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Plane-like minimal surfaces in periodic media with inclusions

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Plane-like minimal surfaces in periodic media with inclusions

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Plane-like minimal surfaces in periodic media with inclusions

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The mathematical areas of minimal surfaces and homogenization of PDE have been subjects of research for many decades. In this work we consider the particular case of minimal surfaces in heterogeneous media. We prove theoretical results and develop numerical algorithms that characterize the behavior of these surfaces. We also show that this work turns out to be related to the theory of homogenization of Hamilton-Jacobi equations.

In this work we think of $\mathbb{R}^n$ as a lattice of points with integer coordinates, where each cube of edge length 1 has an internal inclusion (we can think of an inclusion as a hole or as a part of the domain containing another material). All inclusions are compact and periodic. Within this framework we measure the area of a surface of codimension one by neglecting the parts that are inside the inclusions, and measuring the outside parts in the standard way. We say that the surface is a minimal surface if any compact perturbation of it increases the area (in this degenerate metric).

In this work we prove the existence of minimal surfaces that always stay at a bounded distance (universal) from a given hyperplane. While we know that the surface is smooth outside the inclusions, in this work we prove a result concerning the behaviour of the minimal surfaces at the boundary: that the intersection between the inclusions and the surface locally looks like two perpendicular hyperplanes.

Within this degenerate metric, the smallest distance between two points is no longer a line. We analyze the behavior of this distance when the edge length of the cube goes to zero. In particular, we want to find, for the case $n = 2$ and the inclusions being closed balls, the effective norm in the homogenized limit. The effective norm depends on the radius of the inclusions, and our results suggest that as the radius gets smaller the behavior of the effective norm changes, though it is always polygonal with more and more sides, until it becomes a circle in the limit.

We implement an algorithm to compute our weighted minimal surfaces. We extend the Bence-Merriman-Osher algorithm to the case of heterogeneous domains. We implement the algorithm in 2 and 3 dimensions using adaptive finite element methods.

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Chapter 1

Introduction

1.1 Introduction

In [10], Luis Caffarelli and Rafael de la Llave consider minimizers of periodic geometric variational problems on sets of locally finite perimeter in $\mathbb{R}^n$ and other manifolds. These problems include as a particular case the problem of finding hypersurfaces of codimension one of minimal area. More specifically, they study minimizers of the elliptic integral:

$$J(E) = \int F(x, \nu)|D\varphi_E|$$

over a class of periodic closed Cacciopoli sets (sets of finite perimeter) that contain a semispace. The measure $|D\varphi_E|$ is the total variation of the vector valued Radon measure $D\varphi_E$ (which is the gradient of φ_E, the characteristic function of E, in the sense of distributions). If ∂E is C^2, $|D\varphi_E|$ is simply the (n-1)-dimensional Hausdorff measure H_{n-1}. F satisfies the following properties, among others: it is continuous, convex and homogeneous of degree 1 in the second variable ν, periodic in x and non-degenerate ($\lambda \leq F(x, \nu) \leq \Lambda$). They prove that there are minimizers that stay at a bounded distance from a given plane. This distance is bounded a priori by properties of the metric and independently of the plane. If we take $F(x, \nu) = |\nu|$, [10] includes the case of minimal surfaces for smooth metrics on $\mathbb{R}^n$ and in other manifolds.

The problem of constructing minimal surfaces for periodic metrics was proposed in [26], where similar results are proven for minimizers of a similar (but different) class of those that are considered in [10]. In [26] the connection of these results with the Aubry-Mather theory is shown. Also, [2] contains part of the results (presented in the context of geometric measure theory) proven in [10] for minimal surfaces in $\mathbb{T}^n$ with a Riemannian metric.

In this thesis, we extend [10] and consider the case of a *degenerate metric*. We also work in the context of sets of finite perimeter, and therefore we deal with boundaries of sets. This is useful since we do no need to consider aspects of orientability. Also, this allow us to take inclusions of sets.

Another goal of this thesis is to begin exploring the connections of this theory with the theory of homogenization.

In what follows we consider $\mathbb{R}^n$ as a lattice of cubes. We assume that each cube of edge length 1 has an internal inclusion (an inclusion can be a hole or a part of the domain containing another material). Let Y be a fundamental cube, that is, a cube of edge length 1 and vertices with integer coordinates. Let Z be the inclusion inside Y. We assume:

1. Z is compact.

2. $\partial Z \cap \partial Y = \emptyset$

Any other inclusion is an integer translation of Z. We denote $\alpha = d(\partial Z, \partial Y)$.

Let Σ be a surface in $\mathbb{R}^n$ of codimension 1. We consider the following procedure for measuring the area of Σ: the parts that are inside the inclusions do not contribute to the area, but outside the inclusions the area is measured in the standard way. Let $\mathbb{R}^n_i$ denote $\mathbb{R}^n$ with this way of measuring area. We say that Σ is a *minimal surface in* $\mathbb{R}^n_i$ if Σ minimizes area outside the inclusions. (This is made precise in Section 2).

The next two chapters of the thesis are devoted to constructing plane-like minimal surfaces in $\mathbb{R}^n_i$; that is, minimal surfaces that stay at a bounded distance (universal) from a plane. We say that a constant C is universal if it only depends on the dimension n and α. More precisely, the theorem we prove is:

Theorem 1. There exists a universal constant C, such that for every (n-1)-dimensional hyperplane Π, we can find a minimal surface Σ in $\mathbb{R}^n_i$ such that $d(\Pi, \Sigma) \leq C$. The surface Σ is the boundary of a set that includes a semispace bounded by Π. Moreover, if $n \leq 6$ and the inclusions have C^2 boundary, Σ enters the inclusions orthogonally.

In Theorem 1, by orthogonal we mean that the intersection of Σ with an inclusion looks like two perpendicular hyperplanes when viewed locally. Mathematically, this means that we can trap Σ in a cone forming an arbitrarily small angle θ such that as θ goes to zero, the cone tends to a hyperplane perpendicular to the inclusion.

The surface Σ given by Theorem 1 will be periodic when the slope of the given hyperplane is rational. If the slope of the hyperplane is irrational we will see that we can approximate with rational slopes but in this case Σ will not be periodic.

The first part of Theorem 1 is true for any dimension and for inclusions satisfying the two conditions given at the beginning of the introduction. The second part of Theorem 1 will be proven when the inclusions have C^2 boundary and $n \leq 6$. The orthogonallity will be clearly true for n=2, since as we will see in the last section, for this case Σ consists of pieces of lines connecting inclusions.

For the first part of Theorem 1 the main tool is the *density estimates* that minimizers satisfy. The existence of minimizers is given by a lower semi-continuity. Moreover, due to some subadditivity properties, it is possible to define an *infimal minimizer* which is contained in all the other minimizers. The *infimal minimizer* satisfies several monotonicity properties including a geometric property analogous to the property called Birkhoff in Aubry-Mather theory. This property, together with the density estimates allow us to prove that the infimal minimizer is contained in a band whose width is independent of the direction of the plane. A similar theorem to the first part of Theorem 1 is proven in [10], for C^2 metrics on $\mathbb{R}^n$ and in other manifolds.

The second part of Theorem 1 is obtained by blowing up the minimizer around a point of its boundary that lies on the boundary of an inclusion. The main point here is that we can prove also the *density estimates* for points on the boundary of the minimizer that lie on the boundary of the inclusions. These density estimates allow us to pass to the limit and obtain (after a suitable reflection) a minimal cone. We then use the well known theorem concerning the non-existence of minimal cones for dimension $n \leq 7$. (See [28], [18]).

Later in the dissertation we begin to explore the connections with homogenization. Indeed, in some sense Theorem 1 tell us that, in spite of our heterogeneous media, the *infimal minimizer* looks like a plane (homogeneos media) when seen from a far distance. We know that the smallest distance between two points in $\mathbb{R}^2$ is given by the line that joins the two points. However, in $\mathbb{R}^2_i$ this might not be the case. For example, considering the particular case where the inclusions are closed balls with radius ρ, we can prove that if $\rho > \frac{2-\sqrt{2}}{2}$, the smallest distance in $\mathbb{R}^2_i$ between two points that are centers of balls is given by a path that consists only of vertical and horizantal segments.

An interesting question is to study the behavior of our minimal surfaces when we fix two points in the plane and let the edge length of the cubes tend to zero. If P and Q denote any two fixed points in the plane and $d^\rho_\epsilon(P,Q)$ denotes the smallest distance between P and Q (this length measured in $\mathbb{R}^2_i$) at the scale ϵ (that is, when the size of the cubes is ϵ and the radius of the inclusions is $\rho\epsilon$) then the limit $d^\rho_0(P,Q) := \lim_{\epsilon\to 0} d^\rho_\epsilon(P,Q)$ will be called the *homogenized limit*. The function d^ρ_0 defines a norm and will be called the *effective norm*. In particular, we want to find, for the case $n = 2$, the effective norms for different values of ρ.

Let $|\cdot|_1$ and $|\cdot|_2$ denote the l_1 and Euclidean norms in $\mathbb{R}^2$ respectively. In the last section we prove that the effective norm is $(1-2\rho)|\cdot|_1$ (if ρ is big enough). As ρ decreases, we get other effective norms, but our results suggest that they (unit balls) are polygons with more and more edges. Also, we prove that the effective norms converge to $|\cdot|_2$ as ρ goes to zero.

Chapter 2

Proof of Theorem 1.

2.1 Main definitions and existence of minimizers

We first recall some theory regarding sets of finite perimeter. Let $\Omega \subset \mathbb{R}^n$ be an open set and let $f \in L^1(\Omega)$. Define:

$$\int_\Omega |Df| = \sup\{\int_\Omega f \mathrm{div} g : g \in C_0^1(\Omega; \mathbb{R}^n), |g(x)| \leq 1, \quad \text{for} \quad x \in \Omega\}$$

A function $f \in L^1(\Omega)$ is said to have bounded variation in Ω if $\int_\Omega |Df| < \infty$. We define $BV(\Omega)$ as the space of all functions in $L^1(\Omega)$ with bounded variation. With the norm $|f|_{BV} = |f|_{L^1(\Omega)} + \int_\Omega |Df|$, $BV(\Omega)$ is a Banach space.

Let E be a Borel set. Define the perimeter of E in Ω as:

$$\mathrm{Per}(E, \Omega) = \int_\Omega |D\varphi_E|$$

where φ_E is the characteristic function of the set E.

If a Borel set E has locally finite perimeter, that is, if $\mathrm{Per}(E, \Omega) < \infty$ for every bounded open set Ω, then E is called a Cacciopoli set. If $E \subset \mathbb{R}^n$ has C^2 boundary then $\mathrm{Per}(E, \Omega)$ is the (n-1)-dimensional Hausdorff measure of $\partial E \cap \Omega$; that is, $\mathrm{Per}(E, \Omega) = H_{n-1}(\partial E \cap \Omega)$.

Let $\{f_j\}$ be a sequence of functions in $BV(\Omega)$ that converge in $L^1_{loc}(\Omega)$ to a function f. The following semicontinuity property holds:

$$\int_\Omega |Df| \leq \liminf_{j\to\infty} \int_\Omega |Df_j| \tag{2.1.1}$$

Since $BV(\Omega) \subset L^1(\Omega)$, it makes no sense to talk about the value of a BV function on $\partial\Omega$, which is a set of measure zero. To solve this problem, given $f \in BV(\Omega)$, we can build a function f_{tr} such that:

$$\lim_{\rho\to 0} \int_{B(y,\rho)\cap\Omega} |f(x)| = f_{tr}(y)$$

for every $y \in \partial\Omega$. f_{tr} is called the *trace function.*

Finally, we define the *reduced boudary* of a Cacciopoli set E, which play a fundamental role in the regularity theory (See [18]). We say that $x \in \partial E$ is a point in the *reduced boundary* if:

- $\int_{B(x,\rho)} |D\varphi_E| > 0$ for all $\rho > 0$.
- The limit $\nu(x) = \lim_{\rho\to 0^+} \int_{B(x,\rho)} D\varphi_E/|D\varphi_E|$ exists.
- $|\nu(x)| = 1$.

The reduced boundary is usually denoted by $\partial^* E$. An important fact is that $\partial^* E$ is dense in ∂E. Details about theory of Cacciopoli sets can be found in [16, 18].

Let $O = \mathbb{R}^n \backslash \{\text{Inclusions}\}$. We make the following definitions:

We say that the Cacciopoli set $E \subset \mathbb{R}^n$ has *least area or perimeter in* Ω if

$$\int_\Omega |D\varphi_E| \;=\; \inf\{\int_\Omega |D\varphi_F| : F \text{ is a Cacciopoli set}, \ \operatorname{spt}(\varphi_F - \varphi_E) \subset \Omega\}$$

We say that the Cacciopoli set $E \subset \mathbb{R}^n$ is a *class A minimizer* if it has least area in $B(0,R) \cap O$, where $B(0,R)$ is any open ball. In other words, E *is a class A minimizer* if any perturbation of E inside any open ball $B(0,R)$, (which would be a new set L that has the same trace as E on $\partial B(0,R)$) results in the property that $\mathrm{Per}(E, B(0,R) \cap O) \leq \mathrm{Per}(L, B(0,R) \cap O)$.

Definition 2.1.1. $\Sigma \subset \mathbb{R}^n$ is *a minimal surface in* $\mathbb{R}^n_i$ if $\Sigma = \partial E$, where E is a class A minimizer.

It is proven in [18] that, if $n \leq 7$ and E is a Cacciopoli set that has least area in Ω, then $\partial E \cap \Omega$ is a smooth surface. At times we will use the word "surface" to denote the boundary of a Cacciopoli set, although this boundary could have singularities. These singularities, if ∂E minimizes area, have $(n-8)$-Hausdorff dimension zero (See also [8]).

We now start to prove the existence of minimizers. Let $\omega \in \mathbb{R}^n$. Let us consider first the case when $\omega \in \mathbb{Q}^n$. Let $N \in \mathbb{Z}$, $M \in \mathbb{R}$, with $M, N > 0$. Define:

$S_1 = \Gamma_{\omega,0} = \{x \in \mathbb{R}^n : x \cdot \omega \leq 0\}$

$S_2 = \Gamma_{\omega,M} = \{x \in \mathbb{R}^n : x \cdot \frac{\omega}{|\omega|} \leq M\}$

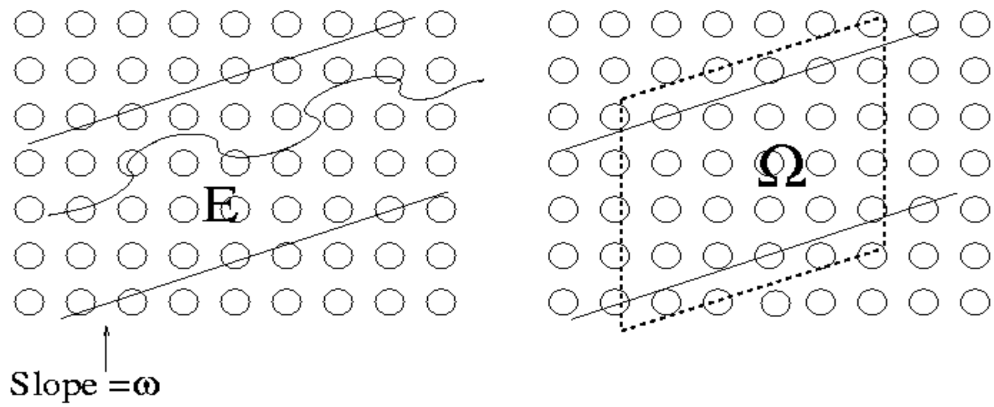

Figure 2.1: Diagram showing parallel plane restrictions and period for minimization.

We denote $\Pi_1 = \{x \in \mathbb{R}^n : x \cdot \omega = 0\}$ and $\Pi_2 = \{x \in \mathbb{R}^n : x \cdot \frac{\omega}{|\omega|} = M\}$ as the parellel plane restrictions. Define now:

$A_{S_1,S_2} = \{E : E$ is a closed Cacciopoli set, $S_1 \subset E \subset S_2$, $T_{Nk}E = E$, $\forall k \in Z^n$ with $\omega \cdot k = 0\}$

Since the sets in A_{S_1,S_2} are periodic, we can minimize the perimeter outside the inclusions in an open parellelogram P, representing one period (Figure 1).

Let $H_1, ..., H_k$ be the inclusions with $H_i \cap P \neq \emptyset$. Define $I = \bigcup_{i=1}^{k} H_i$ and set $\Omega = P - I$. Define, for each $E \in A_{S_1,S_2}$:

$$J(E) = \int_{\Omega} |D\varphi_E| \tag{2.1.2}$$

Let $\beta = \inf_{E \in A_{S_1,S_2}} J(E)$. Let $\{E_j\}$ be a sequence such that $J(E_j) \to \beta$. Therefore, the sequence $\{\int_{\Omega} |D\varphi_{E_j}|\}$ is uniformly bounded. Hence, since $BV(\Omega) \hookrightarrow L^1(\Omega)$, there exists a subsequence, that we detote again by $\{E_j\}$, such that $\{E_j\}$ is convergent in $L^1(\Omega)$. Let $E_0 \in L^1(\Omega)$ be the limit.

By (2.1),

$$\int_{\Omega} |D\varphi_{E_0}| \leq \liminf \int_{\Omega} |D\varphi_{E_j}|$$

Thus,

$$J(E_0) = \inf_{E \in A_{S_1,S_2}} J(E)$$

We make the following definitions:

Definition 2.1.2. If $E \in A_{S_1,S_2}$ satisfies $J(E) = J(E_0)$ then E will be called a minimizer corresponding to the class A_{S_1,S_2}.

Definition 2.1.3. We will say that the minimizer E is an unconstrained minimizer if we can find $M_1, M_2 \in \mathbb{R}$ such that $\Gamma_{\omega,M_1} \subset Int(S_1), S_2 \subset Int(\Gamma_{\omega,M_2})$ and E is a minimizer corresponding to the class A_{T_1,T_2}, where $T_1 = \Gamma_{\omega,M_1}$ and $T_2 = \Gamma_{\omega,M_2}$.

In other words, if E is an unconstrained minimizer then the parallel plane restrictions are irrelavant.

Let E be a minimizer corresponding to the class A_{S_1,S_2}. Since $S_1 \subset E$ and S_1 is connected, it follows that S_1 is contained in a single component of E, say E_1. Notice that $J(E_1) > 0$ since $\partial E_1 \cap O \neq \emptyset$. We now define:

Definition 2.1.4. We will say that the minimizer E is connected if E_1 is the only component satisfying $J(E_1) > 0$.

Notice that if E has a component contained inside an inclusion, then this component does not contribute to (2.2) at all and it can be neglected. The following Lemma will prove that in fact the minimizer E_0 constructed above has only one component that contributes to (2.2) with a nonzero quantity.

Lemma 2.1.1. Let E be a minimizer corresponding to the class A_{S_1,S_2}. Then, E and $\overline{\mathbb{R}^n \backslash E}$ are connected.

Proof: As above, let E_1 denote the component of E that contains S_1. We know that $J(E_1) > 0$. Assume that E has another component, say E_2, with $J(E_2) > 0$. This implies that $J(E_1) < J(E)$, which contradicts the fact that E is a minimizer. We can now consider the class A_{S_2,S_1}, where S_2 and S_1 interchange roles. Notice that $\overline{\mathbb{R}^n \backslash E}$ is a minimizer corresponding to this class. Hence, the same argument above shows that $\overline{\mathbb{R}^n \backslash E}$ is connected. $\square$

Let E be a minimizer corresponding to the class A_{S_1,S_2}. Consider $\partial E \cap O$. It is proven in [9, 18] that for $n < 8$, $\partial E \cap O$ is locally the graph of a smooth function. If $n \geq 8$, $\partial E \cap O$ may have singularities, but in this case, for any $k > n - 8$ the k-dimensional Hausdorff measures of these singularities are zero.

2.2 Infimal minimizer

The minimizer we have just constructed may not be unique. However, we can prove the existence of an infimal minimizer, that is, a minimizer that is contained in any other minimizer. Clearly, if such an infimal minimizer exists, it is unique. In order to prove its existence, we use the following theorem:

Theorem 2.2.1. Let A and B be Caccipoli sets. Let Ω be any open set. Then

$$\mathrm{Per}(A \cap B, \Omega) + \mathrm{Per}(A \cup B, \Omega) \quad \leq \quad \mathrm{Per}(A, \Omega) + \mathrm{Per}(B, \Omega).$$

Proof: Let f, g be two smooth functions with $0 \leq f \leq 1$, $0 \leq g \leq 1$. Define $\Psi = f + g - fg$ and $\Phi = fg$. Notice that:

$$\begin{aligned} \int_\Omega |D\Psi| &\leq \int_\Omega (1-f)|Dg| + \int_\Omega (1-g)|Df| \\ \int_\Omega |D\Phi| &\leq \int_\Omega f|Dg| + \int_\Omega g|Df| \end{aligned}$$

This implies:

$$\int_\Omega |D\Phi| + \int_\Omega |D\Psi| \leq \int_\Omega |Df| + \int_\Omega |Dg| \tag{2.2.3}$$

We can find [18] sequences of smooth functions f_j and g_j such that $f_j \to \varphi_A, g_j \to \varphi_B$, in $L^1(\Omega)$ and $\int_\Omega |Df_j| \to \int_\Omega |D\varphi_A|$, $\int_\Omega |Dg_j| \to \int_\Omega |D\varphi_B|$. Since $\Psi_j = f_j + g_j - f_j g_j \to \varphi_{A\cup B}$, $\Phi_j = f_j g_j \to \varphi_{A\cap B}$, the Theorem follows from (3.6) and (1.1).$\Box$

Now we can prove the existence of the infimal minimizer.

Theorem 2.2.2. There exists $E_* \in A_{S_1,S_2}$, such that E_* is the infimal minimizer corresponding to this class.

Proof: Let $\mathcal{B}$ be the set of all minimizers. Let Ω be the open set defined in section 2. We have that $\mathcal{B} \subset L^1(\Omega)$.

Let $E_1, E_2 \in \mathcal{B}$. By theorem 3.1 we have:

$$\mathrm{Per}(E_1 \cap E_2, \Omega) + \mathrm{Per}(E_1 \cup E_2, \Omega) \leq \mathrm{Per}(E_1, \Omega) + \mathrm{Per}(E_2, \Omega).$$

But $\mathrm{Per}(E_1, \Omega) = \mathrm{Per}(E_2, \Omega)$. Since $E_1 \cup E_2$ is an admissible set, we have that:

$$\mathrm{Per}(E_1 \cup E_2, \Omega) \geq \mathrm{Per}(E_1, \Omega)$$

hence:

$$\mathrm{Per}(E_1 \cap E_2, \Omega) \leq \mathrm{Per}(E_1, \Omega)$$

which implies that $E_1 \cap E_2$ is also a minimizer. Since we can uniformly bound the perimeters of minimizers in Ω, it follows from (1.1) that $\mathcal{B}$ is a compact subset of $L^1(\Omega)$. Since $L^1(\Omega)$ is separable, $\mathcal{B}$ is also separable. Let $\{E_j\}$ be a dense subset of $\mathcal{B}$ and define:

$$\tilde{E_n} = \bigcap_{j=1}^{n} E_j$$

Each $\tilde{E_n}$ is a minimizer. Since $\tilde{E}_{n+1} \subset \tilde{E_n}$ and $|\tilde{E_1} \cap \Omega| < \infty$ we have that:

$$|\tilde{E_n} \cap \Omega| \rightarrow |\bigcap_{n=1}^{\infty} \tilde{E_n} \cap \Omega|$$

thus,

$$\tilde{E_n} \rightarrow \bigcap_{n=1}^{\infty} \tilde{E_n} \quad in \quad L^1(\Omega)$$

Let

$$E_* = \bigcap_{n=1}^{\infty} \tilde{E}_n$$

By (1.1),

$$\mathrm{Per}(E_*, \Omega) \leq \liminf \mathrm{Per}(\tilde{E_n}, \Omega)$$

which implies that E_* is a minimizer.

Let E be any other minimizer. We now want to prove that $|(E_* \backslash E) \cap \Omega| = 0$. If this is not true, then $|(E_* \backslash E) \cap \Omega| > \delta > 0$. Since $\{E_j\}$ is a dense subset of $\mathcal{B}$, we can find E_k such that $|(E_k \backslash E) \cap \Omega| < \frac{\epsilon}{2}, \epsilon < \delta$. Choose N large enough such that $\tilde{E}_N \subset E_k$ and $|(E_* \backslash \tilde{E}_N) \cap \Omega| < \frac{\epsilon}{2}$. We have:

$$\begin{aligned}
|(E_* \backslash E) \cap \Omega| &\leq |(E_* \backslash \tilde{E}_N) \cap \Omega| + |(\tilde{E}_N \backslash E) \cap \Omega| \\
&\leq |(E_* \backslash \tilde{E}_N) \cap \Omega| + |(E_k \backslash E) \cap \Omega| \\
&\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon < \delta
\end{aligned}$$

which is a contradiction. We conclude that E_* is the infimal minimizer. $\square$

Let $M_1, M_2 \in \mathbb{R}$ be such that $T_1 := \Gamma_{\omega, M_1} \subset S_1$ and $T_2 := \Gamma_{\omega, M_2} \subset S_2$, with $T_1 \subset T_2$. The following Proposition will be used later to establish properties of the infimal minimizer.

Proposition 2.2.1. Let E be a minimizer corresponding to the class A_{S_1, S_2} and L a minimizer corresponding to the class A_{T_1, T_2}. Then:

(a) $E \cap L$ is a minimizer corresponding to the class A_{T_1, T_2}.

(b) $E \cup L$ is a minimizer corresponding to the class A_{S_1,S_2}.

(c) $E_{*,T_1,T_2} \subset E_{*,S_1,S_2}$

Proof: Note that $E \cup L \in A_{S1,S2}$ and $E \cap L \in A_{T_1,T_2}$. Since E and L are minimizers, we have:

$$\begin{aligned} \mathrm{Per}(E,\Omega) &\leq \mathrm{Per}(E \cup L,\Omega) \\ \mathrm{Per}(L,\Omega) &\leq \mathrm{Per}(E \cap L,\Omega) \end{aligned}$$

Using theorem 3.1:

$$\begin{aligned} \mathrm{Per}(E \cap L,\Omega) + \mathrm{Per}(E,\Omega) &\leq \mathrm{Per}(E \cap L,\Omega) + \mathrm{Per}(E \cup L,\Omega) \\ &\leq \mathrm{Per}(E,\Omega) + \mathrm{Per}(L,\Omega) \\ \Rightarrow \mathrm{Per}(E \cap L,\Omega) &\leq \mathrm{Per}(L,\Omega) \end{aligned}$$

This implies that $E \cap L$ is a minimizer in the class A_{T_1,T_2}, which proves (a). In the same way we prove (b).

In order to prove (c) notice that, by (a), $E_{*,T_1,T_2} \cap E_{*,S_1,S_2}$ is a minimizer corresponding to the class A_{T_1,T_2}. Hence:

$$\begin{aligned} E_{*,T_1,T_2} &\subset (E_{*,T_1,T_2} \cap E_{*,S_1,S_2}) \\ \Rightarrow E_{*,T_1,T_2} &\subset E_{*,S_1,S_2} \end{aligned}$$

2.3 Birkhoff property

Let E be the infimal minimizer corresponding to the class A_{S_1,S_2}. We will denote by T_k the translation operator by $k \in \mathbb{Z}^n$, that is, $T_k(x) = x + k$, $x \in \mathbb{R}^n$. The Birkhoff property is stated in the following Lemma.

Lemma 2.3.1. Let $k \in \mathbb{Z}^n$.

(a) if $k \cdot \omega \leq 0$ then $T_k E \subset E$.

(b) if $k \cdot \omega \geq 0$ then $E \subset T_k E$.

In particular, if $k \cdot \omega = 0$, we have $T_k E = E$.

Proof:(a) Let $T_1 = T_k(S_1)$ and $T_2 = T_k(S_2)$, where as before $S_1 = \{x \in \mathbb{R}^n : x \cdot \omega \leq 0\}$ and $S_2 = \{x \in \mathbb{R}^n : x \cdot \omega \leq M\}$. If $k \cdot \omega \leq 0$ we have that $T_1 \subset S_1$, $T_2 \subset S_2$, and $T_1 \subset T_2$. Notice that $T_k E$ is the infimal minimizer in A_{T_1,T_2}. By Proposition 3.1 we have $T_k E \subset E$.

(b)If $k \cdot \omega \geq 0$ we have that $S_1 \subset T_1$, $S_2 \subset T_2$, and $T_1 \subset T_2$. Again $T_k E$ is the infimal minimizer in A_{T_1,T_2}. By Proposition 3.1, $E \subset T_k E$.

In what follows we will use the word *slab* to denote the region in between two hyperplanes, both parallel to the parallel plane restrictions Π_1 and Π_2.

We now prove the following Lemma:

Lemma 2.3.2. Let $\mathcal{C}$ be a cube of edge length $l \geq 3$.

(a) if $\mathcal{C} \subset \mathbb{R}^n \backslash E$, then there exists a slab of width at least one contained in $\mathbb{R}^n \backslash E$.

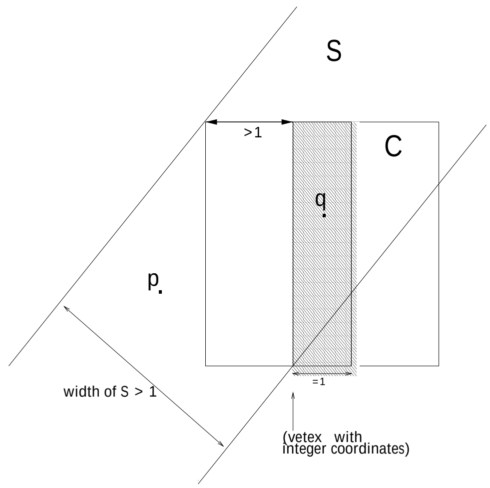

Figure 2.2: Case when $\mathcal{C} \subset \mathbb{R}^n \backslash E$

(b) if $\mathcal{C} \subset E$, then there exists a slab of width at least one contained in E.

Proof: (a) Proceed by contradiction. Let S be the slab as shown in Figure 2. Assume that there exists a point $p \in S$ with $p \in E$. Then, we can find $k \in \mathbb{Z}^n$, $k \cdot \omega \leq 0$, such that $T_k(p) = q$ with q in the shaded region of $\mathcal{C}$. By Lemma 4.1, $T_k E \subset E$. This implies that $q \in E$, which gives a contradiction. Notice that the width of the slab S is larger than 1.

(b)Proceed again by contradiction. Let S be the slab as shown in Figure 3. Assume that there exists $p \in S$ with $p \in \mathbb{R}^n \backslash E$. We can find $k \in \mathbb{Z}^n$, $\omega \cdot k \geq 0$ such that $T_k(p) = q$, with q in the shaded region of $\mathcal{C}$. Now, $\omega \cdot (-k) \leq 0$ and, by Lemma 4.1, $T_{-k}E \subset E$. Since $T_{-k}(q) = p$, this implies that $p \in E$, which is a contradiction. Notice that the width of the slab S is

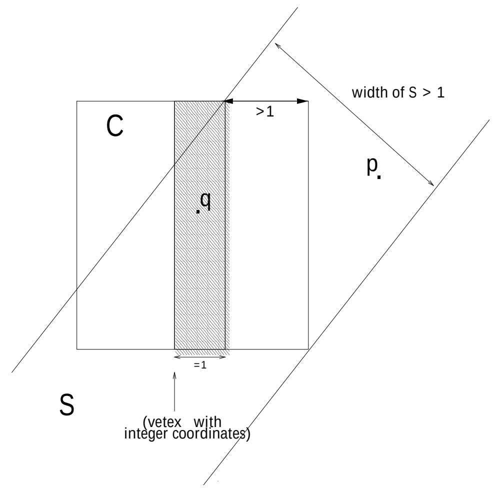

Figure 2.3: Case when $\mathcal{C} \subset E$

larger than 1. $\square$

We will use the previous Lemma to prove the following Proposition, which will be used in the next section to prove Theorem 1.

Proposition 2.3.1. Let $\mathcal{C}$ be a cube of edge length $l \geq 3$. Then, we can not have $\mathcal{C} \subset E \backslash S_1$ where $S_1 = \{x \in \mathbb{R}^n : x \cdot \omega \leq 0\}$.

Proof: Assume that $\mathcal{C} \subset E \backslash S_1$. Then by Lemma 4.2, there exists a slab of width larger than 1 contained in E. Let $\tilde{\Pi}_1$ and $\tilde{\Pi}_2$ be the lower and upper hyperplanes respectively that bound this slab. Since the slab has width larger than 1, it has to contain a point with integer coordinates that does not intersect $\tilde{\Pi}_1$ or $\tilde{\Pi}_2$. Let Π be the plane that contains this point and is parallel to Π_1 and Π_2. This plane Π divides E in two parts. One of this parts, say E_1, contains Π_1, and the other part, say E_2, contains Π_2. We can find $k \in \mathbb{Z}^n$, with $k \cdot \omega \leq 0$, such that $T_k(\tilde{\Pi}_2) = \Pi$. Let S denote the interior of the slab in between Π and $\tilde{\Pi}_2$. Consider now the set $E_1 \cup T_k(E_2 \backslash S)$. Clearly, this set is also a minimizer contained (and not equal) in E. This contradicts the fact that E is the infimal minimizer, that is, a minimizer that is contained in any other minimizer. $\square$

2.4 The case when ω is not rational

Given $\omega \in \mathbb{R}^n \backslash \mathbb{Q}^n$, we can construct a sequence $\{\omega_j\} \in \mathbb{Q}^n$ with $\omega_j \to \omega$. Let $\{E_{\omega_j}\}$ denote the corresponding infimal minimizers constructed as in the previous sections. Consider $B(0, R)$, the open ball of radius R centered at the origin. We have:

$$\text{Per}(E_{\omega_j}, B(0, R) \cap O) \quad \leq \quad CR^{n-1}$$

Thus, $\{E_{\omega_j}\}$ has a subsequence that is convergent in $L^1(B(0, R) \cap O)$. By applying the diagonal procedure, we get a subsequence of $\{E_{\omega_j}\}$ (which we will denote again as $\{E_{\omega_j}\}$) and a set E_ω such that $E_{\omega_j} \to E_\omega$ in $L^1_{loc}(\mathbb{R}^n \cap O)$.

We need to check that E_ω is a class A minimizer. Let $\epsilon > 0$. Consider again the ball $B(0, R)$ and assume that we perturb E_ω inside the ball, resulting in a new set L. Since each E_{ω_j} is a minimizer and since $E_{\omega_j} \to E_\omega$ in $L^1(B(0, R) \cap O)$, we have that for j large enough:

$$\text{Per}(E_{\omega_j}, B(0, R) \cap O) \quad \leq \quad \text{Per}(L, B(0, R) \cap O) + \gamma(\epsilon)$$

where $\gamma(\epsilon)$ goes to zero as $\epsilon \to 0$.

Using (1.1), and since $E_{\omega_j} \to E_\omega$ in $L^1(B(0,R) \cap O)$, we have:

$$\begin{aligned} \text{Per}(E_\omega, B(0,R) \cap O) &\leq \liminf \text{Per}(E_{\omega_j}, B(0,R) \cap O) \\ &\leq \text{Per}(L, B(0,R) \cap O) + \gamma(\epsilon) \end{aligned}$$

Letting $\epsilon \to 0$ we have

$$\text{Per}(E_\omega, B(0,R) \cap O) \leq \text{Per}(L, B(0,R) \cap O)$$

This proves that E_ω is a class A minimizer. Finally, since $d(\partial E_{\omega_j}, \partial \Gamma_{0,\omega_j}) \leq 2\lambda\sqrt{n}$ it follows that $d(\partial E_\omega, \partial \Gamma_{0,\omega}) \leq 2\lambda\sqrt{n}$.

2.5 Proof of Theorem 1

Let E be the infimal minimizer corresponding to the class A_{S_1,S_2}. Recall our notation $O = \mathbb{R}^n \backslash \{\text{Inclusions}\}$. In order to clarify exposition, in the remainder of the thesis we use the same C to denote different universal constants.

The proof of the first part of Theorem 1 will be a covering argument. We will need to prove that we can cover the boundary of E with cubes of edge length 2 in such a way that each of these cubes contains at least some fixed amount of area (that will depend only on the dimension and α). This is a consequence of the following three Lemmas.

Lemma 2.5.1. Let $x \in \partial E \cap O$. Assume that x is contained in one fundamental cube, say Y, that does not intersect the parallel plane restrictions. Then, $\partial E \cap \partial Y \neq \emptyset$.

Proof: If $x \in \partial Y$ the proof is complete. Consider the case $x \in \mathrm{Int}(Y)$. Let $\rho > 0$ such that $B(x, \rho) \subset \mathrm{Int}(Y) \cap O$. Since $x \in \partial E$ and $\partial^* E$ is dense in ∂E, by an approximation argument and using Lemma 5.2 below we obtain $\int_{B(x,\rho)} |D\varphi_E| \geq C\rho^{n-1} > 0$.

Assume that $\partial E \cap \partial Y = \emptyset$. This implies that $\partial Y \subset \mathrm{Int}(E)$ or $\partial Y \subset \mathbb{R}^n \backslash E$, since otherwise we could write:

$$\partial Y = (\mathrm{Int}(E) \cap \partial Y) \cup ((\mathbb{R}^n \backslash E) \cap \partial Y)$$

which is not possible since ∂Y is connected.

Suppose that $\partial Y \subset \mathbb{R}^n \backslash E$. We have:

$$E = (E \cap \mathrm{Int}(Y)) \cup (E \cap (\mathbb{R}^n \backslash Y))$$

Since $S_1 \subset E$ we have that $E \cap (\mathbb{R}^n \backslash Y) \neq \emptyset$. Also, it is clear that $E \cap \mathrm{Int}(Y) \neq \emptyset$. Because $\int_{B(x,\rho)} |D\varphi_E| > 0$ and E is periodic it follows that E has two components that contribute to the perimeter with nonzero quantities. This is a contradiction since E is connected. Thus, $\partial Y \subset \mathrm{Int}(E)$, but in this case:

$$\overline{\mathbb{R}^n \backslash E} = (\overline{\mathbb{R}^n \backslash E} \cap \mathrm{Int}(Y)) \cup (\overline{\mathbb{R}^n \backslash E} \cap (\mathbb{R}^n \backslash Y))$$

and again this is a contradiction since $\overline{\mathbb{R}^n \backslash E}$ is connected. We conclude that $\partial E \cap \partial Y \neq \emptyset$. $\square$

Lemma 2.5.2. Let $y \in \partial^* E$. Assume that $B(y, \frac{\alpha}{2})$ intersects neither the parallel plane restrictions nor the inclusions. Then there exists a universal constant $C > 0$ such that $\int_{B(y,r)} |D\varphi_E| \geq Cr^{n-1}$, for any $r \leq \frac{\alpha}{2}$.

Proof: Since E is a minimizer and $B(y, r)$ does not intersect any inclusion it follow that:

$$\int_{B(y,r)} |D\varphi_E| \leq H_{n-1}(E \cap \partial B(y, r)), \quad r \leq \frac{\alpha}{2}$$

Let $V(r) = |E \cap B(y, r)|$, $r \leq \frac{\alpha}{2}$. Using an isoperimetric inequality, we obtain:

$$\begin{aligned}
|E \cap B(y,r)| &\leq C[\mathrm{Per}(E \cap B(y,r)]^{\frac{n}{n-1}} \\
&= C[|D\varphi_E|(E \cap B(y,r)) + H_{n-1}(E \cap \partial B(y,r))]^{\frac{n}{n-1}} \\
&\leq C[H_{n-1}(E \cap \partial B(y,r))]^{\frac{n}{n-1}}
\end{aligned}$$

Hence:

$$\begin{aligned}
V(r) &\leq CV'(r)^{\frac{n}{n-1}} \\
\Rightarrow C &\leq V(r)^{\frac{1-n}{n}} V'(r) \\
\Rightarrow C &\leq (V(r)^{\frac{1}{n}})' \\
\Rightarrow V(r)^{\frac{1}{n}} &\geq Cr \\
\Rightarrow V(r) &\geq Cr^n
\end{aligned}$$

In the same way we can prove that $|(\mathbb{R}^n \backslash E) \cap B(y, r)| \geq Cr^n$.

Thus, another isoperimetric inequality (the proofs of the two isoperimetric inequalities can be found in [18], page 25) gives us:

$$\begin{aligned} \min\{|(\mathbb{R}^n \backslash E) \cap B(y,r)|, |E \cap B(y,r)|\} &\leq C \left(\int_{B(y,r)} |D\varphi_E| \right)^{\frac{n}{n-1}} \\ &\Rightarrow \\ Cr^n &\leq \left(\int_{B(y,r)} |D\varphi_E| \right)^{\frac{n}{n-1}} \end{aligned}$$

Finally, we conclude:

$$\int_{B(y,r)} |D\varphi_E| \geq Cr^{n-1}$$

This completes the proof of the Lemma 5.2 $\square$.

Lemma 2.5.3. Let $x \in \partial E \cap O$. Assume that x is contained in one fundamental cube, say Y. Assume also that Y is far away from the parallel plane restrictions Π_1 and Π_2; that is, $d(Y, \Pi_1) > 2$ and $d(Y, \Pi_2) > 2$. Then, there exists a cube $\mathcal{C}_x$ of edge length 2 and a universal constant $\beta > 0$, such that $x \in \mathcal{C}_x$ and $\mathcal{C}_x$ contains at least β amount of area.

Proof: Because of Lemma 5.1 we can take $y \in \partial Y \cap \partial E$. If we make a dyadic decomposition of Y, we get 2^n cubes of side $\frac{1}{2}$ contained in Y. The point y must be contained in one of these dyadic cubes, say $\tilde{Y}$. Both Y and $\tilde{Y}$ have a common vertex, say v.

Let $\mathcal{C}_x$ be the cube of edge length 2 with center in v. Clearly, $B(y, \frac{\alpha}{2})$ does not intersect any inclusion.

Using Lemma 5.2 (and an approximation argument if y is not in the reduced boundary) we obtain the existence of the constant β; in fact, $\beta = C(\frac{\alpha}{2})^n$. This completes the proof of the Lemma. $\square$

We will use Vitali's covering Lemma (See [16], chapter 1):

Lemma 2.5.4. Let $\mathcal{F}$ be any collection of non-degenerate closed cubes in $\mathbb{R}^n$ with edges parallel to the coordiante axes and satisfying:

$$\sup\{\quad \text{diag} \quad \mathcal{C} : \mathcal{C} \in \mathcal{F}\} < \infty$$

Then there exists a countable family $\mathcal{G}$ of disjoints cubes in $\mathcal{F}$ such that:

$$\bigcup_{\mathcal{C}\in\mathcal{F}} \mathcal{C} \ \subset \ \bigcup_{\mathcal{C}\in\mathcal{G}} \hat{\mathcal{C}}$$

where $\hat{\mathcal{C}}$ is concentric with $\mathcal{C}$, and with edge length 5 times the edge length of $\mathcal{C}$.

Proof: The proof is the same as with balls, using the fact that the cubes are oriented in the same way as the coordinate axes.$\square$

We will also use the following remark:

Remark 2.5.1. If we have a cube $\mathcal{C}$ in $\mathbb{R}^n$ of edge length l, then we can have at most $3^n - 1$ cubes of edge length l that interset $\mathcal{C}$ without intersecting among themselves in a set of positive measure.

Let us define $\tau = 5$. Fix λ to be a multiple of 2τ and:

$$\lambda > \frac{2^{2n} n \tau^n (3^n - 1)}{\beta} \tag{2.5.4}$$

Let $\tilde{M} = 2\lambda\sqrt{n}$. Notice that $2\lambda\sqrt{n}$ is the length of the diagonal of the cube of edge length 2λ. Let E denote the infimal minimizer corresponding to the class A_{S_1,S_2}, where $S_1 = \Gamma_{\omega,0}$ and $S_2 = \Gamma_{\omega,\tilde{M}}$. Our choice of λ allow us to fit a cube $\tilde{\mathcal{C}}$ of edge length 2λ in between $\Pi_1 = \{x \in \mathbb{R}^n : x \cdot \omega = 0\}$ and $\Pi_2 = \{x \in \mathbb{R}^n : x \cdot \frac{\omega}{|\omega|} = \tilde{M}\}$, with $\tilde{\mathcal{C}}$ having integer vertices, edges parallel to the coordinate axes and intersecting Π_1 in a line. The first part of Theorem 1 is given by the following Lemma:

Lemma 2.5.5. $d(\partial E, \Pi_1) < \tilde{M}$

Proof: Let $\mathcal{C}$ be the cube of edge length λ that is concentric with the cube $\tilde{\mathcal{C}}$. One of the following two things must happen:

1. $\mathcal{C} \subset \mathbb{R}^n \backslash E$. In this case, by Lemma 4.2, we get a slab of width at least 1 contained in $\mathbb{R}^n \backslash E$. Since E is connected the Lemma follows.

2. $\mathcal{C} \cap E \neq \emptyset$. Notice that we can not have $\mathcal{C} \subset E$, since otherwise we would have a slab of width at least 1 completely contained in E, which would contradict that E is the infimal minimizer (Proposition 4.1). Hence, $\mathcal{C}$ must contain points from the complement of E and therefore it must also contain points from ∂E.

Recall our definition $O = \mathbb{R}^n \backslash \{\text{Inclusions}\}$. Let $x \in \partial E \cap O \cap \mathcal{C}$. Let $\mathcal{C}_x$ be the cube of edge length 2 constructed in Lemma 5.3. Therefore, We have a cover $\{\cup \mathcal{C}_x\}$ for $\partial E \cap \mathcal{C}$.

By Lemma 5.3 we can extract a countable disjoint family $\{\mathcal{C}_i\}$ such that:

$$\bigcup \mathcal{C}_x \subset \bigcup \hat{\mathcal{C}}_i \tag{2.5.5}$$

where $\hat{\mathcal{C}}_i$ is concentric with $\mathcal{C}_i$ and has edge length 2τ.

We now use the fact that E is a Cacciopoli set, along with (5.9), to conclude that the disjoint family has a finite number of cubes, say K.

In fact, since:

$$\begin{aligned} \int_{(\cup \mathcal{C}_i) \cap O} |D\varphi_E| &\leq 2n(2\lambda)^{n-1} \\ \Rightarrow K &\leq \frac{2^n n \lambda^{n-1}}{\beta} \end{aligned}$$

Since λ is a multiple of 2τ, we can divide $\mathcal{C}$ in $\frac{\lambda^n}{(2\tau)^n}$ cubes of edge length 2τ which do not intersect in sets of positive measure Let us call $\mathcal{B}$ this collection of cubes. By Remark 5.1, out of this collection $\mathcal{B}$, at most:

$$\frac{(3^n - 1)2^n n \lambda^{n-1}}{\beta}$$

interset ∂E. Because of our choice of λ, we have that:

$$\frac{(3^n - 1)2^n n \lambda^{n-1}}{\beta} < \frac{\lambda^n}{(2\tau)^n}$$

This implies that there exists $\mathcal{C}' \in \mathcal{B}$ such that $\mathcal{C}' \cap \partial E = \emptyset$. We must have $\mathcal{C}' \subset \mathbb{R}^n \backslash E$ (otherwise, by proposition 4.1, we would contradict the fact that E is the infimal minimizer). Now, by Lemma 4.2, there exists a slab of width at least 1 contained in $\mathbb{R}^n \backslash E$. Since E is connected the Lemma follows. $\square$

We can now prove the following propositions:

Proposition 2.5.1. E is an unconstrained minimizer.

Proof: We already proven that the upper constraint is irrelevant. We now want to prove that the lower constraint is also irrelevant. This is true since, given any $a > 0$, we can find $k \in \mathbb{Z}^n$ such that $\Gamma_{\omega,a} \subset T_k(E)$.$\square$

Proposition 2.5.2. E is a class A minimizer.

Proof: By applying a compact perturbation to E and minimizing in a class with a period larger than the diameter of the perturbation (with the distance between the plane restrictions also larger than the diameter of the perturbation), the Proposition follows since E is also a minimizer corresponding to this new class.$\square$

We will now prove the second part of Theorem 1. Let E be a minimizer (not necessarily the infimal minimizer) corresponding to the class A_{S_1,S_2}. If E intersects an inclusion I, we claim that E enters that inclusion orthogonally. By orthogonally we mean that the intersection of E and I locally looks like two perpendicular hyperplanes. We make this statement mathematically precise in a moment.

Since ∂E minimizes perimeter outside the inclusions, the interior regularity theory [9, 18] demands that, for $n \leq 7$, ∂E is a smooth surface. That is, outside the inclusions, $|D\varphi_E|$ is just H_{n-1}.

The next question is whether we have regularity up to the boundary of the inclusions. This is a local problem and can be recast as follows. If a set minimizes area, with part of its boundary fixed and lying on the boundary of an open ball, and the rest being a free boundary lying on a C^2 surface S, can the regularity of the free boundary be characteried?. In [21] it is proven that if $n \leq 6$ there are no singularities, if $n = 6$ the singularities are discrete and if $n > 7$ the dimension of the singularities is at most $n - 7$. Notice that if the boundary of S were a hyperplane, then by reflecting across S we would obtain a set of least area. Then we could apply the interior regularity theory to obtain the regularity of the free boundary. The difficulty is that the boundary of our inclusions is not necessarily a plane. This problem can be solved by noticing that, if we reflect across S and S has C^2 boundary, we do not get a set of least area. Instead we get a set that is "almost a set of least area" in an open ball of radius ρ; (See also [1]) that is, if we apply a compact perturbation to it, we increase the area with an extra factor of the form ρ^α. This idea of "almost minimal set" is used in both [21] and [1] to prove regularity of the free boundary.

The fact that we have regularity up to the boundary allows us to define a unique tangent plane at points in ∂E that lie on the boundary of the inclusions. What we wish to prove now is that the unique tangent plane is orthogonal to the inclusion. The fact that we have a free boundary allows us to prove density estimates at points on the free boundary. We can then make a blow up and prove that ∂E enters the inclusions orthogonally.

We will prove the second part of Theorem 1 when the inclusions have C^2 boundary. Let I denote an inclusion that intersects ∂E. Let $x_0 \in \partial E \cap \partial I$.

Since ∂I is C^2, then in a neighborhood of x_0, ∂I is the graph of a C^2 function f. Upon relabeling and reorienting the coordinate axis if necessary we can assume that, in a neighborhood of x_0, $\partial I = \{(x', f(x'))\}$, where $x' = (x_1, ..., x_{n-1})$. We can assume, by translating and/or rotating if necessary, that $x_0 = 0$, $f(0) = 0$ and $Df(0) = \{x : x_n = 0\}$.

We could have $\partial E \cap \partial I \neq \emptyset$, but recall that the boundary of E inside I does not contribute to the area. Therefore we can assume that $H_{n-1}(\partial I \cap \partial E) = 0$. We will denote the open ball of radius r with center at the origin by B_r.

Choose r_0 small enough such that $H_{n-1}(\partial B_r \cap E \cap O) \neq 0$, for all $r \leq r_0$, and B_{r_0} does not intersect any other inclusion. Let $D_r = E \cap B_r \cap O$, $r \leq r_0$. We need to prove first that if r is small enough, $|D_r| \geq Cr^n$. If ∂I were a hyperplane, we could prove this as we did in Lemma 5.2 by applying directly the isoperimetric inequalites in D_r to prove $|D_r| \geq Cr^n$. Since ∂I is C^2, ∂I is almost a plane inside B_r if r is small enough.

Let us make a change of variables to flatten ∂I. The transformation $T : (x', x_n) \to (y', y_n)$ given by $y' = (y_1, .., y_{n-1}) = x' = (x_1, ..., x_{n-1})$ and $y_n = x_n - f(x')$ flattens ∂I. We can check that det $D_y T(x(y)) = 1$. Also, if S is a Cacciopoli set, then if r is small enough, $|D\varphi_S|(T(S \cap B_r)) \leq C|D\varphi_S|(S \cap B_r)$. Let $S_{1,r} = T(\partial E \cap B_r \cap O)$, $S_{2,r} = T(\partial B_r \cap E \cap O)$, $S_{3,r} = T(\partial I \cap E \cap B_r)$ and $\tilde{D}_r = T(D_r)$. We can compute now $|D_r|$ using the isoperimetric inequalities in the new domain $\tilde{D}_r$.

In fact:

$$\begin{aligned}
|D_r| &= |\tilde{D}_r| \\
&\leq C(\mathrm{Per}(\tilde{D}_r))^{\frac{n}{n-1}} \\
&= C(H_{n-1}(S_{1,r}) + H_{n-1}(S_{2,r}) + H_{n-1}(S_{3,r}))^{\frac{n}{n-1}} \\
&\leq C(H_{n-1}(S_{1,r}) + H_{n-1}(S_{2,r}))^{\frac{n}{n-1}}
\end{aligned}$$

the last inequality being true since $S_3 \subset \{x : x_n = 0\}$.

Since $H_{n-1}(S_{1,r}) \leq CH_{n-1}(\partial E \cap B_r \cap O)$ and $H_{n-1}(S_{2,r}) \leq CH_{n-1}(\partial B_r \cap E \cap O)$ for r small enough, say $r \leq r_1 \leq r_0$, we obtain:

$$|D_r| \leq C(H_{n-1}(\partial E \cap B_r \cap O) + H_{n-1}(\partial B_r \cap E \cap O))^{\frac{n}{n-1}}$$

for all $r \leq r_1$.

Also, because in the minimization process there was no restriction on how ∂E enters the inclusions, we have that:

$$\mathrm{Per}(E, B_r \cap O) \leq H_{n-1}(\partial B_r \cap O \cap E)$$

and hence:

$$|D_r| \leq C(H_{n-1}(\partial B_r \cap O \cap E))^{\frac{n}{n-1}}$$

Let $V(r) = |D_r|, r \leq r_1$. We have that $V'(r) = H_{n-1}(\partial B_r \cap O \cap E)$. Proceeding as in Lemma 5.2 we obtain:

$$V(r) \geq Cr^n, r \leq r_1 \tag{2.5.6}$$

The previous estimate will allow us to make a blow up of E around zero, take the limit and make sure that this limit doesn't vanish. Fix $r \leq r_1$. Define, for $0 < t < 1$, the set $E_t = \{x \in \mathbb{R}^n : tx \in E \cap B_r \cap O\}$. We have that, for any $R > 0$ and t small enough:

$$\int_{B_R} |D\varphi_{E_t}| = t^{1-n} \int_{B_{tR} \cap O} |D\varphi_E| \tag{2.5.7}$$

$$|E_t \cap B_R| = t^{-n} |E \cap B_{tR} \cap O| \tag{2.5.8}$$

Again, since ∂E was free in the minimization, it follows that for t small enough:

$$t^{1-n} \int_{B_{tR} \cap O} |D\varphi_E| \leq Ct^{1-n}(tR)^{n-1} = CR^{n-1}$$

This implies that the sequence $\{E_t\}$ has uniformly bounded perimeter in B_R, and therefore it has a subsequence that converges in $L^1(B_R)$. By applying the diagonal trick we obtain a set E_1 such that $E_t \to E_1$ in $L^1_{loc}(\mathbb{R}^n)$.

From (5.10) and (5.12) we have that for any $R > 0$ and t small enough:

$$|E_t \cap B_R| \geq \tilde{C} R^n \tag{2.5.9}$$

We will now prove that $\partial E_1 \cap \{x : x_n > 0\} \neq \emptyset$. Choose a cone Q with vertex at 0, contained in $\{x : x_n > 0\}$ and containing $\{x : x = (0, x_n), x_n > 0\}$. Choose also $r_2 \leq r_1$ small enough such that:

$$|(\mathbb{R}^n \backslash Q) \cap O \cap B_{r_2}| = \theta r_2^n \tag{2.5.10}$$

where $\theta < \tilde{C}$.

We claim that $(Q \cap \partial E_1) \backslash \{0\} \neq \emptyset$. Assume this is not true. We have two cases, either $|E_1 \cap Q \cap B_{r_2}| = 0$ or $|(\mathbb{R}^n \backslash E_1) \cap Q \cap B_{r_2}| = 0$. We will prove the case $|E_1 \cap Q \cap B_{r_2}| = 0$. The other case is proven in the same way but using the estimate in (5.10) with the complement of E, which also holds. Let ϵ be small enough such that $\theta r_2^n + \epsilon < \tilde{C} r_2^n$. Since $E_t \to E_1$ in $L^1(B_{r_2})$, it follows that for t small enough $|E_t \cap B_{r_2}| \leq |E_1 \cap B_{r_2}| + \epsilon = |E_1 \cap (\mathbb{R}^n \setminus Q) \cap B_{r_2}| + \epsilon < \theta r_2^n + \epsilon < \tilde{C} r_2^n$, by (5.14).

On the other hand, for the same t and from (5.13) we have that $|E_t \cap B_{r_2}| \geq \tilde{C} r_2^n$, which is a contradiction. Hence, we conclude that $(\partial E_1 \cap Q) \setminus \{0\} \neq \emptyset$.

We reflect E_1 across the axis $\{x_n = 0\}$ and call E_2 the reflected set. Let $U = (E_1 \cup E_2) \setminus (\{x : x_n = 0\} \cap \partial E_1)$. We claim that U has least perimeter in $\mathbb{R}^n$. In fact, assume that this is not true. Then there exists $R > 0$ and a set L, having the same trace as E on ∂B_R, such that $\mathrm{Per}(L, B_R) < \mathrm{Per}(U, B_R)$. Since $\mathrm{Per}(L, B_R \cap \{x : x_n > 0\}) + \mathrm{Per}(L, B_R \cap \{x : x_n < 0\}) < \mathrm{Per}(U, B_R) = 2\mathrm{Per}(E_1, \{x : x_n > 0\})$ it follows that either $\mathrm{Per}(L, B_R \cap \{x : x_n > 0\}) < \mathrm{Per}(E_1, B_R \cap \{x : x_n > 0\})$ or $\mathrm{Per}(L, B_R \cap \{x : x_n < 0\}) < \mathrm{Per}(E_2, B_R \cap \{x : x_n < 0\})$. However, either case is a contradiction since $\mathrm{Per}(E_1, B_R \cap \{x : x_n > 0\}) \leq \liminf \mathrm{Per}(E_t, B_R \cap \{x : x_n > 0\}) \leq \mathrm{Per}(L, B_R \cap \{x : x_n > 0\}) + \gamma(\epsilon)$, where $\gamma(\epsilon)$ goes to zero as ϵ goes to zero. The same estimate holds for $\mathrm{Per}(E_1, B_R \cap \{x : x_n < 0\})$.

We conclude that U minimizes area and, if $n \leq 7$, if follows from [18] (Chapter 17) that ∂U is a hyperplane. However, since we obtained U by reflecting E_1, it follows that ∂U is a hyperplane perpendicular to $\{x : x_n = 0\}$. Hence, ∂E_1 is also perpendilcular to $\{x : x_n = 0\}$. Now we return to E and prove that E enters the inclusion I orthogonally. That is, we need to prove that for any given cone Q with vertex at 0 that contains ∂E_1, there exists a radius r such that $\partial E \cap B_r$ is contained inside the cone. We prove now the following Lemma:

Lemma 2.5.1. For almost every $r \leq r_1$ we have:

$$\int_{B_r \cap O} |D\varphi_{E_t}| \to \int_{B_r \cap O} |D\varphi_{E_1}|$$

Proof: Define $A = B_r \cap O$. In this Lemma we want to prove the strong convergence of the measures. We know that, since $E_t \to E_1$ in L^1, the lower semicontinuaity property give us one inequality, namely:

$$\int_A |D\varphi_{E_1}| \leq \liminf \int_A D\varphi_{E_t}$$

The fact that the sets E_t also minimize area will allow us to get the reversed inequality. Approximate the domain A from the interior with smooth domains A_ϵ. Fix t and let F_ϵ be the set that coincides with E_t in the region $A \backslash A_\epsilon$ and coincides with E_1 inside A_ϵ. Since both F_ϵ and E_t have the same trace on ∂A_ϵ and since E_t minimizes area in A it follows that $\int_A |D\varphi_{E_t}| \leq \int_A |D\varphi_{F_\epsilon}|$. In fact, we have:

$$\begin{aligned} \int_A |D\varphi_{E_t}| &\leq \int_A |D\varphi_{F_\epsilon}| \\ &= \int_{A_\epsilon} |D\varphi_{F_\epsilon}| + \int_{A\backslash A_\epsilon} |D\varphi_{F_\epsilon}| + \int_{\partial A_\epsilon} |D\varphi_{F_\epsilon}| \\ &= \int_{A_\epsilon} |D\varphi_{E_1}| + \int_{A\backslash A_\epsilon} |D\varphi_{E_t}| + \int_{\partial A_\epsilon} |D\varphi_{F_\epsilon}| \end{aligned}$$

since F_ϵ coincides with E_1 and E_t in A_ϵ and $A\backslash A_\epsilon$ respectively. But we have that:

$$\int_{\partial A_\epsilon} |D\varphi_{F_\epsilon}| = \int_{\partial A_\epsilon} |(\varphi_{E_t})_{tr} - (\varphi_{E_1})_{tr}|$$

where $(\varphi_{E_t})_{tr}$ and $(\varphi_{E_1})_{tr}$ are the traces of the characteristic functions φ_{E_t} and φ_{E_1} respectively. However, for almost every ϵ, the traces coincide with the functions themselves. Hence we have that:

$$\begin{aligned}\int_A |D\varphi_{E_t}| &\leq \int_A |D\varphi_{F_\epsilon}| \\ &= \int_{A_\epsilon} |D\varphi_{E_1}| + \int_{A\backslash A_\epsilon} |D\varphi_{E_t}| + \int_{\partial A_\epsilon} |D\varphi_{F_\epsilon}| \\ &= \int_{A_\epsilon} |D\varphi_{E_1}| + \int_{A\backslash A_\epsilon} |D\varphi_{E_t}| + \int_{\partial A_\epsilon} |\varphi_{E_1} - \varphi_{E_t}|\end{aligned}$$

for almost every ϵ. If we now let ϵ goes to zero it follows that $\int_{A_\epsilon} |D\varphi_{E_1}| \rightarrow \int_A |D\varphi_{E_1}|$, $\int_{A\backslash A_\epsilon} |D\varphi_F| \rightarrow 0$ and $\int_{\partial A_\epsilon} |\varphi_{E_1} - \varphi_{E_t}| \rightarrow \int_{\partial A} |\varphi_{E_1} - \varphi_{E_t}|$. Also, since $E_t \rightarrow E_1$ in $L^1(B_r \cap O)$ we have that $\int_{B_r \cap O} |\varphi_{E_1} - \varphi_{E_t}| \rightarrow 0$ as t goes to zero. This implies that $\int_{\partial B_r \cap O} |\varphi_{E_1} - \varphi_{E_t}|$ is zero for almost every r. We have proven that for almost every $r \leq r_1$:

$$\int_{B_r \cap O} |D\varphi_{E_t}| \leq \int_{B_r \cap O} |D\varphi_{E_1}|$$

and hence $\limsup \int_{B_r \cap O} |D\varphi_{E_t}| \leq \int_{B_r \cap O} |D\varphi_{E_1}|$. This proves the reversed inequality. Moreover, since $|D\varphi_{E_1}|$ is concentrated inside the cone Q the following also holds:

$$\int_{B_r\cap(\mathbb{R}^n\setminus Q)\cap O} |D\varphi_{E_t}| \;\; \rightarrow \;\; \int_{B_r\cap(\mathbb{R}^n\setminus Q)\cap O} |D\varphi_{E_1}|$$

. $\square$

Remark 2.5.2. Notice that the convergence stated in this claim holds for a subsequence of $\{E_t\}$, where we previously chose a sequence $\{t\}$ going to zero. However if we chose another sequence $\{\tilde{t}\}$ going to zero, then this new sequence will have a subsequence that converges in L^1 to the same set E_1 (because of the regularity result up to the boundary); that is, any blow up will give us the same E_1. This observation is important since in general, if the blow up of a function is a plane, we can not conclude that the function itself is differentiable. Indeed, a different blow up could give us a different plane.

Fix r for which the Lemma holds. Suppose now that ∂E has points outside the cone Q for any radius $\leq$ r . Then there exist sequences $\{t_j\}, \{\rho_j\}, \{x_j\}$ such that $\{x_j\} \subset \partial E$, $t_j \rightarrow 0$, $B(x_j, \rho_j) \subset B_{rt_j} \cap \mathbb{R}^n \setminus Q$ and $\rho_j = \gamma t_j$, where γ is a constant that depends on the angle θ spanned by the cone Q. Because of the claim and Remark 5.2 we have that:

$$r^{1-n}\int_{B_r\cap(\mathbb{R}^n\setminus Q)\cap O} |D\varphi_{E_{t_j}}| \rightarrow r^{1-n}\int_{B_r\cap(\mathbb{R}^n\setminus Q)\cap O} |D\varphi_{E_1}| = 0$$

However:

$$\begin{aligned} r^{1-n}\int_{B_r\cap(\mathbb{R}^n\setminus Q)\cap O}|D\varphi_{E_{t_j}}| &= (t_jr)^{1-n}\int_{B_{t_jr}\cap(\mathbb{R}^n\setminus Q)\cap O}|D\varphi_E| \\ &\geq C(\rho_j)^{1-n}\int_{B(x_j,\rho_j)\cap O}|D\varphi_E| \\ &\geq C \end{aligned}$$

which gives a contradiction. □

The boundary regularity Theorem in [21] and the proof of the orthogonallity given above give us a picture of how the minimizer E intersects the inclusions. Recall the definitions of P, Ω , I and H_i in Section 2.

We can also build minimizers in the following way: Define $\Omega_\epsilon = \{x \in \Omega : d(x, I) > \epsilon\}$. Let F_ϵ be a smooth function satisfying:

$$\begin{aligned} \epsilon &\leq F_\epsilon \quad \leq 1 \\ F_\epsilon(x) &= \epsilon, \quad x \in I \\ F_\epsilon(x) &= 1, \quad x \in \Omega_\epsilon \end{aligned}$$

For each $E \in A_{S_1,S_2}$, consider the quantity:

$$\tilde{J}(E) = \int_P F_\epsilon|D\varphi_E| \tag{2.5.11}$$

Since each F_ϵ is smooth, by [10] there exists $E_\epsilon \in A_{S_1,S_2}$ that minimizes (5.15). We have:

$$\lim_{\epsilon\to 0}\int_{\Omega_\epsilon}|D\varphi_{E_\epsilon}| = \int_\Omega |D\varphi_{E_0}|$$

where E_0 is the minimizer constructed in Section 2.

Let $\tilde{E}_\epsilon$ be a Cacciopoli set having the same trace (recall the definition of trace function given in the Introduction) as E_ϵ on the boundary of each $H_i^\epsilon = \{x : d(x, \partial H_i) < \epsilon\}$, coinciding with E_ϵ on Ω_ϵ and minimizing perimeter in each open set H_i^ϵ. Notice that, for almost every ϵ, $\mathrm{Per}(\tilde{E}_\epsilon, \Omega) = \mathrm{Per}(E_\epsilon, \Omega_\epsilon) + \gamma(\epsilon)$ and $\gamma(\epsilon)$ goes to zero as ϵ goes to zero. Hence, there exist $\tilde{E}_0$ such that $\tilde{E}_\epsilon \to \tilde{E}_0$ in $L^1(\Omega)$. Using (1.1):

$$\begin{aligned}
\int_\Omega |D\varphi_{\tilde{E}_0}| &\leq \liminf_{\epsilon\to 0} \int_\Omega |D\varphi_{\tilde{E}_\epsilon}| \\
&= \liminf_{\epsilon\to 0} (\int_{\Omega_\epsilon} |D\varphi_{E_\epsilon}| + \gamma(\epsilon)) \\
&\leq \limsup_{\epsilon\to 0} (\int_{\Omega_\epsilon} |D\varphi_{E_\epsilon}| + \gamma(\epsilon)) \\
&\leq \limsup_{\epsilon\to 0} \int_{\Omega_\epsilon} |D\varphi_{E_\epsilon}| \\
&= \int_\Omega |D\varphi_{E_0}|
\end{aligned}$$

This implies that $\tilde{E}_0$ is also a minimizer corresponding to the class A_{S_1,S_2}.

Chapter 3

Homogenization for n=2.

3.1 Homogenization for the case $n = 2$

In what follows we will assume that $n = 2$ and that the inclusions are closed balls. Let $0 < \rho < \frac{1}{2}$ denote the radius of each ball.

Let P and Q denote any two fixed points in the plane. Let $l_\epsilon^\rho(P,Q)$ denote the optimal path in $\mathbb{R}_i^2$ joining P and Q at the scale ϵ, that is, when the size of the cubes is ϵ and the radius of each ball is $\epsilon\rho$. Let $d_\epsilon^\rho(P,Q)$ denote the length of $l_\epsilon^\rho(P,Q)$ (this length is measured in $\mathbb{R}_i^2$). The behavior of this quantity will depend on the value of ρ. We are interested in looking at its limit as ϵ goes to zero. More specifically, we wish to find the effective norm in the homogenized limit (recall the definitions of *effective norm* and *homogenized limit* given in the Introduction).

Notice that $l_\epsilon^\rho(P,Q)$ is composed of pieces of lines, and that each piece of line meets two inclusions (or maybe only one inclusion at the end of the path) orthogonally.

We have, for any $0 < \epsilon \leq 1$:

$$0 \leq d_\epsilon^\rho(P,Q) \leq |P-Q|_2 \tag{3.1.1}$$

Thus, as $\epsilon \to 0$, $\{d_\epsilon^\rho(P,Q)\}$ has a convergent subsequence. We will again denote the subsequence as $\{d_\epsilon^\rho(P,Q)\}$. Let $d_0^\rho(P,Q)$ denote its limit.

Let X and Y be centers of balls at the scale ϵ. By replacing ∂E inside the inclusions with lines, we can think of the optimal path $l_\epsilon^\rho(X,Y)$ as being composed of a sequence of segments, and we can classify (after a suitable translation and/or rotation) these segments in the following four categories:

1. A segment joining the points $(0,0)$ and $(\frac{i}{\epsilon}, \frac{j}{\epsilon})$ where $i, j \in \mathbb{Z}^+$ are relatively prime and $j < i$.
2. The segment joining the points $(0,0)$ and $(\frac{1}{\epsilon}, 0)$.
3. The segment joining the points $(0,0)$ and $(0, \frac{1}{\epsilon})$.
4. The segment joining $(0,0)$ and $(\frac{1}{\epsilon}, \frac{1}{\epsilon})$.

Let us identify the first type of segment with the pair $[i,j]$, the second and third types with $[1,0]$ and the last with $[1,1]$. We have then that the set:

$$\mathcal{P} = \{[i,j] : i, j \in \mathbb{Z}^+, i, j \quad \text{are relatively prime, } j < i\} \cup \{[1,1]\} \cup \{[1,0]\}$$

give us the set of all possible segments. We will prove in the next theorem that if $\rho > \frac{2-\sqrt{2}}{2}$ then $l_\epsilon^\rho(X,Y)$ is composed only of segments of the type $[1,0]$.

Theorem 3.1.1. If $\rho > \frac{2-\sqrt{2}}{2}$ then, for any $0 < \epsilon \leq 1$, the optimal path connecting two points that are centers of balls is composed only of segments of

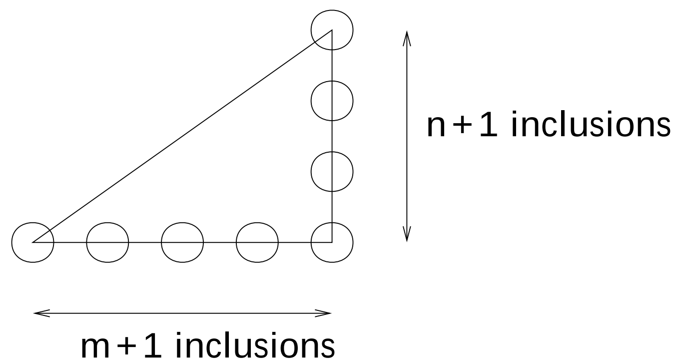

Figure 3.1: $n \leq m$, $n \geq 1$

the type $[1, 0]$. Moreover, if P and Q are any two points we have:

$$\lim_{\epsilon \to 0} d^{\rho}_{\epsilon}(P, Q) \quad = \quad (1 - 2\rho)|P - Q|_1$$

Proof: Fix $\epsilon > 0$.

Let $X = (x_1, x_2)$, $Y = (y_1, y_2)$ be two points that are centers of balls at the scale ϵ. We can assume $y_1 \geq x_1$ and $y_2 \geq x_2$. Assume that $l^{\rho}_{\epsilon}(X, Y)$ has a segment different from $[1, 0]$. Hence, $l^{\rho}_{\epsilon}(X, Y)$ has a line segment as shown in Figure 4. Since $l^{\rho}_{\epsilon}(X, Y)$ is the optimal path we have that:

$$\begin{aligned}
m(\epsilon - 2\epsilon\rho) + n(\epsilon - 2\epsilon\rho) &\geq \sqrt{\epsilon^2 m^2 + \epsilon^2 n^2} - 2\epsilon\rho \\
m(1 - 2\rho) + n(1 - 2\rho) &\geq \sqrt{m^2 + n^2} - 2\rho \\
m + n - \sqrt{m^2 + n^2} &\geq 2(m + n - 1)\rho \\
\Rightarrow \rho &\leq \frac{m + n - \sqrt{m^2 + n^2}}{2(m + n - 1)}
\end{aligned}$$

We claim that $\frac{m+n-\sqrt{m^2+n^2}}{2(m+n-1)} \leq \frac{2-\sqrt{2}}{2}$. To prove this, define the function $f(x) = \frac{x+n-\sqrt{x^2+n^2}}{2(x+n-1)}$. If we compute its derivative we obtain that $f'(x) = \frac{1}{2}.\frac{n^2-\sqrt{x^2+n^2}+x-xn}{(x+n+1)^2}$. Notice that $f'(x) \leq 0$ if $x \geq 0$. This implies that f is decreasing and thus $f(m) \leq f(n)$. We have that: $f(n) = \frac{2n-\sqrt{2}n}{2(2n-1)} = \frac{2-\sqrt{2}}{2}(\frac{n}{2n-1}) \leq \frac{2-\sqrt{2}}{2}($ since $\frac{n}{2n-1} \leq 1)$. Hence, $\rho \leq f(m) \leq \frac{2-\sqrt{2}}{2}$, which contradicts the fact that $\rho > \frac{2-\sqrt{2}}{2}$. This proves the first part of the theorem. Because of the above result, we can explicitly compute $d_\epsilon^\rho(X, Y)$:

$$\begin{aligned}
d_\epsilon^\rho(X, Y) &= \frac{y_2 - x_2}{\epsilon}(\epsilon - 2\epsilon\rho) + \frac{y_1 - x_1}{\epsilon}(\epsilon - 2\epsilon\rho) \\
&= (1 - 2\rho)[(y_2 - x_2) + (y_1 - x_1)] \\
&= (1 - 2\rho)|Y - X|_1 \qquad (3.1.2)
\end{aligned}$$

Now let P and Q be any two points in the plane. Let P' and Q' be the closest centers of balls, at the scale ϵ, to P and Q respectively. We have:

$$d_\epsilon^\rho(P', Q') - 2\epsilon \leq d_\epsilon^\rho(P, Q) \leq d_\epsilon^\rho(P', Q') + 2\epsilon$$

Using (7.17) we have:

$$(1 - 2\rho)|P' - Q'|_1 - 2\epsilon \leq d_\epsilon^\rho(P, Q) \leq (1 - 2\rho)|P' - Q'|_1 + 2\epsilon$$

Using the triangle inequality again:

$$\begin{aligned}(1-2\rho)(|P-Q|_1 - 2\sqrt{2}\epsilon) - 2\epsilon &\leq d_\epsilon^\rho(P,Q) \\ &\leq (1-2\rho)(|P-Q|_1 + 2\sqrt{2}\epsilon) + 2\epsilon\end{aligned}$$

Hence:

$$\begin{aligned}(1-2\rho)(1-\frac{2\sqrt{2}\epsilon}{|P-Q|_1}) - \frac{2\epsilon}{|P-Q|_1} &\leq \frac{d_\epsilon^\rho(P,Q)}{|P-Q|_1} \\ &\leq (1-2\rho)(1+\frac{2\sqrt{2}\epsilon}{|P-Q|_1}) + \frac{2\epsilon}{|P-Q|_1}\end{aligned}$$

Now letting $\epsilon \longrightarrow 0$ we have:

$$\begin{aligned}1-2\rho &\leq \frac{d_0^\rho(P,Q)}{|P-Q|_1} \leq 1-2\rho \\ &\Rightarrow \\ d_0^\rho(P,Q) &= (1-2\rho)|P-Q|_1 \quad \square\end{aligned}$$

We now wish to study the behavior of the optimal path as ρ goes to zero. As ρ decreases, new paths (new segments of the collection $\mathcal{P}$) become available. For each segment $[i,j]$ there exists a critical radius $\rho_{[i,j]}$, that is, the largest radius for which $d_1^{\rho_{[i,j]}}((0,0),(i,j)) = \sqrt{i^2+j^2}$.

Since $\mathcal{P}$ is countable, we can enumerate the sequence $\{\rho_{[i,j]}\}$ in such a way that the coordinate i is always increasing. We have the following Lemma:

Lemma 3.1.1. $\lim_{i\to\infty} \rho_{[i,j]} = 0$

Proof: Recall that $[i,j]$ represents the line joining $(0,0)$ with (i,j). Let $P=(p_1,p_2)$ be the closest point (other than the extremes) with integer coordinates to this line. The point P satisfies the equation $|\frac{i}{j}-\frac{p_1}{p_2}|=\frac{1}{jp_2}$, that is, $|ip_2-jp_1|=1=A(p)$ where $A(p)$ is the area of the parallelogram spanned by (i,j) and (p_1,p_2). Hence we have that $\frac{1}{2}=d(P,[i,j])\frac{\sqrt{i^2+j^2}}{2}$, which implies that $=d(P,[i,j])=\frac{1}{i^2+j^2}$. Let $l=|(i,j)|_2$, $l_1=|P|_2$ and $l_2=|P-(i,j)|_2$. Solving the equation $l-2\rho=l_1+l_2-4\rho$, we have $\rho=\frac{l_1+l_2-l}{2}$. Using the triangle inequality we have:

$$
\begin{aligned}
\rho_{[i,j]} \leq \rho &= \frac{l_1+l_2-l}{2} \\
&\leq \frac{2d(P,[i,j])+l-l}{2} \\
&= d(P,[i,j]) \\
&= \frac{1}{i^2+j^2}
\end{aligned}
$$

which proves the Lemma. □

In the Lemma above we can get a better upper bound for $\rho_{[i,j]}$ if we compute exactly l_1 and l_2. For this, we need to know the exact position of P. There exists an algorithm in number theory that uses continuous fractions to approximate the slope of $[i,j]$ and compute P. This algorithm has been implemented in computer packages and allows to compute the exact value of ρ.

An easy computation gives us $\rho_{[1,1]}=\frac{2-\sqrt{2}}{2}$,$\rho_{[2,1]}=\frac{1+\sqrt{2}-\sqrt{5}}{2}$ and $\rho_{[3,1]}=\frac{1+\sqrt{5}-\sqrt{10}}{2}$. We have the following Theorem:

Theorem 3.1.2. Let P, Q be any two points in the plane. Then,

(a) if $\frac{1+\sqrt{2}-\sqrt{5}}{2} < \rho \leq \frac{2-\sqrt{2}}{2}$, we have:

$$\lim_{\epsilon \to 0} d_\epsilon^\rho(P, Q) = |P - Q|_{1,1}$$

where $|\cdot|_{1,1} : \mathbb{R}^2 \to \mathbb{R}^+$ is given by:

$|(x, y)|_{1,1} = (\sqrt{2} - 1)|x| + (1 - 2\rho)|y|$, if $|x| \leq |y|$, and

$|(x, y)|_{1,1} = (\sqrt{2} - 1)|y| + (1 - 2\rho)|x|$, if $|y| \leq |x|$.

(b) if $\frac{1+\sqrt{5}-\sqrt{10}}{2} < \rho \leq \frac{1+\sqrt{2}-\sqrt{5}}{2}$, we have:

$$\lim_{\epsilon \to 0} d_\epsilon^\rho(P, Q) = |P - Q|_{2,1}$$

where $|\cdot|_{2,1} : \mathbb{R}^2 \to \mathbb{R}^+$ is given by:

$|(x, y)|_{2,1} = (1 - 2\rho)|x| + (\sqrt{5} - 2 + 2\rho)|y|$, if $|y| \leq \frac{|x|}{2}$,

$|(x, y)|_{2,1} = (2\sqrt{2} - \sqrt{5} - 2\rho)|y| + (\sqrt{5} - \sqrt{2})|x|$, if $\frac{|x|}{2} < |y| \leq |x|$,

$|(x, y)|_{2,1} = (1 - 2\rho)|y| + (\sqrt{5} - 2 - 2\rho)|x|$, if $|y| \geq 2|x|$,and

$|(x, y)|_{2,1} = (2\sqrt{2} - \sqrt{5} + 2\rho)|x| + (\sqrt{5} - \sqrt{2})|y|$, if $|x| \leq |y| < 2|x|$.

Moreover, $|\cdot|_{1,1}$ and $|\cdot|_{2,1}$ define norms in $\mathbb{R}^2$.

Proof: Let $X = (x_1, x_2)$ and $Y = (y_1, y_2)$ be centers of balls at the scale ϵ. We can assume that $x_1 \leq y_1$ and $x_2 \leq y_2$. To prove (a) consider first the case when $y_2 - x_2 \leq y_1 - x_1$. If ρ lies in the interval given in (a), the only paths available are $[1, 0]$ and $[1, 1]$, and the optimal path $l_\epsilon^\rho(X, Y)$ will have as many segments $[1,1]$ as possible since for this interval $[1,1]$ is better than two $[1,0]$. Hence, we can compute $d_\epsilon^\rho(X,Y)$ explicitly. In fact:

$$\begin{aligned} d_\epsilon^\rho(X,Y) &= \frac{y_2-x_2}{\epsilon}(\sqrt{2}\epsilon-2\epsilon\rho)+\frac{(y_1-x_1)-(y_2-x_2)}{\epsilon}(\epsilon-2\epsilon\rho) \\ &= (\sqrt{2}-1)(y_2-x_2)+(1-2\rho)(y_1-x_1) \\ &= |Y-X|_{1,1} \end{aligned}$$

The case $y_2-x_2 \geq y_1-x_1$ is computed in the same way but interchanging the roles of the coordinates. To prove (a) we can proceed now in exactly the same way (provided that $|\cdot|_{1,1}$ is a norm) as in Theorem 7.1. Let us check now that $|\cdot|_{1,1}$ defines a norm. We only need to check that the triangle inequality holds. There are three different cases to check but we will only check one case, since the other two are proven in exactly the same way.

Let $(x,y),(w,z)$ be any two points in the plane. Consider the case: $|y| \leq |x|$, $|w| \leq |z|$ and $|x+w| \leq |y+z|$. We need to prove that $(\sqrt{2}-1)|x+w|+(1-2\rho)|y+z| \leq (\sqrt{2}-1)(|y|+|w|)+(1-2\rho)(|x|+|z|)$; that is, $(\sqrt{2}-1)(|x+w|-|y|-|w|)+(1-2\rho)(|y+z|-|x|-|z|) \leq 0$. Using the triangle inequality for real numbers we can see that the last inequality is true because $|x|-|y| \geq 0$ and $1-2\rho > \sqrt{2}-1$ for ρ in the interval given in (a). The unit ball for this norm is a polygon with 8 edges as shown in Figure 5.

To prove (b) notice the following: if $p > 0, q \geq 0$ are two integers satisfying $-p+2q \leq 0$ then, for ρ in the interval given in (b), the best path joining $(0,0)$ with (p,q) consists only of segments of the type $[2,1]$ and $[1,0]$. Furthermore, this path takes as many $[2,1]$ segments as possible and then completes the trajectory with segments $[1,0]$. If $-p+2q>0$ and $q<p$, the best path consists only on segments of the type $[2,1]$ and $[1,1]$ and this path takes as many $[2,1]$ segments as possible and then completes the trajectory with segments $[1,1]$. Thus, we can compute exactly $d_\epsilon^\rho(X,Y)$ as before and proceed as in Theorem (7.1). The unit ball for $|\cdot|_{2,1}$ is a polygon with 16 edges as shown in Figure 5. $\square$

The following Theorem gives an asymptotic behavior of d_0^ρ.

Theorem 3.1.3. Let P, Q be any two points in the plane. Then:

$$\lim_{\rho\to 0} d_0^\rho(P,Q) = |P-Q|_2$$

Proof: Let X and Y be two points that are centers of balls at the scale ϵ. $l_\epsilon^\rho(X,Y)$ intersects a finite numbers of balls, say N. Let:

$$\begin{aligned} \tilde{d}_\epsilon^\rho(X,Y) &= d_\epsilon^\rho(X,Y) + (N-1)(2\epsilon\rho) \\ \Rightarrow d_\epsilon^\rho(X,Y) &= \tilde{d}_\epsilon^\rho(X,Y) - (N-1)(2\epsilon\rho) \end{aligned} \tag{3.1.3}$$

Notice that:

$$\begin{aligned} N &\le \frac{\tilde{d}_\epsilon^\rho(X,Y)}{\epsilon} + 1 \\ \Rightarrow N-1 &\le \frac{\tilde{d}_\epsilon^\rho(X,Y)}{\epsilon} \end{aligned}$$

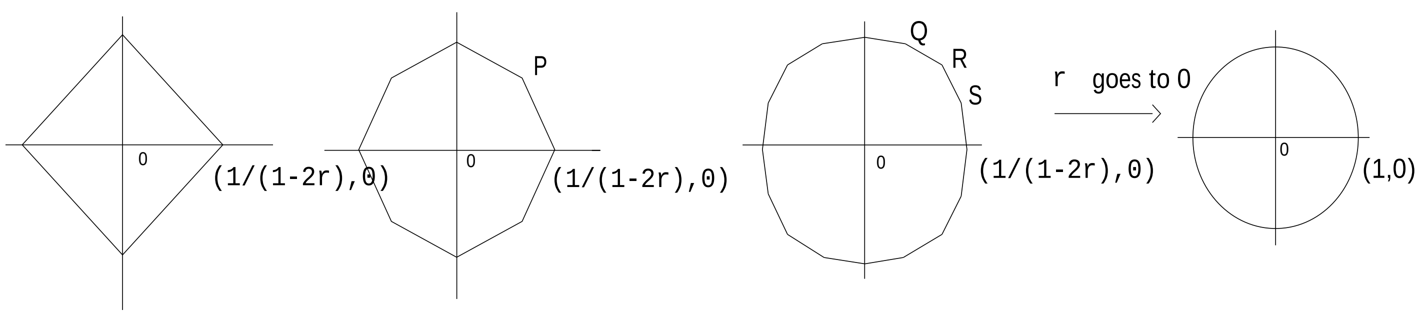

Figure 3.2: Unit balls for limiting norms. $P = R = (\frac{1}{\sqrt{2}-2\rho}, \frac{1}{\sqrt{2}-2\rho})$, $Q = (\frac{1}{\sqrt{5}-2\rho}, \frac{2}{\sqrt{5}-2\rho})$, $S = (\frac{2}{\sqrt{5}-2\rho}, \frac{1}{\sqrt{5}-2\rho})$.

Hence, using (7.18):

$$\begin{aligned} d_\epsilon^\rho(X,Y) &\geq \tilde{d}_\epsilon^\rho(X,Y) - \frac{\tilde{d}_\epsilon^\rho(X,Y)}{\epsilon}(2\epsilon\rho) \\ &= \tilde{d}_\epsilon^\rho(X,Y)(1-2\rho) \\ &\geq (1-2\rho)|X-Y|_2 \end{aligned} \tag{3.1.4}$$

Let P' and Q' be the closest points to P and Q respectively, at the scale ϵ, in such a way that both P' and Q' are centers of balls. We have that:

$$d_\epsilon^\rho(P',Q') - 2\epsilon \leq d_\epsilon^\rho(P,Q) \leq |P-Q|_2$$

Using (7.19), we have:

$$(1-2\rho)|P'-Q'|_2 - 2\epsilon \leq d_\epsilon^\rho(P,Q) \leq |P-Q|_2$$

Using the triangle inequality we have:

$$\begin{aligned} (1-2\rho)(|P-Q|_2 - 2\epsilon) - 2\epsilon &\leq d_\epsilon^\rho(P,Q) \leq |P-Q|_2 \\ \Rightarrow (1-2\rho)(1 - \frac{2\epsilon}{|P-Q|_2}) - \frac{2\epsilon}{|P-Q|_2} &\leq \frac{d_\epsilon^\rho(P,Q)}{|P-Q|_2} \leq 1 \end{aligned}$$

Now letting $\epsilon \to 0$ we have:

$$1 - 2\rho \leq \frac{d_0^\rho(P,Q)}{|P-Q|_2} \leq 1$$

This implies:

$$\lim_{\rho \to 0} d_0^\rho(P,Q) = |P-Q|_2 \quad \square$$

Figure 5 shows the unit balls of the effective norms for the cases $\frac{2-\sqrt{2}}{2} < \rho < 0.5$, $\frac{1+\sqrt{2}-\sqrt{5}}{2} < \rho \leq \frac{2-\sqrt{2}}{2}$ and $\frac{1+\sqrt{5}-\sqrt{10}}{2} < \rho \leq \frac{1+\sqrt{2}-\sqrt{5}}{2}$. Also, as $\rho \to 0$, the effective norms converge to the Euclidean norm. Our results suggest that as ρ gets smaller the behavior of the unit ball changes, though it is always polygonal with more and mores edges, until it becomes a circle in the limit.

3.2 Connection with Effective Hamiltonians

The theory of homogenization studies the asymptotic behaviour of a family of partial differential equations, which oscillate with small period size $\epsilon > 0$. In this section we want to start exploring the connection between the results proven in the previous section and the theory of homogenization of Hamilton-Jacobi equations.

Consider, for each ϵ, the solution u^ϵ to the following problem:

$$\begin{aligned} H(Du^\epsilon, \frac{x}{\epsilon}) &= 0 \text{ in } \Omega \\ u^\epsilon &= 0 \text{ on } \partial\Omega \end{aligned} \tag{3.2.5}$$

where $H : \mathbb{R}^n \times \overline{\Omega} \to \mathbb{R}$ satisfies, among others, the following properties:

- $H(p, x)$ is periodic in the x variable.
- $H(p, x)$ is convex in the p variable.
- H is uniformly continuous in the x variable.
- $\lim_{|p|\to\infty} H(p, x) = \infty$

and Ω is an open set. It can be proven, see for instance [15], that the sequence of solutions $\{u^\epsilon\}$ has a subsequence that converges uniformly to a Lipschitz function u. Moreover, u satisfies the following PDE:

$$\begin{aligned} \overline{H}(Du) &= 0 \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega \end{aligned}$$

where $\overline{H} : \mathbb{R}^n \to \mathbb{R}$ is defined as follows: For each $p \in \mathbb{R}^n$, $\overline{H}(p)$ is the unique number for which the PDE:

$$\begin{aligned} H(p + D_y v, y) &= \overline{H}(p) \text{ in } \mathbb{R}^n \\ v \quad \text{is} &\quad [0,1]^n - \text{periodic} \end{aligned} \tag{3.2.6}$$

has a Lipschitz viscosity solution.

The function $\overline{H}$ is called the effective Hamiltonian and a very interesting question is to study structure of $\overline{H}$ and to find explicit formulas for it. This is still a largely open problem. In [14], Evans gives some partial results in this direction. See also [12].

In the previous section we proved some results concerning the homogenization for the case $n = 2$. We want to recast those results in the context of homogenization for Hamilton-Jacobi equations. Even though the results of the previous section were proven for $n = 2$, let us define the following function:

$$u^\epsilon(x) = d^\rho_\epsilon(0, x), x \in \mathbb{R}^n \tag{3.2.7}$$

We would like to find the Hamilton-Jacobi equation that the previous function solves. Because of the nature of our domain (we have inclusions and inside of these inclusions the area is neglected), we see that we need to find the PDE that u^ϵ solves outside the inclusions together with the condition the u^ϵ satisfies on the boundary of the inclusions. In the following exposition, let $\epsilon = 1$ and fix ρ. Define de function:

$$v(x) = d_1^\rho(x, 0), x \in \mathbb{R}^n \tag{3.2.8}$$

That is, $v(x)$ is the smallest distance from x to the origin at the scale $\epsilon = 1$. Recall our notation $O = \mathbb{R}^n \backslash \{\text{Inclusions}\}$; that is, O is the open set outside the inclusions. Le H denote the union of all inclusions. Therefore, $O = \mathbb{R}^n \backslash H$. Notice that our function v is constant on each inclusion I and denote by $v(I)$ the value of v at any point of the inclusion I. We claim that v satisfies the following properties:

- $|Dv| = 1$ in O (in the viscosity sense)
- v is constant on each inclusion I
- $v(0) = 0$
- $\frac{\partial v}{\partial \nu}(x_I) = -1$ for some $x_I \in \partial I$

Remark 3.2.1. We assume that the the origin is the center of an inclusion (closed ball).

Since we are measuring the lenght of a curve by neglecting the parts inside the inclusions, it follows that v is constant on each inclusion I and it is also clear that $v(0) = 0$. We will prove now formally that, given any inclusion I

there exits a direccion ν an a point x_I on ∂I such that $\frac{\partial v}{\partial \nu}(x_I) = -1$. Take an inclusion I and let $x \in \partial I$. Let I_ϵ be a neighborhood of I; that is, $I_\epsilon = \{z | d(z, I)\} < \epsilon$. We have that, if ϵ is small enough:

$$v(x) = \inf_{\tilde{y} \in \partial I_\epsilon} (d(\tilde{y}, \partial I) + v(\tilde{y})) \tag{3.2.9}$$

For each $y \in \partial I$, let $\nu(y)$ denote the outward unit normal vector at y. From 3.2.9, if follows that:

$$0 = \inf_{\tilde{y} \in \partial I_\epsilon} (d(\tilde{y}, \partial I) + v(\tilde{y}) - v(x))$$

Let $y \in \partial I$ denote the point where $|y - \tilde{y}|$ is minimum. Using the fact that $v(y) = v(x)$ and $|y - \tilde{y}| = \epsilon$ we have:

$$\begin{aligned} 0 &= \inf_{\tilde{y} \in \partial I_\epsilon} (\epsilon + v(\tilde{y}) - v(y)) \\ 0 &= \inf_{y \in \partial I} (\epsilon + v(\tilde{y}) - v(y)) \\ 0 &= \inf_{y \in \partial I} (1 + \frac{v(y + \epsilon\nu(y)) - v(y)}{\epsilon}) \end{aligned}$$

Letting $\epsilon \to 0$ we have $0 = \inf_{\partial I}(1 + \frac{\partial v}{\partial \nu}(y))$. If this infimum is reached at $x_I \in \partial I$, if follows that $\frac{\partial v}{\partial \nu}(x_I) = -1$. That is, in at least one point of ∂I, the graph of v was "growing" just before reaching the constant value $v(I)$ at that point. We will prove now to that our function v solves the eikonal equation $|Dv| = 1$ in the viscosity sense outside the inclusions. Refer to [3] for the definition an properties of viscosity solutions. We prove first that v is a viscosity subsolution of $|Du| = 1$. Let φ be a C^1 function such that $v - \varphi$ has a local maximum at the point $x_0 \in O$. We need to prove that $|D\varphi(x_0)| \leq 1$.

Since $v-\varphi$ has a local maximum at x_0 it follows $v(x)-v(x_0)\leq\varphi(x)-\varphi(x_0)$, for all x in a neighborhood of x_0. We can write a point x in $B(x_0,h)$ (for h small enough) as x_0+hz for some z satisfying $|z|=1$. We have:

$$\begin{aligned}
v(x_0+hz)-v(x_0) &\leq \varphi(x_0+hz)-\varphi(x_0)\\
&= \int_0^h \frac{d}{ds}\varphi(x_0+sz)\\
&= \int_0^h D\varphi(x_0+sz)\cdot zds\\
&\leq \int_0^h D\varphi(x_0)\cdot zds + Ch^2
\end{aligned}$$

for all $|z|=1$. If we chose $z_0=-\frac{D\varphi(x_0)}{|D\varphi(x_0)|}$ we have:

$$\begin{aligned}
v(x_0+hz_0)-v(x_0) &\leq -\int_0^h |D\varphi(x_0)|ds + Ch^2\\
&= -h|D\varphi(x_0)|+Ch^2 \qquad (3.2.10)
\end{aligned}$$

We now use the fact that v is a Lipchitz function and from 3.2.10 we obtain:

$$\begin{aligned}
h|D\varphi(x_0) &\leq v(x_0)-v(x_0+hz_0)+Ch^2\\
&\leq |hz_0|+Ch^2
\end{aligned}$$

And hence:

$$|D\varphi(x_0)| \leq 1+Ch$$

By letting $h\to 0$ we conclude $|D\varphi(x_0)|\leq 1$. We prove now that v is a supersolution. Let φ be a C^1 function such that $v-\varphi$ has a local minimum at the point $x_0\in O$. We need to prove that $|D\varphi(x_0)|\geq 1$. Since $v-\varphi$ has a local minimum at x_0 it follows $v(x) - v(x_0) \geq \varphi(x) - \varphi(x_0)$, for all x in a neighborhood of x_0. As before, we write a point x in $B(x_0, h)$ as $x_0 + hz$ for some z satisfying $|z| = 1$. We have:

$$\begin{aligned} v(x_0 + hz) - v(x_0) &\geq \varphi(x_0 + hz) - \varphi(x_0) \\ &= \int_0^h \frac{d}{ds}\varphi(x_0 + sz) \\ &= \int_0^h D\varphi(x_0 + sz) \cdot z ds \\ &\geq \int_0^h D\varphi(x_0) \cdot z ds - Ch^2 \\ &\geq -h|D\varphi(x_0)| - Ch^2 \end{aligned} \tag{3.2.11}$$

for all $|z| = 1$. Since $v(x_0) = \inf_{|z|=1}\{h + v(x_0 + hz)\}$, there exists a point z_0 such that $v(x_0 + hz_0) + h \leq v(x_0) + h^2$ and thus $v(x_0) - v(x_0 + hz_0) \geq h - h^2$. Using this and 3.2.11 we obtain:

$$\begin{aligned} h|D\varphi(x_0) &\geq v(x_0) - v(x_0 + hz_0) - Ch^2 \\ &\geq h - h^2 - Ch^2 \\ &\Rightarrow \\ |D\varphi(x_0) &\geq 1 - h - Ch \end{aligned}$$

and letting $h \to 0$ we obtain $|D\varphi(x_0)| \geq 1$. Consider now the following Hamilton-Jacobi equation:

$$\begin{gathered} |Du| = 1 \quad in \quad O \\ u = \text{constant on each inclusion } I \\ u(0) = 0 \end{gathered} \tag{3.2.12}$$

We have proved that v is a viscosity solution of 3.2.12. We make the following definition:

Definition 3.2.1. We say that u is a maximal solution of the previous equation if $u(x) \geq w(x)$, for any w solution of the problem and any $x \in \mathbb{R}^n$.

The ecuacion 3.2.12 could have more than one viscosity solution. However, we will see that v is the unique maximal solution. That is, any other solution u satisfies $u(x) \leq v(x)$. We have the following comparison principle.

Theorem 3.2.1. v is the unique maximal viscosity solution of 3.2.12. That is, if u solves 3.2.12 in the viscosity sense we have $u \leq v$.

Proof: Assume that u a solution for 3.2.12. Consider first the case u is bounded. Let $0 < \theta < 1$ and define the function $u_\theta = \theta u$. If we prove that $u_\theta \leq v$ for any $0 < \theta < 1$, then by letting $\theta \to 1$ we obtain $u \leq v$. It can be proven that the function u_θ solves the equation $|Du_\theta| = \theta$ in the viscosity sense. To prove $u_\theta \leq v$ we proceed by contradiction. Assume that $u_\theta - v$ is positive at some point. Since $\lim_{|x|\to\infty} v = \infty$ it follows that $u_\theta - v$ in negative outside a large ball B_R and therefore $u_\theta - v$ attains a positive maximun $M > 0$ inside B_R. We claim that all the points where $u_\theta - v$ attains its maximum M are in the open set O. To see this suppose that $(u_\theta - v)(x_0) = M$ and $x_0 \in \partial I$ for some inclusion I. We proved above that there exists $\tilde{x} \in \partial I$ so that $\frac{\partial v}{\partial \nu}(\tilde{x}) = -1$. Since both u_θ and v are contant on I it follows that $(u_\theta - v)(\tilde{x}) = M$. We have $(u_\theta - v)(x) \leq (u_\theta - v)(\tilde{x}) = M$ for all $x \in \mathbb{R}^n$. If we denote $\tilde{v} = v + M$, we have $(u_\theta - \tilde{v})(x) \leq 0$ for all $x \in \mathbb{R}^n$ and $u_\theta(\tilde{x}) = \tilde{v}(\tilde{x})$.

This tells that that graph of u_θ is below the graph of $\tilde{v}$. On the other hand, since $|Du_\theta| = \theta$ and $\frac{\partial \tilde{v}}{\partial \nu}(\tilde{x}) = -1$, it follows that in the direction ν, the graph of $\tilde{v}$ is strictly below that the graph of u_θ which gives a contraction. We conclude that $u_\theta - v$ can not attain a maximun at a point on ∂O. Define the function:

$$\Phi_\epsilon(x, y) = u_\theta(x) - v(y) - \frac{1}{\epsilon}|x - y|^2 \tag{3.2.13}$$

and let $x_0 \in O$ be such that $(u_\theta - v)(x_0) = M$. Since $\Phi_\epsilon(x_0, x_0) > 0$ and $\Phi_\epsilon(x, y)$ is negative outside a large ball $B_{\tilde{R}} \times B_{\tilde{R}}$ we have that $\Phi_\epsilon(x, y)$ attains a positive maximum at a point $(x_\epsilon, y_\epsilon) \in B_{\tilde{R}} \times B_{\tilde{R}}$. The sequence $\{(x_\epsilon, y\epsilon)\}$ has a convergent subsequence (denoted again by $\{(x_\epsilon, y\epsilon)\}$). Let $(x, y) = \lim_{\epsilon \to 0}(x_\epsilon, y_\epsilon)$. We have:

$$\begin{aligned} M = \Phi_\epsilon(x_0, x_0) &\leq \Phi_\epsilon(x_\epsilon, x_\epsilon) \\ &= u_\theta(x_\epsilon) - v(y_\epsilon) - \frac{1}{\epsilon}|x_\epsilon - y_\epsilon|^2 \\ &\leq u_\theta(x_\epsilon) - v(y_\epsilon) \end{aligned} \tag{3.2.14}$$

Since u_θ and v are Lipchitz continuos, by letting $\epsilon \to 0$ we obtain:

$$M \leq u_\theta(x) - v(y) \tag{3.2.15}$$

From 3.2.14 and 3.2.15 we obtain:

$$\begin{aligned} \frac{1}{\epsilon}|x_\epsilon - y_\epsilon|^2 + M &\leq u_\theta(x_\epsilon) - v(y_\epsilon) \\ &\leq u_\theta(x_\epsilon) - v(x_\epsilon) + v(x_\epsilon) - v(y_\epsilon) \\ &\leq M + |x_\epsilon - y_\epsilon| \end{aligned}$$

From this if follows that if $x_\epsilon \neq y_\epsilon$ then $|x_\epsilon - y_\epsilon| \leq \epsilon$. Therefore $|x - y| \leq |x - x_\epsilon| + |x_\epsilon - y_\epsilon + |y_\epsilon - y| \leq |x - x_\epsilon| + \epsilon + |y_\epsilon - y|$ and by letting $\epsilon \to 0$ it follows that $|x - y| = 0$; that is, $x = y$. From 3.2.15 we have $u_\theta(x) - v(x) = M$ and hence $x \in O$. Thus, for ϵ small enough we have $x_\epsilon \in O$ and $y_\epsilon \in O$. We have:

$$u_\theta(x) - v(y) - \frac{1}{\epsilon}|x - y|^2 \quad \leq \quad u_\theta(x_\epsilon) - v(y_\epsilon) - \frac{1}{\epsilon}|x_\epsilon - y_\epsilon|^2$$

for all $x, y \in \mathbb{R}^n$. If we fix $y = y_\epsilon$ we have:

$$u_\theta(x) - (v(y_\epsilon) + \frac{1}{\epsilon}|x - y_\epsilon|^2) \quad \leq \quad u_\theta(x_\epsilon) - (v(y_\epsilon) + \frac{1}{\epsilon}|x_\epsilon - y_\epsilon|^2)$$

and thus, $x \to u_\theta(x) - (v(y_\epsilon) + \frac{1}{\epsilon}|x - y_\epsilon|^2)$ has a local maximum at the point x_ϵ. Since $|Du_\theta| \leq \theta$ we have:

$$|\frac{2}{\epsilon}(x_\epsilon - y_\epsilon)| \leq \theta \tag{3.2.16}$$

Multiplying 3.2.16 by -1 and fixing $x = x_\epsilon$ we have:

$$v(y) - (u_\theta(x_\epsilon) - \frac{1}{\epsilon}|x_\epsilon - y|^2) \quad \geq \quad v(y_\epsilon) - (u_\theta(x_\epsilon) - \frac{1}{\epsilon}|x_\epsilon - y_\epsilon|^2)$$

and thus $y \to v(y) - (u_\theta(x_\epsilon) - \frac{1}{\epsilon}|x_\epsilon - y|^2)$ has a local minimum at y_ϵ. Since $|Dv| \geq 1$ in the viscosity sense we have:

$$|\frac{2}{\epsilon}(x_\epsilon - y_\epsilon)| \geq 1 \tag{3.2.17}$$

which contradicts 3.2.16. From this contradiction we conclude that $u_\theta \leq v$. To consider the case u is not bounded, we can use an standard argument that consists in composing the function u with a suitable function Ψ so that

$\tilde{u} := \Psi(u)$ is bounded and $|D\tilde{u}| \leq \theta$ in the viscosity sense. We now give an sketch of this procedure. Let $\Psi_n : \mathbb{R} \to \mathbb{R}$ be a C^1 function such that $\Psi_n(r) = \theta r$ if $|r| \leq n$, $\Psi_n(r) = 0$ outside a large ball, $|\Psi_n'(r)| \leq \theta$ and $\Psi_n(r) \to r$ pointwise. Define the function $\tilde{u}_n = \Psi_n(u)$. It can be proven that $|D\tilde{u}_n| \leq \theta$ in the viscosity sense and since $\tilde{u}_n$ is bounded we are in the previous case and we can conclude $\tilde{u}_n \leq v$. Finally, by letting $n \to \infty$ we obtain $u \leq v$. We want to perform now the homogenization of the Hamilton-Jacobi equation 3.2.12. At the scale ϵ, our domain is the open set ϵO and the inclusions now are closed balls or radius $\epsilon\rho$. Recall the definition of u^ϵ in 3.2.7. We have fixed the radius ρ and we are interested in the behaviour of the solutions u^ϵ of the Hamilton-Jacobi equation:

$$|Du| = 1 \quad in \quad O_\epsilon$$

$$u = \text{constant on each inclusion } \epsilon I$$

$$u(0) = 0$$

Using standard theory for viscosity solutions we can prove that the sequence u^ϵ has a subsequence that converges uniformly to a function u. In the previous section we have found explicit formulas for u, at least for some values of ρ and $n = 2$. In fact, $u(x) = |x|_1, |x|_{1,1}$ or $|x|_{2,1}$ depending on the value of ρ (See previous section). Since we have a formula for u, this suggest that we should be able to give a formula for the corresponding effective Hamiltonian. However, we need first to formulate the cell problem (in analogy to 3.2.6) corresponding to our problem.

It would be interesting to be able to find an explicit formula for the effective Hamiltonian corresponding to this problem because there are few examples where this can be done. In fact, another interesting project would be to compute it numerically.

Chapter 4

An algorithm to compute weighted minimal surfaces in 2 and 3 dimensions.

4.1 Preliminaries

In Chapter 2 we constructed periodic surfaces that always stay at a bounded distance from a given plane, and in this thesis we give an algorithm for computing these minimal surfaces. The algorithm involves h-adaptive finite element approximations of the linear convection-diffusion equation. The latter equation has been shown to linearize the governing nonlinear PDE for evolution by weighted mean curvature.

The evolution of hypersurfaces $\mathbb{R}^n$ according to their mean curvature has been investigated thoroughly in the literature. There are many results in this area using parametric methods of differential geometry (See [20], [19], [23]), and also in the setting of varifold theory from geometric measure theory (See [6]).

However, a completely different approach was given by Osher and Sethian in [27]. Their approach, recast slightly, is as follows. Given the initial hypersurface Γ_0, select some continuous function g so that:

$$\Gamma_0 = \{x \in \mathbb{R}^n : g(x) = 0\}$$

Then, the PDE:

$$\begin{aligned} u_t = |Du|\mathrm{div}(\frac{Du}{|Du|}) \quad \text{in} \quad \mathbb{R}^n \times [0, \infty), \\ u = g \quad \text{on} \quad \mathbb{R}^n \times \{t = 0\} \end{aligned} \tag{4.1.1}$$

evolves the level sets of u according to their mean curvature.

Despite the conciseness of the above formulation, solving this equation numerically is computationally expensive due to the presence of the nonlinear elliptic term. Standard methods of solution for equations of this type include direct methods (Newton iterations), and indirect methods (nonlinear conjugate gradient). However, these algorithms are computationaly expensive due to the large number of iterations that are typically needed for convergence. It is desirable to have an algorithm that is relatively simple to implement and inexpensive to compute, so that the cost per time step is low. In such a case many time steps and long-time simulations can be carried out relatively quickly.

Bence, Merriman and Osher proposed in [25] a numerical algorithm for computing mean curvature flow using the heat equation and reinitializing after short time steps. This algorithm is a quasi-linearization of the PDE 4.1.1. The surface Γ_0 is evolved in the following way: let C_0 be an open bounded set with boundary Γ_0. Consider the PDE:

$$\begin{aligned} u_t - \Delta u = 0 \quad \text{in} \quad \mathbb{R}^n \times (0, \infty) \\ u = \varphi_{C_0} \quad \text{on} \quad \mathbb{R}^n \times \{t = 0\} \end{aligned} \tag{4.1.2}$$

where φ_{C_0} is the characteristic function of C_0.

The algorithm involves fixing a time step $t > 0$, defining the new set:

$$C_t = \{x \in \mathbb{R}^n : u(x,t) \geq \frac{1}{2}\} \tag{4.1.3}$$

and solving 4.1.2 again but replacing C_0 by C_t. In [25] heuristics are given to show that the evolution of C_0 in to C_t aproximates for small times the mean curvature motion of the boundary Γ_0 of C_0. Then by repeating this procedure, i.e. solving the heat equation and reinitializing after a short time, we approximate mean curvature flow. This is valid even for large times. The algorithm only involves the heat equation, which is easy to implement numerically and very inexpensive to compute. This allows for long-time simulations with thousands of time steps, with minimal computational cost.

Proofs of the convergence of this algorithm can be found in Evans [15], Barles-Georgelin [4].

4.2 The Algorithm

Chapter 2 contains the theoretical results concerning the construction of our weighted minimal surfaces, and these will be summarized next. Given $\omega \in \mathbb{Q}^n$, $N \in \mathbb{Z}$, $M \in \mathbb{R}$, with $M, N > 0$ we define:

$S_1 = \{x \in \mathbb{R}^n : x \cdot \omega \leq 0\}$

$S_2 = \{x \in \mathbb{R}^n : x \cdot \omega \leq M\}$

We refer to $\Pi_1 = \{x \in \mathbb{R}^n : x \cdot \omega = 0\}$ and $\Pi_2 = \{x \in \mathbb{R}^n : x \cdot \omega = M\}$ as the parallel plane restrictions. We minimize area outside the inclusions in the following class of periodic sets:

$A_{S_1,S_2} = \{E : E \quad \text{is a closed Cacciopoli set}, S_1 \subset E \subset S_2, T_{Nk}E = E, \forall k \in Z^n \quad \text{with} \quad \omega \cdot k = 0\}$

Since the sets in A_{S_1,S_2} are periodic, we minimize the area outside the inclusions in an open parallelogram P, representing one period (see figure 2.1 in chapter 2). By applying the standard argument in the calculus of variations we prove the existence of minimizers.

We now describe an algorithm to compute these minimizers, in the case of a heterogeneous domain. In our problem the area is measured by putting a weight of 0 inside the inclusions and a weight of 1 outside the inclusions. In order to treat the discontinuity of the weight in the computations, we approximate the original minimizer by ϵ-minimizers corresponding to the ϵ-problem (See end of Chapter 2). For the ϵ-problem, the discontinuous weight is smoothed using the function $\epsilon \leq w \leq 1$, with $\epsilon << 1$, where w has value ϵ inside the inclusions and increases smoothly to 1 in a very narrow neigborhood of the inclusion. Since w is smooth, the existence of minimizes for the ϵ-problem is contained in [10] and we proved in Chapter 2 that the area of the ϵ-minimizers outside the inclusions converges to the area of the true minimizer. The minimizers we have constructed are boundary of sets, so they are not the graph of a function. However, we have regularity for these minimizers and in fact each ϵ-minimizer is locally the graph of a smooth function (For references and a discusion about the regularity of the ϵ-minimizer see [10]). Also, as we explained in Chapter 2, the true minimizer is smooth outside the inclusions. Therefore, we can restrict ourselves to a neighborhood of a point in the boundary of the inclusions where the ϵ-minimizer is the graph of a smooth function and derive the corresponding Euler-Lagrange equation. Let $\Omega \subset \mathbb{R}^{n-1}$ be an open set where the ϵ-mininizer (a set contained in $\mathbb{R}^n$) is the graph of the smooth function $f(x')$, $x' \in \mathbb{R}^{n-1}$. Since f minimizes area with this smooth weighted metric it follows that f minimizes the following:

$$\int_\Omega w(x', f(x'))\sqrt{1 + |Df(x')|^2} \tag{4.2.4}$$

over all smooth functions agreeing with f on $\partial\Omega$. Thus f satisfies the Euler-Lagrange equation:

$$\operatorname{div}(w(x', f(x'))\frac{Df(x')}{\sqrt{1 + |Df(x')|^2}}) - w_{y_n}(x', f(x'))\sqrt{1 + |Df(x')|^2} = 0 \tag{4.2.5}$$

The first variation 4.2.5 of the functional 4.2.4 gives a formula for the weighted mean curvature at the point (x',f(x')). However, as we said before, our minimal surfaces are boundaries of sets and hence they are not globally the graph of a function. Therefore, in order to apply the level set approach invented in [27], we need to find the corresponding formula to 4.2.5 for the case when we have a surface defined implicitely. The main idea in the level set approach is to embed the surface we want to evolve as the zero level set of a continuos function and find the PDE that evolves this surface according to a prescribed velocity. In our problem, each point in the surface will move in the normal direction with a velocity $-H_w$, where $H_w(x)$ is the weighted mean curvature at each point x of the surface. More precisely, suppose we have $\Gamma_0 = \{x \in \mathbb{R}^n : g(x) = 0\}$, where g is a smooth function. Notice that by doing this, we have added one more dimension to the problem. We want to evolve Γ_0 according to its weighted mean curvature. Suppose that at each time $t \geq 0$, the evolved hypersurface is given implicitely by the equation $u(x,t) = 0$. Then $u(x,0) = g(x)$. Let $x \in \mathbb{R}^n$ and let $\alpha(s)$, $\alpha(t) = x$, be the trajectory of x under this evolution. We have that $u(\alpha(s), s) = 0$ for all s. Differentiating with respect to s gives that $u_t(\alpha(s), s) + Du(\alpha(s), s) \cdot \alpha'(s) = 0$; that is, $u_t(x,t) + Du(x,t) \cdot \alpha'(t) = 0$. However,

$$\alpha'(t) = -H_w(x)\frac{Du}{|Du|} \tag{4.2.6}$$

Since $\frac{Du}{|Du|}$ is the exterior unit normal vector at x, 4.2.6 says we are evolving the surface in the normal direction with velocity $-H_w(x)$. Since we are evolving the level sets of u and we know that locally these level sets are the graph of a function, we can use 4.2.5 to prove (after a somewhat lenghty computation) that:

$$H_w(x) = \mathrm{div}(w(x)\frac{Du}{|Du|}) \tag{4.2.7}$$

Notice that the second term in 4.2.5 does not appear in H_w. Thus, we obtain:

$$\begin{aligned} u_t(x,t) &= Du \cdot H_w(x)\frac{Du}{|Du|} \\ &= |Du| H_w(x) \\ &= |Du|\mathrm{div}(w(x)\frac{Du}{|Du|}) \end{aligned}$$

Then, the PDE that evolves Γ_0 according to its weighted mean curvature is:

$$\begin{aligned} u_t(x,t) &= |Du|\mathrm{div}(w(x)\frac{Du}{|Du|}) \\ u(x,0) &= g(x) \end{aligned} \tag{4.2.8}$$

We consider a linearization of 4.2.8 analogous to the linearization of equation 4.1.1 by 4.1.2. That is, we want to find a BMO ([25]) version for the equation 4.2.8. This is in fact possible and the equation:

$$\begin{aligned} u_t(x,t) &= \mathrm{div}(w(x)Du) + \frac{1}{2}Dw \cdot Du \\ u(x,0) &= g(x) \end{aligned} \tag{4.2.9}$$

when reinitialized after small times, converges to equation 4.2.8. More specifically, if we let Ω_0 be an open set such that $\partial\Omega_0 = \Gamma_0$, then the following scheme:

- $u_{\Delta t}(x,0) = \varphi_{\Omega_0} - \varphi_{\Omega_0^c}$
- Solve:
  $v_t = \mathrm{div}(w(x)Dv) + \frac{1}{2}Dw \cdot Dv$, in $\mathbb{R}^n \times (0, \Delta t]$
  $v(x,0) = u_{\Delta t}(x,(n-1)\Delta t)$
- $u_{\Delta t}(x,n\Delta t) = 1$, if $v(x,\Delta t) > 0$
  $u_{\Delta t}(x,n\Delta t) = -1$, if $v(x,\Delta t) \leq 0$

converges, as $\Delta t \to 0$, to motion by weigthed mean curvature. Note that this algorithm contains not only the standard *diffusion* term $\mathrm{div}(w(x)Du)$, but also the *convective* term $\frac{1}{2}Dw \cdot Du$.

A proof of this can be found in [24]. Our problem is a particular case of a more general situation considered in [24].

4.3 Numerical implementation

We now implement the BMO algorithm corresponding to motion by weigthed mean curvature. We solve the equation 4.2.9 in the domain P, which is the parallelogram that represents one period (see Figure 4.1), using Dirichlet boundary conditions 1 and 0 on the bottom and the top of P respectively. Our initial condition is the characteristic function of the set below a parallel plane to the restrictions Π_1 and Π_2 (set C in Figure 4.1). We solve the equation 4.2.9 for a small time and then reinitialize, in the same way as for equation 4.1.2. The algorithm can be summarized as follows:

1. Let $u_0 = \varphi_C$

2. Solve:

$$\begin{aligned} u_t(x,t) &= \operatorname{div}(w(x)Du(x,t)) + \frac{1}{2}Dw \cdot Du \\ u(x,0) &= u_0 \end{aligned}$$

3. Reinitialize at $t = T_{chop}$:

   if $u(x_i) \geq 0.5 \Rightarrow$ set $u_0(x_i) = 1$ else set $u_0(x_i) = 0$

4. Repeat step 2

   The location of the front is given by the level set $\{u = \frac{1}{2}\}$

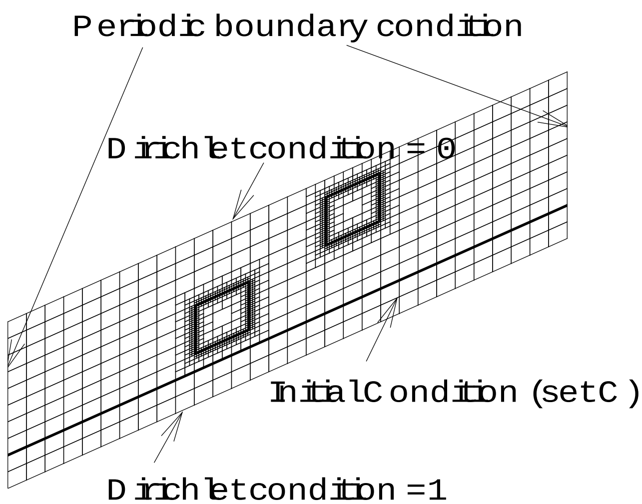

Figure 4.1: Diagram showing the domain with inclusions where we are solving the equation 4.2.9

The algorithm has been implemented in two and three dimensions. The spatial discretization was accomplished using the Fortran 90 codes 2Dhp90 and 3Dhp90, both from the Texas Institute for Computational and Applied Mathematics. These codes solve the the equation $D_j(a_{ij}u_i) = 0$ in two and three dimensions, respectively, with built-in capability for *local* adaptivity. Local adaptivity is very important for resolving the discontinuities along the interfaces of the inclusions, as well as for accurately representing inclusions with complex shapes. The time discretization was a simple backward Euler scheme for the time derivative, and carrying the convective term to the right hand side. The main modifications to the codes include:

1. Implementation of the time dependence.

2. Reinitialization after the chopping time T_{chop}.

3. Periodic boundary conditions and local refinements of the mesh around the inclusions.

4. Implementation of the convective term $\frac{1}{2}Du \cdot Dw$.

4.4 Results

We present results for dimensions 2 and 3 (see figures at the end). Since the algorithm can not differentiate by itself a local minimizer from a global minimizer, we need to consider the posibility of the algorithm converging to a local minimizer; that is, a minimizer that satisfies zero weighted mean curvature but that is not a global minimizer. However, this difficulty can be solved by stoping the chopping process and diffusing for a time until the surface continues its evolution. This is the well known tecnique of *annealing* used in numerical analysis to pertubate a local minimizer while looking for a global one.

The algorithm evolves the initial condition, which is always given by the plane $\omega \cdot x = b$ where ω is the slope of the parallel plane restrictions. As you can see from the pictures, the initial condition is not a global minimizer. There are four diferent parameters that affect the evolution of the surface: Δt denotes the time step, *mod* denotes the number of iterations between any two reinitializations, *iter* denotes the total number of iterations we perform and b tells how far from the lower constraint we want to place our initial condition. We have found experimentally that the value of *mod* is important in order to get a good result. In general, *mod* depends on the number of iterations and the time step but a value between 5 and 8 works well. If *mod* is too small, it is difficult for the surface to evolve and it could not even evolve at all (compare for example Figures 4.8 and 4.9). On the the other hand, if *mod* is too large, then we are practically only diffusing and the resulting surface is a very bad approximation of a surface that must satisfies zero mean curvature outside and inside the inclusions (compare Figures 4.3 and 4.2). Notice that in Figure 4.2, we get a much better approximation of a surface that has zero mean curvature outside the inclusions (for n=2 the minimal surface is composed by pieces of lines).

An interesting question is to determine the role that the convective term plays in the evolution. Our numerical experiments show that the convective term acts as an extra force that *pushes* up the surface near an inclusion. The examples in Figures 4.5, 4.7 and 4.10 were obtained by computing only the diffusion term. Compare these Figures with the Figures 4.4, 4.6 and 4.8 respectively. Figure 4.11 was obtained using the same values of the parameters as Figure 4.8 with the exception that in Figure 4.11 we split the time step by 2 and doble the number of iterations, and since both results are the same we can conclude that the algorithm converges in the time discretization. Also, an advantage of our algorithm is that stability is not an issue, since the reinitializations takes out any growing inestability.

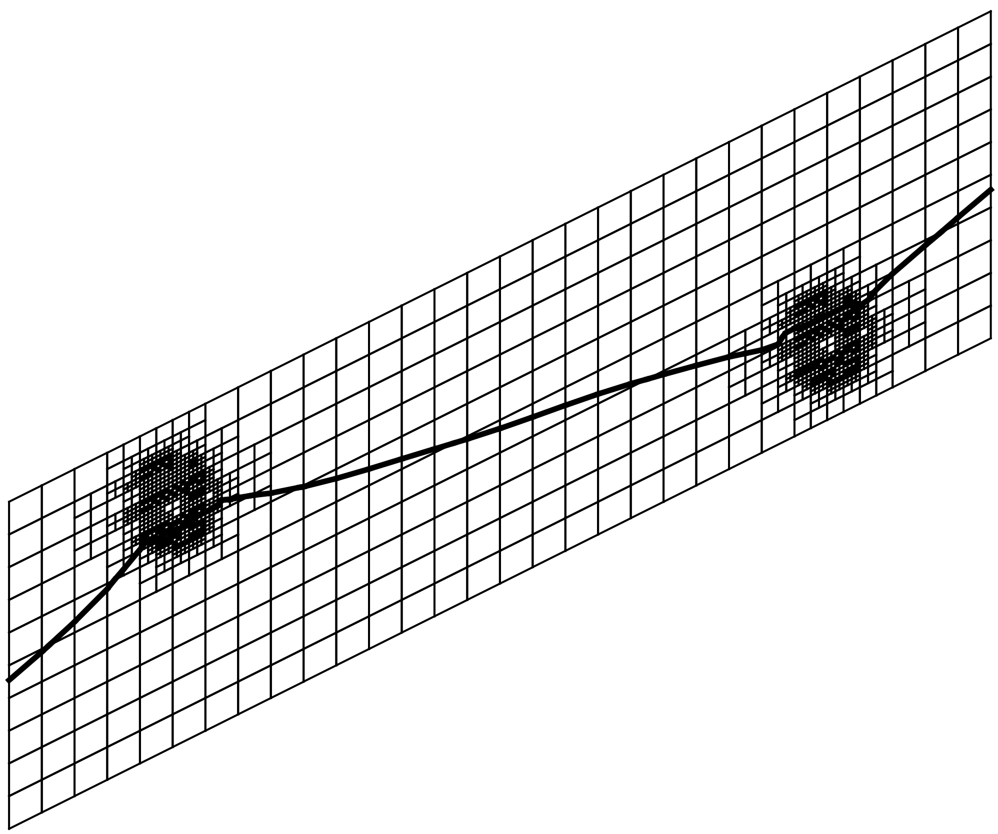

Figure 4.2: A 2-Dimensional h-adaptive finite element simulation of motion by weighted mean curvature. The initial surface was a straight line parallel and close to the bottom of the domain. The grid in the figure is the mesh used to solve the PDE. The parameters are: mod=6, $\Delta t = 0.01$, iter=100, b=0.1

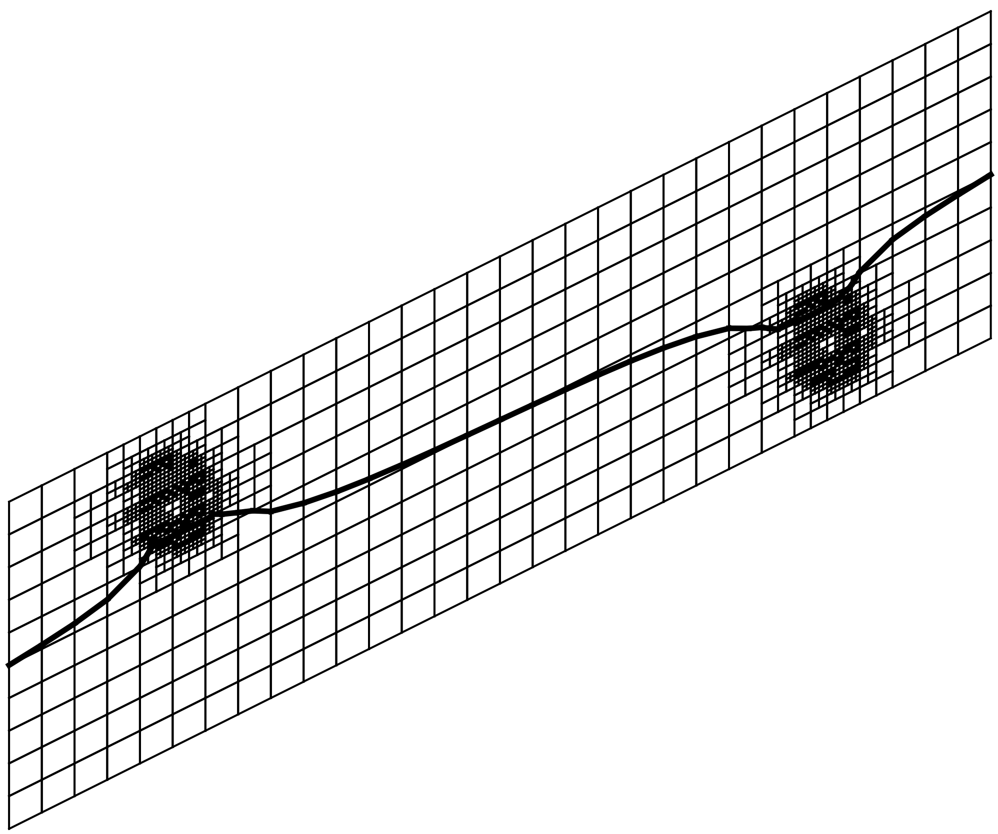

Figure 4.3: A 2-Dimensional h-adaptive finite element simulation of motion by weighted mean curvature. The initial surface was a straight line parallel and close to the bottom of the domain. The grid in the figure is the mesh used to solve the PDE. The parameters are $mod = 12$, $\Delta t = 0.01$, iter=100, b=0.1

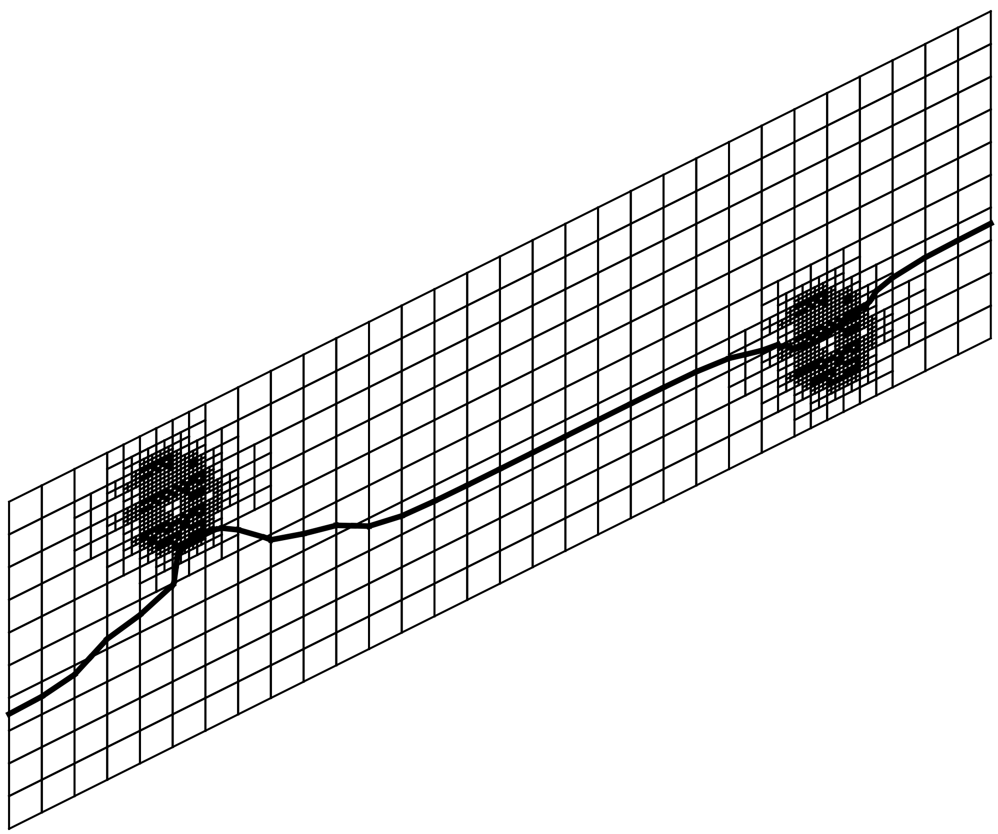

Figure 4.4: A 2-Dimensional h-adaptive finite element simulation of motion by weighted mean curvature. The initial surface was a straight line parallel and close to the bottom of the domain. The grid in the figure is the mesh used to solve the PDE. Compare this result with the next one, which only contains the diffusion term. Compare also with Figure 4.2, in which we performed 100 iterations instead of 25. The parameters are: mod=6, Δt = 0.01, iter=25, b=0.1

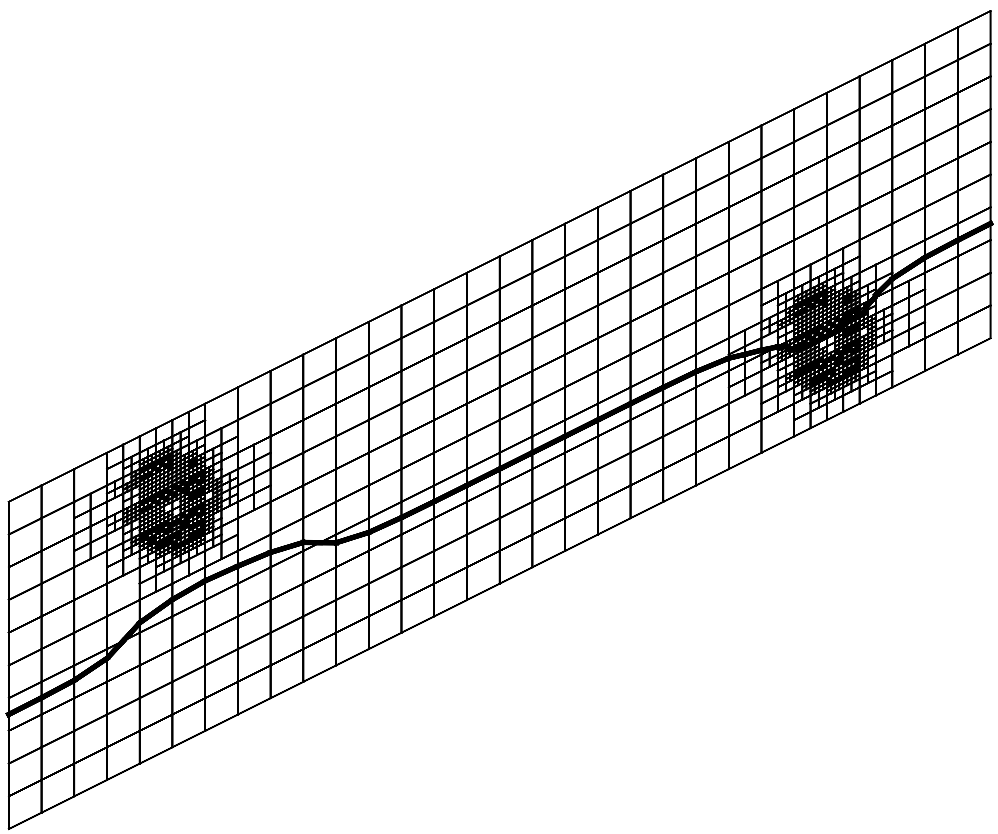

Figure 4.5: A 2-Dimensional h-adaptive finite element simulation of motion by weighted mean curvature. The initial surface was a straight line parallel and close to the bottom of the domain. The grid in the figure is the mesh used to solve the PDE. This example contains only the diffusion term. The parameters are: mod=6, $\Delta t = 0.01$, iter=25, b=0.1

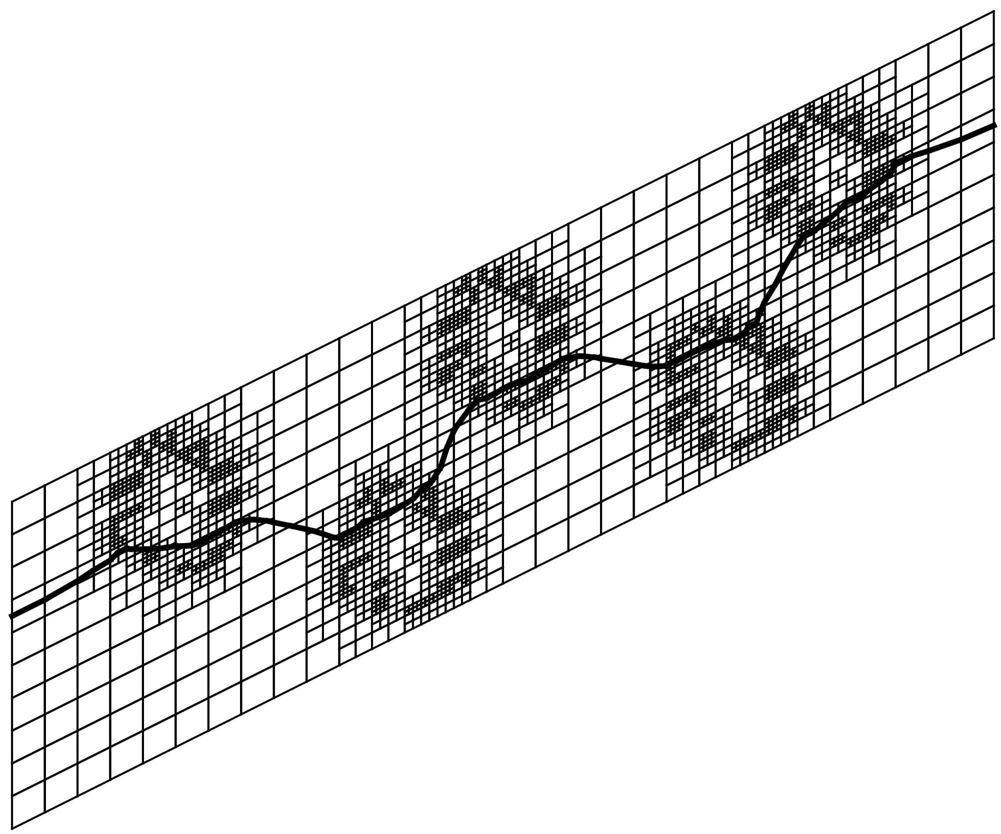

Figure 4.6: A 2-Dimensional h-adaptive finite element simulation of motion by weighted mean curvature. The initial surface was a straight line parallel and close to the bottom of the domain. The grid in the figure is the mesh used to solve the PDE. Note the refinement of the mesh around the inclusions. The parameters are: mod=8, $\Delta t = 0.01$, iter=100, b=0.05

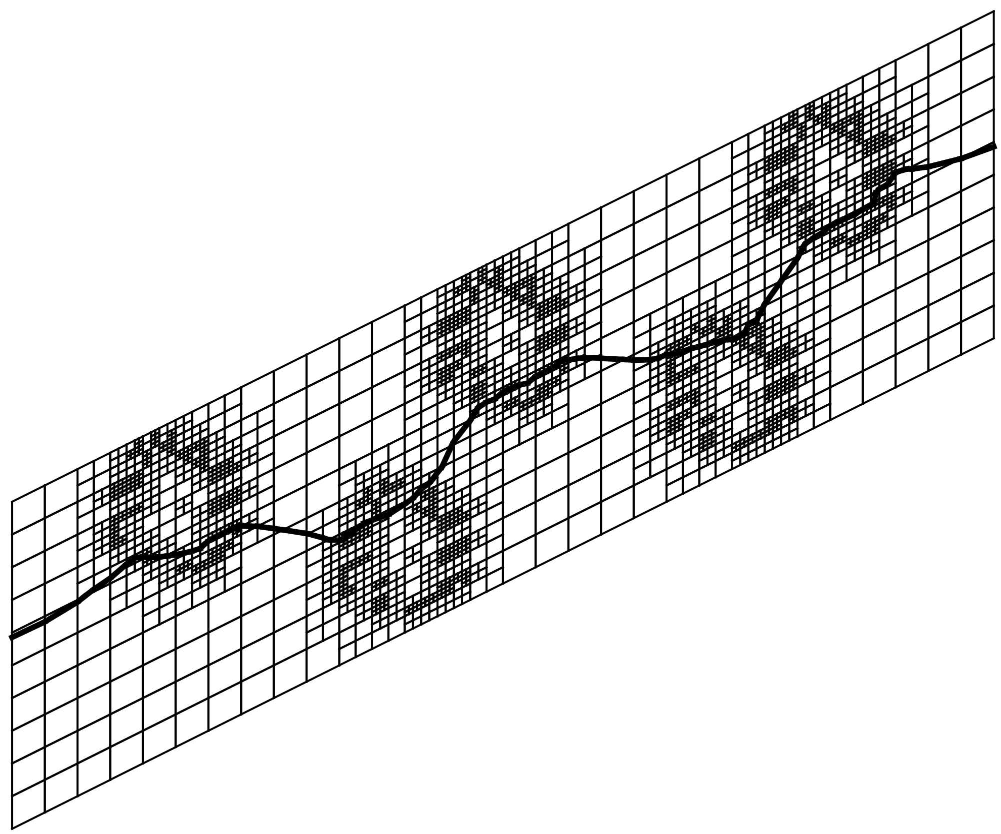

Figure 4.7: A 2-Dimensional h-adaptive finite element simulation of motion by weighted mean curvature. The initial surface was a straight line parallel and close to the bottom of the domain. This simulation contains only the diffusion term. The grid in the figure is the mesh used to solve the PDE. Note the refinement of the mesh around the inclusions. The parameters are: mod=8, $\Delta t = 0.01$, iter=100, b=0.05

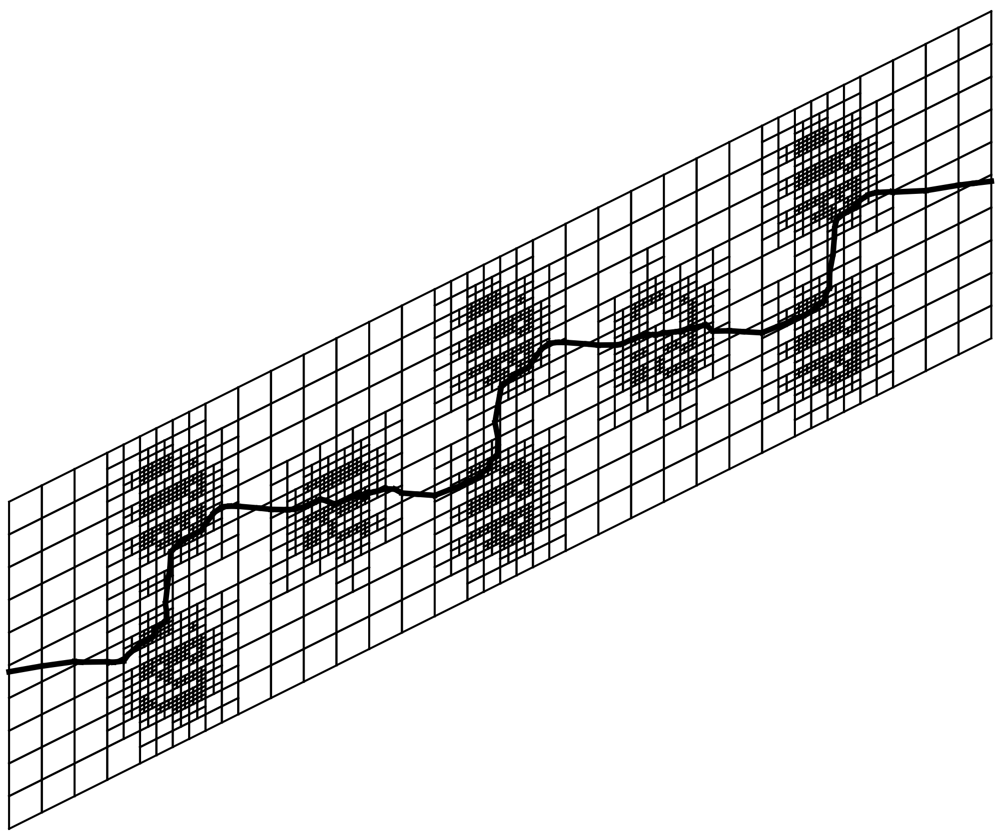

Figure 4.8: A 2-Dimensional h-adaptive finite element simulation of motion by weighted mean curvature. The initial surface was a straight line parallel and close to the bottom of the domain. The grid in the figure is the mesh used to solve the PDE. Note the refinement of the mesh around the inclusions. The parameters are: mod=8, $\Delta t = 0.01$, iter=250, b=0.1

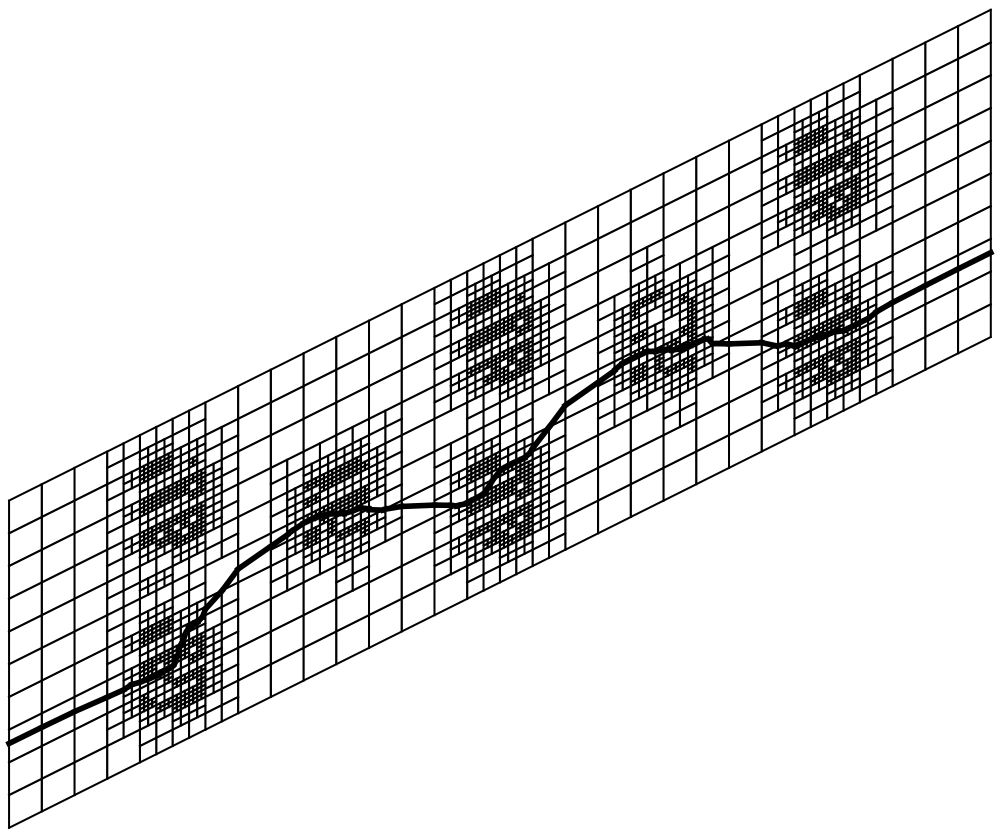

Figure 4.9: A 2-Dimensional h-adaptive finite element simulation of motion by weighted mean curvature. The initial surface was a straight line parallel and close to the bottom of the domain. The grid in the figure is the mesh used to solve the PDE. Note the refinement of the mesh around the inclusions. The parameters are: mod=4, $\Delta t = 0.01$, iter=250, b=0.1

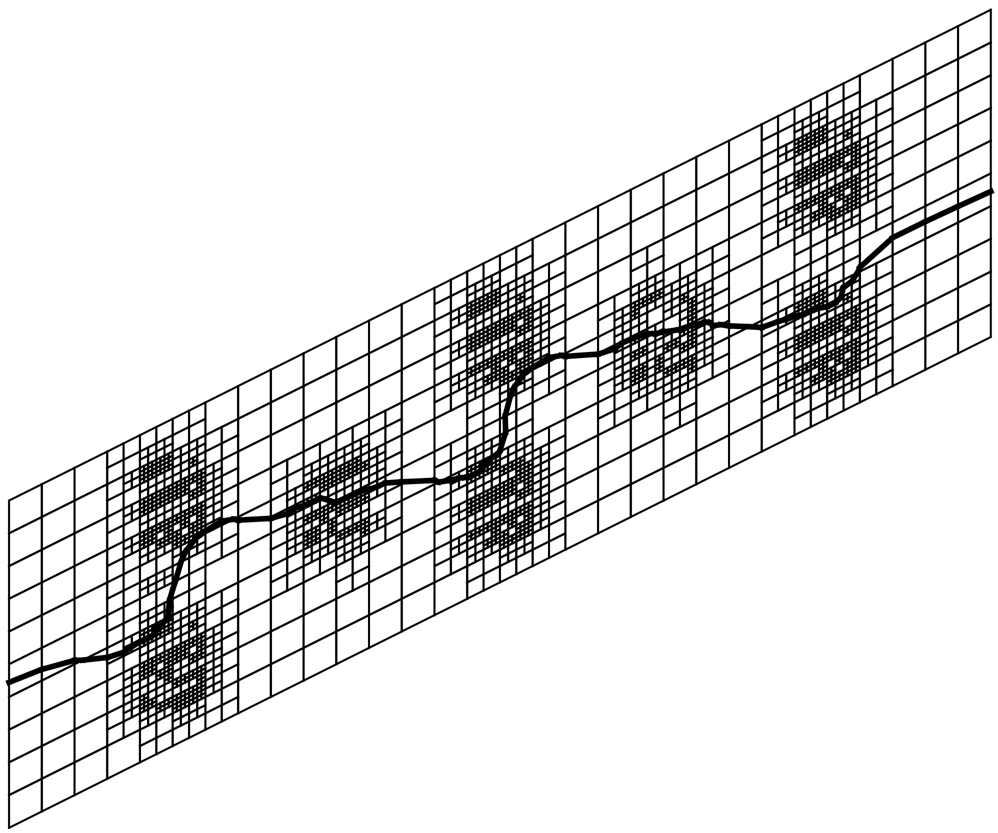

Figure 4.10: A 2-Dimensional h-adaptive finite element simulation of motion by weighted mean curvature. The initial surface was a straight line parallel and close to the bottom of the domain. This simulation contains only the diffusion term. The grid in the figure is the mesh used to solve the PDE. Note the refinement of the mesh around the inclusions. The parameters are: mod=8, $\Delta t = 0.01$, iter=250, b=0.1

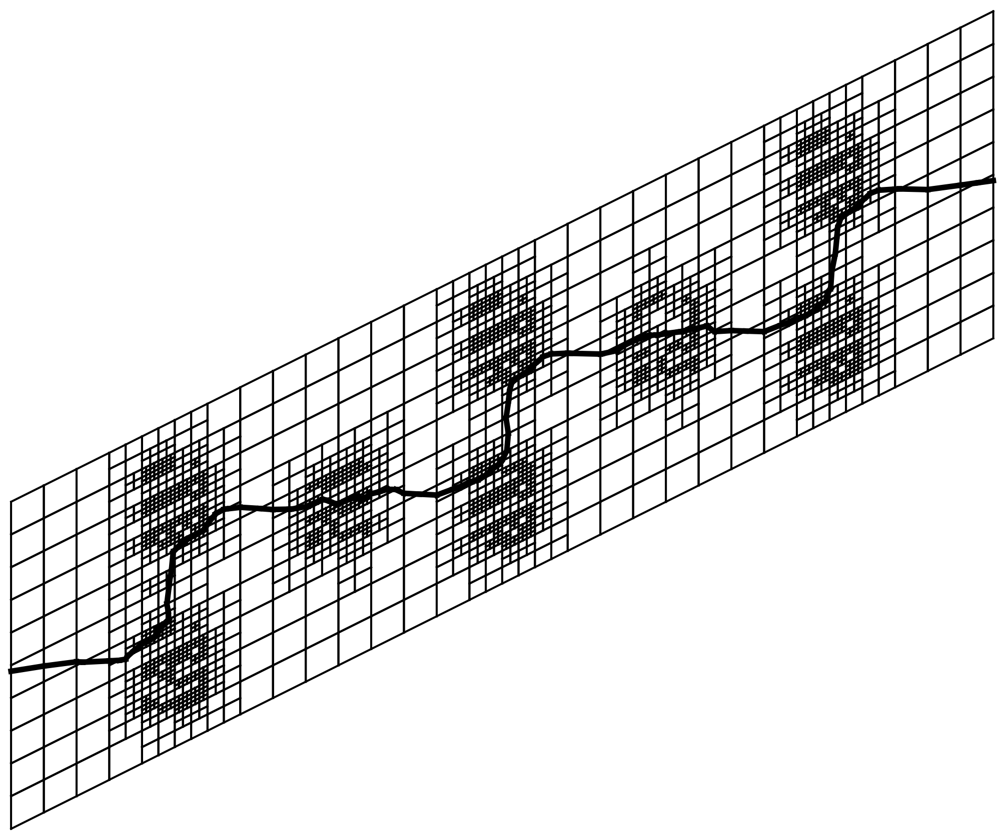

Figure 4.11: A 2-Dimensional h-adaptive finite element simulation of motion by weighted mean curvature. The initial surface was a straight line parallel and close to the bottom of the domain. The grid in the figure is the mesh used to solve the PDE. Note the refinement of the mesh around the inclusions. The parameters are: mod=16, $\Delta t = 0.005$, iter=500, b=0.1

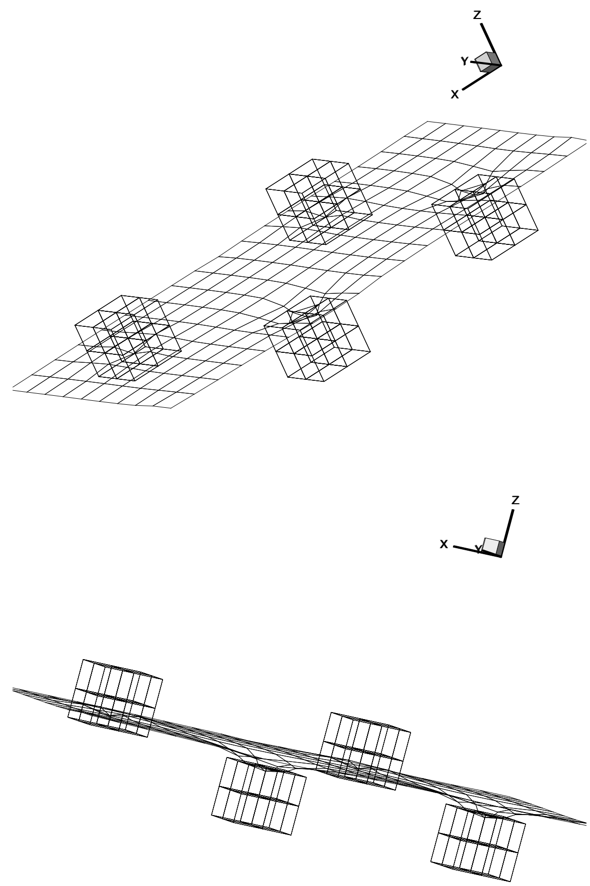

Figure 4.12: A 3-Dimensional h-adaptive finite element simulation of motion by weighted mean curvature. The initial surface was a plane parallel and close to the bottom of the domain. We are showing two different views of the same evolution. For clarity we only show the surface and the elements of the mesh that form the inclusions. The parameters are: mod=6, $\Delta t = 0.01$, iter=350 (we start sharpening after the first 100 iterations), b=0.02

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Vita

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