ON TENSORS AND EXTENSORS HAVING

COMPLEX COMPONENTS

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# ON TENSORS AND EXTENSORS HAVING COMPLEX COMPONENTS

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## PREFACE

Chapter one presents a brief summary of the mathematics and physics context of the theory, recalling the essential topics and notation of tensor and extensor analysis, as well as outlining historically Einstein's transition from classical mechanics to his relativistic formulation thereof and the subsequent developments of relativistic wave equations.

Chapter two is an exposition of some work by Ghosh and by Craig, followed by further developments relating their definitions and theorems to prior theorems in tensor and extensor analysis due to Craig, Corson, van der Waerden, Infeld, and others.

Chapter three is a further development of a generalization of the tensorial theory in chapter two. The term 'spin components of an extensor' is defined.

Chapter four contains examples in which the spin extensor components (complex numbers or complex vectors, i.e., ordered number pairs or ordered (Gibbsian) vector pairs) have properties which are analogous to those of ordinary extensor components (numbers or vectors).

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CHAPTER ONE

## 1. Mathematics Background and Notation

Fundamental in all the following work is a four dimensional Riemannian space, $R_4$, having coordinates $x^a$, in which all the basic developments of tensor and extensor analysis [4; 60]* are valid. For example, recall that for a Riemannian subspace of an Euclidean space, defined by a class $C'$ function $r(x)$ and a parameterized arc $C$: $x^a = x^a(t)$, $r'(t) = r_{;a}\, x'^a$ involves a contraction of extensors; furthermore, the arc length $s$ of the parameterized arc $C$ is given by $[s'(t)]^2 = r' \cdot r' = r_{;a} \cdot r_{;c}\, x'^a\, x'^c$ $= g_{ac}\, x'^a\, x'^c$, which also involves a contraction of extensors.

Generalizing, $r^{(M)}$ is a function of the sets of variables $x^a, \ldots, x^{a(M)}$, and both $r^{(M+1)} = r^{(M)}_{;\alpha a}\, x^{(\alpha+1)a}$ and $[r^{(M+1)}]^2$ $= g_{\alpha a \cdot \gamma c}\, x^{(\alpha+1)} x^{(\gamma+1)}$ involve extensor contractions [3; 280]. In the case $M = 1$, and the parameter being arc length, the last equation is an expression for the (curvature)$^2$ that is invariant in functional form. The theory of the extensors derivable from those in the above contractions, and their relationships to the Christoffel symbols, and the intrinsic and covariant derivatives, is well known.

Generalizing further, extensor components which are either complex numbers (ordered real-number pairs) or complex vectors (ordered directed-stroke pairs) can be defined [5; ]. For a Riemannian subspace pair, each in an Euclidean subspace, respectively, of a given Euclidean space, and a function pair $(r_1, r_2)$, each of class $C'$, satisfying suitable conditions [5; ], we consider ordered vector pairs satisfying the usual rules of

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*[p;q] denotes page q of reference number p listed in the bibliography.

complex arithmetic rephrased in forms having the usual numerical multiplication replaced as needed by scalar, dot or cross multiplication, and in particular we have

$$
\begin{aligned}
&(r_1,r_2)'\cdot(r_1,r_2)' \\
&= (r_1',r_2')\cdot(r_1',r_2') \\
&= (r_1'\cdot r_1' - r_2'\cdot r_2' \ , \ 2\, r_1'\cdot r_2') \\
&= (g_{ac}x'^a x'^c - f_{ac}x'^a x'^c \ , \ h_{ac}x'^a x'^c) \\
&= (d_{ac} \ , \ h_{ac})x'^a x'^c \ .
\end{aligned}
$$

Note that this formulation allows for a metric which is not positive definite.

A generalization of the situation for $r^{(M)}$ will be developed in the sequel.

Tensor indices in $R_4$ are from the set $\{1,2,3,4\}$, or in some instances in the literature from the set $\{0,1,2,3\}$, and are denoted by lower case Latin letters. Extensor indices pertaining to $R_4$ are denoted by lower case Greek letters. Enclosure of a superscript in parentheses denotes differentiation with respect to a curve parameter. A semi-colon (or colon), or a comma, denotes partial or covariant differentiation, respectively. Enclosure of a subscript in parentheses denotes the Kawaguchi components.

Anticipating tensors and extensors other than those in $R_4$, indices on a spin tensor (as distinguished from tensors in the space $R_4$, also called world tensors) are from the set $\{1,2,1^\cdot,2^\cdot\}$ (the significance of the dotted indices will be explained in II 1), and are denoted by upper case Latin letters. This is different from standard usage, in which upper case

letters are used to forestall the Einstein summation convention, and is the reason for modifying this convention to the extent that non-summation is explicitly indicated by the notation $\sim\Sigma$. Recall that the summation convention applies to a pair of repeated indices, one a **superscript** the other a **subscript**, except when one appears in a binomial coefficient.

As usual, indication of components of indexed quantities in a transformed coordinate system will be indicated by letters in the last half (or in the case of three systems, in the middle and last thirds) of the alphabet; i.e., rather than write $\bar{t}^r = t^a X^r_a$, we will write only $t^r = t^a X^r_a$. Similarly, for a function $f(x^a,\ldots)$ whose mate function is constructed by replacing the $x^a$'s and other indexed arguments by their expressions in barred coordinates, i.e., $\bar{f}(x^r,\ldots) = f(x^a(x^r),\ldots)$, $\partial\bar{f}/\partial\bar{x}^r$ will be written $\partial f/\partial x^r$. The function $\bar{f}$ constructed in this manner from $f$ will be called $f$'s transform by invariance. This use of the term invariance should not be confused with invariance in functional form, or with covariance as this term is used in some literature to mean invariance in functional form.

To allow for comparison with various published results, covariant components $T_a$ related to given contravariant components $S^a$ by means of the symmetric metric tensor $g_{ac}$ are denoted by $S_{\underline{a}}$; i.e., $S_{\underline{a}} = g_{ac} S^c$. Similarly, given $S_a$, $S^{\underline{a}} = g^{ac} S_c$. This notation allows a distinction to be made from associated components by means of the Levi-Civita density; e.g., if $T_{ac}$ has type $[(0,2),0]$, then [2;8]

$$T^{\underline{ac}} = g^{ab} g^{cd} T_{bd} ,$$

but

$$T^{\underset{\vee}{ab}} = \frac{1}{2} e^{abcd} T_{cd} ,$$

and

$$\check{T}^{ab} = g^{-1/2}\, T^{\underset{\vee}{ab}} .$$

The term 'type' as used in the preceding statement refers to a symbol $[(p,q),w]$ in which $p$ is the contravariant order, $q$ the covariant order and $w$ the weight of the tensor. In the case of a tensor-first degree extensor, $[(p_e, q_e),M; (p_t, q_t),w]$ is the type in which $p_e$ is the excontravariant order, $q_e$ the excovariant order, $M$ the functional order, $p_t$ the contravariant order, $q_t$ the covariant order and $w$ the weight. A type $[(p,0),w]$ tensor will be called contravariant, and type $[(0,q),w]$ will be called covariant.

A symbol such as $E^{\alpha a.b\ldots}_{\gamma c.d\ldots}$ is used to indicate that there may be more or fewer indices than shown.

If $A = B + iC = (B,C)$, then $A^* = B - iC = (B,-C)$; and $i^P$ will be denoted by ixp P.

## 2. Physics Background

### 2.1 Pre-Quantum Physics

The object of any scientific investigation is to "coordinate our experiences and to bring them into a logical system" [6; 1], i.e., a system of ideas, and in particular ideas of space and time. Early ideas involved the independence of these concepts; the coming of relativity necessitated their unification into space-time, and specialization of the idea of space, too broad a term to be definable, to space of reference.

Two points marked on an ideal rigid body define a line segment, which may be oriented at rest in a space of reference in many ways. A space of reference whose points can be referred to coordinates $x^a$ in such a way that $\Sigma(\Delta x^a)^2$ is the same number for each orientation of the line segment

is called an Euclidean space, and the coordinates Cartesian. A fundamental assumption of pre-relativity physics was that the laws of the orientation of ideal rigid bodies are consistent with Euclidean geometry.

The hypothesis that all directions in a reference space, in particular that of the earth, are physically equivalent, irrespective of the assumption of the validity of Euclidean geometry, is called the 'Principle of Relativity with respect to Direction' [6; 24].

The 'Principle of Special Relativity' pertains to a relativity with respect to the state of motion of the space of reference, i.e., it concerns whether or not there are spaces of reference in motion relatively to each other which are physically equivalent. In order to infer conclusions from this principle, the equations relating the coordinates in one system to those in another must be known. In pre-relativity physics the Galilean transformation was tacitly assumed, and Newton's equations are invariant in functional form under this transformation. However, Maxwell's equations are not, and their consequences are well proven in practical applications. The consequence that in a vacuum light is propagated with a constant speed $c$ is accepted as a principle of physics.

Having redefined the concepts 'time' and 'velocity', the principles of special relativity and constancy of the speed of light allow the determination of the Lorentz equations of transformation between coordinates in relative motion. The Lorentz transformation [3; 376] exemplifies the fact that the time coordinate is not transformed into itself.

A uniform treatment of the coordinates $x, y, z, t$ was developed by Minkowski [6; 32].

The principle of inertia seemingly implies physically objective properties of the space-time continuum. But there are criticisms of the

ordinary conception: that the continuum is something which could act but not be acted on (this problem led to Mach's mechanics), and that the equality of the two types of mass, experimentally determined to be equal, is not explained. An extension of the principle of special relativity follows from the fact that the equality of the inertial and gravitational mass is equivalent to the theorem that the acceleration imparted to a body by a gravitational field is independent of the nature of the body. The 'principle of equivalence' involved in proving the above theorem implies an extension of the principle of relativity to coordinate systems which are in nonuniform motion relative to each other [6; 58]. "The possibility of explaining the numerical equality of inertia and gravitation by the unity of their nature gives to the general theory of relativity, according to my conviction, such a superiority over the conceptions of classical mechanics, that all the difficulties encountered in development must be considered as small in comparison" [6; 58].

Thus although the principle of inertial systems seemed well established experimentally, it had the flaw of involving a circular argument (a mass moves without acceleration if it is sufficiently far from other bodies - and we know that it is sufficiently far from other bodies only by the fact that it moves without acceleration). So although the question of inertial systems for extended parts of the continuum is open, there are finite regions with respect to a suitable chosen space of reference in which the laws of special relativity seem to hold. Such regions are called Galilean regions.

Here the already developed theory of surfaces of Gauss and the succeeding theories of tensor analysis by Riemann, Ricci and Levi-Civita became useful.

## 2.2 Quantum Physics

Relativistic quantum mechanics had its beginning in 1900 at Planck's formulation of the law of black body radiation. In 1905 Einstein attributed corpuscular character to electromagnetic radiation (the photon). He endowed photons, whose frequence of radiation is $\nu$, with an energy of $h\nu$ and momentum of $h\nu/c$. These assignments of energy and momentum to these zero rest mass particles were consistent with the postulates of relativity (note that zero rest mass particles can be understood only within the framework of relativistic dynamics) [13; 484]. The first new theory inspired by special relativity was deBroglie's wave theory of matter [8; 191].

The beginning of the modern phase of the theory of elementary particles was the publication in 1925 by Uhlenbeck and Goudsmit of the discovery that the electron should be considered not purely as a point particle, but also as having an intrinsic angular momentum (spin) [10; 88]. The first treatment after Uhlenbeck and Goudsmit of electron spin was by Heisenberg and Jordan in 1926, and used matrix mechanics [10; 89].

Systematic attempts to develop a technique applicable to wave mechanics were made by Darwin and Pauli [10; 89]. Pauli's method soon became standard for discussion of ordinary problems of atomic and molecular structure, and nuclear problems involving only protons and neutrons. A theory of the origin of electron spin was not offered and the methods do not satisfy requirements of the special theory of relativity. Pauli's theory can be considered a first approach to the more complicated Dirac theory.

Dirac's theory was not obtained from an established non-relativistic theory, but was derived from basic quantum theoretical and relativistic postulates. The crucial quantum part was that the wave equation described

a one-particle system, with a positive definite probability density, and that it could therefore be put in the canonical form $H\psi = (-h/i)\partial\psi/\partial t$. Lorentz invariance required the equation to have only first derivatives with respect to the coordinates. The relativistic energy-momentum relation uniquely determined the form of the Dirac Hamiltonian (at least for the free electron), thus leading to the four components of the wave function and to electron spin. The simplest assumption for the interaction with the electromagnetic field gave the correct value of the electron's magnetic moment. This Dirac Hamiltonian form was later abandoned. From there on there was no longer any relativistic quantum theory apart from quantum field theory, which deals with an indefinite number of particles. In particular, $\psi$ was changed from a state vector to a quantized field variable, and Dirac's equation became one of many possible systems of relativistic field equations [8; 193].

In 1929, van der Waerden [16; 100] developed an algebra for spinors which plays a role in the transformation theory of wave functions (which must have four components to allow for Lorentz invariance of Dirac's equation for the electron) analogous to the role of tensor algebra in special relativity.

In 1930, Dirac presented his positron theory, or theory of holes, which involves solutions of his equation for which the energy of the solution (negative electron or positron) satisfies $E < -mc^2$ [10; 121]. The discovery of the positron (and the anti-proton) removes the asymmetry difficulty referred to in Pauli's 1921 book on relativity [8; 190].

In 1933 Infeld and van der Waerden [11; 380] presented an analysis for spinors which liberated the formalism from the restriction of special relativity and permitted the use of spinors within a background of a general Riemannian metric.

## CHAPTER TWO

## SPIN TENSOR COMPONENTS

### 1. The Transfer Equations

Preliminary to the formulation of spin type extensor components from ordinary extensor components in a Riemannian space $R_4$ having real number coordinates $x^a$, the equations [5; 3][9; 696]

DEFINITION 1'.1 $$Z^1 = (x^4, x^3), \quad Z^2 = (x^1, x^2)$$

define either a transformation from $R_4$ to what can be described as a complex space $S_2$, or a coordinate transformation from number coordinates of $R_4$ to ordered number pair coordinates on $R_4$. Their structure is independent of the coordinate system in the sense that a coordinate transformation $x^a = x^a(x^r)$, $x^r = x^r(x^a)$ in $R_4$ induces a transformation such that if $Z^A = (x^b, x^d)$, then $Z^R = (x^s, x^t)$.

Ghosh [9; 696] defined conjugate coordinates by

DEFINITION 1'.2 $$Z^{1\cdot} = (Z^1)^* = (x^4, -x^3) ,$$
$$Z^{2\cdot} = (Z^2)^* = (x^1, -x^2) .$$

Writing equations 1'.1 and 1'.2 as binomials involving $i = (0,1)$ they define coefficients $T^A_a$ (denoted by Chosh [9; 696] as $S^A_a$), called transfer quantities, implicitly by means of

DEFINITION 1 $$Z^A = T^A_a x^a ,$$

for $A = 1, 2, 1^\cdot, 2^\cdot$. These transfer quantities can be written in the form

$$\begin{bmatrix} T_4^{1\cdot} & T_1^{1\cdot} & T_2^{1\cdot} & T_3^{1\cdot} \\ T_4^{2} & T_1^{2} & T_2^{2} & T_3^{2} \\ T_4^{2\cdot} & T_1^{2\cdot} & T_2^{2\cdot} & T_3^{2\cdot} \\ T_4^{1} & T_1^{1} & T_2^{1} & T_3^{1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -i \\ 0 & 1 & i & 0 \\ 0 & 1 & -i & 0 \\ 1 & 0 & 0 & i \end{bmatrix} = \begin{bmatrix} I & \sigma_2 \\ \sigma_1 & -i\sigma_3 \end{bmatrix}$$

REMARK. The equation of the light cone can be written

$$0 = x^2 + y^2 + z^2 - c^2t^2 = x^2 + y^2 - (-z^2 + c^2t^2) .$$

Hence, if $x^1 = x$, $x^2 = y$, $x^3 = z$, $x^4 = ict$, then

$$x^2 + y^2 + z^2 - c^2t^2 = (x^1 + ix^2)(x^1 - ix^2) - (-ix^4 - x^3)(-ix^4 + x^3) ,$$

and thus the equation of the light cone can be written

$$\begin{vmatrix} -Z^{1\cdot} & Z^{2\cdot} \\ Z^2 & Z^1 \end{vmatrix} = 0 .$$

THEOREM 1. If $s(1\cdot1) = 1\cdot$, $s(1\cdot2) = 2\cdot$, $s(2\cdot1) = 2$, $s(2\cdot2) = 1$ and

and $|A\cdot| = A$ ,

then the quantities defined by

$$\sigma_a^{A\cdot B} = \frac{1}{\sqrt{2}}\, i^{-|s(A\cdot B)|}\, T_a^{s(A\cdot B)} , \quad \sigma_0^{A\cdot B} = i\sigma_4^{A\cdot B}$$

are the four 2x2 matrices defined in [1; 719] (here the natural order for the doublet indices is $1\cdot1$, $1\cdot2$, $2\cdot1$, $2\cdot2$).

Proof.

$$[\sigma_a^{1\cdot1}] = -i[T_a^{1\cdot}]/\sqrt{2} = -i[1,0,0,-i]/\sqrt{2} = [-i,0,0,-1]/\sqrt{2} ,$$

$$[\sigma_a^{1\cdot2}] = -[T_a^{2\cdot}]/\sqrt{2} = -[0,1,-i,0]/\sqrt{2} = [0,-1,i,0]/\sqrt{2}$$

$$[\sigma_a^{2\cdot 1}] = -[T_a^2]/\sqrt{2} = -[0,1,i,0]/\sqrt{2} = [0,-1,-i,0]/\sqrt{2} ,$$

$$[\sigma_a^{2\cdot 2}] = -i[T_a^1]/\sqrt{2} = -i[1,0,0,i]/\sqrt{2} = [-i,0,0,1]/\sqrt{2} ;$$

or

$$[\sigma_4^{A\cdot B}] = \frac{1}{\sqrt{2}} \begin{bmatrix} -i & 0 \\ 0 & -i \end{bmatrix} = -\frac{i}{\sqrt{2}} \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} = -i[\sigma_0^{A\cdot B}] ,$$

$$[\sigma_1^{A\cdot B}] = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} , \quad [\sigma_2^{A\cdot B}] = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$$

$$[\sigma_3^{A\cdot B}] = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} .$$

Note: The order of the above coordinates is 4, 1, 2, 3.

COROLLARY 1. The above matrices are related to the Pauli spin matrices,

$$\sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} , \quad \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} , \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} , \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} .$$

Proof.

$$[\sigma_4^{A\cdot B}] = \frac{-i}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \frac{-i}{\sqrt{2}} \sigma_0$$

$$[\sigma_1^{A\cdot B}] = \frac{-1}{\sqrt{2}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \frac{-1}{\sqrt{2}} \sigma_1$$

$$[\sigma_2^{A\cdot B}] = \frac{-1}{\sqrt{2}} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \frac{-1}{\sqrt{2}} \sigma_2$$

$$[\sigma_3^{A\cdot B}] = \frac{-1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \frac{-1}{\sqrt{2}} \sigma_3$$

The transformation 1'.1, 1'.2 is reversible, because $T = \det T^{A}_{a} = 4$ , and the inverse equations are

$$x^4 = (Z^1 + Z^{1^\bullet})/2 ,$$

$$x^1 = (Z^2 + Z^{2^\bullet})/2 ,$$

2'

$$x^2 = (Z^2 - Z^{2^\bullet})/2i ,$$

$$x^3 = (Z^1 - Z^{1^\bullet})/2i ,$$

because of the properties of sums and differences of complex numbers and their conjugates. These equations define the transfer quantities $T^{a}_{A}$ whose numerical values are

$$\begin{bmatrix} T^4_{1^\bullet} & T^4_{2^\bullet} & T^4_{2} & T^4_{1} \\ T^1_{1^\bullet} & T^1_{2^\bullet} & T^1_{2} & T^1_{1} \\ T^2_{1^\bullet} & T^2_{2^\bullet} & T^2_{2} & T^2_{1} \\ T^3_{1^\bullet} & T^3_{2^\bullet} & T^3_{2} & T^3_{1} \end{bmatrix} = \begin{bmatrix} 1/2 & 0 & 0 & 1/2 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & i/2 & -i/2 & 0 \\ i/2 & 0 & 0 & -i/2 \end{bmatrix} ,$$

such that equations 2' have the form

DEFINITION 2. $$x^a = T^{a}_{A} Z^{A} .$$

Equations 1 can be solved for the $x^a$ to yield equation 2 by a contracted multiplication, and conversely.

THEOREM 2. If the symbols are defined as above,

then $T^{A}_{a} T^{c}_{A} = \delta^{c}_{a}$ and $T^{A}_{a} T^{a}_{B} = \delta^{A}_{B}$ .

Proof. Theorems A16.9 and A16.10 in [3; 90].

For example, $T^1_a\, T^a_1 = T^1_4\, T^4_1 + T^1_3\, T^3_1 = (1 + i(-i))/2 = 1 = \delta^1_1$ ,

$$T^1_a\, T^a_{1^\cdot} = T^1_4\, T^4_{1^\cdot} + T^1_3\, T^3_{1^\cdot} = (1 + ii)/2 = 0 = \delta^1_{1^\cdot} \quad .$$

THEOREM 3. If $s(1^\cdot 1) = 1^\cdot$, $s(1^\cdot 2) = 2^\cdot$, $s(2^\cdot 1) = 2$, $s(2^\cdot 2) = 1$,

and $|A^\cdot| = A$ ,

then the quantities defined by

$$\sigma^a_{A^\cdot B} = \sqrt{2}\ i^{xp|s(A^\cdot B)|}\ T^a_{s(A^\cdot B)} \ , \quad \sigma^0_{A^\cdot B} = -i\sigma^4_{A^\cdot B} \ , \quad \sim\!\sum \ ,$$

are the four 2x2 matrices defined in [1; 719].

Proof.

$$[\sigma^a_{1^\cdot 1}] = \sqrt{2}\ i[T^a_{1^\cdot}] = \sqrt{2}\ \frac{i}{2}\begin{bmatrix}1\\0\\0\\i\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}i\\0\\0\\-1\end{bmatrix} \ ,$$

$$[\sigma^a_{1^\cdot 2}] = -\sqrt{2}[T^a_{2^\cdot}] = -\frac{\sqrt{2}}{2}\begin{bmatrix}0\\1\\i\\0\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\-1\\-i\\0\end{bmatrix} \ ,$$

$$[\sigma^a_{2^\cdot 1}] = -\sqrt{2}[T^a_{2}] = -\frac{\sqrt{2}}{2}\begin{bmatrix}0\\1\\-i\\0\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\-1\\i\\0\end{bmatrix} \ ,$$

$$[\sigma^a_{2^\cdot 2}] = \sqrt{2}\ i[T^a_{1}] = \sqrt{2}\ \frac{i}{2}\begin{bmatrix}1\\0\\0\\-1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}i\\0\\0\\1\end{bmatrix} \ ;$$

or

$$[\sigma^4_{A^\cdot B}] = \frac{1}{\sqrt{2}}\begin{bmatrix}i & 0\\0 & i\end{bmatrix} = \frac{i}{\sqrt{2}}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = i[\sigma^0_{A^\cdot B}] \ ,$$

$$[\sigma^1_{A^\cdot B}] = \frac{1}{\sqrt{2}}\begin{bmatrix}0 & -1\\-1 & 0\end{bmatrix} , \quad [\sigma^2_{A^\cdot B}] = \frac{1}{\sqrt{2}}\begin{bmatrix}0 & -i\\i & 0\end{bmatrix} , \quad [\sigma^3_{A^\cdot B}] = \frac{1}{\sqrt{2}}\begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix} .$$

## 2. The Differential ∂-Operators

In a role analogous to that of the operator $\partial/\partial x^a$, which we will denote by $\partial_a$, in the theory of ordinary tensors and extensors, we have the operator $\partial_A$ (which Ghosh [9; 696] denoted by $\xi_A$) defined by

DEFINITION 1. $$\partial_A = T^a_A \partial_a ,$$

In detail,

1' $$\partial_1 = (\partial_4, -\partial_3)/2 , \quad \partial_2 = (\partial_1, -\partial_2)/2 .$$

*REMARK CONCERNING NOTATION. In general there will not be any confusion between $\partial_a$ and $\partial_A$. However, in specific instances where the components are written out in detail, ambiguity exists for the indices 1 and 2. Here and in similar contexts in the sequel, to distinguish $\partial_a$ for $a = 1$ and $a = 2$ from $\partial_A$ for $A = 1$ and $A = 2$, we will write simply $\partial_1$ and $\partial_2$ in the prior case and $\grave{\partial}_1$ and $\grave{\partial}_2$ in the latter. Thus the equations 1' are*

$$\grave{\partial}_1 = (\partial_4, -\partial_3)/2 , \quad \grave{\partial}_2 = (\partial_1, -\partial_2)/2 .$$

Note here [6; 32] that if $x^0 = ct$ and $x^4 = ict$,

then $\partial_0 = i\partial_4$

and $$\grave{\partial}_1 = (\partial_4 - i\partial_3)/2 = (-i\partial_0 - i\partial_3)/2 = -i(\partial_0 + \partial_3)/2$$

or $$\grave{\partial}_1 = -(0, \partial_0 + \partial_3)/2 ,$$

and $$\grave{\partial}_{1'} = (\partial_4 + i\partial_3)/2 = (-i\partial_0 + i\partial_3)/2 = -i(\partial_0 - \partial_3)/2$$

or $$\grave{\partial}_{1'} = -(0, \partial_0 - \partial_3)/2 .$$

THEOREM 1. If $s(A\dot{}B)$ is as defined in II 1 T3,

then the quantities defined by

$$\partial_{A\dot{}B} = \sqrt{2}\, i^{xp|s(A\dot{}B)|}\, \grave{\partial}_{s(A\dot{}B)}$$

are those defined in [1; 722].

Proof.

$$\partial_{1\dot{}1} = \sqrt{2}\, i\grave{\partial}_{1\dot{}} = (\partial_0 - \partial_3)/\sqrt{2} \quad ,$$

$$\partial_{1\dot{}2} = -\sqrt{2}\, \grave{\partial}_{2\dot{}} = (-\partial_1 - i\partial_2)/\sqrt{2} \quad ,$$

$$\partial_{2\dot{}1} = -\sqrt{2}\, \grave{\partial}_{2} = (-\partial_1 + i\partial_2)/\sqrt{2} \quad ,$$

$$\partial_{2\dot{}2} = \sqrt{2}\, i\grave{\partial}_{1} = (\partial_0 + \partial_3)/\sqrt{2} \quad .$$

COROLLARY 1. If $s(A\dot{}B)$ is as defined above,

then the quantities defined by

$$\partial_{A\dot{}B} = -2\, i^{xp|s(A\dot{}B)|}\, \grave{\partial}_{s(A\dot{}B)}$$

are those defined in [12; ].

THEOREM 2. If the symbols are defined as in II 1 D1, II 1 D2 and II 2 D1

then $\partial_a = T_a^A\, \partial_A$ .

Proof. Because of II 1 T2,

$$T_a^A\, \partial_A = T_a^A\, T_A^c\, \partial_c = \delta_a^c\, \partial_c \, .$$

In detail,

$$\partial_4 = \grave{\partial}_1 + \grave{\partial}_{1\dot{}} \quad ,$$

$$\partial_1 = \grave{\partial}_2 + \grave{\partial}_{2\dot{}} \quad ,$$

$$\partial_2 = i(\grave{}\partial_2 - \grave{}\partial_{2\cdot}) \ ,$$

$$\partial_3 = i(\grave{}\partial_1 - \grave{}\partial_{1\cdot}) \ .$$

THEOREM 3. If each of $F$ and $G$ is a class $C'$ function and $H = (F,G)$ then $H' = Z^{A'} \ \partial_A H$, where $'$ denotes differentiation with respect to a curve parameter.

Proof. $H' = x^{a'} \ \partial_a H = T^a_A \, Z^{A'} \ T^B_a \ \partial_B H = \delta^B_A \, Z^{A'} \ \partial_B H$ .

A corollary follows from this theorem, the equations in which can be considered the basis for a generalization of the definition of tensor components, and ultimately of extensor components.

COROLLARY 3. If $H$ and $Z$ are suitably differentiable functions, then $\partial_R H = \partial_A H \ \partial_R Z^A$ and $Z^{R'} = Z^{A'} \ \partial_A Z^R$ .

DEFINITION 2. $Z^A_R = \partial_R Z^A$, $Z^R_A = \partial_A Z^R$ .

THEOREM 4. $(Z^R_A)^* = Z^{R\cdot}_{A\cdot}$ and $(Z^{R\cdot}_A)^* = Z^R_{A\cdot}$ .

Proof. Because $(T^A_a)^* = T^{A\cdot}_a$, $(T^a_A)^* = T^a_{A\cdot}$, and $A^{\cdot\cdot} = A$ ,

$$(Z^R_A)^* = (T^a_A \ \partial_a \ T^R_r \ x^r)^* = T^a_{A\cdot} \partial_a \ T^{R\cdot}_r \ x^r = \partial_{A\cdot} \ Z^{R\cdot} \quad \text{and}$$

$$(Z^{R\cdot}_A)^* = (T^a_A \ \partial_a \ T^{R\cdot}_r \ x^r)^* = T^a_{A\cdot} \ \partial_a \ T^R_r \ x^r = \partial_{A\cdot} \ Z^R \ .$$

Besides the property in Theorem 4, the quantities in definition 2 have several properties analogous to the tensor transformation coefficients, to which they are related.

THEOREM 5. If the symbols are as defined above,

then $Z^R_A = T^a_A\, T^R_r\, X^r_a$ and $X^s_c = T^s_R\, T^A_c\, Z^R_A$ .

Proof.

$$Z^R_A = \partial_A\, Z^R = T^a_A\, \partial_a\, T^R_r\, x^r = T^a_A\, T^R_r\, \partial_a\, x^r$$

$$X^s_c = \delta^a_c\, \delta^s_r\, X^r_a = T^s_R\, T^A_c\, T^a_A\, T^R_r\, X^r_c = T^s_R\, T^A_c\, Z^R_A \quad .$$

COROLLARY 5. $T^B_b\, Z^R_B = T^R_r\, X^r_b$ .

Proof. $T^B_b\, Z^R_B = \delta^a_b\, T^R_r\, X^r_a$ .

THEOREM 6. If the usual associations of letters in different parts of the alphabet with different coordinate systems are made,

then $Z^R_A = Z^L_A\, Z^R_L$ and $Z^R_S = \delta^R_S$ .

Proof.

$$Z^R_A = \partial_A\, Z^R = \partial_L\, Z^R\, \partial_R\, Z^L$$

$$Z^R_S = T^s_S\, \partial_s\, T^R_r\, x^r = T^s_S\, T^R_r\, \delta^r_s = T^r_s\, T^R_r \quad .$$

## 3. Spin Tensor Components

We are led by Corollary II 2 T3 to study quantities whose transformation law is formally analogous to the tensor transformation law in $R_4$, but in which the coefficients $X^{\cdot}_{\cdot}$ are replaced by $Z^{\cdot}_{\cdot}$ ; such quantities are called spin tensor components [5; ].

REMARK. *The spin tensor transformation law for first order contravariant components,*

$$h^R = h^A Z^R_A ,$$

*as defined here is formally analogous to the spinor transformation law which is common in the literature (e.g., references 1, 2, 12, 16) except with regard to the range of indices; here it is the set* $\{1, 2, 1^\bullet, 2^\bullet\}$, *rather than* $\{1, 2\}$.

THEOREM *If* (a) $Z^1_1 Z^2_2 - Z^1_2 Z^2_1 = 1$

(b) $[Z^{R^\bullet}_A] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ ,

*then* (c) $\det Z^{R^\bullet}_{A^\bullet} = 1$

(d) $[Z^R_{A^\bullet}] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ .

*Proof. Straightforward from II2 T4.*

*Property (a) in the preceding theorem is the condition that the spinor transformation law be unimodular (a common requirement), and therefore this theorem essentially states that spin tensor components as defined here are a generalization of ordinary spinors. Ehrenfest first used the term 'spinor' to refer to 2- or 4-component quantities which do not transform as tensors, but which appear in Dirac's work.*

THEOREM 1. If $J(x/\overline{x})$ denotes the Jacobian of a transformation $x^a = x^a(x^r)$, $x^r = x^r(x^a)$ which induces a transformation of the $Z$'s ,

then $J(Z/\overline{Z}) = J(x/\overline{x})$.

Proof. $J(x/\overline{x}) = e_{abcd}\, X^a_1\, X^b_2\, X^c_3\, X^d_4$

$$= e_{abcd}\, T^a_A\, T^R_1\, Z^A_R \dots Z^D_U$$

$$= e_{ABCD}\, |T| \dots Z^T_U = e_{RSTU}\, |T|\, J(Z/\overline{Z}) \dots$$

$$= J(Z/\overline{Z}) \ ,$$

the intermediate steps following from the facts that $(\det T^a_A)(\det T^A_a) = 1$ and that the theory of the e-systems can be extended to the case in which the indices are not 1, 2, 3, 4 but some other set of symbols such as 1, 2, 1', 2' provided the natural order is prescribed.

THEOREM 2. If $h_a$ are tensor components of type $[(0,1),w]$ and $h_A = T^a_A\, h_a$ then $h_A$ are spin tensor components of the same type.

Proof. $\overline{Z}\, h_R\, Z^R_A = \overline{Z}\, Z^R_A\, T^r_R\, h_r$

$$= \overline{Z}\, X^r_a\, T^a_A\, \underline{X}\, h_c\, X^c_r = \delta^c_a\, T^a_A\, h_c \ ,$$

where $\overline{Z} = J(\overline{Z}/Z)^w$ and $\underline{X} = J(x/\overline{x})^w$ .

In particular,

$$`h_1 = (h_4, -h_3)/2, \quad `h_2 = (h_1, -h_2)/2 \ .$$

COROLLARY 2. Under the hypotheses of II 3 T2,

$$h_a = T^A_a\, h_A \ .$$

Proof. $h_a = \delta^c_a\, h_c = T^c_B\, T^B_a\, h_c = T^B_a\, h_B$ .

In particular,

$$h_4 = \grave{h}_1 + \grave{h}_{1\cdot}$$

$$h_1 = \grave{h}_2 + \grave{h}_{2\cdot}$$

$$h_2 = i(\grave{h}_2 - \grave{h}_{2\cdot})$$

$$h_3 = i(\grave{h}_1 - \grave{h}_{1\cdot})$$

THEOREM 3. If $\sigma^d_{A\cdot B}$ are defined as

$$\sigma^a_{A\cdot B} = \sqrt{2}\ i^{x_p|s(A\cdot B)|}\ T^a_{s(A\cdot B)}\ ,\quad \sigma^0_{A\cdot B} = -i\sigma^4_{A\cdot B}\ ,\quad \sim\!\!\sum$$

and $h_d$ are type $[(0,1),0]$ tensor components

then the quantities defined by $h_{A\cdot B} = \sigma^d_{A\cdot B}\ h_d$

are those defined in [1; 719].

Proof. $\sqrt{2}\ i^{x_p|D|}\ h_D$

$$= \sum_d \sqrt{2}\ i^{x_p|D|}\ T^d_D\ h_d = \sigma^d_{s^{-1}(D)}\ h_d$$

implies

$$h_{1\cdot 1} = \sqrt{2}\ i\ \grave{h}_{1\cdot}$$

$$h_{1\cdot 2} = -\sqrt{2}\ \grave{h}_{2\cdot}$$

$$h_{2\cdot 1} = -\sqrt{2}\ \grave{h}_2$$

$$h_{2\cdot 2} = \sqrt{2}\ i\ \grave{h}_1\ .$$

THEOREM 4. If $h^a$ are tensor components of type $[(1,0),w]$

and $h^A = T^A_a\ h^a$,

then $h^A$ are spin tensor components of type $[(1,0),w]$.

Proof. $\overline{Z}\, h^R\, Z^A_R = \overline{Z}\, Z^A_R\, T^R_r\, h^r$

$= \overline{Z}\, X^a_r\, T^A_a\, \underline{X}\, h^c\, X^r_c = \delta^a_C\, T^A_a\, h^c$ .

In particular,

$$`h^1 = (h^4, h^3) , \quad `h^2 = (h^1, h^2) .$$

COROLLARY 4. Under the hypotheses of II 3 T4

$$h^a = T^a_A\, h^A .$$

Proof. $h^a = \delta^a_C\, h^c = T^a_B\, T^B_c\, h^c = T^a_B\, h^B$ .

In particular,

$$h^4 = (`h^1 + `h^{1^\bullet})/2 ,$$

$$h^1 = (`h^2 + `h^{2^\bullet})/2 ,$$

$$h^2 = i(-`h^2 + `h^{2^\bullet})/2 ,$$

$$h^3 = i(-`h^1 + `h^{1^\bullet})/2 .$$

THEOREM 5. If $\sigma^{A^\bullet B}_a$ are defined as

$$\sigma^{A^\bullet B}_a = \frac{1}{\sqrt{2}}\, ixp(-|s(A^\bullet B)|) T^{s(A^\bullet B)}_a , \quad \sigma^{A^\bullet B}_0 = i\sigma^{A^\bullet B}_4$$

and $h^a$ are tensor components of type $[(1,0),0]$

then the quantities defined by $h^{A^\bullet B} = \sigma^{A^\bullet B}_a\, h^a$

are those defined in [1; 719].

Proof. $\frac{1}{\sqrt{2}}\, ixp(-|D|)\, h^D$

$= \frac{1}{\sqrt{2}}\, ixp(-|D|)\, T^D_d\, h^d = \sigma^{s^{-1}(D)}_d\, h^d$

implies

$$h^{1\dot{1}} = -i\,\grave{h}^{1\dot{1}}/\sqrt{2} = (-i\,h^4 - h^3)/\sqrt{2}\ ,$$

$$h^{1\dot{2}} = -\grave{h}^{2\dot{2}}/\sqrt{2} = (-h^1 + i\,h^2)/\sqrt{2}\ ,$$

$$h^{2\dot{1}} = -\grave{h}^{2}/\sqrt{2} = (-h^1 - i\,h^2)/\sqrt{2}\ ,$$

$$h^{2\dot{2}} = -i\,\grave{h}^{1}/\sqrt{2} = (-i\,h^4 + h^3)/\sqrt{2}\ .$$

THEOREM 6. If $h_c^a$ are tensor components of type $[(1,1),0]$

and $h_c^A = T_a^A\, h_c^a$, $h_C^a = T_C^c\, h_c^a$, $h_C^A = T_C^c\, h_c^A$,

then $h_c^A = T_a^A\, h_C^a$, $h_R^r = h_C^a\, X_a^r\, Z_R^C$

$$h_r^R = h_a^A\, X_r^a\, Z_A^R\ ,\quad h_S^R = h_B^A\, Z_A^R\, Z_S^B\ .$$

Proof. Straightforward.

THEOREM 7. If $h_{ac}$ are tensor components of type $[(2,0),0]$

and $h_{AC} = T_A^a\, T_C^c\, h_{ac}$,

then $h_{RS} = h_{AC}\, Z_R^A\, Z_S^C$.

Proof. Straightforward.

In detail,

$$4h_{11\dot{1}} = 4h_{1\dot{1}1}^{\ *} = h_{44} + h_{33} - i(-h_{43} + h_{34})\ ,$$

$$4h_{12\dot{1}} = 4h_{1\dot{1}2}^{\ *} = h_{41} + h_{32} - i(-h_{42} + h_{31})\ ,$$

$$4h_{12} = 4h_{1\dot{1}2\dot{1}}^{\ *} = h_{41} - h_{32} - i(h_{42} + h_{31})\ ,$$

$$4h_{11} = 4h_{1\dot{1}1\dot{1}}^{\ *} = h_{44} - h_{33} - i(h_{43} + h_{34})\ ,$$

$$4h_{21\dot{1}} = 4h_{2\dot{1}1}^{\ *} = h_{14} + h_{23} - i(-h_{13} + h_{24})\ ,$$

$$4h_{22\dot{1}} = 4h_{2\dot{1}2}^{\ *} = h_{11} + h_{22} - i(-h_{12} + h_{21})\ ,$$

$$4h_{22} = 4h_{2\cdot 2\cdot}{}^{*} = h_{11} - h_{22} - i(h_{12} + h_{21}) \ ,$$

$$4h_{21} = 4h_{2\cdot 1\cdot}{}^{*} = h_{14} - h_{23} - i(h_{13} + h_{24}) \ .$$

THEOREM 8. If $\sigma_a^{A\cdot B}$ are defined as in theorem II 1 T1

and if $h^{ac}$ are type $[(2,0),0]$ tensor components,

then the quantities defined by $h^{A\cdot BC\cdot D} = \sigma_a^{A\cdot B}\ \sigma_b^{C\cdot D}\ h^{ab}$

are those defined in [1; 719].

THEOREM 9. If $\sigma_a^{A\cdot B}$ are defined as in II 1 T1

and $h^{P1\cdots Pa}$ are of the type indicated by the indices,

then $h^{A_1\cdot B_1\cdots A_a\cdot B_a} = 2^{-a/2}\ ixp(-E)\ h^{P_1\cdots P_a}$ ,

where $E = \sum_{c=1}^{a} |s(A_c{}^{\cdot}B_c)|$ and $p_c = s(A_c{}^{\cdot}B_c)$ .

## 4. Metric Spin Tensors

An analog in $S_2$ for the metric tensor $g_{ac}$ in $R$ is now at hand. For if $h^A$ is a contravariant spin tensor, then

$$h^B = {}_A^B h^A = T_a^B\ T_A^a\ h^A = T_a^B\ h^a = T_a^B\ g^{ac}\ h_c = T_a^B\ T_c^A\ g^{ac}\ h_A = g^{AB}\ h_A \ .$$

Hence it is natural to call

$$g^{AB} = T_a^A\ T_b^B\ g^{ab} \quad \text{and} \quad g_{AB} = T_A^a\ T_B^b\ g_{ab}$$

the metric spin tensors.

THEOREM 1. If each of $h_a$ and $k_a$ is covariant,

and $h_A$, $k_A$ the corresponding spin components,

then $h_a k_c g^{ac} = h_A k_C g^{AC}$ .

Proof. $h_a k_c g^{ac} = T^A_a h_A T^C_c k_C g^{ac} = h_A k_C T^A_a T^C_c g^{ac}$ .

DEFINITION 1. For $h_A$ spin covariant and $k^A$ spin contravariant,

$$h^{\underline{A}} = g^{AC} h_C \quad \text{and} \quad k_{\underline{A}} = g_{AC} k^C .$$

THEOREM 2. If $g_{ac} = \eta_{ac} = -\delta_{ac}$ , $a \neq 0$ , $c \neq 0$ , $g_{00} = \eta_{00} = 1$ ,

then $`\eta_{11} = `\eta_{22} = -2$ , all others are zero.

Proof. Directly from II 3 T7.

COROLLARY 2. If $h_a$ is covariant and $h_A$ spin covariant,

then $(h^{\underline{0}}, h^{\underline{1}}, h^{\underline{2}}, h^{\underline{3}}) = (h_0, -h_1, -h_2, -h_3)$

$(h^{\underline{1}}, h^{\underline{2}}, h^{\underline{1}^\cdot}, h^{\underline{2}^\cdot}) = (2h_1, -2h_{2^\cdot}, 2h_{1^\cdot}, -2h_2)$ .

THEOREM 3. If $[ab,c] = -\partial_c g_{ab} + \partial_b g_{ca} + \partial_a g_{bc}$ ,

$[AB,C] = -\partial_C g_{AB} + \partial_B g_{CA} + \partial_A g_{BC}$ ,

then $[ab,c] = T^A_a T^B_b T^A_c [AB,C]$ .

THEOREM 4. If $\{ {e \atop a\ b} \} = g^{ec}[ab,c]$ ,

$\{ {E \atop A\ B} \} = g^{CE}[AB,C]$ ,

then $\{ {e \atop a\ b} \} = T^e_E T^A_a T^B_b \{ {E \atop A\ B} \}$ .

THEOREM 5. If $\{{}^{e}_{a}\} = \{{}^{e}_{a\,b}\}\, x^{b}{}_{,}$ ,

$$\{{}^{E}_{A}\} = \{{}^{E}_{A\,B}\}\, Z^{B}{}_{,} \quad ,$$

then $\{\{{}^{e}_{a}\} = T^{e}_{E}\, T^{A}_{a}\, \{{}^{E}_{A}\}$ .

THEOREM 6. If $R^{a}_{bcd} = \{{}^{a}_{b\ [c}\} + \{{}^{e}_{b\ [c}\}\{{}^{a}_{e]\ d}\}$

$$R^{A}_{BCD} = \partial_{[D}\{{}^{A}_{B\ C]}\} + \{{}^{E}_{B\ [C}\}\{{}^{A}_{E]\ D}\} \ ,$$

then $R^{a}_{bcd} = T^{a}_{A}\, T^{B}_{b}\, T^{C}_{c}\, T^{D}_{d}\, R^{A}_{BCD}$ .

THEOREM 7. If $\delta V^{a} = V^{a}{}_{,} + V^{b}\{{}^{a}_{b}\}$ ,

$$\delta V^{A} = V^{A}{}_{,} + V^{B}\{{}^{A}_{B}\}$$

then $\delta V^{a} = T^{a}_{A}\, \delta v^{A}$ .

CHAPTER THREE

SPIN EXTENSOR COMPONENTS

## 1. Extended Transformation Coefficients

Analogously with the fundamental quantities of extensor analysis (assuming the coordinate transformation involved is of class $C^{(M)}$), the introduction of a parameterized arc $x^a = x^a(t)$ of class $C^{(M)}$ makes not only the variables $x^a$ and $x^r$ class $C^{(M)}$ functions of $t$, but also the variables $Z^A$ and $Z^R$.

DEFINITION 1. $Z^{(\sigma)S} = Z^{S(\sigma)}$ .

THEOREM 1. If $x \in C^{\sigma}$

then $Z^{(\sigma)S} = T^S_s \, x^{(\sigma)s}$ .

Proof. $Z^{(\sigma)S} = Z^{S(\sigma)} = (T^S_s \, x^s)^{(\sigma)} = T^S_s \, x^{s(\sigma)}$ .

DEFINITION 2. $\partial_{\alpha a} = \partial / \partial \, x^{\alpha a}$ , $\partial_{\alpha A} = T^a_A \, \partial_{\alpha a}$ .

DEFINITION 3. If $Z \in C^{\alpha}$ ,

then $Z^{\alpha A}_{\sigma S} = \partial_{\sigma S} \, Z^{(\alpha)A}$ and $Z^{\sigma S}_{\alpha A} = \partial_{\alpha A} \, Z^{(\sigma)S}$ .

SPECIAL CONVENTION: Henceforth, all quantities involving differentiation symbols will be assumed to have the necessary differentiability for their existence.

THEOREM 2. If the symbols are those defined above and $Z \in C^{\alpha}$

then $\partial_{\beta b} \, Z^{(\alpha)A} = \delta^{\alpha}_{\beta} \, T^A_b$ , $\partial_{\beta B} \, x^{(\alpha)a} = \delta^{\alpha}_{\beta} \, T^a_B$ .

Proof. $\partial_{\beta b} \, T^A_a \, x^{(\alpha)a} = T^A_a \, X^{\alpha a}_{\beta b} = T^A_a \, \delta^{\alpha}_{\beta} \, \delta^a_b$

$$\partial_{\beta B}\, x^{(\alpha)a} = T^{b}_{B}\, \partial_{\beta b}\, x^{(\alpha)a} = T^{b}_{B}\, \delta^{\alpha}_{\beta}\, \delta^{a}_{b} .$$

THEOREM 3. If the symbols $Z^{\sigma S}_{\alpha A}$ are as defined above,

then $Z^{\sigma S}_{\alpha A} = T^{a}_{A}\, T^{S}_{s}\, X^{\sigma s}_{\alpha a}$ , $X^{\sigma s}_{\alpha a} = T^{s}_{R}\, T^{A}_{\alpha}\, Z^{\sigma R}_{\alpha A}$

$$Z^{\sigma S}_{\alpha A}\, T^{A}_{a} = T^{S}_{s}\, X^{\sigma s}_{\alpha a} , \quad Z^{\sigma S}_{\alpha A}\, T^{s}_{S} = T^{a}_{A}\, X^{\sigma s}_{\alpha a} .$$

Proof. $\partial_{\alpha A}\, Z^{(\sigma)S} = T^{a}_{A}\, \partial_{\alpha a}\, T^{S}_{s}\, x^{(\sigma)s} = T^{a}_{A}\, T^{S}_{s}\, X^{\sigma s}_{\alpha a}$

$$X^{\sigma s}_{\alpha a} = \delta^{s}_{r}\, \delta^{c}_{a}\, X^{\sigma r}_{\alpha c} = T^{s}_{R}\, T^{R}_{r}\, T^{c}_{A}\, T^{A}_{a}\, X^{\sigma r}_{\alpha c} = T^{s}_{R}\, T^{A}_{a}\, Z^{\sigma R}_{\alpha A}$$

$$Z^{\sigma S}_{\alpha A}\, T^{A}_{a} = \delta^{S}_{R}\, Z^{\sigma R}_{\alpha A}\, T^{A}_{a} = T^{S}_{s}\, T^{s}_{R}\, T^{A}_{a}\, Z^{\sigma R}_{\alpha A} = T^{S}_{s}\, X^{\sigma s}_{\alpha a} .$$

COROLLARY 3. $Z^{\sigma S}_{\alpha A} = \binom{\sigma}{\alpha}\, Z^{S(\sigma-\alpha)}_{A}$ .

Proof. $Z^{\sigma S}_{\alpha A} = T^{a}_{A}\, T^{S}_{s}\, \binom{\sigma}{\alpha}\, X^{s(\sigma-\alpha)}_{a} = \binom{\sigma}{\alpha}\left(T^{a}_{A}\, T^{S}_{s}\, X^{s}_{a}\right)^{(\sigma-\alpha)}$ .

THEOREM 4. If $x$, $\overline{x}$, $\bar{x}$ are admissible coordinate systems,

and the associated $Z$ coordinates are defined as above,

then $Z^{\sigma S}_{\mu M}\, Z^{\alpha A}_{\sigma S} = Z^{\alpha A}_{\mu M}$ and $Z^{\sigma S}_{\alpha A}\, Z^{\beta B}_{\sigma S} = \delta^{\beta}_{\alpha}\, \delta^{B}_{A}$ .

Proof. $Z^{\sigma S}_{\mu M}\, Z^{\alpha A}_{\sigma S} = T^{m}_{M}\, T^{S}_{s}\, X^{\sigma s}_{\mu m}\, T^{r}_{S}\, T^{A}_{c}\, X^{\alpha c}_{\sigma r}$

$$= T^{m}_{M}\, \delta^{r}_{s}\, X^{\sigma s}_{\mu m}\, T^{A}_{c}\, X^{\alpha c}_{\sigma r} = T^{m}_{M}\, X^{\alpha c}_{\mu m}\, T^{A}_{c}$$

$$Z^{\sigma S}_{\alpha A}\, Z^{\beta B}_{\sigma S} = Z^{\beta B}_{\alpha A} = T^{a}_{A}\, \partial_{\alpha a}\, T^{B}_{b}\, x^{(\beta)b}$$

$$= T^{a}_{A}\, T^{B}_{b}\, \delta^{\beta}_{\alpha}\, \delta^{b}_{a} = T^{b}_{A}\, T^{B}_{b}\, \delta^{\beta}_{\alpha} = \delta^{B}_{A}\, \delta^{\beta}_{\alpha} .$$

## 2. Spin Extensor Components

As a generalization of the formula in II 2 C3, we have the following.

THEOREM 1. If $F(Z^{A},\ldots,Z^{(M)A}) \in C'$ and $\overline{F} = F(Z^{A}(Z^{R}),\ldots)$ then $\partial_{\sigma S}\,\overline{F} = \partial_{\alpha A}\,F\,\partial_{\sigma S}\,Z^{\alpha A}$.

Proof. $\partial_{\sigma S}\,F = T^{s}_{S}\,\partial_{\sigma s}\,F = T^{s}_{S}\,\partial_{\alpha a}\,F\,X^{\alpha a}_{\sigma s} = T^{s}_{S}\,T^{A}_{a}\,\partial_{\alpha A}\,F\,X^{\alpha a}_{\sigma S}$.

This result suggests indexed quantities that transform according to the extensor-like law

$$E_{\sigma S} = E_{\alpha A}\,Z^{\alpha A}_{\sigma S}.$$

THEOREM 2. If $h^{A} = h^{R}\,Z^{A}_{R}$, then $h^{A(\alpha)} = h^{R(\rho)}\,Z^{\alpha A}_{\rho R}$.

Proof. $$(h^{R}\,Z^{A}_{R})^{(\alpha)} = \sum_{\theta=0}^{\alpha} \binom{\alpha}{\theta}\, h^{R(\theta)}\, Z^{A(\alpha-\theta)}_{R}$$

$$= \sum_{\theta=0}^{\alpha} h^{R(\theta)}\, Z^{\alpha A}_{\theta R} + \sum_{\theta=\alpha+1}^{M} h^{R(\theta)}\, Z^{\alpha A}_{\theta R}.$$

THEOREM 3. If $h_{A} = h_{R}\,Z^{R}_{A}$, then $h^{(\alpha)}_{A} = h^{(\rho)}_{R}\,Z^{\rho R}_{\alpha A}$.

Proof. $$(h_{R}\,Z^{R}_{A})^{(\alpha)} = \sum_{\theta=0}^{\alpha} \binom{\alpha}{\theta}\, h_{R}^{(\theta)}\, Z^{R(\alpha-\theta)}_{A}.$$

THEOREM 4. If $h_{A} = h_{R}\,Z^{R}_{A}$, and $h_{(\alpha)A} = \binom{M}{\alpha}\, h^{(M-\alpha)}_{A}$ then $h_{(\alpha)A} = h_{(\rho)R}\,Z^{\rho R}_{\alpha A}$.

Proof. $\binom{M}{\alpha}(h_R Z_A^R)^{(M-\alpha)} = \binom{M}{\alpha}\sum_{\theta=0}^{M-\alpha}\binom{M-\alpha}{\theta} h_R^{(M-\alpha-\theta)} Z_A^{R(\theta)}$

$$= \sum_{\rho=\alpha}^{M}\binom{M}{\alpha}\binom{M-\alpha}{\rho-\alpha} h_R^{(M-\rho)} Z_A^{R(\rho-\alpha)}$$

$$= \sum_{\rho=\alpha}^{M}\binom{M}{\rho} h_R^{(M-\rho)}\binom{\rho}{\alpha} Z_A^{R(\rho-\alpha)}$$

$$+ \sum_{\rho=0}^{\alpha-1} h_{(\rho)r} Z_{\alpha A}^{\rho R} .$$

Application of the relationships in definitions II 1 D1 and II 1 D2 to the equations of a parameterized arc $C: x^a = x^a(t)$ may be regarded either as determining the equations $Z^A = Z^A(t) = T_a^A x^a(t)$ of the mate arc in the complex space $S_2$ or as defining $C$ in terms of the ordered number pair coordinates: in either case, a set of extensor components at $P[C(x)]$ in a coordinate system $x$ defines a set of mate quantities associated with $z$ by means of the equations in II D1. For a pair of related coordinate systems $x$ and $z$, the quantities $E_{\gamma C.D\ldots}^{\alpha A.B\ldots}$ defined by the equation

$$E_{\gamma C.D\ldots}^{\alpha A.B\ldots} = T_a^A T_b^B \ldots T_C^c T_D^d \ldots E_{\gamma c.d\ldots}^{\alpha a.b\ldots}$$

with a similar equation for the transform quantities $E_{\tau T.U\ldots}^{\rho R.S\ldots}$ in terms of $E_{\tau t.u}^{\rho r.s}$ for the coordinate systems $\bar{x}$ and $\bar{z}$, will be called the spin components of the given ordinary extensor $E_{\gamma c.d\ldots}^{\alpha a.b\ldots}$ and will be referred to as spin extensor components. These quantities will be said to be of the type indicated by their free indices.

Clearly the ordinary extensor components can be solved for in the usual manner by contracted multiplication by suitable transfer coefficients, specifically

$$T_A^e\, T_B^f \cdots T_g^C\, T_h^D\, E_{\gamma C.D\ldots}^{\alpha A.B\ldots} = \delta_a^e\, \delta_b^f \cdots \delta_g^c\, \delta_h^d \cdots E_{\gamma c.d\ldots}^{\alpha a.b\ldots} = E_{\gamma g.h\ldots}^{\alpha e.f\ldots}$$

Furthermore, given a coordinate transformation $x, \bar{x}$ and the related transformation $z, \bar{z}$, the transformation equations relating $\bar{E}$ of $\bar{z}$ and $E$ of $z$ may be obtained from the extensor transformation relating $\bar{E}$ of $\bar{x}$ and $E$ of $x$.

THEOREM 5. If $E_{\gamma C.D\ldots}^{\alpha A.B\ldots}$ and $E_{\tau T.U\ldots}^{\rho R.S\ldots}$ are spin extensor components of the type indicated by their free indices in systems $z$ and $\bar{z}$, respectively, and $z, \bar{z}$ are coordinate systems induced by a transformation $x, \bar{x}$ in $R_4$, respectively

then $E_{\gamma C.D\ldots}^{\alpha A.B\ldots} = E_{\tau T.U\ldots}^{\rho R.S\ldots}\, Z_{\rho R}^{\alpha A}\, Z_S^B \cdots Z_{\gamma c}^{\tau T}\, Z_D^U$ .

Proof. $E_{\gamma C.D\ldots}^{\alpha A.B\ldots} = T_a^A\, T_b^B \cdots T_C^c\, T_D^d\, E_{\tau t.u\ldots}^{\rho r.s\ldots}\, X_{\rho r}^{\alpha a}\, X_s^b \cdots X_{\gamma c}^{\tau t}\, X_d^u \cdots$

$$= Z_{\rho R}^{\alpha A}\, T_r^R\, Z_S^B\, T_s^S \cdots Z_{\gamma C}^{\tau T}\, T_T^t\, Z_D^U\, T_U^u \cdots E_{\tau t.u\ldots}^{\rho r.s\ldots}$$

$$= T_r^R\, T_s^S \cdots T_T^t\, T_U^u\, E_{\tau t.u\ldots}^{\rho r.s\ldots}\, Z_{\rho R}^{\alpha A}\, Z_S^B \cdots Z_{\gamma C}^{\tau T}\, Z_D^U \cdots$$

$$= E_{\tau T.U\ldots}^{\rho R.S\ldots}\, Z_{\rho R}^{\alpha A}\, Z_S^B \cdots Z_{\gamma C}^{\tau T}\, Z_D^U \cdots \quad .$$

THEOREM 6. If at a point $P[C(x)]$ one set of quantities $E_{\gamma C.D\ldots}^{\alpha A.B\ldots}$ is given for each coordinate system $z$ of the set of $z$ coordinates obtainable by II 1 D1 from the given $x$ coordinate

systems, and for each coordinate system pair $z,x$ $\bar{z},\bar{x}$ the equation in the conclusion of III 2 T5 holds, then the quantities $E^{\alpha a.b\ldots}_{\gamma c.d\ldots}$ defined by the equation following II 2 D and similar equations for other coordinate systems, are extensor components.

Proof. $$E^{\alpha e.f\ldots}_{\gamma g.h\ldots} X^{\gamma g}_{\rho r} X^{h}_{s} \ldots X^{\tau t}_{\alpha e} X^{u}_{f} \ldots$$

$$= E^{\alpha A.B\ldots}_{\gamma C.D\ldots} T^{e}_{A} T^{f}_{B} \ldots T^{C}_{g} T^{D}_{h} X^{\gamma g}_{\rho r} X^{h}_{s} \ldots X^{\tau t}_{\alpha e} X^{u}_{f} \ldots$$

$$= E^{\alpha A.B\ldots}_{\gamma C.D\ldots} T^{t}_{R} Z^{\tau R}_{\alpha A} T^{u}_{S} Z^{S}_{B} \ldots T^{T}_{r} Z^{\gamma C}_{\rho T} T^{U}_{s} Z^{D}_{U}$$

$$= E^{\tau R.S}_{\rho T.U} T^{t}_{R} T^{u}_{S} T^{T}_{r} T^{U}_{s}$$

$$= E^{\tau t.u\ldots}_{\rho r.s\ldots} .$$

Note that for spin components thus defined, the examples at the beginning of this section, III 2 T2, T3 and T4, follow from the invariance of the transfer quantities.

COROLLARY 6.1. If $h^{a}$ are contravariant tensor components, and $T^{A}_{a} h^{a} = h^{A}$ then $h^{A(\alpha)}$ are spin extensor components.

Proof. $$h^{A(\alpha)} = (T^{A}_{a} h^{a})^{(\alpha)} = T^{A}_{a} h^{a(\alpha)} = T^{A}_{a} h^{r(\rho)} X^{\alpha a}_{\rho r}$$

$$= T^{R}_{r} Z^{\alpha A}_{\rho R} h^{r(\rho)} = h^{R(\rho)} Z^{\alpha A}_{\rho R} .$$

COROLLARY 6.2. If $h_{a}$ are covariant tensor components, and $T^{a}_{A} h_{a} = h_{A}$ then the Kawaguchi components for $h_{A}$ are spin extensor components.

Proof. $h_{(\alpha)A} = T^a_A \, h_{(\alpha)a} = T^a_A \, h_{(\rho)r} \, X^{\rho r}_{\alpha a}$

$$= h_{(\rho)r} \, T^r_R \, Z^{\rho R}_{\alpha A} = h_{(\rho)R} \, Z^{\rho R}_{\alpha A} \, .$$

# CHAPTER FOUR

## GENERALIZED GEOMETRICAL QUANTITIES AND PRIMARY EXTENSORS

In this chapter we develop a generalization of the extensors associated with the base vectors, the intrinsic derivative, the spin components to a space pair, and primary extensors.

### 1. Some Quantities in a Riemannian space $R_4$ immersed in an Euclidean J-space, $E_J$, for which $J \geq 4$

The Riemannian space $R_4$ is determined by equations

$$y^i = y^i(x^a), \quad i = 1,2,\ldots,J, \quad a = 1,2,3,4$$

in which the functions $y^i(x^a)$ are of class $C^{(M+1)}$; and a set of unit orthogonal base vectors $\hat{\imath}_1, \hat{\imath}_2, \ldots, \hat{\imath}_J$ associated with the rectangular cartesian coordinate system $y$ from which is defined a radius vector $\rho = \overrightarrow{OP}$, by

$$\rho = y^i \hat{\imath}_i = y^i(x) \hat{\imath}_i = \rho(x) ,$$

which is a function of the $x^a$'s.

Introduction of the set of parameterized arcs $x^a = x^a(t)$ (class $C^{(M)}$ functions) makes $\rho$ a function of $t$ of class $C^{(M)}$. The derivatives $\rho', \ldots, \rho^{(M)}$ are functions of the variables $x^{(\alpha)a} = x^{a(\alpha)}$ for $\alpha \leq M$.

THEOREM 1. $\rho^{(M)}_{;\alpha a}$ are components of an extensor whose tensor rank is $\rho_{;a}$.

Proof. [3; 280].

DEFINITION 1. $\rho_{(\alpha)a} = \rho^{(M)}{}_{;\alpha a}$ .

Definition II 1 assigns number pair coordinates $Z^A$ to the point P whose number coordinates are $x^a$, and equations II 3 assign a base vector pair, $\rho_A = T^a_A \rho_{;a}$ .

THEOREM 2. $\grave{\rho}_1 = (\rho_{;4}, -\rho_{;3})/2$ , $\grave{\rho}_2 = (\rho_{;1}, -\rho_{;2})/2$ ,

$$\rho_{(\alpha)A} = \rho^{(M)}{}_{\alpha A}, \qquad \rho_{(M)A} = \rho_A .$$

Proof. $\rho_{(\alpha)A} = T^a_A \rho_{(\alpha)a} = T^a_A \rho^{(M)}{}_{;\alpha a} = T^a_A \binom{M}{\alpha} \rho^{(M-\alpha)}_{;a}$

$$= \binom{M}{\alpha}(T^a_A \rho_{;a})^{(M-\alpha)} = \binom{M}{\alpha} \rho_A^{(M-\alpha)} = \rho^{(M)}{}_{\alpha A}$$

$$\rho_{(M)A} = T^a_A \rho_{(M)a} = T^a_A \rho_{;a} = \rho_A .$$

DEFINITION 2. $g_{\alpha a \cdot c} = \rho_{(\alpha)a} \cdot \rho_{;c}$ .

THEOREM 3. $g_{\alpha a \cdot c}$ are extensor components of type $[(0,1),M;(0,1),0]$ and $g_{Ma \cdot c} = g_{ac}$ , for $M = 1$ $g_{0a \cdot c} = [ab,c]\, x'^b$ .

Proof. [3; 280].

THEOREM 4. If $g_{AC}$ are the spin components defined by II 3 T7 then $g_{AC} = \rho_A \cdot \rho_C$ .

Proof. $g_{AC} = g_{ac} T^a_A T^c_C = \rho_{;a} \cdot \rho_{;c} T^a_A T^c_C$

$$= (\rho_{;a} T^a_A) \cdot (\rho_{;c} T^c_C) = \rho_A \cdot \rho_C .$$

In detail, for example

$$g_{11} = \tfrac{1}{4}(\rho_{;4}, -\rho_{;3}) \cdot (\rho_{;4}, -\rho_{;3})$$

$$= \tfrac{1}{4}(\rho_{;4} \cdot \rho_{;4} - \rho_{;3} \cdot \rho_{;3}, -\rho_{;4} \cdot \rho_{;3} - \rho_{;3} \cdot \rho_{;4})$$

$$= \tfrac{1}{4}(g_{44} - g_{33}, -2g_{43})$$

$$= g_{1\cdot 1^{\cdot *}} \cdot$$

THEOREM 5. If the symbols are as defined above,

then $\rho_{AB} \cdot \rho_{C} = [AB,C]$.

Proof. $\rho_{AB} \cdot \rho_{C} = \rho_{;ab} \cdot \rho_{;c}\ T^{a}_{A}\ T^{b}_{B}\ T^{c}_{C}$

$$= (-g_{ab;c} + g_{bc;a} + g_{ca;b})\ T^{a}_{A}\ T^{b}_{B}\ T^{c}_{C}$$

$$= -g_{AB;C} + g_{BC;A} + g_{CA;B}$$

where $g_{ab;c}\ T^{a}_{A}\ T^{b}_{B}\ T^{c}_{C} = \partial_{c}\,(g_{ab}\ T^{a}_{A}\ T^{b}_{B})\ T^{c}_{C} = \partial_{C}\, g_{AB} = g_{AB;C}$ .

THEOREM 6. If the symbols are defined as in Chapter II, for $M = 1$,

then $\rho_{(\alpha)A} \cdot \rho_{C}$ are spin extensor components.

Proof. $\rho_{(\alpha)A} \cdot \rho_{C} = \rho_{(\alpha)a} \cdot \rho_{;c}\ T^{a}_{A}\ T^{c}_{C}$

$$= g_{\rho r\cdot s}\ X^{\rho r}_{\alpha a}\ X^{s}_{c}\ T^{a}_{A}\ T^{c}_{C}$$

$$= g_{\rho r\cdot s}\ Z^{\rho R}_{\alpha A}\ T^{r}_{R}\ Z^{S}_{C}\ T^{s}_{S}$$

$$= \rho_{(\rho)R} \cdot \rho_{;S}\ Z^{\rho R}_{\alpha A}\ Z^{S}_{C}$$ .

THEOREM 7. If $g^{ab}$ is the normalized cofactor of $g_{ab}$, and $g^{AB}$ are the spin components, then $g^{AC} g_{BC} = \delta^{A}_{B}$.

Proof. $g^{AC} g_{BC} = g^{ac} T^{A}_{a} T^{C}_{c} g_{bd} T^{b}_{B} T^{d}_{C}$

$= g^{ac} g_{bd} T^{A}_{a} T^{b}_{B} \delta^{d}_{c} = \delta^{a}_{b} T^{A}_{a} T^{b}_{B}$

$= \delta^{A}_{B}$.

COROLLARY 7. If $g^{ab}$ is the normalized cofactor of $g_{ab}$, then $g^{AB}$ is the normalized cofactor of $g_{AB}$.

Proof. Denoting the normalized cofactor of $g_{AB}$ by $C^{AB}$,

$$C^{AD} = \delta^{A}_{B} C^{BD} = g^{AC} g_{BC} C^{BD} = g^{AC} \delta^{D}_{C} = g^{AD}.$$

THEOREM 8. The quantities defined by $\rho^{a} = g^{ac} \rho_{;c}$ are contravariant tensor components.

Proof. [3; 202].

THEOREM 9. If $\rho^{A}$ are the spin components for $\rho^{a}$ then $\rho^{A} = g^{AC} \rho_{C}$.

Proof. $\rho^{A} = \rho^{a} T^{A}_{a} = g^{ab} \rho_{;b} T^{A}_{a} = g^{CD} T^{a}_{C} T^{b}_{D} \rho_{;b} T^{A}_{a}$

$= g^{CD} \delta^{A}_{C} \rho_{D} = g^{AD} \rho_{D}$.

THEOREM 10. The quantities defined by $\rho^{(\alpha)a} = \rho^{a(\alpha)}$ are excontravariant extensor components.

Proof. [3; 297].

THEOREM 11. If $\rho^{(\alpha)A}$ are the spin components for $\rho^{(\alpha)a}$ then $\rho^{(\alpha)A} = \rho^{A(\alpha)}$.

Proof. $\rho^{(\alpha)A} = \rho^{(\alpha)a}\, T_a^A = (\rho^a\, T_a^A)^{(\alpha)} = \rho^{A(\alpha)}$.

From the basic extensors and tensors $\rho_{;a}$, $\rho^a$, $\rho_{(\alpha)a}$ and $\rho^{(\alpha)a}$, we have besides the quantities $g_{\alpha a\cdot c}$ also the mixed quantitied defined below.

DEFINITION 3. $g_b^{\alpha a} = \rho^{(\alpha)a} \cdot \rho_{;b}$ and $g_{\beta b}^{a} = \rho^a \cdot \rho_{(\beta)b}$.

THEOREM 12. The quantities $g_b^{\alpha a}$ and $g_{\beta b}^{a}$ are extensors of types [(1,0),M; (0,1),0] and [(0,1),M; (1,0),0], respectively, and

$$g_{0b}^{a} = \rho^a \cdot \rho_{;b}{}' = \left\{ {a \atop b\ c} \right\} x'^c = \left\{ {a \atop b} \right\},$$

$$g_{1b}^{a} = \delta_b^a = g_b^{0a}$$

$$g_b^{1a} = \rho^{a'} \cdot \rho_{;b} = -\rho^a \cdot \rho_{;b}{}' = -\left\{ {a \atop b} \right\}.$$

Proof. [3; 281], [3; 296].

THEOREM 13. The spin components for the mixed g-extensors satisfy

$$g_B^{\alpha A} = \rho^{(\alpha)A} \cdot \rho_B \quad \text{and} \quad g_{\beta B}^{A} = \rho^A \cdot \rho_{(\beta)B}.$$

Proof. $g_B^{\alpha A} = \rho^{(\alpha)a} \cdot \rho_{;b}\, T_a^A\, T_B^b = (\rho^{(\alpha)a}\, T_a^A) \cdot (\rho_{;b}\, T_B^b)$

$$g_{\beta B}^{A} = \rho^a \cdot \rho_{(\beta)b}\, T_a^A\, T_B^b = (\rho^a\, T_a^A) \cdot (\rho_{(\beta)b}\, T_B^b).$$

THEOREM 14. The spin components for the mixed $g$ extensors

$$g_{\alpha a \cdot \beta b} = \rho_{(\alpha)a} \cdot \rho_{(\beta)b} \quad \text{and} \quad g^{\alpha a}_{\beta b} = \rho^{(\alpha)a} \cdot \rho_{(\beta)b}$$

satisfy the equation

$$g_{\alpha A \cdot \beta B} = \rho_{(\alpha)A} \cdot \rho_{(\beta)B} \quad \text{and} \quad g^{\alpha A}_{\beta B} = \rho^{(\alpha)A} \cdot \rho_{(\beta)B} .$$

Proof. Straightforward.

Recall that if $v^a = v^r X^a_r$, then $v^{a'}$ are not tensor components, because

$$v^{a'} = v^{r'} X^a_r + v^r X^{a'}_r .$$

Hence the problem of tensor differentiation, i.e., the problem of finding tensor components which involve $v^{a'}$, is solved one way by means of the tensor character of the extensor contraction (with $M = 1$),

$$v^{(\beta)b} g^a_{\beta b} .$$

With these two facts in mind we give:

DEFINITION 4. For $v^a = v^r X^a_r$ and $v_a = v_r X^r_a$, $M = 1$,

$$\delta v^a = v^{(\beta)b} g^a_{\beta b} \quad \text{and} \quad \delta v_a = v_{(\beta)b} g^{\beta b}_a ,$$

respectively.

THEOREM 15. If the symbols are as defined above,

then $\delta v^A = v^{(\beta)B} g^A_{\beta B}$ and $\delta v_A = v_{(\beta)B} g^{\beta B}_A$.

Proof. $\delta v^A = \delta v^a T^A_a = v^{(\beta)b} g^a_{\beta b} T^A_a$

$$= v^{(\beta)b} g^a_{\beta d} T^A_a T^d_B T^B_b = v^{(\beta)B} g^A_{\beta B}$$

$$\delta v_A = v_{(\beta)b} g^{\beta b}_a T^a_A = v_{(\beta)b} g^{\beta d}_a T^a_A T^b_B T^B_d$$

$$= v_{(\beta)B} g^{\beta B}_A .$$

## 2. Some Quantities in a Riemannian space pair R,S, each immersed in an Euclidean subspace $E_1,E_2$ of an Euclidean space E

In this section each of R and S denote a Riemannian space with $R \subseteq E_1 \subseteq E$, $S \subseteq E_2 \subseteq E$. R and S are determined by equations

$$w^i = w^i(x), \quad i = 1,\ldots, I$$

$$y^j = y^j(x), \quad j = 1,\ldots, J,$$

respectively, and base vector sets $\alpha_i$, $\beta_j$ in $E_1,E_2$, respectively, which define a radius vector pair

$$\rho = w^i \alpha_i = w^i(x) \alpha_i = \rho(x) ,$$

$$\sigma = y^j \beta_j = y^j(x) \beta_j = \sigma(x) .$$

Introduction of the set of parameterized arcs $x^a = x^a(t)$ into the auxiliary x-space then determines an arc pair $C_1$ in R, $C_2$ in S [5; ].

DEFINITION 1. Denoting $(\rho,\sigma)$ by $\vec{\rho}$, we have

$$\vec{\rho}' = (\rho', \sigma') \quad \text{and} \quad \vec{\rho}_{;a} = (\rho_{;a}, \sigma_{;a}) .$$

As this definition indicates, we will be concerned with ordered pairs, whose coordinates are either numbers or vectors. The algebraic properties for ordered number pairs are well known. We will use analogous operational properties for ordered vector pairs, but with vector addition instead of ordinary addition in the usual term-by-term sum of ordered pairs, and with dot, cross or scalar multiplication suitably used in ordinary complex multi-

plication of ordered pairs. In detail, if each of A, B, C, and D denotes a vector, and each of x and y denotes a number, then $(A,B) = (C,D)$ means $A = C$ and $B = D$, $(A,B) + (C,D) = (A+C, B+D)$, $(x,y)(A,B) = (xA-yB, xB+yA)$, $(A,B)\cdot(C,D) = (A\cdot C-B\cdot D, A\cdot D+B\cdot C)$, $(A,0) = A$ .

As in section IV 1, the transfer equations assign z coordinates to a point P in R whose x coordinates are $x^a$ and also the same z-coordinates to the point Q in S having $x^a$ as coordinates. Using the same definitions for spin extensor components for ordered vector pair extensors as were used for ordinary extensors, we have the following.

LEMMA 1. If $\vec{\rho}_A$ are the spin components for $\vec{\rho}_{;a}$ then $\vec{\rho}_A = (\rho_A, \sigma_A)$.

Proof. $\vec{\rho}_A = \vec{\rho}_{;a}\, T^a_A = (\rho_{;a}, \sigma_{;a})\, T^a_A = (\rho_{;a}\, T^a_A, \sigma_{;a}\, T^a_A)$ .

In detail $\grave{\vec{\rho}}_1 = (\grave{\rho}_1, \grave{\sigma}_1) = (\rho_{;4} - i\,\rho_{;3}, \sigma_{;4} - i\,\sigma_{;3})/2$ ,

$\grave{\vec{\rho}}_{1\cdot} = (\rho_{;4} + i\,\rho_{;3}, \sigma_{;4} + i\,\sigma_{;3})/2$ ,

$\grave{\vec{\rho}}_2 = (\rho_{;1} - i\,\rho_{;2}, \sigma_{;1} - i\,\sigma_{;3})/2$ ,

$\grave{\vec{\rho}}_{2\cdot} = (\rho_{;1} + i\,\rho_{;2}, \sigma_{;1} + i\,\sigma_{;3})/2$ .

THEOREM 1. If $\vec{\rho} = (\rho,\sigma)$, then $\vec{\rho}' = \vec{\rho}_{;a}\, x'^a = \vec{\rho}_A\, Z'^A$ .

Proof. $\vec{\rho}' = (\rho', \sigma') = (\rho_{;a}, \sigma_{;a})\, x'^a = \vec{\rho}_{;a}\, x'^a = \vec{\rho}_A\, T^A_a\, T^a_B\, Z'^B$ .

DEFINITION 2. $\vec{g}_{ac} = \vec{\rho}_{;a} \cdot \vec{\rho}_{;c}$ .

THEOREM 2. If $d_{ac} = g_{ac} - f_{ac} = \rho_{;a} \cdot \rho_{;c} - \sigma_{;a} \cdot \sigma_{;c}$

and $h_{ac} = \rho_{;a} \cdot \sigma_{;c} + \sigma_{;a} \cdot \rho_{;c}$

then $\vec{g}_{ac} = (d_{ac}, h_{ac})$, $\vec{g}_{ac}$ are tensor components,

and $\vec{g}_{AC} = \vec{\rho}_A \cdot \vec{\rho}_C$ .

Proof. $\vec{\rho}_{;a} \cdot \vec{\rho}_{;c} = (\rho_{;a}, \sigma_{;a}) \cdot (\rho_{;c}, \sigma_{;c})$

$$= (\rho_{;a}\cdot\rho_{;c} - \sigma_{;a}\cdot\sigma_{;c}, \rho_{;a}\cdot\sigma_{;c} + \sigma_{;a}\cdot\rho_{;c})$$

$$\vec{g}_{ac}\, x^a_r\, x^c_s = (g_{ac}\, x^a_r\, x^c_s - f_{ac}\, x^a_r\, x^c_s, h_{ac}\, x^a_r\, x^c_s)$$

$$= (\rho_{;r}\cdot\rho_{;s} - \sigma_{;r}\cdot\sigma_{;s}, \rho_{;r}\cdot\sigma_{;s} + \sigma_{;r}\cdot\rho_{;s})$$

$$= (d_{rs}, h_{rs})$$

$$\vec{g}_{AC} = \vec{g}_{ac}\, T^a_A\, T^c_C = \vec{\rho}_a\, T^a_A \cdot \vec{\rho}_c\, T^c_C = \vec{\rho}_A \cdot \vec{\rho}_C \cdot$$

THEOREM 3. $\vec{g}_{ac}\, x'^a\, x'^c = \vec{g}_{AC}\, Z'^A\, Z'^C$ .

Proof. $\vec{\rho}' \cdot \vec{\rho}' = \vec{\rho}_{;a}\, x'^a \cdot \vec{\rho}_{;c}\, x'^c = \vec{g}_{ab}\, x'^a\, x'^c$

$$= \vec{\rho}_A\, Z'^A \cdot \vec{\rho}_C\, Z'^C = \vec{g}_{AC}\, Z'^A\, Z'^C \cdot$$

DEFINITION 3. $[\overrightarrow{ab,c}] = \vec{\rho}_{;ab} \cdot \vec{\rho}_{;c}$ .

LEMMA 2. $\rho_{;ab} \cdot \sigma_{;c} + \sigma_{;ab} \cdot \rho_{;c} = [ab,c]_h$ .

Proof. $2[ab,c]_h = -h_{ab;c} + h_{bc;a} + h_{ca;b}$

$$= -(\rho_{;a}\cdot\sigma_{;b} + \sigma_{;a}\cdot\rho_{;b})_{;c}$$

$$+ (\rho_{;b}\cdot\sigma_{;c} + \sigma_{;b}\cdot\rho_{;c})_{;a} + (\rho_{;c}\cdot\sigma_{;a} + \sigma_{;c}\cdot\rho_{;a})_{;b}$$

$$= 2\rho_{;ab}\cdot\sigma_{;c} + 2\sigma_{;ab}\cdot\rho_{;c} .$$

LEMMA 3. $[\vec{ab},c] = ([ab,c]_g - [ab,c]_f, [ab,c]_h)$ .

Proof. $\vec{\rho}_{;ab}\cdot\vec{\rho}_{;c} = (\rho_{;ab}, \sigma_{;ab})\cdot(\rho_{;c}, \sigma_{;c})$

$$= (\rho_{;ab}\cdot\rho_{;c} - \sigma_{;ab}\cdot\sigma_{;c}, \rho_{;ab}\cdot\sigma_{;c} + \sigma_{;ab}\cdot\rho_{;c})$$

$$= ([ab,c]_g - [ab,c]_f, [ab,c]_h) .$$

LEMMA 4. $\rho_{AB}\cdot\sigma_C + \sigma_{AB}\cdot\rho_C = [AB,C]_h$

and $[\vec{AB},C] = ([AB,C]_g - [AB,C]_f, [AB,C]_h)$ .

Proof. $2[AB,C]_h = -h_{AB;C} + h_{BC;A} + h_{CA;B}$

$$= (-h_{ab;c} + h_{bc;a} + h_{ca;b})\, T_A^a\, T_B^b\, T_C^c$$

$$= (2\rho_{;ab}\cdot\sigma_{;c} + 2\sigma_{;ab}\cdot\rho_{;c})\, T_A^a\, T_B^b\, T_C^c$$

$$= 2(\rho_{AB}\cdot\sigma_C + \sigma_{AB}\cdot\rho_C)$$

$$[\vec{AB},C] = \sum [\vec{ab},c]\, T_A^a\, T_B^b\, T_C^c$$

$$= (\rho_{;ab}\cdot\rho_{;c} - \sigma_{;ab}\cdot\sigma_{;c}, \rho_{;ab}\cdot\sigma_{;c} + \sigma_{;ab}\cdot\rho_{;c})T_A^a\, T_B^b\, T_C^c$$

$$= (\rho_{AB}\cdot\rho_C - \sigma_{AB}\cdot\sigma_C, \rho_{AB}\cdot\sigma_C + \sigma_{AB}\cdot\rho_C) .$$

THEOREM 4. $[\vec{ab},c] = \tfrac{1}{2}(-\vec{g}_{ab;c} + \vec{g}_{bc;a} + \vec{g}_{ca;b})$

$[\vec{AB},C] = \tfrac{1}{2}(-\vec{g}_{AB;C} + \vec{g}_{BC;A} + \vec{g}_{CA;B})$ .

Proof. $\vec{\rho}_{;ab} \cdot \vec{\rho}_{;c} = ([ab,c]_g - [ab,c]_f, [ab,c]_h)$

$$= (\tfrac{1}{2}(-g_{ab;c} + g_{bc;a} + g_{ca;b}) - \tfrac{1}{2}(-f_{ab;c} + f_{bc;a}$$

$$+ f_{ca;b}), \tfrac{1}{2}(-h_{ab;c} + h_{bc;a} + h_{ca;b}))$$

$$= -\tfrac{1}{2}(g_{ab;c} - f_{ab;c}, h_{ab;c})$$

$$+ \tfrac{1}{2}(g_{bc;a} - f_{bc;a}, h_{bc;a})$$

$$+ \tfrac{1}{2}(g_{ca;b} - f_{ca;b}, h_{ca;b})$$

$$= \tfrac{1}{2}(-\vec{g}_{ab;c} + \vec{g}_{bc;a} + \vec{g}_{ca;b})$$

$$[A\vec{B},C] = \vec{\rho}_{;ab} \cdot \vec{\rho}_{;c}\ T^a_A\ T^b_B\ T^c_C$$

$$= \tfrac{1}{2}(-\vec{g}_{ab;c}\ T^a_A\ T^b_B\ T^c_C + \vec{g}_{bc;a}\ T^a_A\ T^b_B\ T^c_C$$

$$+ \vec{g}_{ca;b}\ T^a_A\ T^b_B\ T^c_C)\ .$$

DEFINITION 4. $\vec{\rho}_{(\alpha)a} = \vec{\rho}^{(M)}_{;\alpha a}$ .

THEOREM 5. $\vec{\rho}_{(\alpha)a}$ are extensor components of the type $[(0,1),M,$

$(0,0),0]$ $\vec{\rho}_{(\alpha)a} = \binom{M}{\alpha} \vec{\rho}_{;a}^{(M-\alpha)}$ and $\vec{\rho}_{(\alpha)A} = \binom{M}{\alpha} \vec{\rho}_{A}^{(M-\alpha)}$ .

Proof. $\vec{\rho}_{(\alpha)a}\ x^{\alpha a}_{\rho r} = (\rho^{(M)}, \sigma^{(M)})_{;\alpha a}\ x^{\alpha a}_{\rho r}$

$$= (\rho^{(M)}_{;\alpha a}\ x^{\alpha a}_{\rho r}, \sigma^{(M)}_{;\alpha a}\ x^{\alpha a}_{\rho r}) = (\rho_{(\rho)r}, \sigma_{(\rho)r})$$

$$= \vec{\rho}_{(\rho)r} \cdot \vec{\rho}_{(\alpha)a} = (\rho^{(M)}_{;\alpha a}, \sigma^{(M)}_{;\alpha a})$$

$$= \binom{M}{\alpha}(\rho_{;a}^{(M-\alpha)}, \sigma_{;a}^{(M-\alpha)}) = \binom{M}{\alpha}(\rho_{;a}, \sigma_{;a})^{(M-\alpha)}$$

$$= \binom{M}{\alpha} \vec{\rho}_{;a}^{\,(M-\alpha)} \cdot \vec{\rho}_{(\alpha)A} + T^a_A \vec{\rho}_{(\alpha)a}$$

$$= \binom{M}{\alpha} (T^a_A \vec{\rho}_{;a})^{(M-\alpha)} .$$

In detail, $M = 1$,

$$\vec{\rho}_{(1)a} = \vec{\rho}_{;a}, \quad \vec{\rho}_{(0)a} = \vec{\rho}_{;a}' .$$

DEFINITION 5. $\vec{g}_{\alpha a \cdot c} = \vec{\rho}_{(\alpha)a} \cdot \vec{\rho}_{;c}$ .

THEOREM 6. $\vec{g}_{\alpha a \cdot c}$ are extensor components, $\vec{g}_{1a \cdot c} = \vec{g}_{ac}$,

$\vec{g}_{0a \cdot c} = [a\vec{b},c]\, x'^b$ and $\vec{g}_{\alpha A \cdot C} = \vec{\rho}_{(\alpha)A} \cdot \vec{\rho}_C$ .

Proof. $\vec{\rho}_{(\alpha)a} \cdot \vec{\rho}_{;c}\, x^{\alpha a}_{\rho r}\, x^c_s = (\rho_{(\alpha)a}, \sigma_{(\alpha)a}) \cdot (\rho_{;c}, \sigma_{;c})\, x^{\alpha a}_{\rho r}\, x^c_s$

$$= (\rho_{(\alpha)a}\, x^{\alpha a \cdot}_{\rho r}\, \rho_{;c}\, x^c_s - \sigma_{(\alpha)a}\, x^{\alpha a}_{\rho r \cdot}\, \sigma_{;c}\, x^c_s,$$

$$\rho_{(\alpha)a}\, x^{\alpha a \cdot}_{\rho r}\, \sigma_{;c}\, x^c_s + \sigma_{(\alpha)a}\, x^{\alpha a \cdot}_{\rho r}\, \rho_{;c}\, x^c_s)$$

$$= (\rho_{(\rho)r}, \sigma_{(\rho)r}) \cdot (\rho_{;s}, \sigma_{;s}) = \vec{\rho}_{(\rho)r} \cdot \vec{\rho}_{;s}$$

$$\vec{g}_{1a \cdot c} = \vec{\rho}_{(1)a} \cdot \vec{\rho}_{;c} = \vec{\rho}_{;a} \cdot \vec{\rho}_{;c} = \vec{g}_{ac}$$

$$\vec{g}_{0a \cdot c} = \vec{\rho}_{(0)a} \cdot \vec{\rho}_{;c} = \vec{\rho}_{;ab}\, x'^c \cdot \vec{\rho}_{;c}$$

$$= [a\vec{b},c]\, x'^b$$

$$\vec{g}_{\alpha A \cdot C} = \vec{\rho}_{(\alpha)a} \cdot \vec{\rho}_{;c}\, T^a_A\, T^c_C = \vec{\rho}_{(\alpha)a}\, T^a_A \cdot \vec{\rho}_{;c}\, T^c_C .$$

DEFINITION 6. $\vec{\rho}^a = (\rho^a, -\sigma^a)$ .

LEMMA 5. $\vec{\rho}^A = (\rho^A, -\sigma^A)$ .

Proof. $\vec{\rho}^A = T^A_a\,(\rho^a, -\sigma^a) = (T^A_a\,\rho^a, -T^A_a\,\sigma^a)$ .

THEOREM 7. If $\rho^a \cdot \sigma_{;c} = \sigma^a \cdot \rho_{;c}$

then $\vec{\rho}^a \cdot \vec{\rho}_{;c} = (2\delta^a_c, 0)$, $\rho^A \cdot \sigma_B = \sigma^A \cdot \rho_B$, and $\vec{\rho}^A \cdot \vec{\rho}_B = (2\delta^A_B, 0)$.

Proof. $\vec{\rho}^a \cdot \vec{\rho}_{;c} = (\rho^a, -\sigma^a) \cdot (\rho_{;c}, \sigma_{;c})$

$$= (\rho^a\cdot\rho_{;c} + \sigma^a\cdot\rho_{;c},\ \rho^a\cdot\sigma_{;c} - \sigma^a\cdot\rho_{;c}) = (2\delta^a_c, 0)$$

$$\rho^A \cdot \sigma_B - \sigma^A \cdot \rho_B = (\rho^a\cdot\sigma_{;b} - \sigma^a\cdot\rho_{;b})\ T^A_a\ T^b_B$$

$$(\rho^A, -\sigma^A)\cdot(\rho_B, \sigma_B) = (\rho^A\cdot\rho_B + \sigma^A\cdot\sigma_B, 0) .$$

REMARK. The condition $\rho^a \cdot \sigma_{;c} = \sigma^a \cdot \rho_{;c}$ is somewhat stringent. Some important special cases for which it holds are developed as examples.

Example A. $\sigma = \rho + \alpha$, $\alpha$ constant.

Verification: It follows upon observing that

$$\sigma^a = \rho^a \quad \text{and} \quad \sigma_{;a} = \rho_{;a} .$$

Example B. $\rho = x\hat{i} + y\hat{j}$, $\sigma = (ax + by)\hat{i} + (cx - ay)\hat{j}$

for $a^2 + bc = 1$

$$\sigma = \pm\,(x\hat{i} + y\hat{j}) .$$

Verification: Consider $\rho = x\hat{i} + y\hat{j}$ and

$$\sigma = (ax + by)\hat{i} + (cx + dy)\hat{j} .$$

Then $f_{11} = a^2 + c^2$, $f_{12} = f_{21} = ab + cd$, $f_{22} = b^2 + d^2$

$$f = (ad - bc)^2 = \Delta^2 .$$

$$\rho^1 \cdot \sigma_{;1} - \sigma^1 \cdot \rho_{;1} = a - \frac{d}{\Delta}\,,\ \rho^2 \cdot \sigma_{;2} - \sigma^2 \cdot \rho_{;2} = d - \frac{a}{\Delta}\,;$$

$$\rho^1 \cdot \sigma_{;2} - \sigma^2 \cdot \rho_{;1} = b(1 + \frac{1}{\Delta}),$$

$$\rho^2 \cdot \sigma_{;1} - \sigma^2 \cdot \rho_{;1} = c(1 + \frac{1}{\Delta}) .$$

Case 1. $\Delta = -1$.

If $\Delta = -1$, we have $d = -a$, $-(a^2 + bc) = -1$, with $b$ and $c$ otherwise arbitrary. It is interesting to note that if

(1) $a = 1$, $b = c = 0$, the transformation gives a reflection across $\hat{i}$;

(2) $a = -1$, $b = c = 0$, the transformation gives a reflection across $\hat{j}$;

(3) $a = 0$, $b = c = 1$, the transformation gives a reflection across $\hat{i} + \hat{j}$;

(4) $a = 0$, $b = c = -1$, the transformation gives a reflection across $\hat{i} - \hat{j}$;

(5) $a = \cos\theta$, $b = c = \sin\theta$, the transformation is related to rotation by

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} ;$$

(6) $a = \sin\theta$, $b = c = \cos\theta$, the transformation is related to rotation by

$$\begin{bmatrix} \sin\theta & \cos\theta \\ \cos\theta & -\sin\theta \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.$$

Case 2. $0 \neq \Delta \neq -1$.

Clearly, $b = c = 0$, $\Delta = ad$, $a = \frac{d}{\Delta}$, $d = \frac{a}{\Delta}$

$\Delta^2 = 1$, $\Delta = 1$, $a = d = \pm 1$ and hence

$a = d = 1$ (same as example A for $\alpha = 0$), or

$a = d = -1$.

Example C. Reflection across $y = mx$, $m = \tan\theta$, $\rho = x\hat{i} + y\hat{j}$,

$$\sigma = \left(\frac{1-m^2}{1+m^2}\,x + \frac{2m}{1+m^2}\,y\right)\hat{i} + \left(\frac{2m}{1+m^2}\,x - \frac{1-m^2}{1+m^2}\,y\right)\hat{j} \ .$$

Verification: $f_{11} = f_{22} = 1$, $f_{12} = f_{21} = 0$, $\sigma^a = \rho^a$,

$\sigma_{;c} = \rho_{;c}$, and hence, $\rho^a \cdot \sigma_{;c} - \sigma^a \cdot \rho_{;c} \equiv 0$ .

DEFINITION 7. The statement $\rho$ and $\sigma$ are compatible means the lower and upper $\rho$ base vectors can be expressed in terms of the lower and upper $\sigma$ base vectors, respectively.

LEMMA 6. If $\rho^a \cdot \sigma_{;c} = \sigma^a \cdot \rho_{;c}$, and $\rho$, $\sigma$ are compatible then $\rho^a = \sigma^a \cdot \rho_{;c}\ \sigma^c$, $\rho_{;c} = \rho^a \cdot \sigma_{;c}\ \sigma_{;a}$,

$\sigma^a = \rho^a \cdot \sigma_{;c}\ \rho^c$, $\sigma_{;c} = \sigma^a \cdot \rho_{;c}\ \rho_{;a}$,

$\rho^A = \sigma^A \cdot \rho_B\ \sigma^B$, $\rho_B = \rho^A \cdot \sigma_B\ \sigma_A$,

$\sigma^A = \rho^A \cdot \sigma_B\ \rho^B$, $\sigma_B = \sigma^A \cdot \rho_B\ \rho_A$ .

Proof. $\rho^a = \rho^a \cdot \sigma_{;c}\ \sigma^c = \sigma^a \cdot \rho_{;c}\ \sigma^c$ ,

$\rho_{;c} = \rho_{;c} \cdot \sigma^a\ \sigma_{;a} = \sigma_{;c} \cdot \rho^a\ \sigma_{;a}$ ,

$\rho^A = T^A_a\ \rho^a \cdot \sigma_{;c}\ T^c_B\ T^B_b\ \sigma^b = \rho^A \cdot \sigma_B\ \sigma^B$ .

The remaining equations follow similarly.

LEMMA 7. If $\rho^a \cdot \sigma_{;c} = \sigma^a \cdot \rho_c$, and $\rho, \sigma$ are compatible,

then $f_{ab}\,\rho^a = \sigma_{;b} \cdot \rho_{;a}\,\sigma^a$ and $g_{ab}\,\sigma^a = \rho_{;b} \cdot \sigma_{;a}\,\rho^a$,

$$f_{AB}\,\rho^A = \sigma_B \cdot \rho_A\,\sigma^A \quad \text{and} \quad g_{AB}\,\sigma^A = \rho_B \cdot \sigma_A\,\rho^A .$$

Proof. $f_{ab}\,\rho^a \cdot \sigma_{;c} = f_{ab}\,\sigma^a \cdot \rho_{;c} = \sigma_{;b} \cdot \rho_{;c}$

$$\Rightarrow f_{ab}\,\rho^a = \sigma_{;b} \cdot \rho_{;c}\,\sigma^c .$$

$$g_{ab}\,\sigma^a \cdot \rho_{;c} = g_{ab}\,\rho^a \cdot \sigma_{;c} = \rho_{;b} \cdot \sigma_{;c}$$

$$\Rightarrow g_{ab}\,\sigma^a = \rho_{;b} \cdot \sigma_{;c}\,\rho^c .$$

$$f_{AB}\,\rho^A = T^a_A\,T^b_B\,T^A_c\,\rho^C f_{ab} = T^b_B\,\delta^a_C\,\rho^C f_{ab}$$

$$= T^a_A\,T^b_B\,T^A_c\,\sigma_{;b} \cdot \rho_{;a}\,\sigma^C = \sigma_B \cdot \rho_A\,\sigma^A .$$

A similar argument gives the remaining equations.

LEMMA 8. If $\rho^a \cdot \sigma_{;c} = \sigma^a \cdot \rho_{;c}$, and $\rho, \sigma$ are compatible

then $(-\sigma_{;c}, \rho_{;c}) \cdot (\sigma_{;a}\,\rho^a - \rho_{;a}\,\sigma^a) = (0,0)$, and

$$(-\sigma_B, \rho_B) \cdot (\sigma_A\,\rho^A - \rho_A\,\sigma^A) = ((0,0),(0,0)) .$$

Proof. $\sigma_{;a} \cdot \sigma_{;b}\,\rho^a = \sigma_{;b} \cdot \rho_{;a}\,\sigma^a$ and

$$\rho_{;a} \cdot \rho_{;b}\,\sigma^a = \rho_{;b} \cdot \sigma_{;a}\,\rho^a \Rightarrow \sigma_{;b} \cdot (\sigma_{;a}\,\rho^a - \rho_{;a}\,\sigma^a) = 0,$$

and $\rho_{;b} \cdot (\rho_{;a}\,\sigma^a - \sigma_{;a}\,\rho^a) = 0$ .

Similarly, $\sigma_A \cdot \sigma_B\,\rho^A - \sigma_B \cdot \rho_A\,\sigma^A = (0,0)$,

$$\rho_A \cdot \rho_B\,\sigma^A - \rho_B \cdot \sigma_A\,\rho^A = (0,0)$$

$$\Rightarrow (-\sigma_{;b} \cdot (-\sigma_{;a}\rho^a + \rho_{;a}\sigma^a), \rho_{;b} \cdot (\rho_{;a}\sigma^a - \sigma_{;a}\rho^a)) = (0,0)$$

$$(-\sigma_B \cdot (-\sigma_A \rho^A + \rho_A \sigma^A), \rho_B \cdot (\rho_A \sigma^A - \sigma_A \rho^A)) = ((0,0),(0,0)).$$

THEOREM 8. If $\rho^a \cdot \sigma_{;c} = \sigma^a \cdot \rho_{;c}$, and $\rho$, $\sigma$ are compatible, then $\vec{g}_{ac}\,\vec{\rho}^a = 2\vec{\rho}_{;c}$ and $\vec{g}_{AB}\,\vec{\rho}^A = 2\vec{\rho}_B$.

Proof. $(g_{ac} - f_{ac}, h_{ac})(\rho^a, -\sigma^a)$

$$= (\rho_{;c} - \sigma_{;a} \cdot \sigma_{;c}\rho^a + \rho_{;a} \cdot \sigma_{;c}\sigma^a + \sigma_{;a} \cdot \rho_{;c}\sigma^a,$$

$$-\rho_{;a} \cdot \rho_{;c}\sigma^a + \sigma_{;a} + \rho_{;a} \cdot \sigma_{;c}\rho^a + \sigma_{;a} \cdot \rho_{;c}\rho^a)$$

$$= (2\rho_{;c}, 2\sigma_{;c}) + (-\sigma_{;c}, \rho_{;c}) \cdot (\sigma_{;a}\rho^a - \rho_a \sigma^a)$$

$$T^a_A\, T^b_B\, T^A_c\, (g_{ab} - f_{ab}, h_{ab})(\rho^c, -\sigma^c)$$

$$= T^b_B\, (g_{ab} - f_{ab}, h_{ab})(\rho^a, -\sigma^a)$$

$$= T^b_B\, (2\rho_{;b}, 2\sigma_{;b}) = 2(T^b_B\, \rho_{;b}, T^b_B\, \sigma_{;b}).$$

DEFINITION 8. $\vec{g}^{ac} = \vec{\rho}^a \cdot \vec{\rho}^c$.

THEOREM 9. $\vec{g}^{ac} = (g^{ac} - f^{ac}, -h^{ac})$ and $\vec{g}^{AB} = \vec{\rho}^A \cdot \vec{\rho}^B = (g^{AB} - f^{AB}, -h^{AB})$.

Proof. $\vec{g}^{ac} = (\rho^a, -\sigma^a) \cdot (\rho^c, -\sigma^c)$

$$= (\rho^a \cdot \rho^c - \sigma^a \cdot \sigma^c, -\rho^a \cdot \sigma^c - \sigma^a \cdot \rho^c)$$

$$= (g^{ac} - f^{ac}, -h^{ac}).$$

$$\vec{g}^{AB} = T^A_a\, T^B_b\, \vec{\rho}^a \cdot \vec{\rho}^b = T^A_a\, \vec{\rho}^a \cdot T^B_b\, \vec{\rho}^b$$

$$= \vec{\rho}^A \cdot \vec{\rho}^B = (\rho^A, -\sigma^A) \cdot (\rho^B, -\sigma^B)$$

$$= (\rho^A \cdot \rho^B - \sigma^A \cdot \sigma^B, -\rho^A \cdot \sigma^B - \sigma^A \cdot \rho^B)$$

$$= (g^{AB} - f^{AB}, -h^{AB}) .$$

LEMMA 9. If $\rho^a \cdot \sigma_{;c} = \sigma^a \cdot \rho_{;c}$ and $\rho, \sigma$ are compatible,

then $f^{ab}\ \rho_{;a} = \rho^a \cdot \sigma^b\ \sigma_{;a}$, $g^{ab}\ \sigma_{;a} = \sigma^a \cdot \rho^b\ \rho_{;a}$,

$f^{AB}\ \rho_A = \rho^A \cdot \sigma^B\ \sigma_A$ and $g^{AB}\ \sigma_A = \sigma^A \cdot \rho^B\ \rho_A$.

Proof. $f^{ab}\ \rho_{;a} = f^{ab}\ \rho_{;a} \cdot \sigma^c\ \sigma_{;c} = f^{ab}\ \sigma_{;a} \cdot \rho^c\ \sigma_{;c}$

$$= \sigma^b \cdot \rho^c\ \sigma_{;c}$$

$$f^{AB}\ \rho_A = T_a^A\ T_b^B\ T_A^c\ f^{ab}\ \rho_{;c} = T_b^B\ f^{ab}\ \rho_{;a}$$

$$= T_b^B\ \sigma^b \cdot \rho^a\ T_a^A\ T_A^c\ \sigma_{;c} = \sigma^B \cdot \rho^A\ \sigma_A$$

and the other two follow similarly.

LEMMA 10. If $\rho^a \cdot \sigma_{;c} = \sigma^a \cdot \rho_{;c}$, and $\rho, \sigma$ are compatible

then $(\sigma^a \cdot \rho^a) \cdot (\rho^c\ \sigma_{;c} - \sigma^c\ \rho_{;c}) = (0,0)$ and

$(\sigma^A, \rho^A) \cdot (\rho^B\ \sigma_B - \sigma^B\ \rho_B) = ((0,0), (0,0))$ .

Proof. $\sigma^a \cdot \sigma^b\ \rho_{;a} = f^{ab}\ \rho_{;b} = \sigma^b \cdot \rho^a\ \sigma_{;a}$

and $\rho^a \cdot \rho^b\ \sigma_{;a} = g^{ab}\ \sigma_{;a} = \sigma^a \cdot \rho^b\ \rho_{;a}$

$\Rightarrow \sigma^b \cdot (\rho^a \cdot \sigma_{;a} - \sigma^a \cdot \rho_{;a}) = \rho^b \cdot (\rho^a\ \sigma_{;a} - \sigma^a\ \rho_{;a}) = 0$ .

Similarly, from $\sigma^A \cdot \sigma^B\ \rho_A = \rho^A \cdot \sigma^B\ \sigma_a$ and $\rho^A \cdot \rho^B\ \sigma_A = \sigma^A \cdot \rho^B\ \rho_A$

the other result follows.

THEOREM 10. If $\rho^a \cdot \sigma_{;c} = \sigma^a \cdot \rho_{;c}$ and $\rho, \sigma$ are compatible,

then $\vec{g}^{ac}\ \vec{\rho}_{;c} = 2\vec{\rho}^a$ and $\vec{g}^{AB}\ \vec{\rho}_B = 2\vec{\rho}^A$ .

Proof. $\vec{g}^{ac}\,\vec{\rho}_{;c} = (g^{ac} - f^{ac}, -h^{ac})(\rho_{;c}, \sigma_{;c})$

$= (\rho^a - f^{ac}\,\rho_{;c} + h^{ac}\,\sigma_{;c}, g^{ac}\,\sigma_{;c} - \sigma^a - h^{ac}\,\rho_{;c})$

$= (\rho^a - \sigma^a\cdot\sigma^c\,\rho_{;c} + \rho^a\cdot\sigma^c\,\sigma_{;c} + \sigma^a\cdot\rho^c\,\sigma_{;c},$

$\rho^a\cdot\rho^c\,\sigma_{;c} - \sigma^a - \rho^a\cdot\sigma^c\,\rho_{;c} - \sigma^a\cdot\rho^c\,\rho_{;c})$

$= (2\rho^a, -2\sigma^a) + (\sigma^a \cdot (\rho^c\,\sigma_{;c} - \sigma^c\,\rho_{;c}),$

$\rho^a \cdot (\rho^c\,\sigma_{;c} - \sigma^c\,\rho_{;c}))$ .

Also, $\vec{g}^{AB}\,\vec{\rho}_{B} = T^A_a\,T^B_b\,T^c_B\,\vec{g}^{ab}\,\vec{\rho}_{;c} = T^A_b \cdot 2\vec{\rho}^b = 2\vec{\rho}^A$ .

THEOREM 11. If $\rho^a \cdot \sigma_{;c} = \sigma^a \cdot \rho_{;c}$ and $\rho$, $\sigma$ are compatible, then $\vec{g}^{ac}\,\vec{g}_{ab} = (4\delta^c_b, 0)$ and $\vec{g}^{AC}\,\vec{g}_{AB} = (4\delta^C_B, 0)$ .

Proof. $\vec{g}^{ac}\,\vec{g}_{ab} = (g^{ac} - f^{ac}, -h^{ac})(g_{ab} - f_{ab}, h_{ab})$

$= (2\delta^c_b, 0) + (2\delta^c_b, 2(\rho^c\cdot\sigma_{;b} - \sigma^c\cdot\rho_{;b}))$

$+ (\sigma^c \cdot (\rho^a\,\sigma_{;a} - \sigma^a\,\rho_{;a}), \rho^c \cdot (\rho^a\,\sigma_{;a} - \sigma^a\,\rho_{;a}))$

$\cdot (\rho_{;b}, \sigma_{;b}) = (4\delta^c_b, 0)$ .

Also, $\vec{g}^{AC}\,\vec{g}_{AB} = T^A_a\,T^C_d\,T^b_A\,T^e_B\,\vec{g}^{ad}\,\vec{g}_{be} = T^C_d\,T^e_B\,\vec{g}^{ad}\,\vec{g}_{ae}$

$= (T^C_d\,T^e_B\,4\delta^d_e, 0) = (4T^C_e\,T^e_B, 0) = (4\delta^c_B, 0)$ .

DEFINITION 9. $\{\vec{a}_{b\,c}\} = \vec{\rho}^a \cdot \vec{\rho}_{;bc}$ , $\{\vec{a}_b\} = \vec{\rho}^a \cdot \vec{\rho}_{;b}{}'$ .

THEOREM 12. $\{a_b\} = (\{a_b\}_g + \{a_b\}_f\,, \Gamma^a_b)$, where $\Gamma^a_b = \rho^a \cdot \sigma_{;b}{}' - \sigma^a \cdot \rho_{;b}{}'$ ,

and $\{\vec{A}_B\} = (\{A_B\}_g + \{A_B\}_f\,, \Gamma^A_B)$ .

Proof. $\{\overset{\rightarrow}{\underset{b}{a}}\} = \overset{\rightarrow}{\rho}^{a} \cdot \overset{\rightarrow}{\rho}_{;b}' = (\rho^{a}, -\sigma^{a}) \cdot (\rho_{;b}', \sigma_{;b}')$

$= (\rho^{a}\cdot\rho_{;b}' + \sigma^{a}\cdot\sigma_{;b}', \rho^{a}\cdot\sigma_{;b}' - \sigma^{a}\cdot\rho_{;b}')$

$= (\{{}^{a}_{b}\}_{g} + \{{}^{a}_{b}\}_{f}, \Gamma^{a}_{b})$ .

Also, $\{\overset{\rightarrow}{\underset{B}{A}}\} = \overset{\rightarrow}{\rho}^{A} \cdot \overset{\rightarrow}{\rho}_{B}' = T^{A}_{a}\,\overset{\rightarrow}{\rho}^{a} \cdot T^{b}_{B}\,\overset{\rightarrow}{\rho}_{;b}' = \overset{\rightarrow}{\rho}^{A} \cdot \overset{\rightarrow}{\rho}_{B}'$ .

THEOREM 13. If $\rho^{a} \cdot \sigma_{;c} = \sigma^{a} \cdot \rho_{;c}$ and $\rho$, $\sigma$ are compatible,

then $\overset{\rightarrow}{g}^{ad}\,[b\vec{c},d] = 2\{\overset{\rightarrow}{\underset{b\ c}{a}}\}$ and $\overset{\rightarrow}{g}^{AD}\,[B\vec{C},D] = 2\{\overset{\rightarrow}{\underset{B\ C}{A}}\}$ .

Proof. $(g^{ad} - f^{ad}, -h^{ad})([bc,d]_{g} - [bc,d]_{f}, [bc,d]_{h})$

$= 2(\rho^{a}\cdot\rho_{;bc} + \sigma^{a}\cdot\sigma_{;bc}\ ,\ \rho^{a}\cdot\sigma_{;bc} - \sigma^{a}\cdot\rho_{;bc})$

$+ (\rho_{;bc}, \sigma_{;bc}) \cdot ((\sigma_{;d}\,\rho^{d} - \rho_{;d}\,\sigma^{d}) \cdot \sigma^{a}\ ,$

$(\sigma_{;d}\,\rho^{d} - \rho_{;d}\,\sigma^{d}) \cdot \rho^{a})$ .

$\overset{\rightarrow}{g}^{AD}\,[B\vec{C},D] = T^{A}_{a}\,T^{D}_{d}\,\overset{\rightarrow}{g}^{ad}\,\overset{\rightarrow}{\rho}_{;bc} \cdot \overset{\rightarrow}{\rho}_{;e}\,T^{b}_{B}\,T^{c}_{C}\,T^{e}_{D}$

$= T^{A}_{a}\,T^{b}_{B}\,T^{c}_{C}\,2\overset{\rightarrow}{\rho}_{;bc} \cdot \overset{\rightarrow}{\rho}^{a} = 2\overset{\rightarrow}{\rho}_{BC} \cdot \overset{\rightarrow}{\rho}^{A}$ .

DEFINITION 10. $\overset{\rightarrow}{\rho}^{(\alpha)a} = \overset{\rightarrow}{\rho}^{a(\alpha)}$ .

THEOREM 14. $\overset{\rightarrow}{\rho}^{(\alpha)a}$ are extensor components of the type [(1,0),M; (0,0),0].

Proof. $\overset{\rightarrow}{\rho}^{(\alpha)a}\ x^{\rho r}_{\alpha a} = (\rho^{a}, -\sigma^{a})^{(\alpha)}\ x^{\rho r}_{\alpha a}$

$= (\rho^{(\alpha)a}\ x^{\rho r}_{\alpha a}, -\sigma^{(\alpha)a}\ x^{\rho r}_{\alpha a}) = (\rho^{r(\rho)}, -\sigma^{r(\rho)})$

$= (\rho^{r}, -\sigma^{r})^{(\rho)} = \overset{\rightarrow}{\rho}^{r(\rho)}$ .

DEFINITION 11. $\overset{\rightarrow}{g}^{a}_{\beta b} = \overset{\rightarrow}{\rho}^{a} \cdot \overset{\rightarrow}{\rho}_{(\beta)b}$ .

THEOREM 15. $\overset{\rightarrow}{g}{}^{a}_{\beta b}$ are extensor components of the type $[(0,1),M;\ (1,0),0]$

$\overset{\rightarrow}{g}{}^{a}_{0b} = \{\overset{\rightarrow}{{}^{a}_{b}}\}$, and for $M = 1$ $\overset{\rightarrow}{g}{}^{a}_{1b} = (2\delta^{a}_{b},\ 0)$, and

$$\overset{\rightarrow}{g}{}^{A}_{\beta B} = \overset{\rightarrow}{\rho}{}^{A} \cdot \overset{\rightarrow}{\rho}_{(\beta)B} .$$

Proof. $g^{a}_{\beta b}\ x^{r}_{a}\ x^{\beta b}_{\sigma s} = \overset{\rightarrow}{\rho}{}^{a} \cdot \overset{\rightarrow}{\rho}_{(\beta)b}\ x^{r}_{a}\ x^{\beta b}_{\sigma s}$

$$= (\rho^{a}\ x^{r}_{a},\ -\sigma^{a}\ x^{r}_{a}) \cdot (\rho_{(\beta)b}\ x^{\beta b}_{\sigma s},\ \sigma_{(\beta)b}\ x^{\beta b}_{\sigma s})$$

$$= (\rho^{r},\ -\sigma^{r}) \cdot (\rho_{(\sigma)s},\ \sigma_{(\sigma)s})$$

$$= \overset{\rightarrow}{\rho}{}^{r} \cdot \overset{\rightarrow}{\rho}_{(\sigma)s} = g^{r}_{\sigma s} .$$

$$\overset{\rightarrow}{g}{}^{a}_{0b} = \overset{\rightarrow}{\rho}{}^{a} \cdot \overset{\rightarrow}{\rho}_{;b}{}' = \{\overset{\rightarrow}{{}^{a}_{b}}\},\quad \overset{\rightarrow}{g}{}^{a}_{1b} = \overset{\rightarrow}{\rho}{}^{a} \cdot \overset{\rightarrow}{\rho}_{;b} = (2\delta^{a}_{b},\ 0) .$$

Also, $\overset{\rightarrow}{g}{}^{A}_{\beta B} = T^{A}_{a}\ T^{b}_{B}\ \overset{\rightarrow}{g}{}^{a}_{\beta b} = T^{A}_{a}\ \overset{\rightarrow}{\rho}{}^{a} \cdot\ T^{b}_{B}\ \overset{\rightarrow}{\rho}_{(\beta)B}$ ,

DEFINITION 12. $\delta\overset{\rightarrow}{V}{}^{a} = \sum_{\beta=1}^{0} \binom{2}{\beta}^{-1} \overset{\rightarrow}{g}{}^{a}_{\beta b}\ \overset{\rightarrow}{V}{}^{(\beta)b}$ .

THEOREM 16. If $\overset{\rightarrow}{V}{}^{a} = (u^{a},\ w^{a})$ then $\delta\overset{\rightarrow}{V}{}^{a} = (\delta u^{a},\ \delta w^{a}) + \Omega(u^{a},\ w^{a})$

where $\Omega(u^{a},\ w^{a}) = (\{{}^{a}_{b}\}_{f}\ u^{b} - \Gamma^{a}_{b}\ w^{b},\ \{{}^{a}_{b}\}_{g}\ w^{b} + \Gamma^{a}_{b}\ u^{b})$ .

Proof. $\sum_{\beta=1}^{0} \binom{2}{\beta}^{-1} g^{a}_{\beta b}\ (u^{(\beta)b},\ w^{(\beta)b})$

$$= \tfrac{1}{2}(2\delta^{a}_{b},\ 0)(u^{b}{}',\ w^{b}{}') + (\{{}^{a}_{b}\}_{g} + \{{}^{a}_{b}\}_{f},\ \Gamma^{a}_{b})(u^{b},\ w^{b})$$

$$= (u^{a}{}',\ w^{a}{}') + (\{{}^{a}_{b}\}_{g}\ u^{b} + \{{}^{a}_{b}\}_{f}\ u^{b} - \Gamma^{a}_{b}\ w^{b},$$

$$\{{}^{a}_{b}\}_{g}\ w^{b} + \{{}^{a}_{b}\}_{f}\ w^{b} + \Gamma^{a}_{b}\ u^{b})$$

$$= (u^{a}{}' + \{{}^{a}_{b}\}_{g}\ u^{b},\ w^{a}{}' + \{{}^{a}_{b}\}_{f}\ w^{b}) + (\{{}^{a}_{b}\}_{f}\ u^{b} - \Gamma^{a}_{b}\ w^{b},$$

$$\{{}^{a}_{b}\}_{g}\ w^{b} + \Gamma^{a}_{b}\ u^{b}) .$$

THEOREM 17. If $\vec{V}^b$ has type ((1,0), 0)

and $\vec{G}_M|^a_{\beta b}$ has type ((0,1),M; (1,0),0)

then there is a type ((0,1), M+1; (1,0),0) extensor

$\vec{G}_{M+1}|^a_{\beta_b}$ so that for $\vec{V}_{M+1}|^{(\beta)b}$ of type ((1,0), M+1; (0,0),0),

$$\delta\vec{G}_M|^a_{\beta b}\ \vec{V}_M|^{(\beta)b} = \vec{G}_{M+1}|^a_{\beta b}\ \vec{V}_{M+1}|^{(\beta)b} .$$

Proof. $\delta\vec{G}_M|^a_{\beta b}\ \vec{V}_M|^{(\beta)b}$

$$= \vec{G}_M|^a_{\beta b}\ \vec{V}_M|^{b(\beta+1)} + \vec{G}_M|^{a'}_{\beta b}\ \vec{V}_M|^{b(\beta)} + \{\vec{a}_c\}\ \vec{G}_M|^c_{\beta b}\ \vec{V}^{b(\beta)}$$

$$= \vec{G}_M|^a_{Mb}\ \vec{V}_M|^{b(M+1)} + \sum_{\beta=0}^{M} [\vec{G}_M|^a_{\beta-1,b} + \vec{G}_M|^{a'}_{\beta b} + \{\vec{a}_c\}\ \vec{G}_M|^c_{\beta b}]\ \vec{V}_M|^{b(\beta)}$$

$$= \sum_{\beta=0}^{M+1} \vec{G}_{M+1}|^a_{\beta b}\ \vec{V}_{M+1}|^{b(\beta)}$$

where $\vec{G}_{M+1}|^a_{M+1,b} = \vec{G}_M|^a_{Mb}$

$$\vec{G}_{M+1}|^a_{\beta b} = \vec{G}_M|^a_{\beta-1,b} + \vec{G}_M|^{a'}_{\beta b} + \{\vec{a}_c\}\ \vec{G}_M|^c_{\beta b},\quad 0 \leq \beta \leq M .$$

DEFINITION 13. The extensor $\vec{G}_{M+1}|^a_{\beta b}$ defined by

$$\vec{G}_{M+1}|^a_{M+1,b} = \vec{G}_M|^a_{Mb}$$

$$\vec{G}_{M+1}|^a_{\beta b} = \vec{G}_M|^a_{\beta-1,b} + \vec{G}_M|^{a'}_{\beta b} + \{\vec{a}_c\}\ \vec{G}_M|^c_{\beta b},\quad 0 \leq \beta \leq M$$

is called the extensive derivative of $\vec{G}_M|^a_{\beta b}$ .

DEFINITION 14. If $\vec{G}_0|^a_{\beta b} = \vec{G}_0|^a_{0b} = (\delta^a_b, 0)$, then $\vec{G}_M|^a_{\beta b}\ \vec{V}_M|^{(\beta)b}$

is called the $M^{th}$ intrinsic derivative of $\vec{V}^a$, and is denoted by $\delta^M \vec{V}^a$ .

DEFINITION 15. $(\Omega u^a, \Omega w^a) = \Omega(u^a, w^a)$ .

THEOREM 18. $\delta^M(u^a, w^a) = ((\delta + \Omega)^M u^a, (\delta + \Omega)^M w^a)$ .

Proof. The theorem is valid for $M = 1$ since

$$\delta^1(u^a, w^a) = (\delta u^a, \delta w^a) + \Omega(u^a, w^a)$$
$$= ((\delta + \Omega)^1 u^a, (\delta + \Omega)^1 w^a),$$

which agrees with Theorem 16 upon using Definition 15.

We assume the theorem is valid for $M = K$, i.e.,

$$\delta^K(u^a, w^a) = ((\delta + \Omega)^K u^a, (\delta + \Omega)^K w^a) . \qquad (A)$$

Hence applying $\delta$ to the left hand side of equation A and using in turn IV 2 D14, D12, and D13, we obtain

$$\delta^1 \delta^K(u^a, w^a) = \vec{G}_1|^a_{\beta b} (\vec{G}_M|^b_{\gamma c} \vec{V}^{c(\gamma)})^{(\beta)}$$
$$= \sum_{\beta=1}^{0} \vec{G}_1|^a_{\beta b} \sum_{\theta=\beta}^{0} \binom{\beta}{\theta} \vec{G}_M|^{b(\beta-\theta)}_{\gamma c} \vec{V}^{c(\gamma)(\theta)}$$
$$= \vec{G}_M|^a_{Mc} V^{c(M+1)} + \sum_{\gamma=M-1}^{0} \vec{G}_M|^b_{\gamma c} V^{c(\gamma+1)}$$
$$+ \sum_{\gamma=M}^{0} [\vec{G}_M|^{b'}_{\gamma c} + \{\vec{a}_b\} \vec{G}_M|^b_{\gamma c}] \vec{V}^{c(\gamma)} , \quad \text{or}$$

$$\delta^{K+1}(u^a, w^a) = \sum_{\gamma=M+1}^{0} \vec{G}_{M+1}|^a_{\gamma c} \vec{V}_{M+1}|^{c(\gamma)} \qquad (B)$$

where $\vec{V}^a = (u^a, w^a)$ .

Applying $\delta$ to the right hand side of equation A and using in turn IV 2 T16 and IV 2 D15, we obtain

$$\delta^1((\delta + \Omega)^K u^a, (\delta + \Omega)^K w^a) = (\delta(\delta + \Omega)^K u^a, \delta(\delta + \Omega)^K w^a)$$
$$+ \Omega((\delta + \Omega)^K u^a, (\delta + \Omega)^K w^a)$$

$$= ((\delta + \Omega)^{K+1} u^a, (\delta + \Omega)^{K+1} w^a) . \quad (C)$$

Now equating the left hand side of equation (B) to the right hand side of equation (C), we see that the result is just the equation of Theorem 18 with M replaced by K+1.

Hence, by the postulate of mathematical induction the result follows.

## 3. Primary Spin Extensors

In this section we generalize the ideas of primary extensors associated with functions whose arguments are coordinates and extensors to functions whose arguments are z-coordinates and spin extensors.

THEOREM E1. If $F(x^a,\ldots,x^{a(M)}, E^{0a},\ldots,E^{Ma})$ is a class $C'$ function associated with a coordinate system, of the $2(M+1)N$ arguments, the E's being extensor components, and $\overline{F}$ is the mate function in another coordinate system such that

$$\overline{F}(x^r,\ldots, E^{Mr}) = F(x^a(x^r),\ldots, E^{Ma} X^{\ r}_{Ma})$$

then $\partial\overline{F}/\partial E^{\rho r}$, $\partial F/\partial E^{\alpha a}$ are excovariant.

Proof. [4a; 160].

An example of such a function occurs in the theory of dynamical systems having coordinates $q^a$, namely the kinetic energy being given by a positive definite quadratic form

$T = \frac{1}{2} g_{ac}(q)\, q'^{a} q'^{c}$. In general, a function $F(x,x')$ is this type, and the zero rank, $E^{0a} = x'^{a}$, are tensor components.

If in $F(x^a,x'^a)$ each $x'^a$ is replaced by a tensor symbol $E^a$, and $\overline{F}$ is constructed as above, then $(\partial F/\partial E^r, \partial F/\partial E^a)$ are covariant. If from $F(x^a,x'^a)$, $\overline{F}^*(x^r,x'^r)$ is constructed as above, then $\partial_{\alpha a} F$, $\partial_{\rho r} \overline{F}^*$ are excovariant. There is not any essential distinction between $\partial\overline{F}^*/\partial x'^r$ and $\partial\overline{F}/\partial E^r$, so that $(\partial F/\partial x^a, \partial F/\partial E^a)$, $(\partial\overline{F}/\partial x^r, \partial\overline{F}/\partial E^r)$ are excovariant. Because of the covariance above, the following notation is natural.

DEFINITION 1. $F_{:a} = \Delta_a F = \partial F/\partial E^a$.

For spin components, $\Delta_A = \Delta_a T^a_A$.

REMARK. It is convenient to let the free indices denote the type of tensor, extensor, spin tensor, or spin extensor involved in the sequel. The weight is invariably zero.

LEMMA 1. If $E^r$ are tensor components, and the spin quantities are defined as above, then $E^S_{:R} = \delta^S_R$.

Proof. $E^S_{:R} = \Delta_R E^S = T^r_R \Delta_r T^S_s E^s = T^r_R T^S_s \delta^s_r = T^s_R T^S_s = \delta^S_R$.

LEMMA 2. If $E^R$ are spin tensor components, then $E^R_{:A} = Z^R$.

Proof. $\Delta_A E^R = \Delta_A E^B Z^R_B = \delta^B_A Z^R_B = Z^R_A$ .

LEMMA 3. If $E^A$ are spin tensor components,

then $\Delta_R E^A \Delta_A = \Delta_R$ .

THEOREM S1. If $F = F(x^a, E^a)$, $\overline{F}(x^r, E^r) = F(x^a(x^r), E^a X^r_a)$,

$E^A$ are the spin components,

then $F_{:A} = F_{:R} Z^R_A$ .

Proof. $\Delta_A F = \Delta_a F T^a_A = \Delta_r F T^a_A X^r_a = \Delta_r F T^r_R Z^R_A = \Delta_R F Z^R_A$ .

THEOREM E2. If $F(x^a,\ldots,x^{(M)a}, E_{Ma},\ldots,E_{0a})$ is a class C'

function, the E's are extensor components, $\overline{F}$

is the mate function

then $\partial F/\partial E_r$, $\partial F/\partial E_a$ are excontravariant components.

Proof. [4a; 159].

An example of such a function occurs in the theory of dynamical systems, where if $T(q^a,q'^a)$ is the kinetic energy, then the momenta $p_a = T_{;1a}$ are covariant, and $T$ can be expressed as $t(q^a,p_a) = \frac{1}{2} g^{ac}(q)\, p_a\, p_c$.

In general, if $E_a$ are covariant tensor components and $F = F(x^a,E_a)$, then $\partial F/\partial E_a$ are contravariant tensor components.

DEFINITION 2. For tensor components, $F^{:a} = \Delta^a F = \partial F/\partial E_a$ .

For spin components, $F^{:A} = \Delta^A F = \partial F/\partial E_A$ .

LEMMA 4. If $E_A$ are spin components,

then $E_A{}^{:R} = Z^R_{;A}$ .

THEOREM S2. If $F = G(x^a, E_a) + i\, H(x^a, E_a)$, the $E$'s are covariant extensor components, $E_A$ are the corresponding spin components, $\overline{F}$ is the mate function,

then $F^{:A} = F^{:R}\, Z^A_R$.

Proof. $\Delta^A F = T^A_a\, \Delta^a F = T^A_a\, \Delta^r F\, X^a_r = T^A_a\, T^r_R\, \Delta^R F\, X^a_r = \Delta^R F\, T^A_a\, T^a_B\, Z^B_R$

$= \Delta^R F\, \delta^A_B\, Z^B_R$.

DEFINITION 3. $F^{:\alpha a} = \Delta^{\alpha a} F = \partial F / \partial E_{\alpha a}$ and $\Delta^{\alpha A} = \Delta^{\alpha a}\, T^A_a$.

THEOREM S3. If $E_{\alpha a}$ are excovariant extensor components, $E_{\alpha A}$ are their spin components $\overline{F}(x^{\rho r}, E_{\rho r}) = F(x^{\alpha a}, E_{\alpha a})$,

then $F^{:\rho R} = F^{:\alpha A}\, Z^{\rho R}_{\alpha A}$.

Proof. $\Delta^{\rho R} F = T^R_r\, \Delta^{\rho r} F = T^R_r\, \Delta^{\alpha a} F\, X^{\rho r}_{\alpha a} = T^A_a\, Z^{\rho R}_{\alpha A}\, \Delta^{\alpha a} F = \Delta^{\alpha A} F\, Z^{\rho R}_{\alpha A}$.

THEOREM E4. If $F = F(x^a, y_a, f(x))$ is a class $C'$ function of the $2n+1$ independent variables, $f$ is a class $c'$ function of the $x^a$, the $y$'s are absolute covariant tensor components, $\overline{F}$ is the mate function, $F^{:a} = x'^a$,

then $E_{0a} = F_{;a} + \frac{\partial F}{\partial f} f_{;a}$, $E_{1a} = -y_a$ define extensor components.

Proof. [4a; 161].

DEFINITION 4. The extensor whose components are $E_{0a}, E_{1a}$ in IV 3 TE4 is denoted by $F_{\alpha a}$.

In case $y_a = f_{;a}$ and $f$ does not explicitly occur in $F$, then $F_{0a} = F_{;a}$, $F_{1a} = -y_a$, and the Kawaguchi

extensor is $F_{(0)a} = F_{la}{}' = -y_a{}'$, $F_{(1)a} = F_{la} = -y_a$ .

THEOREM S4. If $F = G(x^a, h_a) + i\,H(x^a, h_a)$ the h's are absolute covariant tensor components, $h_A$ are the corresponding spin components $\overline{F}$ is the mate function $F^{:A} = Z'^{A}$, then $F_{0A} = F_{;A}$, $F_{1A} = -h_A$ define spin extensor components.

Proof. $$\partial_A F = \partial_A Z^R \partial_R F + \partial_A h_R \Delta^R F$$
$$= \partial_R F Z^R_A + \partial_A h_B Z^B_R \Delta^R F .$$

$$\partial_A h_B Z^B_R = h_B \partial_A Z^B_R = h_B \partial_A Z^S \partial_S Z^B_R = h_B \partial_A Z^S \partial_R Z^B_S$$

and $$Z^S_A \partial_R Z^B_S = - \partial_R Z^C \partial_C Z^S_A Z^B_S$$

imply $$\partial_A h_B Z^B_R = -h_B Z^C_R \partial_C Z^S_A Z^B_S = -h_S \partial_C Z^S_A Z^C_R .$$

Hence $$\partial_A F = \partial_R F Z^R_A - h_S \partial_C Z^S_A Z^C_R \Delta^R F .$$

If $\Delta^R F = Z^{R'}$ then $Z^C_R \Delta^R F = Z^C_R Z^{R'} = Z^{C'}$

and $$\partial_C Z^S_A Z^{C'} = Z^{S'}_A = Z^{1S}_{0A} .$$

From this result and $h_A = h_R Z^R_A$,

$$\partial_A F = \partial_R F Z^{0R}_{0A} - h_R Z^{1R}_{0A}$$

$$-h_A = -h_R Z^{1R}_{1A} ,$$

which are the desired results.

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