The Dissertation Committee for Katherine Sue Socha
Certifies that this is the approved version of the following dissertation:

MODAL ANALYSIS OF LONG WAVE EQUATIONS

Committee:

Misha Vishik, Supervisor

Jerry L. Bona, Co-Supervisor

Irene Gamba

Ronald B. Guenther

Philip Morrison

Ralph Showalter

MODAL ANALYSIS OF LONG WAVE EQUATIONS

by

KATHERINE SUE SOCHA, B.A., M.S.

DISSERTATION

Presented to the Faculty of the Graduate School of

The University of Texas at Austin

in Partial Fulfillment

of the Requirements

for the Degree of

DOCTOR OF PHILOSOPHY

THE UNIVERSITY OF TEXAS AT AUSTIN

May 2002

MODAL ANALYSIS OF LONG WAVE EQUATIONS

Publication No. _____

Katherine Sue Socha, Ph.D.
The University of Texas at Austin, 2002

Supervisors: Misha Vishik
Jerry L. Bona

This work studies the use of modal expansion approximations of solutions of model long wave equations. Such model equations are of interest to oceanographers and engineers because they describe the propagation of surface water waves, used in near-shore models of sandbar formation.

General theoretical results are derived for standard long wave models in the form of dispersive, nonlinear partial differential equations. Particular numerical results are computed for such model equations, including the Korteweg-de Vries equation, the Benjamin-Bona-Mahony equation, and the Benjamin-Ono equation, among others.

Table of Contents

List of Figures

Chapter 1

Introduction

1.1 Motivation

Figure 1.1: A beach on the Australian Gold Coast, before erosion

The first photograph (Figure 1.1) shows a beach on the Australian Gold Coast on the date February 1, 1988. Nearly one year later in January, 1989, the beach had eroded beyond viable use (Figure 1.2). After the failure of restoration through direct nourishment of the beach, the local community called in oceanographer B. Boczar-Karakiewicz and ocean engineer L. Jackson to help restore the beach and prevent further erosion.

Through their efforts [11], an artificial sandbar was created at some distance offshore. (See Figure 1.3.) This sandbar altered the near-shore dynamics

Figure 1.2: The same beach, one year later, after total erosion

Figure 1.3: Ship discharging tons of sand to create artificial sandbar

sufficiently to protect the newly nourished beach from further substantial erosion.

Ocean engineering problems of this nature contributed to modern interest in modeling and understanding the dynamics of near-shore processes, specifically the interaction of surface waves with the sandy beach bottom. In 1981, Boczar-Karakiewicz and J. Bona developed such a model of wave-bottom interaction, and continued refining it through the latter part of the decade.

(See, for example, [9], [8], [10].)

As this model provides the motivation for the analytical and numerical studies in the present work, we will describe it in some detail.

1.2 A Model of Sandbar Formation

A sandbar is a large underwater structure found in near-shore zones. Shaped like gentle hills, they are often hundreds of meters long, several meters high, and formed of loose granular sediment. [29]

The model of sandbar formation developed by Boczar-Karakiewicz et al. [10] describes the interaction between incoming surface water waves and the evolving sandy bottom. The model may be described verbally as follows.

1. A regular wavetrain, incoming from deep water, and fixed bottom topography provide boundary conditions for

2. a nonlinear partial differential equation or system that describes the one-way propagation of (dispersive) waves. This surface motion determines the velocity field of the fluid in the interior of the region between the surface and a thin boundary layer at the bottom. The internal velocity field provides a boundary condition for

3. a differential equation that characterizes the motion of the sediment-laden, viscous boundary layer at the bottom. Longuet-Higgins' boundary layer theory determines the velocity profile in this thin region. The velocity profile is averaged over the depth of the layer and over the fundamental period of the incoming surface waves to determine

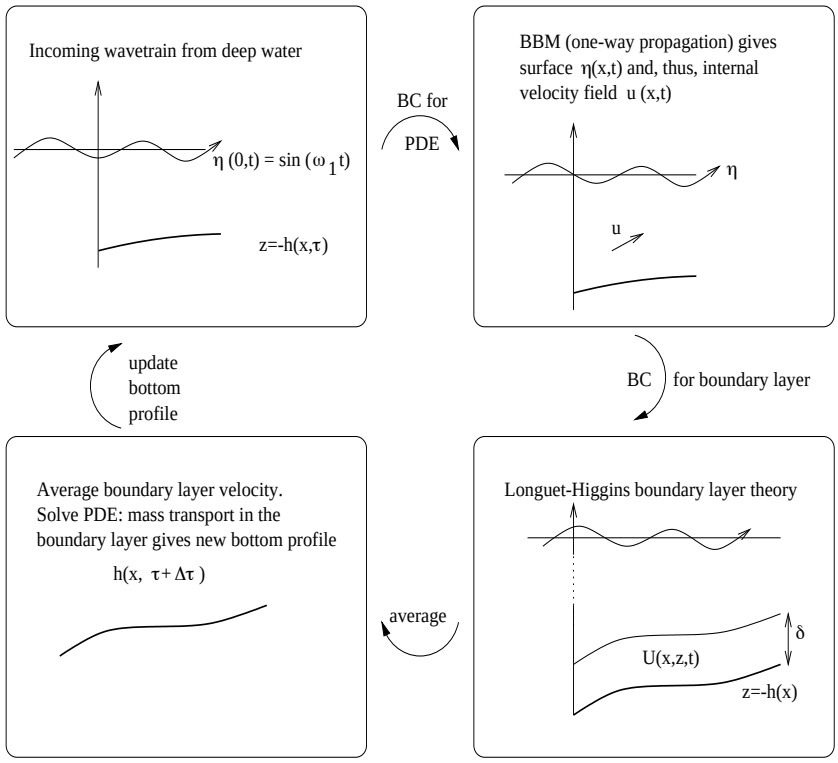

Figure 1.4: Flowchart describing sandbar model

4. the mass transport within the boundary layer. The bottom topography evolves according to the mass transport equation, which simply expresses conservation of mass. This new configuration, coupled with the regular incoming wavetrain, provides the starting point (1. above) for another iteration of the model.

Figure (1.4) shows a flow chart of one cycle in this iterative process.

1.2.1 Physical context

The definition sketch shown in Figure (1.5) describes the physical context of the problem.

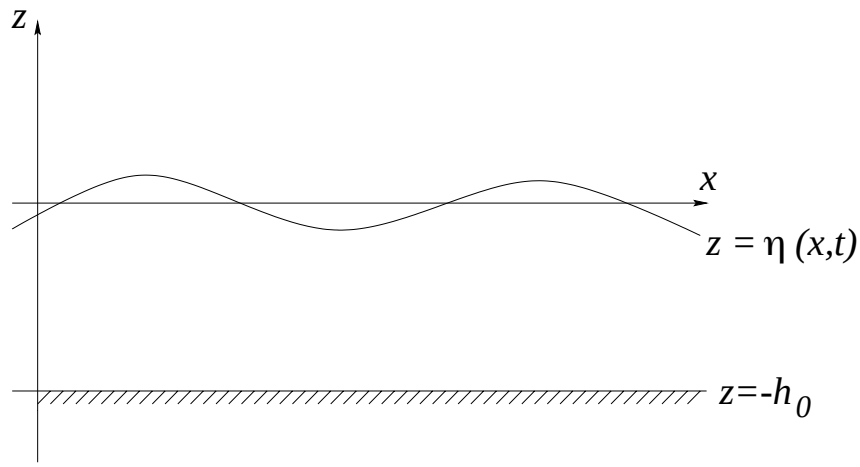

Figure 1.5: Definition sketch

The spatial variables are (x, z) with the still water level at $z = 0$, the flat bottom at $z = -h_0$, and the free surface at $z = \eta(x, t)$ where t is the time variable. A typical assumption is that the problem is independent of one horizontal (the longshore) direction so that we may neglect this third dimension. Further assume that the fluid is inviscid and incompressible, moving irrotationally. The model characterizes the interplay between the evolving sandy bottom and the evolving velocity field of the fluid and its free surface.

1.3 Modeling Considerations

1.3.1 The governing partial differential equation

Boczar-Karakiewicz et al. began their work using the Boussinesq Approximation to the Euler system (BAE), a system of the form

$$u_t + \zeta_x + \alpha u u_x - \frac{1}{3}\beta h^2 u_{xxt} = 0 \tag{1.1}$$

$$\zeta_t + [u(\alpha\zeta + h)]_x = 0 \tag{1.2}$$

where the nondimensional quantities are $u = u(x,t)$ for the velocity field, $\zeta = \zeta(x,t)$ for the parametrization of the free surface, $h = h(x)$ for the bottom topography (which is parametrized by $z = -h(x)$), and parameters α (ratio of characteristic wave amplitude to characteristic water depth) and β (squared ratio of characteristic water depth to characteristic wave length).

Boczar-Karakiewicz et al. later adapted their model, using the BBM equation as the partial differential equation governing propagation of the surface waves. The remainder of this section is devoted to BBM, discussing its derivation in broad terms following the work of Benjamin, Bona, and Mahony [7], as described in Bona (lecture notes—in preparation, [13]). For details, refer to Appendix A.

The physical framework of the discussion begins with the study of a fluid (say, water) in an idealized format that is inviscid and incompressible. The fluid is of finite depth and is bounded below by an impermeable surface. The only external force acting on the fluid is gravity, and the fluid is presumed to flow in an irrotational manner.

This well-studied situation is governed by the Euler equations, first described by Euler in 1755 (see, for example, [13]) with boundary conditions imposed at the impermeable bottom and at the air-water interface (the surface). The Euler equations describe the conservation of mass (combined with the incompressibility assumption) and the conservation of momentum. From vector calculus identities, it is possible to show the existence of a velocity potential ϕ such that the velocity field of the fluid flow is given by $u = \nabla\phi$. Then the conservation of mass equation reduces to the Laplace equation for

the velocity potential, $\Delta\phi = 0$. That is, the solution of the Laplace equation gives the velocity field directly and subsequently, through the conservation of momentum equation, determines the hitherto unknown pressure.

The next step is to relate the above description of the fluid flow to the description of the motion of the surface (which will result in the BBM equation): there are two boundary conditions at the surface linking ϕ (hence u) and the surface. Assume that the surface is described by a function of the spatial variable x and the time variable t; that is, eliminate the case of overturning waves from our consideration, describing the surface by a function $\eta(x,t)$. The two boundary conditions are

1. "kinematic," meaning the velocity of the fluid at the surface is equated to the velocity of the surface, and

2. "dynamic," meaning (after neglecting surface tension's small effect) the pressure is continuous across the surface; that is, the air pressure at the surface is equated to the water pressure at the surface, and thus Bernoulli's equation may be used.

Use of the kinematic and dynamic boundary conditions leads to (one version of) the Boussinesq system of equations, linking the free surface η and the horizontal velocity w of the fluid at the bottom. This derivation depends on parameters α and β introduced in a nondimensionalization process: assuming typical wavelengths l, wave amplitudes a, and constant fluid depth h_0, we introduce $\alpha = a/h_0$ and $\beta = h_0^2/l^2$. Requiring that α and β be small indicates that we are studying the physical context of *small amplitude, large wavelength* waves.

The Boussinesq system may further be reduced by assuming the waves propagate in only one direction. Such an assumption forces a relationship between η and w which at lowest order is simply $\eta = w$. At the order α and β, this is corrected by taking $w = \eta + \alpha A + \beta B$, where $A = A(\eta, \eta_x, \eta_t, \ldots)$ and $B = B(\eta, \eta_x, \eta_t, \ldots)$, in the full Boussinesq system. Forcing consistency of the equations places restrictions on A and B, so that only 8 possible model equations can be developed. Of these, two in particular are selected (on mathematical grounds) for study. One is the famous KdV equation,

$$\eta_t + \eta_x + \frac{3}{2}\alpha\eta\eta_x + \frac{\beta}{6}\eta_{xxx} = 0 \tag{1.3}$$

first derived by Boussinesq in 1871 (see [13]) and later by Korteweg and de Vries in 1895 [21]. The second equation is the well-known BBM equation (originally called the Regularized Long Wave equation by Benjamin, Bona, and Mahony in 1972 [7]),

$$\eta_t + \eta_x + \frac{3}{2}\alpha\eta\eta_x - \frac{\beta}{6}\eta_{xxt} = 0 \tag{1.4}$$

chosen by Boczar-Karakiewicz and Bona for their development of a model of sandbar-oceanwave interactions. For full details of this derivation, refer to Appendix A.

1.3.2 Information obtained from BBM

The BBM equation gives information about both the evolution of the surface waves and the evolution of the internal velocity field (through the velocity potential ϕ). According to inviscid theory, this internal velocity field is assumed to drive the upper boundary of the laminar, sediment-laden, viscous bottom

boundary layer. Due to the associated drift velocity (known as Stokes drift), the motion of the boundary layer imparts a mean transport of sediment mass.

1.3.3 Boundary Layer Theory and Mass Transport

In a fundamental 1953 paper, M. Longuet-Higgins [23] derived a mathematical theory describing the behavior of fluid near a boundary. In particular, he described a general method for finding the mass transport velocity in arbitrary small amplitude oscillations of a perfect fluid (given knowledge of the first order motion). This mass transport is determined by only the first order motion and the local boundary conditions.

For their model of wave-topography interaction, Boczar-Karakiewicz et al. used the theory of Longuet-Higgins to calculate the mass transport velocity in the boundary layer of the bed. Subsequently, they calculated the sediment flux (as a function of the spatial variable). This quantity is necessary to described the conservation of mass within the boundary layer. Conservation of mass is the physical principle underlying the erosion equation, which then determines the time evolution of the bottom topography.

1.4 Goal of Present Work

The Boczar-Karakiewicz-Bona model appears to give results that agree fairly well with actual physical data, for example, describing the location and number of sandbars at the U. S. Army Corps of Engineers Field Research Facility at Duck, North Carolina. (See [12] and [30].) However, implementation of the model is computationally quite intensive [14]. Hence, a reasonable goal is to

reduce the computational burden by introducing an analytical approximation to the governing partial differential equation. The point of such an approximation procedure is to eliminate the work required to solve numerically a partial differential equation, while retaining a minimal error margin.

This work aims to address the latter issue, applying the idea of the modal expansion approximation to the specific nonlinear, dispersive equations used by Boczar-Karakiewicz et al. in their model. The results obtained in these contexts are easily extended to more general nonlinear, dispersive wave equations. Both analytical and numerical results will be exhibited, providing validation of such an approach.

Chapter 2

The Flat Bottom BBM Equation

2.1 The Modal Expansion

The modal expansion is a well-established approach to the solution of nonlinear partial differential equations, combining the ideas of Fourier series and of separation of variables. In a 1962 paper studying light waves in a nonlinear dielectric, Armstrong, Bloembergen, Ducuing, and Pershan [5] used this approach (that they called quantum-mechanical perturbation theory) to develop coupled amplitude equations, describing for instance the interaction between a plane light wave and its second harmonic. In 1972, Mei and Ünlüata [25] continued studying the phenomenon of second harmonic generation, but in the context of the shallow water theory instead of nonlinear optics. Their overview points out the applicability of these ideas to other contexts including deep water waves (as studied by Phillips and others). Refer to [22] for further details.

Also in 1972, Lau and Barcilon published a fundamental paper [22] in which they analyze the impact of bed topography on incoming shallow water waves, thus characterizing the behavior of wave energy in near shore processes.

The application of the modal expansion to the model of wave-sandbar

interaction described in Chapter 1 was developed by Boczar-Karakiewicz et al. [10] in 1987. The details of this work are described in the next section.

2.2 Derivation of Lau–Barcilon Equations for the Boussinesq Approximation to the Euler Equations

In this section, the work of Boczar-Karakiewicz, Bona, and Cohen [10] is summarized, in order to provide a foundation for further work with these ideas. Following Boczar-Karakiewicz et al., begin with the nondimensional Boussinesq approximation to the Euler equations in one spatial dimension,

$$u_t + \zeta_x + \alpha u u_x - \frac{1}{3}\beta h^2 u_{xxt} = 0 \tag{2.1}$$

$$\zeta_t + [u(\alpha\zeta + h)]_x = 0 \tag{2.2}$$

where the nondimensional quantities are $u = u(x,t)$ for the velocity field, $\zeta = \zeta(x,t)$ for the parametrization of the free surface, $h = h(x)$ for the bottom topography (which is parametrized by $z = -h(x)$), and parameters α (ratio of characteristic wave amplitude to characteristic water depth) and β (ratio of characteristic water depth to characteristic wave length).

Introduce a new horizontal variable, which will represent a long spatial scale, $X = \alpha x$. Introduce a two-scale expansion, in which x and X will be treated as independent; that is, the above differentiations with respect to x will be replaced via the chain rule by

$$\frac{d}{dx} = \frac{\partial}{\partial x} + \alpha\frac{\partial}{\partial x}.$$

The use of such a change of variables is well-known as a reasonable approach to studying problems characterized by physical effects occurring at different physical scales.

Suppose that the bottom is varying gradually, and write (taking account of scaling) $h(X) = 1 + \alpha G(X)$. Apply the chain rule and this assumption to the above system, keeping only terms which are $O(\alpha)$. This process gives the two equations

$$\begin{aligned} u_t + \zeta_x - \frac{\beta}{3} u_{xxt} = & - \alpha\zeta_x - \alpha u u_x + \frac{2\alpha\beta}{3} G u_{xxt} \\ & + \frac{2\alpha\beta}{3} u_{xXt} + O(\alpha^2) \qquad (2.3) \\ \zeta_t + u_x = & - \alpha u_X - \alpha(u\zeta)_x - \alpha G u_x + O(\alpha^2) \qquad (2.4) \end{aligned}$$

under the assumption that the incoming wavetrain is periodic with frequency ω_1. If the system were linear, the next step would be to assume the form of the solution was a linear combination of plane waves (in complex form, $e^{i(k_j x - \omega_j t)}$). Since the system is nonlinear, modify this approach by assuming that the amplitudes for these plane waves are slowly-varying, meaning that the amplitudes are functions of X only. Thus, suppose

$$\begin{aligned} \zeta(x, X, t) &= \sum_j \zeta_j(X) e^{i(k_j x - \omega_j t)} + \zeta_j^* e^{-i(k_j x - \omega_j t)} \qquad (2.5) \\ u(x, X, t) &= \sum_j u_j(X) e^{i(k_j x - \omega_j t)} + u_j^* e^{-i(k_j x - \omega_j t)} \qquad (2.6) \end{aligned}$$

where the asterisk (*) indicates complex conjugate. Notice that the assumption that ζ and u are of order one implies that the amplitudes ζ_j and u_j are also of order one.

Introduce some helpful notation: let $E_j = e^{i(k_j x - \omega_j t)}$ so that $E_j^* = e^{-i(k_j x - \omega_j t)}$. Next, derive an order-$\alpha$ relationship between ζ_j and u_j. Using the second of the derived equations, (2.4) above, compute the derivatives and

keep the right side as $O(\alpha)$:

$$\sum_j -i\omega_j\zeta_j E_j + i\omega_j\zeta_j^* E_j^* + ik_j u_j E_j - ik_j u_j^* E_j^* = O(\alpha)$$

which, after grouping, gives

$$\sum_j (-\omega_j\zeta_j + k_j u_j)E_j + (\omega_j\zeta_j^* - k_j u_j^*)E_j^* = O(\alpha).$$

Thus, compute

$$-\omega_j\zeta_j + k_j u_j = O(\alpha) \tag{2.7}$$

$$k_j u_j = \omega_j\zeta_j + O(\alpha) \tag{2.8}$$

which shows that

$$u_j(X) = \frac{\omega_j}{k_j}\zeta_j(X) + O(\alpha).$$

The next step is to eliminate ζ from the left side of the above system (2.3, 2.4) of PDEs by computing $\frac{\partial}{\partial t}$ of the first and $-\frac{\partial}{\partial x}$ of the second to obtain

$$\begin{aligned} u_{tt} - u_{xx} - \frac{\beta}{3}u_{xxtt} = \quad & \alpha \quad \Big\{-\zeta_{Xt} + u_{Xx} + Gu_{xx} \\ & + \quad \frac{2\beta}{3}[Gu_{xxtt} + u_{xXtt}] - \frac{1}{2}(u^2)_{xt} + (\zeta u)_{xx}\Big\} + O(\alpha^2) \end{aligned} \tag{2.9}$$

Working with this equation, substitute in the assumed forms for u and ζ, neglect $O(\alpha^2)$ terms, and ignore all but the first three harmonics. A helpful notation here is to introduce the detuning parameter, $\Delta_k = k_2 - 2k_1$. With this notation, for example, $E_1^2 = e^{-i\Delta_k x}E_2$. Thus, it is easy to determine which harmonics interact to yield effects in the first three harmonics. After somewhat long computations (using the chain rule over and over), one determines that

the left-hand side of the partial differential equation has the form

$$
\begin{aligned}
&E_1(-\omega_1^2 u_1 + k_1^2 u_1 - \frac{\beta}{3} k_1^2 \omega_1^2 u_1) \\
&+E_2(-\omega_2^2 u_2 + k_2^2 u_2 - \frac{\beta}{3} k_2^2 \omega_2^2 u_2) \\
&+E_3(-\omega_3^2 u_3 + k_3^2 u_3 - \frac{\beta}{3} k_3^2 \omega_3^2 u_3) \\
&+\text{conj}
\end{aligned}
$$

where 'conj' represents the complex conjugate of all the previous terms. This is reassuring, as it gives the usual (first-order) dispersion relation

$$-\omega_j^2 + k_j^2 - \frac{\beta}{3} k_j^2 \omega_j^2$$

that is,

$$\omega_j^2 = \frac{k_j^2}{1 + \frac{\beta}{3} k_j^2} \qquad \text{and} \qquad k_j^2 = \frac{\omega_j^2}{1 - \frac{\beta}{3} \omega_j^2}$$

which is plotted in Figure (2.1).

The order α coefficient of the right-hand side of the partial differential equation has the form

$$
\begin{aligned}
&E_1\Bigg(i\omega_1\zeta_1' + ik_1u_1' - Gk_1^2u_1 + \frac{2\beta}{3}[Gk_1^2\omega_1^2u_1 - ik_1\omega_1^2u_1'] \\
&-[(k_2-k_1)\omega_1u_1^*u_2e^{i\Delta_k x} + (k_3-k_2)\omega_1u_2^*u_3e^{-i(k_1+k_2-k_3)x}] \\
&-[(k_2-k_1)^2(u_1^*\zeta_2 + u_2\zeta_1^*)e^{i\Delta_k x} + (k_2-k_3)^2(u_2^*\zeta_3 + u_3\zeta_2^*)e^{-i(k_1+k_2-k_3)x}]\Bigg)\alpha \\
&+E_2\Bigg(i\omega_2\zeta_2' + ik_2u_2' - Gk_2^2u_2 + \frac{2\beta}{3}[Gk_2^2\omega_2^2u_2 - ik_2\omega_2^2u_2'] \\
&-[k_1\omega_2u_1^2e^{-i\Delta_k x} + (k_3-k_1)\omega_2u_1^*u_3e^{-i(k_1+k_2-k_3)x}] \\
&-[4k_1^2u_1\zeta_1e^{-i\Delta_k x} + (k_1-k_3)^2(u_1^*\zeta_3 + u_3\zeta_1^*)e^{-i(k_1+k_2-k_3)x}]\Bigg)\alpha \\
&+E_3\Bigg(i\omega_3\zeta_3' + ik_3u_3' - Gk_3^2u_3 + \frac{2\beta}{3}[Gk_3^2\omega_3^2u_3 - ik_3\omega_3^2u_3']
\end{aligned}
$$

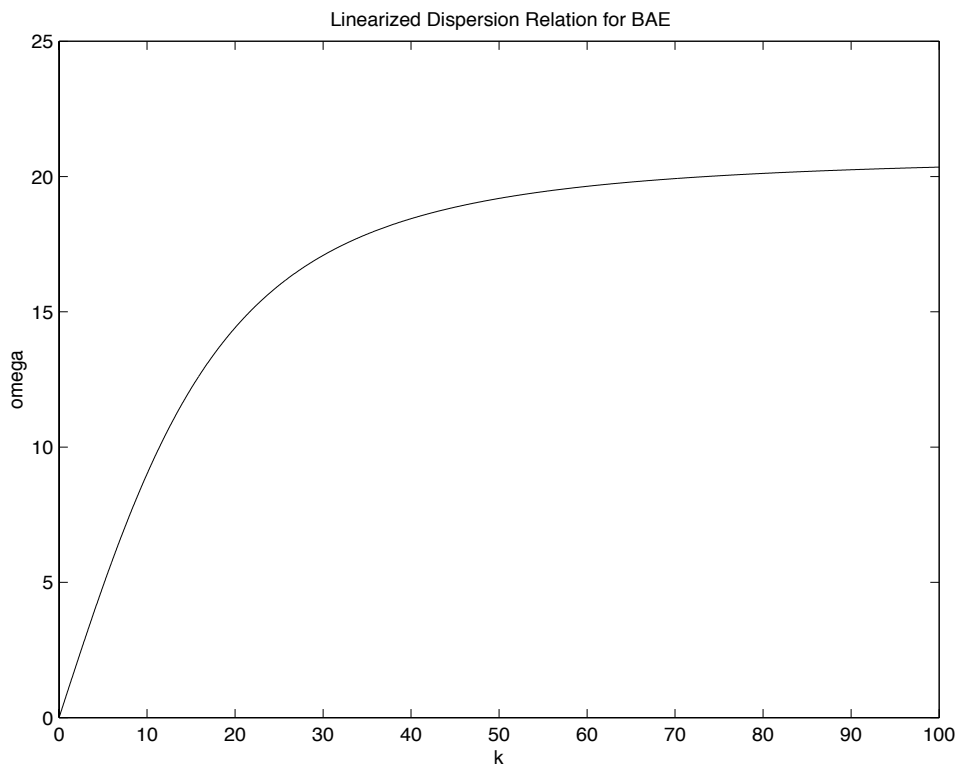

Figure 2.1: Dispersion Relation, $\omega(k)$

$$+(k_1+k_2)\omega_3 u_1 u_2 e^{i(k_1+k_2-k_3)x} - (k_1+k_2)^2(u_1\zeta_2+u_2\zeta_1)e^{i(k_1+k_2-k_3)x}\Big)\alpha$$
$$+\text{conj} + O(\alpha^2)$$

Notice that neglecting the third mode (by setting $u_3 = \zeta_3 = 0$) reduces this form to the corresponding result for the case of two modes, namely

$$\begin{aligned}
&E_1\Big(i\omega_1\zeta_1' + ik_1u_1' - Gk_1^2u_1 + \frac{2\beta}{3}(Gk_1^2\omega_1^2u_1 - ik_1\omega_1^2u_1') \\
&-(k_2-k_1)\omega_1 u_1^* u_2 e^{i\Delta_k x} - (k_2-k_1)^2(u_2\zeta_1^* + u_1^*\zeta_2)e^{i\Delta_k x}\Big)\alpha \\
&+E_2\Big(i\omega_2\zeta_2' + ik_2u_2' - Gk_2^2u_2 + \frac{2\beta}{3}(Gk_2^2\omega_2^2u_2 - ik_2\omega_2^2u_2') \\
&-k_1\omega_2u_1^2e^{-i\Delta_k x} - 4k_1^2u_1\zeta_1e^{-i\Delta_k x}\Big)\alpha + \text{ conj } + O(\alpha^2)
\end{aligned}$$

The Lau-Barcilon equations are simply the ordinary differential equations which result from setting each coefficient of E_j equal to zero. Simplify these equations by putting

$$u_j = \frac{\omega_j}{k_j}\zeta_j + O(\alpha)$$

and by carrying out some straightforward algebraic manipulation. Here are the equations which result:

$$\zeta_1' + iF_1\zeta_1 + iQ_1 e^{i\Delta_k x}\zeta_1^*\zeta_2 + iR_1 e^{-i(k_1+k_2-k_3)x}\zeta_2^*\zeta_3 = 0 \tag{2.10}$$

$$\zeta_2' + iF_2\zeta_2 + iQ_2 e^{-i\Delta_k x}\zeta_1^2 + iR_2 e^{-i(k_1+k_2-k_3)x}\zeta_1^*\zeta_3 = 0 \tag{2.11}$$

$$\zeta_3' + iF_3\zeta_3 + iR_3 e^{i(k_1+k_2-k_3)x}\zeta_1\zeta_2 = 0 \tag{2.12}$$

where

$$R_1 = \frac{3k_1^2}{\omega_1}\frac{(k_3-k_2)}{k_2k_3}\cdot\left[\omega_1 + (k_3-k_2)\left(\frac{k_2}{\omega_2}+\frac{k_3}{\omega_3}\right)\right] \tag{2.13}$$

$$R_2 = \frac{3k_2^2}{16\omega_1}\frac{(k_3-k_1)}{k_1k_3}\cdot\left[\omega_2 + (k_3-k_1)\left(\frac{k_1}{\omega_1}+\frac{k_3}{\omega_3}\right)\right] \tag{2.14}$$

$$R_3 = \frac{k_3^2}{9\omega_3}\frac{(k_1+k_2)}{k_1k_2}\cdot\left[\omega_3 + (k_1+k_2)\left(\frac{k_1}{\omega_1}+\frac{k_2}{\omega_2}\right)\right] \tag{2.15}$$

After changing variables to eliminate the dependence on X, rewrite the flat bed version of this system as

$$a_1' = -i\alpha Q_1 e^{i\Delta_k x}a_1^*a_2 - i\alpha R_1 e^{-ik_H x}a_2^*a_3 \tag{2.16}$$

$$a_2' = -i\alpha Q_2 e^{-i\Delta_k x}a_1^2 - i\alpha R_2 e^{-ik_H x}a_1^*a_3 \tag{2.17}$$

$$a_3' = -i\alpha R_3 e^{ik_H x}a_1a_2, \tag{2.18}$$

where $k_H = k_1 + k_2 - k_3$. The flat bed two modes case is characterized by the following system of two coupled ordinary differential equations.

$$a_1' = -i\alpha Q_1 e^{i\Delta_k x}a_1^*a_2 \tag{2.19}$$

$$a_2' = -i\alpha Q_2 e^{-i\Delta_k x} a_1^2, \tag{2.20}$$

with

$$Q_1 = (k_2 - k_1)\frac{k_1}{k_2\omega_1}\left[\omega_1 + (k_2 - k_1)\left(\frac{k_1}{\omega_1} + \frac{k_2}{\omega_2}\right)\right] \tag{2.21}$$

$$Q_2 = \frac{k_2^2}{8k_1}\left(1 + \frac{2k_1^2}{\omega_1^2}\right). \tag{2.22}$$

2.3 Derivation of the Lau-Barcilon Equations for BBM

Recall the BBM equation from Chapter 1,

$$u_t + u_x + \frac{3}{2}\alpha u u_x - \frac{1}{6}\beta u_{xxt} = 0. \tag{2.23}$$

Suppose that u is characterized by a slowly varying amplitude. This requires two steps. First, carry out a two-scale expansion by setting $X = \alpha x$ for the small parameter α and $u(x, X, t) = u(x, t)$, so that in the above form of BBM ∂_x is replaced with $\partial_x + \alpha\partial_X$. The process to follow is commonly (though casually) described as 'assuming that X and x are independent,' but what does it mean to treat X and x as independent variables? Consider the following two calculations:

$$\begin{aligned}(\partial_x + \alpha\partial_X)\left[f(X)g(x)\right] &= f(X)g'(x) + \alpha f'(X)g(x)\\ \partial_x\left[f(\alpha x)g(x)\right] &= f(\alpha x)g'(x) + \alpha f'(\alpha x)g(x).\end{aligned}$$

These are identical for $X = \alpha x$! So, in fact, we need not *assume* the independence of X and x to replace ∂_x by $\partial_x + \alpha\partial_X$ in our governing partial differential equation. This is simply a consequence of the separation of variables and the product rule from calculus.

Apply the two-scale expansion to the BBM equation (2.23) gives

$$u_t + (\partial_x + \alpha\partial_X)u + \frac{3}{2}\alpha u(\partial_x + \alpha\partial_X)u - \frac{\beta}{6}(\partial_x + \alpha\partial_X)(u_{xt} + \alpha u_{Xt}) = 0$$

which reduces to

$$u_t + u_x - \frac{\beta}{6}u_{xxt} + \alpha\left[u_X + \frac{3}{2}uu_x - \frac{\beta}{3}u_{xXt}\right] = O(\alpha^2). \tag{2.24}$$

Second, set

$$u(x, X, t) = \sum_{j=-N}^{N} u_j(X)e^{i(k_j x - \omega_j t)}, \tag{2.25}$$

where $\omega_j = j\omega_1$, $\omega_{-j} = -\omega_j$, $k_{-j} = -k_j$, $\omega_0 = 0$, $k_0 = 0$, $u_0(X) = 0$, and $u_{-j} = u_j^*$ represents the complex conjugate. Using the notation $E_j = e^{i(k_j x - \omega_j t)}$ introduced in the previous section, consider the order one terms in the two-scale version of BBM:

$$\begin{aligned} u_t &= \sum_{j=-N}^{N} -i\omega_j u_j E_j \\ u_x &= \sum_{j=-N}^{N} ik_j u_j E_j \\ u_{xxt} &= \sum_{j=-N}^{N} ik_j^2\omega_j u_j E_j. \end{aligned}$$

These terms combine to give the $O(\alpha^0)$ expression

$$u_t + u_x - \frac{\beta}{6}u_{xxt} = \sum_j i\left(k_j - \omega_j - \frac{\beta}{6}k_j^2\omega_j\right)E_j.$$

Setting this equal to zero gives the linearized dispersion relation for BBM:

$$k_j - \omega_j - \frac{\beta}{6}k_j^2\omega_j = 0$$

which is typically written as

$$\omega = \frac{k}{1 + \frac{\beta}{6}k^2}.$$

This linearized BBM dispersion relation agrees with the full linearized dispersion relation for the Euler equation to order β. For $\beta = \left(\frac{1}{12}\right)^2$, this relation has the form shown in Figure (2.2).

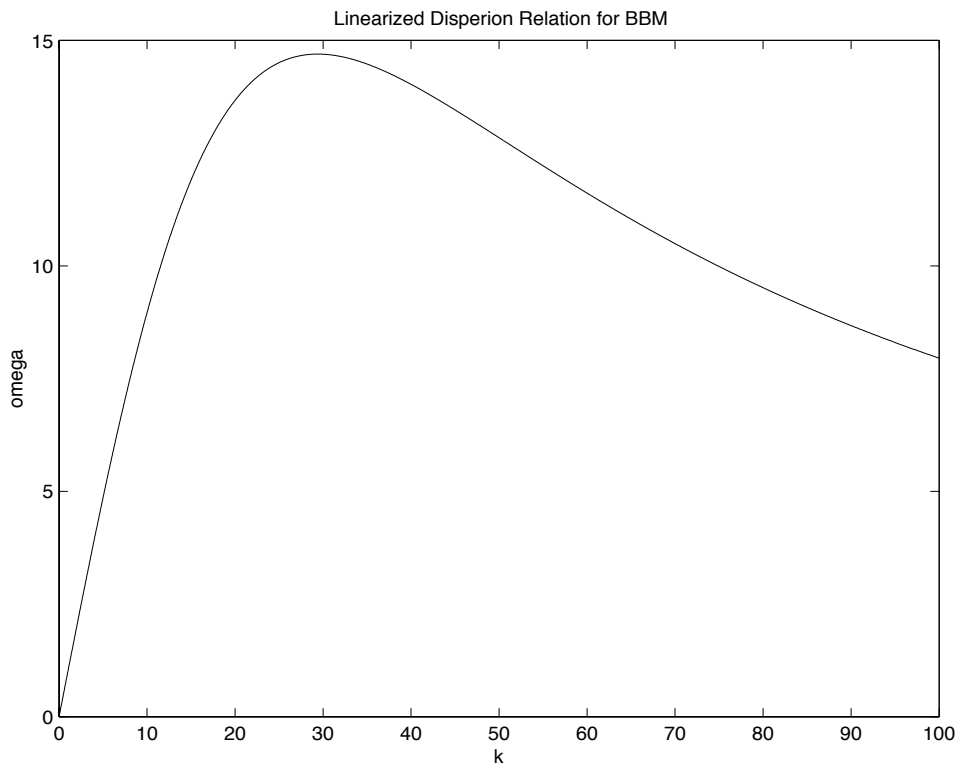

Figure 2.2: Dispersion Relation, $\omega(k)$

Refer to Figure 3.2 in Chapter 3 for comparison with another linearized dispersion relation, for the case of the KdV equation.

Another issue is highlighted by the linearized dispersion relation: how can the relation between β and ω be used in the (later) numerical analysis

of this situation? More simply, how do the magnitudes of β and ω compare? First, this requires an understanding of the *physical* meaning of the quantity k_1 in the above modal expansion. In physical variables, the wavenumber is

$$k = \frac{2\pi}{\lambda_{ph}}$$

where $\lambda_{ph} = \lambda/h$ is the physical variable representing a typical wavelength. Thus, in the nondimensional version of the model,

$$k = \frac{2\pi h}{\lambda} = 2\pi\beta.$$

Now, inserting this expression into the dispersion relation gives

$$\omega = \frac{k}{1 + \frac{\beta}{6}k^2} = \frac{2\pi\beta}{1 + \frac{2}{3}\pi^2\beta^4},$$

and the assumption that β is small yields the approximation $\omega \approx 2\pi\beta$. For the typical value $\beta = (1/12)^2$ (to be discussed later in the development of the numerical results), one finds $\omega_1 \approx 2\pi\beta = 0.5236$.

Returning to the expansion (in powers of α), consider the pieces which contribute to the order α terms in the two-scale version of BBM:

$$\begin{aligned} u_X &= \sum_{j=-N}^{N} u_j' E_j \\ u_{xXt} &= \sum_{j=-N}^{N} k_j \omega_j u_j' E_j. \end{aligned}$$

The expansion and simplification of the nonlinear term depends on how many modes (N) are being used to approximate u. The nonlinearity generates higher order modes which must be neglected. For example, in the case of two modes

$(N = 2)$,

$$\begin{aligned} uu_x &= \left(\sum_{j=-2}^{2} u_j E_j\right)\left(\sum_{n=-2}^{2} ik_n u_n E_n\right) \\ &= ik_1 u_1^2 E_1^2 + ik_2 u_2^2 E_2^2 \\ &\quad + E_1 E_2 \left(i(k_1+k_2)u_1 u_2\right) + E_1^* E_2 \left(i(k_2-k_1)u_1^* u_2\right) \\ &\quad + \text{conj} \end{aligned}$$

Here, $E_2^2 = e^{2i(k_2 x - \omega_2 t)} = e^{i(2k_2 x - \omega_4 t)}$, but this gives a fourth mode, at frequency $\omega_4 = 4\omega_1$, which must be neglected in the two modes approximation. Hence, after simplification,

$$uu_x = ik_1 u_1^2 e^{-i\Delta_k x} E_2 + i(k_2 - k_1) u_1^* u_2 e^{i\Delta_k x} E_1 + \text{conj}$$

where $\Delta_k = k_2 - 2k_1$.

Putting together these pieces and simplifying gives the expression for the $O(\alpha)$ portion of the two-scale BBM equation in the case of 2 modes:

$$\begin{aligned} u_X + \frac{3}{2}uu_x - \frac{\beta}{3}u_{xXt} &= E_1\left[u_1' - \frac{\beta}{3}k_1\omega_1 u_1' + i\frac{3}{2}(k_2-k_1)u_1^* u_2 e^{i\Delta_k x}\right] \\ &+ E_2\left[u_2' - \frac{\beta}{3}k_2\omega_2 u_2' + i\frac{3}{2}k_1 u_1^2 e^{-i\Delta_k x}\right] \\ &+ \text{conj.} \end{aligned}$$

This equation provides the Lau-Barcilon equations:

$$\begin{aligned} u_1'(X) &= -i\frac{3}{2}\frac{k_1 - k_2}{1 - \frac{\beta}{3}k_1\omega_1} e^{i\Delta_k x} u_1^*(X) u_2(X) \\ u_2'(X) &= -i\frac{3}{2}\frac{k_1}{1 - \frac{\beta}{3}k_2\omega_2} e^{-i\Delta_k x} u_1^2(X). \end{aligned}$$

Change variables back to the original (x,t) coordinates by setting $a_j(x) = u_j(X) = u_j(\alpha x)$ so that ∂_X is replaced by $\frac{1}{\alpha}\partial_x$. Thus the system is

$$a_1' = -i\alpha\frac{3}{2}\frac{k_1-k_2}{1-\frac{\beta}{3}k_1\omega_1}e^{i\Delta_k x}a_1^* a_2 \tag{2.26}$$

$$a_2' = -i\alpha\frac{3}{2}\frac{k_1}{1-\frac{\beta}{3}k_2\omega_2}e^{-i\Delta_k x}a_1^2. \tag{2.27}$$

The Lau-Barcilon equations for the case of three modes may be computed similarly, with the nonlinear term having the form

$$\begin{aligned} uu_x = \quad & E_1\left(i(k_2-k_1)u_1^* u_2 e^{i\Delta_k x} + i(k_3-k_2)u_2^* u_3 e^{-i(k_1+k_2-k_3)x}\right) \\ & +E_2\left(ik_1 u_1^2 e^{-i\Delta_k x} + i(k_3-k_1)u_1^* u_3 e^{-i(k_1+k_2-k_3)x}\right) \\ & +E_3\left(i(k_1+k_2)u_1 u_2 e^{i(k_1+k_2-k_3)x}\right) \\ & +\text{conj} \end{aligned}$$

after neglecting higher harmonics. As in the two modes case, the $O(\alpha)$ terms yield the Lau-Barcilon equations:

$$\begin{aligned} u_1'(X) &= -i\frac{3}{2}\frac{k_2-k_1}{1-\frac{\beta}{3}k_1\omega_1}e^{i\Delta_k x}u_1^*(X)u_2(X) \\ &\quad -i\frac{3}{2}\frac{k_3-k_2}{1-\frac{\beta}{3}k_1\omega_1}e^{-i(k_1+k_2-k_3)x}u_2^*(X)u_3(X) \end{aligned} \tag{2.28}$$

$$\begin{aligned} u_2'(X) &= -i\frac{3}{2}\frac{k_1}{1-\frac{\beta}{3}k_2\omega_2}e^{-i\Delta_k x}u_1^2(X) \\ &\quad -\frac{3}{2}i\frac{k_3-k_1}{1-\frac{\beta}{3}k_2\omega_2}e^{-i(k_1+k_2-k_3)x}u_1^*(X)u_3(X) \end{aligned} \tag{2.29}$$

$$u_3'(X) = -i\frac{3}{2}\frac{k_1+k_2}{1-\frac{\beta}{3}k_3\omega_3}e^{i(k_1+k_2-k_3)x}u_1(X)u_2(X). \tag{2.30}$$

After a change of variables back to the usual (x,t) coordinates by taking, as before, $a_j(x) = u_j(X) = u_j(\alpha x)$, the Lau-Barcilon equations become

$$a_1' = -i\alpha\frac{3}{2}\frac{k_2-k_1}{1-\frac{\beta}{3}k_1\omega_1}e^{i\Delta_k x}a_1^* a_2$$

$$-i\alpha\frac{3}{2}\frac{k_3-k_2}{1-\frac{\beta}{3}k_1\omega_1}e^{-i(k_1+k_2-k_3)x}a_2^*a_3 \tag{2.31}$$

$$a_2' = -i\alpha\frac{3}{2}\frac{k_1}{1-\frac{\beta}{3}k_2\omega_2}e^{-i\Delta_k x}a_1^2$$

$$-i\alpha\frac{3}{2}\frac{k_3-k_1}{1-\frac{\beta}{3}k_2\omega_2}e^{-i(k_1+k_2-k_3)x}a_1^*a_3 \tag{2.32}$$

$$a_3' = -i\alpha\frac{3}{2}\frac{k_1+k_2}{1-\frac{\beta}{3}k_3\omega_3}e^{i(k_1+k_2-k_3)x}a_1a_2. \tag{2.33}$$

Introducing notation for the constants makes the structure of these equations easier to observe. Set

$$Q_1 = \frac{3}{2}\frac{k_2-k_1}{1-\frac{\beta}{3}k_1\omega_1} \tag{2.34}$$

$$Q_2 = \frac{3}{2}\frac{k_1}{1-\frac{\beta}{3}k_2\omega_2} \tag{2.35}$$

and

$$R_1 = \frac{3}{2}\frac{k_3-k_2}{1-\frac{\beta}{3}k_1\omega_1} \tag{2.36}$$

$$R_2 = \frac{3}{2}\frac{k_3-k_1}{1-\frac{\beta}{3}k_2\omega_2} \tag{2.37}$$

$$R_3 = \frac{3}{2}\frac{k_1+k_2}{1-\frac{\beta}{3}k_3\omega_3}. \tag{2.38}$$

Introducing the further notation $k_H = k_1+k_2-k_3$ then yields the three modes Lau-Barcilon equations:

$$a_1' = -\alpha i Q_1 e^{i\Delta_k x}a_1^*a_2 - \alpha i R_1 e^{-ik_H x}a_2^*a_3 \tag{2.39}$$

$$a_2' = -\alpha i Q_2 e^{-i\Delta_k x}a_1^2 - \alpha i R_2 e^{-ik_H x}a_1^*a_3 \tag{2.40}$$

$$a_3' = -\alpha i R_3 e^{ik_H x}a_1a_2. \tag{2.41}$$

2.4 The Modal Expansion for Linear BBM

The linear BBM equation is simply equation (2.23) without the nonlinear term, that is

$$u_t + u_x - \frac{\beta}{6} u_{xxt} = 0.$$

This equation models traveling waves in which dispersive effects dominate nonlinear effects.

Apply the two-scale expansion, taking $X = \alpha x$ and, consequently, replacing ∂_x with $\partial_x + \alpha \partial_X$ in the above equation. After simplification, linear BBM written in the two-scale variables becomes

$$u_t + u_x - \frac{\beta}{6} u_{xxt} + \alpha \left[u_X - \frac{\beta}{3} u_{Xxt} \right] = O(\alpha^2).$$

After the modal expansion is applied to u, the $O(\alpha^0)$ terms will reduce to the usual linearized dispersion relation for each mode:

$$-i\omega_j + ik_j - \frac{\beta}{6}(ik_j)^2(-i\omega_j) = 0 \tag{2.42}$$

which may be rewritten as

$$\omega_j = \frac{k_j}{1 + \frac{\beta}{6} k_j^2}. \tag{2.43}$$

The next step is to apply the modal expansion to the $O(\alpha)$ terms: substitute the form

$$u(x, X, t) = \sum_{j=-\infty}^{\infty} u_j(X) e^{i(k_j x - \omega_j t)}$$

into the equation

$$u_X - \frac{\beta}{3} u_{Xxt} = 0. \tag{2.44}$$

The coefficients of each mode then provide the relationship

$$u_j' e^{-i(k_j x - \omega_j t)} - \frac{\beta}{3}(ik_j)(-i\omega_j)u_j' e^{-i(k_j x - \omega_j t)} = 0$$

where the prime refers to differentiation with respect to X. This relation reduces to

$$\left(1 + \frac{\beta}{3}k_j\omega_j\right)u_j' = 0$$

which shows that

$$u_j(X) = C_j,$$

meaning each coefficient is constant. Therefore, in the case of linear BBM, the modal expansion simply reduces to the standard Fourier series expansion:

$$u(x,t) = \sum_{j=-\infty}^{\infty} C_j e^{-i(k_j x - \omega_j t)}$$

subject to the dispersion relation for each ($j-$th) mode. The coefficients are, as usual, determined once boundary data has been posed for the problem.

2.5 Conservation of Energy for BBM

Begin with the Lau-Barcilon equations for the case of the BBM equation. These are

$$\frac{da_1}{dx} = -i\alpha Q_1 e^{i\Delta_k x} a_1^* a_2 \tag{2.45}$$

$$\frac{da_2}{dx} = -i\alpha Q_2 e^{-i\Delta_k x} a_1^2 \tag{2.46}$$

Write these equations in complex form by setting $a_j(x) = r_j(x)e^{i\theta_j(x)}$. The system then becomes

$$r_1' + ir_1\theta_1' = -i\alpha Q_1 r_1 r_2 e^{i[\theta_2 - 2\theta_1 + \Delta_k x]} \tag{2.47}$$

$$r_2' + ir_2\theta_2' = -i\alpha Q_2 r_1^2 e^{-i[\theta_2 - 2\theta_1 + \Delta_k x]} \tag{2.48}$$

where the prime indicates differentiation with respect to the variable x. The next step is to separate these equations into real and imaginary components, first by setting $\phi = \theta_2 - 2\theta_1 + \Delta_k x$. Expanding the exponential term in sine and cosine yields

$$r_1' + ir_1\theta_1' = -i\alpha Q_1 r_1 r_2 \cos\phi + \alpha Q_1 r_1 r_2 \sin\phi \tag{2.49}$$

$$r_2' + ir_2\theta_2' = -i\alpha Q_2 r_1^2 \cos\phi - \alpha Q_2 r_1^2 \sin\phi. \tag{2.50}$$

So, sorting into real and imaginary parts gives the following four ordinary differential equations.

$$r_1' = \alpha Q_1 r_1 r_2 \sin\phi \qquad \theta_1' = -\alpha Q_1 r_2 \cos\phi \tag{2.51}$$

$$r_2' = -\alpha Q_2 r_1^2 \sin\phi \qquad \theta_2' = -\alpha Q_2 \frac{r_1^2}{r_2} \cos\phi \tag{2.52}$$

Reduce the number of equations by noting that the equations for θ_1 and θ_2 may be written as one equation for the phase ϕ, since

$$\phi' = \theta_2' - 2\theta_1' + \Delta_k = -\alpha Q_2 \frac{r_1^2}{r_2} \cos\phi + 2\alpha Q_1 r_2 \cos\phi + \Delta_k.$$

This reduces the system to

$$r_1' = \alpha Q_1 r_1 r_2 \sin\phi \tag{2.53}$$

$$r_2' = -\alpha Q_2 r_1^2 \sin\phi \tag{2.54}$$

$$\phi' = \Delta_k - \alpha\left[Q_2 \frac{r_1^2}{r_2} - 2Q_1 r_2\right] \cos\phi. \tag{2.55}$$

Notice that this system must have a periodic solution.

From equations 2.53 and 2.54, compute

$$\frac{r_1'}{r_2'} = -\frac{Q_1}{Q_2}\frac{r_2}{r_1} \tag{2.56}$$

$$r_1 r_1' = -\frac{Q_1}{Q_2} r_2 r_2' \tag{2.57}$$

$$Q_2 \left(\frac{1}{2} r_1^2\right)' = -Q_1 \left(\frac{1}{2} r_2^2\right)' \tag{2.58}$$

which integrates to give the conservation of energy expression

$$Q_2 r_1^2 + Q_1 r_2^2 = R^2.$$

One could choose to re-scale, so that this becomes $r_1^2 + r_2^2 = C$, where C is some positive constant.

To derive a conservation law for the case of three modes, follow a nearly identical calculation. Begin with the three modes Lau-Barcilon equations for BBM:

$$a_1' = -i\alpha Q_1 e^{i\Delta_k x} a_1^* a_2 - i\alpha R_1 e^{-ik_H x} a_2^* a_3 \tag{2.59}$$

$$a_2' = -i\alpha Q_2 e^{-i\Delta_k x} a_1^2 - i\alpha R_2 e^{-ik_H x} a_1^* a_3 \tag{2.60}$$

$$a_3' = -i\alpha R_3 e^{ik_H x} a_1 a_2 \tag{2.61}$$

including the conjugate equations to these, where the previously introduced notational conveniences $\Delta_k = k_2 - 2k_1$ and $k_H = k_1 + k_2 - k_3$ are used.

As in the two modes case, set $a_j(x) = r_j(x) e^{i\theta_j(x)}$ so that

$$a_j' = r_j' e^{i\theta_j} + i r_j \theta_j' e^{i\theta_j}.$$

Substituting these forms into the first of the above ordinary differential equations gives

$$r_1' e^{i\theta_1} + i r_1 \theta_1' e^{i\theta_1} = -i\alpha Q_1 e^{i\Delta_k x} r_1 e^{-i\theta_1} r_2 e^{i\theta_2} - i\alpha R_1 e^{-ik_H x} r_2 e^{-i\theta_2} r_3 e^{i\theta_3}.$$

Computing similarly for the second and third equations and grouping the exponential terms gives the following system,

$$r_1' + ir_1\theta_1' = -i\alpha Q_1 r_1 r_2 e^{i[\theta_2 - 2\theta_1 + \Delta_k x]} - i\alpha R_1 r_2 r_3 e^{-i[\theta_1+\theta_2-\theta_3+k_H x]} \quad (2.62)$$

$$r_2' + ir_2\theta_2' = -i\alpha Q_2 r_1^2 e^{-i[\theta_2 - 2\theta_1 + \Delta_k x]} - i\alpha R_2 r_1 r_3 e^{-i[\theta_1+\theta_2-\theta_3+k_H x]} \quad (2.63)$$

$$r_3' + ir_3\theta_3' = -i\alpha R_3 r_1 r_2 e^{i[\theta_1+\theta_2-\theta_3+k_H x]} \quad (2.64)$$

together with conjugate equations.

Now set $\phi = \theta_2 - 2\theta_1 + \Delta_k x$ and $\psi = \theta_1 + \theta_2 - \theta_3 + k_H x$. Using this notation, the above system becomes

$$r_1' + ir_1\theta_1' = -i\alpha Q_1 r_1 r_2 e^{i\phi} - i\alpha R_1 r_2 r_3 e^{-i\psi} \quad (2.65)$$

$$r_2' + ir_2\theta_2' = -i\alpha Q_2 r_1^2 e^{-i\phi} - i\alpha R_2 r_1 r_3 e^{-i\psi} \quad (2.66)$$

$$r_3' + ir_3\theta_3' = -i\alpha R_3 r_1 r_2 e^{i\psi}. \quad (2.67)$$

Breaking this system into real and imaginary parts yields the following six ordinary differential equations,

$$r_1' = \alpha Q_1 r_1 r_2 \sin\phi - \alpha R_1 r_2 r_3 \sin\psi \quad (2.68)$$

$$\theta_1' = -\alpha Q_1 r_2 \cos\phi - \alpha R_1 \frac{r_2 r_3}{r_1} \cos\psi \quad (2.69)$$

$$r_2' = -\alpha Q_2 r_1^2 \sin\phi - \alpha R_2 r_1 r_3 \sin\psi \quad (2.70)$$

$$\theta_2' = -\alpha Q_2 \frac{r_1^2}{r_2} \cos\phi - \alpha R_2 \frac{r_1 r_3}{r_2} \cos\psi \quad (2.71)$$

$$r_3' = \alpha R_3 r_1 r_2 \sin\psi \quad (2.72)$$

$$\theta_3' = -\alpha R_3 \frac{r_1 r_2}{r_3} \cos\psi. \quad (2.73)$$

The next step is to write these equations in terms of the variables ϕ and ψ, eliminating explicit dependence on the θ_j. Using the forms

$$\phi' = \theta_2' - 2\theta_1' + \Delta_k \quad (2.74)$$

$$\psi' = \theta_1' + \theta_2' - \theta_3' + k_H \tag{2.75}$$

gives the following system in terms of $r_1, r_2, r_3, \phi, \psi$:

$$\begin{aligned}
r_1' &= \alpha Q_1 r_1 r_2 \sin\phi - \alpha R_1 r_2 r_3 \sin\psi \\
r_2' &= -\alpha Q_2 r_1^2 \sin\phi - \alpha R_2 r_1 r_3 \sin\psi \\
r_3' &= \alpha R_3 r_1 r_2 \sin\psi \\
\phi' &= \Delta_k + \alpha\cos\phi\left(2Q_1 r_2 - Q_2 r_1^2/r_2\right) + \alpha\cos\psi\left(2R_1\frac{r_2 r_3}{r_1} - R_2\frac{r_1 r_3}{r_2}\right) \\
\psi' &= k_H - \alpha\cos\phi\left(Q_1 r2 + Q_2 r_1^2/r_2\right) \\
&\qquad + \alpha\cos\psi\left(R_3\frac{r_1 r_2}{r_3} - R_1\frac{r_2 r_3}{r_1} - R_2\frac{r_1 r_3}{r_2}\right).
\end{aligned}$$

The next step is to carry out the algebra needed to obtain a conservation law—this requires only the first three equations in the above system. First, compute

$$(Q_2 r_1 r_1' + Q_1 r_2 r_2')R_3 = (-\alpha\left(Q_2 R_1 + Q_1 R_2\right) r_1 r_2 r_3 \sin\psi)R_3$$

and then add this to the quantity $\left(Q_2 R_1 + Q_1 R_2\right) r_3 r_3'$ to obtain the result that

$$Q_2 R_3 r_1 r_1' + Q_1 R_3 r_2 r_2' + \left(Q_2 R_1 + Q_1 R_2\right) r_3 r_3' = 0.$$

This may be written as

$$\left[\frac{1}{2}Q_2 R_3 r_1^2 + \frac{1}{2}Q_1 R_3 r_2^2 + \frac{1}{2}(Q_2 R_1 + Q_1 R_2) r_3^2\right]' = 0$$

which integrates to the conservation law

$$Q_2 R_3 r_1^2 + Q_1 R_3 r_2^2 + (Q_2 R_1 + Q_1 R_2) r_3^2 = \text{constant}.$$

2.6 Closed-form Solution: 2 Modes Case

Following Armstrong et al. [5], derive the analytical solution to the two modes Lau-Barcilon equations as follows. Rewrite the ordinary differential equation for ϕ,

$$\frac{d\phi}{dx} = \Delta_k - \alpha \left[Q_2 \frac{r_1^2}{r_2} - 2Q_1 r_2 \right] \cos\phi$$

by substitution, first noting that

$$\frac{d}{dx} \ln\left(r_1^2 r_2\right) = \frac{1}{r_1^2 r_2} \left(2r_1 \frac{dr_1}{dx} r_2 + r_1^2 \frac{dr_2}{dx} \right) = \frac{2}{r_1}\frac{dr_1}{dx} + \frac{1}{r_2}\frac{dr_2}{dx}$$

and so,

$$\begin{aligned}
\frac{\cos\phi}{\sin\phi}\,\frac{d}{dx}\ln(r_1^2 r_2) &= \frac{\cos\phi}{\sin\phi}\frac{2}{r_1}\frac{dr_1}{dx} + \frac{\cos\phi}{\sin\phi}\frac{1}{r_2}\frac{dr_2}{dx} \\
&= \frac{\cos\phi}{\sin\phi}\frac{2}{r_1}\alpha Q_1 r_1 r_2 \sin\phi - \frac{\cos\phi}{\sin\phi}\frac{1}{r_2}\alpha Q_2 r_1^2 \sin\phi \\
&= 2\alpha Q_1 \cos\phi\; r_2 - \alpha Q_2 \cos\phi\; \frac{r_1^2}{r_2} \\
&= -\alpha \left[Q_2 \frac{r_1^2}{r_2} - 2Q_1 r_2 \right] \cos\phi
\end{aligned}$$

Then rewrite the differential equation for ϕ as

$$\begin{aligned}
\frac{d\phi}{dx} &= \Delta_k - \alpha \left[Q_2 \frac{r_1^2}{r_2} - 2Q_1 r_2 \right] \cos\phi \\
&= \Delta_k + \frac{\cos\phi}{\sin\phi}\,\frac{d}{dx}\ln(r_1^2 r_2). \qquad (2.76)
\end{aligned}$$

Now continue with the three differential equations for the two modes case,

$$\frac{dr_1}{dx} = \alpha Q_1 r_1 r_2 \sin\phi \qquad (2.77)$$

$$\frac{dr_2}{dx} = -\alpha Q_2 r_1^2 \sin\phi \qquad (2.78)$$

$$\frac{d\phi}{dx} = \Delta_k + \frac{\cos\phi}{\sin\phi}\,\frac{d}{dx}\ln(r_1^2 r_2). \qquad (2.79)$$

Apply the change of variables

$$\rho_1 = \sqrt{Q_2}\, r_1 \tag{2.80}$$

$$\rho_2 = \sqrt{Q_1}\, r_2 \tag{2.81}$$

$$\xi = -\alpha\sqrt{Q_1}x \tag{2.82}$$

so that differentiation with respect to x is replaced by

$$\frac{d}{dx} = \frac{d\xi}{dx}\,\frac{d}{d\xi} = -\alpha\sqrt{Q_1}\frac{d}{d\xi}$$

and, further,

$$\alpha Q_1 r_1 r_2 \sin\phi = \alpha Q_1 \frac{\rho_1}{\sqrt{Q_2}}\,\frac{\rho_2}{\sqrt{Q_1}}\sin\phi = \alpha\frac{\sqrt{Q_1}}{\sqrt{Q_2}}\rho_1\rho_2\sin\phi.$$

Thus,

$$\frac{d\rho_1}{d\xi} = -\rho_1\rho_2\sin\phi.$$

Similarly,

$$\frac{dr_2}{dx} = -\alpha\sqrt{Q_1}\,\frac{d}{d\xi}\left(\frac{1}{\sqrt{Q_1}}\rho_2\right) = -\alpha\frac{d\rho_2}{d\xi}$$

and

$$-\alpha Q_2 r_1^2\sin\phi = -\alpha Q_2\cdot\frac{1}{Q_2}\rho_1^2\sin\phi = -\alpha\rho_1^2\sin\phi$$

yield

$$\frac{d\rho_2}{d\xi} = \rho_1^2\sin\phi.$$

Finally,

$$\begin{aligned}\Delta_k + \frac{\cos\phi}{\sin\phi}\,\frac{d}{dx}\ln(r_1^2 r_2) &= \Delta_k + \frac{\cos\phi}{\sin\phi}\,(-\alpha\sqrt{Q_1})\frac{d}{d\xi}\ln\left(\frac{\rho_1^2}{Q_2}\frac{\rho_2}{\sqrt{Q_1}}\right)\\ &= \Delta_k - \alpha\sqrt{Q_1}\frac{\cos\phi}{\sin\phi}\,\frac{d}{d\xi}\ln(\rho_1^2\rho_2) \qquad (2.83)\end{aligned}$$

so that the ordinary differential equation for ϕ becomes

$$\frac{d\phi}{d\xi} = -\frac{\Delta_k}{\alpha\sqrt{Q_1}} + \frac{\cos\phi}{\sin\phi}\,\frac{d}{d\xi}\ln(\rho_1^2\rho_2). \tag{2.84}$$

In summary, the new coordinates give

$$\frac{d\rho_1}{d\xi} = -\rho_1\rho_2\sin\phi \tag{2.85}$$

$$\frac{d\rho_2}{d\xi} = \rho_1^2\sin\phi \tag{2.86}$$

$$\frac{d\phi}{d\xi} = \Delta_s + \frac{\cos\phi}{\sin\phi}\,\frac{d}{d\xi}\ln(\rho_1^2\rho_2) \tag{2.87}$$

where $\Delta_s = -\Delta_k/(\alpha\sqrt{Q_1})$.

The conservation law in terms of the new variables is

$$\rho_1^2 + \rho_2^2 = R^2.$$

Rescaling by the constant R gives the form

$$\rho_1^2 + \rho_2^2 = 1. \tag{2.88}$$

As in Armstrong et al., if $\Delta_k = 0$, then the ordinary differential equation (2.84) for ϕ may be integrated:

$$\begin{aligned}
\frac{d\phi}{d\xi} &= \frac{\cos\phi}{\sin\phi}\,\frac{d}{d\xi}\ln(\rho_1^2\rho_2) \\
\frac{\sin\phi}{\cos\phi}\,\frac{d\phi}{d\xi} &= \frac{d}{d\xi}\ln(\rho_1^2\rho_2) \\
-\ln(\cos\phi) &= \ln(\rho_1^2\rho_2) + c_1 \\
\frac{1}{\cos\phi} &= c_2\rho_1^2\rho_2
\end{aligned}$$

Thus,

$$\rho_1^2\rho_2\cos\phi = c \quad \text{for all } \xi.$$

Using this result, derive a formula for the constant c:

$$\begin{aligned} c &= \rho_1^2(0)\rho_2(0)\cos\phi(0) \\ &= Q_2 r_1^2(0)\sqrt{Q_1}\; r_2(0)\cos\left[\theta_2(0) - 2\theta_1(0)\right] \\ &= Q_2\sqrt{Q_1}\; r_1^2(0) r_2(0)\cos\left[\theta_2(0) - 2\theta_1(0)\right] \end{aligned} \tag{2.89}$$

The next goal is to use the first two ordinary differential equations to derive an elliptic integral, as in Armstrong et al. Notice:

$$\begin{aligned} \frac{d}{d\xi}\left(\rho_2^2\right) &= 2\rho_2\rho_2' \\ &= 2\rho_1^2\rho_2\sin\phi \\ &= \pm 2\rho_1^2\rho_2\sqrt{1-\cos^2\phi} \\ &= \pm 2\sqrt{\rho_1^4\rho_2^2 - \rho_1^4\rho_2^2\cos^2\phi} \\ &= \pm 2\sqrt{(1-\rho_2^2)^2\rho_2^2 - (\rho_1^2\rho_2\cos\phi)^2} \\ &= \pm 2\sqrt{\rho_2^2(1-\rho_2^2)^2 - c^2} \end{aligned}$$

Rewrite this formulation using separation of variables to find

$$\frac{d(\rho_2^2)}{d\xi} = \pm 2\sqrt{\rho_2^2(1-\rho_2^2)^2 - c^2} \tag{2.90}$$

$$d\xi = \pm\frac{1}{2}\frac{d(\rho_2^2)}{\sqrt{\rho_2^2(1-\rho_2^2)^2 - c^2}} \tag{2.91}$$

Thus derive the elliptic integral,

$$\xi = \pm\frac{1}{2}\int_{\rho_2^2(0)}^{\rho_2^2(\xi)} \frac{d(\rho_2^2)}{\sqrt{\rho_2^2(1-\rho_2^2)^2 - c^2}} \tag{2.92}$$

that represents the general solution of the differential equation (2.90) for ρ_2.

Now consider the polynomial $\rho_2^2(1-\rho_2^2)^2 - c^2$. Since $1 = \rho_1^2 + \rho_2^2$ and $\rho_2(\xi) \in \mathbb{R}$, it must follow that $0 \le \rho_2^2 \le 1$, and, thus, $\rho_2^2(1-\rho_2^2)^2 \ge 0$. Study

this polynomial more closely, considering the graph of

$$f(x) = x(1-x)^2 - c^2 = 0 \quad \text{for } x \in [0,1].$$

It is an elementary calculus exercise to show that the local maximum and minimum occur at $x = 1/3$ and $x = 1$, respectively. For what values of c is $x(1-x)^2 - c^2 > 0$ at the value $x = 1/3$? Solving the inequality gives $c < \sqrt{4/27}$. This is realistic in the context of the modal problem, as the starting values $r_1(0)$ and $r_2(0)$ are typically small; thus, in fact, one obtains three real roots.

These roots of f are named $\sigma_a^2 \leq \sigma_b^2 \leq \sigma_c^2$. If $x \in (0, \sigma_a^2)$, then $f(x) = x^3 - 2x^2 + x - c^2 < 0$. This forces the integrand of the elliptic integral (2.92) to be strictly imaginary, which is not allowed. Similarly, $x \in (\sigma_b^2, 1)$ forces $f(x) < 0$, which is not allowed. Thus, as Armstrong et al. note, require

$$\sigma_a^2 < \rho_2^2 < \sigma_b^2$$

where σ_a^2 and σ_b^2 are the smallest two roots of $\rho_2^2(1-\rho_2^2)^2 - c^2 = 0$.

In the case where $c = 0$, return to the differential equation

$$\frac{d(\rho_2^2)}{d\xi} = \pm 2\left(\rho_2^2(1-\rho_2^2)^2\right)^{1/2}$$

and recognize that the sign $\pm$ is determined only by $\sin\phi|_{z=0}$. Without loss of generality, assume the sign is positive. Then, since $\rho_2 \neq 0$,

$$\frac{d(\rho_2^2(\xi))}{d\xi} = 2\rho_2(\xi)(1-\rho_2^2(\xi)) \tag{2.93}$$

$$2\rho_2(\xi)\frac{d\rho_2}{d\xi} = 2\rho_2(\xi)(1-\rho_2^2(\xi)) \tag{2.94}$$

$$\frac{d\rho_2}{d\xi} = 1 - \rho_2^2 \tag{2.95}$$

This equation is separable and may be integrated to find

$$\frac{1}{2}\int\left[\frac{1}{1+\rho_2}+\frac{1}{1-\rho_2}\right]d\rho_2=\int\frac{d\rho_2}{1-\rho_2^2}=\int d\xi=\xi+c_1$$

So,

$$\frac{1}{2}\left[\ln(1+\rho_2)-\ln(1-\rho_2)\right] = \xi+c_1 \tag{2.96}$$

$$\ln\left[\sqrt{\frac{1+\rho_2}{1-\rho_2}}\right] = \xi+c_1 \tag{2.97}$$

$$\sqrt{\frac{1+\rho_2}{1-\rho_2}} = e^{\xi+c_1} \tag{2.98}$$

$$\frac{1+\rho_2}{1-\rho_2} = e^{2\xi+2c_1} \tag{2.99}$$

Setting $\rho_2(0)=0$ then forces $c_1=0$, and so one may rewrite the solution ρ_2 as follows:

$$1+\rho_2 = e^{2\xi}-\rho_2 e^{2\xi} \tag{2.100}$$

$$\rho_2(e^{2\xi}+1) = e^{2\xi}-1 \tag{2.101}$$

$$\rho_2(\xi) = \frac{e^{\xi}-e^{-\xi}}{e^{\xi}+e^{-\xi}}=\tanh(\xi) \tag{2.102}$$

and, subsequently, from the conservation law (equation 2.88) find

$$\rho_1(\xi)=\text{sech }(\xi).$$

The next step is to analyze the case where $c\neq 0$ in equation (2.92). First introduce a change of variables, writing

$$w^2=\frac{\rho_2^2-\sigma_a^2}{\sigma_b^2-\sigma_a^2} \qquad \text{and} \qquad \gamma^2=\frac{\sigma_b^2-\sigma_a^2}{\sigma_c^2-\sigma_a^2}.$$

This allows reformulation of the elliptic integral as

$$\xi = \frac{\pm 1}{(\sigma_c^2 - \sigma_a^2)^{1/2}} \int_{w(0)}^{w(\xi)} \frac{dw}{\left[(1-w^2)(1-\gamma^2 w^2)\right]^{1/2}}$$

(see Appendix B for this calculation). Manipulation of this expression will eventually yield a formulation for ρ_2^2:

$$\begin{aligned}
(\sigma_c^2 - \sigma_a^2)^{1/2}\xi &= \int_{w(0)}^{w(\xi)} \frac{dw}{\left[(1-w^2)(1-\gamma^2 w^2)\right]^{1/2}} \\
(\sigma_c^2 - \sigma_a^2)^{1/2}\xi &= \int_{w(0)}^{w(\xi)} \frac{dw}{\left[(1-w^2)(1-\gamma^2 w^2)\right]^{1/2}} \\
&\quad + \int_{0}^{w(0)} \frac{dw}{\left[(1-w^2)(1-\gamma^2 w^2)\right]^{1/2}} \\
&\quad - \int_{0}^{w(0)} \frac{dw}{\left[(1-w^2)(1-\gamma^2 w^2)\right]^{1/2}} \\
(\sigma_c^2 - \sigma_a^2)\xi &= \int_{0}^{w(\xi)} \frac{dw}{\left[(1-w^2)(1-\gamma^2 w^2)\right]^{1/2}} \\
&\quad - \int_{0}^{w(0)} \frac{dw}{\left[(1-w^2)(1-\gamma^2 w^2)\right]^{1/2}}.
\end{aligned}$$

Setting $K = (\sigma_c^2 - \sigma_a^2)^{1/2}\xi_0$, compute the solution by taking the inverse:

$$\begin{aligned}
(\sigma_c^2 - \sigma_a^2)^{1/2}\xi + K &= \mathrm{sn}^{-1}(w, \gamma) \\
\mathrm{sn}((\sigma_c^2 - \sigma_a^2)^{1/2}(\xi + \xi_0), \gamma) &= w \\
&= \sqrt{\frac{\rho_2^2 - \sigma_a^2}{\sigma_b^2 - \sigma_a^2}}
\end{aligned}$$

so that

$$\rho_2^2 - \sigma_a^2 = (\sigma_b^2 - \sigma_a^2)\mathrm{sn}^2\left[(\sigma_c^2 - \sigma_a^2)^{1/2}(\xi + \xi_0), \gamma\right]$$

or

$$\rho_2^2 = \sigma_a^2 + (\sigma_b^2 - \sigma_a^2)\mathrm{sn}^2\left[(\sigma_c^2 - \sigma_a^2)^{1/2}(\xi + \xi_0), \gamma\right].$$

By the conservation of energy expression 2.88, one obtains

$$\rho_1^2 = 1 - \rho_2^2 = 1 - \sigma_a^2 - (\sigma_b^2 - \sigma_a^2)\mathrm{sn}^2\left[(\sigma_c^2 - \sigma_a^2)^{1/2}(\xi + \xi_0), \gamma\right].$$

2.7 The Lau-Barcilon Equations for Infinitely Many Modes

Begin with the $O(\alpha)$ equation obtained by setting equal to zero the coefficient of α in equation (2.24):

$$u_X + \frac{3}{2}uu_x - \frac{\beta}{3}u_{xXt} = 0.$$

Take

$$u(x, X, t) = \sum_{j=1}^{\infty} u_j(X)e^{i(k_j x - \omega_j t)} + u_j^*(X)e^{-i(k_j x - \omega_j t)}$$

and compute the relevant terms. Again using the notation $E_j = e^{i(k_j x - \omega_j t)}$,

$$\begin{aligned} u_X &= \sum_j u_j' E_j + u_j^* E_j^* && (2.103) \\ u_{xXt} &= \sum_j k_j\omega_j u_j' E_j + k_j\omega_j u_j^{*\prime} E_j^* && (2.104) \\ uu_x &= \sum_{m,n}(ik_n u_m u_n E_m E_n - ik_n u_m^* u_n^* E_m^* E_n^* \\ &\qquad + ik_n u_m^* u_n E_m^* E_n - ik_n u_m u_n^* E_m E_n^*) && (2.105) \end{aligned}$$

The latter expression may be simplified, since E_n:

$$\begin{aligned} E_m E_n &= e^{i(k_m + k_n)x - (\omega_m + \omega_n)t} e^{i(k_{m+n} - k_{m+n})x} \\ &= e^{i(k_m + k_n - k_{m+n})x} E_{m+n} \end{aligned}$$

where $\omega_m + \omega_n = m\omega_1 + n\omega_1 = (m+n)\omega_1 = \omega_{m+n}$. Thus, the conjugate equation,

$$E_m^* E_n^* = e^{-i(k_m + k_n - k_{m+n})x} E_{m+n}^*$$

holds.

Similarly, compute $E_m^* E_n = e^{i((k_n-k_m)x-(\omega_n-\omega_m)t)}$ but $\omega_n - \omega_m = (n - m)\omega_1$. So, examine the following cases

$$\text{if } n > m: \qquad E_m^* E_n = e^{i(k_n-k_m-k_{n-m})x} E_{n-m} \tag{2.106}$$

$$\text{if } n < m: \qquad E_m^* E_n = e^{i(k_n-k_m+k_{m-n})x} E_{m-m}^* \tag{2.107}$$

$$\text{if } n = m: \qquad E_m^* E_n = 1 \tag{2.108}$$

The nonlinear term may be written as

$$\begin{aligned} uu_x = \quad & \sum_{m,n} ik_n u_m u_n e^{i(k_m+k_n-k_{m+n})x} E_{m+n} - ik_n u_m^* u_n^* e^{-i(k_m+k_n-k_{m+n})x} E_{m+n}^* \\ & + \begin{cases} ik_n u_m^* u_n e^{i(k_n-k_m-k_{n-m})x} E_{n-m} & \text{if } n > m \\ ik_n u_m^* u_n e^{i(k_n-k_m+k_{m-n})x} E_{m-n}^* & \text{if } n < m \end{cases} \\ & - \begin{cases} ik_n u_m u_n^* e^{-i(k_n-k_m-k_{n-m})x} E_{n-m}^* & \text{if } n > m \\ ik_n u_m u_n^* e^{-i(k_n-k_m+k_{m-n})x} E_{m-n} & \text{if } n < m \end{cases} \end{aligned}$$

Notice that if $n = m$ the two cases cancel each other. Closer examination of these two cases shows that if $n > m$ then

$$ik_n u_m^* u_n e^{i(k_n-k_m-k_{n-m})x} E_{n-m} - ik_n u_m u_n^* e^{-i(k_n-k_m-k_{n-m})x} E_{n-m}^*$$

while if $n < m$ then

$$ik_n u_m^* u_n e^{i(k_n-k_m+k_{m-n})x} E_{m-n}^* - ik_n u_m u_n^* e^{-i(k_n-k_m+k_{m-n})x} E_{m-n}.$$

Then notice that

$$E_j^* = e^{-i(k_j x-\omega_j t)} = e^{i(-k_j x+\omega_j t)} = e^{i(k_{-j} x-\omega_{-j} t)} = E_{-j}$$

Hence the expressions from the two cases above are identical: the same form may be used independently of the sign of $m - n$. It is also necessary to

take $u_{-j} = u_j^*$, as will be seen later in comparing this derivation with hand computations of the 2 modes case and the 3 modes case.

However, the large expression for uu_x is still not in the form needed in order to combine it with the series terms from u_X and u_{xXt}. Notice that this is simply the Fourier transform of a product, hence a convolution; thus the coefficient of E_j has the form

$$\left(\sum_{m=1}^{\infty} ik_{j-m} u_m u_{j-m} e^{i(k_m + k_{j-m} - k_j)x} + ik_{j+m} u_m^* u_{j+m} e^{i(k_{j+m} - k_m - k_j)x} \right).$$

The series expressions for uu_x and $u_X - \frac{\beta}{3} u_{xXt}$ may be substituted into the $O(\alpha)$ equation,

$$u_X + \frac{3}{2} uu_x - \frac{\beta}{3} u_{xXt} = 0.$$

Setting the coefficient of each mode (that is, of each E_j) equal to zero gives the Lau-Barcilon equations, indexed by $j = 1 \ldots \infty$:

$$\begin{aligned} &(1 - \frac{\beta}{3} k_j \omega_j) u_j' \\ &= -\frac{3}{2} \sum_{m=1}^{\infty} ik_{j-m} u_m u_{j-m} e^{i(k_m + k_{j-m} - k_j)x} + ik_{j+m} u_m^* u_{j+m} e^{i(k_{j+m} - k_m - k_j)x} \end{aligned}$$

However, differentiation in these equations is with respect to X, with $u_j = u_j(X)$. Change variables so that these are all ordinary differential equations with respect to the independent variable x to reach the expression

$$\begin{aligned} &(1 - \frac{\beta}{3} k_j \omega_j) u_j' \\ &= -\frac{3}{2} \alpha \sum_{m=1}^{\infty} ik_{j-m} u_m u_{j-m} e^{i(k_m + k_{j-m} - k_j)x} + ik_{j+m} u_m^* u_{j+m} e^{i(k_{j+m} - k_m - k_j)x} \end{aligned}$$

where $j = 1, 2, \ldots$.

A quick check of this computation can be carried out by comparing with the 2 modes case computed earlier. To do so, take $j = 1, 2$ so that $m = 1, 2$ and $u_j = 0$ for $j > 2$. The first equation comes from the $j = 1$ case:

$$\begin{aligned} 0 &= (1 - \frac{\beta}{3}k_1\omega_1)u_1' \\ &\quad + \frac{3}{2}\alpha\left(0 + ik_2u_1^*u_2e^{i(k_2-k_1-k_1)x} + ik_{-1}u_2u_{-1}e^{i(k_2+k_{-1}-k_1)x}\right) \\ &= (1 - \frac{\beta}{3}k_1\omega_1)u_1' + \frac{3}{2}i(k_2 - k_1)u_1^*u_2e^{i(k_2-2k_1)x} \end{aligned}$$

which gives

$$u_1' = -i\frac{3}{2}\frac{k_2 - k_1}{1 - \frac{\beta}{3}k_1\omega_1}e^{i\Delta_k x}u_1^*u_2,$$

the first Lau-Barcilon equation. Similarly, taking $j = 2$ gives

$$\begin{aligned} 0 &= (1 - \frac{\beta}{3}k_2\omega_2)u_2' + \frac{3}{2}ik_1u_1u_1e^{i(k_1+k_1-k_2)x} \\ &= (1 - \frac{\beta}{3}k_2\omega_2)u_2' + \frac{3}{2}ik_1u_1^2e^{-i(k_2-2k_1)x} \end{aligned}$$

which gives the second Lau-Barcilon equation,

$$u_2' = -i\frac{3}{2}\frac{k_1}{1 - \frac{\beta}{3}k_2\omega_2}u_1^2e^{-i\Delta_k x}.$$

While the process of determining the mathematical form taken by the ordinary differential equations in the case of infinite modes has been informative, it is actually true that, for the model at hand, only a finite number of modes is necessary. In fact, to preserve the dispersion relation only a finite number of modes is even permitted! Consider the function

$$\omega(k) = \frac{k}{1 + \frac{\beta}{6}k^2}$$

where $\beta = (1/12)^2$. Notice that by a freshman calculus argument we may compute the maximum value of ω:

$$\omega'(k) = \frac{1 - \frac{\beta}{6}k^2}{\left(1 + \frac{\beta}{6}k^2\right)^2} = 0 \qquad \text{when} \qquad k = \frac{\sqrt{6}}{\beta} = 12\sqrt{6}$$

and $\omega(12\sqrt{6}) = 6\sqrt{6} < 15$. The graph of ω was shown in Section 4, in context of the first derivation of the Lau-Barcilon equations.

Now let $\omega_1 =$ constant and recall that $\omega_j = j\omega_1$ for positive integers j. Once $j > \frac{15}{\omega_1}$, there exists no value k_j for which the linearized dispersion relation

$$\omega_j = \frac{k_j}{1 + \frac{\beta}{6}k_j^2}$$

holds. In fact, $j > 6\sqrt{6}/\omega_1$ implies that

$$\omega_j = j\omega_1 > 6\sqrt{6} > \max_{k \in \mathbb{R}^+}[\omega(k)].$$

Thus, after a finite number of terms, the modes in the expansion

$$\sum_{j=-\infty}^{\infty} a_j(X)e^{i(k_j x - \omega_j t)}$$

must fail to satisfy the linearized dispersion relation.

For the typical value of $\beta = (1/12)^2$, as we saw earlier $\omega_1 \approx 2\pi\beta = 0.5236$. Thus, if

$$j > 6\sqrt{6}/0.5236 = 28.0690,$$

the dispersion relation between ω_j and k_j fails. This forces the requirement that j is no greater than 28, and hence the number of modes permitted is no greater that 28.

The infinite modal expansion is not useless, however! The general form for the jth mode may be used to write down easily the Lau-Barcilon system for these modes as verified above for the $N = 2$ modes case. For example, here is the $N = 4$ modes system:

$$\begin{aligned}
u_1' &= -i\alpha\left[Q(1)e^{i\Delta_k x}u_1^* u_2 + S(1)e^{-ik_H x}u_2^* u_3 + T(1)e^{ik_G x}u_3^* u_4\right] \\
u_2' &= -i\alpha\left[Q(2)e^{-i\Delta_k x}u_1^2 + S(2)e^{-ik_H x}u_1^* u_3 + T(2)e^{i(k_4-2k_2)x}u_2^* u_4\right] \\
u_3' &= -i\alpha\left[S(3)e^{ik_H x}u_1 u_2 + T(3)e^{ik_G x}u_1^* u_4\right] \\
u_4' &= -i\alpha\left[T(4)e^{-ik_G x}u_1 u_3 + T(5)e^{-i(k_4-2k_2)x}u_2^2\right]
\end{aligned}$$

Similarly, here is the $N = 5$ modes system:

$$\begin{aligned}
u_1' &= -i\alpha[Q(1)e^{i\Delta_k x}u_1^* u_2 + S(1)e^{-ik_H x}u_2^* u_3 + T(1)e^{ik_G x}u_3^* u_4 \\
&\qquad +P(1)e^{ik_F x}u_4^* u_5] \\
u_2' &= -i\alpha[Q(2)e^{-i\Delta_k x}u_1^2 + S(2)e^{-ik_H x}u_1^* u_3 + T(2)e^{i(k_4-2k_2)x}u_2^* u_4 \\
&\qquad +P(2)e^{ik_F x}u_3^* u_5] \\
u_3' &= -i\alpha[S(3)e^{ik_H x}u_1 u_2 + T(3)e^{ik_G x}u_1^* u_4 + P(3)e^{ik_E x}u_2^* u_5] \\
u_4' &= -i\alpha[T(4)e^{-ik_G x}u_1 u_3 + T(5)e^{-i(k_4-2k_2)x}u_2^2 + P(4)e^{ik_F x}u_1^* u_5] \\
u_5' &= -i\alpha[P(5)e^{-ik_F x}u_1 u_4 + P(6)e^{-ik_E x}u_2 u_3]
\end{aligned}$$

The coefficients of these systems are given by

$$Q(1) = \frac{3}{2}\frac{k_2 - k_1}{(1 - \frac{\beta}{3}k_1\omega_1)} \qquad \text{and} \qquad Q(2) = \frac{3}{2}\frac{k_1}{(1 - \frac{\beta}{3}k_2\omega_2)}$$

and, similarly,

$$S(1) = \frac{3}{2}\frac{k_3 - k_2}{(1 - \frac{\beta}{3}k_1\omega_1)}; \quad S(2) = \frac{3}{2}\frac{k_3 - k_1}{(1 - \frac{\beta}{3}k_2\omega_2)}; \quad S(3) = \frac{3}{2}\frac{k_1 + k_2}{(1 - \frac{\beta}{3}k_3\omega_3)}$$

and, also,

$$T(1) = \frac{3}{2}\frac{k_4 - k_3}{(1 - \frac{\beta}{3}k_1\omega_1)}; \quad T(2) = \frac{3}{2}\frac{k_4 - k_2}{(1 - \frac{\beta}{3}k_2\omega_2)}; \quad T(3) = \frac{3}{2}\frac{k_4 - k_1}{(1 - \frac{\beta}{3}k_3\omega_3)};$$

$$T(4) = \frac{3}{2}\frac{k_1 + k_3}{(1 - \frac{\beta}{3}k_4\omega_4)}; \qquad T(5) = \frac{3}{2}\frac{k_2}{(1 - \frac{\beta}{3}k_4\omega_4)}$$

and, finally,

$$P(1) = \frac{3}{2}\frac{k_5 - k_4}{(1 - \frac{\beta}{3}k_1\omega_1)}; \quad P(2) = \frac{3}{2}\frac{k_5 - k_3}{(1 - \frac{\beta}{3}k_2\omega_2)}; \quad P(3) = \frac{3}{2}\frac{k_5 - k_2}{(1 - \frac{\beta}{3}k_3\omega_3)};$$

$$P(4) = \frac{3}{2}\frac{k_5 - k_1}{(1 - \frac{\beta}{3}k_4\omega_4)}; \quad P(5) = \frac{3}{2}\frac{k_1 + k_4}{(1 - \frac{\beta}{3}k_5\omega_5)}; \quad P(6) = \frac{3}{2}\frac{k_2 + k_3}{(1 - \frac{\beta}{3}k_5\omega_5)}.$$

Perhaps as was only to be expected, the theory derived in the previous sections does not hold for these higher mode solutions. In particular, conservation of energy in the form

$$\sum_j c_j \cdot r_j r_j' = 0 \tag{2.109}$$

(where the c_j are constants involving the Lau-Barcilon coefficients) does not hold in the $N = 4$ and higher mode cases. The next subsection sketches the argument supporting this claim.

2.7.1 Sketch of argument

To demonstrate that a conservation law of the form (2.109) above does not hold, begin by following the procedure used to derive the conservation law in

the $N = 2$ and $N = 3$ modes cases. That is, write the equations in the 4 modes system in real and imaginary parts, by setting

$$u_j(x) = r_j(x)e^{i\theta_j(x)}.$$

The result is a system of the form

$$\begin{aligned}
r_1' &= \alpha Q_1 r_1 r_2 \sin A + \alpha S_1 r_2 r_3 \sin B + \alpha T_1 r_3 r_4 \sin C \\
r_2' &= -\alpha Q_2 r_1^2 \sin A + \alpha S_2 r_1 r_3 \sin B + \alpha T_2 r_2 r_4 \sin D \\
r_3' &= -\alpha S_3 r_1 r_2 \sin B + \alpha T_3 r_1 r_4 \sin C \\
r_4' &= -\alpha T_4 r_1 r_3 \sin C - \alpha T_5 r_2^2 \sin D
\end{aligned}$$

where A, B, C, D are quantities depending on the θ_j terms and the detuning-type parameters $\Delta_k x$, $k_H x$, and so on. Forming the system of equations for $r_j r_j'$ gives the array

$$\begin{aligned}
r_1 r_1' &= Q_1 \mathcal{A} + S_1 \mathcal{B} + T_1 \mathcal{C} \\
r_2 r_2' &= -Q_2 \mathcal{A} + S_2 \mathcal{B} + T_2 \mathcal{D} \\
r_3 r_3' &= -S_3 \mathcal{B} + T_3 \mathcal{C} \\
r_4 r_4' &= -T_4 \mathcal{C} + T_5 \mathcal{D}
\end{aligned}$$

where the calligraphic quantities (such as $\mathcal{A}$) represent terms of the form $(r_i r_j r_k/\alpha)\sin A$. The goal, then, is to show that there do not exist nonzero quantities c_j such that

$$\sum_{j=1}^{4} c_j r_j r_j' = 0 \tag{2.110}$$

Equation (2.110) may be written in the form

$$0 = c_1 r_1 r_1' + c_2 r_2 r_2' + c_3 r_3 r_3' + c_4 r_4 r_4'$$

$$
\begin{aligned}
&= \mathcal{A} \cdot (c_1 Q_1 - c_2 Q_2) + \mathcal{B} \cdot (c_1 S_1 + c_2 S_2 - c_3 S_3) \\
&\qquad + \mathcal{C} \cdot (c_1 T_1 + c_3 T_3 - c_4 T_4) + \mathcal{D} \cdot (c_2 T_2 - c_5 T_5) \qquad (2.111)
\end{aligned}
$$

Setting each coefficient of the calligraphed quantities equal to zero gives the system of equations

$$
\begin{aligned}
c_1 Q_1 - c_2 Q_2 &= 0 \\
c_1 S_1 + c_2 S_2 - c_3 S_3 &= 0 \\
c_1 T_1 + c_3 T_3 - c_4 T_4 &= 0 \\
c_2 T_2 - c_5 T_5 &= 0
\end{aligned}
$$

which may be written as a single matrix equation,

$$
\begin{pmatrix} Q_1 & -Q_2 & 0 & 0 \\ S_1 & S_2 & -S_3 & 0 \\ T_1 & 0 & T_3 & -T_4 \\ 0 & T_2 & 0 & -T_5 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad (2.112)
$$

Alas for the goal of finding a conservation law in the form of (2.109) above, this matrix in equation (2.112) has nonzero determinant, indicating that the only solution to the equation is $c_j = 0$ for $j = 1 \ldots 4$. Of course, this result does not preclude the existence of a conservation law in another form, perhaps one including cross terms of the form $r_j r_k$ where $j \neq k$.

An important question is, how good an approximation is the modal expansion for the lower modes?

2.8 Error Calculations

The goal here is to apply the BBM operator to the approximate solution to the BBM equation which was obtained via the Lau-Barcilon equations. The

BBM operator, after the two-scale expansion, has the following form:

$$\begin{aligned} BBM &= \partial_t + (\partial_x + \alpha\partial_X) + \frac{3}{2}\alpha(\cdot, \partial_x + \alpha\partial_X) - \frac{\beta}{6}(\partial_x + \alpha\partial_X)^2\partial_t \\ &= \partial_t + \partial_x + \alpha\partial_X + \frac{3}{2}\alpha(\cdot, \partial_x) + \frac{3}{2}\alpha^2(\cdot, \partial_X) - \frac{\beta}{6}\partial_x^2\partial_t \\ &\quad -\frac{\beta}{6}\alpha^2\partial_X^2\partial_t - \frac{\beta}{3}\alpha\partial_x\partial_X\partial_t. \end{aligned}$$

Apply this operator to the N modes solution

$$u(x, X, t) = \sum_{j=-N}^{N} u_j(X)e^{i(k_j x - \omega_j t)}$$

where the $u_j(X)$ satisfy the Lau-Barcilon equations.

Carrying out the computations as in the derivation of the Lau-Barcilon equations yields the following expressions:

$$u_t = \sum_{j=-N}^{N} -i\omega_j u_j E_j \tag{2.113}$$

$$u_x = \sum_{j=-N}^{N} ik_j u_j E_j \tag{2.114}$$

$$\alpha u_X = \sum_{j=-N}^{N} \alpha u_j' E_j \tag{2.115}$$

$$-\frac{\beta}{6}u_{xxt} = \sum_{j=-N}^{N} -\frac{\beta}{6}i\omega_j k_j^2 u_j E_j \tag{2.116}$$

$$-\frac{\beta}{6}\alpha^2 u_{XXt} = \sum_{j=-N}^{N} \frac{\beta}{6}\alpha^2 i\omega_j u_j'' E_j \tag{2.117}$$

$$-\frac{\beta}{3}\alpha u_{xXt} = \sum_{j=-N}^{N} -\frac{\beta}{3}\alpha k_j\omega_j u_j' E_j. \tag{2.118}$$

The portions of the nonlinear terms retained depend on the number of modes used to characterize the approximate solution. In the case of two

modes, the Lau-Barcilon equations are

$$u_1'(X) = -iQ_1 u_1^* u_2 e^{i\Delta_k x} \tag{2.119}$$

$$u_2'(X) = -iQ_2 u_1^2 e^{-i\Delta_k x} \tag{2.120}$$

where

$$Q_1 = \frac{3}{2}\frac{k_2 - k_1}{1 - \frac{\beta}{3}k_1\omega_1} \qquad \text{and} \qquad Q_2 = \frac{3}{2}\frac{k_1}{1 - \frac{\beta}{3}k_2\omega_2}.$$

The first nonlinear term becomes

$$\begin{aligned}
\frac{3}{2}\alpha u u_x = \; & E_1 \left[-i\frac{3\alpha}{2}(k_1 - k_2)u_1^* u_2 e^{i\Delta_k x}\right] + E_1^* \left[i\frac{3\alpha}{2}(k_1 - k_2)u_1 u_2^* e^{-i\Delta_k x}\right] \\
+ \; & E_2 \left[i\frac{3\alpha}{2}k_1 u_1^2 e^{-i\Delta_k x}\right] + E_2^* \left[-i\frac{3\alpha}{2}k_1 u_1^{*2} e^{i\Delta_k x}\right] \\
+ \; & E_3 \left[i\frac{3\alpha}{2}(k_1 + k_2)u_1 u_2 e^{i(k_1+k_2-k_3)x}\right] \\
+ \; & E_3^* \left[-i\frac{3\alpha}{2}(k_1 + k_2)u_1^* u_2^* e^{i(k_1+k_2-k_3)x}\right] \\
+ \; & E_4 \left[-i\frac{3\alpha}{2}k_2 u_2^{*2} e^{-i(k_4-2k_2)x}\right] + E_4^* \left[i\frac{3\alpha}{2}k_2 u_2^2 e^{i(k_4-2k_2)x}\right]
\end{aligned}$$

and the second nonlinear term becomes

$$\begin{aligned}
\frac{3\alpha^2}{2}u u_X = \; & E_1 \left[\frac{3\alpha^2}{2}(u_1^* u_2)' e^{i\Delta_k x}\right] + E_1^* \left[\frac{3\alpha^2}{2}(u_1 u_2^*)' e^{-i\Delta_k x}\right] \\
+ \; & E_2 \left[\frac{3\alpha^2}{2}\left(\frac{1}{2}u_1^2\right)' e^{-i\Delta_k x}\right] + E_2^* \left[\frac{3\alpha^2}{2}\left(\frac{1}{2}u_1^{*2}\right)' e^{i\Delta_k x}\right] \\
+ \; & E_3 \left[\frac{3\alpha^2}{2}(u_1 u_2)' e^{i(k_1+k_2-k_3)x}\right] + E_3^* \left[\frac{3\alpha^2}{2}(u_1^* u_2^*)' e^{-i(k_1+k_2-k_3)x}\right] \\
+ \; & E_4 \left[\frac{3\alpha^2}{2}\left(\frac{1}{2}u_2^{*2}\right)' e^{-i(k_4-2k_2)x}\right] + E_4^* \left[\frac{3\alpha^2}{2}\left(\frac{1}{2}u_2^2\right)' e^{i(k_4-2k_2)x}\right].
\end{aligned}$$

Again, as in the derivation of the Lau-Barcilon equations, these expressions have used the fact that E_j^2, E_1E_2, $E_1E_2^*$ et cetera may be written in terms of E_1, E_2, E_3, E_4. The next step is to put all these pieces together

and determine the quantity $BBM(u)$. To simplify the final result, use the first order dispersion relation, $k_j - \omega_j - \frac{\beta}{6}\omega_j k_j^2 = 0$, and use the Lau-Barcilon equations to eliminate the first order derivatives. The result is

$$\begin{aligned}
BBM(u) &= E_1\alpha^2\left[\frac{3}{2}e^{i\Delta_k x}(u_1^* u_2)' + i\frac{\beta}{6}\omega_1 u_1''\right] \\
&+ E_2\alpha^2\left[\frac{3}{2}e^{-i\Delta_k x}\left(\frac{1}{2}u_1^2\right)' + i\frac{\beta}{6}\omega_2 u_2''\right] \\
&+ E_3\left(\frac{3}{2}\alpha e^{i(k_1+k_2-k_3)x}\right)\left[i(k_1+k_2)u_1u_2 + \alpha(u_1u_2)'\right] \\
&+ E_4\left(\frac{3}{2}\alpha e^{-i(k_4-2k_2)x}\right)\left[-ik_2u_2^{*2} + \alpha\left(\frac{1}{2}u_2^{*2}\right)'\right] \\
&+ \text{conj.}
\end{aligned}$$

At this point, change variables to eliminate the long spatial scale X. This provides the appropriate error estimate:

$$\begin{aligned}
BBM(u) &= E_1\cdot\alpha^2\left[\frac{3}{2}\alpha e^{i\Delta_k x}(u_1^* u_2)' + i\frac{\beta}{6}\omega_1\alpha^2 u_1''\right] \\
&+ E_2\cdot\alpha^2\left[\frac{3}{2}e^{-i\Delta_k x}\alpha\left(\frac{1}{2}u_1^2\right)' + i\frac{\beta}{6}\omega_2\alpha^2 u_2''\right] \\
&+ E_3\cdot\left(\frac{3}{2}\alpha e^{i(k_1+k_2-k_3)x}\right)\left[i(k_1+k_2)u_1u_2 + \alpha^2(u_1u_2)'\right] \\
&+ E_4\cdot\left(\frac{3}{2}\alpha e^{-i(k_4-2k_2)x}\right)\left[-ik_2u_2^{*2} + \alpha^2\left(\frac{1}{2}u_2^{*2}\right)'\right] \\
&+ \text{conj} \\
&= E_1\cdot\alpha^3\left[\frac{3}{2}e^{i\Delta_k x}(u_1^* u_2)' + i\frac{\beta}{6}\omega_1\alpha u_1''\right] \\
&+ E_2\cdot\alpha^3\left[\frac{3}{2}e^{-i\Delta_k x}\left(\frac{1}{2}u_1^2\right)' + i\frac{\beta}{6}\omega_2\alpha u_2''\right] \\
&+ E_3\cdot\alpha\left(\frac{3}{2}e^{i(k_1+k_2-k_3)x}\right)\left[i(k_1+k_2)u_1u_2 + \alpha^2(u_1u_2)'\right] \\
&+ E_4\cdot\alpha\left(\frac{3}{2}e^{-i(k_4-2k_2)x}\right)\left[-ik_2u_2^{*2} + \alpha^2\left(\frac{1}{2}u_2^{*2}\right)'\right]
\end{aligned}$$

$$+ \quad \text{conj}$$

It is interesting to notice that the error in the first two modes is of order α^3, while in the third and fourth modes the error terms are of order α.

The next step is to carry out the computation for the three modes case and to determine a general form for larger numbers of modes.

For three modes, apply the two-scale BBM operator to the approximate solution

$$u(x, X, t) = \sum_{j=-3}^{3} u_j(X) E_j$$

where

$$u_1' = -i\alpha Q_1 e^{i\Delta_k x} u_1^* u_2 - i\alpha R_1 e^{-ik_H x} u_2^* u_3 \tag{2.121}$$

$$u_2' = -\alpha Q_2 e^{-i\Delta_k x} u_1^2 - i\alpha R_2 e^{-ik_H x} u_1^* u_3 \tag{2.122}$$

$$u_3' = -i\alpha R_3 e^{ik_H x} u_1 u_2 \tag{2.123}$$

and the constants are given by

$$R_1 = \frac{k_3 - k_2}{1 - \frac{\beta}{3} k_1 \omega_1} \tag{2.124}$$

$$R_2 = \frac{k_3 - k_1}{1 - \frac{\beta}{3} k_2 \omega_2} \tag{2.125}$$

$$R_3 = \frac{k_1 + k_2}{1 - \frac{\beta}{3} k_3 \omega_3} \tag{2.126}$$

$$Q_1 = \frac{k_2 - k_1}{1 - \frac{\beta}{3} k_1 \omega_1} \tag{2.127}$$

$$Q_2 = \frac{k_1}{1 - \frac{\beta}{3} k_2 \omega_2}. \tag{2.128}$$

The next step is to compute all the pieces required to put together $BBM(u)$. The linear terms are as in the two modes case.

After some computation and simplification, in which terms of the form $E_m E_n$ are re-written in the form E_j (exactly as described in the derivation of the Lau-Barcilon equations for the case of infinitely many modes), the nonlinear terms have the following forms:

$$\begin{aligned}
\frac{3}{2}\alpha u u_x &= E_1\left[-\frac{3}{2}\alpha i(k_1-k_2)u_1^* u_2 e^{i\Delta_k x} - \frac{3}{2}\alpha i(k_2-k_3)u_2^* u_3 e^{-i(k_1+k_2-k_3)x}\right] \\
&+ E_2\left[\frac{3}{2}\alpha i k_1 u_2^2 e^{i\Delta_k x} - \frac{3}{2}\alpha i(k_1-k_3)u_1^* u_3 e^{-i(k_1+k_2-k_3)x}\right] \\
&+ E_3\left[\frac{3}{2}\alpha i(k_1+k_2)u_1 u_2 e^{i(k_1+k_2-k_3)x}\right] \\
&+ E_4\left[\frac{3}{2}\alpha i k_2 u_2^2 e^{i(2k_2-k_4)x} + \frac{3}{2}\alpha i(k_1+k_3)u_1 u_3 e^{i(k_1+k_3-k_4)x}\right] \\
&+ E_5\left[\frac{3}{2}\alpha i(k_2+k_3)u_2 u_3 e^{i(k_2+k_3-k_5)x}\right] \\
&+ E_6\left[\frac{3}{2}\alpha i k_3 u_3^2 e^{i(2k_3-k_6)x}\right] \\
&+ \text{conj}
\end{aligned}$$

and

$$\begin{aligned}
\frac{3}{2}\alpha^2 u u_X &= E_1\left[\frac{3}{2}\alpha^2(u_1^* u_2)' e^{i\Delta_k x} + \frac{3}{2}\alpha^2(u_2^* u_3)' e^{-i(k_1+k_2-k_3)x}\right] \\
&+ E_2\left[\frac{3}{2}\alpha^2(\frac{1}{2}u_1^2)' e^{-i\Delta_k x} + \frac{3}{2}\alpha^2(u_1^* u_3)' e^{-i(k_1+k_2-k_3)x}\right] \\
&+ E_3\left[\frac{3}{2}\alpha^2(u_1 u_2)' e^{i(k_1+k_2-k_3)x}\right] \\
&+ E_4\left[\frac{3}{2}\alpha^2(\frac{1}{2}u_2^2)' e^{i(2k_2-k_4)x} + \frac{3}{2}\alpha^2(u_1 u_3)' e^{i(k_1+k_3-k_4)x}\right] \\
&+ E_5\left[\frac{3}{2}\alpha^2(u_2 u_3)' e^{i(k_2+k_3-k_5)x}\right] \\
&+ E_6\left[\frac{3}{2}\alpha^2(\frac{1}{2}u_1^{*2})' e^{i(2k_3-k_6)x}\right] \\
&+ \text{conj.}
\end{aligned}$$

The next step is to put all these pieces together, sorting them by mode E_j. During this process, cancel terms which correspond to the first order dispersion relation,

$$-i\omega_j u_j + ik_j u_j - \frac{\beta}{6} i\omega_j k_j^2 u_j = 0,$$

and to the Lau-Barcilon equations for u_j'. What remains is the following relation:

$$\begin{aligned}
BBM(u) &= E_1 \cdot \alpha^2 \left[\frac{\beta}{6} i\omega_1 u_1'' + \frac{3}{2}(u_1^* u_2)' e^{i\Delta_k x} + \frac{3}{2}(u_2^* u_3)' e^{-i(k_1+k_2-k_3)x} \right] \\
&+ E_2 \cdot \alpha^2 \left[\frac{\beta}{6} i\omega_2 u_2'' + \frac{3}{2}\left(\frac{1}{2}u_1^2\right)' e^{-i\Delta_k x} + \frac{3}{2}(u_1^* u_3)' e^{-i(k_1+k_2-k_3)x} \right] \\
&+ E_3 \cdot \alpha^2 \left[\frac{\beta}{6} i\omega_3 u_3'' + \frac{3}{2}(u_1 u_2)' e^{i(k_1+k_2-k_3)x} \right] \\
&+ E_4 \Bigg[\left(\alpha i k_2 u_2^2 + \alpha^2 (\frac{1}{2}u_2^2)' \right) \frac{3}{2} e^{i(2k_2-k_4)x} \\
&+ \qquad \left(\alpha i (k_1+k_3) u_1 u_3 + \alpha^2 (u_1 u_3)' \right) \frac{3}{2} e^{i(k_1+k_3-k_4)x} \Bigg] \\
&+ E_5 \left[\alpha i (k_2+k_3) u_2 u_3 + \alpha^2 (u_2 u_3)' \right] \frac{3}{2} e^{i(k_2+k_3-k_5)x} \\
&+ E_6 \left[\alpha i k_3 u_3^2 + \alpha^2 (\frac{1}{2} u_1^{*2})' \right] \frac{3}{2} e^{i(2k_3-k_6)x} \\
&+ \text{conj.}
\end{aligned}$$

After changing variables from the long spatial X to x, the error terms have the form

$$\begin{aligned}
BBM(u) &= E_1 \cdot \alpha^2 \Bigg[\frac{\beta}{6} i\omega_1 \alpha^2 u_1'' + \frac{3}{2}\alpha (u_1^* u_2)' e^{i\Delta_k x} \\
&\qquad + \frac{3}{2}\alpha (u_2^* u_3)' e^{-i(k_1+k_2-k_3)x} \Bigg] \\
&+ E_2 \cdot \alpha^2 \Bigg[\frac{\beta}{6} i\omega_2 \alpha^2 u_2'' + \frac{3}{2}\alpha (\frac{1}{2}u_1^2)' e^{-i\Delta_k x}
\end{aligned}$$

$$
\begin{aligned}
& \quad + \frac{3}{2}\alpha(u_1^* u_3)' e^{-i(k_1+k_2-k_3)x} \Big] \\
+ \quad & E_3 \cdot \alpha^2 \left[\frac{\beta}{6} i\omega_3 \alpha^2 u_3'' + \frac{3}{2}\alpha(u_1 u_2)' e^{i(k_1+k_2-k_3)x} \right] \\
+ \quad & E_4 \Bigg[\left(\alpha i k_2 u_2^2 + \alpha^3 (\frac{1}{2} u_2^2)' \right) \frac{3}{2} e^{i(2k_2-k_4)x} \\
+ \quad & \qquad \left(\alpha i (k_1+k_3) u_1 u_3 + \alpha^3 (u_1 u_3)' \right) \frac{3}{2} e^{i(k_1+k_3-k_4)x} \Bigg] \\
+ \quad & E_5 \left[\alpha i (k_2+k_3) u_2 u_3 + \alpha^3 (u_2 u_3)' \right] \frac{3}{2} e^{i(k_2+k_3-k_5)x} \\
+ \quad & E_6 \left[\alpha i k_3 u_3^2 + \alpha^3 (\frac{1}{2} u_1^{*2})' \right] \frac{3}{2} e^{i(2k_3-k_6)x} \\
+ \quad & \text{conj.}
\end{aligned}
$$

This may be re-written as

$$
\begin{aligned}
BBM(u) \quad = \quad & E_1 \cdot \alpha^3 \left[\frac{\beta}{6} i\omega_1 \alpha u_1'' + \frac{3}{2}(u_1^* u_2)' e^{i\Delta_k x} + \frac{3}{2}(u_2^* u_3)' e^{-i(k_1+k_2-k_3)x} \right] \\
+ \quad & E_2 \cdot \alpha^3 \left[\frac{\beta}{6} i\omega_2 \alpha u_2'' + \frac{3}{2}(\frac{1}{2} u_1^2)' e^{-i\Delta_k x} + \frac{3}{2}(u_1^* u_3)' e^{-i(k_1+k_2-k_3)x} \right] \\
+ \quad & E_3 \cdot \alpha^3 \left[\frac{\beta}{6} i\omega_3 \alpha u_3'' + \frac{3}{2}(u_1 u_2)' e^{i(k_1+k_2-k_3)x} \right] \\
+ \quad & E_4 \cdot \alpha \Bigg[\left(i k_2 u_2^2 + \alpha^2 (\frac{1}{2} u_2^2)' \right) \frac{3}{2} e^{i(2k_2-k_4)x} \\
+ \quad & \qquad \left(i (k_1+k_3) u_1 u_3 + \alpha^2 (u_1 u_3)' \right) \frac{3}{2} e^{i(k_1+k_3-k_4)x} \Bigg] \\
+ \quad & E_5 \cdot \alpha \left[i (k_2+k_3) u_2 u_3 + \alpha^2 (u_2 u_3)' \right] \frac{3}{2} e^{i(k_2+k_3-k_5)x} \\
+ \quad & E_6 \cdot \alpha \left[i k_3 u_3^2 + \alpha^2 (\frac{1}{2} u_1^{*2})' \right] \frac{3}{2} e^{i(2k_3-k_6)x} \\
+ \quad & \text{conj.}
\end{aligned} \tag{2.129}
$$

In summary,

$$
\begin{aligned}
E_j : \qquad & \alpha^3 \left[O(1) + O(\alpha) \right] \qquad \text{for } j = 1, 2, 3, \\
E_j : \qquad & \alpha \left[O(1) + O(\alpha^2) \right] \qquad \text{for } j = 4, 5, 6.
\end{aligned} \tag{2.130}
$$

So the approximate solution $u(x,t) = \sum_{j=-3}^{3} u_j(x)E_j$ provides a very good approximation of the behavior in the first three modes, with error order of α^3; but the approximation is good in the last three modes only if little energy is present in those modes. That is, not much energy may cascade into these higher modes from the dominant modes.

The obvious generalization of the error calculation to N modes is to obtain

$$BBM(u) = E_1 + \ldots + E_N + E_{N+1} \ldots E_{2N}$$

where the first N terms are $O(\alpha^3)$ from $\alpha^2 \cdot (u_j u_k)'$ and the last N terms are $O(\alpha)$ from $\alpha u u_x$ and $\alpha^2 u u_X$. For example, the two-scale BBM operator applied to the $N = 3$ modes $u(x, X, t)$ yielded terms organized as

$$\underbrace{\partial_t + \partial_x}_{\substack{\text{disp.}\\ \text{rel.}}} \underbrace{+\alpha\partial_X + \frac{3}{2}\alpha(\cdot, \partial_x)}_{\substack{E_{1,2,3} \to \text{LB}\\ E_{4,5,6} \to \text{error}}} \underbrace{+\frac{3}{2}\alpha^2(\cdot, \partial_X)}_{\text{error}} \underbrace{-\frac{\beta}{6}\partial_x^2\partial_t}_{\substack{\text{disp.}\\ \text{rel.}}} \underbrace{-\frac{\beta}{6}\alpha^2\partial_X^2\partial_t}_{\text{error}} \underbrace{-\frac{\beta^3}{3}\alpha\partial_x\partial_X\partial_t}_{\substack{E_{1,2,3} \to \text{LB}\\ E_{4,5,6} \to \text{error}}}$$

where 'LB' and 'disp. rel.' indicate use of the Lau-Barcilon equations and of the dispersion relation, respectively, to eliminate terms.

2.9 Numerical Analysis

2.9.1 Numerics

The reduction from a fully nonlinear evolution equation to a system of simple ordinary differential equations provides a remarkable computational advantage. Numerical solutions to the modal equations may be computed with the most simple-minded of numerical schemes, specifically, the Euler difference method implemented in the environment of the commercial package MATLAB.

MATLAB's 'ode45' command implements a standard Runge-Kutta method that computed results remarkably faster than the Euler method. However, the efficiency of the Runge-Kutta method gave large x-spacing which made some results (such as the graph of the conservation law) too sparsely sampled for good graphical representation. In all that follows, results were generated with the simple Euler difference scheme. For results of a convergence study for this implementation, refer to [17], in preparation. The following figures clearly demonstrate the mathematical results obtained earlier, namely the transfer of energy between modes and the conservation of energy.

All numerical results presented in this section used the parameter values

$$\alpha = .15 \qquad \beta = \left(\frac{1}{12}\right)^2 \qquad \omega_1 = \frac{\pi}{6} \tag{2.131}$$

Boundary data values $u_j(0)$ in vector form $[u_1(0)u_2(0)\ldots]$ are indicated in the heading of each figure.

2.9.2 Numerical results for BBM

For the case of two modes, the solution of the Lau-Barcilon differential equations (2.45, 2.46) for the BBM equation is given in Figure (2.3). For the case of three modes, the solution of the Lau-Barcilon differential equations is given in Figure 2.4.

The second part of each figure shows the behavior of the conservation law. Its small, linear growth provides one indication of the level of numerical error for the modes shown in the first part of the figure. Notice that the energy transfer between modes is clearly visible. A further point to note is the cascade of energy: the first mode exhibits more energy than the second mode.

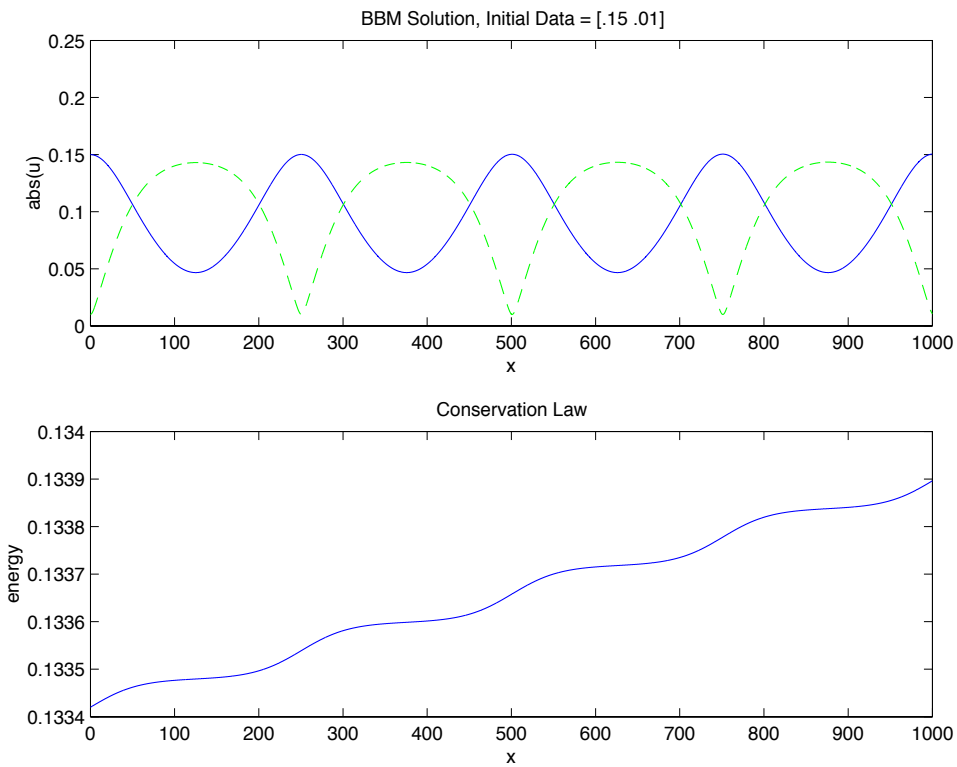

Figure 2.3: BBM modal amplitudes, $N = 2$: u_1 solid blue, u_2 dashed green

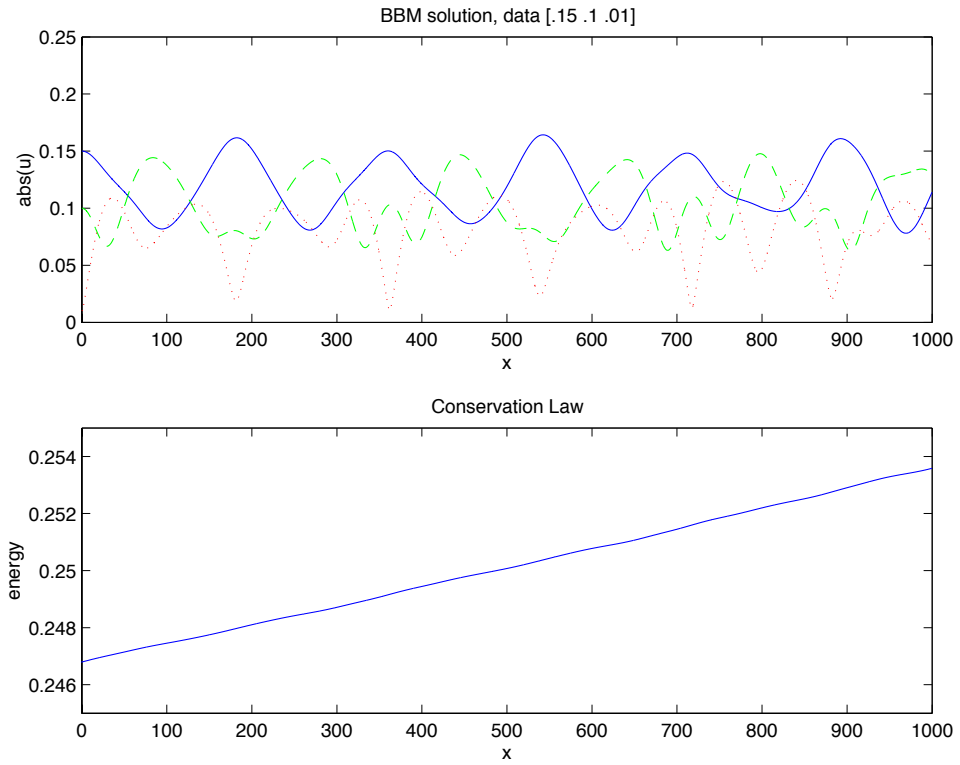

Figure 2.4: BBM modal amplitudes, $N = 3$: u_1 solid blue, u_2 dashed green, u_3 dotted red

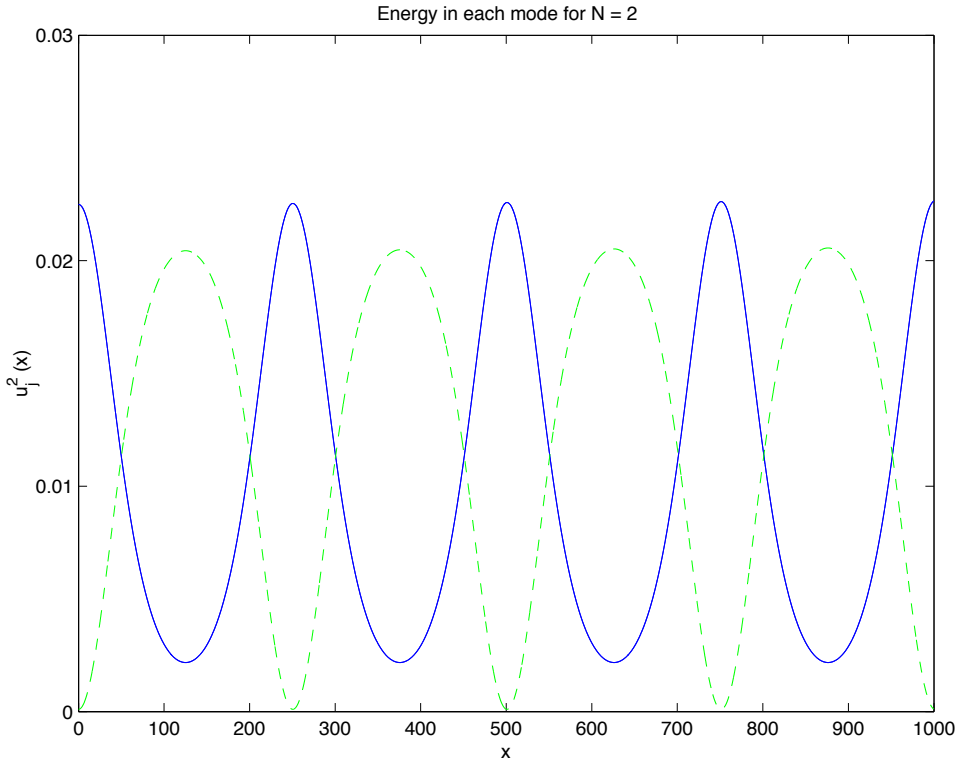

Figure 2.5: Squares of modal amplitudes for BBM, $N = 2$

In this context, energy in the higher modes is proportional to the square of the modulus of the modal amplitudes, as demonstrated in the error calculation (2.129). Plots showing these squares of the modal amplitudes ($N = 2, 3$ cases) are given in Figures 2.5 and 2.6.

Also, it is possible to compute the solution for $N = 4$ and $N = 5$ modes. The graphs of these solutions suggest that chaotic behavior may develop in these cases (see Figures (2.7) and (2.8)), but further study is indicated.

The point of the modal expansion is to obtain an approximate solution η of the governing nonlinear partial differential equation; hence, use the for-

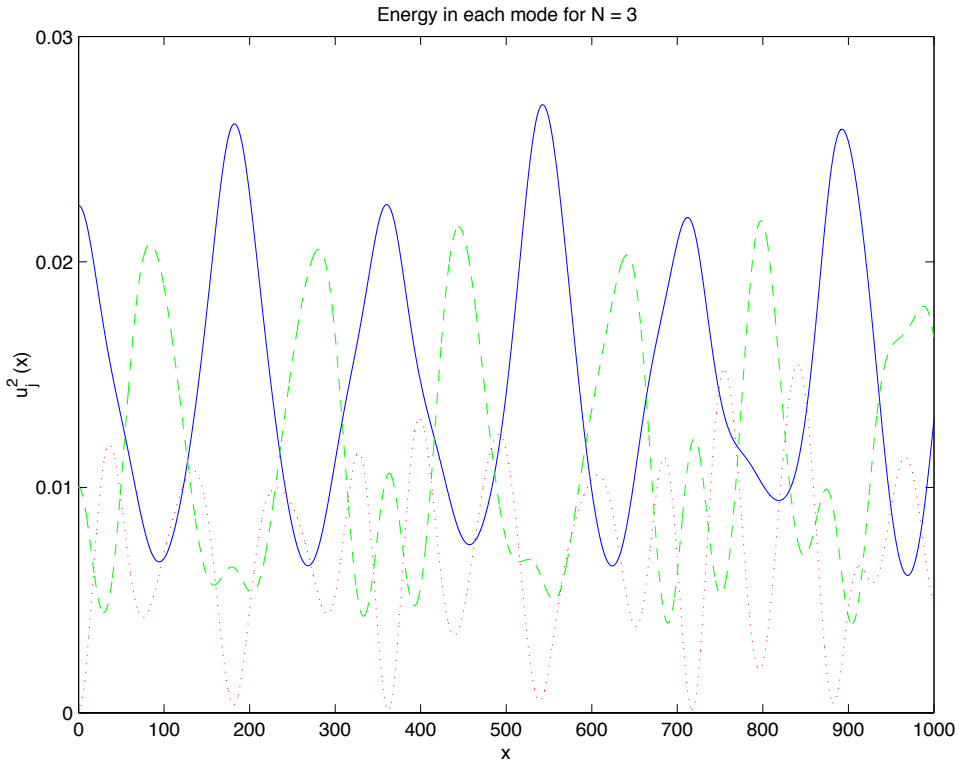

Figure 2.6: Squares of modal amplitudes for BBM, $N = 3$

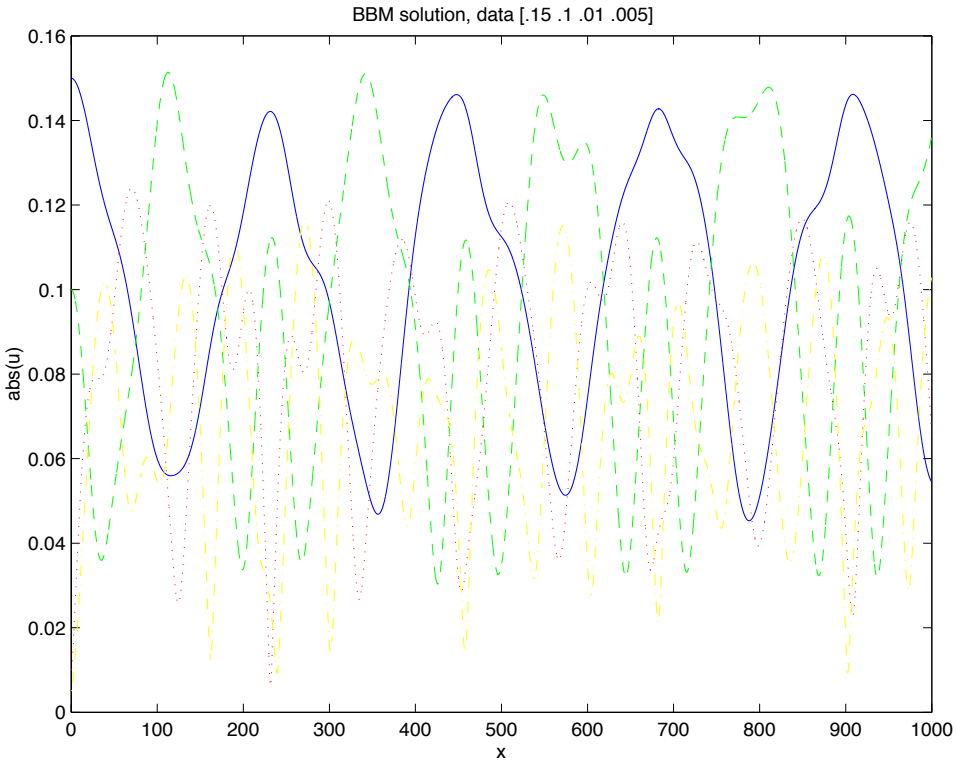

Figure 2.7: modal amplitudes, $N = 4$: u_1 solid blue, u_2 dashed green, u_3 dotted red, u_4 dash-dot yellow

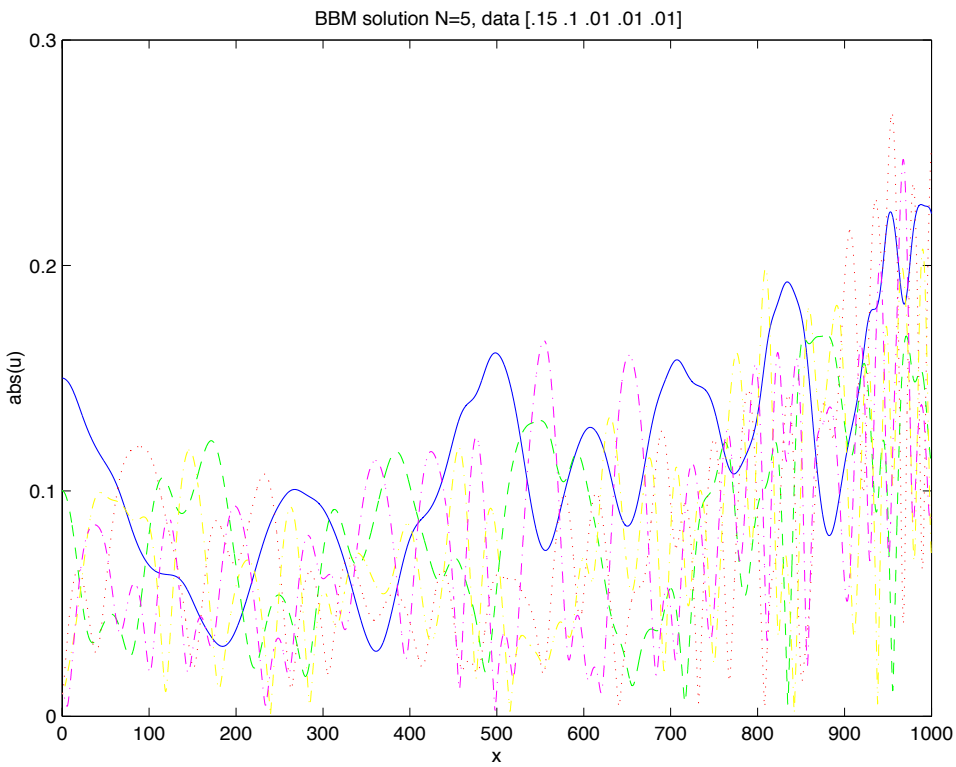

Figure 2.8: Modal amplitudes for BBM, $N = 5$: u_1 solid blue, u_2 dashed green, u_3 dotted red, u_4 dash-dot yellow, u_5 dash-dot magenta

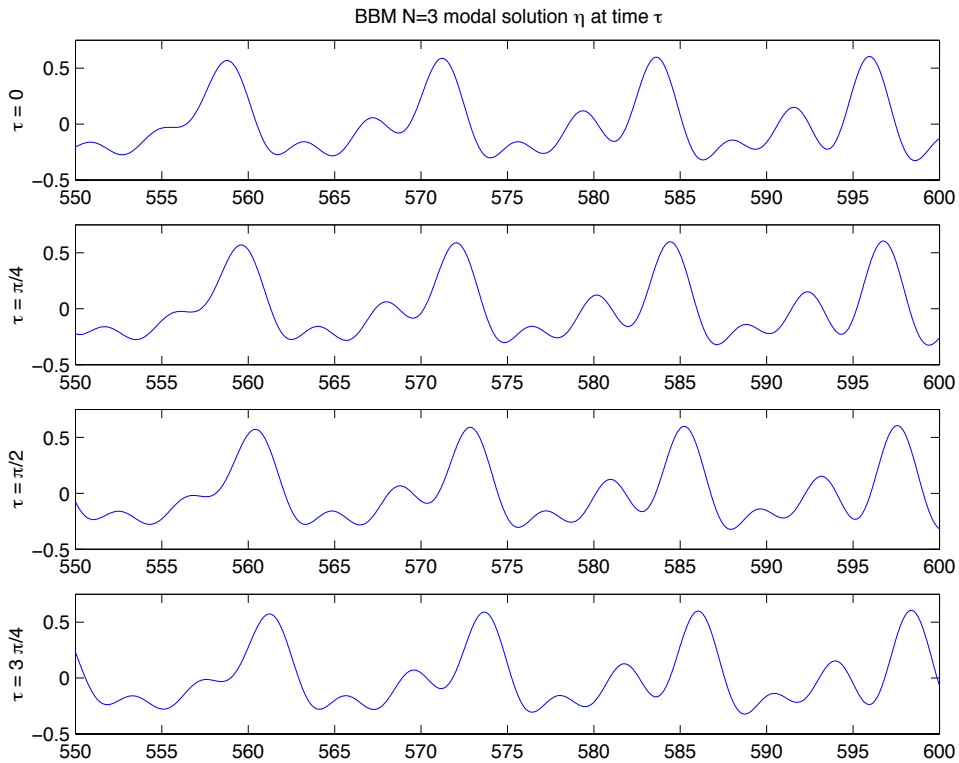

Figure 2.9: Modal approximations to BBM, $N = 3$

mulation (2.25) with the computed modal amplitudes to determine $u(x, t)$ at various time values. See Figures 2.9 and 2.10 and note the rightward propagation of waves along the $x-$axis.

2.9.3 Numerical results for BAE

As studied by Boczar-Karakiewicz et al., one may easily solve the Lau-Barcilon equations corresponding to the Boussinesq approximation to the Euler equation (BAE). In the case of two modes, this solution has the form shown in Figure (2.11).

Similarly, it is easy to solve the Lau-Barcilon equations for BAE in the

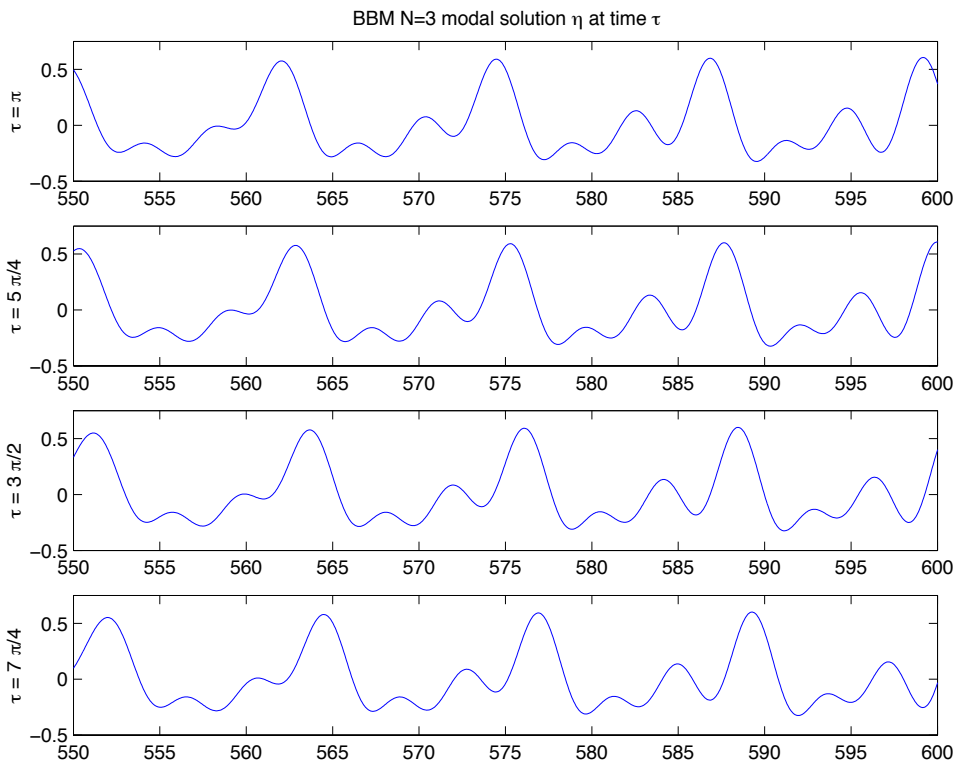

Figure 2.10: Modal approximations to BBM, $N = 3$

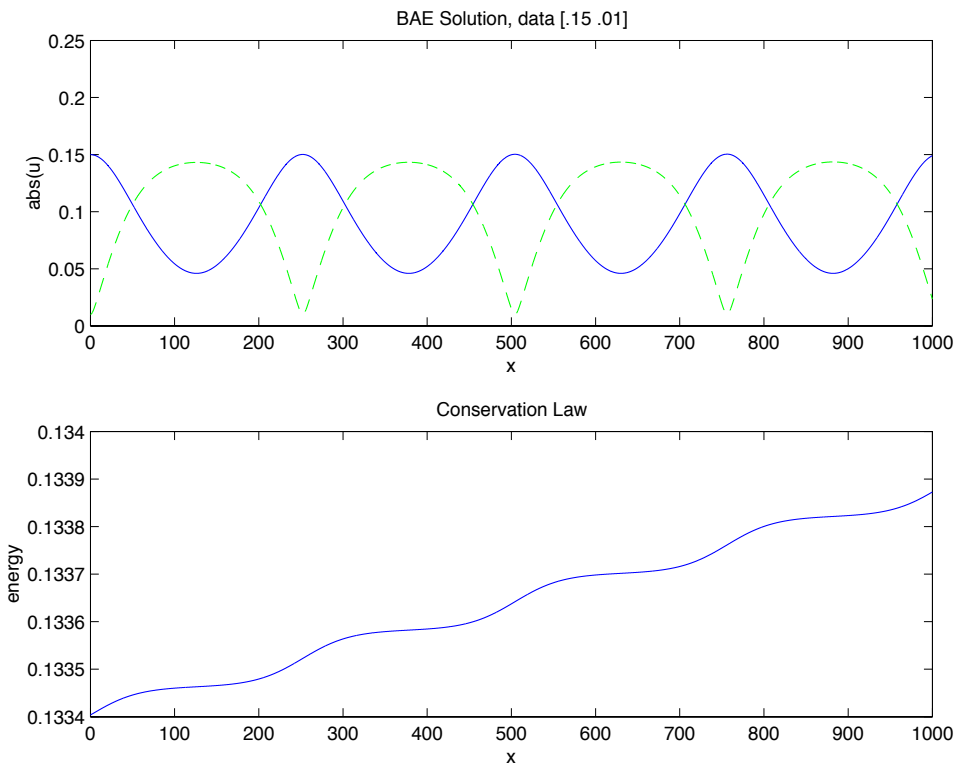

Figure 2.11: BAE modal amplitudes, $N = 2$: u_1 solid blue, u_2 dashed green

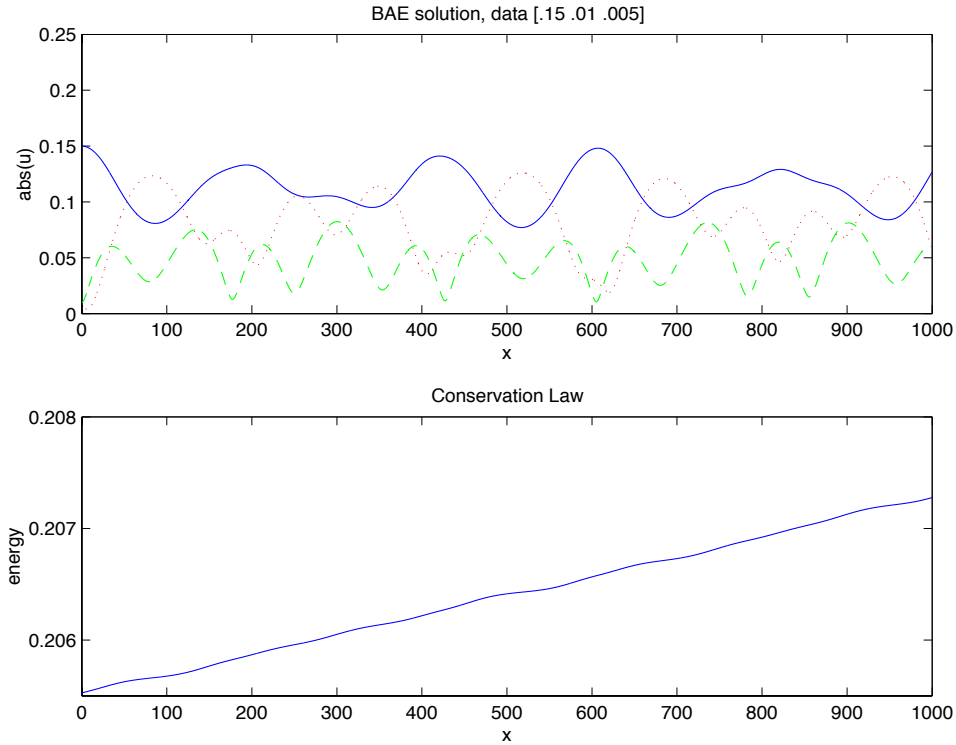

Figure 2.12: Modal amplitudes for BAE, $N = 3$: u_1 solid blue, u_2 dashed green, u_3 dotted red

case of three modes. This solution is shown in Figure (2.12).

2.9.4 Numerical comparison between BAE and BBM

The analytical studies described in earlier sections suggest that the modal expansion technique provides a reasonable approximate solution to the BBM equation in the context of the wave-sandbar interaction model of Karakiewicz and Bona. Comparison of the numerical results validates this conclusion. The absolute differences of the modal solutions are plotted in Figures (2.13) and (2.14).

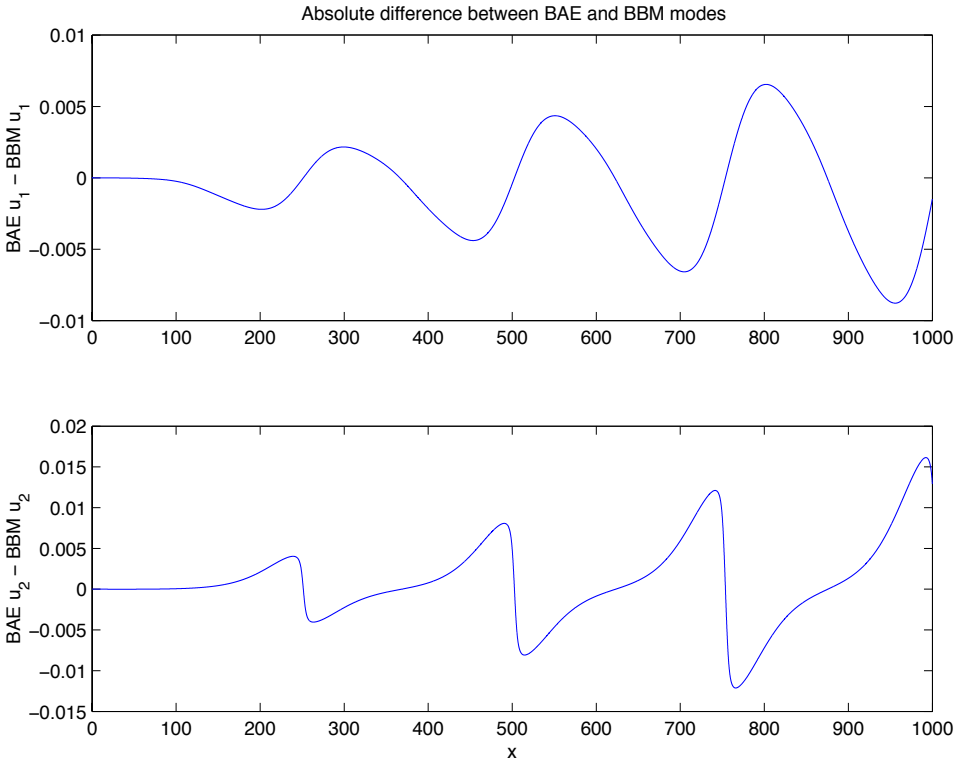

Figure 2.13: Modulus of difference of modal amplitudes for BAE and BBM, $N = 2$

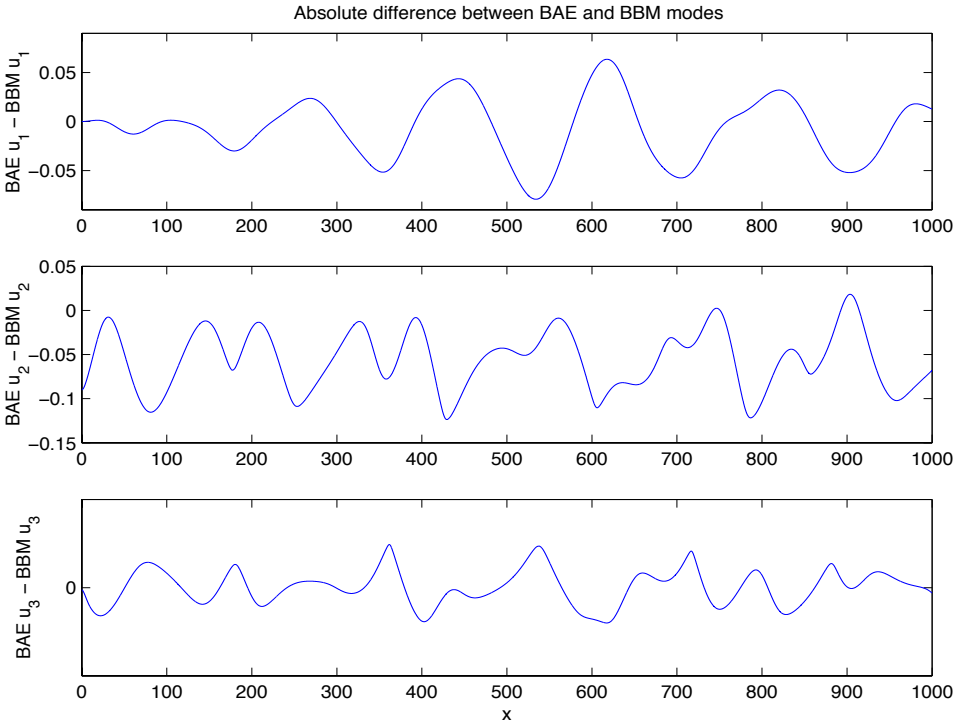

Figure 2.14: Modulus of difference of modal amplitudes for BAE and BBM, $N = 3$

Both of these modal approximations agree with each other very well, as they should!

Chapter 3

Modal Expansion of the KdV Equation

3.1 Introduction

The Korteweg-de Vries equation (KdV) first derived by Boussinesq in 1871, later published by Korteweg and de Vries in a seminal 1895 paper, has been well studied (see [21], [13], [7], for example) during the past century, generating a wealth of mathematics.

As a description of small amplitude, large wavelength surface water waves, KdV may also be used as the governing equation in the model of wave-bottom interaction developed by Boczar-Karakiewicz and Bona (refer to Chapter 1). Hence, it is interesting to study the results of the modal approximation applied to this partial differential equation. The goal of this chapter is to carry out such a study and to compare with the results for BBM derived in the previous chapter.

3.2 Derivation of the Lau-Barcilon Equations for KdV

Begin with the KdV equation,

$$u_t + u_x + \frac{3}{2}\alpha u u_x + \frac{\beta}{6} u_{xxx} = 0. \tag{3.1}$$

Just as in the previous chapter, suppose that u may be characterized by a slowly varying amplitude that carries more rapidly oscillating information. Carry out a two-scale expansion by setting $X = \alpha x$ for the small parameter α and $u(x, X, t) = u(x, t)$, so that in equation (3.1) above ∂_x is replaced by $\partial_x + \alpha\partial_X$. This process yields

$$u_t + (\partial_x + \alpha\partial_X)u + \frac{3}{2}\alpha u(\partial_x + \alpha\partial_X)u + \frac{\beta}{6}(\partial_x + \alpha\partial_X)^2(u_x + \alpha u_X) = 0$$

which reduces to

$$u_t + u_x + \frac{\beta}{6}u_{xxx} + \alpha\left[u_X + \frac{3}{2}uu_x + \frac{\beta}{2}u_{xxX}\right] = O(\alpha^2). \tag{3.2}$$

Setting to zero the coefficient at each order of α gives the following equations.

$$O(\alpha^0): \quad 0 = u_t + u_x + \frac{\beta}{6}u_{xxx} \tag{3.3}$$

$$O(\alpha): \quad 0 = u_X + \frac{3}{2}uu_x + \frac{\beta}{2}u_{xxX} \tag{3.4}$$

Now, set

$$u(x, X, t) = \sum_{j=-N}^{N} u_j(X)e^{i(k_jx-\omega_jt)},$$

where as before $\omega_j = j\omega_1$, $\omega_{-j} = -\omega_j$, $k_{-j} = -k_j$, $\omega_0 = 0$, $k_0 = 0$, $u_0(X) = 0$, and $u_{-j} = u_j^*$ represents the complex conjugate. Notice that these conditions on ω_j and k_j force u to be real-valued. Again, use the notation $E_j = e^{i(k_jx-\omega_jt)}$.

Now consider the order one terms in the two-scale version of KdV:

$$\begin{aligned} u_t &= \sum_{j=-N}^{N} -i\omega_j u_j E_j \\ u_x &= \sum_{j=-N}^{N} ik_j u_j E_j \\ u_{xxx} &= \sum_{j=-N}^{N} -ik_j^3 u_j E_j. \end{aligned}$$

These terms combine to give the $O(\alpha^0)$ expression

$$u_t + u_x + \frac{\beta}{6} u_{xxx} = \sum_j i \left(k_j - \omega_j - \frac{\beta}{6} k_j^3 \right) E_j.$$

Setting this equal to zero gives the linearized dispersion relation for KdV:

$$k_j - \omega_j - \frac{\beta}{6} k_j^3 = 0$$

which is typically written as

$$\omega = k \left(1 - \frac{\beta}{6} k^2 \right).$$

For $\beta = (1/12)^2$, this relation has the form shown in Figure (3.1).

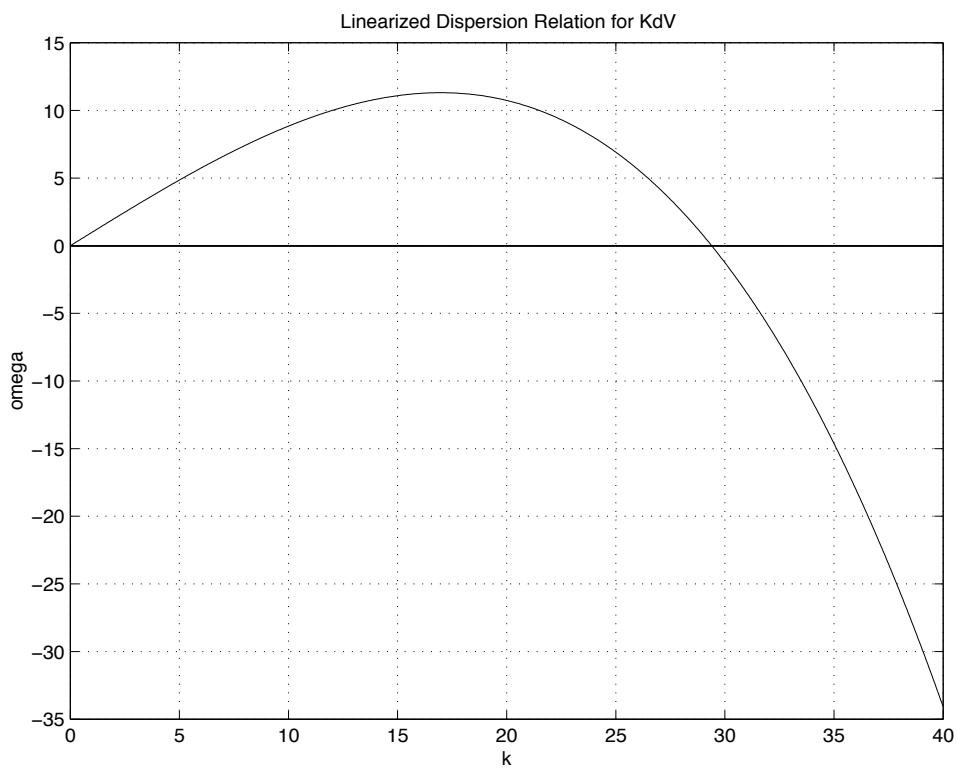

Figure 3.1: Dispersion Relation for KdV

As for BBM, the dispersion relation may be used to show an approximate size for the values of ω,

$$\omega = k\left(1 - \frac{\beta}{6}k^2\right) \approx 2\pi\beta + O(\beta^4) \tag{3.5}$$

as long as β remains small.

Figure (3.2) shows a comparison of this dispersion relation with dispersion relations for BAE, BBM, and the Euler equation (as discussed in Chapter 2).

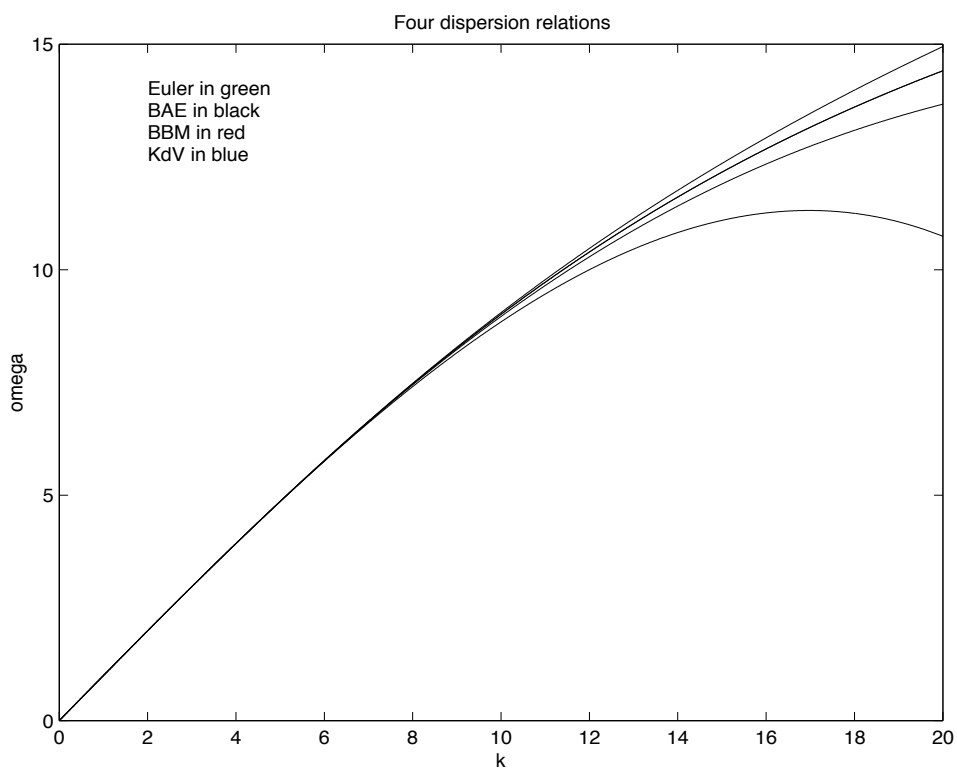

Figure 3.2: A plot of the four relevant dispersion relations

As in Chapter 2, the dispersion relation also provides a mathematical restriction on the number of modes permitted in this expansion. This point

will be discussed later in Section 6.

Returning to the expansion (in powers of α), consider the terms which contribute to the order α equation from the two-scale version of KdV:

$$\begin{aligned} u_X &= \sum_{j=-N}^{N} u_j' E_j \\ u_{xxX} &= \sum_{j=-N}^{N} -k_j^2 u_j' E_j. \end{aligned}$$

As in the previous chapter, the expansion and simplification of the nonlinear term depends on how many modes (N) are being used to approximate u. The nonlinearity typically generates higher order modes which must be neglected. In the case of two modes ($N = 2$),

$$\begin{aligned} uu_x &= \left(\sum_{j=-2}^{2} u_j E_j\right)\left(\sum_{n=-2}^{2} ik_n u_n E_n\right) \\ &= ik_1 u_1^2 E_1^2 + ik_2 u_2^2 E_2^2 \\ &\quad + E_1 E_2 \left(i(k_1+k_2)u_1 u_2\right) + E_1^* E_2 \left(i(k_2-k_1)u_1^* u_2\right) \\ &\quad + \text{conj} \end{aligned}$$

with the same notation as used in Chapter 2. Neglect the higher harmonics to write the nonlinear term as

$$uu_x = ik_1 u_1^2 e^{-i\Delta_k x} E_2 + i(k_2 - k_1) u_1^* u_2 e^{i\Delta_k x} E_1 + \text{conj}$$

where $\Delta_k = k_2 - 2k_1$.

Put together these quantities and simplify to obtain the expression for the $O(\alpha)$ portion of the two-scale KdV equation in the case of 2 modes. Setting

the coefficient of each mode equal to zero gives

$$u_1' - \frac{\beta}{2}k_1^2 u_1' + i\frac{3}{2}(k_2 - k_1)u_1^* u_2 e^{i\Delta_k x} = 0 \tag{3.6}$$

$$u_2' - \frac{\beta}{2}k_2^2 u_2' + i\frac{3}{2}k_1 u_1^2 e^{-i\Delta_k x} = 0 \tag{3.7}$$

This equation yields the Lau-Barcilon equations for KdV:

$$u_1'(X) = -i\frac{3}{2}\frac{k_2 - k_1}{1 - \frac{\beta}{2}k_1^2}e^{i\Delta_k x}u_1^*(X)u_2(X) \tag{3.8}$$

$$u_2'(X) = -i\frac{3}{2}\frac{k_1}{1 - \frac{\beta}{2}k_2^2}e^{-i\Delta_k x}u_1^2(X). \tag{3.9}$$

Notice that these differ from the Lau-Barcilon equations for BBM only in the denominator of the (constant) coefficient. Thus, all the analysis developed in Chapter 2 for BBM carries over to the KdV case nearly identically.

Change variables back to the original (x, t) coordinates by setting $a_j(x) = u_j(X) = u_j(\alpha x)$ so that ∂_X is replaced by $\frac{1}{\alpha}\partial_x$. Thus the Lau-Barcilon equations are

$$a_1' = -i\alpha\frac{3}{2}\frac{k_2 - k_1}{1 - \frac{\beta}{2}k_1^2}e^{i\Delta_k x}a_1^* a_2 \tag{3.10}$$

$$a_2' = -i\alpha\frac{3}{2}\frac{k_1}{1 - \frac{\beta}{2}k_2^2}e^{-i\Delta_k x}a_1^2. \tag{3.11}$$

The Lau-Barcilon equations for the case of three modes may be computed similarly, with the nonlinear term having the form

$$\begin{aligned} uu_x = \quad & E_1\left(i(k_2 - k_1)u_1^* u_2 e^{i\Delta_k x} + i(k_3 - k_2)u_2^* u_3 e^{-i(k_1+k_2-k_3)x}\right) \\ & + E_2\left(ik_1 u_1^2 e^{-i\Delta_k x} + i(k_3 - k_1)u_1^* u_3 e^{-i(k_1+k_2-k_3)x}\right) \\ & + E_3\left(i(k_1 + k_2)u_1 u_2 e^{i(k_1+k_2-k_3)x}\right) \\ & + \text{conj} \end{aligned}$$

after neglecting higher harmonics. As in the two modes case, the $O(\alpha)$ terms yield the Lau-Barcilon equations:

$$\begin{aligned} u_1'(X) &= -i\frac{3}{2}\frac{k_2-k_1}{1-\frac{\beta}{2}k_1^2}e^{i\Delta_k x}u_1^*(X)u_2(X) \\ &\quad -i\frac{3}{2}\frac{k_3-k_2}{1-\frac{\beta}{2}k_1^2}e^{-i(k_1+k_2-k_3)x}u_2^*(X)u_3(X) \qquad (3.12) \\ u_2'(X) &= -i\frac{3}{2}\frac{k_1}{1-\frac{\beta}{2}k_2^2}e^{-i\Delta_k x}u_1^2(X) \\ &\quad -i\frac{3}{2}\frac{k_3-k_1}{1-\frac{\beta}{2}k_2^2}e^{-i(k_1+k_2-k_3)x}u_1^*(X)u_3(X) \qquad (3.13) \\ u_3'(X) &= -i\frac{3}{2}\frac{k_1+k_2}{1-\frac{\beta}{2}k_3^2}e^{i(k_1+k_2-k_3)x}u_1(X)u_2(X). \qquad (3.14) \end{aligned}$$

After a change of variables back to the usual (x,t) coordinates by taking $a_j(x) = u_j(X) = u_j(\alpha x)$, the three modes Lau-Barcilon equations for the KdV model become

$$\begin{aligned} a_1' &= -i\alpha\frac{3}{2}\frac{k_2-k_1}{1-\frac{\beta}{2}k_1^2}e^{i\Delta_k x}a_1^*a_2 \\ &\quad -i\alpha\frac{3}{2}\frac{k_3-k_2}{1-\frac{\beta}{2}k_1^2}e^{-i(k_1+k_2-k_3)x}a_2^*a_3 \qquad (3.15) \\ a_2' &= -i\alpha\frac{3}{2}\frac{k_1}{1-\frac{\beta}{2}k_2^2}e^{-i\Delta_k x}a_1^2 \\ &\quad -i\alpha\frac{3}{2}\frac{k_3-k_1}{1-\frac{\beta}{2}k_2^2}e^{-i(k_1+k_2-k_3)x}a_1^*a_3 \qquad (3.16) \\ a_3' &= -i\alpha\frac{3}{2}\frac{k_1+k_2}{1-\frac{\beta}{2}k_3^2}e^{i(k_1+k_2-k_3)x}a_1a_2. \qquad (3.17) \end{aligned}$$

Again, introducing notation for the constants makes the structure of these equations easier to observe. Similarly to the definitions in the previous chapter, set

$$\tilde{Q}_1 = \frac{3}{2}\frac{k_2-k_1}{1-\frac{\beta}{2}k_1^2} \qquad \tilde{Q}_2 = \frac{3}{2}\frac{k_1}{1-\frac{\beta}{2}k_2^2} \qquad (3.18)$$

and

$$\tilde{R}_1 = \frac{3}{2}\frac{k_3 - k_2}{1 - \frac{\beta}{2}k_1^2} \qquad \tilde{R}_2 = \frac{3}{2}\frac{k_3 - k_1}{1 - \frac{\beta}{2}k_2^2} \qquad \tilde{R}_3 = \frac{3}{2}\frac{k_1 + k_2}{1 - \frac{\beta}{2}k_3^2}. \tag{3.19}$$

Introducing the further notation $k_H = k_1 + k_2 - k_3$ then yields the three modes Lau-Barcilon equations:

$$a_1' = -\alpha i \tilde{Q}_1 e^{i\Delta_k x} a_1^* a_2 - \alpha i \tilde{R}_1 e^{-ik_H x} a_2^* a_3 \tag{3.20}$$

$$a_2' = -\alpha i \tilde{Q}_2 e^{-i\Delta_k x} a_1^2 - \alpha i \tilde{R}_2 e^{-ik_H x} a_1^* a_3 \tag{3.21}$$

$$a_3' = -\alpha i \tilde{R}_3 e^{ik_H x} a_1 a_2. \tag{3.22}$$

Later sections describe the analytical structure of these equations, including a derivation of the conservation law they obey.

3.3 The Modal Expansion for Linear KdV

The linear KdV equation is

$$u_t + u_x + \frac{\beta}{6} u_{xxx} = 0.$$

Apply the two-scale expansion, taking $X = \alpha x$, consequently replacing ∂_x with $\partial_x + \alpha \partial_X$ in the above equation. After simplification, the KdV equation written in the two-scale variables becomes

$$u_t + u_x + \frac{\beta}{6} u_{xxx} + \alpha \left[u_X + \frac{\beta}{2} u_{xxX} \right] = O(\alpha^2).$$

After the modal expansion is applied to u, the $O(\alpha^0)$ terms will yield the usual linearized dispersion relation for each mode, as described in the previous section.

The next step is to apply the modal expansion to the $O(\alpha)$ terms: substitute the form

$$u(x, X, t) = \sum_{j=-\infty}^{\infty} u_j(X) e^{i(k_j x - \omega_j t)}$$

into the equation

$$u_X + \frac{\beta}{2} u_{xxX} = 0.$$

The coefficients of each mode then must satisfy

$$u_j' e^{i(k_j x - \omega_j t)} + \frac{\beta}{2}(ik_j)^2 u_j' e^{i(k_j x - \omega_j t)} = 0$$

where the prime refers to differentiation with respect to X. This relation reduces to

$$\left(1 - \frac{\beta}{2} k_j^2\right) u_j' = 0.$$

Thus,

$$u_j(X) = \tilde{C}_j,$$

meaning each coefficient in the modal expansion is constant. Therefore, in the case of linear KdV (as in the case of linear BBM), the modal expansion reduces to the standard Fourier series solution:

$$u(x, t) = \sum_{j=-\infty}^{\infty} \tilde{C}_j e^{i(k_j x - \omega_j t)}$$

subject to the dispersion relation for each ($j-$th) mode. The coefficients are determined once boundary data has been posed for the problem, just as for the BBM case.

3.4 Conservation of Energy

Begin with the Lau-Barcilon equations for the case of the KdV equation:

$$\frac{da_1}{dx} = -i\alpha\tilde{Q}_1 e^{i\Delta_k x} a_1^* a_2 \tag{3.23}$$

$$\frac{da_2}{dx} = -i\alpha\tilde{Q}_2 e^{-i\Delta_k x} a_1^2 \tag{3.24}$$

Exactly as in the BBM case, derive conservation laws via the following calculation. Write these equations in complex form by setting $a_j(x) = r_j(x)e^{i\theta_j(x)}$. The system then becomes

$$r_1' + ir_1\theta_1' = -i\alpha\tilde{Q}_1 r_1 r_2 e^{i[\theta_2 - 2\theta_1 + \Delta_k x]} \tag{3.25}$$

$$r_2' + ir_2\theta_2' = -i\alpha\tilde{Q}_2 r_1^2 e^{-i[\theta_2 - 2\theta_1 + \Delta_k x]} \tag{3.26}$$

where the prime indicates differentiation with respect to the variable x. Separate these equations into real and imaginary components, setting $\phi = \theta_2 - 2\theta_1 + \Delta_k x$. Expand the exponential in terms of sine and cosine to find

$$r_1' + ir_1\theta_1' = -i\alpha\tilde{Q}_1 r_1 r_2 \cos\phi + \alpha\tilde{Q}_1 r_1 r_2 \sin\phi \tag{3.27}$$

$$r_2' + ir_2\theta_2' = -i\alpha\tilde{Q}_2 r_1^2 \cos\phi - \alpha\tilde{Q}_2 r_1^2 \sin\phi, \tag{3.28}$$

then sort these into real and imaginary parts to obtain the ordinary differential equations

$$r_1' = \alpha\tilde{Q}_1 r_1 r_2 \sin\phi \qquad \theta_1' = -\alpha\tilde{Q}_1 r_2 \cos\phi \tag{3.29}$$

$$r_2' = -\alpha\tilde{Q}_2 r_1^2 \sin\phi \qquad \theta_2' = -\alpha\tilde{Q}_2 \frac{r_1^2}{r_2} \cos\phi. \tag{3.30}$$

As before, reduce the number of equations: since

$$\phi' = \theta_2' - 2\theta_1' + \Delta_k = -\alpha\tilde{Q}_2 \frac{r_1^2}{r_2} \cos\phi + 2\alpha\tilde{Q}_1 r_2 \cos\phi + \Delta_k,$$

the system is simply

$$r_1' = \alpha \tilde{Q}_1 r_1 r_2 \sin\phi \tag{3.31}$$

$$r_2' = -\alpha \tilde{Q}_2 r_1^2 \sin\phi \tag{3.32}$$

$$\phi' = \Delta_k - \alpha\left[\tilde{Q}_2 \frac{r_1^2}{r_2} - 2\tilde{Q}_1 r_2\right]\cos\phi \tag{3.33}$$

From the first two of these, compute

$$\frac{r_1'}{r_2'} = -\frac{\tilde{Q}_1}{\tilde{Q}_2}\frac{r_2}{r_1} \tag{3.34}$$

$$\tilde{Q}_2 r_1 r_1' = -\tilde{Q}_1 r_2 r_2' \tag{3.35}$$

$$\tilde{Q}_2\left(\frac{1}{2}r_1^2\right)' + \tilde{Q}_1\left(\frac{1}{2}r_2^2\right)' = 0 \tag{3.36}$$

which integrates to give the conservation of energy expression

$$\tilde{Q}_2 r_1^2 + \tilde{Q}_1 r_2^2 = R^2.$$

Again, one could choose to re-scale, so that this becomes $r_1^2 + r_2^2 = C$, where C is some positive constant.

To derive a conservation law for the case of three modes, follow a nearly identical calculation. Begin with the three modes Lau-Barcilon equations for KdV:

$$a_1' = -i\alpha\tilde{Q}_1 e^{i\Delta_k x} a_1^* a_2 - i\alpha\tilde{R}_1 e^{-ik_H x} a_2^* a_3 \tag{3.37}$$

$$a_2' = -i\alpha\tilde{Q}_2 e^{-i\Delta_k x} a_1^2 - i\alpha\tilde{R}_2 e^{-ik_H x} a_1^* a_3 \tag{3.38}$$

$$a_3' = -i\alpha\tilde{R}_3 e^{ik_H x} a_1 a_2 \tag{3.39}$$

including the conjugate equations to these, where the notational conveniences $\Delta_k = k_2 - 2k_1$ and $k_H = k_1 + k_2 - k_3$ are used.

Again, set $a_j(x) = r_j(x)e^{i\theta_j(x)}$ so that $a_j' = r_j' e^{i\theta_j} + i r_j \theta_j' e^{i\theta_j}$. Substituting these forms into the first of the above ordinary differential equations gives

$$r_1' e^{i\theta_1} + i r_1 \theta_1' e^{i\theta_1} = -i\alpha \tilde{Q}_1 e^{i\Delta_k x} r_1 e^{-i\theta_1} r_2 e^{i\theta_2} - i\alpha \tilde{R}_1 e^{-ik_H x} r_2 e^{-i\theta_2} r_3 e^{i\theta_3}.$$

Computing similarly for the second and third equations and grouping the exponential terms gives the following system,

$$\begin{aligned} r_1' + i r_1 \theta_1' &= -i\alpha \tilde{Q}_1 r_1 r_2 e^{i[\theta_2 - 2\theta_1 + \Delta_k x]} - i\alpha \tilde{R}_1 r_2 r_3 e^{-i[\theta_1 + \theta_2 - \theta_3 + k_H x]} && (3.40) \\ r_2' + i r_2 \theta_2' &= -i\alpha \tilde{Q}_2 r_1^2 e^{-i[\theta_2 - 2\theta_1 + \Delta_k x]} - i\alpha \tilde{R}_2 r_1 r_3 e^{-i[\theta_1 + \theta_2 - \theta_3 + k_H x]} && (3.41) \\ r_3' + i r_3 \theta_3' &= -i\alpha \tilde{R}_3 r_1 r_2 e^{i[\theta_1 + \theta_2 - \theta_3 + k_H x]} && (3.42) \end{aligned}$$

together with conjugate equations.

Now set $\phi = \theta_2 - 2\theta_1 + \Delta_k x$ and $\psi = \theta_1 + \theta_2 - \theta_3 + k_H x$. With this notation, the above system becomes

$$\begin{aligned} r_1' + i r_1 \theta_1' &= -i\alpha \tilde{Q}_1 r_1 r_2 e^{i\phi} - i\alpha \tilde{R}_1 r_2 r_3 e^{-i\psi} && (3.43) \\ r_2' + i r_2 \theta_2' &= -i\alpha Q_2 r_1^2 e^{-i\phi} - i\alpha R_2 r_1 r_3 e^{-i\psi} && (3.44) \\ r_3' + i r_3 \theta_3' &= -i\alpha R_3 r_1 r_2 e^{i\psi}, && (3.45) \end{aligned}$$

or in terms of the real and imaginary parts,

$$\begin{aligned} r_1' &= \alpha \tilde{Q}_1 r_1 r_2 \sin\phi - \alpha \tilde{R}_1 r_2 r_3 \sin\psi && (3.46) \\ \theta_1' &= -\alpha \tilde{Q}_1 r_2 \cos\phi - \alpha \tilde{R}_1 \frac{r_2 r_3}{r_1} \cos\psi && (3.47) \\ r_2' &= -\alpha \tilde{Q}_2 r_1^2 \sin\phi - \alpha \tilde{R}_2 r_1 r_3 \sin\psi && (3.48) \\ \theta_2' &= -\alpha \tilde{Q}_2 \frac{r_1^2}{r_2} \cos\phi - \alpha \tilde{R}_2 \frac{r_1 r_3}{r_2} \cos\psi && (3.49) \\ r_3' &= \alpha \tilde{R}_3 r_1 r_2 \sin\psi && (3.50) \\ \theta_3' &= -\alpha \tilde{R}_3 \frac{r_1 r_2}{r_3} \cos\psi. && (3.51) \end{aligned}$$

Write these equations in terms of the variables ϕ and ψ, eliminating explicit dependence on the θ_j. Use the forms

$$\phi' = \theta_2' - 2\theta_1' + \Delta_k \tag{3.52}$$

$$\psi' = \theta_1' + \theta_2' - \theta_3' + k_H \tag{3.53}$$

to obtain the following system in terms of $r_1, r_2, r_3, \phi, \psi$:

$$\begin{aligned}
r_1' &= \alpha\tilde{Q}_1 r_1 r_2 \sin\phi - \alpha\tilde{R}_1 r_2 r_3 \sin\psi \\
r_2' &= -\alpha\tilde{Q}_2 r_1^2 \sin\phi - \alpha\tilde{R}_2 r_1 r_3 \sin\psi \\
r_3' &= \alpha\tilde{R}_3 r_1 r_2 \sin\psi \\
\phi' &= \Delta_k + \alpha\cos\phi\left(2\tilde{Q}_1 r_2 - \tilde{Q}_2 r_1^2/r_2\right) + \alpha\cos\psi\left(2\tilde{R}_1\frac{r_2 r_3}{r_1} - \tilde{R}_2\frac{r_1 r_3}{r_2}\right) \\
\psi' &= k_H - \alpha\cos\phi\left(\tilde{Q}_1 r2 + \tilde{Q}_2 r_1^2/r_2\right) \\
&\quad +\alpha\cos\psi\left(\tilde{R}_3\frac{r_1 r_2}{r_3} - \tilde{R}_1\frac{r_2 r_3}{r_1} - \tilde{R}_2\frac{r_1 r_3}{r_2}\right).
\end{aligned}$$

The next step is to carry out the algebra needed to obtain a conservation law—this utilizes only the first three equations in the above system. First, compute

$$(\tilde{Q}_2 r_1 r_1' + \tilde{Q}_1 r_2 r_2')\tilde{R}_3 = (-\alpha\left(\tilde{Q}_2\tilde{R}_1 + \tilde{Q}_1\tilde{R}_2\right) r_1 r_2 r_3 \sin\psi)\tilde{R}_3$$

and then add this to the quantity $\left(\tilde{Q}_2\tilde{R}_1 + \tilde{Q}_1\tilde{R}_2\right) r_3 r_3'$ to obtain the result that

$$\tilde{Q}_2\tilde{R}_3 r_1 r_1' + \tilde{Q}_1\tilde{R}_3 r_2 r_2' + \left(\tilde{Q}_2\tilde{R}_1 + \tilde{Q}_1\tilde{R}_2\right) r_3 r_3' = 0.$$

This may be written as

$$\left[\frac{1}{2}\tilde{Q}_2\tilde{R}_3 r_1^2 + \frac{1}{2}\tilde{Q}_1\tilde{R}_3 r_2^2 + \frac{1}{2}(\tilde{Q}_2\tilde{R}_1 + \tilde{Q}_1\tilde{R}_2) r_3^2\right]' = 0$$

which integrates to the conservation law

$$\tilde{Q}_2\tilde{R}_3 r_1^2 + \tilde{Q}_1\tilde{R}_3 r_2^2 + (\tilde{Q}_2\tilde{R}_1 + \tilde{Q}_1\tilde{R}_2) r_3^2 = \text{constant}.$$

Again, this derivation has been identical to the derivation of the conservation laws for the BBM modal expansion.

3.5 Closed-form Solution: 2 Modes Case

The analytical solution to the two modes Lau-Barcilon equations for KdV may be derived exactly as in Chapter 2 for the two modes Lau-Barcilon equations for BBM, only replacing the coefficients Q_j, R_j with $\tilde{Q}_j, \tilde{R}_j$, as defined in equations 3.18 and 3.19.

3.6 Derive the Lau-Barcilon Equations for Infinitely Many Modes

Exactly as in Chapter 2 for the BBM equation, one may derive the Lau-Barcilon equations in the case $N \to \infty$. These have the form

$$\begin{aligned} 0 = \quad & (1 - \frac{\beta}{2}k_j^2)u_j' \\ & + \alpha\frac{3}{2}\sum_{m=1}^{\infty} ik_{j-m}u_m u_{j-m} e^{i(k_m + k_{j-m} - k_j)x} + ik_{j+m}u_m^* u_{j+m} e^{i(k_{j+m} - k_m - k_j)x}. \end{aligned}$$

Again, a quick check of this computation can be carried out by comparing with the 2 modes case computed earlier. To do so, take $j = 1, 2$ so that $m = 1, 2$ and $u_j = 0$ for $j > 2$. The first equation comes from the $j = 1$ case:

$$0 = \left(1 - \frac{\beta}{2}k_1^2\right) u_1'$$

$$
\begin{aligned}
&+\alpha\frac{3}{2}\left(0+ik_2u_1^*u_2e^{i(k_2-k_1-k_1)x}+ik_{-1}u_2u_{-1}e^{i(k_2+k_{-1}-k_1)x}\right)\\
=\quad&\left(1-\frac{\beta}{2}k_1^2\right)u_1'+i\frac{3}{2}(k_2-k_1)u_1^*u_2e^{i(k_2-2k_1)x}
\end{aligned}
$$

which gives

$$u_1'=-i\alpha\frac{3}{2}\frac{k_2-k_1}{1-\frac{\beta}{2}k_1^2}e^{i\Delta_k x}u_1^*u_2,$$

the first Lau-Barcilon equation. Similarly, taking $j=2$ gives

$$
\begin{aligned}
0 &= \left(1-\frac{\beta}{2}k_2^2\right)u_2'+i\alpha\frac{3}{2}k_1u_1u_1e^{i(k_1+k_1-k_2)x}\\
&= \left(1-\frac{\beta}{2}k_2^2\right)u_2'+i\alpha\frac{3}{2}k_1u_1^2e^{-i(k_2-2k_1)x}
\end{aligned}
$$

which gives the second Lau-Barcilon equation,

$$u_2'=-i\alpha\frac{k_1}{1-\frac{\beta}{2}k_2^2}u_1^2e^{-i\Delta kx}.$$

Recall from the discussion of the BBM long wave model that, in fact, to preserve the dispersion relation only a finite number of modes is allowed. Consider the function

$$\omega(k)=k\left(1-\frac{\beta}{6}k^2\right)$$

where $\beta=1/12$. Notice that by a freshman calculus argument we may compute the maximum value of ω:

$$\omega'(k)=1-\frac{\beta}{2}k^2=0\qquad\text{when}\qquad k=\frac{\sqrt{2}}{\beta}=12\sqrt{2}$$

and $\omega(4\sqrt{3})=8\sqrt{2}<12$. The graph of ω was given in Figure (3.1).

Now let $\omega_1=$ constant and recall that $\omega_j=j\omega_1$ for positive integers j. Once $j>\frac{12}{\omega_1}$, there exists no value k_j for which the linearized dispersion relation

$$\omega_j=k_j\left(1-\frac{\beta}{6}k_j^2\right)$$

holds. Thus, after a finite number of terms, the modes in the expansion

$$\sum_{j=-\infty}^{\infty} a_j(X) e^{i(k_j x - \omega_j t)}$$

must fail to satisfy the linearized dispersion relation. For our typical values of $\beta = (1/12)^2$ and $\omega_1 = \pi/6$, one finds that $j > 23$ violates the dispersion relation.

As in the BBM case, the general form for the jth mode may be used to write easily the Lau-Barcilon system for these modes as verified above for the $N = 2$ modes case. For example, here is the $N = 4$ modes system:

$$\begin{aligned}
u_1' &= -i\alpha \left[\tilde{Q}(1) e^{i\Delta_k x} u_1^* u_2 + \tilde{S}(1) e^{-ik_H x} u_2^* u_3 + \tilde{T}(1) e^{ik_G x} u_3^* u_4\right] \\
u_2' &= -i\alpha \left[\tilde{Q}(2) e^{-i\Delta_k x} u_1^2 + \tilde{S}(2) e^{-ik_H x} u_1^* u_3 + \tilde{T}(2) e^{i(k_4 - 2k_2)x} u_2^* u_4\right] \\
u_3' &= -i\alpha \left[\tilde{S}(3) e^{ik_H x} u_1 u_2 + \tilde{T}(3) e^{ik_G x} u_1^* u_4\right] \\
u_4' &= -i\alpha \left[\tilde{T}(4) e^{-ik_G x} u_1 u_3 + \tilde{T}(5) e^{-i(k_4 - 2k_2)x} u_2^2\right]
\end{aligned}$$

Similarly, here is the $N = 5$ modes system:

$$\begin{aligned}
u_1' &= -i\alpha [\tilde{Q}(1) e^{i\Delta_k x} u_1^* u_2 + \tilde{S}(1) e^{-ik_H x} u_2^* u_3 + \tilde{T}(1) e^{ik_G x} u_3^* u_4 \\
&\qquad + \tilde{P}(1) e^{ik_F x} u_4^* u_5] \\
u_2' &= -i\alpha [\tilde{Q}(2) e^{-i\Delta_k x} u_1^2 + \tilde{S}(2) e^{-ik_H x} u_1^* u_3 + \tilde{T}(2) e^{i(k_4 - 2k_2)x} u_2^* u_4 \\
&\qquad + \tilde{P}(2) e^{ik_F x} u_3^* u_5] \\
u_3' &= -i\alpha \left[\tilde{S}(3) e^{ik_H x} u_1 u_2 + \tilde{T}(3) e^{ik_G x} u_1^* u_4 + \tilde{P}(3) e^{ik_E x} u_2^* u_5\right] \\
u_4' &= -i\alpha \left[\tilde{T}(4) e^{-ik_G x} u_1 u_3 + \tilde{T}(5) e^{-i(k_4 - 2k_2)x} u_2^2 + \tilde{P}(4) e^{ik_F x} u_1^* u_5\right] \\
u_5' &= -i\alpha \left[\tilde{P}(5) e^{-ik_F x} u_1 u_4 + \tilde{P}(6) e^{-ik_E x} u_2 u_3\right]
\end{aligned}$$

The coefficients of these systems are given by

$$\tilde{Q}(1) = \frac{3}{2}\frac{k_2 - k_1}{(1 - \frac{\beta}{2}k_1^2)} \qquad \text{and} \qquad \tilde{Q}(2) = \frac{3}{2}\frac{k_1}{(1 - \frac{\beta}{2}k_2^2)}$$

and, similarly,

$$\tilde{S}(1) = \frac{3}{2}\frac{k_3 - k_2}{(1 - \frac{\beta}{2}k_1^2)}; \quad \tilde{S}(2) = \frac{3}{2}\frac{k_3 - k_1}{(1 - \frac{\beta}{2}k_2^2)}; \quad \tilde{S}(3) = \frac{3}{2}\frac{k_1 + k_2}{(1 - \frac{\beta}{2}k_3^2)}$$

and, also,

$$\tilde{T}(1) = \frac{3}{2}\frac{k_4 - k_3}{(1 - \frac{\beta}{2}k_1^2)}; \quad \tilde{T}(2) = \frac{3}{2}\frac{k_4 - k_2}{(1 - \frac{\beta}{2}k_2^2)}; \quad \tilde{T}(3) = \frac{3}{2}\frac{k_4 - k_1}{(1 - \frac{\beta}{2}k_3^2)};$$

$$\tilde{T}(4) = \frac{3}{2}\frac{k_1 + k_3}{(1 - \frac{\beta}{2}k_4^2)}; \qquad \tilde{T}(5) = \frac{3}{2}\frac{k_2}{(1 - \frac{\beta}{2}k_4^2)}$$

and, finally,

$$\tilde{P}(1) = \frac{3}{2}\frac{k_5 - k_4}{(1 - \frac{\beta}{2}k_1^2)}; \quad \tilde{P}(2) = \frac{3}{2}\frac{k_5 - k_3}{(1 - \frac{\beta}{2}k_2^2)}; \quad \tilde{P}(3) = \frac{k_5 - k_2}{(1 - \frac{\beta}{2}k_3^2)};$$

$$P(4) = \frac{3}{2}\frac{k_5 - k_1}{(1 - \frac{\beta}{2}k_4^2)}; \quad P(5) = \frac{3}{2}\frac{k_1 + k_4}{(1 - \frac{\beta}{2}k_5^2)}; \quad P(6) = \frac{3}{2}\frac{k_2 + k_3}{(1 - \frac{\beta}{2}k_5^2)}.$$

As in the BBM case, the theory derived in the previous sections does not hold for these higher mode solutions. In particular, conservation of energy in the form

$$\sum_j c_j \cdot r_j r_j' = 0 \tag{3.54}$$

(where the c_j are constants involving the Lau-Barcilon coefficients) does not hold in the $N = 4$ and higher mode cases. Refer to the identical argument given in the corresponding section in Chapter 2 to verify this assertion.

The next issue to study is the accuracy of the approximation.

3.7 Error Calculations

The goal here is to apply the KdV operator to the approximate solution to the KdV equation which was obtained via the Lau-Barcilon equations. The KdV operator, after the two-scale expansion, has the following form:

$$\begin{aligned} KdV &= \partial_t + (\partial_x + \alpha\partial_X) + \frac{3}{2}\alpha(\cdot, \partial_x + \alpha\partial_X) + \frac{\beta}{6}(\partial_x + \alpha\partial_X)^3 \\ &= \partial_t + \partial_x + \alpha\partial_X + \frac{3}{2}\alpha(\cdot, \partial_x) + \frac{3}{2}\alpha^2(\cdot, \partial_X) + \frac{\beta}{6}\partial_x^3 \\ &\quad + \frac{\alpha\beta}{2}\partial_x^2\partial_X + \frac{\alpha^2\beta}{2}\partial_x\partial_X^2 + \frac{\alpha^3\beta}{6}\partial_X^3. \end{aligned}$$

Apply this operator to the N modes solution

$$u(x, X, t) = \sum_{j=-N}^{N} u_j(X) e^{i(k_j x - \omega_j t)}$$

where the $u_j(X)$ satisfy the Lau-Barcilon equations for KdV. Carry out the computations exactly as in Chapter 2, only replacing Q_j, R_j with $\tilde{Q}_j, \tilde{R}_j$ as given in equations 3.18 and 3.19.

The resulting error estimate is

$$\begin{aligned} KdV(u) = \quad & E_1\alpha^2\left[\frac{3}{2}e^{i\Delta_k x}\alpha(u_1^* u_2)' - i\alpha^2\frac{\beta}{2}k_1 u_1'' + \alpha^4\frac{\beta}{6}u_1'''\right] \\ + \quad & E_2\alpha^2\left[\frac{3}{2}e^{-i\Delta_k x}\alpha\left(\frac{1}{2}u_1^2\right)' - i\alpha^2\frac{\beta}{2}k_2 u_2'' + \alpha^4\frac{\beta}{6}u_2'''\right] \\ + \quad & E_3\left(\alpha e^{ik_H x}\right)\left[i\frac{3}{2}(k_1 + k_2)u_1 u_2 + \alpha^2\frac{3}{2}(u_1 u_2)'\right] \\ + \quad & E_4\left(\alpha e^{-i(k_4 - 2k_2)x}\right)\left[-i\frac{3}{2}k_2 u_2^{*2} + \alpha^2\frac{3}{2}\left(\frac{1}{2}u_2^{*2}\right)'\right] \\ + \quad & \text{conj} \end{aligned}$$

$$
\begin{aligned}
= \quad & E_1\alpha^3\left[\frac{3}{2}e^{i\Delta_k x}(u_1^*u_2)' - i\alpha\frac{\beta}{2}k_1u_1'' + \alpha^3\frac{\beta}{6}u_1'''\right] \\
+ \quad & E_2\alpha^3\left[\frac{3}{2}e^{-i\Delta_k x}\left(\frac{1}{2}u_1^2\right)' - i\alpha\frac{\beta}{2}k_2u_2'' + \alpha^3\frac{\beta}{6}u_2'''\right] \\
+ \quad & E_3\alpha\left(e^{ik_Hx}\right)\frac{3}{2}\left[i(k_1+k_2)u_1u_2 + \alpha^2(u_1u_2)'\right] \\
+ \quad & E_4\alpha\left(e^{-i(k_4-2k_2)x}\right)\frac{3}{2}\left[-ik_2u_2^{*2} + \alpha^2\left(\frac{1}{2}u_2^{*2}\right)'\right] \\
+ \quad & \text{conj}
\end{aligned}
$$

case and to determine a general form for larger numbers of modes.

For the three modes case, the error terms have the form

$$
\begin{aligned}
KdV(u) \quad = \quad & E_1\cdot\alpha^3\Bigg[\alpha^3\frac{\beta}{6}u_1''' + \frac{3}{2}(u_1^*u_2)'e^{i\Delta_k x} + \frac{3}{2}(u_2^*u_3)'e^{-ik_Hx} \\
& \qquad -i\alpha\frac{\beta}{2}k_1u_1''\Bigg] \\
+ \quad & E_2\cdot\alpha^3\Bigg[\alpha^3\frac{\beta}{6}u_2''' + \frac{3}{2}\left(\frac{1}{2}u_1^2\right)'e^{-i\Delta_k x} + \frac{3}{2}(u_1^*u_3)'e^{-ik_Hx} \\
& \qquad -i\alpha\frac{\beta}{2}k_2u_2''\Bigg] \\
+ \quad & E_3\cdot\alpha^3\left[\alpha^3\frac{\beta}{6}u_2''' + \frac{3}{2}(u_1u_2)'e^{ik_Hx} - i\alpha\frac{\beta}{2}k_3u_3''\right] \\
+ \quad & E_4\cdot\frac{3}{2}\alpha\Bigg[\left(ik_2u_2^2 + \alpha^2\left(\frac{1}{2}u_2^2\right)'\right)e^{i(2k_2-k_4)x} \\
+ \quad & \qquad \left(i(k_1+k_3)u_1u_3 + \alpha^2(u_1u_3)'\right)e^{i(k_1+k_3-k_4)x}\Bigg] \\
+ \quad & E_5\cdot\frac{3}{2}\alpha\left[i(k_2+k_3)u_2u_3 + \alpha^2(u_2u_3)'\right]e^{i(k_2+k_3-k_5)x} \\
+ \quad & E_6\cdot\frac{3}{2}\alpha\left[ik_3u_3^2 + \alpha^2\left(\frac{1}{2}u_1^{*2}\right)'\right]e^{i(2k_3-k_6)x} \\
+ \quad & \text{conj.}
\end{aligned}
$$

In summary,

$$
E_j: \qquad \alpha^3\left[O(1) + O(\alpha)\right] \qquad \text{for } j = 1, 2, 3,
$$

$$E_j : \qquad \alpha\left[O(1) + O(\alpha^2)\right] \qquad \text{for } j = 4, 5, 6. \tag{3.55}$$

The obvious generalization of the error calculation to N modes is to obtain

$$KdV(u) = E_1 + \ldots + E_N + E_{N+1} \ldots E_{2N}$$

where the first N terms are $O(\alpha^3)$ from $\alpha^2 \cdot (u_j u_k)'$ and the last N terms are $O(\alpha)$ from $\alpha u u_x$ and $\alpha^2 u u_X$. As in Chapter 2, the terms in $KdV(u)$ may be viewed according to their contribution to the error. The approximation is good in the last half of the modes only if not much energy cascades into these modes from the dominant modes.

3.8 Numerical Analysis

3.8.1 Numerics

Again, the reduction from a fully nonlinear evolution equation to a system of simple ordinary differential equations provides a remarkable computational advantage. Numerical solutions to the model equation were computed with the Euler difference method implemented in the environment of the commercial package MATLAB, just as described in Chapte 2 for the BAE and BBM equations.

All numerical results presented in this section used the parameter values

$$\alpha = .15 \qquad \beta = \left(\frac{1}{12}\right)^2 \qquad \omega_1 = \frac{\pi}{6} \tag{3.56}$$

Boundary data values $u_j(0)$ are indicated in the heading of each figure.

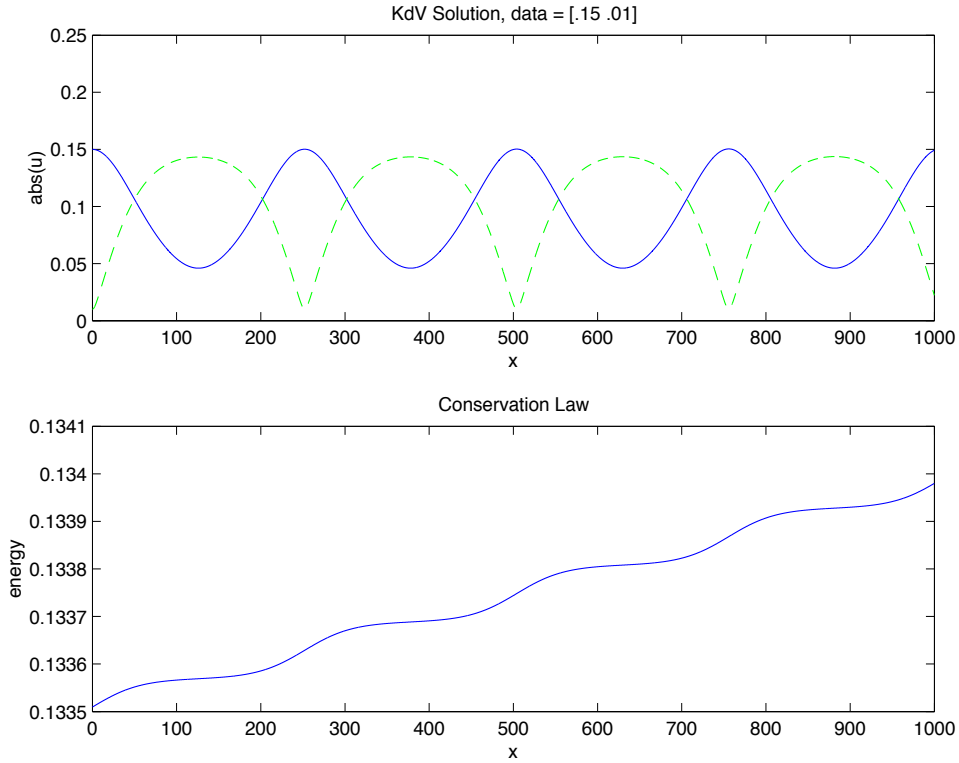

Figure 3.3: KdV modal amplitudes, $N = 2$: u_1 blue solid, u_2 green dashed

3.8.2 Numerical results for KdV, $N = 2$

For the case of two modes, the solution of the Lau-Barcilon differential equations for KdV is given in Figure (3.3).

The second part of the figure shows the behavior of the two modes conservation law. Its small, linear growth provides one indication of the level of numerical error.

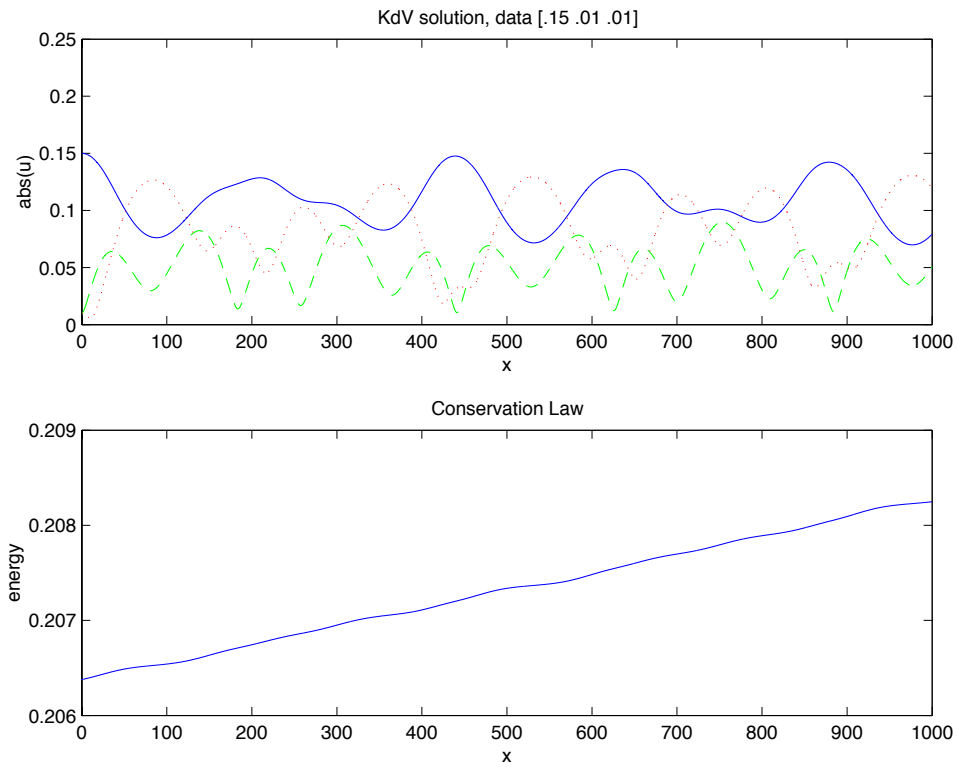

Figure 3.4: KdV modal amplitudes, $N = 3$: u_1 blue solid, u_2 green dashed, u_3 red dotted

3.8.3 Numerical results for KdV, $N = 3$

Figure (3.4) shows the solution of the $N = 3$ modes Lau-Barcilon equations for KdV. Just as in the BBM case, the cascade of energy from lower to higher modes is evident in the relative magnitudes of the modal amplitudes.

3.8.4 Numerical comparison of KdV and BBM

It is reasonable to wonder, at this point, how the results for the BBM model and the KdV model compare in the modal expansion. Numerically, they are nearly identical: Figure (3.5) shows the magnitude of the differences of each modal solution in the $N = 2$ case, computed as described in Chapter 2 for the comparison between BBM and BAE.

Notice the small scale of the vertical axis. Similarly, figure (3.6) shows the magnitude of the differences in the $N = 3$ case.

3.8.5 Numerical conclusions

These analytical studies suggest that the modal expansion technique provides a reasonable approximate solution to the KdV equation, as for the BBM equation, in the context of the wave-sandbar interaction model of Karakiewicz and Bona.

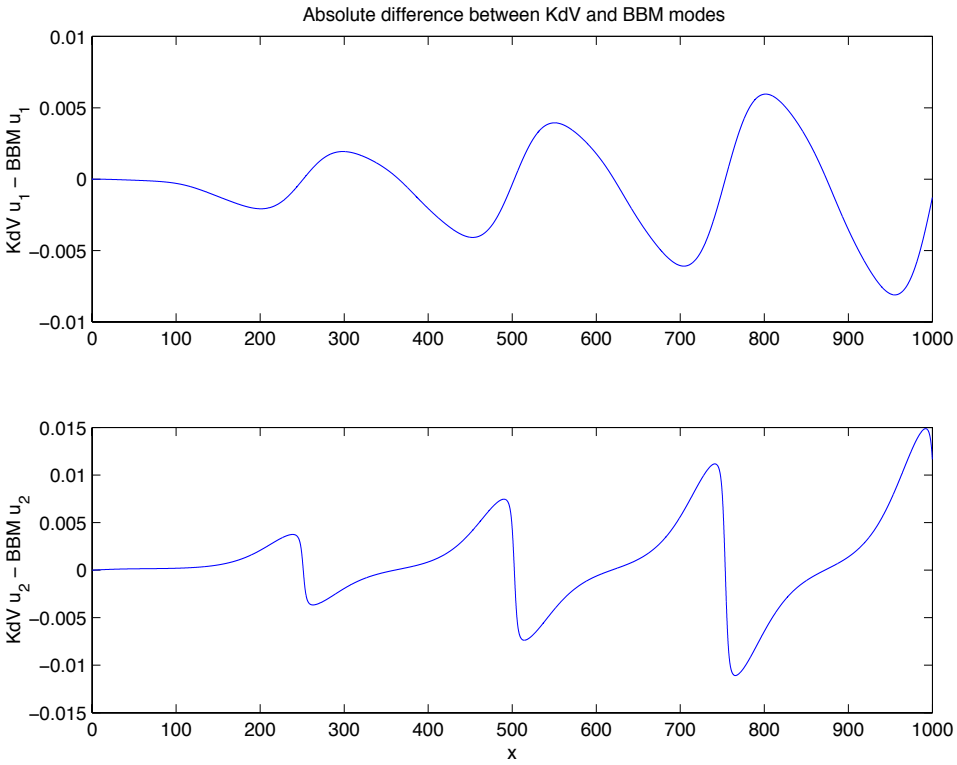

Figure 3.5: Comparison of the two modes KdV and BBM amplitudes

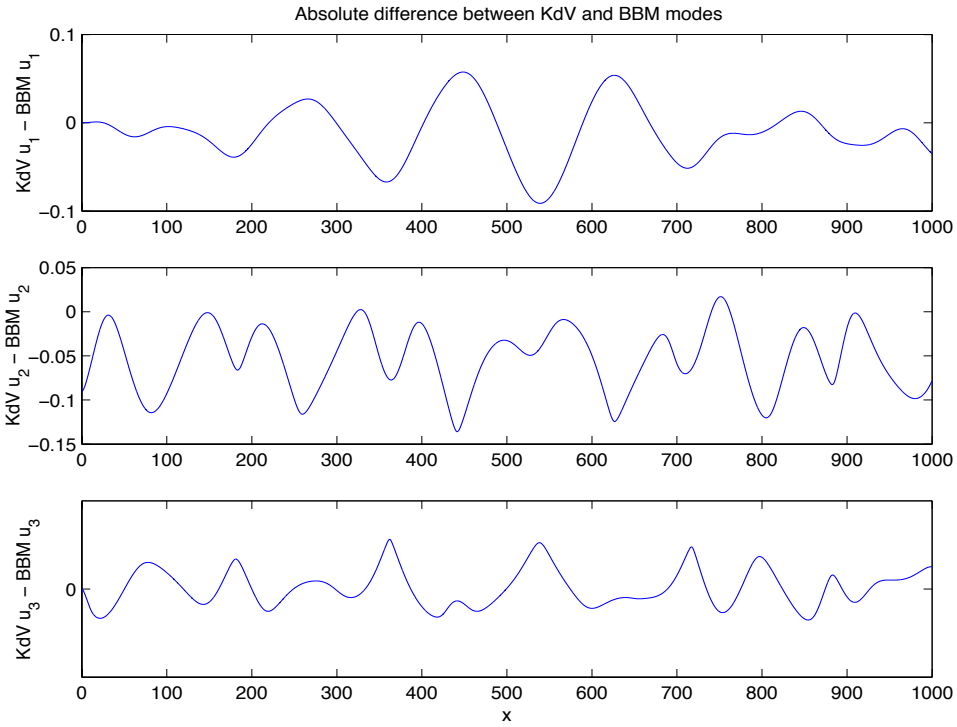

Figure 3.6: Comparison of the three modes KdV and BBM amplitudes

Chapter 4

Modal Analysis of KdV-Type Long Wave Models

The previous chapters applied the Lau-Barcilon derivation to gain a deeper understanding of an effective approximation procedure (the modal expansion) applied to the BBM and KdV equations. The first two sections of this chapter describe the Lau-Barcilon theory as applied to long wave models incorporating general dispersion relations.

4.1 The Two-Scale Form

Begin with a KdV-type model of long wave behavior:

$$u_t + u_x + \alpha u u_x - \beta M(\partial_x) u_x = 0 \tag{4.1}$$

where M is a differential (or pseudo-differential) operator.

Now apply a two-scaling argument by taking $X = \alpha x$, so that ∂_x is replaced in the scaled BBM equation by $\partial_x + \alpha \partial_X$. This procedure yields

$$u_t + u_x + \alpha u_X + \alpha u u_x - \beta M(\partial_x + \alpha \partial_X)(u_x + \alpha u_X) = 0, \tag{4.2}$$

neglecting $O(\alpha^2)$.

In order to sort these terms by orders of α, we need to consider the properties of M and, hence, of its symbol $m(k)$.

The symbol $m(k)$ corresponds to the dispersion relation determined by the governing PDE, in that knowledge of the dispersion relation uniquely determines the symbol and vice versa.

4.2 Linear KdV

For example, the linear KdV equation

$$u_t + u_x + u_{xxx} = 0$$

corresponds to the case in which

$$M(\partial_x) = -\partial_x^2 \qquad \text{and} \qquad m(k) = M(ik) = k^2. \tag{4.3}$$

In this case, the dispersion term is given by

$$\begin{aligned} \beta M(\partial_x + \alpha\partial_X)(u_x + \alpha u_X) &= \beta(\partial_x^2 + 2\alpha\partial_x\partial_X + \alpha^2\partial_X^2)(u_x + \alpha u_X) \\ &= \beta(\partial_x^2 u_x + 3\alpha\partial_x^2 u_X) + O(\alpha^2) \end{aligned} \tag{4.4}$$

Notice that the $O(\alpha^0)$ terms give the dispersion relation $\omega(k) = k(1 - k^2)$. Thus, the linear KdV dispersion relation has the form

$$\omega(k) = k(1 - k^2) = k(1 - m(k)).$$

This is, of course, also the form of the dispersion relation for the generalized linear KdV form,

$$u_t + u_x - M(\partial_x)u_x = 0,$$

since substitution of $u = e^{i(kx-\omega t)}$ gives $-i\omega + ik - m(k)ik = 0$ so that

$$\omega(k) = k - km(k) = k(1 - m(k)).$$

4.3 Polynomial operators M

For an operator $M(\partial_x) = c\partial_x^p$, the corresponding symbol is $m(k) = c(ik)^p$. The dispersive term in (4.2) above has the form

$$\begin{aligned} c(\partial_x + \alpha\partial_X)^p(u_x + \alpha u_X) &= c\partial_x^p u_x + \alpha c\partial_x^p u_X + \alpha pc\partial_x^{p-1}\partial_X u_x + O(\alpha^2) \\ &= c\partial_x^p u_x + c(p+1)\alpha\partial_x^p u_X + O(\alpha^2) \end{aligned}$$

so that the two-scale form is

$$u_t + u_x + \alpha u_X - \beta c\partial_x^p u_x - \alpha c(p+1)\beta\partial_x^p u_X = 0$$

$$u_t + u_x - \beta c\partial_x^p u_x + \alpha\left[u_X - c(p+1)\beta\partial_x^p u_X\right] = 0 \tag{4.5}$$

At the lowest order, $O(\alpha^0)$, this equation gives the dispersion relation $-i\omega + ik - \beta c(ik)^p ik = 0$, that is,

$$\omega = k(1 - \beta m(k)).$$

Substitution of the modal expansion

$$u(x, X, t) = \sum_j u_j(X)e^{i(k_j x - \omega_j t)}$$

into the $O(\alpha)$ equation yields essentially the left-hand side of the Lau-Barcilon equations, $u_j' - \beta(p+1)m(k)u_j' = 0$, which simplifies to

$$\left(1 - (p+1)\beta m(k)\right)u_j' = 0 \tag{4.6}$$

Hence, applying this analysis to the nonlinear equation, in which a term of the form uu_x is present at the beginning (as in Chapters 2 and 3), yields Lau-Barcilon equations of the form

$$(1-(p+1)\beta m(k))\,u_j' = -\,(\text{nonlinear terms}) \tag{4.7}$$

Now, consider the case in which M is actually a polynomial with zero constant term. (A nonzero constant term in M would give only another term of the form u_x, already present in the model equation. One could simply change variables and scale out the factor of $(c+1)$ in front of u_x.) Thus, in general, set

$$M(\partial_x) = \sum_{\kappa=1}^{n} c_\kappa(\partial_x^{p_\kappa}).$$

Apply this form of M to the two-scaled linear KdV equation, (4.2) above, and use the preceding analysis of the $c\partial_x^p$ case to find that

$$0 = u_t + u_x + \alpha u_X - \beta \sum_{\kappa=1}^{n} c_\kappa\left(\partial_x^{p_\kappa} u_x + (p_\kappa+1)\alpha\partial_x^{p_\kappa} u_X\right) + O(\alpha^2).$$

Sorting these terms in orders of α gives

$$u_t + u_x - \beta\sum_{\kappa=1}^{n} c_\kappa(\partial_x^{p_\kappa})u_x + \alpha\left[u_X - \beta\sum_{\kappa=1}^{n} c_\kappa(p_\kappa+1)\partial_x^{p_\kappa}u_X\right] = 0 \tag{4.8}$$

4.4 Smooth operators M

In the general case, given in two-scaled form as (4.2) above, it is necessary to expand the expression $M(\partial_x + \alpha\partial_X)$. Using Taylor's formula and assuming sufficiently smooth behavior of M,

$$M(\partial_x + \alpha\partial_X) = M(\partial_x) + \alpha M'(\partial_x)\partial_X + O(\alpha^2).$$

For particular choices of M, Taylor's Remainder Theorem must be applied to guarantee that the tail of this expansion really remains $O(\alpha^2)$.

Apply this expansion to the two-scaled dispersion term, in (4.2) above, gives

$$\begin{aligned} 0 &= u_t + u_x + \alpha u_X - \beta\left(M(\partial_x) + \alpha M'(\partial_x)\partial_X\right)(u_x + \alpha u_X) \\ &= u_t + u_x - \beta M(\partial_x)u_x + \alpha\left[u_X - \beta\left(M(\partial_x)u_X + M'(\partial_x)\partial_x u_X\right)\right] + O(\alpha^2) \end{aligned}$$

Thus, the $O(\alpha^0)$ terms yield the usual dispersion relation,

$$u_t + u_x - \beta M(\partial_x)u_x = 0. \tag{4.9}$$

For $u = e^{i(kx-\omega t)}$, $-\omega + k - k\beta m(k) = 0$ and, hence,

$$\omega = k - \beta k m(k). \tag{4.10}$$

The $O(\alpha)$ terms yield the following

$$[1 - \beta\left(M(\partial_x) + M'(\partial_x)\partial_x\right)]u_X = 0 \tag{4.11}$$

This is the general formulation, so the corresponding Lau-Barcilon equations for the nonlinear case will look like

$$[1 - \beta\left(m(k) + km'(k)\right)]u_j' = -(\text{nonlinear terms}) \tag{4.12}$$

Note that there is no factor of i in the term $km'(k)$ (a result of the $M'(\partial_x)\partial_x$ term) since $m'(k) = \frac{d}{dk}M(ik) = iM'(ik)$; thus, the factor of i from the Fourier transform of ∂_x is absorbed into $m(k)$.

4.4.1 The Benjamin-Ono equation

One example of interest is the Benjamin-Ono equation, discussed in Abdelouhab et al. [1] The Benjamin-Ono equation has the general form as in (4.1) above, where the symbol of the dispersion operator is given by

$$m(k) = 2\pi|k| \tag{4.13}$$

Note that, for the purposes of the modal expansion, $k \in \mathbb{R}^+$. In this case, then, the dispersion relation is

$$\omega = k - 2\pi\beta k^2, \tag{4.14}$$

as graphed in Figure (4.1).

Note, as for BAE, BBM, and KdV, the dispersion relation may be used to estimate the size of ω: $k \approx 2\pi\beta$ forces $\omega \approx 2\pi\beta$ to $O(\beta^3)$.

The Lau-Barcilon equations (4.12) then become

$$[1 - 4\pi\beta k]u_j' = -(\text{nonlinear terms}) \tag{4.15}$$

The parameters used to compute the numerical solution of these equations are

$$\alpha = .15 \qquad \beta = \left(\frac{1}{12}\right) \qquad \omega_1 = \frac{\pi}{24}. \tag{4.16}$$

Boundary data values $u_j(0)$ are indicated in the heading of each figure.

The two modes solution of these equations is very interesting, clearly demonstrating the oscillation of energy between the two modal amplitudes. However, the second mode exhibits little growth, remaining small with small initial data.

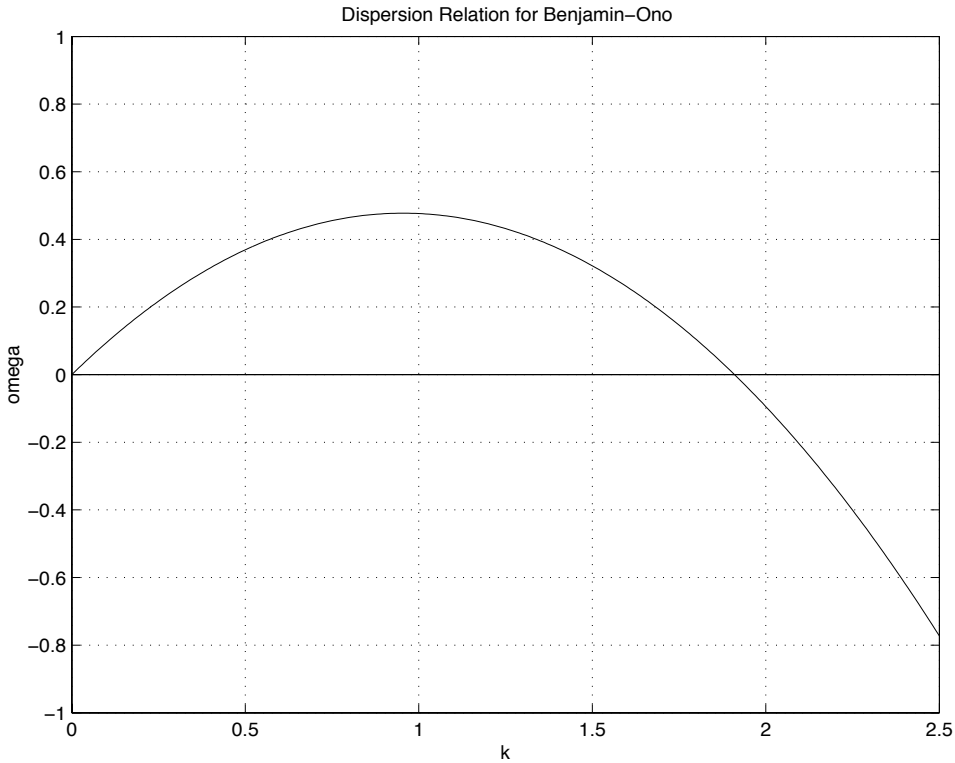

Figure 4.1: Benjamin-Ono dispersion relation

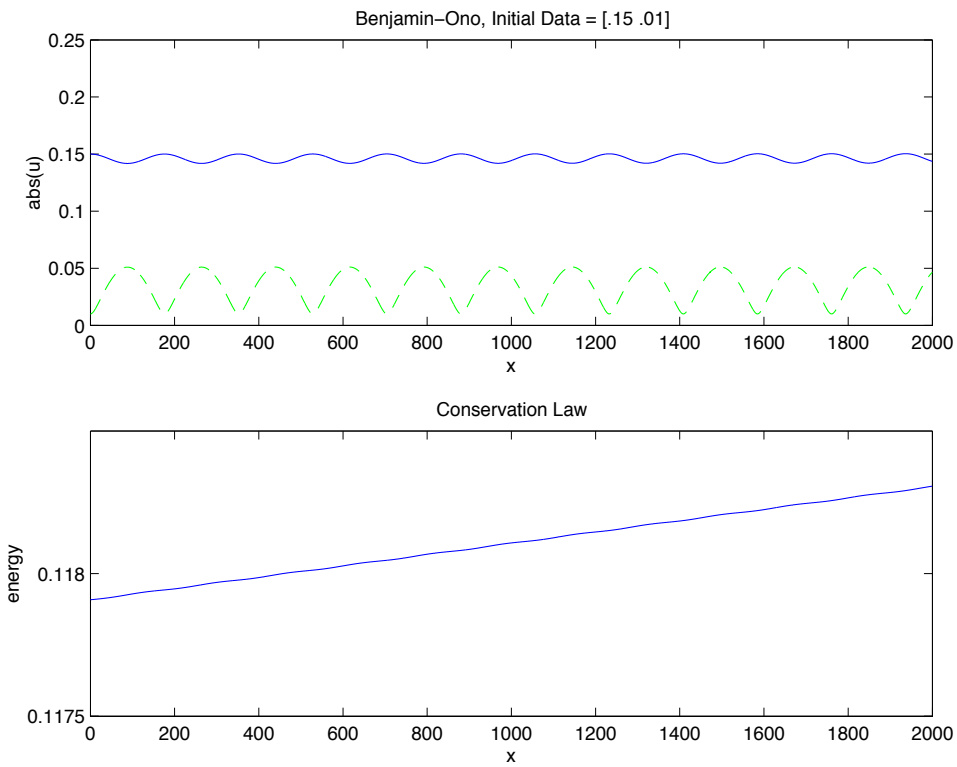

Figure 4.2: Benjamin-Ono solution $N = 2$, small data in second mode: u_1 blue solid, u_2 green dashed

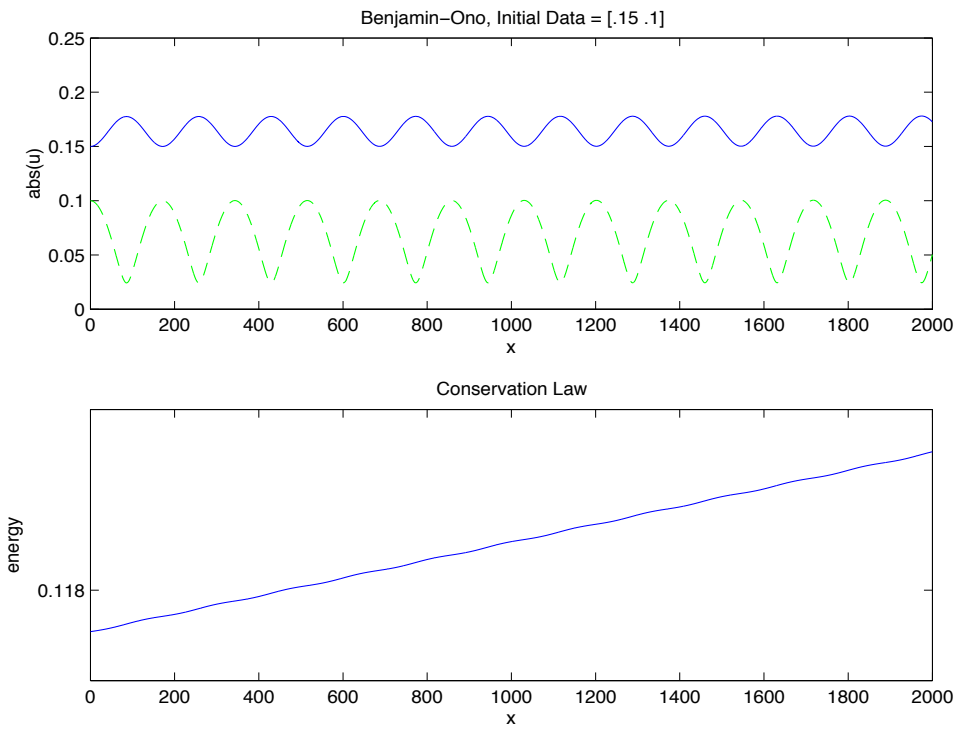

Figure 4.3: Benjamin-Ono solution $N = 2$, medium data in second mode: u_1 blue solid, u_2 green dashed

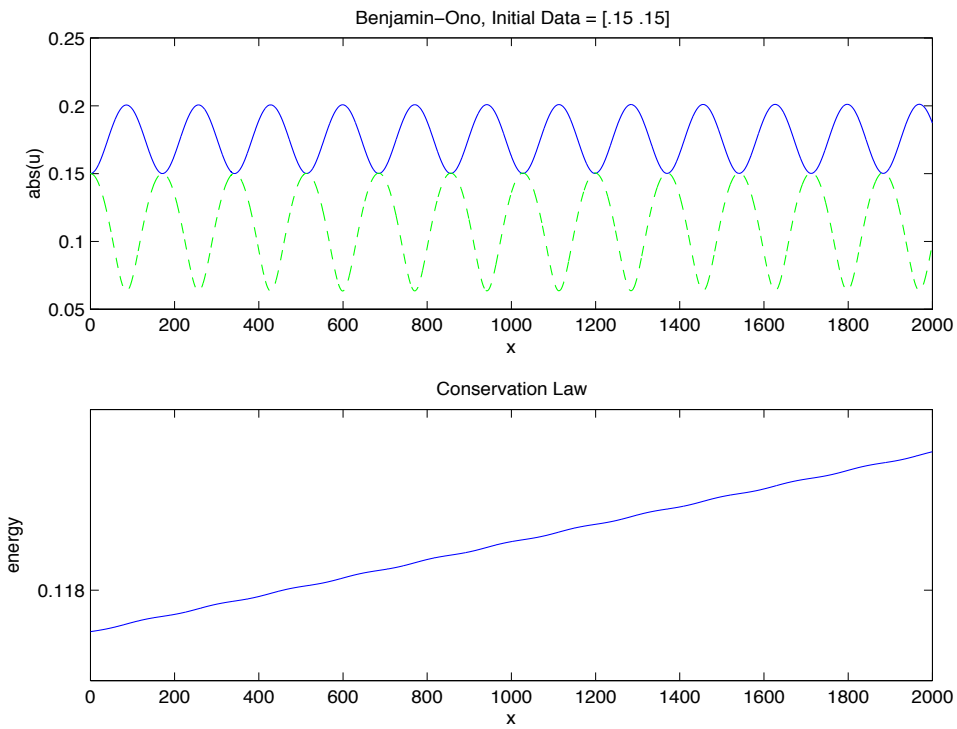

Figure 4.4: Benjamin-Ono solution $N = 2$, large data in second mode: u_1 blue solid, u_2 green dashed

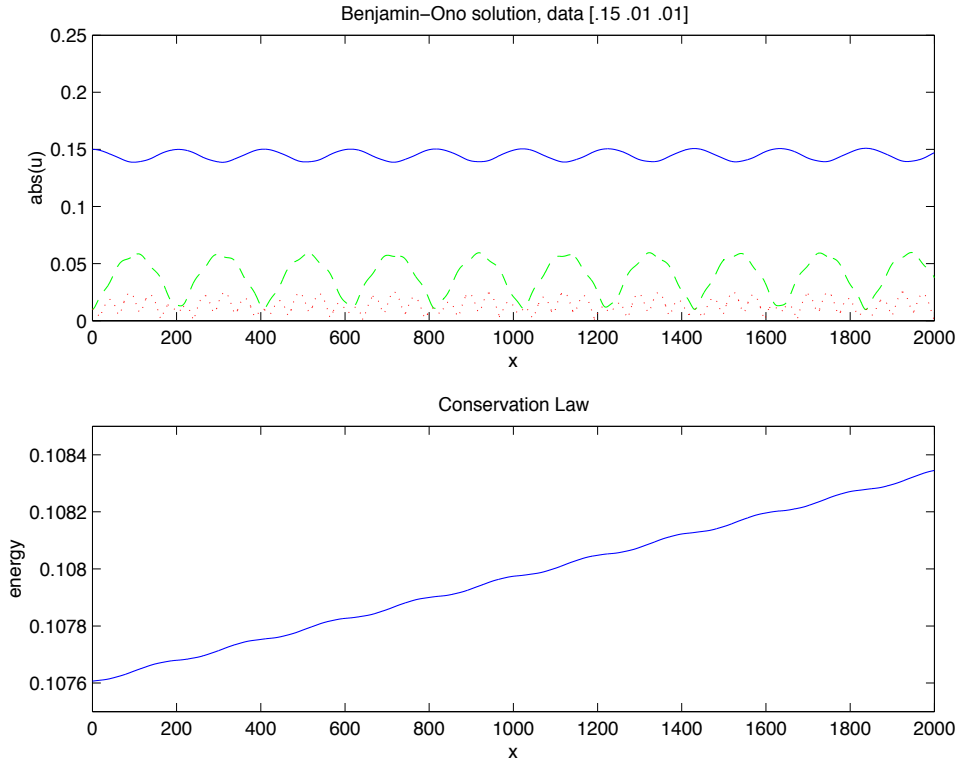

Figure 4.5: Benjamin-Ono solution $N = 3$: u_1 blue solid, u_2 green dashed, u_3 red dotted

Figures (4.5) and (4.6) show the modal amplitudes for the $N = 3$ Lau-Barcilon equations modeling the Benjamin-Ono equation, using different sizes of boundary data. Note that the energy levels are, of course, greatly affected by the boundary data.

4.4.2 Intermediate Long Wave equation

The intermediate long wave equation (ILW) is another example of great interest, as it may be used to describe two-layer fluid flow.

The ILW equation has the form in (4.1) above, where the symbol of

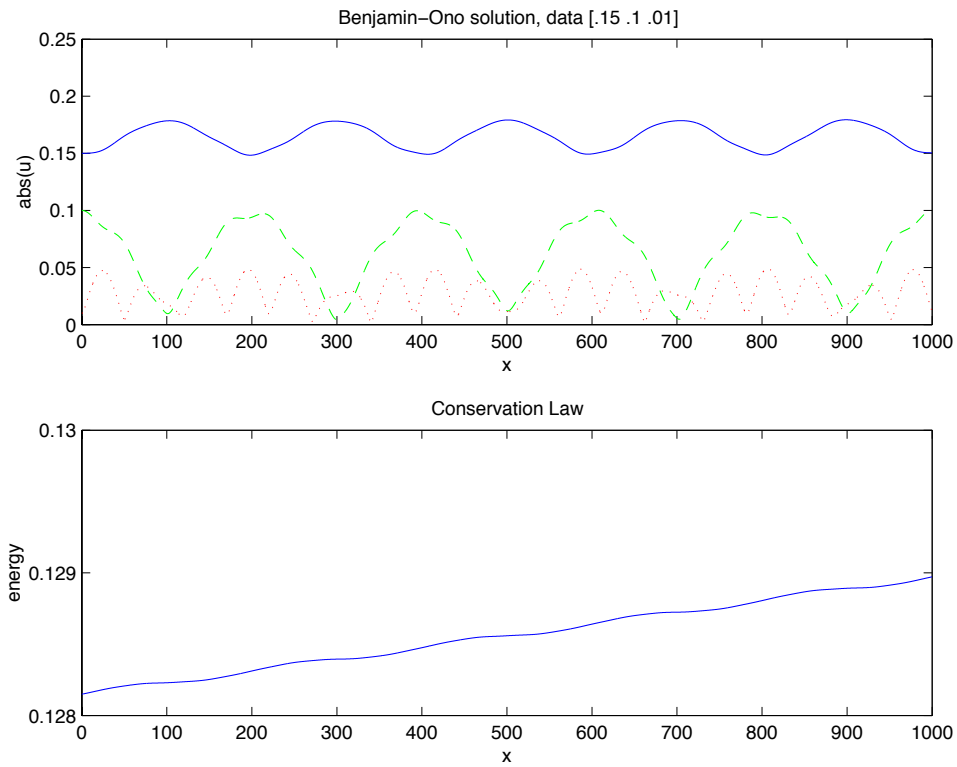

Figure 4.6: Benjamin-Ono solution $N = 3$: u_1 blue solid, u_2 green dashed, u_3 red dotted

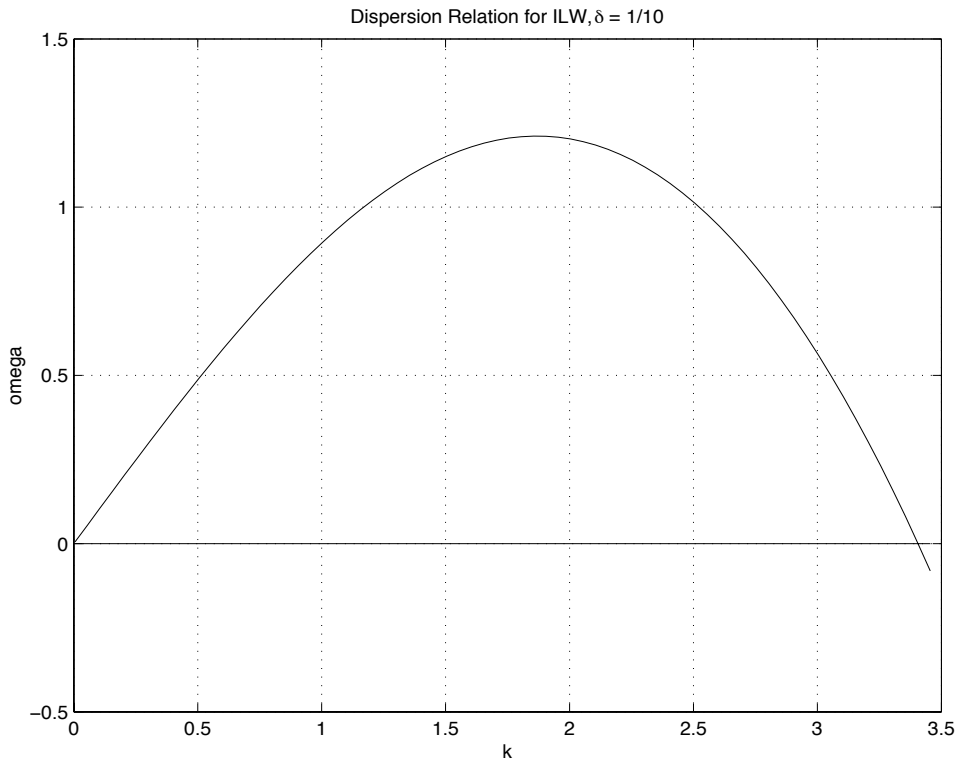

Figure 4.7: ILW Dispersion Relation with $\delta = .1$

the dispersion operator is given by

$$m_\delta(k) = 2\pi k \coth(2\pi\delta k) - \frac{1}{\delta}. \tag{4.17}$$

The parameter δ characterizes the depth of the lighter, upper fluid layer in the system. The dispersion relation for ILW is

$$\omega = k - \beta k \left(2\pi k \coth(2\pi\delta k) - \frac{1}{\delta} \right). \tag{4.18}$$

A graph of this dispersion relation for $\delta = .1$ is shown in Figure (4.7).

As for the other operators discussed here, $k \approx 2\pi\beta$ forces $\omega \approx 2\pi\beta$ to

$O(\beta^2)$ as long as $\frac{1}{\delta} \approx O(\beta)$. The derivative of m_δ is

$$m'_\delta(k) = 2\pi \coth(2\pi\delta k) - \frac{4\pi^2\delta k}{\operatorname{csch}^2(2\pi\delta k)} \tag{4.19}$$

so that the Lau-Barcilon equations for ILW have the form

$$\left[1 - \beta\left(2\pi k \coth(2\pi\delta k) - \frac{1}{\delta} + k m'_\delta(k)\right)\right] u'_j = -(\text{nonlinear terms}) \tag{4.20}$$

The parameters used to compute the numerical solution of these equations are

$$\alpha = .15 \qquad \beta = \left(\frac{1}{12}\right) \qquad \omega_1 = \frac{\pi}{6}. \tag{4.21}$$

Boundary data values $u_j(0)$ are indicated in the heading of each figure.

An interesting line of future research may develop from the work of Abdelouhab et al. They established rigorously the fact that the KdV and Benjamin-Ono equations may be obtained as limiting forms of ILW as δ tends to zero and infinity, respectively. It would be interesting to compare numerical results of the modal approximation to ILW for increasingly small and large values of δ with numerical results of the modal approximations to KdV and Benjamin-Ono.

4.4.3 The Smith equation

The final example in this section is the Smith equation, having the form in (4.1) above, where the symbol of the dispersion operator is given by

$$m_s(k) = 2\pi\left(\sqrt{k^2+1} - 1\right). \tag{4.22}$$

The dispersion relation for the Smith equation is, then,

$$\omega = k - \beta k \cdot 2\pi\left(\sqrt{k^2+1} - 1\right). \tag{4.23}$$

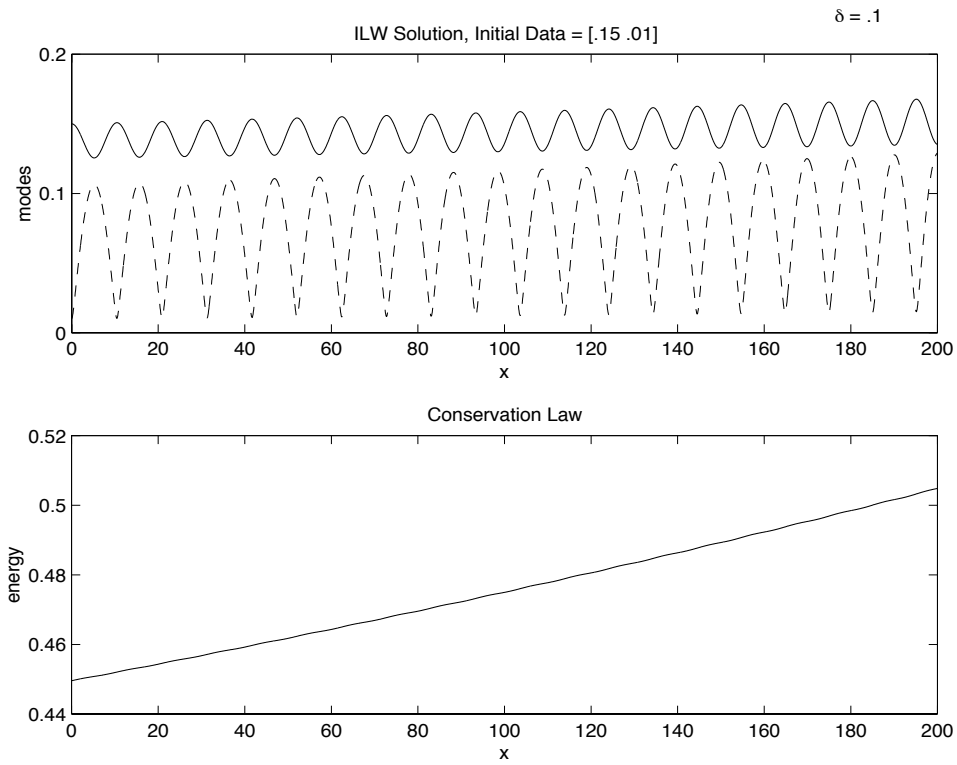

Figure 4.8: ILW solution $N = 2$: u_1 solid, u_2 dashed

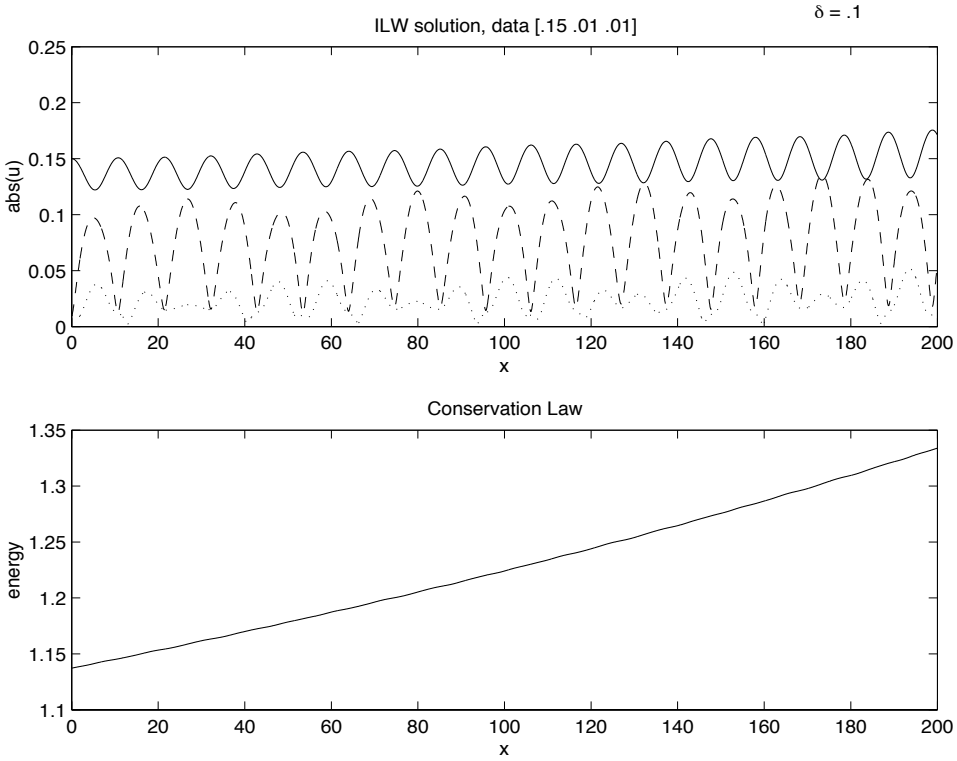

Figure 4.9: ILW solution $N = 3$: u_1 solid, u_2 dashed, u_3 dotted

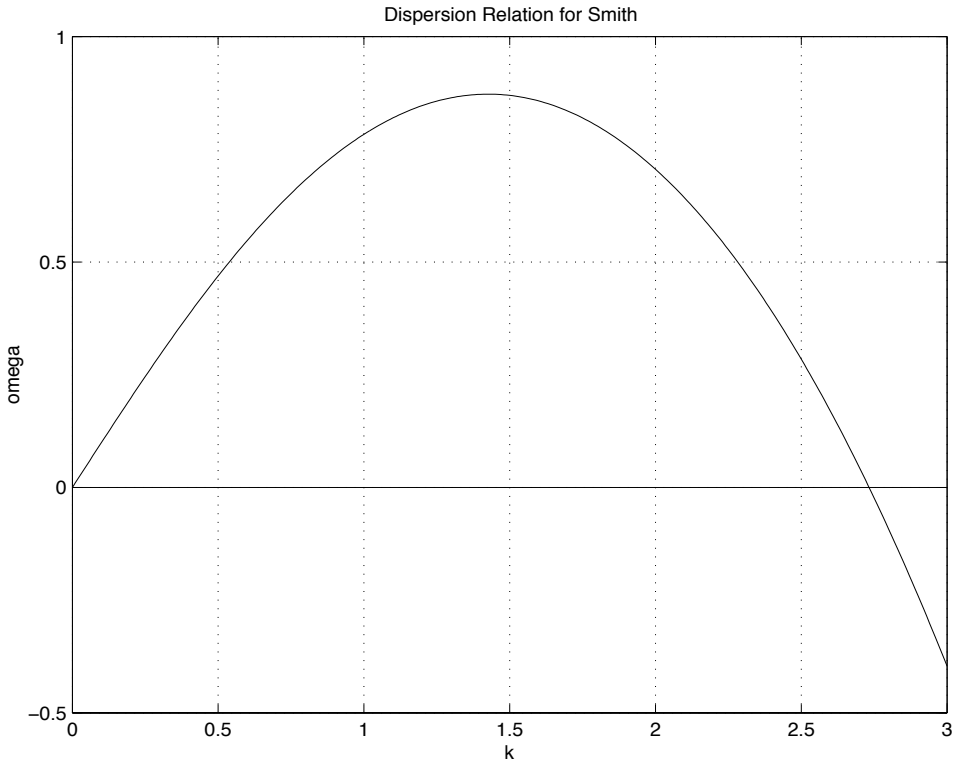

Figure 4.10: Smith equation dispersion relation

The graph of this dispersion relation is given in Figure (4.10). Again, since $k \approx 2\pi\beta$, it follows that $\omega \approx 2\pi\beta$ to $O(\beta^2)$.

Since

$$m_s'(k) = \frac{2\pi k}{\sqrt{k^2+1}} \tag{4.24}$$

the Lau-Barcilon equations for the Smith equation have the form

$$\left[1 - \beta\left(2\pi\sqrt{k^2+1} - 2\pi + \frac{2\pi k^2}{\sqrt{k^2+1}}\right)\right] u_j' = -(\text{nonlinear terms}) \tag{4.25}$$

The solutions of the two modes Lau-Barcilon equations for the Smith equation are displayed in Figure (4.11). The parameters used to compute the numerical solution of these equations are

$$\alpha = .15 \qquad \beta = \left(\frac{1}{12}\right) \qquad \omega_1 = \frac{\pi}{24}. \tag{4.26}$$

Boundary data values $u_j(0)$ are indicated in the heading of each figure.

The three modes solution is displayed in Figure (4.12).

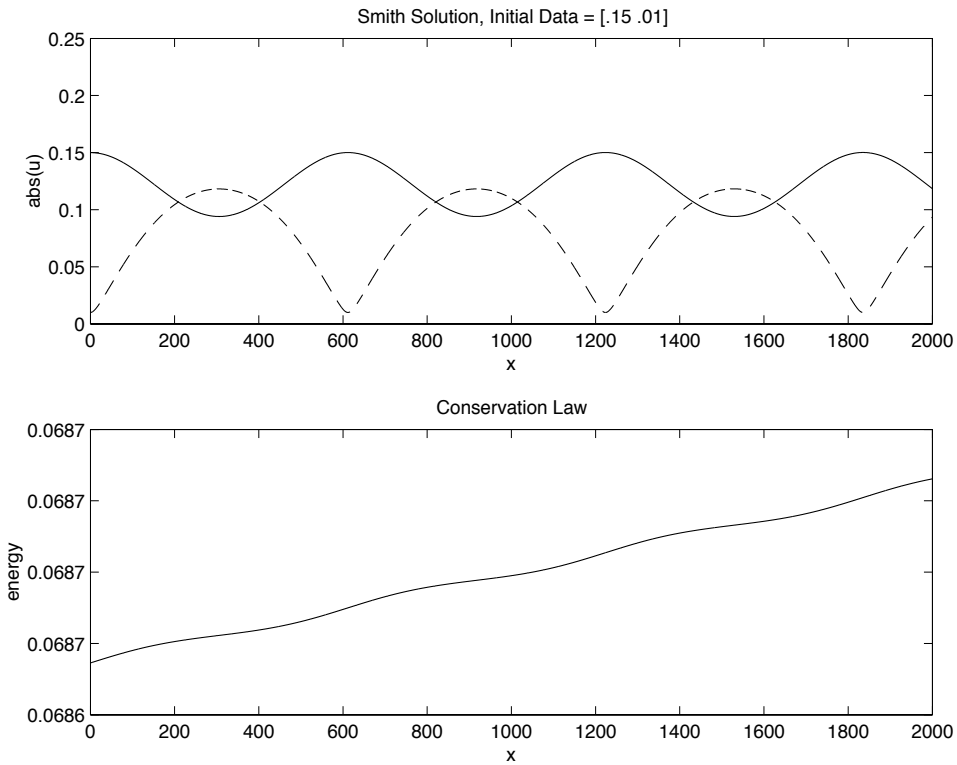

Figure 4.11: Smith solution $N = 2$: u_1 solid, u_2 dashed

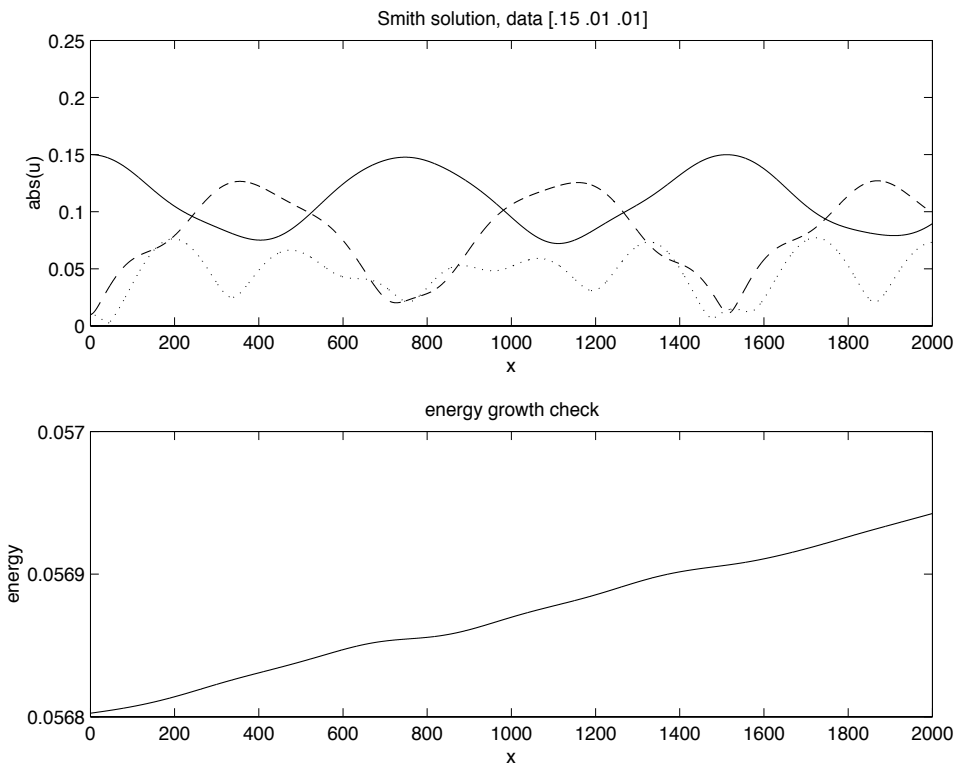

Figure 4.12: Smith solution $N = 2$: u_1 solid, u_2 dashed, u_3 dotted

Chapter 5

Concluding Comments

5.1 Summary

This work has demonstrated the utility and efficiency of the modal expansion as an approximate solution to nonlinear partial differential equations that include dispersive effects. Such equations are relevant in modeling physical phenomena, particularly traveling, nonlinear dispersive surface water waves.

The modal expansion provides a viable computational alternative to a full partial differential equations solver, as it reduces the problem to solving numerically a system of ordinary differential equations. This approach has been demonstrated for well-known nonlinear dispersive equations such as the Korteweg-deVries, Benjamin-Bona-Mahony, Intermediate Long Wave, and Benjamin-Ono equations. The results have also been demonstrated in the context of simple nonlinear partial differential equations for which the dispersion term is characterized by a smooth operator.

5.2 Future Directions

Several interesting lines of research may follow from this work.

1. As mentioned in Chapter 4, the Korteweg-deVries and Benjamin-Ono

equations may be obtained as limiting forms of the Intermediate Long Wave equation. It would be interesting to study rigorously the limiting behavior of the associated modal expansion approximations.

2. The Korteweg-deVries equation has an infinite number of conserved quantities. While the Benjamin-Bona-Mahony equation is not integrable, it too has several associated conserved quantities. A question of interest is, what (if any) new information results from applying a modal expansion to these quantities?

3. Now that the Lau-Barcilon machinery is in place, the next obvious step is to work directly with the wave-sandbar model of Boczar-Karakiewicz et al., continuing a comparison of model predictions with the enormous amount of data collected at various oceanographic research sites worldwide.

4. Yet another direction is to work on improvements to the wave-sandbar model directly, incorporating neglected physical effects such as run-up and reflection, possibly even finding a way to include wave-breaking.

5. Finally, other areas of interest are closely related to this work, including study of the nonlinear Schrödinger equation in context of developing a theory of fully two-dimensional surface wave patterns.

Appendices

Appendix A

Derivation of BBM and KdV

A.1 The Governing Partial Differential Equation

Following Benjamin, Bona, and Mahony [7] begin with conservation of mass,

$$\nabla \cdot u = 0,$$

conservation of momentum,

$$\frac{\partial}{\partial t} u + u \cdot \nabla u = -\frac{1}{\rho} \nabla P - g\vec{k},$$

and the assumption of irrotational flow,

$$\nabla \times u = 0,$$

of an incompressible, inviscid fluid. As usual, when the curl of a vector field u is zero, there exists a potential function $\phi(x, y, z, t)$ such that $u = \nabla\phi$. Thus, the incompressibility condition $\nabla \cdot u = 0$ yields

$$\Delta\phi = 0,$$

the famed Laplace equation.

In the above descriptions, P represents the pressure, g the force of gravity, $\vec{k}$ the unit vector in the vertical (z) direction, and u the velocity field.

Rewriting conservation of momentum in terms of ϕ yields

$$\frac{\partial}{\partial t}\nabla\phi + \frac{1}{2}\nabla(\nabla\phi\cdot\nabla\phi) + \frac{1}{\rho}\nabla P + g\vec{k} = 0$$

from which

$$\nabla\left[\frac{\partial\phi}{\partial t} + \frac{1}{2}\nabla\phi\cdot\nabla\phi + \frac{1}{\rho}P + gz\right] = 0.$$

Since this gradient is zero everywhere, the function of (x, z, t) in the square brackets must be independent of x and z. Thus, it is a function of t alone,

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}\nabla\phi\cdot\nabla\phi + \frac{1}{\rho}(P - P_0) + gz = B(t)$$

where P_0 is the (constant) pressure in the air near the fluid surface. Rewrite this formulation by taking

$$\tilde{\phi}(x, z, t) = \phi(x, z, t) - \int_0^t B(\tau)d\tau$$

so that

$$\frac{\partial\tilde{\phi}}{\partial t} + \frac{1}{2}\nabla\tilde{\phi}\cdot\nabla\tilde{\phi} + \frac{1}{\rho}(P - P_0) + gz = 0.$$

Dropping the tilde and rearranging yields

$$\frac{P - P_0}{\rho} = -\frac{\partial\phi}{\partial t} - \frac{1}{2}\nabla\phi\cdot\nabla\phi - gz.$$

So the problem is to solve the Laplace equation $\Delta\phi = 0$: this yields the velocity field u from $u = \nabla\phi$ and, subsequently, the pressure P from the above formulation. The next task is to determine appropriate boundary conditions. Begin by assuming that the free surface of the fluid is described by the level set of a function $f(x, z, t) = 0$.

A.2 The Kinematic Boundary Condition

The physical meaning of the kinematic boundary condition is simply that the velocity of the fluid at the surface matches the velocity of the surface. Mathematically, this means for $u = (u_1, u_2, v)$

$$\frac{u_1 f_x + u_2 f_y + v f_z}{(f_x^2 + f_y^2 + f_z^2)^{1/2}} = u \cdot \vec{n} = \frac{-f_t}{||\nabla f||}$$

where $\vec{n}$ is the unit normal vector to the surface and where the subscripts indicate differentiation. This gives the condition

$$f_t + u_1 f_x + u_2 f_y + v f_z = 0.$$

If the free surface can be described by a single-valued function of (x, t), namely $f(x, z, t) = \eta(x, t) - z$, then the equation

$$z = \eta(x, t)$$

parametrizes the surface. (This assumption eliminates breaking waves from the model.) Applying this assumption and writing the boundary condition in terms of the velocity potential ϕ, yields the *kinematic boundary condition*

$$\eta_t + \phi_x \eta_x + \phi_y \eta_y = \phi_z.$$

A.3 The Dynamic Boundary Condition

The dynamic boundary condition, following from the conservation of momentum, describes the pressure jump across the free surface. Reference to any standard physics text reminds one that the pressure jump is proportional to the curvature of the surface:

$$\frac{P - P_0}{\rho} = -\frac{\sigma}{\rho} \frac{\eta_{xx}}{\sqrt{(1 + \eta_x^2)^3}}$$

thus the conservation of momentum becomes

$$\frac{\sigma}{\rho}\frac{\eta_{xx}}{\sqrt{(1+\eta_x^2)^3}} + \phi_t + (\nabla\phi)^2 + g\eta = 0.$$

For the case in which surface tension is absent, $\sigma = 0$. The dynamic boundary condition is then

$$\phi_t + (\nabla\phi)^2 + g\eta = 0.$$

A.4 The Boundary Value Problem

The full boundary value problem is the governing partial differential equation within the flow domain together with two boundary conditions at the surface and the condition of zero vertical velocity at the bottom. In three dimensions, this problem is

$$\Delta\phi = 0 \qquad \text{in} \qquad -h_0 < z < \eta \tag{A.1}$$

$$\phi_z = 0 \qquad \text{at} \qquad z = -h_0 \tag{A.2}$$

$$\eta_t + \phi_x\eta_x + \phi_y\eta_y - \phi_z = 0 \qquad \text{at} \qquad z = \eta \tag{A.3}$$

$$\phi_t + \frac{1}{2}(\phi_x^2 + \phi_y^2 + \phi_z^2) + g\eta = 0 \qquad \text{at} \qquad z = \eta \tag{A.4}$$

For the purposes of this derivation, assume uniform behavior in the y direction and work in only two dimensions. The two-dimensional formulation of this problem is

$$\phi_{xx} + \phi_{zz} = 0 \qquad \text{in} \qquad -h_0 < z < \eta \tag{A.5}$$

$$\phi_z = 0 \qquad \text{at} \qquad z = -h_0 \tag{A.6}$$

$$\eta_t + \phi_x\eta_x - \phi_z = 0 \qquad \text{at} \qquad z = \eta \tag{A.7}$$

$$\phi_t + \frac{1}{2}(\phi_x^2 + \phi_z^2) + g\eta = 0 \qquad \text{at} \qquad z = \eta \tag{A.8}$$

Notice that the boundary conditions are partial differential equations that are applied at the unknown and variable free surface, η. Thus, the next step is to determine a change of variables to make this problem more tractable mathematically.

A.5 A Change of Variables

Let the old variables from the previous sections (e.g. x) be replaced by the same variables with tildes (e.g. $\tilde{x}$). Then impose the general change of variables

$$\tilde{x} = Xx; \quad \tilde{z} = Zz - h_0; \quad \tilde{t} = Tt \quad \tilde{\eta} = N\eta; \quad \tilde{\phi} = F\phi$$

After some algebra, the differential equations in the boundary value problem may be written in the dimensionless form

$$\left(\frac{Z}{X}\right)^2 \phi_{xx} + \phi_{zz} = 0 \tag{A.9}$$

$$\phi_z = 0 \tag{A.10}$$

$$\eta_t + \frac{FT}{X^2}\phi_x\eta_x - \frac{FT}{NZ}\phi_z = 0 \tag{A.11}$$

$$\phi_t + \frac{gNT}{F}\eta + \frac{FT}{X^2}\left(\frac{1}{2}\phi_x^2\right) + \frac{FT}{Z^2}\left(\frac{1}{2}\phi_z^2\right) = 0 \tag{A.12}$$

where the units of the scaling quantities are as follows:

$$(X) = (Z) = (N) = \text{length} \tag{A.13}$$

$$(T) = \text{time} \tag{A.14}$$

$$(F) = \text{length}^2/\text{time} \tag{A.15}$$

Notice that these choices make all of the quantities in the scaled boundary value problem *unitless!*

Now, impose the change of variables given by

$$F = \left(\frac{gla}{\sqrt{gh_0}}\right); \qquad T = \left(\frac{l}{\sqrt{gh_0}}\right); \tag{A.16}$$

$$\tag{A.17}$$

$$X = l; \qquad Z = h_0; \qquad N = a \tag{A.18}$$

With these choices, the boundary value problem is transformed as follows.

1. The governing equation:

$$\frac{Z^2}{X^2} = \left(\frac{h_0}{l}\right)^2 = \beta$$

so the partial differential equation becomes

$$\beta\phi_{xx} + \phi_{zz} = 0.$$

2. The flow domain: beginning with $-h_0 < \tilde{z} < \tilde{\eta}$ yields

$$-h_0 \quad < \quad h_0 z - h_0 \quad < \quad a\eta \tag{A.19}$$

$$0 \quad < \quad h_0 z \quad < \quad a\eta + h_0 \tag{A.20}$$

$$0 \quad < \quad z \quad < \quad \frac{a}{h_0}\eta + 1 \tag{A.21}$$

Putting $\alpha = \frac{a}{h_0}$ gives the flow domain in the form

$$0 < z < \alpha\eta + 1.$$

3. The bottom boundary condition: this is the easiest to describe, as it becomes simply

$$\phi_z = 0 \qquad \text{at} \qquad z = 0.$$

4. The kinematic boundary condition: first sort through some algebra.

$$\frac{FT}{X^2} = \frac{\frac{gla}{\sqrt{gh_0}}\frac{l}{\sqrt{gh_0}}}{l^2} = \frac{a}{h_0} = \alpha \tag{A.22}$$

$$\frac{FT}{NZ} = \frac{\frac{gla}{\sqrt{gh_0}}\frac{l}{\sqrt{gh_0}}}{ah_0} = \frac{l^2}{h_0^2} = \frac{1}{\beta} \tag{A.23}$$

Thus, the kinematic boundary condition may be written in terms of the small parameters α and β as follows:

$$\eta_t + \alpha\phi_x\eta_x - \frac{1}{\beta}\phi_z = 0$$

at the free surface.

5. The dynamic boundary condition: again sort through some algebra.

$$\frac{gNT}{F} = \frac{ga\frac{l}{\sqrt{gh_0}}}{\frac{gla}{\sqrt{gh_0}}} = 1 \tag{A.24}$$

$$\frac{FT}{Z^2} = \frac{\frac{gla}{\sqrt{gh_0}}\frac{l}{\sqrt{gh_0}}}{h_0^2} = \frac{l^2}{h_0^2}\,\frac{a}{h_0} = \frac{\alpha}{\beta} \tag{A.25}$$

Hence, the dynamic boundary condition is

$$\phi_t + \eta + \alpha\frac{1}{2}\phi_x^2 + \frac{\alpha}{\beta}\frac{1}{2}\phi_z^2 = 0$$

at the free surface.

A.6 Apply the Governing Equation and Bottom Boundary Condition

Begin with the formulation of the dimensionless boundary value problem,

$$\begin{aligned}
\beta\phi_{xx} + \phi_{zz} &= 0 && \text{in} && 0 < z < \alpha\eta + 1 \\
\eta_t + \alpha\phi_x\eta_x - \frac{1}{\beta}\phi_z &= 0 && \text{at} && z = \alpha\eta + 1 \\
\phi_t + \eta + \alpha\frac{1}{2}\phi_x^2 + \frac{\alpha}{\beta}\frac{1}{2}\phi_z^2 &= 0 && \text{at} && z = \alpha\eta + 1 \\
\phi_z &= 0 && \text{at} && z = 0
\end{aligned}$$

Expand the velocity potential ϕ in a power series (with a nod to the differential equations technique called separation of variables):

$$\phi(x,z,t) = \sum_{m=0}^{\infty} f_m(x,t) z^m.$$

Substitution of this form of ϕ into the governing equation yields the following calculation:

$$\begin{aligned} 0 &= \beta \sum_{m=0}^{\infty} f_m''(x,t) z^m + \sum_{m=2}^{\infty} m(m-1) f_m(x,t) z^{m-2} && \text{(A.26)} \\ &= \sum_{m=0}^{\infty} \left(\beta f_m'' + (m+2)(m+1) f_{m+2}\right) z^m && \text{(A.27)} \end{aligned}$$

which yields the recursion relation

$$f_{m+2} = \frac{-\beta}{(m+2)(m+1)} f_m''$$

where the primes indicate differentiation with respect to x.

Now the bottom boundary condition $\phi_z = 0$ at $z = 0$ forces the coefficient of z to be zero; namely, $0 = f_1(x,t)$. Application of the recursion relation easily shows that all odd-indexed terms in the expansion of ϕ must be identically zero. Thus, ϕ may be rewritten as

$$\phi(x,z,t) = \sum_{m=0}^{\infty} (-1)^m \beta^m \frac{z^{2m}}{(2m)!} f^{(2m)}(x,t)$$

where $f(x,t) = f_0(x,t)$ and $f^{(2m)}$ indicates the $2m$-th derivative with respect to x.

A.7 Apply the Kinematic and Dynamic Boundary Conditions

Now substitute the quantity

$$\phi(x,z,t) = \sum_{m=0}^{\infty}(-1)^m\beta^m\frac{z^{2m}}{(2m)!}f^{(2m)}(x,t)$$

into each of the kinematic and dynamic boundary conditions, in hope of simplifying the relations between ϕ and η.

1. The kinematic condition: straightforward substitution yields

$$\begin{aligned} 0 = \eta_t \quad &+ \quad \alpha\eta_x\left(\sum_{m=0}^{\infty}(-1)^m\beta^m\frac{z^{2m}}{(2m)!}f^{(2m+1)}(x,t)\right) \\ &- \quad \frac{1}{\beta}\left(\sum_{m=1}^{\infty}(-1)^m\beta^m\frac{z^{2m-1}}{(2m-1)!}f^{(2m)}(x,t)\right) \end{aligned} \tag{A.28}$$

   Writing out several terms of these series and substituting $z = \alpha\eta+1$ gives a long relation in terms of α and β. Neglecting terms that are quadratic in these parameters (namely, α^2, β^2, and $\alpha\beta$) yields the equation

$$\eta_t + [(\alpha\eta+1)f_x]_x - \frac{\beta}{6}f_{xxxx} = O(\text{quadratics}).$$

2. The dynamic condition: as before, simply substitute the series form of ϕ into the dynamic boundary condition and evaluate at $z = \alpha\eta+1$, which gives

$$\begin{aligned} 0 \quad &= \quad \left(\sum_{m=0}^{\infty}(-1)^m\beta^m\frac{z^{2m}}{(2m)!}f_t^{(2m)}(x,t)\right) + \eta \\ &+ \quad \frac{\alpha}{2}\left(\sum_{m=0}^{\infty}(-1)^m\beta^m\frac{z^{2m}}{(2m)!}f^{(2m+1)}(x,t)\right)^2 \\ &+ \quad \frac{\alpha}{2\beta}\left(\sum_{m=1}^{\infty}(-1)^m\beta^m\frac{z^{2m-1}}{(2m-1)!}f^{(2m)}(x,t)\right)^2 \end{aligned} \tag{A.29}$$

Again, writing out several terms in the series expansion, substituting $z = \alpha\eta + 1$, and neglecting terms that are quadratic in α and β yield

$$\eta + f_t + \alpha\frac{1}{2}f_x^2 - \beta\frac{1}{2}f_{xxt} = O(\text{quadratics})$$

In summary, the equations resulting from the surface boundary conditions are

$$\eta_t + [(\alpha\eta + 1)f_x]_x - \frac{\beta}{6}f_{xxxx} = 0 \tag{A.30}$$

$$\eta + f_t + \alpha\frac{1}{2}f_x^2 - \beta\frac{1}{2}f_{xxt} = 0 \tag{A.31}$$

Introduce a new variable, $w = f_x$: this is a physically relevant variable, as it represents the horizontal velocity at the bottom of the flow domain. Further, differentiate the dynamic condition with respect to x. With these definitions, the above system becomes

$$\eta_t + [(\alpha\eta + 1)w]_x - \frac{\beta}{6}w_{xxx} = 0 \tag{A.32}$$

$$\eta_x + w_t + \alpha w w_x - \frac{\beta}{2}w_{xxt} = 0 \tag{A.33}$$

A.8 Obtaining BBM and KdV

Rewrite the above system to find that

$$\eta_t + w_x + \alpha(\eta w)_x - \frac{\beta}{6}w_{xxx} = 0 \tag{A.34}$$

$$\eta_x + w_t + \alpha\left(\frac{1}{2}w^2\right)_x - \frac{\beta}{2}w_{xxt} = 0 \tag{A.35}$$

Notice that at lowest order this system is the factored one-dimensional wave equation. In that case, it is easy to find that $w = \eta$ and $\eta_t + \eta_x = 0$.

(Note that this means $\eta_t = -\eta_x$ and $w_t = -w_x$.) Here, then, it is natural to try the Ansatz $w = \eta + \alpha A + \beta B$, where A and B are functions of η and all of its derivatives. Substitute of this form of w into the above system and retain only terms quadratic in α and β:

1. The kinematic condition becomes

$$\begin{aligned} 0 = & \quad \eta_t + \eta_x + \alpha A_x + \beta B_x + \alpha(\eta^2 + \alpha\eta A + \beta\eta B)_x \\ & - \frac{\beta}{6}(\eta_{xxx} + \alpha A_{xxx} + \beta B_{xxx}) \qquad \text{(A.36)} \\ = & \quad \eta_t + \eta_x + \alpha A_x + \beta B_x + 2\alpha\eta\eta_x - \frac{\beta}{6}\eta_{xxx} \qquad \text{(A.37)} \end{aligned}$$

   and grouping terms in powers of α and β gives

$$\eta_t + \eta_x + \alpha(A_x + 2\eta\eta_x) + \beta\left(B_x - \frac{1}{6}\eta_{xxx}\right) = 0$$

2. The dynamic boundary condition becomes

$$\begin{aligned} 0 = \ \eta_x & + \eta_t + \alpha A_t + \beta B_t + \alpha(\eta + \alpha A + \beta B)(\eta_x + \alpha A_x + \beta B_x) \\ & - \frac{\beta}{2}(\eta_{xxt} + \alpha A_{xxt} + \beta B_{xxt}) \\ & = \eta_t + \eta_x + \alpha(A_t + \eta\eta_x) + \beta\left(B_t - \frac{1}{2}\eta_{xxt}\right) \qquad \text{(A.38)} \end{aligned}$$

   which may be written as

$$\eta_t + \eta_x + \alpha(A_t + \eta\eta_x) + \beta\left(B_t - \frac{1}{2}\eta_{xxt}\right) = 0 \qquad \text{(A.39)}$$

The task now is to reconcile these two equations, choosing A and B so that the terms of order α and the terms of order β respectively correspond. That is, force the equalities

$$\begin{aligned} A_x + 2\eta\eta_x &= -A_x + \eta\eta_x \\ B_x - \frac{1}{6}\eta_{xxx} &= -B_x - \frac{1}{2}\eta_{xxt}, \end{aligned}$$

where the fact that $w_t = w_x$ forced $A_t = -A_x$ and $B_t = -B_x$. Choosing A and B that satisfy these requirements, for example,

$$A = -\frac{1}{4}\eta^2 \qquad \text{and} \qquad B = \frac{1}{12}\eta_{xxx} - \frac{1}{4}\eta_{xt}$$

leads to a choice of model equations, among which are two of particular interest, the KdV equation

$$\eta_t + \eta_x + \frac{3}{2}\alpha\eta\eta_x + \frac{\beta}{6}\eta_{xxx} = 0,$$

and the BBM equation

$$\eta_t + \eta_x + \frac{3}{2}\alpha\eta\eta_x - \frac{\beta}{6}\eta_{xxt} = 0,$$

where the parameters α and β are assumed small.

Appendix B

Jacobian Elliptic Functions Solution of $N = 2$ Modal Equations

These are the details for the change of variables described in the derivation of the analytical solution to the two modes Lau-Barcilon equations, where

$$w^2 = \frac{\rho_2^2 - \sigma_a^2}{\sigma_b^2 - \sigma_a^2} \qquad \text{and} \qquad \gamma^2 = \frac{\sigma_b^2 - \sigma_a^2}{\sigma_c^2 - \sigma_a^2}$$

are applied to the elliptic integral

$$\xi = \pm\frac{1}{2}\int_{\rho_2^2(0)}^{\rho_2^2(\xi)} \frac{d(\rho_2^2)}{\sqrt{\rho_2^2(1-\rho_2^2)^2 - c^2}}.$$

Notice that $\rho_2^2(1-\rho_2^2)^2 - c^2 = 0$ if and only if

$$0 = (\rho_2^2 - \sigma_a^2)(\rho_2^2 - \sigma_b^2)(\rho_2^2 - \sigma_c^2).$$

Since $\rho_2^2 = (\sigma_b^2 - \sigma_a^2)w^2 + \sigma_a^2$, we note that $d(\rho_2^2) = 2(\sigma_b^2 - \sigma_a^2)wdw$ and, thus

$$\frac{d(\rho_2^2)}{2(\sigma_b^2 - \sigma_a^2)} = \frac{(\rho_2^2 - \sigma_a^2)^{1/2}}{(\sigma_b^2 - \sigma_a^2)^{1/2}}\, dw \tag{B.1}$$

$$dw = \frac{d(\rho_2^2)}{2(\sigma_b^2 - \sigma_a^2)^{1/2}} \cdot \frac{(\sigma_b^2 - \sigma_a^2)^{1/2}}{(\rho_2^2 - \sigma_a^2)^{1/2}} \tag{B.2}$$

$$= \frac{1}{2}\frac{d(\rho_2^2)}{\left[(\sigma_b^2 - \sigma_a^2)(\rho_2^2 - \sigma_a^2)\right]^{1/2}} \tag{B.3}$$

Now, sort through some algebra:

$$\begin{aligned}
(1-w^2)(1-\gamma^2 w^2)(\sigma_c^2-\sigma_a^2) &= \left(1-\frac{\rho_2^2-\sigma_a^2}{\sigma_b^2-\sigma_a^2}\right) \\
&\quad \cdot \left(1-\frac{\sigma_b^2-\sigma_a^2}{\sigma_c^2-\sigma_a^2}\cdot\frac{\rho_2^2-\sigma_a^2}{\sigma_b^2-\sigma_a^2}\right)(\sigma_c^2-\sigma_a^2) \\
&= \left(1-\frac{\rho_2^2-\sigma_a^2}{\sigma_b^2-\sigma_a^2}\right)\left(1-\frac{\rho_2^2-\sigma_a^2}{\sigma_c^2-\sigma_a^2}\right)(\sigma_c^2-\sigma_a^2) \\
&= \left(\frac{\sigma_b^2-\rho_2^2}{\sigma_b^2-\sigma_a^2}\right)\left(\sigma_c^2-\rho_2^2\right) \\
&= \left[\frac{\sigma_b^2-\rho_2^2}{\sigma_b^2-\sigma_a^2}\right](\sigma_c^2-\rho_2^2)
\end{aligned}$$

Hence,

$$\begin{aligned}
\frac{dw}{\left[(\sigma_c^2-\sigma_a^2)(1-w^2)(1-\gamma^2w^2)\right]^{1/2}} &= \frac{1}{2}\frac{d(\rho_2^2)}{\left[(\sigma_b^2-\sigma_a^2)(\rho_2^2-\sigma_a^2)\right]^{1/2}} \\
&\quad \cdot \left[\frac{\sigma_b^2-\sigma_a^2}{(\sigma_b^2-\rho_2^2)(\sigma_c^2-\rho_2^2)}\right]^{1/2} \\
&= \frac{1}{2}\frac{d(\rho_2^2)}{\left[(\rho_2^2-\sigma_a^2)(\rho_2^2-\sigma_b^2)(\rho_2^2-\sigma_c^2)\right]^{1/2}} \\
&= \frac{1}{2}\frac{d(\rho_2^2)}{\left[\rho_2^2(1-\rho_2^2)^2-\gamma^2\right]^{1/2}}
\end{aligned}$$

Thus, w is a Jacobian elliptic function of ξ, in particular,

$$\xi = \frac{\pm 1}{(\sigma_c^2-\sigma_a^2)^{1/2}}\int_{w(0)}^{w(\xi)}\frac{dw}{\left[(1-w^2)(1-\gamma^2w^2)\right]^{1/2}}.$$

Simply write down the solution, change variables back to ρ_2^2 and use the conservation law to determine $\rho_1^2 = 1-\rho_2^2$.

Bibliography

[1] L. Abdelouhab, J. L. Bona, M. Felland, and J.-C. Saut. Non-local models for nonlinear, dispersive waves. *Physica D*, 40:360–392, 1989.

[2] M. J. Ablowitz and H. Segur. *Solitons and the Inverse Scattering Transform.* SIAM, 1981.

[3] J. P. Albert and J. L. Bona. Comparisons between model equations for long waves. *J. Nonlinear Sci.*, 1:345–374, 1991.

[4] J. P. Albert, J. L. Bona, and J.-C. Saut. Model equations for waves in stratified fluids. *Proc. R. Soc. Lond. A*, 453:1233–1260, 1997.

[5] J. A. Armstrong, N. Bloembergen, J. Ducuing, and P. S. Pershan. Interactions between light waves in a nonlinear dielectric. *Phys. Rev.*, 127:1918–1939, 1962.

[6] W. Bascom. *Waves and Beaches: The Dynamics of the Ocean Surface.* Cambridge University Press, 1980.

[7] T. B. Benjamin, J. L. Bona, and J. J. Mahony. Model equations for long waves in nonlinear dispersive systems. *Proc. R. Soc. Lond. A*, 272:47–78, 1972.

[8] B. Boczar-Karakiewicz and J. L. Bona. Action of shallow water waves on gently sloping beaches. In B. G. Holden, editor, *Proceedings of the Canadian Coastal Conference*, pages 323–324, Ottawa, 1983. National Research Council.

[9] B. Boczar-Karakiewicz and J. L. Bona. Wave-dominated shelves: a model of sand-ridge formation by progressive, infragravity waves. In R. J. Knight and J. R. McLean, editors, *Shelf sands and sandstones*, 11, pages 163–179. Canadian Society of Petroleum Geologists, 1986.

[10] B. Boczar-Karakiewicz, J. L. Bona, and D. C. Cohen. Interaction of shallow-water waves and bottom topography. In J. Bona, C. Dafermos, J. Erickson, and D. Kinderlehrer, editors, *Dynamical Problems in Continuum Physics*, number 4 in IMA series in mathematics and its applications, pages 131–176. Springer-Verlag, New York, 1987.

[11] B. Boczar-Karakiewicz and L. A. Jackson. On the protection of a beach system, gold coast, queensland, australia. In *22nd International Conference on Coastal Engineering*, pages 2268–2275, Delft, The Netherlands, 1991.

[12] B. Boczar-Karakiewicz, W. Romańczyk, J. L. Bona, and E. B. Thornton. Modeling the dynamics of a bar system at Duck, NC, USA. In *Proceedings of the 26th International Conference on Coastal Engineering*, volume 3, pages 2877–2887. ASCE, 1998.

[13] J. L. Bona. *Lectures on Nonlinear Wave Phenomena.* In preparation.

[14] J. L. Bona and M. Chen. A boussinesq system for two-way propagation of nonlinear dispersive waves. *Physica D: Nonlinear Phenomena*, 116:191–224, 1998.

[15] J. L. Bona, W. G. Pritchard, and L. R. Scott. An evaluation of a model equation for water waves. *Phil. Trans. R. Soc. Lond. A*, 302:457–510, 1981.

[16] J. L. Bona and M. Scialom. The effect of change in the nonlinearity and the dispersion relation of model equations for long waves. *Canadian Appl. Math. Quarterly*, 3:1–41, 1995.

[17] J. L. Bona and K. Socha. Lau-Barcilon analysis of the reguarlized long wave equation. In preparation.

[18] F. Bowman. *Introduction to Elliptic Functions with applications*. English Universities Press Ltd., St. Paul's House, Warwick Square, London, 1953.

[19] J. D. Cole. *Perturbation methods in applied mathematics*. Blaisdell, Waltham, Mass., 1968.

[20] D. Halliday and R. Resnick. *Fundamentals of Physics*. Wiley, 3 edition, 1988.

[21] D. J. Korteweg and G. de Vries. On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves. *Phil. Mag.*, 39(5), 1895.

[22] J. Lau and A. Barcilon. Harmonic generation of shallow water waves over topography. *J. Phys. Oceanogr.*, 2:405–410, 1972.

[23] M. S. Longuet-Higgins. Mass transport in water waves. *Phil. Trans. R. Soc. Lond. A*, 903:535–581, 1953.

[24] M. S. Longuet-Higgins. Resonant interaction between two trains of gravity waves. *J. Fluid Mech.*, 12:321–332, 1962.

[25] C. C. Mei and Ü. Ünlüata. Harmonic generation in shallow water waves. In R. E. Meyer, editor, *Waves on Beaches and Resulting Sediment Transport*, pages 181–202. Academic Press, 1972.

[26] H. D. Peregrine. Equations for water waves and the approximation behind them. In R. E. Meyer, editor, *Waves on beaches and related sediment transport*, pages 95–121. Academic Press, New York, 1972.

[27] O. M. Phillips. The dynamics of unsteady gravity waves of finite amplitude. Part I: elementary interactions. *J. Fluid Mech.*, 9:193–217, 1960.

[28] O. M. Phillips. *The Dynamics of the Upper Ocean.* Cambridge University Press, 1966.

[29] J. M. Restrepo and J. L. Bona. Three-dimensional model for the formation of longshore sand structures on the continental shelf. *Nonlinearity*, 8:781–820, 1995.

[30] W. Romańczyk, B. Boczar-Karakiewicz, E. B. Thornton, and J. L. Bona. Sand bars at Duck, NC, USA. Observation and model predictions. In *Proceedings of the International Conference 'Coastal Sediments '99'*, pages 491–504. ASCE, 1999.

[31] G. G. Stokes. On the theory of oscillatory waves. *Transactions of the Cambridge Philosophical Society*, 8:441–473, 1847.

Vita

Katherine Sue Socha was born in Hempstead, New York, on February 7, 1968, the daughter of John Socha and Toni Wallace. After completing high school in Hollywood, California, where she gave a graduation speech at the Hollywood Bowl, she matriculated at Reed College, in Portland, Oregon. She received her Bachelor of Arts degree from Reed College in 1990 and continued her studies at Portland State University, earning a Master of Science in mathematics in 1993. After studying mathematics and oceanography at Oregon State University for two years, Ms. Socha entered the doctoral program in mathematics at The University of Texas at Austin. Ms. Socha has complemented her academic studies with various employment experiences, including free-lance writing, providing technical support at Blue Sky Research (creators of Textures, an implementation of TEX for the Macintosh), holding teaching and research assistantships, and coordinating Saturday Morning Math Group for The University of Texas at Austin Department of Mathematics.

Permanent address: Department of Mathematics
The University of Texas at Austin
Austin, TX 78712

This dissertation was typeset with LATEX[‡] by the author.

[‡]LATEX is a document preparation system developed by Leslie Lamport as a special version of Donald Knuth's TEX Program.