Spherical and Hyperbolic Geometry
In the High School Curriculum

by

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Abstract

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The structure of Euclidean, spherical, and hyperbolic geometries are compared, considering specifically postulates, curvature of the plane, and visual models. Implications for distance, the sum of the angles of triangles and circumference to diameter ratios are investigated.

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INTRODUCTION

In the state of Texas the Texas Essential Knowledge and Skills (TEKS) delineate the high school curriculum for mathematics (TAC, 2009). For geometry the TEKS cover six broad areas: basic understandings, geometric structure, geometric patterns, dimensionality, congruence, and similarity. Each of these six areas includes from two to six topics that must be taught, for a total of 22 topics among all of the areas. Additionally, eight of these topics describe 26 student performance expectations. These 40 teaching objectives for geometry at the high school level build on the geometry taught in middle school and elementary school, and all 40 objectives are grounded in Euclidean geometry. Although not referred to specifically as Euclidean in the TEKS, the geometry identified in the teaching areas and topics is Euclidean nonetheless. It is the geometry of the world around us and of objects that can be held and measured. Despite this clear focus on Euclidean geometry, only one of these 40 objectives mentions Euclidean geometry by name: "The student compares and contrasts the structures and implications of Euclidean and non-Euclidean geometries" (Texas Administrative Code, 2009, Ch.111.34.b.1.C). This objective mentions Euclidean geometry, but it also introduces non-Euclidean geometries to the teaching objectives.

Geometry in high school is the study of points, lines, planes, and angles; their properties and relationships; and questions of size, position and shape. Euclidean geometry is an axiomatic system based on the axioms of Euclid; a non-Euclidean geometry is any other axiomatic system of geometry. Both geometries study the same objects and ask the same questions. However, in a typical secondary geometry course only a small amount of instructional time is spent on alternate geometries. Instruction about non-Euclidean geometries is constrained by limited time and the limited knowledge of the teacher.

While the TEKS are not clear about the depth of understanding of multiple geometries required of students, this paper asserts that if teachers have a more than minimal knowledge of these geometries, students will benefit. This paper looks for accessible and useful information about the structure and implications of multiple geometries that would serve high school geometry teachers as foundational knowledge.

STRUCTURE OF GEOMETRY

The TEKS require students to compare and contrast the structure of different geometries. Structure in geometry is the system of axioms, postulates, and undefined basic terms. Euclid's five postulates are the basis for the geometry of space taught throughout elementary, middle and most of high school. The postulates are rarely stated in the classroom until briefly in high school, and then approximations are used.

Euclid's five postulates state:

> Let the following be postulated:
> 1. To draw a straight line from any point to any point.
> 2. To produce a finite straight line continuously in a straight line.
> 3. To describe a circle with any centre and distance.
> 4. That all right angles are equal to one another.
> 5. That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles. (Euclid, 2002, p.2)

These postulates are understood to mean:

1. Any two points describe a line segment.

2. Extend the two ends of a line segment indefinitely to describe a line.

3. A circle is described by its center and radius.

4. All right angles are equal.

5. Through a point not on a given line, exactly one line can be drawn parallel to the given line.

Postulates are elementary mathematical statements, so obviously true that they can be accepted as true without proof. Euclid's fifth postulate, the parallel postulate, is the only one of the postulates considered controversial. The fifth postulate is less elementary, less obviously true than the other four postulates. Efforts to prove, or disprove, the truth of Euclid's parallel postulate were unsuccessful. Changes to this postulate, however, are compatible with the remaining four postulates. The structures of spherical and hyperbolic geometries involve a change to this parallel postulate, thus creating non-Euclidean geometries. If the fifth postulate is changed to read: through a point not on a given line, no lines can be drawn parallel to the given line, spherical geometry results, having no parallel lines. If the postulate reads: through a point not on a given line, infinitely many lines can be drawn parallel to the given line, hyperbolic geometry results.

Spherical and hyperbolic geometry did not originate from this change to the parallel postulate. According to Rickey (1992) navigation on the globe and cartography involved manipulating geometric figures on a sphere centuries before spherical geometry developed as an axiomatic system. In Barnett's (2004) history of hyperbolic functions, hyperbolic geometry was found necessary to account for the objects and properties of hyperbolic trigonometry. The change to the parallel postulate allowed alternate geometries, consistent with Euclidean geometry, to develop as axiomatic systems.

Euclid's basic terms remain unchanged in the non-Euclidean geometries. A point is still zero-dimensional, location with no size, "that which has no part" (Euclid, 2002, p.1). A line is a one-dimensional object of "breadthless length" (p.1). A plane is a two-dimensional object with length and width but no thickness, a surface composed of lines, "a surface which lies evenly with the lines on itself" (p.1). These basic objects are modeled adequately in Euclidean geometry as a poppy seed, a piece of uncooked spaghetti, and a stiff piece of paper.

Lines in Euclidean geometry are straight, extend indefinitely, and are flat. In spherical geometry lines comprise the surface of the sphere; spherical lines are straight, curved, and extend indefinitely. These spherical lines are great circles whose plane passes through the center of the sphere. On our globe, longitude lines and the equator model great circles; other latitude lines do not. Note that any two distinct great circles intersect twice, thus there are no parallel lines on the spherical plane. In hyperbolic geometry lines appear to be either segments of circles or straight lines. Note, however, that 'flat' is not a requirement for the line, or the resultant plane. A change in the curvature of the plane entails a change to the parallel postulate, or conversely, a change in the parallel postulate entails a change in the curvature of the plane. In Euclidean the curvature is zero, the plane is flat; in spherical the curvature is greater than zero; and in hyperbolic the curvature of the plane is less than zero.

The visual model of the *Euclidean plane* E^2 is a desktop, a blackboard, or a flat sheet of paper on which lines are drawn. These approximations of a flat plane are used in the classroom from elementary school through high school. As analogies for E^2, a plane with zero curvature, they are familiar and adequate. For spherical geometry, the visual model for the plane is a sphere. The outer surface of the sphere has a constant positive curvature. As a visual model the sphere is accessible and intuitive. A visual model for the *hyperbolic plane* H^2 is more difficult to obtain. The hyperbolic plane is an imaginary object in the sense that it is an abstract surface in complex space. Mykytuik and Shenitzer (1995) note that Euclidean figures and their properties, Euclidean geometry, arose as abstractions or simplifications of physical objects while hyperbolic geometry arose as a logical necessity for abstract relationships. The development of a visual model, the Beltram-Klein disk, was essential to the acceptance of hyperbolic geometry as an alternative geometry according to Mykytuik and Shenitzer. This paper will consider three visual models of H^2: the hyperboloid, the Poincaré disk, and the upper half-plane.

The hyperbolic plane when projected into *Minkowski space* M^3 is the inside surface of a curved but non-spherical half ball, the pseudosphere or hyperboloid (Figure 1). This half ball has an ever-increasing radius, such that points on its 'edge' are points at infinity. Reynolds (1993) finds this model useful and accessible in introductory hyperbolic geometry. For Reynolds this model yields results for distance, angle measure, area, and trigonometric ratios that are consistent with the abstract results of hyperbolic geometry.

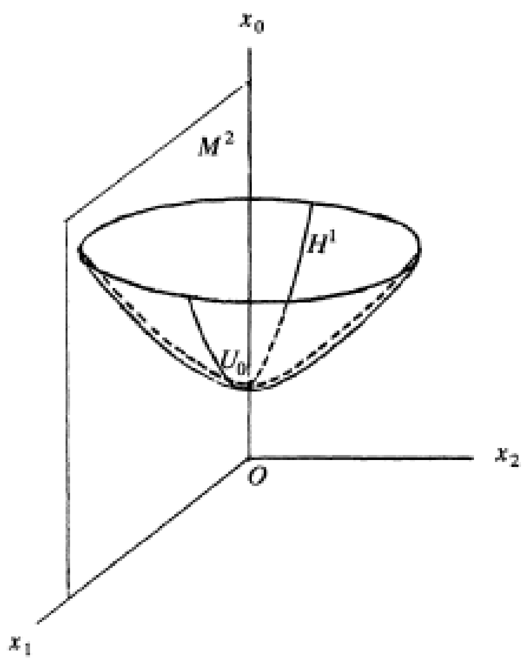

Figure 1 Hyperboloid Model Reynolds (1993, p. 444)

The Poincaré disk results from a projection of certain three-dimensional objects (Beltram-Klein disk or pseudosphere) onto a plane (Figure 2). This model of H^2 is easy to provide in the classroom – just draw a circle! The difficulty in the high school classroom is to differentiate the disk from a Euclidean circle. The Euclidean circle is not the edge of the disk; once again, points at this 'edge' are points at infinity. Measurement in this disk will require an unusual ruler. Lines are now either arcs of circles perpendicular to the 'edge' or diameters of the disk. Fenn (1983) uses a Poincaré disk to examine the geometry of the surface of H^2, specifically transformations and tiling. Since distance on the disk is very different from distance on E^2, the tiling of H^2 as a Poincaré disk looks very different from the tiling of E^2.

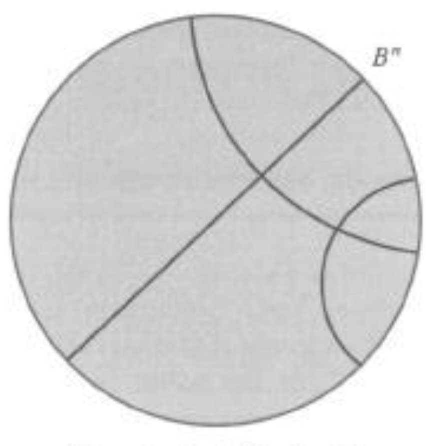

Figure 2 Poincaré Disk Model Jeffers (2000, p. 802)

A stereographic projection of the Poincaré disk yields the Poincaré half-plane or upper half plane visual model for H^2 (Figure 3). Millam (1980) advocates this model as accessible and useful to high school geometry students. The plane is the upper half of the complex plane; points on this plane have a positive imaginary part. Millam identifies two types of lines: semicircles perpendicular to the horizontal axis or lines vertical to the horizontal axis. Since an infinite number of distinct semicircles can share a point not on a given line, those infinitely many semicircles are now the infinitely many parallel lines of the parallel postulate. Millam shows that on this part of the complex plane for H^2 distance is now measured on a logarithmic scale and angles are measured from the line tangent to the curve.

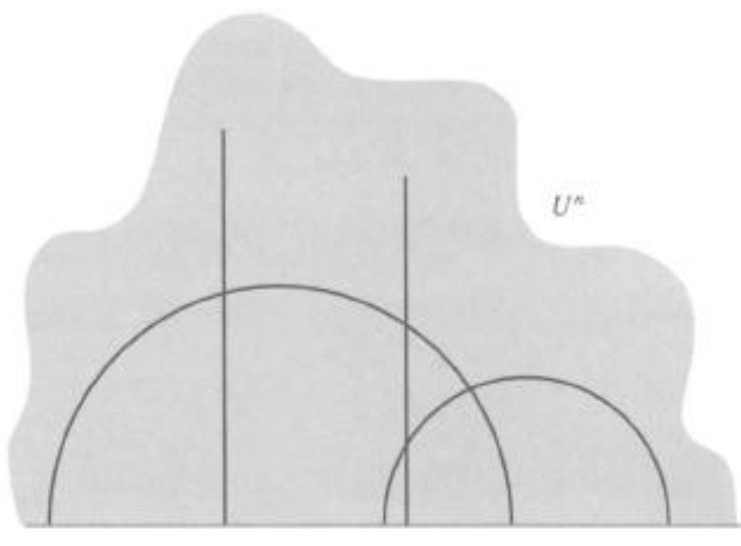

Figure 3 Upper Half-Plane Model Jeffers (2000, p. 801)

Spherical and hyperbolic geometry as axiomatic systems involve a change to Euclid's parallel postulate, reflecting a change in the shape of the plane from flat to curved. Not all theorems in geometry derive from the parallel postulate. According to

Hartshorne (2003), theorems involving triangle congruence, the concurrence of altitudes and of angle bisectors in triangles, and the congruence of the base angles of isosceles triangles remain unchanged by changes to the parallel postulate. These are the theorems of neutral geometry.

In Euclidean geometry the circumference to diameter ratio is π; the sum of the angles of any triangle is 180˚ or π radians; and the length of a segment on the Cartesian plane, as derived from the Pythagorean Theorem, is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. Euclidean ratio, sum, and length have different values in spherical and hyperbolic geometry.

IMPLICATIONS

In spherical geometry the angle between the great lines is the angle of the planes (through the center of the sphere) defining the lines at the point of intersection with the plane tangent to the sphere. The formula for the sum of the angles of a triangle is determined by McCleary (1994) using the area of the triangle as derived from the area of the lunes. Since the area of the sphere is $4\pi r^2$, if θ is the angle from the center of the sphere to two great circles, the area of the lune is:

$$\frac{\theta}{2\pi} \cdot 4\pi r^2 = 2\theta r^2.$$

Three great circles intersect to form a triangle, so three lunes overlap to form a triangle. If the lunes have central angles of α, β, and γ, then the area of the hemisphere is

$$2\pi r^2 = 2\alpha r^2 + 2\beta r^2 + 2\gamma r^2 - 3(\text{area of } \Delta) + 1(\text{area of } \Delta).$$

Therefore the area of a triangle on the sphere is $r^2(\alpha + \beta + \gamma - \pi)$. Note that the sum of the angles of the triangle is greater than π radians or 180˚.

To derive the Pythagorean Theorem for spherical geometry, according to McCleary (1994), consider the sides of the triangle, a, b, c, as defined by their central angles, α, β, γ, and the vertices, A, B, C, as vectors from the center of the sphere. The length of a side of the triangle is the measure of the angle between the vertex vectors scaled by the radius of the sphere. The angle between the vertex angles can be found using the Law of Cosines:

$$\|A - B\|^2 = \|A\|^2 + \|B\|^2 - 2\|A\|\|B\| \cos\gamma$$

$$A \cdot B = \|A\|\|B\| \cos\gamma$$

$$\cos\gamma = \frac{A \cdot B}{\|A\|\|B\|}.$$

Using the basic vectors i, j, and k, let the vertex vectors in component form be

$$A = ri$$

$$B = r\cos\beta\cos\alpha\, i + r\sin\beta\cos\alpha\, j + r\sin\alpha\, k$$

$$C = r\cos\beta\, i + r\sin\beta\, j, \text{ such that}$$

$$A \cdot B = r(r\cos\beta\cos\alpha)$$

$$\|A\| = r$$

$$\|B\| = \sqrt{(r\cos\beta\cos\alpha)^2 + (r\sin\beta\cos\alpha)^2 + (r\sin\alpha)^2} = r\,.$$

Therefore

$$\cos\gamma = \cos\alpha\cos\beta,$$

$$a = \alpha r \text{ or } \alpha = \frac{a}{r}, b = \beta r \text{ or } \beta = \frac{b}{r}, c = \gamma r \text{ or } \gamma = \frac{c}{r}.$$

Thus $\cos\frac{c}{r} = \cos\frac{a}{r}\cos\frac{b}{r}$, the spherical version of the Pythagorean Theorem.

To connect this spherical version to its more familiar Euclidean form, $a^2 + b^2 = c^2$, Veljan (2000) uses the power series expansion of $\cos x$:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

$$1 - \frac{1}{2!}\left(\frac{c}{r}\right)^2 + \frac{1}{4!}\left(\frac{c}{r}\right)^4 - \cdots =$$

$$\left(1 - \frac{1}{2!}\left(\frac{a}{r}\right)^2 + \frac{1}{4!}\left(\frac{a}{r}\right)^4 - \cdots\right)\left(1 - \frac{1}{2!}\left(\frac{b}{r}\right)^2 + \frac{1}{4!}\left(\frac{b}{r}\right)^4 - \cdots\right)$$

$$c^2 - \frac{1}{12}\cdot\frac{c^4}{r^2} + \cdots = a^2 + b^2 - \frac{a^2b^2}{2r^2} - \frac{a^4}{12r^2} - \frac{b^4}{12r^2} + \cdots$$

Notice that as r becomes large, this power expansion converges to $c^2 = a^2 + b^2$, showing that as the radius of the sphere becomes very large, the curvature of the sphere becomes flatter and approaches Euclidean space.

Andersen, Stumpf, & Tiller (2003) place a Euclidean circle on a sphere and use Euclidean and spherical features to compare circumference to diameter ratios in Euclidean and spherical geometry (Figure 4). In Euclidean space this ratio of the circle is always π; in spherical space this ratio of a circle has a range of values from π to - .6824595706 (Andersen, Stumpf & Tiller, 2003, p. 229). Consider an example of a great circle as a circle on the sphere. The circumference is the arc length of the great circle and the diameter is the arc length between antipodal points, or half of the circumference. The circumference to diameter ratio is $\frac{2}{1}$, not π.

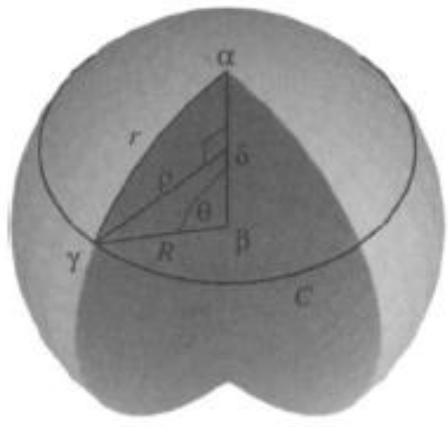

Figure 4 Euclidean Circle on a Sphere Andersen, Stumpf & Tiller (2003, p. 227)

The circle considered in Euclidean space has a center and radius that lie on the flat plane of Euclidean space. The radius can be as large as the radius of the sphere, but no larger. The circle in spherical space has a center and radius that lie on the curved surface of the sphere. The radius of the circle in spherical space is an arc length and is not limited by the radius of the sphere. In fact, the expansion of a circle can begin at one pole, expand beyond the equator, and expand even beyond the opposite pole, giving rise to the possibility of negative values for the ratio.

Following Andersen, Stumpf, & Tiller (2003) consider a sphere with radius R and a circle of circumference C with radius r on the sphere and with a Euclidean radius ρ. Let the center of the circle on the sphere be α, the center of the sphere β, a point on the circle γ, the Euclidean center of the circle δ, and angle $\gamma\beta\alpha$ is θ. Then $r = R\theta$, $\sin\theta = \frac{\rho}{R}$ and $C = 2\rho\pi$, so that $C = 2\pi\frac{r}{\theta}\sin\theta$. Let the circumference to diameter ratio be Π, so that the ratio in terms of θ is

$$\Pi(\theta) = \frac{2\pi\left(\frac{r}{\theta}\sin\theta\right)}{2r} = \pi\frac{\sin\theta}{\theta}.$$

As $\theta \to 0, \Pi \to \pi$ and the sphere is locally Euclidean. As the circle expands and approaches being a great circle, $\theta \to \frac{\pi}{2}$ and $\Pi \to 2$. As the circle expands toward the point opposite α, $\theta \to \pi, \Pi \to \pi\frac{\sin\pi}{\pi} = 1$. As the circle wraps around the sphere and back over itself, $\theta \to 2\pi, \Pi \to \pi\frac{\sin 2\pi}{2\pi} = 0$. Note that although both θ and the diameter continue to increase the absolute value of the circumference has a maximum value, the arc length of a great circle. As the circle begins to wrap around the sphere, θ moves from π to 2π, thus the circumference can assume a negative value; therefore negative values for Π occur. To find the minimum value for the ratio, set the first derivative of Π equal to zero.

$$\Pi'(\theta) = \pi\frac{(\theta\cos\theta - \sin\theta)}{\theta^2} = 0$$

$$\tan\theta = \theta$$

Using the bisection method, $\theta \approx \pm 4.4934095$, such that the minimum value for the ratio is

$$\Pi(4.4934095) \approx -.682459705.$$

Therefore, the range of values for the circumference-diameter ration on the sphere is

$$-.682459705 \leq \Pi \leq \pi.$$

The spherical plane has a constant positive curvature; the hyperbolic plane has a constant negative curvature. In spherical geometry the sum of the angles of a triangle is greater than π and the area of the triangle is proportional to the angle excess; in hyperbolic geometry the sum of the angles of a triangle is less than π and the area of the triangle is proportional to this angle defect. The sides of triangles on the spherical plane have the relationship $\cos\left(\frac{c}{r}\right) = \cos\left(\frac{a}{r}\right)\cos\left(\frac{b}{r}\right)$; on the hyperbolic plane this relationship becomes $\cosh c = \cosh a \cosh b$.

The surface of a sphere, the plane of spherical space, is described by

$$x^2 + y^2 + z^2 = R^2.$$

Delman and Galperin (2003) compare this spherical plane to H^2. Let this pseudosphere or hyperboloid have an imaginary radius iR and an imaginary z coordinate it. The hyperbolic plane can be described by

$$x^2 + y^2 + (it)^2 = (iR)^2$$

$$x^2 + y^2 - t^2 = -R^2.$$

The length of the vertex vectors is now $\sqrt{x^2 + y^2 - t^2}$ and the length of every radial vector is iR. The tangent vectors are expressed in terms of differential expressions so that the length of a tangent vector is $\sqrt{dx^2 + dy^2 - dt^2}$. In terms of cylindrical coordinates $(dr, rd\theta, dt)$, this expression becomes $\sqrt{r^2 d\theta^2 + (dr^2 - dt^2)}$. The distance between points on the pseudosphere is found by integrating the lengths of the tangent vectors. Reynolds (1993), working in Minkowski space M^3, shows this integral as

$$d(A,B) = \int_{a_1}^{b_1} \sqrt{-\frac{t^2}{1+t^2} + 1}\,dt = \left[\ln\left(t + \sqrt{t^2+1}\right)\right]_{a_1}^{b_1} = \sinh^{-1} b_1 - \sinh^{-1} a_1.$$

Fenn (1983) looks at distance between points on the Poincaré disk. In this model $A, B, C,$ and D are points on the complex plane such that the line through A and B meets infinity at C and D. Fenn defines distance on the disk as the logarithm (natural log) of the cross ratio:

$$d(A,B) = \log \frac{(A-C)(B-D)}{(A-D)(B-C)}.$$

Millam (1980) uses the upper half-plane model for hyperbolic space where $A\ (x_1, y_1)$ and $B(x_2, y_2)$ are points on a line, a is the center of the semicircle, and r is the radius of the semicircle. Now hyperbolic distance is defined as

$$d(A,B) = \left|\ln \frac{\frac{x_2-a+r}{y_2}}{\frac{x_1-a+r}{y_1}}\right|.$$

McCleary (1994) connects the distance as defined by Fenn (1983) and Millam (1980). Let $A = z_1 = x_1 + iy_1, B = z_2 = x_2 + iy_2, C = a - r,$ and $D = a + r$. Now

$$d(A,B) = \log \frac{(A-C)(B-D)}{(A-D)(B-C)} = \left|\ln \frac{z_1-(a-r)}{z_1-(a+r)} \cdot \frac{z_2-(a+r)}{z_2-(a-r)}\right|$$

$$d(A,B) = \left|\ln \frac{x_2-a+r}{x_1-a+r} \cdot \frac{y_1}{y_2}\right|.$$

In all cases, though, the hyperbolic version of the Pythagorean theorem is

$$\cosh c = \cosh a \cosh b.$$

This version of the Pythagorean theorem is connected by Veljan (2000) to the familiar Euclidean version once again by considering the power series expansion

$$\cosh x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

Note that in hyperbolic space when the distances a and b are very small, the plane is locally Euclidean.

Andersen, Stumpf, and Tiller (2003) consider the circumference to diameter ratio on the pseudosphere of imaginary radius R and curvature $-\frac{1}{R^2}$. Now circumference is

$$2\pi iR \sin\frac{r}{iR} = 2\pi ir\frac{\sin\theta}{\theta} = 2\pi ir\frac{e^{-i^2\theta} - e^{i^2\theta}}{2i}\cdot\frac{1}{\theta} = 2\pi r\frac{\sinh\theta}{\theta}$$

The circumference to diameter ratio $\Pi(\theta)$ in H^2 is

$$2\pi r\frac{\sinh\theta}{\theta}\cdot\frac{1}{2r} = \pi\frac{\sinh\theta}{\theta}$$

In H^2 as $\theta \to 0, \Pi \to \pi$ and the plane is locally Euclidean; for all other values of $\theta, \Pi > \pi$.

CONCLUSION

Euclidean geometry provides one way to describe and measure space. Change in the curvature of the plane leads to different angle and length relationships, resulting in spherical and hyperbolic geometries. Considering ΔABC with sides a, b, c and angles α, β, γ:

in S^2 curvature is positive, $\alpha + \beta + \gamma > \pi$ and area $= R^2(\alpha + \beta + \gamma - \pi)$;
in E^2 curvature is zero, $\alpha + \beta + \gamma = \pi$ and area$= ab\sin\gamma$ or $\frac{1}{2}$base $\cdot$ height;
in H^2 curvature is negative, $\alpha + \beta + \gamma < \pi$ and area $= \pi - \alpha + \beta + \gamma$.

The Pythagorean theorem in S^2 is

$$\cos c = \cos a \cos b,$$

in E^2,

$$a^2 + b^2 = c^2$$

and in H^2

$$\cosh c = \cosh a \cosh b.$$

The circumference to diameter ratio in S^2 is less than π, in E^2 is equal to π, and in H^2 is greater than π. Derivations on H^2 vary according to the model used.

The visual model can be chosen to facilitate presentation, calculations, derivations, or intuitive understanding. Derivations on S^2 connect easily to Euclidean geometry; the sphere is seen embedded in E^3. This foundational knowledge about spherical and hyperbolic geometry would benefit high school teachers and students and could serve as a basis for an investigation into map projections, differential geometry, the history of geometry, and neutral geometry.

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Vita

Corrie Schaffer Cowley was born in Harrisburg, Pennsylvania on January 10, 1949, the daughter of Ruth Bender Cowley and Harvey Paul Cowley. She graduated from Wells College in 1970 with a Bachelor of Arts, majoring in Economics. She attended graduate school at Mankato State University (now Minnesota State University) in Environmental Studies from 1978 to 1980. At Tarleton State University she studied mathematics from 1998 to 1999 and in 2000 completed the T-MATE program for teacher certification. She has been a mathematics teacher at Eastland High School since 1999. In June, 2007, she entered the Graduate School at the University of Texas in Austin.

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