.23. *A twist of a closed immersion of quasiprojective varieties is a closed immersion.*

Proof. Let $X \hookrightarrow Y$ be a G equivariant closed immersion. Closed immersions are affine, and the construction via Galois descent is local, so we may assume X and Y are affine. Then $X = Spec\, B$, $Y = Spec\, A$, and the closed immersion is a surjective, G invariant ring homomorphism $\phi : A \to B$. Replace B with A/I. Then we have an exact sequence

$$0 \longrightarrow I \longrightarrow A \xrightarrow{\phi} A/I \longrightarrow 0 \tag{5.1}$$

and saying the closed immersion is G invariant is equivalent to saying I is a G invariant ideal. Via Galois descent we take the G invariants of the above sequence to get a sequence

$$0 \longrightarrow I^G \longrightarrow A^G \xrightarrow{\phi^G} (A/I)^G \longrightarrow 0 \tag{5.2}$$

such that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I^G \otimes_F K & \longrightarrow & A^G \otimes_F K & \xrightarrow{\phi^G \otimes id_K} & (A/I)^G \otimes_F K & \longrightarrow & 0 \\ & & \downarrow \wr & & \downarrow \wr & & \downarrow \wr & & \\ 0 & \longrightarrow & I & \longrightarrow & A & \xrightarrow{\phi} & A/I & \longrightarrow & 0 \end{array}$$

is an isomorphism of short exact sequences. Since sequence 5.1 is exact, and K/F is faithfully flat, sequence 5.2 is exact, and ϕ^G is a surjection. Thus the twist is a closed immersion. $\square$

Lemma 5.0.24. *A twist of a surjection of quasiprojective varieties is a surjection.*

Proof. Given a G equivariant surjection $X \to Y$, with twist $X_G \to Y_G$, Galois descent yields a commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X_G & \longrightarrow & Y_G \end{array}$$

and the surjectivity of $X_G \to Y_G$ will follow from the surjectivity of $Y \to Y_G$, which we prove now. The construction of Y_G from Y via descent is performed on an affine cover of Y, in such a way that $Y \to Y_G$ is affine. Thus we may assume that Y and Y_G are affine, and we have $Y = Spec\,A$, $Y_G = Spec\,A^G$, and $Y \to Y_G$ is the natural inclusion $\phi : A^G \to A$ with $\phi \otimes id_K : A^G \otimes_F K \tilde{\to} A$. Since K is finite dimensional over F, A is module finite over A^G, thus the map $Y \to Y_G$ is finite, and thus surjective. □

Lemma 5.0.25. *Given a twisted G action on X over $Spec\,K$, and the descent variety X_G over $Spec\,F$, the K-rational, G fixed points of X biject with the F-rational points of X_G under the natural map $X \to X_G$.*

Proof. Since $X \to X_G$ is surjective and affine, the question becomes local and we may assume both are affine. Thus $X = Spec\,R \to X_G = Spec\,R^G$ is given by the inclusion map $R^G \hookrightarrow R$. An F-rational point of X_G is a maximal ideal $\mathscr{M} \subseteq R^G$ with the following short exact diagram:

$$0 \to \mathscr{M} \to R^G \to F \to 0.$$

A K-rational point of X is a maximal ideal $M \subseteq R$ with the following short exact sequence:

$$0 \to M \to R \to K \to 0,$$

and being G invariant means M is a G invariant ideal. We apply Galois descent from to get a map $M \mapsto \mathscr{M} = M^G$ and $\mathscr{M} \mapsto \mathscr{M} \otimes_F L = M$, and it is clear that these operations are inverse to each other. Thus the sets of ideals biject. $\square$

Earlier we described $SB_r(A)$ via its embedding $SB_r(A) \hookrightarrow Gr(nr, A)$. Consider this embedding for the split algebra $M_n(K)$. The G action on $M_n(K)$ yielding A induces a G action on $Gr(nr, M_n(K))$. Explicitly, if a_σ lifts to $C_\sigma \in GL_n(K)$ such that the twisted action on $M_n(K)$ is $(\sigma, M) \mapsto C_\sigma(\sigma(M))C_\sigma^{-1}$, then the action on any subspace W of $M_n(K)$ is $(\sigma, W) \mapsto C_\sigma(\sigma(W))C_\sigma^{-1}$. This action preserves the dimension of the subspace, and induces an action on every Grassmannian. Since these are algebra automorphisms, the action will preserve ideals, both left and right, of any rank, thus the embedding

$$SB_r(M_n(K)) \hookrightarrow Gr(nr, M_n(K))$$

is a G invariant, closed immersion of quasiprojective varieties, whose twist is

$$SB_r(A) \hookrightarrow Gr(nr, A),$$

the usual embedding of the generalized Severi-Brauer variety into projective space.

If we consider the algebra $M_n(K)$ as a variety, namely affine n^2 space over K, then the G action on $M_n(K)$ gives a G action on the variety, and descent will yield a variety which must be isomorphic to affine n^2 space over F. This variety is not given by looking at the invariants, but by looking at the induced action on the affine ring and taking the ring of invariants. Never the less, by Lemma 5.0.25, we have a bijection between the G invariant, K-rational points of $M_n(K)$ and the F-rational points of the twist, $\mathbb{A}_F^{n^2}$. Since the K-rational points are exactly $M_n(K)$ as an algebra, the F rational points of $\mathbb{A}_F^{n^2}$ are exactly $({}_aM_n(K))^G = A$. So thinking of $M_n(K)$ as a variety, and taking the twist of the induced G action, is the same as taking the twist of the G action on $M_n(K)$, A, and considering it as a variety. In summary, $({}_aM_n(K))^G = A$ implies $({}_aM_n(K))_G = A$.

Now we may induce a G action on the product $Gr(nr, M_n(K)) \times M_n(K)$ by letting G act on each coordinate. Let $\mathscr{T}_r(M_n(K))$ be the tautological bundle over $Gr(nr, M_n(K))$. It is a subbundle of the trivial bundle $Gr(nr, M_n(K)) \times M_n(K)$, and we have the following commutative diagram:

$$\begin{array}{ccc} \mathscr{T}_r(M_n(K)) & \hookrightarrow & Gr(nr, M_n(K)) \times M_n(K) \\ \downarrow & \swarrow & \\ Gr(nr, M_n(K)) & & \end{array}$$

We want to show that $\mathscr{T}_r(M_n(K))$ is G invariant. The K-rational points of $\mathscr{T}_r(M_n(K))$ are of the form $\{(V, v) \mid V \in Gr(nr, M_n(K)), v \in V\}$. Given $\sigma \in G$, $(\sigma)(V, v) \mapsto (C_\sigma(\sigma(V))C_\sigma^{-1}, C_\sigma(\sigma(v))C_\sigma^{-1})$, thus $C_\sigma(\sigma(v))C_\sigma^{-1} \in C_\sigma(\sigma(V))C_\sigma^{-1}$ and the image is still in $\mathscr{T}_r(M_n(K))$. The variety $\mathscr{T}_r(M_n(K))$ is rational, since $Gr(nr, M_n(K))$ is, and thus the K points are dense. Since σ takes the K points of $\mathscr{T}_r(M_n(K))$ to $\mathscr{T}_r(M_n(K))$, it takes the closure,

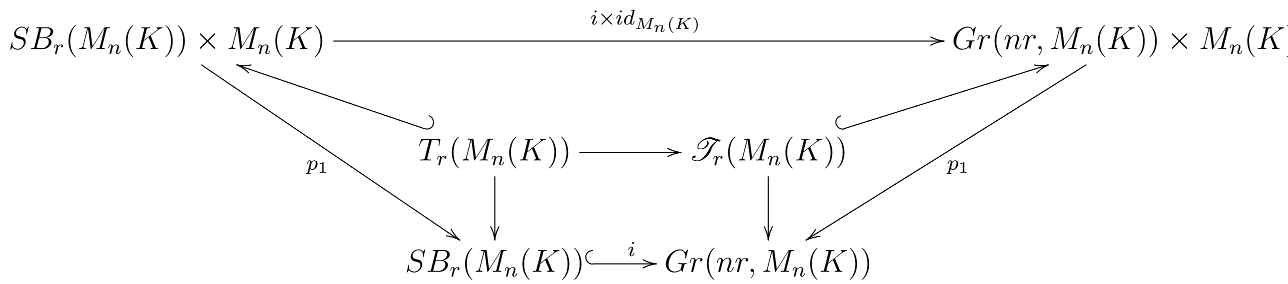
$\mathscr{T}_r(M_n(K))$, to itself. Thus $\mathscr{T}_r(M_n(K))$ is G invariant, and the above diagram is a diagram of G invariant morphisms. Applying Galois descent, we obtain a commutative diagram over F:

$$\begin{array}{ccc} \mathscr{T}_r(M_n(K))_G & \hookrightarrow & Gr(nr, A) \times A \\ \downarrow & \swarrow & \\ Gr(nr, A) & & \end{array}$$

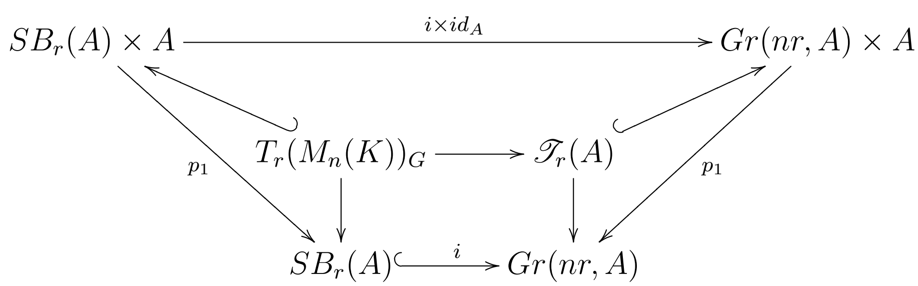
I claim that $\mathscr{T}_r(M_n(K))_G = \mathscr{T}_r(A)$, the tautological bundle over $Gr(nr, A)$. Note that $\mathscr{T}_r(A)$ is rational, since $Gr(nr, A)$ is, and its F-points are dense. Given an F-point, (W, w), this bijects with a G fixed, K-point $(V, v') \in Gr(nr, M_n(K)) \times M_n(K)$, which must be $(W \otimes_F K, w \otimes 1)$. Thus $v' = w \otimes 1 \in W \otimes_F K = V$ and $(V, v') \in \mathscr{T}_r(M_n(K))$ is G fixed. Projecting back down we have $(W, w) \in \mathscr{T}_r(M_n(K))_G$, thus the F-rational points of $\mathscr{T}_r(A)$ are inside the closed subvariety $\mathscr{T}_r(M_n(K))_G$, so $\mathscr{T}_r(A) \hookrightarrow \mathscr{T}_r(M_n(K))_G$. Now $\mathscr{T}_r(M_n(K))_G$ is reduced and irreducible, by Jahnel, and of the same dimension as $\mathscr{T}_r(A)$, so they are in fact equal.

Now we have the following commutative diagram, where every object and arrow is G invariant. Here $T_r(M_n(K))$ is the pullback of the tautological bundle, $i^*\mathscr{T}_r(M_n(K))$. It is G invariant using the fact that $SB_r(M_n(K))$ is G invariant, followed by an argument similar to the one showing that $\mathscr{T}_r(M_n(K))$

is G invariant.

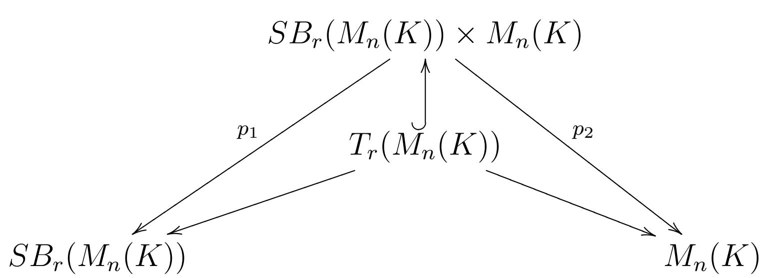


Applying Galois descent and using the results above, we have the following commutative diagram:

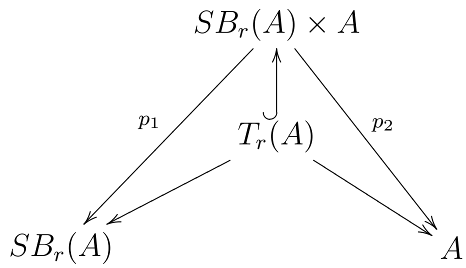


The twist $T_r(M_n(K))_G$ is therefore the pullback of the tautological bundle $\mathscr{T}_r(A)$, $T_r(M_n(K))_G = i^*\mathscr{T}_r(A)$. We will define this bundle to be $T_r(A)$.

We now have a G invariant diagram

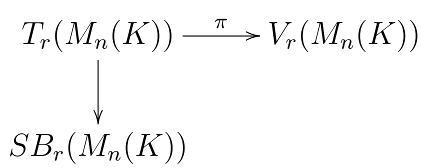


whose twist is



and we want to consider the second projection. To this end, consider the varieties $V_r(M_n(K)) = \{M \in M_n(K) \,|\, rank(M) \leq r\} \subseteq M_n(K)$. These are the so-called determinantal varieties ([6] Lecture 9). They are defined by the ideal of polynomials given by taking the $(r+1) \times (r+1)$ minors of the generic matrix $\sum X_{i,j}e_{i,j}$. Call this ideal $I_r \subseteq K[X_{i,j}]$, the coordinate ring of $M_n(K)$. We have natural inclusions $V_{r-1}(M_n(K)) \hookrightarrow V_r(M_n(K))$, and we will denote $V_r^+(M_n(K)) = V_r(M_n(K)) \setminus V_{r-1}(M_n(K))$, the elements of rank exactly r, which is a dense, open subset of $V_r(M_n(K))$.

The second projection takes a point $(I, i) \in T_r(M_n(K))$ to its second coordinate $i \in I \subseteq M_n(K)$. Since each ideal I has rank r, each element i has rank less than or equal to r. Thus the second projection factors through $V_r(M_n(K))$:



(5.3)

Since each element of rank less than or equal to r is in some ideal of rank r, the map $\pi : T_r(M_n(K)) \to V_r(M_n(K))$ is surjective. Consider $\pi^{-1}(V_r^+(M_n(K))) =$

$\{(I,i)\,|\,i \in I, rank(i) = r\}$. This is a dense, open subset of $T_r(M_n(K))$, which we will denote $T_r^+(M_n(K))$. Any element $i \in I$ of rank r is a generator of I, thus i is in exactly one ideal of rank r. This gives a map back, $\mu : V_r^+(M_n(K)) \to T_r^+(M_n(K))$, $i \mapsto (i \cdot M_n(K), i)$, and composing with π in both directions gives the identity. So μ is G invariant and π is birational:

$$\begin{array}{ccc} T_r^+(M_n(K)) & \xrightarrow{\sim} & V_r^+(M_n(K)) \\ \downarrow & & \\ SB_r(M_n(K)) & & \end{array} \tag{5.4}$$

We can identify $SB_r(M_n(K))$ with a Grassmannian, so it is smooth, and so the vector bundle $T_r(M_n(K))$ is smooth. Thus π is a desingularization of the determinantal variety $V_r(M_n(K))$.

I would like to apply Galois descent to diagrams 5.3 and 5.4. Since π is the restriction of the G invariant projection map, I simply need to show that $V_r(M_n(K))$ and $V_r^+(M_n(K))$ are G invariant. Consider the twisted G action given by $\sigma \cdot M = C_\sigma(\sigma(M))C_\sigma^{-1}$, where $\sigma(M)$ is the usual G action. For the moment, consider only the usual G action on $M_n(K)$, and its induced action on the polynomial ring $K[X_{i,j}]$. The generators of I_r are the $r \times r$ minors of the generic matrix. These polynomials have coefficients ± 1, and so are invariant under the induced action on $K[X_{i,j}]$. In other words, given a minor polynomial $P(X_{i,j})$, $P(\sigma(M)) = \sigma(P(M))$. Thus the rank of any matrix M does not change after applying σ. Conjugation by the matrix C_σ also does not alter the rank of any matrix. So the rank of a matrix does not change under the twisted G action on $M_n(K)$, implying that the varieties $V_r(M_n(K))$ and

$V_r^+(M_n(K))$ are G invariant. We will denote their twists by $V_r(A)$ and $V_r^+(A)$. Note that $V_{n-1}(A)$ is the usual norm hypersurface ([5] Construction 2.6.1, p. 37).

Now we can apply Galois descent to diagrams 5.3 and 5.4 to prove:

Theorem 5.0.26. *Denote the tautological bundle over $Gr(nr, A)$ as $\mathscr{T}_r(A)$. The pullback of this bundle under $i : SB_r(A) \hookrightarrow Gr(nr, A)$, $T_r(A)$, is a twist of $T_r(M_n(K))$ over $SB_r(M_n(K))$. It is a desingularization of the closed subvariety $V_r(A) \hookrightarrow A$ via the natural projection map, where $V_r(A)$ is a twist of the determinantal variety $V_r(M_n(K))$ of elements of rank less than or equal to r in $M_n(K)$. The same statement is true replacing right ideals with left ideals, and replacing $SB_r(A)$ with ${}_rSB(A)$.*

Proof. By the above, the following diagrams are G invariant:

$$V_r(M_n(K)) \hookrightarrow M_n(K) \qquad V_r^+(M_n(K)) \hookrightarrow V_r(M_n(K)) \hookrightarrow M_n(K)\,.$$

Applying Galois descent and Lemma 5.0.23, we obtain quasiprojective subvarieties

$$V_r(A) \hookrightarrow A \qquad V_r^+(A) \hookrightarrow V_r(A) \hookrightarrow A\,.$$

As shown above, $(T_r(M_n(K)))_G = i^*\mathscr{T}_r(A)$, which we define to be $T_r(A)$. Applying Galois descent to diagram 5.3 yields

$$\begin{array}{ccc} T_r(M_n(K)) \xrightarrow{\pi} V_r(M_n(K)) & \overset{(\,\cdot\,)_G}{\rightarrow} & T_r(A) \xrightarrow{\pi} V_r(A) \\ \downarrow & & \downarrow \\ SB_r(M_n(K)) & & SB_r(A) \end{array}$$

where both arrows in the descent diagram are surjections by Lemma 5.0.24. Applying Galois descent to diagram 5.4 yields

$$\begin{array}{ccccc} T_r^+(M_n(K)) & \xrightarrow{\sim} & V_r^+(M_n(K)) & \overset{(\,\cdot\,)_G}{\longrightarrow} & T_r^+(A) & \xrightarrow{\sim} & V_r^+(A) \\ \downarrow & & & & \downarrow & & \\ SB_r(M_n(K)) & & & & SB_r(A) & & \end{array}$$

where the isomorphism is preserved ([10]). Thus $\pi : T_r(A) \to V_r(A)$ is a desingularization. □

Corollary 5.0.27. *The variety of rank r elements of A, $V_r(A)$, is rational over the generalized Severi-Brauer variety $SB_r(A)$ of transcendence degree $n \cdot r$.*

Proof. The variety $V_r(A)$ is birational over $SB_r(A)$ with $T_r(A)$, which is a vector bundle of rank $n \cdot r$. Thus as a variety, its function field is rational over $F_r(A)$ of transcendence degree $n \cdot r$. □

Denote π^+ to be the restriction of $\pi : T_r(A) \to V_r(A)$ to $T_r^+(A)$, thus $\pi^+ : T_r^+(A) \to V_r^+(A)$ is an isomorphism with inverse μ. Denote $p_1 \circ \mu : V_r^+(A) \to SB_r(A)$ by Φ_r. In the case $r = n$, this is the map Φ from Chapter 4, and in general it can be described as taking each point of rank r to the unique, rank r, right ideal it generates, as a point of $SB_r(A)$. The maps $\Psi_{N,d}$ from Chapter 4 are given by taking each ideal in a dense subset of rank $n-1$ right ideals to its unique intersection with $N+d$, which has rank $n-1$. These maps are proven to be rational embeddings of $SB_{n-1}(A)$ into the norm hypersurface $V = V_{n-1}(A)$ by showing that they are rational sections of the map Φ. Let

$\Phi_r : V_r^+(A) \to SB_r(A)$ take a rank r element to the right ideal it generates. Via π^+, $p_1 : T_r^+(A) \to SB_r(A)$ and $\Phi_r : V_r^+(A) \to SB_r(A)$ are isomorphic over $SB_r(A)$ as fiber bundles. Given a K rational point I of $SB_r(A)$, both fibers are isomorphic to I^+, the dense open subset of generators of I. In fact, this is $I \cap V_r^+(A)$. So the rational sections of Φ_r biject with the rational sections of $p_1 : T_r^+(A) \to SB_r(A)$, which form a subset of the rational sections of $p_1 : T_r(A) \to SB_r(A)$. These are rational sections of a vector bundle, and biject with the rational points of the generic fiber. Since $SB_r(A)$ is absolutely irreducible (Theorem 2.0.4, part 3), it has a unique generic point. The generic fiber of the bundle $T_r(A)$ over $SB_r(A)$ is the pullback of the bundle via the generic point:

$$\begin{array}{ccccc} M_r & \longrightarrow & T_r(A) & \xrightarrow{p_2} & V_r(A) \\ \downarrow & & \downarrow{\scriptstyle p_1} & & \\ Spec\, F_r(A) & \longrightarrow & SB_r(A) & & \end{array}$$

where M_r is the generic, rank r right ideal, $M_r \subseteq A \otimes_F F_r(A)$. For $r = n-1$, this is the generic, maximal right ideal from Chapter 4. The rational sections of p_1 biject with $F_r(A)$ rational points of M_r, that is, rational sections of the map $M_r \to Spec\, F_r(A)$.

Given an $F_r(A)$ rational point $m \in M_r$, m is both a rational section of $T_r(A) \to SB_r(A)$ and an $F_r(A)$ rational point of $V_r(A)$. As a rational section of $T_r(A)$, we will refer to the point as $\tilde{m} : SB_r(A) \dashrightarrow T_r(A)$. As a point of $V_r(A)$, it is a point of $M_r \to V_r(A)$, and we have $\tilde{m}(M_r) = (M_r, m)$. Given such a section, we obtain a rational map $p_2 \circ \tilde{m} : SB_r(A) \dashrightarrow V_r(A)$, and we have the generic point of the image is $p_2 \circ \tilde{m}(M_r) = p_2(M_r, m) = m$. This generic point has rank s, that is, $m \in V_s^+(A)$, if and only if $p_2 \circ \tilde{m}$ restricts to a rational map $SB_r(A) \dashrightarrow V_s^+(A)$. We will define this to be the rank of $\tilde{m}$. Thus a rational section $\tilde{m}$ gives a rational section $p_2 \circ \tilde{m}$ of $\Phi_r : V_r^+(A) \dashrightarrow SB_r(A)$ if and only the rank of m is r, that is, if and only if $m \in M_r^+$. We have the following:

Theorem 5.0.28. *The rational sections of $\Phi_r : V_r^+(A) \to SB_r(A)$ are exactly the rank r, rational elements in $M_r \subseteq A \otimes_F F(A)$.*

Consider the construction of the maps in Chapter 4. They were given by choosing N from a dense subset of $Gr(n, A)$ such that $(N \otimes_F F(A)) \cap M = (0)$, and then choosing d from a dense subset of A such that $(N \otimes_F F(A) + d) \cap M = \{m_{N,d}\}$ was a singleton of rank $n-1$. We see from the above theorem that it is exactly the element $m_{N,d}$ which produces the rational section.

Corollary 5.0.29. *Each of the rational embeddings of $SB_{n-1}(A)$ into the norm hypersurface from [12], [13], and from Chapter 4 are given by $F(A)$ rational elements $m \in M_{n-1}$ under the above construction.*

We now have several generalizations of the results in [12].

Theorem 5.0.30.

1. *$F(V_r(A))$ is a generic $1/r$ splitting field of A.*

2. *$F(V_r(A)) \cong F(V_r(B)) \Leftrightarrow \langle [A]^r \rangle = \langle [B]^r \rangle$.*

Proof. For 1, $F(V_r(A))$ is a rational extension of $F_r(A)$, which is a $1/r$ splitting field of A. The index doesn't change under rational extensions, so $ind(A \otimes_F F(V_r(A))) = ind(A \otimes_F F_r(A))|r$ implies $F(V_r(A))$ is a $1/r$ splitting field. Now suppose L is a $1/r$ splitting field of A. Then $LF_r(A)/L$ is rational, so $LF(V_r(A))/L$ is rational. Conversely, suppose $LF(V_r(A))/L$ is rational. Then $V_r(A)$ has an L rational point. Follow this point $Spec\, L \to V_r(A)$ by the projection $p_1 : V_r(A) \to SB_r(A)$, showing an L rational point for $SB_r(A)$. Thus $ind(A \otimes_F L)|r$, and L is a $1/r$ splitting field for A.

For 2, if $F(V_r(A)) \cong F(V_r(B))$, then $Br(F(V_r(A))/F) = Br(F(V_r(B))/F)$. Since $F(V_r(A))$ is a rational extension of $F_r(A)$, $Br(F(V_r(A))/F) = Br(F_r(A)/F) = \langle [A]^r \rangle$, and similarly for B. Thus $\langle [A]^r \rangle = \langle [B]^r \rangle$.

If $\langle [A]^r \rangle = \langle [B]^r \rangle$, then $B^r \in Br(F_r(A)/F)$, which means $F_r(A)$ is a $1/r$ splitting field for B. Then $F_r(B)F_r(A)/F_r(A)$ is rational, of transcendence degree $r(n-r)$. Similarly for $F_r(B)F_r(A)/F_r(B)$. So we have

$$F_r(A)(Z_1, ..., Z_{r(n-r)}) \cong F_r(A)F_r(B) \cong F_r(B)(Y_1, ..., Y_{r(n-r)}).$$

Since $V_r(A)$ is birational over $SB_r(A)$ with a vector bundle, its functions field is rational over $F_r(A)$ of transcendence degree $dim(I) = n \cdot r$. For all $r = 1, ..., n-1$, we have $nr > r(n-r)$, so $F(V_r(A)) \cong F(V_r(B))$. □

One immediate question that arises from this construction is given a rational section of M_r of rank less than r, what properties does it have as a rational map? The next chapter address the question for ranks dividing the index of A. Here we look at an example of an element of rank 1.

Theorem 5.0.31. *There exist rank 1, rational elements in M which give rational embeddings of $SB(A)$ into the norm hypersurface V.*

Proof. Consider the isomorphism $ann : SB_{n-1}(A) \to {}_1SB(A)$ which takes a maximal right ideal to its left annihilator (Theorem 2.0.4). We take the pullback of the tautological bundle ${}_1T(A)$ via ann, and consider it as a subbundle of $SB_{n-1}(A) \times A$.

$$\begin{array}{ccccc} A \otimes_F F(A) & \longrightarrow & SB(A) \times A & \xrightarrow{ann \times id} & {}_1SB(A) \times A \\ \uparrow & & \uparrow & & \uparrow \\ ann(M) & \longrightarrow & ann^*\,{}_1T(A) & \longrightarrow & {}_1T(A) \\ \downarrow & & \downarrow & & \downarrow \\ Spec\,F(A) & \longrightarrow & SB(A) & \xrightarrow[ann]{\sim} & {}_1SB(A) \end{array} \tag{5.5}$$

The fiber over the generic point is the annihilator of the generic maximal right ideal $M_{n-1} = M$. That is, we have the following diagram:

$$\begin{array}{ccc} & A \otimes_F F(A) & \\ \nearrow & & \nwarrow \\ M & & ann(M) \\ \searrow & & \swarrow \\ & Spec\,F(A) & \end{array} \quad \to \quad \begin{array}{ccc} & SB(A) \times A & \\ \nearrow & & \nwarrow \\ T(A) & & ann^*({}_1T(A)) \\ \searrow & & \swarrow \\ & SB(A) & \end{array}$$

Now $A \otimes_F F(A)$ is split, so M is a maximal right ideal and $ann(M)$ is a minimal left ideal. Any minimal left ideal and maximal right ideal intersect nontrivially in a split algebra. Let $m_1 \in M \cap ann(M)$ be any nonzero element, and thus a generator of $ann(M)$. As a rational section, it is a section of both

$T(A)$ and $ann^*{}_1T(A)$. Note that $ann^*{}_1T(A) \to {}_1T(A)$ is the identity in the second factor, so $m \mapsto m$, and we can investigate m as a rational section of ${}_1T(A) \to {}_1SB(A)$ (diagram 5.5). Here we see that $\tilde{m}$ is a rational embedding of ${}_1SB(A)$ into $V_1(A) \subseteq A$. Pulling this rational section back to $SB(A)$ gives a rank one element of M which rationally embeds $SB(A)$ into $V_1(A)$. $\square$

Chapter 6

Separable, Splitting Subrings

Saltman's Results

For an arbitrary F-CSA, A, I have the usual rational map $\Phi : V \dashrightarrow SB(A)$ taking an element of rank $n-1$ to the maximal right ideal it generates. Let L be a separable, splitting subring of A (Definition 2.0.7), and choose any $\alpha \in A$, $\alpha \notin L$. Then I can define a map back, $\Psi : SB(A) \dashrightarrow V$, as follows. We know that $M \cap L = (0)$ as in the proof of Theorem 4.0.18, so for almost every maximal right ideal I of A, we have $I \cap L = (0)$. Thus $I \cap (L + \alpha)$ must be a singleton, which we define to be $\Psi(I)$. Notice that we have chosen a specific L, not an arbitrary n-dimensional subspace of A. However, we have made no assumptions about α. One goal in this chapter is to present Saltman's classification of the α for which this map Ψ is a rational section of Φ, and thus a rational embedding $SB(A) \dashrightarrow V$. Then we use this construction to produce rational embeddings of $SB(A)$ into $V(A)$ with fibers of positive dimension.

Notice that Ψ actually factors through the closed subvariety $(L + \alpha) \cap V = V(f|_{L+\alpha})$. Let us denote $p_\alpha = f|_{L+\alpha}$. If we have a basis of L, $\{e_1, ..., e_n\}$, then $p_\alpha(X_1, ..., X_n) = n_A(\sum X_i e_i + \alpha) \in F[X_1, ..., X_n]$, where $\sum X_i e_i + \alpha$ is the generic point of $L + \alpha$. We consider the map $\Psi : SB(A) \dashrightarrow V(p_\alpha)$ and we will classify the α for which this is birational. These are the α which produce rational sections of Φ, but it is not yet clear that $V(p_\alpha)$ even intersects the domain of Φ, V^+. We will extend scalars to $\overline{F}$ and see that $V(p_\alpha)$ and V^+ have nontrivial intersection for all α (Lemma 6.0.33), so they have nontrivial intersection over F, and the map Φ restricts to a rational map $\Phi : V(p_\alpha) \dashrightarrow SB(A)$.

If Ψ maps an ideal I to an element in the domain of Φ, $V^+ \cap V(p_\alpha)$, then the image is an element of I with rank $n-1$, thus a generator of I, and Φ will map it back to I. A similar statement can be made when Φ maps an element of $V(p_\alpha)$ to the domain of Ψ. That is to say, where it makes sense, Φ and Ψ are inverses. We have made this argument before, but for completeness let us repeat it here:

Lemma 6.0.32. *Let $U = \{I \in SB(A) | I \cap L = (0)\}$ be the domain of Ψ. If $\Phi(v) \in U$, then $\Psi(\Phi(v)) = v$, and if $\Psi(I) \in V^+$, then $\Phi(\Psi(I)) = I$.*

Proof. We showed above that if $\Psi(I) \in V^+$, then $\Psi(I)$ is a generator of I, and Φ will map that element back to the ideal it generates, I. For the first statement, if v is in the domain of Φ, then v is an element of $L+\alpha$ of rank $n-1$, and $\Phi(v) = I$ is the ideal v generates. So $v \in (L+\alpha) \cap I$. If $\Phi(v) = I \in U$, then $I \cap L = (0)$, and $\Psi(I)$ is the unique element in the intersection $(L+\alpha) \cap I$, which must be v. So $\Psi(\Phi(v)) = v$. □

To prove that Ψ is birational, we need to show that the images of Φ and Ψ are dense. There is one obstruction: $V(p_\alpha)$ may not be geometrically irreducible. That is to say, p_α may not be absolutely irreducible. But $U \subseteq SB(A)$ is open, thus irreducible. Then Ψ can not possibly be birational. Saltman proves that if p_α *is* absolutely irreducible, then the images of Φ and Ψ are dense, and by Lemma 6.0.32, they are birational inverses (reference theorem). So one criteria for building rational sections of Φ is choosing α such that p_α is absolutely irreducible. Later in this chapter we will see another criterion for finding these α that is easier to use in practice.

To prove that Φ and Ψ are birational inverses to each other, we will take the maps and varieties defined above and extend scalars to an algebraic closure, $\overline{F}$. Assume for the moment that we have proven the case over $\overline{F}$ for our choice of α. Then $V(p_\alpha) \times_F \overline{F}$ is irreducible, that is, p_α is absolutely irreducible, and $\Phi : SB(A) \times_F \overline{F} \dashrightarrow V(p_\alpha) \times_F \overline{F}$ and $\Psi : V(p_\alpha) \times_F \overline{F} \dashrightarrow SB(A) \times_F \overline{F}$ are birational inverses. Then their images are dense, and so the images of Ψ and Φ are dense over F, and by Lemma 6.0.32, they are birational inverses over F.

So we need only prove the case where $F = \overline{F}$. In this case A is split, so let $A = End_F(W)$. We can choose matrix elements $e_{i,j}$ so that L is the set of diagonal matrices (see the comment after Definition 2.0.7). For simplicity, we will assume α has zeros along the diagonal, but we will show later that this assumption is not necessary. For ease of notation we will use $\triangle(X_1, ..., X_r)$ to be the $r \times r$ diagonal matrix

$$\begin{pmatrix} X_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & X_r \end{pmatrix}.$$

Then $\triangle(X_1, ..., X_n) + \alpha$ is the generic point of $L + \alpha$, and p_α is the determinant of this matrix. Factor this polynomial into a product of primes, $p_\alpha = \prod p_i$, and define $V_i = V(p_i)$. Then $V(p_\alpha) = \cup V_i$ expresses $V(p_\alpha)$ uniquely as a union of irreducible subvarieties.

As noted above, in order to define Φ on $V(p_\alpha)$, we need $V(p_\alpha)$ to intersect the open set of rank $n-1$ elements, V^+. Something even stronger is true:

Lemma 6.0.33. *With $V(p_\alpha) = \cup V_i$ as above, $V_i \cap V^+$ is nonempty for all i.*

Saltman proves this by analyzing the factorization of p_α. The proofs do not rely on any other results, but they are somewhat technical and we will omit them. Now that we know these sets are nontrivial, let's define $V_i^+ = V_i \cap V^+$, and define Φ_i to be the restriction of Φ to V_i.

Over $\overline{F}$ we can identify $SB(A)$ with $\mathbb{P}(W)$ by taking a maximal right ideal to its common range, which is a hyperplane in W ([13] Proposition 13.4, p. 91). I identify this hyperplane, H, with the degree one, homogeneous polynomial that defines it, up to scalar multiplication. That is, $H = \{a_1 w_1 + \cdots + a_n w_n \,|\, c_1 a_1 + \cdots + c_n a_n = 0\}$ where w_i is the basis of W induced by the matrix units $e_{i,j}$, and I identify H with this tuple $[c_1 : \cdots : c_n] \in \mathbb{P}(W)$. An element a of $A = End_{\overline{F}}(W)$ has image in H if and only if $\begin{pmatrix} c_1 & \cdots & c_n \end{pmatrix} \cdot a = 0$ (the product here is matrix multiplication) so we can recover the maximal ideal I from H be defining $I = \{a \in A \,|\, \begin{pmatrix} c_1 & \cdots & c_n \end{pmatrix} \cdot a = 0\}$. Now we can write down the map $V^+ \to SB(A)$ as follows: a rank $n-1$ matrix a will have a unique point $[c_1 : \cdots : c_n]$ such that $\begin{pmatrix} c_1 & \cdots & c_n \end{pmatrix} \cdot a = 0$, and this point is the image of a.

Suppose $[c_1 : \cdots : c_n]$ is in the image of Ψ. We want to compute the dimension of the fiber over $[c_1 : \cdots : c_n]$. Consider two points in the fiber, $\triangle(a_1, ..., a_n) + \alpha$ and $\triangle(b_1, ..., b_n) + \alpha$ where

$$\begin{pmatrix} c_1 & \cdots & c_n \end{pmatrix} (\triangle(a_1, ..., a_n) + \alpha) = \begin{pmatrix} c_1 & \cdots & c_n \end{pmatrix} (\triangle(b_1, ..., b_n) + \alpha) = 0.$$

We see that both points are in the fiber if and only if their difference, $\triangle(a_1 - b_1, ..., a_n - b_n)$, satisfies:

$$\begin{pmatrix} c_1 & \cdots & c_n \end{pmatrix} \triangle(a_1 - b_1, ..., a_n - b_n) = 0 \Leftrightarrow c_j(a_j - b_j) = 0 \text{ for all } j.$$

Thus for c_j nonzero, a_j is fixed, and for $c_j = 0$, a_j is free. So the dimension of the fiber over $[c_1 : \cdots : c_n]$ is the cardinality of $\{j \,|\, c_j = 0\}$.

Now we want to compute the closure of the image of each Φ_i. Take the generic point of V_i, and let $[c_1 : \cdots : c_n]$ be its image. Let T be the set of j such that $c_j = 0$, and let s be the cardinality of T. Then the dimension of this generic fiber is s. For $W = \{\lambda_1 w_1 + \cdots + \lambda_n w_n\}$, (where $A = End_{\overline{F}}(W)$, and the basis w_i is induced by L) let $W_T \subseteq W$ be the subspace defined by setting $\lambda_j = 0$ for $j \in T$. Then the generic point of V_i has image in $\mathbb{P}(W_T)$, so the closure of the image of Φ_i is in $\mathbb{P}(W_T)$. Also, the dimension of W_T is $n - s$. Since the dimension of the generic fiber is s, and the dimension of V_i is $n - 1$, the dimension of the image is $n - s - 1$, exactly the dimension of $\mathbb{P}(W_T)$. We have proved:

Lemma 6.0.34. *The image of Φ_i is dense in $\mathbb{P}(W_T)$.*

We can now give a description of the α such that p_α is irreducible. A corollary will be that for these α, the corresponding Ψ is birational. As discussed at the beginning of the chapter, the same conclusion will follow for an arbitrary F-CSA. Specifically, for those α such that p_α is absolutely irreducible, p_α remains irreducible at the algebraic closure. Thus over $\overline{F}$, Ψ is birational and has dense image, so over F, Ψ has dense image, and is birational. The only problem is that we have assumed α has zeros along the diagonal, so we will remove that hypothesis in the next theorem.

Theorem 6.0.35. *The polynomial p_α is reducible, with factorization $p_\alpha = \prod p_i$, if and only if by permuting the standard basis w_i of W, α can be put in lower block diagonal form:*

$$\begin{pmatrix} \alpha_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ * & \cdots & \alpha_r \end{pmatrix}$$

where $p_i = p_{\alpha_i}$. Furthermore, this is true for $\alpha \notin L$, regardless of its entries along the diagonal.

Proof. We first consider the case where α has zeros along the diagonal. Since U is irreducible, $\Psi(U) \subseteq V_i$ for some i, and we can take i to be 1. Now choose $i \neq 1$. The image of Ψ may or may not be in V_i for $i \neq 1$, but we know it definitely can not be dense in this V_i. If the image of Ψ is not dense in V_i, then by Lemma 6.0.32, the image of Φ_i is not dense in $\mathbb{P}(W)$. But by

Lemma 6.0.34, it is dense in some $\mathbb{P}(W_T)$ with $W_T \subsetneq W$. The generic point of $\mathbb{P}(W_T)$ is $[c_1 : \cdots : c_n]$ with $c_j = 0$ for $j \in T$, and it is in the image of Φ_i since the image is dense. Let $\triangle(a_1, ..., a_n) + \alpha$ be in the fiber of the generic point, then

$$\begin{pmatrix} c_1 & \cdots & c_n \end{pmatrix} (\triangle(a_1, ..., a_n) + \alpha) = 0$$

which implies

$$\begin{pmatrix} c_1 & \cdots & c_n \end{pmatrix} \alpha = -\begin{pmatrix} c_1 & \cdots & c_n \end{pmatrix} \triangle(a_1, ..., a_n) = -\begin{pmatrix} c_1 a_1 & \cdots & c_n a_n \end{pmatrix}$$

which is in W_T since $c_j = 0$ for $j \in T$. Since α takes the generic point of W_T back to W_T, it must preserve the entire subspace W_T, that is, $\alpha(W_T) \subseteq W_T$. By renumbering the indices, α can be put in block diagonal form

$$\begin{pmatrix} \alpha_1 & 0 \\ * & \alpha_2 \end{pmatrix}.$$

Then the generic point of $L + \alpha$ has the form

$$\begin{pmatrix} \triangle(X_1, ..., X_s) + \alpha_1 & 0 \\ * & \triangle(X_{s+1}, ..., X_n) + \alpha_2 \end{pmatrix}$$

and so $p_\alpha(X_1, ..., X_n) = p_{\alpha_1}(X_1, ..., X_s) p_{\alpha_2}(X_{s+1}, ..., X_n)$. We are done by induction on the size of the blocks α_i and unique factorization of the polynomial p_α.

For arbitrary $\alpha \notin L$, α can be written uniquely as $\alpha_\triangle + \alpha_0$ where $\alpha_\triangle \in L$ is the diagonal matrix with the same entries as α along the diagonal, and α_0 has zeros along the diagonal. □

We now see that over $\overline{F}$, p_α is reducible if and only if α can be lower block diagonalized by permuting the standard basis of W. We also have the following corollary:

Corollary 6.0.36. *If p_α is irreducible, then $\Psi : SB(A) \dashrightarrow V(p_\alpha) \subseteq V$ is birational with rational inverse Φ.*

Proof. As noted above, by Lemma 6.0.32 we simply need to show that the image of Ψ is dense. If not, then the image of Φ is not dense in $SB(A) \cong \mathbb{P}(W)$, again by Lemma 6.0.32. By Lemma 6.0.34, the image of Φ is dense in $\mathbb{P}(W_T)$ for some proper subspace $W_T \subsetneq W$, and just as in the proof of Theorem 6.0.35, α preserves the subspace W_T, and can be lower block diagonalized. By Theorem 6.0.35, this implies p_α is reducible, a contradiction. □

We now give an alternative description of these α. Using Schur's Lemma and the Denstiy Theorem, along with Theorem 6.0.35, we have the following.

Theorem 6.0.37.

1. *$p_{L,\alpha}$ is absolutely irreducible if and only if $\langle L, \alpha \rangle$ generates the entire algebra A.*

2. *For fixed L, the $\alpha \in A$ such that $\langle L, \alpha \rangle = A$ form a Zariski open subset of A with F points.*

3. *For fixed L, there is a dense set of $\alpha \in A$ such that $SB(A)$ is birational to $V(p_{L,\alpha}) \hookrightarrow A$.*

New Results

First, let us give an explicit description of the intersection map Ψ in the case A is split.

Take a representative of $[a_1 : \cdots : a_n] \in U$ written as a $1 \times n$ matrix $(a_1 \cdots a_n)$. We know α is zero along the diagonal, $\alpha = (\alpha_{i,j})$ with $\alpha_{i,i} = 0$, so consider the equation

$$\begin{pmatrix} a_1 & \cdots & a_n \end{pmatrix} \begin{pmatrix} x_1 & \alpha_{1,2} & \cdots & \alpha_{1,n} \\ \alpha_{2,1} & x_2 & \cdots & \alpha_{2,n} \\ \vdots & & \ddots & \vdots \\ \alpha_{n,1} & \cdots & \alpha_{n,n-1} & x_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

There is a unique choice of x_i solving this system of equations, for example

$$x_1 = \frac{-a_2\alpha_{2,1} - \cdots - a_n\alpha_{n,1}}{a_1}.$$

Another way to see this is that $(a_1 \cdots a_n) \cdot \alpha$ is a column vector (c_i) and our solution is $x_i = -c_i/a_i$. Geometrically, we are taking a hyperplane H to the unique matrix in $L + \alpha$, call it $\psi_{L,\alpha}(H)$, whose range, $\psi_{L,\alpha}(H)(W)$, is contained in the hyperplane H. This immediately forces the matrix $\psi_{L,\alpha}(H)$ to be singular, and thus in $V \cap (L + \alpha)$.

For a very simple example, let's take $A = M_2(F)$ and

$$\alpha = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then for every $[a : b] \in \mathbb{P}^2$, we have

$$\begin{pmatrix} a & b \end{pmatrix} \cdot \begin{pmatrix} x & 1 \\ 1 & y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow x = -b/a, y = -a/b$$

which in local coordinates $[a : b] = [1 : b/a] = [1 : z]$ gives the birational isomorphism

$$\begin{array}{ccc} \mathbb{P}^1 & \dashrightarrow & V(ad - bc = 0) \cap (L + \alpha) \\ z & \mapsto & \begin{pmatrix} -z & 1 \\ 1 & -1/z \end{pmatrix} \end{array}$$

The map $\psi_{L,\alpha}$ is defined for any α not in L. In 6.1, we are interested in those α which can not be lower block diagonalized by permuting the standard basis of W. Here we will consider α which *can* be lower block diagonalized. Some α may still yield rational embeddings, like the following.

Example 1. Just as a very brief example of a situation like the above, let

$$\alpha = \sum_{i=2}^{n} -1 \cdot e_{1,i} = \begin{pmatrix} 0 & -1 & \cdots & -1 \\ 0 & 0 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

Then $\alpha_i = 0$ for all i and $p_{L,\alpha} = X_1 \cdots X_n$, with $V_i = V(X_i)$. We see that the solution to

$$\begin{pmatrix} a_1 & \cdots & a_n \end{pmatrix} \cdot \begin{pmatrix} X_1 & -1 & \cdots & -1 \\ 0 & X_2 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & X_n \end{pmatrix} \quad \text{is} \quad \begin{pmatrix} 0 & -1 & \cdots & -1 \\ 0 & a_1/a_2 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & a_1/a_n \end{pmatrix} \in V_1$$

yielding the rational embedding

$$\begin{array}{ccc} SB(A) & \dashrightarrow & V_1 = V(X_1) \\ (z_1, ..., z_n) & \mapsto & \begin{pmatrix} 0 & -1 & \cdots & -1 \\ 0 & 1/z_1 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & 1/z_n \end{pmatrix} \end{array}$$

Note that for any α, the map ϕ restricted to $V_2 \cup \cdots \cup V_n$, taking a rank $n-1$ element to the ideal it generates, or the hyperplane that is its range, must map into the complement of the domain of definition of ψ. In the simple example we have above, any rank $n-1$ element in V_i, $i \neq 1$, has the form

$$\begin{pmatrix} x_1 & -1 & \cdots & -1 \\ 0 & x_2 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & x_n \end{pmatrix}$$

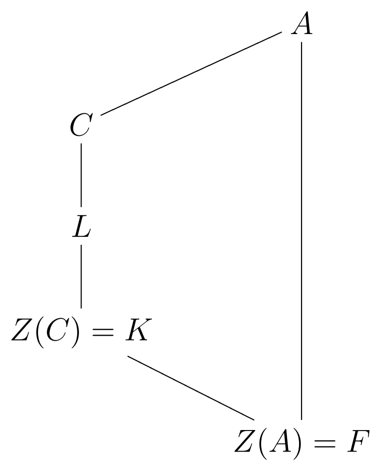
where $x_i = 0$ and $x_j \neq 0$ for $j \neq i$. The image of this element will be the hyperplane given by $a_i x_i = 0$, that is the point $[0 : \cdots : 0 : 1 : 0 : \cdots : 0]$ in the ith coordinate in $\mathbb{P}^{n-1}$, which is clearly in the complement of the domain of ψ.

Now let us consider an α which can be block diagonalized, and which yields a rational map $SB(A) \dashrightarrow A$ with fibers of positive dimension.

Theorem 6.0.38. *There are elements in M of rank r not relatively prime to n that give rational maps with fibers of positive dimension.*

Proof. Let A/F be a division algebra, with maximal, separable subfield L/F. Let K be an intermediate field, not L or F, such that L/K is normal. Then $C = Cent_A(K)$ is a simple subalgebra of A with center K, and containing L

as a maximal, separable subfield. We have the following diagram:



Suppose $[L : K] = l$ and $[K : F] = k$, so $lk = n$. Choose $\alpha \in C$ such that $C = \langle L, \alpha \rangle$. We have that $A \otimes_F K \sim C$, which is a division algebra, so $A \otimes_F K \cong M_k(C)$. Choose matrix units so that the splitting subring $L \otimes_F K$ is diagonal:

$$L \oplus \cdots \oplus L \cong L \otimes_F K \cong \begin{pmatrix} L & & \\ & \ddots & \\ & & L \end{pmatrix}.$$

Then we have

$$C \oplus \cdots \oplus C \cong C \otimes_F K \cong \begin{pmatrix} C & & \\ & \ddots & \\ & & C \end{pmatrix}$$

and $C \hookrightarrow C \otimes_F K \hookrightarrow M_k(C)$ takes $c \mapsto \triangle(c, ..., c)$.

We want to compute the rank of $m_{L,\alpha}$, where $M \cap (L \otimes_F F(A) + \alpha) = \{m_{L,\alpha}\}$. This rank does not change after extending scalars to $\overline{L} = \overline{F}$. Similarly, we can first extend scalars by $\overline{F}$, and then by the function field of the

Severi-Brauer variety. As in 6.1, we identify the Severi-Brauer variety $SB(A)$ with $\mathbb{P}^{n-1}_{\overline{F}}$ by choosing matrix units so that $L \otimes_F \overline{F}$ forms the diagonal matrices. This induces a basis of W, where we have $A \otimes_F \overline{F} \cong End_{\overline{F}}(W)$. To calculate $m_{L,\alpha}$, we find its image after extending scalars to $\overline{F}$ and applying these identifications. Then $m_{L,\alpha}$ is the image of the generic element $[c_1 : \cdots : c_n] \in \mathbb{P}^{n-1}_{\overline{F}}$ under Ψ.

We identify a copy of L in $\overline{F}$. To extend scalars by $\overline{F}$, we first extend by K, then L. Over L, both A and $SB(A)$ split, and $L \subseteq A$ can be diagonalized. So it suffices to consider the image of $m_{L,\alpha}$ after extending scalars to L. So we must calculate the image of the generic element $[c_1 : \cdots : c_n] \in \mathbb{P}^{n-1}_{L}$ under Ψ. First, we consider the image of $L + \alpha$ after extending scalars by K. Here

$$\alpha \mapsto \begin{pmatrix} \alpha & & \\ & \ddots & \\ & & \alpha \end{pmatrix} \in \begin{pmatrix} C & & \\ & \ddots & \\ & & C \end{pmatrix}$$

and

$$L \otimes_F K \to \left\{ \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_k \end{pmatrix} \in M_k(C) \,\middle|\, \lambda_i \in L \right\}.$$

We further extend scalars by L, splitting C, and identify α with its image in $M_l(L) \cong C \otimes_K L$. Then

$$\alpha \mapsto \begin{pmatrix} \alpha & & \\ & \ddots & \\ & & \alpha \end{pmatrix} \in \begin{pmatrix} M_l(L) & & \\ & \ddots & \\ & & M_l(L) \end{pmatrix}$$

and

$$L \otimes_F L \to \left\{ \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_{lk} \end{pmatrix} \in M_{lk}(L) \,\middle|\, \lambda_i \in L \right\}.$$

We have chosen α so that $\langle L, \alpha \rangle$ generate C. Thus their images will generate the respective subalgebras. For example, after extending by K,

$$\left\langle \begin{pmatrix} L & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}, \begin{pmatrix} \alpha & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix} \right\rangle = \begin{pmatrix} C & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}$$

and similarly α and the diagonal matrices in $M_l(L)$ will generate $M_l(L)$. We see from this that the polynomial $p_{L,\alpha}$ factors over L into k geometrically irreducible polynomials, each of which is $q_{L,\alpha}(X_j, ..., X_{j'})$, for some subset $X_j, ..., X_{j'}$ of $X_1, ..., X_n$, where $q_{L,\alpha}$ is the reduced norm for C, n_C, restricted to $L + \alpha \subseteq C$ over K.

To calculate the image of $[c_1 : \cdots : c_n] \in \mathbb{P}_L^{n-1}$ under Ψ, we find the unique matrix $m \in L \otimes_F L + \alpha$ such that $\begin{pmatrix} c_1 & \cdots & c_n \end{pmatrix} \cdot m = 0$. Relabelling the c_i so that

$$\begin{pmatrix} c_1 & \cdots & c_n \end{pmatrix} = \begin{pmatrix} c_{1,1} & \cdots & c_{1,l} & \cdots & c_{k,1} & \cdots & c_{k,l} \end{pmatrix},$$

we are solving for $a_{i,j}$ such that

$$\begin{pmatrix} c_{1,1} & \cdots & c_{1,l} & \cdots & c_{k,1} & \cdots & c_{k,l} \end{pmatrix}$$
$$\cdot \begin{pmatrix} \begin{pmatrix} a_{1,1} & & \\ & \ddots & \\ & & a_{1,l} \end{pmatrix} + \alpha & & \\ & \ddots & \\ & & \begin{pmatrix} a_{k,1} & & \\ & \ddots & \\ & & a_{k,l} \end{pmatrix} + \alpha \end{pmatrix} = 0.$$

This is equivalent to solving each of the k matrix equations

$$\begin{pmatrix} c_{j,1} & \cdots & c_{j,l} \end{pmatrix} \cdot \left(\begin{pmatrix} a_{j,1} & & \\ & \ddots & \\ & & a_{j,l} \end{pmatrix} + \alpha \right) = 0.$$

Since $\langle L, \alpha \rangle$ generate C, we know that each of these k solutions, say $m_k \in M_l(L)$, has rank $l-1$. Thus the full solution

$$m = \begin{pmatrix} m_1 & & \\ & \ddots & \\ & & m_k \end{pmatrix}$$

has rank $k(l-1) = n-k$. We also see that each m_j is singular, thus $m \in V(q_1) \cap \cdots \cap V(q_k)$, a variety of dimension $k(l-1) = n-k$. Since $dim(SB(A)) = n-1 < n-k$, we see that this map must have fibers of positive dimension.

In fact, we can see that this map is dominant onto the intersection by calculating the dimension of the generic fiber. This is the dimension of the subvariety of $\begin{pmatrix} \gamma_1 & \cdots & \gamma_n \end{pmatrix}$ such that $\begin{pmatrix} \gamma_1 & \cdots & \gamma_n \end{pmatrix} \cdot m = 0$. Since m has rank $k(l-1)$, this is the dimension of the subvariety of hyperplanes in W containing a subspace W' of dimension $k(l-1) = n-k$. This is isomorphic to the variety of hyperplanes in W/W', a vector space of dimension $n-(n-k) = k$. Thus the dimension of the generic fiber is the dimension of $Gr(k-1, W/W')$, $k-1$. This means that the image must have dimension $n-1-(k-1) = n-k$, the dimension of the intersection $V(q_1) \cap \cdots \cap V(q_k)$.

$\square$

Chapter 7

Classes of Examples

We now give two results about the subspaces N which satisfy the results from Chapter 4. Let M be the generic, maximal right ideal inside $A \otimes_F F(A)$, and $N \in Gr(n, A)$. In Chapter 4, the set of N in our dense subset of $Gr(n, A)$, from which we built intersection maps, had the property that $N \otimes_F F(A) \cap M = (0)$. Our first result shows that if N intersects M in an element of rank $n-1$, then the algebra is split. Let us recall the map defined earlier, $\Phi : V^+ \to SB(A)$ taking an element of rank $n-1$ in A to the maximal right ideal it generates.

Theorem 7.0.39. *If a subspace $N \subseteq A$ intersects M in an element of rank $n-1$, then the algebra is split and $\Phi|_{N \cap V^+}$ induces a birational isomorphism $\mathbb{P}(N) \dashrightarrow SB(A)$.*

Proof. Given $m \in (N \otimes_F F(A)) \cap M$, we have m is an $F(A)$ point of $N \cap V^+$. Let us restrict the map Φ from above, which takes an element of V^+ to the maximal right ideal it generates, to $\Phi : N \cap V^+ \to SB(A)$. Now this map is dominant, since $\Phi(m) = M$. This implies that $dim(N \cap V^+) \geq dim(SB(A)) = n-1$. I want to bound the dimension of the generic fiber, so change base to $Spec\, F(A)$. Here m is a rational point, which generates M, and for any $\alpha \in F(A)^*$, αm

will also generate M. So we have that the fiber over $F(A)$ has dimension greater than or equal to 1. Thus the generic fiber of $N \cap V^+ \to SB(A)$ has dimension greater than or equal to 1. Thus

$$n \geq dim(N \cap V^+) \geq dim((N \cap V^+) \times_{SB(A)} M) + dim(SB(A)) \geq 1 + (n-1) = n$$

and we have that $N \cap V^+$ is dense in N, so $N \subseteq V = \overline{V^+}$. So we immediately know that A can not be a division algebra, since it now has nonzero F points which are singular. In fact, since N is rational, and $N \cap V^+$ is dense, its F points are dense, so there are F points of rank $n-1$ and the algebra must be split.

The map $N \dashrightarrow SB(A)$ can be given by $n \mapsto n \cdot A$ for F points of N. We see that for all $\alpha \in F^*$, $\alpha n \cdot A = n \cdot A$, so the morphism is invariant on lines and induces the dominant morphism $\mathbb{P}(N) \dashrightarrow SB(A)$. Using the techniques from Chapter 5, we view m as a rational map $\tilde{m} : SB(A) \dashrightarrow N$ given by $M \mapsto m$, then for the sake of this argument compose it with $N \dashrightarrow \mathbb{P}(N)$. Now we can see that the composition $SB(A) \dashrightarrow \mathbb{P}(N) \dashrightarrow SB(A)$ taking a maximal right ideal to a line of generators, and taking the line of generators to the ideal they generate, is clearly an identity where defined. Thus the map $\mathbb{P}(N) \dashrightarrow SB(A)$ is a birational isomorphism. □

For examples of N with this property, consider the situation in Chapter 6 where L is diagonal matrices and α is such that $\alpha_1 = (0)$, a 1×1 zero matrix, and $\beta \neq 0$. See Example 1. Then $V_1 = V(X_1) \cap V^+$ is birational to $SB(A)$. Now consider the cone of lines through V_1, $N = F \cdot e_{2,2} + \cdots + Fe_{n,n} + F \cdot \alpha$.

Then N satisfies the situation of the theorem. In this particular case, $\mathbb{P}(N)$ is actually isomorphic to $SB(A)$.

A similar situation arises when we consider a minimal left ideal I. This does not intersect M in a rank $n-1$ element, indeed I is contained in the subvariety of $rank \leq 1$ elements of V. However, it does intersect every minimal right ideal in a line. This gives an isomorphism $\mathbb{P}(I) \cong SB(A)$ thought of as minimal right ideals. Since I here and N above are cones, we can identify their projectivization with the exceptional divisor of $Bl_{(0)}(N)$ or $Bl_{(0)}(I)$. This is again isomorphic to $SB(A)$. This construction only works in the split case, where I is defined.

The next theorem shows that the generalizations of Chapter 4 actually produce new examples of rational embeddings $SB(A) \dashrightarrow V$. Previous examples involved taking a separable, splitting subring $L \subseteq A$ and an element $\alpha \notin L$, and forming the map $\Psi = \Psi_{L,\alpha}$, taking a maximal right ideal to its unique intersection with $L + \alpha$. There are two obvious ways to create new intersection maps from L and α. One is to choose another n-dimensional affine subspace of $L + F\alpha$, which amounts to scaling the map Ψ. These maps all come from L a separable, splitting subring, and α an element in A such that $L + F\alpha$ intersects M in a 1-dimensional subspace, a line, which contains a generator of M. If the line contains a generator of M, every nonzero element of the line is a generator of M. A second way to generate maps is to choose one of these affine subspaces, and scale by an element $\beta \in A^*$, a unit in A. We want to show that not all elements $\Psi_{N,d}$ from Chapter 4 come from $\Psi_{L,\alpha}$

in this manner. To this end, we will define two subsets of $Gr(n, A) \times A$ with the properties discussed above. In particular, these subsets will be sets of F-rational points of $Gr(n, A) \times A$, and thus topological subspaces, but they will have no scheme structure.

First, define $E \subseteq Gr(n, A) \times A$ as follows:

$$E = \{(N, d) \,|\, (N \otimes_F F(A)) \cap M = (0) \text{ and } (N \otimes_F F(A) + d) \cap M \subseteq M^+\}.$$

Next, we define E' as a subset of E. These will be the points $(N, d) \in E$ such that there exists a separable, splitting subring $L \subseteq A$, an element $\alpha \notin L$, and an element $\beta \in A^*$ such that $N + Fd = (L + F\alpha) \cdot \beta$. Note that this will include those (N, d) that come from scaling by β on the right as follows: if $N + Fd = \beta(L + F\alpha)$, then $N + Fd = (\beta L \beta^{-1} + F\beta\alpha\beta^{-1})\beta$, where $\beta L \beta^{-1}$ is again a separable, splitting subring of A, $\beta\alpha\beta^{-1} \notin \beta\alpha\beta^{1}$, and $\beta \in A^*$.

Theorem 7.0.40. *Both E and $E - E'$ are dense subsets of $Gr(n, A) \times A$ for $n \geq 4$.*

Proof. There is a natural surjective map

$$p : Gr(n, A) \times A \dashrightarrow Gr(n + 1, A)$$

taking an n-dimensional subspace N and a vector $d \in A - N$ to the $n + 1$ dimensional subspace $N + Fd$. The domain of definition is the complement of the tautological bundle over $Gr(n, A)$, $\mathscr{T}$, in $Gr(n, A) \times A$, and if we consider the representation of the Grassmannian as wedges, then the map is just the wedge of a basis of N with d:

$$\begin{array}{rcl} Gr(n,A)\times A-\mathscr{T} & \rightarrow & Gr(n+1,A) \\ ([b_1\wedge\cdots\wedge b_n],d) & \mapsto & [b_1\wedge\cdots\wedge b_n\wedge d]. \end{array}$$

We will first change base to $Spec\,\overline{F(A)}$. For ease of notation, denote $\overline{F(A)}=F'$ and $A\otimes_F\overline{F(A)}=A'$. By their functoriality, we have that $(Gr(n,A)\times A-\mathscr{T})\times_F F'\cong Gr(n,A')\times A'-\mathscr{T}'$, where $\mathscr{T}'\cong\mathscr{T}\times_F F'$ is the tautological bundle on $Gr(n,A')$. This is a dense, open subset of $Gr(n,A')\times A'$, and the domain of definition of $p'=p\times_F F' : Gr(n,A')\times A'-\mathscr{T}'\rightarrow Gr(n+1,A')$. Define $M'=M\otimes_{F(A)}F'\subseteq A'$, a maximal right ideal, and consider the open subset of $Gr(n,A')\times A'-\mathscr{T}'$ defined over F' as the elements (N,d) such that $N\cap M'=(0)$. On this dense, open subset of $Gr(n,A')\times A'$ we have a well-defined morphism to M' given by intersection: $(N,d)\rightarrow(N+d)\cap M'$. This morphism will surject M' (simply take any complement to M', N, and any element $d\in M'$), and so the inverse of the generators of M', $(M')^+$, will be a dense, open subset of $Gr(n,A')\times A'$. Call this dense, open subset U, and we can describe it as the set of $(N,d)\in Gr(n,A')\times A'$ such that $d\notin N$, $N\cap M'=(0)$, and $(N+d)\cap M'$ has rank $n-1$.

I claim that the set of F-rational points of U is equal to E. Certainly E is a subset of the F-rational points of U, since these are the F-rational points (N,d) which have the aforementioned properties over $F(A)$. Conversely, given an F-rational point (N,d) of U, then $N\otimes_F F(A)$ must have trivial intersection with M, since $N\otimes_F F'$ has trivial intersection with M'. Thus

$(N \otimes_F F(A) + d) \cap M$ is a singleton, $m_{N,d}$, whose rank over F' is $n-1$. Rank does not change after extending scalars, thus the rank of $m_{N,d}$ over $F(A)$ is $n-1$, and $(N,d) \in E$. Now $Gr(n, A') \times A'$ is rational, and U is a dense, open subset, so its set of F-points, the set E, is dense. It remains to show that $E - E'$ is dense.

Let L be a separable, splitting subring of A', $\alpha \notin L$, and $\beta \in A'^*$. Consider the point $(L\beta, \alpha\beta) \in Gr(n, A') \times A'$. Then $N + Fd = (L + F\alpha) \cdot \beta$ if and only if (N,d) and $(L\beta, \alpha\beta)$ are in the same fiber over p'. We want to show the collection of such $(L\beta, \alpha\beta)$ lie inside a closed subvariety of $Gr(n, A') \times A' - \mathscr{T}'$. Each separable, splitting subring of A' is isomorphic to $\oplus_{i=1}^n F'$, and is conjugate to the diagonal matrices in $A' \cong M_n(F')$. Let $\triangle$ be the separable, splitting subring of diagonal matrices, then for every separable, splitting subring L of A', there exists $g \in A'^*$ such that $g \triangle g^{-1} = L$. Then every $L\beta = g\triangle g^{-1}\beta$. Let $h = g^{-1}\beta \in A'^*$, then these subspaces $L\beta$ are of the form $g\triangle h$, with $g, h \in A'^*$. Then the set of all $(L\beta, \alpha\beta)$ will lie inside the set of $(g\triangle h, a)$, where $g, h \in A'^*$ and $a \in A'$. Note that this set is in $Gr(n, A') \times A'$, and not the complement of $\mathscr{T}'$, since we could choose $a = 0$. However, it is an orbit under the action of a linear, algebraic group. Let $G = A'^* \times A'^* \times A'$. Then G acts on $Gr(n, A') \times A'$ via $(g, h, a) \cdot (N, d) = (gNh, d + a)$. Now the set of $(g\triangle h, a)$ is in the orbit of $(\triangle, 0)$ under this action. Since G is a linear, algebraic group, this orbit is a quasi-projective variety, and is open in its closure. Thus the intersection of this orbit with $Gr(n, A') \times A' - \mathscr{T}'$ has the same property, namely it is open in its closure. Label this subvariety $\mathscr{O}$.

Given a point of U, (N, d), assume there exists a separable, splitting subring L of A', an $\alpha \notin L$, and a $\beta \in A'^*$ such that $N + F'd = (L + F'\alpha)\beta$. Then $(L\beta, \alpha\beta) \in \mathscr{O}$, and (N, d) is in the fiber over $p(\mathscr{O})$. If $(N, d) \in E'$ is an F-rational point in E such that there exists a separable, splitting subring L of A, an $\alpha \notin L$, and a $\beta \in A^*$ with $N + Fd = (L + F\alpha)\beta$, then $L' = L \otimes_F F'$ is a separable, splitting subring of A', $\alpha \notin L \otimes_F F'$, $\beta \in A'^*$, and $N \otimes_F F' + F'd = (L' + F\alpha)\beta$. Thus (N, d) and $(L'\beta, \alpha\beta)$ are in the same fiber over $Gr(n+1, A')$. That is to say, the points of E' are contained in the fiber over $p(\mathscr{O})$. Now, we show that the fiber over $p(\mathscr{O})$ is contained in a closed subvariety of strictly smaller dimension than the dimension of $Gr(n, A') \times A'$. This implies that $E - E'$ contains the F rational points of a dense, open subset of $Gr(n, A') \times A'$, and so is dense.

We know that p is surjective, so to show that the fiber over $p(\mathscr{O})$ is contained in a proper, closed subvariety, we will show that the image, the closure of $p(\mathscr{O})$, has dimension strictly smaller than the dimension of $Gr(n+1, A')$. Let $Y \hookrightarrow Gr(n+1, A')$ be the image of $p(\mathscr{O})$, then

$$dim(Y) \leq dim(\mathscr{O}) \leq dim(A'^* \times A'^* \times A') = n^2 + n^2 + n^2 = 3n^2.$$

The dimension of $Gr(n+1, A')$ is $(n+1)(n^2 - (n+1)) = n^3 - 2n - 1$ and thus we need

$$n^3 - 2n - 1 > 3n^2$$

which for n a positive integer is exactly when $n \geq 4$. $\square$

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This dissertation was typeset with LaTeX[†] by the author.

[†] LaTeX is a document preparation system developed by Leslie Lamport as a special version of Donald Knuth's TeX Program.