The Importance of Teaching Applicable Mathematics

by

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Abstract

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While exploring unfamiliar concepts and striving to grasp higher level mathematics, secondary and postsecondary mathematics students often ask, "When will we ever use this?" Although this question typically stems from students' frustration, skepticism, and confusion, the question has great potential for teachable moments. Mathematics has countless applications in people's daily lives, but the common person often fails to recognize this; those who realize the worldly importance of applicable mathematics often cannot provide specific examples nor understand the rigorous mathematics involved. It is important for mathematics teachers to have a conceptual understanding of the subject, and to be able to provide specific examples of applicable mathematics to students. Although the limit of applicable mathematics examples is infinite, a few cases are explored in this report.

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Introduction

This paper provides specific examples of how mathematics can be applied. Specific applications are discussed from numerous sources, and four mathematical examples are reviewed from: Hahn (1998), Teets (2003), and Klima (1999). These mathematical examples could be beneficial for advanced high school students, calculus students, or students taking other undergraduate courses in mathematics.

"Mathematics now touches people's lives in ways that matter. From symmetry and chaos, computers and cosmology, AIDS epidemiology and nuclear risks, to political polls and ozone depletion, mathematics lurks behind most manifestations of science and technology" (Steen, 1988, p. 422). Since mathematics is such an important field, with infinitely many cases of applicability, it should not be taught as random strands of knowledge, patterns, or theorems, but rather as the ever connected discipline that it is. Teachers should provide students with specific examples of how mathematics can be applied no matter the complexity of the content or the age of the student. When young children learn basic concepts of mathematics, teachers constantly draw pictures or relate the concept to students' lives; these instructional methods should continue to be utilized when teaching higher level mathematics.

As people study mathematics, it is often noted that the ideas and concepts build from one to the next; the level, complexity, and application of mathematics seems infinite. "Once a topic is introduced, applications to other problems will help the students master the new ideas. One of the new problems can lead to a model the students cannot handle, a new topic is introduced, and the cycle begins anew" (Bender, 1973, p. 303).

Unfortunately, in secondary classrooms there are various obstacles preventing students from gaining purposeful insight into higher level mathematics. As a result, frustrated students often give up in mathematics classes and interrupt instruction with the infamous question, “When will we ever use this?” Although this question is typically asked out of boredom or frustration, it could have great potential. Due to lack of time, the complexity of real world examples, and the lack of mathematical understanding, secondary mathematics teachers often give students insufficient, generic answers to this potentially great question.

As a result, “this [applicable mathematics] is *not* the mathematics taught in typical classrooms. Far too often, mathematics in the classroom is a freeze-dried mathematics. Instead of exploration, there is drill; instead of investigation, imitation” (Steen, 1988, p. 418). Additional theory and advanced mathematics courses are often needed for precision in solving problems applicable to the real world. When mathematics is applied, one cannot vaguely solve a problem or ignore tedious details for the sake of getting a point across. For instance, if the precise mathematics of a bridge is not carefully calculated, the results could be devastating. Although secondary teachers may not be able to explore real world examples in entirety with student comprehension, examples of how mathematics can be applied should be discussed in class. Although mathematics is intricate and fun, many people consider that it serves no purpose unless it is applied! Along with teaching mathematics in context, students need to understand the complexity of the discipline and realize that solutions may be challenging and require patience to obtain. “Mathematicians encounter problems daily which they do not know how to solve. Moreover, solution of some of these may possibly require several centuries…the solution of some of which, will, perhaps, be the beginning of a new branch of mathematics." (Frechet, 1948, p. 199)

The answer to the infamous question of how and when mathematics can be used is overwhelming and would require too many pages to answer in full. While mathematics can be applied in many areas, professionals with expertise in a specific field often require assistance from mathematicians in order to effectively apply proper mathematics when solving a problem. "More and more we find business men, professional men, and men interested in such fields as economics, sociology, biology and many others turning to mathematicians for the solutions of their difficulties" (Sleight, 1935, p. 219).

Mathematics can be applied to economics. "Economists never have been able to treat their science without speaking of prices, of numbers, to consider purchase or renting of land without taking into account its area" (Frechet, 1948, p. 205). Frechet continues, "In more modern times they have made more and more use of graphical representations to study, for instance, the variation of prices, consumption, and production, with time" (p. 205). This description of the use of mathematics in economics may be too broad; Franklin (1983) explains the role of mathematics in economics more specifically:

> One surprising thing I found was this: the mathematics was delightful. I knew it was useful, but I hadn't expected it to be beautiful. I was surprised to find that linear programming wasn't just business mathematics or engineering mathematics. Linear programming is one of the many mathematical methods of economics. Here are a few others: quadratic programming, geometric programming, general nonlinear programming, fixed-point theorems—especially the Kakutani theorem; calculus of variations, control theory, dynamics programming; theory of convex sets—especially convex cones; probability, statistics, stochastic processes; finite structures (graph theory, lattice theory); matrix theory; calculus, ordinary differential equations; and special topics like game theory and Arrow's theory of rational preference orderings. (p. 229)

Luce (1964) considers mathematical psychology, which is a field containing many models and a great deal of ongoing research. Luce explains, “Fundamental mathematical notions and results are used throughout mathematical psychology; without a moderate grounding in them one cannot read much of the literature. Many problems are formulated initially in terms of sets, relations, and nonnumerical functions” (p. 374). Luce continues by naming specific examples of mathematics for which mathematical psychology students should understand.

> One cannot penetrate deeply into probability theory, stochastic processes, or advanced statistics without a firm grip on classical analysis, especially the theory of real variables…Elementary probability concepts occur extensively in psychology. Some of the major theorems—e.g., the central limit theorem—are used frequently. (p. 374)

In the field of biomedicine, “Calculus, probability, statistics, operations research, numerical analysis, and logic are used in computational tools, experimental design, data analysis, curve fitting, regression, inferences, correlation, data processing, hospital planning, and medical care,” require mathematics (Wang, 1979, p.498).

In the aircraft industry, specific types of problems that mathematicians and aeronautical engineers collaborate on include:

> 1) Flutter and vibration of airplanes, propellers, and engines, and analysis of servo systems requires:
> a) Solution of a system of linear differential equations with constant coefficients.
> b) Matrix algebra.
> c) Operational calculus and Laplace transforms.
> 2) Pressure distribution around airfoils requires:
> a) Conformal mapping.
> 3) Lift distribution over wings requires:
> a) Solution of an integral equation by expansion of unknown into Fourier Series. (Bollay, 1947, p. 107)

There appears to be a happy marriage between mathematics and engineering, but mathematics is obviously not limited to areas that apply physics. Frechet (1948) explains,"Stereochemistry and crystallography make use of geometry and the theory of groups. Physical chemistry rests upon the theory of differential equations. Genetics is founded on the theory of probability" (p. 199). While on the topic of geometry and science, the ever growing, modern branches of Topology and Combinatorics deserve mentioning.

> Using computational methods, biologists can portray on a computer screen the geometry of a cold virus and search its surface for molecular footholds on which to secure their biological assault. Geneticists are beginning the monumental effort to map the entire human genome, requiring expertise in statistics, combinatorics, artificial intelligence, and data management to organize billions of bits of information. Ecologists continue to use the extensive theories of population dynamics to predict the behavior and interaction of species. Neurologists now use the theory of graphs to model networks of nerves in the body and the neural tangle in the brain. Cell biologists study the replication of DNA using the newly discovered algebraic classification of knots. And, finally, physiologists employ contemporary algorithms applied to nineteenth-century equations of fluid dynamics to determine such things as the effects of turbulence in the blood caused by cholesterol or swollen heart valves. (Steen, 1988, p. 416)

Finally, before reviewing the four selected mathematics problems, it is worth noting a handful of awards earned over the last 30 years in areas where mathematics has successfully been applied. In 1979, the Nobel Prize in medicine went to Allan Cormack for his application of the Radon transform to the development of tomography and CAT scanners. Five years later, the 1984 Nobel Prize in chemistry went to biophysicist Herbert Hauptman for fundamental work in Fourier analysis related to X-ray crystallography (Steen, 1988, p. 416). In 1986, two of the three Fields Medals were awarded in geometry to Michael Freedman and Simon Donaldson for their work in the geometry of four dimensional manifolds. Together Freedman and Donaldson found a deeper understanding of four-dimensional manifolds, and insight from their work has led to applications in string theory (Steen, p. 417).

Examples of Applicable Mathematics

Over the centuries, calculus has contributed to the development of science, engineering, and economics. Hahn (1998) focuses on two historical applications of calculus, both dating back to the seventeenth century. The first application is a pulley problem from Marquis De L'Hospital's 1696 calculus book, *Analyse des Infiniment Petits*. After brief analysis of the problem, one will realize the need to find the maximum value of an algebraic function on a closed interval. The second problem Hahn analyzes comes from a page in Galileo's 1608 notebook; it is an experiment involving a ball rolling down an inclined plane, across a short horizontal (a tabletop), and free falling to the ground.

De L'Hospital's Pulley Problem

In this problem it is assumed that the weight of the pulley and cords can be ignored because the weight at F is much larger. Also in this situation, only consider equilibrium when $r < d$, as shown in Figure 1. If $r \geq d$, the weight at F will hang directly below A.

> At a point B on the horizontal ceiling of a room, attach a cord of length r that has a pulley affixed at its other end, the point C. At a second point A on the ceiling, a distance d from point B, attach a cord of length l, pass it through the pulley at C and connect a weight W at the other end of l, at F. Release the weight and allow the system to achieve its equilibrium position, as shown in Figure 1. (Hahn, 1998, p. 94)

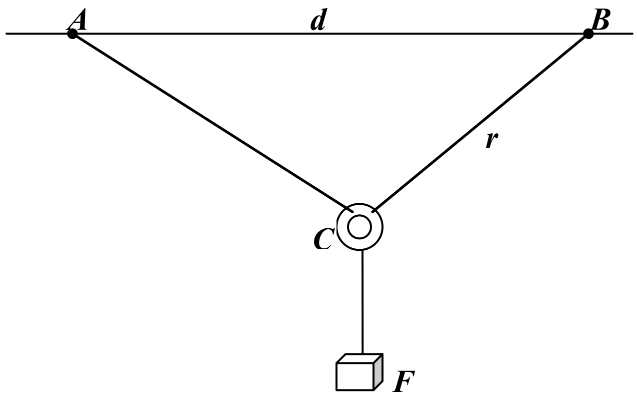

Figure 1. Equilibrium Configuration, $r < d$

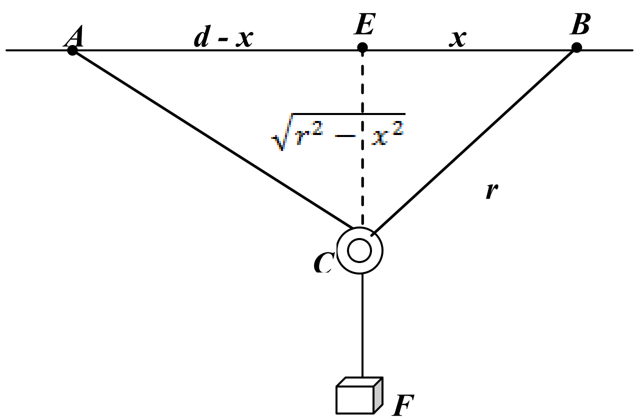

Figure 2. Extension of Equilibrium Configuration

In Figure 2, let *E* be the intersection of the extension of segment *FC* with *AB*. Let *x* denote the distance *EB*. Using the Pythagorean Theorem,

$$AC = \sqrt{AE^2 + EC^2}$$

$$AC = \sqrt{(d-x)^2 + r^2 - x^2}.$$

Thus,

$$CF = l - \sqrt{(d-x)^2 + r^2 - x^2}.$$

Since this system is in equilibrium, the weight at *F* must hang at the lowest possible point, meaning the distance *EF* must be as large as possible.

$$EF = EC + CF$$

$$EF = \sqrt{r^2 - x^2} + l - \sqrt{(d-x)^2 + r^2 - x^2}.$$

De L'Hospital wanted to find the value of x for which *EF* is a maximum on $0 \le x \le r$, in terms of r, d, l, and W, for the function

$$f(x) = (r^2 - x^2)^{1/2} + l - [(d-x)^2 + r^2 - x^2]^{1/2}.$$

In computing the derivative, one can determine where on the interval $[0, r]$ the function fails to exist or is zero. Using the chain rule,

$$f'(x) = \frac{1}{2}(r^2 - x^2)^{-1/2}(-2x) - \frac{1}{2}[(d-x)^2 + r^2 - x^2]^{-1/2}[-2(d-x) - 2x]$$

$$f'(x) = \frac{d(r^2 - x^2)^{1/2} - x[(d-x)^2 + r^2 - x^2]^{1/2}}{(r^2 - x^2)^{1/2}[(d-x)^2 + r^2 - x^2]^{1/2}}.$$

Since the denominator in $f'(x)$ is never zero for $0 < x < r$, the only critical points for this open interval are when the numerator in $f'(x)$ is zero. Thus,

$$d(r^2 - x^2)^{1/2} = x[(d-x)^2 + r^2 - x^2]^{1/2}.$$

Simplifying yields

$$2dx^3 - 2d^2x^2 - r^2x^2 + r^2d^2 = 0.$$

One root of this polynomial is $x = d$, but when only considering $d > r$; this root is outside the interval $(0, r)$. Using long division, divide the numerator of $f'(x)$ by $(x - d)$ to show other roots will satisfy the quadratic equation

$$0 = 2dx^2 - r^2x - dr^2.$$

The quadratic formula shows two other zeros exist:

$$x = \left(\frac{r}{4d}\right)\left(r \pm \sqrt{r^2 + 8d^2}\right).$$

One zero is $x = r - \sqrt{r^2 + 8d^2} < 0$, with this root falling outside the interval $(0, r)$. However, the second root $x = \left(\frac{r}{4d}\right)(r + \sqrt{r^2 + 8d^2}) < r$, is in the interval $(0, r)$, so the maximum value of $f(x)$ on $[0, r]$ occurs at this point. Thus, the equilibrium configuration of L'Hospital's pulley and cords occurs when

$$x = \left(\frac{r}{4d}\right)\left(r + \sqrt{r^2 + 8d^2}\right).$$

Notice the weight W and length of the cord l are not present in this expression of x, implying the equilibrium configuration will be the same regardless of the weight of length of cord.

GALILEO'S EXPERIMENT

In 1608, Galileo performed experiments using an inclined plane with a shallow groove running along it, and a bronze ball with radius of about one centimeter. Galileo would roll the ball down the incline, briefly along a horizontal table, and allow it to fall to the ground through a parabolic path. Various heights were tested.

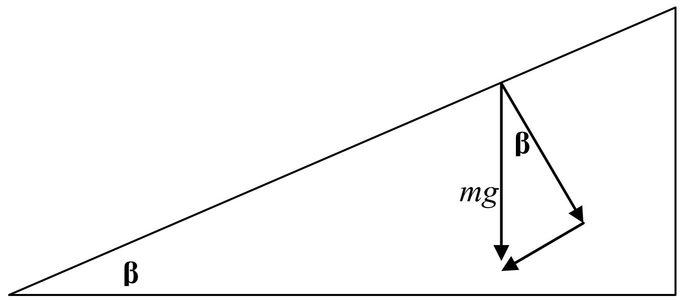

Figure 3. Design of Galileo's Incline

> If the angle of elevation of the inclined plane is *β*, as shown in Figure 3, the gravitational force can be decomposed into a component perpendicular to the plane and a component $mg \sin\beta$ parallel to the plane. Let m be the mass of the ball and g the constant acceleration experienced by falling objects near the surface of the earth. (Hahn, 1998, p.98)

The component of the gravitational force on the ball that is perpendicular to the plane is equal and opposite to the normal force exerted by the plane on the ball, thus these two perpendicular forces cancel out. Due to friction, the ball rolls down the incline rather than sliding down. Denote frictional force by $f(t)$ where *t* is the time that has elapsed since the ball's release. Thus, the total force on the ball from the components parallel to the incline is $mg\sin\beta - f(t)$.

Let $s(t)$ represent the distance the ball has traveled during the time interval [0, *t*]. By Newton's second law of motion (*F* = *ma*),

$$ma(t) = mg\sin\beta - f(t).$$

To understand the motion of the ball, one must recognize and account for the rolling that is involved. The ball rolls a distance *s* represented by

$$s = r\theta,$$

where θ is the angle through which it has turned and *r* is the radius of the ball. Differentiating both sides twice results in

$$s''(t) = a(t) = r\alpha(t),$$

where $\alpha(t)$ is the angular acceleration of the ball. The introduction of angular acceleration, as opposed to just linear acceleration, initiates the discussion of *torque* given by the equation $r \cdot f = I\alpha,$ where *I* represents the *moment of inertia* of a body about the rotational axis. Further discussion can be had, and derivations can be performed to determine the moment of inertia for various shaped objects. In the situation of Galileo's experiment, *I* for a uniform ball can be considered as a solid sphere.

Think of slicing a ball of mass *m* and radius *r* into very thin uniform disks centered on the interval [-*r, r*] of the *x*-axis, as in Figure 4.

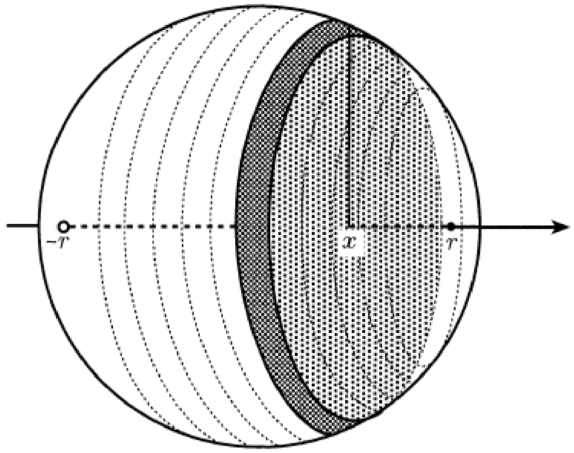

Figure 4. “Sliced” Ball Used in Galileo’s Experiment (Hahn, 1998, p. 100)

The disk centered at *x* has radius $\sqrt{r^2 - x^2}$. Assuming its thickness is *dx*, its volume is approximately $\pi(r^2 - x^2)dx$. “If the mass of the whole ball is *m*, then the density of the material forming the ball is $\frac{m}{\left(\frac{4}{3}\pi r^3\right)}$ mass units per unit volume” (Hahn, p. 100). The mass of the disk at *x* is

$$\frac{3m}{4\pi r^3} \cdot \pi(r^2 - x^2)dx = \frac{3m}{4r^3} \cdot (r^2 - x^2)dx.$$

By taking the integral over the interval [-*r*, *r*], one finds the moment of inertia of a uniform ball to be I = 2/5mr^2. Therefore $r \cdot f$ =2/5$mr^2\alpha$, and simple substitutions yields

$$f(t) = \frac{2}{5}\, ma(t).$$

Finally, substituting the new $f(t)$ into the original equation $ma(t) = mg \sin \beta - f(t)$,

$$a(t) = \frac{5g}{7} \sin \beta.$$

Continuing to use integration, and using the fact that the ball starts at rest, it is known

$$v(t) = \left(\frac{5g}{7} \sin \beta\right) t.$$

Through a second integration, and defining the initial position of the ball as $s(0) = 0$,

$$s(t) = \left(\frac{5g}{14} \sin \beta\right) t^2.$$

Hahn (1998) shows additional calculations to allow one to find the velocity of the ball once it reaches the bottom of the incline (its new, horizontal speed). Using these results, mathematical predictions can be made as to the horizontal and vertical distances the ball will travel off the table in a parabolic path. Table 1 (p. 102) shows how Galileo's experimental results compare with the theoretical calculations made by Hahn.

Table 1. Comparison of Data: Model vs. Galileo

h	*Calculated value of R*	*Galileo's value of R*	*Difference (in inches)*
0.282	0.792	0.752	1.6
0.564	1.120	1.102	0.7
0.752	1.293	1.248	1.8
0.940	1.446	1.410	1.4

These two examples, presented by Hahn (1998), are useful in demonstrating the importance, application, and history of calculus. The methods and ideas developed during the seventeenth century are still significant to calculus in the present day, and these historical applications are good examples for secondary mathematics teachers to understand. Following the NCTM standards (2004, p. 10), "recognizing and applying mathematics in contexts outside of mathematics," teachers should be able to share mathematical history and mathematical applications with their students.

PREDICTING SUNRISE AND SUNSET TIMES

Teets (2003) explores varying levels of mathematics to predict sunrise and sunset times. The author uses basic methods of trigonometry and analytic geometry, then continues further into the Fourier sine series and deeper celestial mechanics. Teets begins his discussion with the *geocentric equatorial* coordinate system shown in Figure 5.

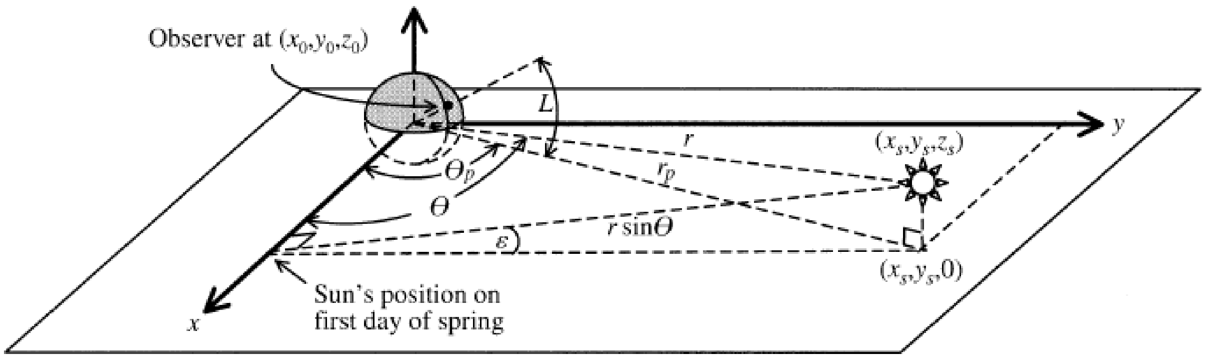

Figure 5. The Earth & Sun in the Geocentric Equatorial Coordinate System (Teets, 2003, p. 317)

Teets (2003) explains,

> …the *xy*-plane contains the earth's equator, with the positive *x*-axis chosen such that it passes through the sun's center on the first day of spring. The angle ϵ is the inclination between the *xy*-plane and the *ecliptic* plane containing the earth's orbit about the sun. At noon on the day of observation, the sun appears directly over the observer's meridian at $\langle x_s, y_s, z_s \rangle$. θ_p is the angle between the projection vector $\langle x_s, y_s, 0 \rangle$ and the positive *x*-axis, L is the observer's latitude, and R is the radius of the earth. Let r be the length of the earth-sun vector $\langle x_s, y_s, z_s \rangle$, θ the angle between this vector and the positive *x*-axis, and r_p the length of the projection vector. (p. 317)

The position of the observer is defined by $\langle x_0, y_0, z_0 \rangle$ where

$$x_0 = R \cos L \cos \theta_p,$$

$$y_0 = R \cos L \sin \theta_p,$$

$$z_0 = R \sin L.$$

Since the earth rotates 2π radians in 1440 minutes, the observer's coordinate t minutes from noon is given by

$$x_0 = R\cos L\cos\left(\theta_p + \frac{2\pi}{1440}t\right),$$

$$y_0 = R\cos L\sin\left(\theta_p + \frac{2\pi}{1440}\right)t,$$

$$z_0 = R\sin L,$$

with $t < 0$ before noon, and $t > 0$ after noon.

Teets (2003) notes that the observer's location with respect to different time zones matters, and assumes the observer to be on a *standard time meridian* corresponding roughly to the middle of a time zone. Sunrise and sunset times for observers not on the meridian, must be adjusted by 1440minutes/360° = 4 minutes per degree of longitude, with earlier sunrise and sunset times for observers to the east and later times for observers to the west within a time zone.

The observer's location $\langle x_0, y_0, z_0\rangle$ can be thought of as a normal vector for the plane tangent to the earth's surface at the observer's location. The equation of the tangent plane is

$$x_0(x - x_0) + y_0(y - y_0) + z_0(z - z_0) = 0.$$

"The sunrise and sunset times can be approximated as *the times when the center of the sun lies in the tangent plane*," (Teets, 2003, p. 318). The times at which

$$x_0(x_s - x_0) + y_0(y_s - y_0) + z_0(z_s - z_0) = 0.$$

By using the fact that $x_0^2 + y_0^2 + z_0^2 = R^2$, simplify the above equation to

$$x_0x_s + y_0y_s + z_0z_s = R^2.$$

By substituting the values previously defined for x_0, y_0, z_0,

$$x_s R\cos L cos\left(\theta_p + \frac{2\pi}{1440}t\right) + y_s R\cos L\ sin\left(\theta_p + \frac{2\pi}{1440}t\right) + z_s R\sin L = R^2.$$

Using basic angle addition formulas for sine and cosine, one can solve for t such that

$$t_0 = \frac{1440}{2\pi}\cos^{-1}\left(\frac{R - z_s\sin L}{r_p\cos L}\right).$$

The value t_0 represents the number of minutes after noon that sunset occurs, while $-t_0$ represents the number of minutes before noon that sunrise occurs. In assuming that the earth's orbit is circular and letting *r* equal the earth's average distance from the sun, then

$$\theta = \frac{2\pi}{365.25}(d - 80),$$

with *d* the day of the year. To account for these calculations being made from the center of the sun, t_0 can be replaced with $\hat{t} = t_0 + 5$ minutes.

Teets (2003) recognizes the assumptions and causes of potential error with the basic calculations performed above. One of the most obvious errors is previously assuming the orbit of the sun to be *circular*. However, according to Kepler's first law, the earth's orbit is elliptical. Secondly, "according to Kepler's second law, the vector from the sun to earth sweeps out equal areas in equal times, so θ does not change at a constant rate" (p. 319). The third source of error is an effect from the *equation of time*. The basic idea is that, "the length of time between noon one day and noon the next is not constant" (p. 319).

Teets (2003) explains two main reasons for the variability in the *equation of time,* and accounts for these errors by using more complex levels of mathematics.

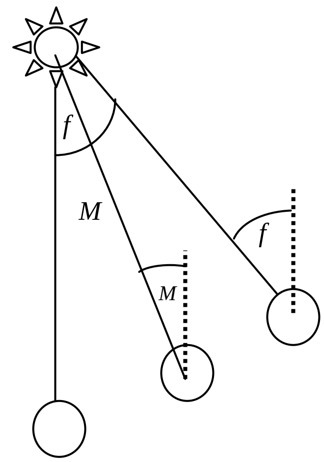

Figure 6. Solar Noon on Successive Days

In Figure 6, *M* denotes the angle at which the earth moves from noon one day until noon the next, while assuming a constant orbital speed. The first reason for variability is due to Kepler's second law (varying speeds) causing the angle to be slightly larger than *M* and is instead *f*. Thus, "the earth must rotate *f* – *M* radians *more* than average from noon to noon, saying that noon is *delayed* $\frac{1400}{2\pi}\ (f - M)$ minutes" (p.320).

By accurately taking *f* and *M* into account and expanding their difference in a Fourier sine series, the delay can be approximated for any given day *d* as
$8\sin\left(\frac{2\pi}{365.25}d\right)$ minutes.

The second cause of variability is that the earth's orbital motion and daily rotation take place in different planes; the extra amount of 2π radians that the earth must rotate in order for the sun to be in the overhead position each day, must take into account a term whose Fourier approximation is $-10\sin\left(\frac{4\pi}{365.25}(d-80)\right)$.

By following more complex mathematics and accounting for the errors, "noon" will vary according to

$$n = 720 - 10\sin\left(\frac{4\pi}{365.25}(d - 80)\right) + 8\sin\left(\frac{2\pi}{365.25}d\right)$$

minutes past midnight.

From the work of Teets (2003) it is important to recognize that real world ideas can be analyzed with varying levels of mathematics. As a result, the accuracy and value of one's calculations depends on the level of mathematics used. This article can enhance secondary mathematics by showing different levels of mathematics, so younger students can see there is much to learn in the field of mathematics. The analysis Teets provides regarding error implies there is "reflection on the process of mathematical problem solving," which is an NCTM (2004, p. 6) standard. Two other NCTM standards that Teets's article exhibits include, "understanding how mathematical ideas interconnect and build on one another" and "recognizing and applying mathematics in contexts outside of mathematics classes" (2004, p. 6).

AMORTIZATION OF LOANS

Klima & Donnelly (1999) provide a useful application of the Intermediate Value Theorem and the Monotonicity Theorem using the amortization of loans. Amortization is thought of as regularly spaced, equal payments made towards the repayment of a loan that has interest adding on at some fixed rate. Car loans and mortgages are common examples of amortization. Klima & Donnelly consider examples for which the time between loan payments is equal to the time between applications of interest (e.g. a loan being repaid in monthly installments which also accrues interest at a fixed monthly rate).

Using the variables P = original loan principal, M = periodic loan payment, n = number of payments to repay the loan, and r = periodic interest rate. It is assumed that all of these variables represent positive quantities and n is an integer. During the repayment of amortized loans, part of a payment is applied directly to the remaining balance of the loan, while the remaining part of a payment is applied toward the interest. This is denoted by P_m, the principal part of the m^{th} payment and I_m, the interest part of the m^{th} payment, where

$$P_m = M - I_m \text{ and } I_m = r\cdot(\text{remaining balance}).$$

Note that the sum of the principal parts must equal the original loan, i.e. $\sum_{i=1}^{n} P_i = P$.

Klima & Donnelly (1999) consider a problem where the values of P, M, and n are fixed and valid, corresponding to a unique positive interest rate r. Then for $1 < m \leq n$,

$$P_m = M - r\,(P - \textstyle\sum_{i=1}^{m-1} P_i).$$

Induction can easily be used to show that for $1 < m \leq n$,

$$P_m = (1+r)^{m-1}(M - rP).$$

Thus, the equation relating P, M, n, and r is given by:

$$P = \sum_{i=1}^{n} P_i = \sum_{i=1}^{n} (1+r)^{i-1}(M - rP)$$

$$= (M - rP)\cdot\frac{(1+r)^n - 1}{(1+r) - 1} = \frac{(M - rP)}{r}.$$

Hence

$$\frac{rP}{M - rP} + 1 = (1+r)^n,$$

which can be rewritten as

$$\frac{M}{M - rP} = (1 + r)^n. \ (1)$$

As one may find, it is easy to solve this equation for P, M, or n in terms of the other variables, but it is difficult to solve for r. The Intermediate Value Theorem and Monotonicity Theorem can be applied to show that for fixed values of P, M, and n there is a unique r that satisfies (1) with $0 < r < \frac{M}{P}$.

Start by letting $f(x) = (M - xP)(1 + x)^n - M$, with $x > 0$. Then $r > 0$ satisfies (1) if and only if r is a zero of f. Next, show that f takes on positive and negative values for $x \in \left(0, \frac{M}{P}\right]$. Since f is continuous, by the Intermediate Value Theorem there must be a zero of f in $\left(0, \frac{M}{P}\right]$. It does not make sense in (1) for $r = \frac{M}{P}$, but $f\left(\frac{M}{P}\right) = -M < 0$. Since $x > 0$ and n is a positive integer, the Binomial Theorem allows

$$(1 + x)^n = 1 + xn + \cdots + x^n \geq 1 + xn.$$

For all $x \in \left(0, \frac{M}{P}\right]$, $M - xP \geq 0$, and hence

$$(M - xP)(1 + x)^n - M \geq (M - xP)(1 + xn) - M.$$

Define

$$g(x) = (M - xP)(1 + xn) - M;$$

then

$$f(r) \geq g(r) \text{ for all } r \in \left(0, \frac{M}{P}\right].$$

Furthermore, $nM - P$, is positive, and so $\frac{M}{P} - \frac{1}{n} > 0$. Then for $\hat{r} = \frac{M}{P} - \frac{1}{n} \in \left(0, \frac{M}{P}\right)$, and $g(\hat{r}) = 0$. $f(\hat{r}) \geq g(\hat{r})$, and hence $f(\hat{r}) \geq 0$.

The goal is to find a value of r in $\left(0, \frac{M}{P}\right)$ for which $f(r) = 0$. If $f(\hat{r}) = 0$ then $r = \hat{r}$. If $f(\hat{r}) > 0$, then along with $\frac{M}{P}$, the values of r occur in $\left(0, \frac{M}{P}\right]$ for which f takes on both positive and negative values. By the Intermediate Value Theorem there is a value of r in $\left(0, \frac{M}{P}\right)$, for which $f(r) = 0$. Thus such an r exists of $r \in \left(0, \frac{M}{P}\right)$, satisfying (1). (Klima & Donnelly, 1999, p. 390)

To complete the proof, Klima & Donnelly (1999) use the Monotonicity Theorem to show the uniqueness of r and to verify that f is monotone decreasing. The derivative of $f(x) = (M - xP)(1+x)^n - M$ is simply

$$f'(x) = (M - xP)n(1+x)^{n-1} - P(1+x)^n$$

$$= (1+x)^{n-1}[n(M - xP) - P(1+x)],$$

and is negative for all $x > 0$. Naturally, $M - rP$ is part of the first payment applied to the principal. The principal part of each payment increases with each payment, so it must be true that $n(M - rP) \leq P$. Mathematically, for $1 \leq m \leq n$,

$$M - rP \leq (1+r)^{m-1}(M - rP), \text{ so}$$

$$M - rP \leq P_m, \text{ and}$$

$$n(M - rP) \leq \sum_{i=1}^{n} P_i = P.$$

Since r is positive, then $P < P(1+r)$, implying $n(M - rP) < P(1+r)$. Thus

$$(1+r)^{n-1}[n(M - rP) - P(1+r)] < 0.$$

This shows that $f'(r) < 0$ for all $r > 0$, and it can be concluded that f is monotone decreasing on its domain. Since Klima & Donnelly already showed the existence of

$r \in \left(0, \frac{M}{P}\right)$ for which $f(r) = 0$, it must follow that r is the unique interest rate corresponding to the fixed values of P, M, and n.

This amortization example discussed by Klima & Donnelly (1999) is yet another way to show mathematics students, or people in general, an application of mathematics in the real world. Mortgages and loans are prevalent in today's society and the amortization within is important for people to understand.

Conclusion

Mathematicians would agree with Steen (1988) that, “Mathematics itself is beautiful, powerful, and deep; the process of doing mathematics is personally stimulating and intellectually rewarding” (p. 421). The power of mathematics is immeasurable because, “We find mathematics hidden away in the most unexpected places and in most ordinary things. It is wedging itself into so many new phases of life” (Sleight, 1935, p. 223). Mathematics can be applied to so many aspects of life and at such varying levels. Therefore, it is important as teachers and mathematicians to educate oneself, students, and the public regarding the field of mathematics and the dependence of the world on mathematics. “It should be kept in mind that *mathematics retained* in a usable form, *not mathematics taught*, is the important variable” (Bender, 1973, p. 304). When mathematics is understood it serves as a persuasive tool; in reality, mathematics does little by itself, but once applied to problems it can do wonders.

> I believe all mathematics is applicable, although some [mathematics] may take a couple hundred years before it can actually be applied to a real problem. For example, although the theory of functions of complex variables was an abstraction when invented by Gauss, it is now an integral part of applied mathematics. (Wang, 1979, p. 498)

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