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GHK mirror symmetry, the Knutson-Tao hive cone, and Littlewood-Richardson coefficients

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GHK mirror symmetry, the Knutson-Tao hive cone, and Littlewood-Richardson coefficients

by

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GHK mirror symmetry, the Knutson-Tao hive cone, and Littlewood-Richardson coefficients

by

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I prove that the full Fock-Goncharov conjecture holds for $\text{Conf}^\times_3 (\mathcal{A})$—the configuration space of triples of decorated flags in generic position. As a key ingredient of this proof, I exhibit a maximal green sequence for the quiver of the initial seed. I compute the Landau-Ginzburg potential $W$ on $\text{Conf}^\times_3 (\mathcal{A})^\vee$ associated to the partial minimal model $\text{Conf}^\times_3 (\mathcal{A}) \subset \text{Conf}_3 (\mathcal{A})$. The integral points of the associated “cone”

$$\Xi := \{W^T \geq 0\} \subset \text{Conf}^\times_3 (\mathcal{A})^\vee (\mathbb{R}^T)$$

parametrize a basis for $\mathcal{O}(\text{Conf}_3 (\mathcal{A})) = \bigoplus (V_\alpha \otimes V_\beta \otimes V_\gamma)^G$ and encode the Littlewood-Richardson coefficients $c^\gamma_{\alpha\beta}$. I exhibit a unimodular $p^*$ map that identifies $W$ with the potential of Goncharov-Shen on $\text{Conf}^\times_3 (\mathcal{A})$ [GS14] and $\Xi$ with the Knutson-Tao hive cone [KT98].
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1 Introduction

1.a Summary of results

In this paper I prove Corollary 0.21 of [GHKK15], and recover Corollary 0.20 as well. Let $G = \text{GL}_n$, and, following [FG06, GS15], let $\mathcal{A}$ be the base affine space $G/U$ and define $\text{Conf}_3(\mathcal{A}) := G\backslash\mathcal{A}^{\times 3}$ and $\text{Conf}_3^\times(\mathcal{A})$ the subvariety where pairs of underlying flags intersect generically.

**Theorem 1.** The full Fock-Goncharov conjecture holds for $\text{Conf}_3^\times(\mathcal{A})$.

The full Fock-Goncharov conjecture is defined in [GHKK15, Definition 0.6] and will be described in the next subsection. $\text{Conf}_3(\mathcal{A})$ is a partial compactification of $\text{Conf}_3^\times(\mathcal{A})$, which is a cluster $\mathcal{A}$-variety, and this compactification gives a Landau-Ginzburg potential $W$ on the mirror $\mathcal{X}$-variety $\text{Conf}_3^\times(\mathcal{A})^\vee$. We can tropicalize $W$ to get a subset $\Xi := \{W^\mathbb{T} \geq 0\}$ of $\text{Conf}_3^\times(\mathcal{A})^\vee(\mathbb{R}^T)$. A choice of seed identifies $\text{Conf}_3^\times(\mathcal{A})^\vee(\mathbb{R}^T)$ with a real vector space and $\Xi$ with a rational polyhedral cone in this vector space.

**Theorem 2.** In the initial seed, the cone $\Xi$ defined by the Landau-Ginzburg potential $W$ on the mirror to $\text{Conf}_3^\times(\mathcal{A})$ is unimodularly equivalent to the Knutson-Tao hive cone. Furthermore, a particular choice of the map $p^* : \mathcal{N} \to \mathcal{M}$ identifies $W$ with the potential of Goncharov-Shen on $\text{Conf}_3^\times(\mathcal{A})$.

The base affine space $\mathcal{A}$ is a torus bundle over the flag variety $\mathcal{B}$. Let $H := B/U$, where $B$ is the Borel subgroup of upper triangular matrices in $G$ and $U$ its unipotent radical– upper triangular matrices with 1’s along the diagonal. Then $\mathcal{A} = G/U$ is naturally a principal $H$-bundle over $\mathcal{B} = G/B$. In the same way, $\text{Conf}_3(\mathcal{A})$ is a principal $H^{\times 3}$-bundle over $\text{Conf}_3(\mathcal{B})$. In both cases, the base is Fano and the total space is the universal torsor for the base. It has a very special property:

$$\mathcal{O}(\text{Conf}_3(\mathcal{A})) = \text{Cox}(\text{Conf}_3(\mathcal{B})) := \bigoplus_{\mathcal{L} \in \text{Pic}(\text{Conf}_3(\mathcal{B}))} \Gamma(\text{Conf}_3(\mathcal{B}), \mathcal{L}).$$

The mirror $\text{Conf}_3^\times(\mathcal{A})^\vee$ to $\text{Conf}_3(\mathcal{A})$ comes with a map to the dual torus $(H^{\times 3})^\vee$, tropicalizing to a map

$$w : \text{Conf}_3^\times(\mathcal{A})^\vee(\mathbb{Z}^T) \to (H^{\times 3})^\vee(\mathbb{Z}^T).$$

The integral tropicalization of a torus $T$ is just its cocharacter lattice $\chi_*(T)$, so $(H^{\times 3})^\vee(\mathbb{Z}^T)$ is the character lattice $\chi^*(H^{\times 3})$. Theorem 1 together with the existence of an optimized seed for each frozen variable (Proposition 16) and existence of a unimodular $p^*$ map (Proposition 22), implies that points in $\Xi(\mathbb{Z}^T) := \Xi \cap \text{Conf}_3^\times(\mathcal{A})^\vee(\mathbb{Z}^T)$ are regular functions on $\text{Conf}_3(\mathcal{A})$ invariant under the $H^{\times 3}$ action. If $f \in \Xi(\mathbb{Z}^T)$, then $w(f)$ is the weight of $f$ under the $H^{\times 3}$ action. Given a weight $(\alpha, \beta, \gamma)$ of this action, $w^{-1}(\alpha, \beta, \gamma) \cap \Xi(\mathbb{Z}^T)$ is a basis for the $(\alpha, \beta, \gamma)$-weight space of $\mathcal{O}(\text{Conf}_3(\mathcal{A}))$. Since $\mathcal{O}(\text{Conf}_3(\mathcal{A})) = \bigoplus_{\alpha, \beta, \gamma} \langle V_\alpha \otimes V_\beta \otimes V_\gamma \rangle^G$, counting these points gives the Littlewood-Richardson coefficients. This is described in more detail in Section 2.

**Remark 3.** $\text{Conf}_3(\mathcal{A})$ and $\mathcal{A}$ have very similar cluster structures. As a result, many of the proofs in [Mag15] apply here as well. For completeness and convenience, I have provided them here. Since the current paper encompasses the main results in [Mag15], I’ll seek publication of this paper and not [Mag15].
1.b Full Fock-Goncharov conjecture

This subsection provides a bit of background on the full Fock-Goncharov conjecture following [GHKK16].

Let $V$ be a cluster variety, and $V^\vee$ its Fock-Goncharov dual, e.g. if $V$ is an $\mathcal{A}$-variety defined in terms of fixed data $\Gamma$, then $V^\vee$ is the $X$-variety defined using the Langlands dual fixed data $\Gamma^\vee$. See [FG09, Section 1.2] and [GHKK16, Appendix A]. We think of $V^\vee$ as mirror to $V$.

In [GHKK16], several algebras are associated to $V$. First, there is the familiar notion of the upper cluster algebra $\text{up}(V) = H^0(V, \mathcal{O}_V)$. Its subalgebra generated by global monomials, i.e. global regular functions restricting to a character on some torus in the atlas for $V$, is the ordinary cluster algebra $\text{ord}(V)$. In the case of an $\mathcal{A}$-type cluster variety, this corresponds to the usual notion of a cluster algebra. If $V^\vee$ is the Fock-Goncharov dual of $V$, $\text{can}(V)$ is a vector space with basis parametrized by $V^\vee(\mathbb{Z}^T)$. Scattering diagrams and broken lines are used to associate to each $m \in V^\vee(\mathbb{Z}^T)$ a (possibly infinite) sum of characters on each torus in $V$’s atlas, the result denoted by $\vartheta_m$, and to define a multiplication rule for the $\vartheta_m$. The details of this construction are beyond the scope of this paper. In situations where the full Fock-Goncharov conjecture holds, $\text{can}(V)$ will be identified with $\text{up}(V)$, and the $\vartheta_m$ will form a canonical basis for $\text{up}(V)$.

More generally, $\text{can}(V)$ has a subspace $\text{mid}(V)$ parametrized by the subset $\Theta \subset V^\vee(\mathbb{Z}^T)$ consisting of $\vartheta_m$ which restrict to finite sums of characters, i.e. Laurent polynomials, on tori from the atlas. Then each element of $\text{mid}(V)$ naturally corresponds to an element of $\text{up}(V)$, but there is no reason a priori that distinct elements of $\text{mid}(V)$ must correspond to distinct elements of $\text{up}(V)$. More formally, there is a canonical algebra homomorphism $\nu : \text{mid}(V) \to \text{up}(V)$.

**Definition 4.** [GHKK16, Definition 0.6] We say that the full Fock-Goncharov conjecture holds for $V$ if $\nu : \text{mid}(V) \to \text{up}(V)$ is injective, $\text{up}(V) = \text{can}(V)$, and $\Theta = V^\vee(\mathbb{Z}^T)$.

Many conditions implying the full Fock-Goncharov conjecture holds for a given cluster variety are provided in [GHKK16]. I will use [GHKK16, Proposition 8.28]: If $\mathcal{A}$ has large cluster complex, then $\mathcal{A}_{\text{prin}}$ has Enough Global Monomials, $\Theta = \mathcal{A}_{\text{prin}}^\vee(\mathbb{Z}^T)$, and the full Fock-Goncharov conjecture holds for $\mathcal{A}_{\text{prin}}$, $\mathcal{X}$, very general $\mathcal{A}_t$ and, if the convexity condition (7) of [GHKK16, Theorem 0.3] holds, for $\mathcal{A}$.

To show that Conf$_3^\times(\mathcal{A})$ has large cluster complex, I exhibit a maximal green sequence [BDP13, Definition 2.8] for the quiver of the initial seed. The existence of a maximal green sequence implies Conf$_3^\times(\mathcal{A})$ has large cluster complex by [GHKK16, Corollary 8.30].

**Theorem 5.** The quiver for the initial seed of Conf$_3^\times(\mathcal{A})$ has a maximal green sequence, and therefore Conf$_3^\times(\mathcal{A})$ has large cluster complex.

The convexity condition (7) of [GHKK16, Theorem 0.3] is the following:

There is a seed $s = (e_1, \ldots, e_n)$ for which all the covectors $\{e_i, \cdot\}$, $i \in I_{ut}$, lie in a strictly convex cone.

I show that this holds for the initial seed. Together with Theorem 5, this shows Theorem 5: the full Fock-Goncharov conjecture holds for Conf$_3^\times(\mathcal{A})$. 

2
1.c Partial compactifications and potentials

The space we are really interested in is \( \text{Conf}_3(A) \), rather than \( \text{Conf}_3(A) \times \text{Conf}_3(A) \). It is \( \text{Conf}_3(A) \), not \( \text{Conf}_3(A) \times \text{Conf}_3(A) \), that gives the decomposition

\[
\mathcal{O}(\text{Conf}_3(A)) = \bigoplus_{\alpha, \beta, \gamma} (V_\alpha \otimes V_\beta \otimes V_\gamma)^G.
\]

But Subsection 1.b gives a canonical basis for \( \mathcal{O}(\text{Conf}_3(A)) \).

This situation is typical. Generally spaces we are interested in, say for representation theoretic reasons, will not be cluster varieties, or even log Calabi-Yau varieties. However, many representation theoretically interesting spaces are partial compactifications of cluster varieties or log Calabi-Yaus in a nice way. For log Calabi-Yaus, the “nice” type of partial compactification we’re interested in is called a \textit{partial minimal model}. Take \( U \) to be a log Calabi-Yau with canonical volume form \( \Omega \). Then an inclusion \( U \subset Y \) as an open subset is a \textit{partial minimal model} if \( \Omega \) has a simple pole along every irreducible divisor of \( Y \setminus U \). In the special case that \( U \) is a cluster \( A \)-variety with frozen variables, there is a simple way these partial minimal models may arise—by taking \( Y \) to be the partial compactification given by allowing some frozen variables to vanish. [GHKK16, Section 0.3] \( \text{Conf}_3(A) \) and \( \text{Conf}_3(A) \) are related in precisely this way.

In this situation, each irreducible divisor in \( D := Y \setminus U \) defines a \textit{divisorial discrete valuation} pairing negatively with \( \Omega \), so by definition a point in \( U^\text{trop}(Z) \). [GHK15a, Definition 1.7] A \textit{divisorial discrete valuation} is a discrete valuation \( v : k(U) \setminus \{0\} \to \mathbb{Z} \) on the field of rational functions of \( U \) given by order of vanishing along a divisor on some variety birational to \( U \). When \( U^\vee \) is an affine log Calabi-Yau with maximal boundary, \( U^\text{trop}(Z) \) is conjectured to be a canonical basis for \( \Gamma(U^\vee, \mathcal{O}_{U^\vee}) \). [GHK15a, Conjecture 0.6] For cluster varieties, we have an identification \( i : U(Z^T) \to U^\text{trop}(Z) \) induced by sign change. See [GHKK16, Section 2]. And when the full Fock-Goncharov conjecture holds for \( U^\vee \), then \( U(Z^T) \) is indeed a basis for \( \Gamma(U^\vee, \mathcal{O}_{U^\vee}) \). With this in mind, \( D \) defines a potential \( W \) on \( U^\vee \). Each irreducible component \( D_k \) of \( D \) is a function \( \vartheta_k \) on \( U^\vee \), and \( W \) is the sum of these functions. \( W \) is known as a \textit{Landau-Ginzburg potential}.

Evaluation gives a pairing between \( U^\text{trop}(Z) \) and \( k(U) \). When points in \( (U^\vee)^\text{trop}(Z) \) are functions on \( U \), we can restrict the evaluation pairing to get a pairing

\[
\langle \cdot, \cdot \rangle : U^\text{trop}(Z) \times (U^\vee)^\text{trop}(Z) \to \mathbb{Z}
\]

\[
(v, w) \mapsto v(\vartheta_w).
\]

We could just as well start with the evaluation pairing between \( (U^\vee)^\text{trop}(Z) \) and \( k(U^\vee) \), which would restrict to the pairing

\[
\langle \cdot, \cdot \rangle^\vee : U^\text{trop}(Z) \times (U^\vee)^\text{trop}(Z) \to \mathbb{Z}
\]

\[
(v, w) \mapsto w(\vartheta_v).
\]

These two pairings are conjectured to agree for affine log Calabi-Yaus with maximal boundary, and for cluster

\[\text{See the last paragraph on page 12 of [GHK15a] for the definition of log Calabi-Yau with maximal boundary.}\]
varieties they are known to agree when either $v$ or $w$ is in the cluster complex\footnote{See \cite[Definition 2.9]{GHKK16} for the definition of the cluster complex.}. In Section 4a I show that each frozen index for $\text{Conf}_3^x(A)$ has an optimized seed, which implies that the associated point in $\text{Conf}_3^x(A)^\vee \times (\mathbb{Z}^T)$ is in the cluster complex\footnote{We can use \cite[Equation (2.5)]{GHKK16} to translate to the Fock-Goncharov tropicalization (max-plus convention). For $x \in \text{Conf}_3^x(A)^\vee \times (\mathbb{R}^T)$, we have $W(x) = W^{\text{trop}}(i(x))$.}. So for each summand $\vartheta_k$ of $W$ and each $v \in (\text{Conf}_3^x(A)^\vee)^{\text{trop}}(\mathbb{Z})$, we have $v(\vartheta_k) = \text{ord}_{D_k}(\vartheta_v)$. This means that $\vartheta_v$ extends to $D_k$ if and only if
\[
\vartheta_k^{\text{trop}}(v) := v(\vartheta_k) \geq 0,
\]
and it extends to all of $D$ if and only if
\[
\min_k \{\text{ord}_{D_k}(\vartheta_v)\} = v(W) =: W^{\text{trop}}(v) \geq 0.
\]
This gives a candidate basis for $\mathcal{O}(\text{Conf}_3(A))$, namely $\{\vartheta_v | W^{\text{trop}}(v) \geq 0\}$\footnote{Definition in \cite[final paragraph of page 12]{GHK15b}.}. This need not be a basis though. Poles can cancel when we add functions, so in principal we could have $\vartheta_p + \vartheta_q$ regular on $\text{Conf}_3(A)$ even if $\vartheta_p$ and $\vartheta_q$ have poles along $D$. This scenario is also prevented by the existence of an optimized seed for each frozen index.\cite[Proposition 9.7]{GHKK16}

**Theorem 6.** Let $\text{Conf}_3^x(A)^\vee$ denote the Fock-Goncharov dual of $\text{Conf}_3^x(A)$, and let $W$ be the Landau-Ginzburg potential on $\text{Conf}_3^x(A)^\vee$ associated to the partial minimal model $\text{Conf}_3^x(A) \subset \text{Conf}_3(A)$. Then
\[
\Xi(\mathbb{Z}^T) := \{W^T \geq 0\} \cap \text{Conf}_3^x(A)^\vee \times (\mathbb{Z}^T)
\]
is a basis for $\mathcal{O}(\text{Conf}_3(A))$, canonically determined by the pair $\text{Conf}_3^x(A) \subset \text{Conf}_3(A)$.

In general if we have some partial minimal model $U \subset Y$ for an affine log Calabi-Yau with maximal boundary, we can’t expect to get a canonical basis for $\mathcal{O}(Y)$ itself. The basis will be determined by the geometry of the pair $U \subset Y$ rather than $Y$’s geometry alone. However, in the particular case $Y = \text{Conf}_3(A)$, no choices need to be made to pick out the log Calabi-Yau open subset $U = \text{Conf}_3^x(A)$. It is simply the locus where underlying flags intersect generically, described in more detail in Section 2. In this sense, $\Xi(\mathbb{Z}^T)$ can be viewed as a canonical basis for $\mathcal{O}(\text{Conf}_3(A))$ itself— it is a basis determined entirely by $\text{Conf}_3(A)$’s own geometry.

1.d Mirror symmetry motivation

The picture described in this paper is motivated geometrically by \cite[Conjecture 1.9]{GHK15a} and \cite[Conjecture 0.6]{GHK15b}. Let $(Y, D)$ be a Looijenga pair\footnote{Definition in \cite[final paragraph of page 12]{GHK15b}} with $D$ ample, and let $U = Y \setminus D$. $U$ is an affine log Calabi-Yau with maximal boundary. Conjecturally, we have the following construction. Set $R = \mathbb{A} \left[ \text{Pic}(Y)^\vee \right]$, and take $V$ to be the free $R$-module on $U^{\text{trop}}(\mathbb{Z})$. Then $V$ has a natural $R$-algebra structure with multiplication coming from counts of rational curves in $U$. The fibration
\[
\text{Spec}(V) =: \mathcal{X} \to \text{Pic}(Y) := \text{Spec}(R)
\]
is a flat family of affine log Calabi-Yaus with maximal boundary, and when $\text{Pic}(U)$ is trivial this is the mirror family to $U$. The mirror family to $U$ does not depend on the choice of minimal model $(Y, D)$. Now repeat this construction replacing $U$ with a fiber $U^\vee$ of the mirror family. This should produce a family $\mathcal{Y}$ of deformations of $U$, and $\mathcal{Y}$ comes equipped with a canonical basis— the integer tropical points of the mirror.

In this paper, $\text{Conf}_3(B)$ plays the role of $Y$, $\text{Conf}_3^x(B)$ the role of $U$, and $\text{Conf}_3^x(A)$ the role of $\mathcal{Y}$. So we have

\[
\begin{align*}
\text{Conf}_3^x(A) & \xleftarrow{T_{\text{Pic}(\text{Conf}_3(B))}} \text{Conf}_3(A) \\
\text{Conf}_3^x(B) & \xleftarrow{T_{\text{Pic}(\text{Conf}_3(B))}} \text{Conf}_3^x(A)^\vee \\
\text{Conf}_3^x(B) & \xleftarrow{T_{\text{Pic}(\text{Conf}_3(B))}} \text{Conf}_3^x(A)^\vee \\
\text{Conf}_3^x(A) & \xleftarrow{T_{\text{Pic}(\text{Conf}_3(B))}} \text{Conf}_3(A) \\
\text{Conf}_3^x(B) & \xleftarrow{T_{\text{Pic}(\text{Conf}_3(B))}} \text{Conf}_3^x(A)^\vee \\
\text{Conf}_3^x(B) & \xleftarrow{T_{\text{Pic}(\text{Conf}_3(B))}} \text{Conf}_3^x(A)^\vee \\
\end{align*}
\]

Note that the base on one side is dual to the fiber on the other. To account for the (partial) compactifications, we include the Landau-Ginzburg potential mentioned in the previous subsection.

\[
\begin{align*}
\text{Conf}_3^x(A) & \xleftarrow{T_{\text{Pic}(\text{Conf}_3(B))}} \text{Conf}_3(A) \\
\text{Conf}_3^x(B) & \xleftarrow{T_{\text{Pic}(\text{Conf}_3(B))}} \text{Conf}_3^x(A)^\vee \\
\text{Conf}_3^x(B) & \xleftarrow{T_{\text{Pic}(\text{Conf}_3(B))}} \text{Conf}_3^x(A)^\vee \\
\text{Conf}_3^x(A) & \xleftarrow{T_{\text{Pic}(\text{Conf}_3(B))}} \text{Conf}_3(A) \\
\text{Conf}_3^x(B) & \xleftarrow{T_{\text{Pic}(\text{Conf}_3(B))}} \text{Conf}_3^x(A)^\vee \\
\text{Conf}_3^x(B) & \xleftarrow{T_{\text{Pic}(\text{Conf}_3(B))}} \text{Conf}_3^x(A)^\vee \\
\end{align*}
\]

$(\text{Conf}_3^x(A)^\vee)^{\text{trop}}(\mathbb{Z})$ gives a $k$-basis for $\text{Conf}_3^x(A)$. (We could get an $R$-basis from $(\text{Conf}_3^x(B)^\vee)^{\text{trop}}(\mathbb{Z})$.) The subset pairing non-negatively with $W^{\text{trop}}$ gives the desired basis for $\text{Conf}_3(A)$.

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\textsuperscript{6}More generally the mirror family should be a quotient of this family by the action of $T_K$, where $K$ is the kernel of the surjection $\text{Pic}(Y) \to \text{Pic}(U)$.
2 Discussion of $\text{Conf}_3(A)$

2.a Representation theory background

Interest in $\text{Conf}_3(A)$ has its roots in representation theory. The starting point is the Peter-Weyl theorem. A group $G$ acts on itself both by left and right multiplication, and this action gives $\mathcal{O}(G)$ the structure of a $G \times G$-bimodule. The following statement of the Peter-Weyl theorem comes from [Pro07].

**Theorem 7.** (Peter-Weyl) Let $G$ be a linearly reductive group. Then as $G \times G$-bimodules

$$\mathcal{O}(G) = \bigoplus_{\lambda} V_{\lambda} \otimes V_{\lambda}^*,$$

where the sum is over isomorphism classes of irreducible representations of $G$.

For $\text{GL}_n$, the span of the highest weight vector $v_{\lambda}$ in the irreducible representation $V_{\lambda}$ of highest weight $\lambda$ is the one dimensional subspace fixed by $U$—the subgroup of upper triangular matrices with 1’s along the diagonal. So

$$\mathcal{O}(G)^{1 \times U} = \bigoplus_{\lambda} V_{\lambda} \otimes \mathbb{C} \cdot u,$$

where $u$ is the highest weight vector for $V_{\lambda}^*$. The weight of $u$ is $-w_0(\lambda)$, where $w_0$ is the longest element of the Weyl group $W$ of $G$.

So the copy of $V_{\lambda}$ appearing in Equation (1) is a weight space for the right action of the maximal torus $H$ in $G$, and its weight is $-w_0(\lambda)$. To stress this point, the left action of $H$ splits $V_{\lambda}$ into weight spaces, the highest weight being $\lambda$, but under the right action $V_{\lambda}$ is the $-w_0(\lambda)$-weight space.

The next thing to observe is that functions on $G$ that are fixed by $U$—so $f(xu) = f(x)$ for all $u \in U$—are the same as functions on $A = G/U$. Then

$$\mathcal{O}(A) = \bigoplus_{\lambda} V_{\lambda},$$

and this is a weight space decomposition for the right action of $H$.

Now if we were to take three copies of $A$ instead of one, we would have

$$\mathcal{O}(A^{\times 3}) = \bigoplus_{\alpha, \beta, \gamma} V_{\alpha} \otimes V_{\beta} \otimes V_{\gamma}.$$

The $G$-fixed subspace of $V_{\alpha} \otimes V_{\beta} \otimes V_{\gamma}$ (with $G$ acting on the left diagonally) is identified with $\text{hom}_G(V_{\gamma}^*, V_{\alpha} \otimes V_{\beta})$. By Schur’s lemma, this is just a copy of the trivial representation for every copy of $V_{\gamma}^*$ appearing in $V_{\alpha} \otimes V_{\beta}$.

Then

$$\mathcal{O}(\text{Conf}_3(A)) = \bigoplus_{\alpha, \beta, \gamma} (V_{\alpha} \otimes V_{\beta} \otimes V_{\gamma})^G,$$

---

The Weyl group for $\text{GL}_n$ is just $S_n$, and $w_0$ is the permutation sending $1, 2, \ldots, n$ to $n, n - 1, \ldots, 1$.
with \((V_\alpha \otimes V_\beta \otimes V_\gamma)^G\) the \((-w_0(\alpha), -w_0(\beta), -w_0(\gamma))\)-weight space of the right \(H^{x3}\) action, and

\[
\dim (V_\alpha \otimes V_\beta \otimes V_\gamma)^G = c_{\alpha\beta\gamma}.
\]

The term on the right is a \textit{Littlewood-Richardson coefficient}. These are the structure constants giving the decomposition

\[
V_\alpha \otimes V_\beta = \bigoplus \gamma V_\gamma^{c_{\alpha\beta\gamma}}.
\]

This is the connection between \(\text{Conf}_3(A)\) and the Littlewood-Richardson coefficients.

### 2.2 Geometric background

As mentioned in Section 1.a, \(\text{Conf}_3(B)\) is Fano and \(\pi : \text{Conf}_3(A) \rightarrow \text{Conf}_3(B)\) is naturally an \(H^{x3}\)-bundle. Points in \(\text{Conf}_3(B)\) are triples of complete flags, defined up to an overall \(G\)-action. Given two arbitrary flags \(X_\bullet = (X_1 \subset \cdots \subset X_n)\) and \(Y_\bullet = (Y_1 \subset \cdots \subset Y_n)\), we expect the \(i\)-dimensional subspace \(X_i\) and the \((n-i)\)-dimensional subspace \(Y_{n-i}\) to intersect transversely. A triple of flags \((X_\bullet, Y_\bullet, Z_\bullet)\) \(\in \text{Conf}_3(B)\) is in \textit{generic configuration} if each pairwise transversality condition is satisfied. \(\text{Conf}_3^x(B) \subset \text{Conf}_3(B)\) is the subset consisting of such triples of flags. It is log Calabi-Yau – its complement is an anticanonical divisor \(D\) in \(\text{Conf}_3(B)\). Furthermore, the canonical volume form on \(\text{Conf}_3^x(B)\) has a pole along all of \(D\). We could in principle use the log Calabi-Yau mirror symmetry machinery to study the pair \(\text{Conf}_3^x(B) \subset \text{Conf}_3(B)\). After all, the vector spaces of interest \((V_\alpha \otimes V_\beta \otimes V_\gamma)^G\) from Subsection 2.a are spaces of sections of line bundles over \(\text{Conf}_3(B)\). However, lifting to \(\text{Conf}_3(A)\) will allow us to tackle all of the line bundles, and so all of the vector spaces \((V_\alpha \otimes V_\beta \otimes V_\gamma)^G\), at once. \(\text{Conf}_3^x(A)\) is precisely \(\pi^{-1}(\text{Conf}_3^x(B))\) and \(\text{Conf}_3(A)\) is again a (partial) minimal model for \(\text{Conf}_3^x(A)\).

### 2.3 Cluster structure

\(\text{Conf}_3^x(A)\) is not just log Calabi-Yau. Fock and Goncharov described a cluster structure for it in \([FG06]\). The discussion here is based on \([FG06]\) and \([GS14]\).

Define \(\widetilde{\text{Conf}}_3(A) := \text{SL}_n \setminus (\text{GL}_n / U)^{x3}\), and define \(\widetilde{\text{Conf}}_3(A), \widetilde{\text{Conf}}_3(B)\), and \(\widetilde{\text{Conf}}_3^x(B)\) analogously. It will also be handy later to define \(\widetilde{W}\) and \(\widetilde{\Xi}\) to be the Landau-Ginzburg potential on \(\text{Conf}_3^x(A)\) and the cone given by its tropicalization. I’ll describe the initial seed of \(\widetilde{\text{Conf}}_3^x(A)\), viewed as a cluster \(A\)-variety, and we’ll view \(\text{Conf}_3^x(A)\) as a quotient of \(\text{Conf}_3^x(A)\).

The quiver for the initial seed comes from the “\(n\)-triangulation” of a triangle, illustrated below for \(n = 4\).
The vertices in the $n$-triangulation will be the vertices of our quiver. For the arrows, we need to orient the edges of the $n$-triangulation. First, the boundary of the original triangle is given a clockwise orientation. The edges of the $n$-triangulation inherit their orientation from this one in the manner illustrated below.

The vertices on the boundary of the original triangle are frozen vertices of the quiver, and the vertices in the interior are unfrozen. We ignore arrows between frozen vertices, so the quiver we are after is

The vertices of the quiver can be indexed by triples of non-negative integers $(a, b, c)$ satisfying $a + b + c = n$. 

Figure 1: 4-triangulation of the triangle.

Figure 2: Oriented 4-triangulation.

Figure 3: Quiver for the initial seed of $\text{Conf}^\times_3(\mathcal{A})$ for $G = \text{GL}_4$. Frozen vertices are blue and unfrozen vertices are orange.
Take $V$ to be an $n$-dimensional vector space. A point $X$ in $\mathcal{A} = \text{GL}(V)/U(V)$ is a complete flag $X_\bullet = (X_1 \subset \cdots \subset X_n)$ of subspaces of $V$ together with non-zero vector $x_1$ in each successive quotient $X_i/X_{i-1}$. I'll denote this by $x_\bullet = (x_1, \ldots, x_n)$. Now choose a volume form $\omega$ on $V$. The cluster variables in the initial seed of $\widetilde{\text{Conf}}_3^\times (\mathcal{A})$ are defined as follows:

$$A_{(a,b,c)} : (X, Y, Z) \mapsto \omega (x_1 \wedge \cdots \wedge x_a \wedge y_1 \wedge \cdots \wedge y_b \wedge z_1 \wedge \cdots \wedge z_c).$$

Note that by definition a linear transformation $T : V \to V$ is in $\text{SL}(V)$ if and only if $\wedge^n T$ acts by the identity on $\wedge^n V$. So $A_{(a,b,c)}$ is indeed a well defined function on $\widetilde{\text{Conf}}_3^\times (\mathcal{A})$—it respects the quotient by the diagonal $\text{SL}(V)$ action. None of these cluster variables are invariant under the diagonal action of $\text{GL}(V)$—they are not functions on $\text{Conf}_3^\times (\mathcal{A})$—but rational functions in these variables can still be $\text{GL}(V)$ invariant. In fact, take a Laurent monomial in these variables:

$$f = \prod_{a+b+c = n} A_{(a,b,c)}^{r_{(a,b,c)}}.$$ 

Then for $g \in \text{GL}(V)$,

$$g \cdot f (X, Y, Z) = (\det g) \sum r_{(a,b,c)} f (X, Y, Z).$$

So $f$ is $\text{GL}(V)$ invariant if and only if

$$\sum_{a+b+c = n} r_{(a,b,c)} = 0.$$

This will lead to a condition on $g$-vectors. See Proposition 19.

Remark 8. The initial data I have described is for $\widetilde{\text{Conf}}_3^\times (\mathcal{A})$ rather than $\text{Conf}_3^\times (\mathcal{A})$. That said, it can easily be translated into initial data for $\text{Conf}_3^\times (\mathcal{A})$. That is, we can view $\text{Conf}_3^\times (\mathcal{A})$ as a cluster variety in its own right, rather than as a quotient of a cluster variety. One way to do this is to replace all cluster variables $A_{(a,b,c)}$ of the initial seed with a new collection of variables, say $\overline{A}_{(a,b,c)} = A_{(a,b,c)}/A_{(n,0,0)}$. Upon doing
so, the proofs I give in the following sections using $\tilde{\text{Conf}}_3^\times (A)$'s cluster structure translate immediately to $\text{Conf}_3^\times (A)$. However, I find it more natural to avoid such choices. In what follows, I will freely use $\text{Conf}_3^\times (A)$'s cluster structure without further comment.
3 Full Fock-Goncharov conjectures holds for Conf$_3^x$ ($\mathcal{A}$)

I will show that the full Fock-Goncharov conjecture holds for Conf$_3^x$ ($\mathcal{A}$) by proving the following two conditions:

1. The quiver $Q_{s_0}$ for the initial seed of Conf$_3^x$ ($\mathcal{A}$) has a maximal green sequence.

2. In the initial seed $s_0 = (e_1, \ldots, e_n)$, all of the covectors $\{e_i, \cdot\}, i \in I_{uf}$, lie in a strictly convex cone.$^9$

Together, (1) and (2) imply that the full Fock-Goncharov conjecture holds for Conf$_3^x$ ($\mathcal{A}$).$^{[GHKK16, Proposition 8.28]}$ We’ll begin with (2) as its proof is much shorter.

**Proposition 9.** In the initial seed $s_0 = (e_1, \ldots, e_n)$, all of the covectors $\{e_i, \cdot\}, i \in I_{uf}$, lie in a strictly convex cone.

**Proof.** This is implied by the existence of a unimodular $p^*$ map. In Section 4.b.3 I construct a particular $p^*$ map and prove its unimodularity in Proposition 22. \hfill \Box

3.a Maximal green sequence

Let’s first review what maximal green sequences are. Recall that the $\mathcal{A}_{prin}$ construction involves a “doubled” quiver, where a new frozen vertex $w_i$ is introduced for each vertex $v_i$ of the original quiver for $\mathcal{A}$, and for each unfrozen vertex $v_i$ we introduce an arrow $v_i \to w_i$. See $^{[GHKK16, Construction 2.11]}$ for a more complete discussion.$^{10}$ This quiver is called the framed quiver $\hat{Q}$ associated to $Q$ in $^{[BDP13]}$. As an example, if we take principal coefficients at $s_0$ for Conf$_3^x$ ($\mathcal{A}$), we would replace the quiver $Q_{s_0}$ of Figure 3 with

![Figure 5: Quiver for Conf$_3^x$ ($\mathcal{A}$) with principal coefficients at $s_0$ for $n = 4$. The new vertices and arrows that have been introduced are in full color while old portions are faded. Only the faded orange vertices are unfrozen.](image)

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$^9$I$_{uf}$ is the unfrozen subset of the indexing set $I$—the subset corresponding to mutable vertices.

$^{10}$The description I am giving here is for the skew-symmetric case, so at first glance it could look different from the more general construction in $^{[GHKK16]}$. 

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Now let \( Q' \) be an arbitrary quiver mutation equivalent to \( \hat{Q} \). An unfrozen vertex \( v_i \) of \( Q' \) is said to be green if all arrows between \( v_i \) and any \( w_j \) are outgoing: \( v_i \to w_j \). On the other hand, \( v_i \) is red if all arrows between \( v_i \) and any \( w_j \) are incoming: \( v_i \leftarrow w_j \). Sign coherence of \( c \)-vectors implies that all unfrozen vertices are either green or red. A sequence of mutations is called a green sequence if each mutation in the sequence is mutation at a green vertex. It is a maximal green sequence if every unfrozen vertex in the resulting quiver is red.

Let \( \triangle_r \) be the top \( r \) rows of unfrozen vertices in \( Q_{s_0} \), and let \( \iota_{\triangle_r} \) be mutation at each of these vertices in order—left to right, top to bottom. For example, for \( n = 6 \), \( \iota_{\triangle_3} \) is the following sequence:

![Diagram](image)

Figure 6: The mutation sequence \( \iota_{\triangle_3} \) for \( n = 6 \). We mutate at the indicated vertices in the indicated order.

Note that \( r \) can be at most \( n - 2 \) (the number of rows of unfrozen vertices).

**Proposition 10.** The sequence \( \iota_{\triangle_{n-2}} \), followed by \( \iota_{\triangle_{n-3}} \), \( \iota_{\triangle_{n-4}} \), \ldots, \( \iota_{\triangle_{1}} \) is a maximal green sequence.

**Remark 11.** This maximal green sequence induces a simple involution on \( \mathcal{O}(\text{Conf}_3(A)) \) that I think is worth mentioning. It sends \( (V_\alpha \otimes V_\beta \otimes V_\gamma)^G \) to \( (V_\gamma^* \otimes V_\alpha^* \otimes V_\beta^*)^G \). See Corollary 30 for details.

Let’s start by looking at \( \iota_{\triangle_r} \). Define \( \triangle'_r := \{ w_i | v_i \in \triangle_r \} \) and let \( F \) be the frozen vertices of \( Q_{s_0} \). We’ll split up the effects of \( \iota_{\triangle_r} \) into three parts—

1. how it affects the full subquiver with vertices \( \triangle_{n-2} \),
2. how it affects the collection of arrows between \( F \) and \( \triangle_{n-2} \), and
3. how it affects the collection of arrows between the \( \triangle'_{n-2} \) and \( \triangle_{n-2} \).

Note that we can split up the analysis this way. Since we never mutate at frozen vertices, arrows between the vertices of \( \triangle_{n-2} \) are unaffected by the presence of the frozen vertices. There is never a composition with the center vertex frozen. Additionally, since we never introduce arrows between frozen vertices, we could in principle treat each frozen vertex separately if we wanted to.

**Lemma 12.** The mutation sequence \( \iota_{\triangle_r} \) sends the subquiver

\[\text{(1)} \quad \text{(2)} \quad \text{(3)} \]
\[ Q_{\triangle_{n-2}} = \]

\[ \cdots \cdots \cdots \]

\[ \cdots \]

\[ \cdots \]

Row \( n-2 \)

To

\[ Q_{\triangle_{r}} = \]

\[ \cdots \cdots \cdots \]

\[ \cdots \]

Row \( r \)

Row \( n-2 \).

So \( Q_{\triangle_{r-1}} \) remains unchanged, \( Q_{\triangle_{r}} \) only has its bottom horizontal arrows deleted, \( Q_{\triangle_{r+1}} \) additionally has its bottom horizontal arrows deleted and its bottom diagonal arrows reversed, and this accounts for all changes to \( Q_{\triangle_{n-2}} \).

Proof. It is immediate that the claim holds for \( r = 1 \)– there is only one mutation to perform. Suppose it holds for all \( q < r \). Then the quiver after performing \( i_{\triangle_{r-1}} \) is \( Q_{i_{\triangle_{r-1}}} \). All that remains is mutation through row \( r \). We start with the leftmost vertex:
This is followed by

and so forth. Mutation at the $k^{th}$ vertex $v_k$ of row $r$, $1 < k < r$, sends

After mutation at $v_{r-1}$, we have the quiver
Finally, mutation at \( v_r \) yields \( Q_{i_{\Delta r}} \).

Now let’s move on to how \( i_{\Delta r} \) affects the collection of arrows between \( F \) and \( \Delta_{n-2} \). This isn’t really necessary to prove Proposition 10, these arrows aren’t considered when determining if an unfrozen vertex is red or green— but it’s worth knowing in any case, and it will provide a nice sanity check later. See Remark 13.

**Lemma 13.** For each \( Q_s' \) mutation equivalent to \( Q_{s_0} \), let \( A_{s'} \) be the subquiver having all vertices of \( Q_{s'} \) but only those arrows for which either the head or tail is in \( F \). Then \( A_{i_{\Delta r}(s_0)} \subset Q_{i_{\Delta r}(s_0)} \) can be constructed from \( A_{s_0} \) as follows:

1. **Rearrange frozen vertices, keeping arrows fixed to their original positions.** (Vertices are being relabeled.)
   
   Send \( v_{(n-1,0)} \) to the \( v_{(n-r-1,0,r+1)} \) position, \( v_{(n-1,0,1)} \) to the \( v_{(n-r-1,r+1,0)} \) position, \( v_{(a,b,0)} \) to the \( v_{(a+1,b-1,0)} \) position for \( 1 < b < r + 1 \), and \( v_{(a,0,c)} \) to the \( v_{(a+1,0,c-1)} \) position for \( 1 < c < r + 1 \).

2. **Reverse arrows involving the vertices now in the \( v_{(n-r-1,r+1,0)} \) and \( v_{(n-r-1,0,r+1)} \) positions.**

I’ll illustrate the claim with an example before proving it. For \( n = 6 \), \( A_{s_0} \) is

and \( A_{i_{\Delta r}(s_0)} \) is
Proof. Mutating \( Q_{s_0} \) at the top unfrozen vertex \( v_{(n-2,1,1)} \) produces

\[
\begin{array}{c}
\vdots \\
\vdots
\end{array}
\]

which we can rearrange as

\[
\begin{array}{c}
\vdots \\
\vdots
\end{array}
\]

So the claim holds for \( A_{i\Delta_1(s_0)} \). Suppose it holds for all \( q < r \). Then, using Lemma 12, \( Q_{i\Delta_{r-1}(s_0)} \) is
It remains to mutate through row $r$. Mutating at the leftmost unfrozen vertex of row $r$ gives

which we rearrange as
The next vertex of mutation sharing an arrow with some frozen vertex is the rightmost vertex of row $r$—the final vertex in our sequence. The penultimate quiver in the sequence is

$$\begin{align*}
v(n-r,0,r) & \quad \cdots \quad v(n-r,r,0) \\
v(n-r-1,0,r+1) & \quad \cdots \quad v(n-1,0,1) \\
v(n-1,1,0) & \quad \cdots \quad v(n-r-1,r+1,0) \\
v(n-r-2,0,r+2) & \quad \cdots \quad v(n-r-2,r+2,0)
\end{align*}$$

Row $r$,

which rearranges to

$$\begin{align*}
v(n-r,0,r) & \quad \cdots \quad v(n-r,0) \\
v(n-r-1,0,r+1) & \quad \cdots \quad v(n-1,0,1) \\
v(n-1,1,0) & \quad \cdots \quad v(n-r-1,r+1,0) \\
v(n-r-2,0,r+2) & \quad \cdots \quad v(n-r-2,r+2,0)
\end{align*}$$

Row $r$,

completing the proof.

Now onto the arrows between $\triangle_{n-2}'$ and $\triangle_{n-2}$. \qed
Lemma 14. For each $Q_{s'}$ mutation equivalent to $\overrightarrow{Q_{s_0}}$, let $R_{s'}$ be the subquiver with vertex set $\triangle_{n-2} \cup \triangle'_{n-2}$ but only those arrows for which either the head or tail is in $\triangle'_{n-2}$. Then $R_{i_{\triangle_{r}(s_0)}}$ can be constructed from $R_{s_0}$ as follows:

(1) Rearrange $\triangle_{n-2}$, keeping arrows fixed to their original positions. (Vertices are being relabeled.) Send $w_{(n-b-1, b, 1)}$ to the $w_{(n-r-1, b, r-b+1)}$ position for $b \leq r$ and $w_{(a, b, n-a-b)}$ to the $w_{(a+1, b, n-a-b-1)}$ position for $b \leq r$.

(2) Reverse the arrow between $v_{(n-r-1, b, r-b+1)}$ and the vertex now in the $w_{(n-r-1, b, r-b+1)}$ position for $b \leq r$.

(3) Introduce a new arrow from the vertex now in the $w_{(a+1, b, n-a-b-1)}$ position to $v_{(n-r-1, b, r-b+1)}$ for $b \leq r$.

(4) If $r < n-2$, introduce a new arrow from $v_{(n-r-2, b, r-b+2)}$ to the vertex now in the $w_{(a,b,n-a-b)}$ position for $b \leq r$.

To illustrate the claim, if we take $n = 7$, then $R_{s_0}$ is

![Diagram](image1.png)

Figure 7: $R_{s_0}$ for $n = 7$. Here the faded orange vertices belong to $\triangle_5$, and the rest to $\triangle'_5$.

and $R_{i_{\triangle_{3}(s_0)}}$ is

![Diagram](image2.png)

Figure 8: $R_{i_{\triangle_{3}(s_0)}}$ for $n = 7$. For visual clarity, arrows from $\triangle_5$ to $\triangle'_5$ are colored cyan and arrows from $\triangle'_5$ to $\triangle_5$ are colored magenta.

Proof. The first mutation gives
which agrees with the statement for \( r = 1 \). Assume it holds for all \( q < r \). Then after mutating through \( i_{\triangle_{r-1}} \) and rearranging the vertices as described, we have

and we just have to mutate through row \( r \). The unfrozen portion of the quiver for each of the remaining mutations is given in the proof of Lemma 12. Note that there is a cyan arrow emanating from \( v_{(n-r-1,b,r-b+1)} \) corresponding to each magenta arrow terminating at \( v_{(n-r-2,b,r-b+2)} \), and there is one additional cyan arrow \( v_{(n-r-1,b,r-b+1)} \rightarrow w_{(n-r-1,b,r-b+1)} \). Now, \( v_{(n-r-1,b,r-b+1)} \) is the \( b \)th vertex of mutation in this row, and each of the magenta arrows are killed by a composition

\[
 v_{(n-r-2,b,r-b+2)} \rightarrow v_{(n-r-1,b,r-b+1)} \rightarrow \bullet
\]

while a cyan arrow \( v_{(n-r-2,b,r-b+2)} \rightarrow w_{(n-r-1,b,r-b+1)} \) is created. Meanwhile, if \( r < n - 2 \), a new cyan arrow is created by the compositions

\[
 v_{(n-r,b,r-b)} \rightarrow v_{(n-r-1,b,r-b+1)} \rightarrow \bullet \quad \Rightarrow \quad v_{(n-r,b,r-b)} \rightarrow \bullet,
\]

and each of the cyan arrows

\[
 v_{(n-r-1,b,r-b+1)} \rightarrow \bullet
\]
is reversed, becoming the magenta arrow

\[ v_{(n-r-1, b, r-b+1)} \rightarrow \]

So after performing \( i_{\Delta_r} \) we obtain the quiver

which we rearrange to

finishing the proof.

We now have all of the ingredients we need to tackle Proposition [10]
Proof. We start with $i_{\vartriangle n-2}$. From the proof of Lemma 14, we see that each time we mutate at a vertex $v_k$ in $i_{\vartriangle n-2}$, all arrows between $v_k$ and $\vartriangle'_{n-2}$ terminate at $v_k$—so $v_k$ is green. Then $i_{\vartriangle n-2}$ is a green sequence. Using Lemmas 12, 13, and 14, performing $i_{\vartriangle n-2}$ on $\hat{Q}_{s_0}$ and rearranging vertices as indicated in the lemmas results in the quiver

The vertices of the bottom unfrozen row are now red, while the remaining unfrozen vertices are all green. For consistency with the $\vartriangle_r$ notation, let’s only consider unfrozen vertices when indexing the rows. So the bottom unfrozen row we’ll call row $n-2$, the one above it row $n-3$, and so forth. This quiver is very similar to the one we started with. Above row $n-2$ the only relevant difference is the introduction of the magenta arrows from $\vartriangle'_{n-3}$ to row $n-2$. Referring to the proof of Lemma 12, we note that no vertex of mutation in the sequence $i_{\vartriangle n-4}$ shares an arrow with row $n-2$. As a result, no composition affecting these arrows can occur until we mutate at row $n-3$. That is, the subsequence $i_{\vartriangle n-4}$ of $i_{\vartriangle n-3}$ proceeds exactly as before, with these magenta arrows tagging along for the ride. Then prior to mutation through row $n-3$, there is a cyan arrow terminating at $v_{(2,b,n-b-2)}$ for all but one of the magenta arrows emanating from $v_{(1,b,n-b-1)}$. These paired magenta arrows are canceled upon mutation at $v_{(2,b,n-b-2)}$. So after performing $i_{\vartriangle n-3}$ and rearranging frozen vertices, we have
Now the unfrozen vertices of rows $n-2$ and $n-3$ are red and the remaining unfrozen vertices are green. We can employ the reasoning just used for $i_{\Delta_{n-3}}$ to the remaining subsequences $i_{\Delta_{n-4}}, i_{\Delta_{n-5}}, \ldots, i_{\Delta_1}$. The resulting quiver is

![Diagram of a quiver with unfrozen vertices of rows $n-2$ and $n-3$ red, and remaining unfrozen vertices green.]

and the sequence $i_{\Delta_{n-2}}$, followed by $i_{\Delta_{n-3}}, i_{\Delta_{n-4}}, \ldots, i_{\Delta_1}$ is a maximal green sequence. 

Remark 15. With the indicated rearranging of frozen vertices, the final quiver we obtained is the same as the original framed quiver with every arrow reversed. Imagine each $u_{(a,b,c)}$ as lying above $v_{(a,b,c)}$. Now ignore temporarily all vertices that aren’t attached to any arrows, and reflect the rest of the final quiver over the plane given by $a = c$. The quiver itself is obviously the same. We’ve just changed its embedding into $\mathbb{R}^3$ and
returned each $w_{(a,b,c)}$ to its original position. Note that this also gives an isomorphism of the full subquiver whose vertex set is all of the $v_{(a,b,c)}$’s with the quiver $Q_{s_0}$. So there is an isomorphism of the final quiver with the coframed quiver\footnote{This differs from the framed quiver in that arrows $w_i \to v_i$ are introduced rather than $v_i \to w_i$.} $\overline{Q}_{s_0}$ fixing the $w_{(a,b,c)}$’s. This is what we expect by \cite[Proposition 2.10]{BDP13}, and it is a sanity check for the work in this section.
4 From $\Conf^x_3(\mathcal{A})$ to $\Conf_3(\mathcal{A})$

4.a Existence of optimized seeds

The main result of this subsection is that every frozen index for $\Conf^x_3(\mathcal{A})$ has an optimized seed. So far we have a basis $\mathbf{B}^x$ for $\mathcal{O}(\Conf^x_3(\mathcal{A}))$. What we really want is a basis $\mathbf{B}$ for $\mathcal{O}(\Conf_3(\mathcal{A}))$. A natural candidate for $\mathbf{B}$ is the subset of $\mathbf{B}^x$ that extends to the divisors we’ve added, i.e. $\mathbf{B}^x \cap \mathcal{O}(\Conf_3(\mathcal{A}))$. But this candidate isn’t automatically a basis for $\mathcal{O}(\Conf_3(\mathcal{A}))$. Maybe $\vartheta_p, \vartheta_q \in \mathbf{B}^x$ both have a pole along some component $D_i$, but these poles cancel in their sum $\vartheta_p + \vartheta_q$. Then we would have $\vartheta_p + \vartheta_q \in \mathcal{O}(\Conf_3(\mathcal{A}))$, but $\vartheta_p, \vartheta_q \not\in \mathcal{O}(\Conf_3(\mathcal{A})) \subset \mathcal{O}(\Conf^x_3(\mathcal{A}))$. The existence of an optimized seed for each frozen index ensures that this does not happen– if a linear combination of $\vartheta$-functions extends to $D_i$, then each $\vartheta$-function in the sum extends as well. [GHKK16, Proposition 9.7] This condition is needed to utilize [GHKK16, Theorem 0.19], which will be used in the coming subsection on the potential $W$ and cone $\Xi$ for $\Conf_3(\mathcal{A})$.

Proposition 16. Every frozen index for $\Conf^x_3(\mathcal{A})$ has an optimized seed.

Proof. For cluster varieties with skew-symmetric exchange matrix, a seed $s$ is optimized for the frozen index $f$ if and only if the vertex $v_f$ is a sink in the quiver $Q_s$. [GHKK16, Lemma 9.2] Consider a quiver $Q_L$ of the form

$$v_f \rightarrow v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_{r-1} \rightarrow v_r.$$ 

The sequence of mutations $v_1, v_2, \ldots, v_r$ yields the quiver

$$Q_{L_f} = v_f \leftarrow v_1 \rightarrow v_2 \rightarrow v_3 \rightarrow \cdots \rightarrow v_{r-1} \rightarrow v_r,$$

making $v_f$ a sink. The initial seed quiver for $\Conf^x_3(\mathcal{A})$ is shown in Figure 3. Call it $Q_{s_0}$. Since there are no arrows to or from the corner vertices $v_{(n,0,0)}$, $v_{(0,n,0)}$, and $v_{(0,0,n)}$, every quiver mutation equivalent to $Q_{s_0}$ will trivially be optimized for these three vertices. Beyond that, $Q_{s_0}$ is optimized for $v_{(n-1,0,0)}$, $v_{(0,n-1,0)}$, and $v_{(1,0,n-1)}$. For the remaining frozen vertices $v_f$, there is a subquiver of $Q_{s_0}$ isomorphic to $Q_L$. Performing these mutations on $Q_{s_0}$ only affects the subquiver whose vertices are either in $Q_L$ or connected to $Q_L$ by an arrow. As arrows between frozen vertices are deleted, any frozen vertices besides $v_f$ can be ignored when determining if $v_f$ becomes a sink. Then the relevant subquiver of $Q_{s_0}$ has the form

Figure 9: Subquiver of $Q_{s_0}$. The frozen vertex $v_f$ in question is blue. All other vertices in this subquiver are unfrozen and have been colored orange. The faded portion is displayed for reference only– it is not part of the subquiver in question.
possibly with the top or bottom row deleted and with the middle row being the subquiver $Q_L$. (Of course, depending upon the position of $v_f$, it may be necessary to rotate Figure 9.) The key observation is that, for every quiver in the sequence, each vertex connected to $v_f$ by an arrow is a vertex of the subquiver $Q_L$. The cycles prevent any new arrows involving $v_f$ from developing via some composition with an arrow not in $Q_L$.

The explicit mutations, ending with $v_f$ as a sink, are shown below.
4.b The potential $W$ and cone $\Xi$ for $\text{Conf}_3(A)$

In this subsection I compute the Landau-Ginzburg potential $W$ and corresponding cone $\Xi := \{W^T \geq 0\} \subset \text{Conf}_3^\times (A)^\vee (\mathbb{R}^T)$. By [GHKK16, Theorem 0.19], the analogous cone for $\mathcal{T}_{\text{prin}}$ gives a canonical basis for the finitely generated algebra $\text{mid}(\mathcal{T}_{\text{prin}}) = \text{up}(\mathcal{T}_{\text{prin}})$. However, the exchange matrix for $\text{Conf}_3^\times (A)$ is full rank over $\mathbb{Z}$. This is immediate from the stronger result Proposition 22. Then, as explained in [GHKK16, Proofs of Corollaries 0.20 and 0.21, page 110], the desired results for $\text{Conf}_3(A)$ are implied by the results for $\mathcal{T}_{\text{prin}}$. I’ll say a few more words about this in Subsection 4.c.

I give an explicit description of $W$ and $\Xi$ in the initial seed and exhibit a map $p^* : N \to M$\footnote{See [GHK15a, Section 2] for a general discussion of $p^*$ maps.} that identifies $W$ with the representation theoretically defined potential $W_{\text{GS}}$ of [GS14] on $\text{Conf}_3^\times (A)$ and identifies $\Xi$ with the Knutson-Tao hive cone.\footnote{Note that $1_\mathcal{H}$ spans a linear subspace of $\Xi$.} Before doing this, let’s recall what $W_{\text{GS}}$ and the Knutson-Tao hive cone are.

4.b.1 Knutson-Tao hive cone

Consider a triangular array of vertices indexed by triples $(a, b, c) \in (\mathbb{Z}_{\geq 0})^3$ with $a + b + c = n$, just like in Figure 3. Let $\mathcal{H}$ be the set of these vertices. $\mathbb{R}^\mathcal{H}$ is the possible labelings of these vertices by real numbers. Now take any pair of neighboring triangles, together forming a rhombus. This rhombus defines a linear inequality in $\mathbb{R}^\mathcal{H}$ by requiring the sum of the labels on the obtuse vertices to be greater than or equal to the sum of the labels on the acute vertices.

![Figure 10: This rhombus gives the inequality $y + z - x - w \geq 0$.](image)

Now denote by $1_\mathcal{H}$ the labeling where each entry is 1. The “Knutson-Tao hive cone” generally refers to one of the following three cones:

1. the polyhedral cone in $\mathbb{R}^\mathcal{H}$ satisfying all rhombus inequalities
2. the slice of (1) having top entry 0
3. the quotient of (1) by $\mathbb{R} \cdot 1_\mathcal{H}$\footnote{\(1_\mathcal{H}\) spans a linear subspace of (1).}

(2) and (3) clearly have completely equivalent combinatorics, with each point in (2) giving a representative
of one of the equivalence classes in \([3]\). The points in the Knutson-Tao hive cone are called **hives**.

The Knutson-Tao hive cone encodes the Littlewood-Richardson coefficients in a really beautiful way. Suppose we want to know \(\dim (V_\alpha \otimes V_\beta \otimes V_\gamma)^G\). The choice of weights \((\alpha, \beta, \gamma)\) determines the border of a hive, which I’ll illustrate in terms of \([3]\). If \(\lambda = (\lambda_1, \ldots, \lambda_n)\), define \(|\lambda| = \lambda_1 + \cdots + \lambda_n\). Now take \(|\alpha| + |\beta| + |\gamma| = 0\)– otherwise \(\dim (V_\alpha \otimes V_\beta \otimes V_\gamma)^G = 0\). Then we label the border of the hive as follows:

\[
\begin{align*}
\gamma_4 &= x_{(4,0,0)} - x_{(3,0,1)} \\
\gamma_3 &= x_{(3,0,1)} - x_{(2,0,2)} \\
\gamma_2 &= x_{(2,0,2)} - x_{(1,0,3)} \\
\gamma_1 &= x_{(1,0,3)} - x_{(0,0,4)}
\end{align*}
\]

Figure 11: Labeling the border of a hive for \(n = 4\), with obvious generalization to arbitrary \(n\).

Note that the condition \(|\alpha| + |\beta| + |\gamma| = 0\) is exactly what we need to be able to fill in the border this way. Also note that, since we are working only up to translations in the \(1_R\) direction, the border is completely determined by the choice of \((\alpha, \beta, \gamma)\). Furthermore, this picture is manifestly symmetric under cyclically permuting \(\alpha\), \(\beta\), and \(\gamma\). Knutson and Tao showed that the number of integral hives with this border is precisely \(\dim (V_\alpha \otimes V_\beta \otimes V_\gamma)^G\). [KT98, Buc98]

### 4.4.2 Goncharov-Shen potential \(\mathcal{W}_{GS}\)

In [GS14], Goncharov and Shen gave a new construction of the Knutson-Tao hive cone, which I describe briefly here. Goncharov and Shen describe points in \(\mathcal{A}\) as pairs \((U, \chi)\), where \(U\) is a maximal unipotent subgroup in \(G\) and \(\chi\) is a non-degenerate additive character on \(U\)– meaning a group homomorphism \(U \to \mathbb{C}_{\chi}\) such that the stabilizer of \((U, \chi)\) under the conjugation action of \(G\) is precisely \(U\). Each triple \(((U_1, \chi_1), (U_2, \chi_2), (U_3, \chi_3)) \in \text{Conf}_3^+ (\mathcal{A})\) has a unique element \(u_{jk} \in U_i\) conjugating \(U_j\) to \(U_k\). This gives a natural function on \(\text{Conf}_3^+ (\mathcal{A})\):

\[
\mathcal{W}_{GS} \left((U_1, \chi_1), (U_2, \chi_2), (U_3, \chi_3)\right) := \chi_1 (u_{21}) + \chi_2 (u_{31}) + \chi_3 (u_{12}).
\]
They then show that in the initial seed of the cluster variety, $W_{GS}^T \geq 0$ gives exactly the rhombus inequalities cutting out the Knutson-Tao hive cone.

4.b.3 $W$, $\Xi$, and $p^*$

The Landau-Ginzburg potential $W$ is the sum of $\vartheta$-functions associated to the irreducible components of $D := \text{Conf}_3(\mathcal{A}) \setminus \text{Conf}_3^\vee(\mathcal{A})$. $D$ is given by

$$
\sum_{i=1}^{n-1} \left( D_{(i,n-i,0)} + D_{(0,i,n-i)} + D_{(n-i,0,i)} \right),
$$

where, for example,

$$
D_{(i,n-i,0)} := \{ (X,Y,Z) \in \text{Conf}_3(\mathcal{A}) \mid X_i \not\in Y_{n-i} \},
$$

which, for example,

$$
D_{(i,n-i,0)} = \{ A_{(i,n-i,0)} = 0 \}.
$$

Suppose the seed $s$ is optimized for the frozen index $(i, n - i, 0)$. Then on the torus $T_{M,s}$ in the atlas for $\text{Conf}_3^\vee(\mathcal{A})^\vee$, $\vartheta_{(i,n-i,0)}$ is given by $z^{-e_{(i,n-i,0)}}$. We can express $\vartheta_{(i,n-i,0)}$ on other tori in the atlas by pulling back $z^{-e_{(i,n-i,0)}}$ via the birational gluing maps. The formula for mutation at $v_k$ is

$$
\mu_k^* (z^n) = z^n \left( 1 + z^{e_k} \right)^{-\{n,e_k\}},
$$

where $n \in \mathbb{N}$ – the lattice of the fixed data used to define the cluster structure. If $s = (e_1, \ldots, e_n)$, then $\mu_k(s) = (e'_1, \ldots, e'_n)$ where

$$
e'_i = \begin{cases} 
e_i + \lfloor e_k \rfloor + e_k & \text{if } i \neq k, \\ -e_k & \text{if } i = k. \end{cases}
$$

I’ll express each of the $\vartheta$-functions, and hence $W$, in the initial seed $s_0$ using this mutation formula.

For each frozen index $f$, we have an explicit sequence of mutations from $s_0$ to a seed $s_f$ optimized for $f$ from the proof of Proposition 16. We want to pullback $\vartheta_f$ from $T_{M,s_f}$ to $T_{M,s_0}$, so we reverse this sequence. All of the mutations occur at vertices of the subquiver $Q_L$, so, by the mutation formula given above, only indices of $Q_L$ will come into play in computing the pullback of $\vartheta_f$.

**Proposition 17.** Recall the quivers $Q_L$ and $Q_L_f$ of Proposition 16. Call the seeds associated to these quivers $s_0$ and $s$. Then the pullback of $z^{-e_f}$ from $T_{M,s}$ to $T_{M,s_0}$ is

$$
z^{-e_f} + z^{-e_f-e_1} + z^{-e_f-e_1-e_2} + \ldots + z^{-e_f-e_1-e_2-\ldots-e_r}.
$$

---

15 The symbol $\not\in$ denotes a non-transverse intersection.

16 The negative sign comes from the sign change identification $i$ of $\text{Conf}_3^\vee(\mathcal{A})^\vee(Z^T)$ and $(\text{Conf}_3^\vee(\mathcal{A})^\vee)^{\text{trop}}(Z)$. 

29
**Proof.** The quiver for the first mutation is

\[
\begin{align*}
v_f & \rightarrow v_1 \rightarrow v_2 \rightarrow v_3 \rightarrow \cdots \rightarrow v_{r-2} \rightarrow v_{r-1} \rightarrow v_r.
\end{align*}
\]

So

\[
\mu_r^* (z^{-e'_f}) = z^{-e'_f} (1 + z^{e_r})^{-1} e_{r,r}
\]

\[
= z^{-e_f - e_r} (1 + z^{e_r}) \{ e_f + [e_r, e_r] \} e_{r,r}
\]

\[
= z^{-e_f - e_r} (1 + z^{e_r}).
\]

The next quiver is

\[
\begin{align*}
v_f & \rightarrow v_1 \rightarrow v_2 \rightarrow v_3 \rightarrow \cdots \rightarrow v_{r-2} \rightarrow v_{r-1} \rightarrow v_r.
\end{align*}
\]

\[
\mu_{r-1}^* \left( z^{-e_f - e_r} \right) = z^{-e_f - e_r} (1 + z^{e_r})^{-1} e_{r,r-1} \left( 1 + z^{e_r} \right) \{ e_f + [e_r, e_{r-1}] \} e_{r,r-1}
\]

\[
= z^{-e_f - e_r - e_{r-1}} (1 + z^{e_r})^0 \left( 1 + z^{e_r} \right) \left( 1 + z^{e_{r-1}} \right)
\]

\[
= z^{-e_f - e_{r-1} - e_r} (1 + z^{e_r} (1 + z^{e_{r-1}})).
\]

This pattern continues with the \( i \)th mutation yielding

\[
\begin{align*}
z^{-e_f - e_{r-i+1} - e_{r-i+2} - \cdots - e_{r}} (1 + z^{e_r} (1 + z^{e_{r-i+1}}) \cdots).
\end{align*}
\]

The result after all \( n \) mutations is

\[
\begin{align*}
z^{-e_f - e_1 - \cdots - e_r} (1 + z^{e_r} (1 + z^{e_{r-1}}) \cdots (1 + z^{e_1}) \cdots)
\end{align*}
\]

\[
= z^{-e_f - e_1 - \cdots - e_r} + z^{-e_f - e_1 - \cdots - e_{r-1}} + \cdots + z^{-e_f},
\]

as claimed. \( \square \)

Using Proposition 17, we can immediately express \( W \) on the torus \( T_{M,s_0} \) of the initial seed of Conf\(^3\) \((A)^\vee\).

**Corollary 18.** Take \( a, b, c \in \mathbb{Z}_{>0} \). The restriction of \( W \) to \( T_{M,s_0} \) is

\[
W = \sum_{a+b=n} \vartheta_{(a,b,0)} + \sum_{b+c=n} \vartheta_{(0,b,c)} + \sum_{a+c=n} \vartheta_{(a,0,c)},
\]

where

\[
\vartheta_{(a,b,0)} = \sum_{i=0}^{n-a-1} z^{-\sum_{j=0}^{i} e_{(a,b-j,j)}},
\]

30
\[ \vartheta_{(0,b,c)} = \sum_{i=0}^{n-b-1} z^{-\sum_{j=0}^{i} \epsilon_{j,b,c-j}}, \]

and

\[ \vartheta_{(a,0,c)} = \sum_{i=0}^{n-c-1} z^{-\sum_{j=0}^{i} \epsilon_{a-j,j,c}}. \]

Note that we now have the basis \( \mathcal{B} \) of \( \mathcal{O}(\text{Conf}_3(\mathcal{A})) \) that we were after—

\[ \Xi(Z^T) = \{ W^T \geq 0 \} \cap \text{Conf}_3^\times(\mathcal{A})^\vee(Z^T). \]

This basis is canonically determined by the pair \( \text{Conf}_3^\times(\mathcal{A}) \subset \text{Conf}_3(\mathcal{A}) \). The subset \( \text{Conf}_3^\times(\mathcal{A}) \) is invariant under the \( H^{\times 3} \) action on \( \text{Conf}_3(\mathcal{A}) \), so \( \mathcal{B} \) must be preserved by this action. Since \( \mathcal{B} \) is a discrete set and \( H^{\times 3} \) acts continuously, the only possibility is that each element of \( \mathcal{B} \) is fixed by \( H^{\times 3} \). The elements of \( \mathcal{B} \) are defined up to scaling, so this means that every element of \( \mathcal{B} \) is an \( H^{\times 3} \)-eigenfunction. The \( H^{\times 3} \) action and the weights of basis elements under this action are discussed further in Subsection 4.c.

At this point we’d like to see if \( W \) to pulls back to the Goncharov-Shen potential \( W_{\text{GS}} \) on \( \text{Conf}_3^\times(\mathcal{A}) \) for some carefully chosen \( p^* \). The guideline for writing down this map will be the representation theoretic interpretation of the cones on both sides. For this, we’ll compare version \( 3 \) of the Knutson-Tao hive cone to \( \Xi \). To have a nice representation theoretic interpretation of \( \Xi \), we need to relate the \( g \)-vector of a \( \vartheta \)-function to its weight under the \( H^{\times 3} \) action.

First let’s fix some notation. If \( S \) is a subset of some real tropical space \( U(\mathbb{R}^T) \), define \( S(Z^T) \) to be its \( Z^T \) points—\( S(Z^T) := S \cap U(Z^T) \). Now let \( \vartheta_p \in \Xi(Z^T) \) and express its \( g \)-vector at \( s_0 \) as

\[ g_{s_0}(\vartheta_p) = \sum_{a+b+c=n} g(a,b,c) e^*_s(a,b,c). \]

Here \( \{ e^*_s(a,b,c) \}_{a+b+c=n} \) is the dual basis to the ordered basis \( s_0 \) of \( N \). Now, \( g_{s_0}(\vartheta_p) \) is the exponent of the leading term of \( \vartheta_p \) expressed as a Laurent polynomial on \( T_{N;s_0} \). Since \( \vartheta_p \) is an eigenfunction of the \( H^{\times 3} \) action on \( \text{Conf}_3(\mathcal{A}) \), the other summands must have the same weight as \( z^{g_{s_0}(\vartheta)} \) under this action.

Represent \( g_{s_0}(\vartheta_p) \) pictorially in the following way, illustrated for \( n = 4 \):

\[ ^{17}\text{The partial ordering on terms comes from the monoid of bending parameters. See } \text{GHKK16, Section 3}. \]
Figure 12: Pictorial representation of $g_{s_0}(\vartheta_p)$ for $n = 4$, with obvious generalization to arbitrary $n$.

Note that

$$z g_{s_0}(\vartheta_p) = \prod_{a+b+c=n} A^{g_{(a,b,c)}}_{(a,b,c)}.$$  

Let $h_1 = \text{diag}(h_{i_1}, \ldots, h_{i_n}) \in H$. Then

$$(h_1, h_2, h_3) \cdot A_{(a,b,c)} = h_{1_{a}} \cdot h_{2_{b}} \cdot h_{3_{c}} A_{(a,b,c)}.$$  

Decompose $\lambda \in \chi^*(H)$ by $\lambda(h) = h_{1}^{\lambda_1} \cdots h_{n}^{\lambda_n}$. Then the following picture lets us read off the $H^{\times 3}$ weight $(\alpha, \beta, \gamma)$ of $z g_{s_0}(\vartheta_p)$, and in turn $\vartheta_p$ (denoted $w(\vartheta_p)$).

Figure 13: If $z g_{s_0}(\vartheta_p)$ has $H^{\times 3}$ weight $(\alpha, \beta, \gamma)$, then all entries on the indicated side of a line sum to the given value. Shown for $n = 4$. The obvious generalization to arbitrary $n$ holds.

There are two immediate consequences. First,
Proposition 19. Let $\vartheta_p \in \Xi (\mathbb{Z}^T)$ and write
\[ g_{s_0} (\vartheta_p) = \sum_{a+b+c=n} g_{(a,b,c)} e^{*}_{(a,b,c)}. \]

Then $\vartheta_p$ is $GL_n$-invariant (and hence in $\Xi (\mathbb{Z}^T)$), if and only if
\[ \sum_{a+b+c=n} g_{(a,b,c)} = 0. \]

Next,

Proposition 20. For each $(\alpha, \beta, \gamma) \in \chi^* (H^{\times 3})$, define $P_{\alpha,\beta,\gamma} \subset \Xi$ to be the subset cut out by the hyperplanes described below:

\[ X \]

\[ Z \]

\[ Y \]

Figure 14: Hyperplanes defining $P_{\alpha,\beta,\gamma}$ for $n = 4$, with obvious generalization to arbitrary $n$. All entries on the indicated side of a line sum to the given value, so each line corresponds to one hyperplane.

Then $P_{\alpha,\beta,\gamma} (\mathbb{Z}^T)$ is a basis for $(V_{\alpha} \otimes V_{\beta} \otimes V_{\gamma})^G$, and $c_{\alpha,\beta}^\gamma = |P_{\alpha,\beta,-w_0(\gamma)} (\mathbb{Z}^T)|$.

We now have a representation theoretic interpretation of $\Xi$. We’ll pictorially represent the inequalities cutting out $\Xi$ in the initial seed as follows, bearing in mind that the sum of all entries must be 0.
Figure 15: Pictorial representation of the inequalities cutting out $\Xi$ in the initial seed for $n = 4$. The boxes indicate that the contained entry must be non-negative. Arrows indicate that the sum of the entries along the line of the arrow, starting with the boxed entry and ending with the entry at the tip of the arrow, must be non-negative. Each inequality corresponds to an exponent appearing in Corollary 18.

Before finding the map $p^* : N \to M$, let’s recall briefly the properties it must satisfy: [GHK15a, p. 14] 18

1. $p^*|_{N_{uf}} : n \mapsto \{n, \cdot\}$ and
2. if $\pi : M \to M/N_{uf}$ is the canonical projection, then $\pi \circ p^* : n \mapsto [\{n, \cdot\} : N_{uf} \to \mathbb{Z}]$.

So we know what properties $p^*$ must satisfy, and we can compare Figures 11 and 14 to further guide our efforts to write down a candidate $p^*$ map. First, take $e_{(a,b,c)} \in N_{uf}$. Using 11 we can immediately write down $p^* (e_{(a,b,c)}) = \{e_{(a,b,c)}, \cdot\}$. For example, if we take $e_{(2,1,1)}$:

$$
\begin{array}{ccccccc}
0 & & & & & & \\
0 & 0 & & & & & \\
0 & 1 & 0 & & & & \\
0 & 0 & 0 & 0 & & & \\
0 & 0 & 0 & 0 & 0 & & \\
& & & & & & \\
\end{array}
$$

Figure 16: $e_{(2,1,1)}$.

then $p^* (e_{(2,1,1)})$ must be

$$
\begin{array}{ccccccc}
0 & & & & & & \\
& & & & & & \\
& & & & & & \\
& & & & & & \\
& & & & & & \\
& & & & & & \\
\end{array}
$$

Figure 17: $p^* (e_{(2,1,1)})$.

18Note that $M = M^\circ$ here— all multipliers $d_i$ are 1.
Note that the sum of the entries of $p^* (e_{(2,1,1)})$ is 0, as are all sums indicated in Figure 14. So things look good so far.

Next, take $e_{(a,b,c)} \in \mathcal{N}_t$. Decompose $p^* (e_{(a,b,c)})$ as $p^* (e_{(a,b,c)})_{uf} + p^* (e_{(a,b,c)})_f$. Then (2) gives us the unfrozen portion $p^* (e_{(a,b,c)})_{uf}$, and comparing Figures 11 and 14 will suggest a candidate for the frozen portion. Take for instance $e_{(3,1,0)}$:

\[
\begin{array}{cccccccc}
& 0 & & & & & & \\
0 & 1 & & & & & & \\
0 & 0 & 0 & & & & & \\
0 & 0 & 0 & 0 & & & & \\
0 & 0 & 0 & 0 & 0 & & & \\
0 & 0 & 0 & 0 & 0 & 0 & & \\
\end{array}
\]

Figure 18: $e_{(3,1,0)}$.

Then (2) gives the unfrozen portion of $p^* (e_{(3,1,0)})$ as

\[
\begin{array}{cccccccc}
\bullet \\
\bullet \\
\bullet \\
\bullet & -1 & \bullet \\
\bullet & 0 & 0 & \bullet \\
\bullet & & & & & & \\
\bullet & & & & & & \\
\bullet & & & & & & \\
\end{array}
\]

Figure 19: $p^* (e_{(3,1,0)})_{uf}$.

Now we use Figure 11 to find that $e_{(3,1,0)}$ has $\alpha = (1, -1, 0, 0)$, $\beta = (0, 0, 0, 0)$, and $\gamma = (0, 0, 0, 0)$. So $p^* (e_{(3,1,0)})$ should have top entry $-\alpha_1 = -1$, entries of the top two rows summing to $-\alpha_2 = 1$, and all other sums from Figure 14 equal to 0. The obvious candidate then is

\[
\begin{array}{cccccccc}
-1 \\
1 & 1 & & & & & & \\
0 & -1 & 0 & & & & & \\
0 & 0 & 0 & 0 & & & & \\
0 & 0 & 0 & 0 & 0 & & & \\
\end{array}
\]

Figure 20: Candidate for $p^* (e_{(3,1,0)})$.

Note again that the sum of the entries is 0.
We can do the same procedure for every $e_{(a,b,c)}$. The resulting map is given by

\[
\begin{pmatrix}
1 & 1 \\
0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \vdots & \vdots \\
\end{pmatrix}
\mapsto
\begin{pmatrix}
-1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \vdots & \vdots \\
1 & -1 & \cdots & \vdots & \vdots \\
0 & -1 & 1 & \cdots & \vdots & \vdots \\
\end{pmatrix}
\]

Figure 21: $p^*$ of a corner.

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & \vdots & \vdots \\
1 & 0 & \cdots & \vdots & \vdots \\
0 & 0 & -1 & 1 & \cdots & \vdots & \vdots \\
\end{pmatrix}
\mapsto
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & \vdots & \vdots \\
1 & -1 & \cdots & \vdots & \vdots \\
0 & -1 & 1 & \cdots & \vdots & \vdots \\
\end{pmatrix}
\]

Figure 22: $p^*$ of an edge entry.

\[
\begin{pmatrix}
0 & 0 & -1 & 1 \\
0 & 1 & 0 & \cdots & \vdots & \vdots \\
0 & 0 & -1 & 1 & \cdots & \vdots & \vdots \\
\end{pmatrix}
\mapsto
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & \vdots & \vdots \\
1 & 0 & -1 & 1 & \cdots & \vdots & \vdots \\
\end{pmatrix}
\]

Figure 23: $p^*$ of an interior entry.

and rotations of these. Every entry not explicitly given is 0.

So this is our proposed map $p^*$. It certainly satisfies (1) and (2). What we want to show now is that $p^*$ gives a unimodular equivalence between version (3) of the Knutson-Tao hive cone and $\Xi$ in the initial seed, and furthermore that $p^*W = W_{GS}$—so the representation theoretic Goncharov-Shen potential has a purely geometric description.

**Proposition 21.**

\[p^*W = W_{GS}\]
Proof. Consider the \( \vartheta \)-function \( \vartheta_{(a,b,0)} = \left( \sum_{i=0}^{n-a-1} z^{-\sum_{j=0}^{i} e(a,b-j,j)} \right) \).

\[
p^\ast \vartheta_{(a,b,0)} = p^\ast \left( \sum_{i=0}^{n-a-1} z^{-\sum_{j=0}^{i} e(a,b-j,j)} \right)
= \sum_{i=0}^{n-a-1} z^{-\sum_{j=0}^{i} p^\ast (e(a,b-j,j))}
= \sum_{i=0}^{n-a-1} z^{-e^\ast(a,b-i,i) + e^\ast(a-1,b-i,i+1) - e^\ast(a,b-i-1,1,1) + e^\ast(a+1,b-i-1,1,1)}
= \sum_{i=0}^{n-a-1} \frac{A((a-1,b-i,i+1)) A((a+1,b-i-1,1,1))}{A(a,b-i,i) A(a,b-i-1,1,i+1)}.
\]

The last line above is expressed in [GS14] as

\[
\sum_{i=0}^{n-a-1} \frac{\Delta_{a-1,b-i,i+1} \Delta_{a+1,b-i-1,1,1}}{\Delta_{a,b-i,i} \Delta_{a,b-i-1,1,i+1}}.
\]

Summing over all \( \vartheta \)-functions in \( W \) yields the potential \( W \) in [GS14, Section 3.1], with each monomial summand corresponding to a different rhombus inequality.

Pictorially, \( p^\ast \) identifies the inequalities defining \( \Xi \) in the initial seed with those defining the Knutson-Tao hive cone in the following way:

Proposition 22. The map \( p^\ast \) is unimodular, so in the initial seed \( \Xi \) is unimodularly equivalent to the Knutson-Tao hive cone.

Remark 23. Keep in mind that upon identifying our tropical spaces with real vector spaces, the domain of \( p^\ast \) will look like \( \mathbb{R}^H / \mathbb{R} \cdot 1_H \) and the codomain will look like the subspace \( V \) of \( \mathbb{R}^H \) in which the sum of all entries is 0. Note that if we view the two copies of \( \mathbb{R}^H \) as dual spaces in the obvious way, then \( 1_H \perp \) is exactly \( V \), and so the domain and codomain of \( p^\ast \) are also dual spaces.

Proof. First note that

\[
\sum_{(a,b,c) \in H} p^\ast (e_{(a,b,c)}) = p^\ast (1_H) = 0.
\]

(2)
Next, I claim that
\[
\text{span}_\mathbb{Z} \left\{ e^*_\left( (n,0,0), p^* (e_{(a,b,c)}) \right) \right\}_{(a,b,c) \in \mathcal{H}} = \mathbb{Z}^\mathcal{H}.
\]

On account of (2), an immediate corollary of this claim would be that
\[
\left\{ e^*_\left( (n,0,0), p^* (e_{(a,b,c)}) \right) \right\}_{\text{All except one } (a,b,c) \in \mathcal{H}}
\]
is a basis for $\mathbb{Z}^\mathcal{H}$, and so
\[
\left\{ p^* (e_{(a,b,c)}) \right\}_{\text{All except one } (a,b,c) \in \mathcal{H}}
\]
would have to be a basis for $\mathbb{Z}^\mathcal{H} \cap V$. Since
\[
\left\{ e_{(a,b,c)} \right\}_{\text{All except one } (a,b,c) \in \mathcal{H}}
\]
is a basis for $\mathbb{R}^\mathcal{H} / \mathbb{R} \cdot 1_{\mathcal{H}}$, this would establish unimodularity of $p^*$. On to the claim.

As seen in the proof of Proposition 21, for each rhombus defining an inequality of the Knutson-Tao hive cone, we get a vector in the image of $p^*$ having 1’s as the entries of the obtuse vertices, −1’s as the entries of the acute vertices, and 0’s elsewhere. In addition, $p^* \left( e_{(n,0,0)} \right) = e^*_\left( (n,0,0), e_{(n-1,0,1)} \right)$, displayed in Figure 21. Adding this to the vector we’ve associated to the top vertical rhombus just shifts its non-zero entries 1 position southeast, giving $e^*_\left( (n-1,1,0), e^*_\left( (n-2,1,1) \right) \right)$.

\[
\begin{array}{ccc}
1 & -1 & 0 \\
-1 & 0 & + & 1 & 1 & = & 0 & 1 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{array}
\]

We can use the other vertical rhombi along the northeast border to shift these entries along the rest of the border, ending with the vector $e^*_\left( (1,n-1,0), e^*_\left( (0,n-1,1) \right) \right)$. Now take the image of the southeast corner: $p^* \left( e_{(0,n,0)} \right) = e^*_\left( (0,n,0), e^*_\left( 1,n-1,0) \right) \right)$. Adding this to our previous result of $e^*_\left( (1,n-1,0), e^*_\left( (0,n-1,1) \right) \right)$ gives $e^*_\left( (0,n,0) - e^*_\left( (0,n-1,1) \right) \right)$.

\[
\begin{array}{ccc}
\vdots & 0 & 0 \\
0 & 1 & + & 0 & -1 & = & \cdots & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 1 & 0 & -1 & 1
\end{array}
\]

We can use the other collection of rhombi along the northeast border to shift the non-zero entries of this
vector northwest along the border, starting with the rhombus containing the southeast corner \((0, n, 0)\).

\[
\begin{pmatrix}
0 & 0 & 0 \\
\vdots & \vdots & \vdots \\
0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 \\
-1 & 1 & = \cdot \cdot \cdot \\
0 & 1 & -1
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0
\end{pmatrix}
\]

So we’ve found two vectors in the image of \(p^*\) to associate to each vertex \(v_{(a,b,1)}\) along the \(c = 1\) line of \(\mathcal{H}\):

\[e^*_{(a+1,b,0)} - e^*_{(a,b,1)}\] oriented diagonally and \(e^*_{(a,b+1,0)} - e^*_{(a,b,1)}\) oriented horizontally. Now introduce \(e^*_{(n,0,0)}\).

Since \(e^*_{(n,0,0)} - e^*_{(n-1,0,1)}\) and \(e^*_{(n,0,0)}\) are in

\[
\Lambda := \text{span}_\mathbb{Z} \left\{ e^*_{(n,0,0)}, p^* \left( e_{(a,b,c)} \right) \right\}_{(a,b,c) \in \mathcal{H}},
\]

so is \(e^*_{(n-1,0,1)}\). Next we use the other vector associated to \((n-1,0,1)\) (the horizontally oriented one \(e^*_{(n-1,1,0)} - e^*_{(n-1,0,1)}\) this time) to see that \(e^*_{(n-1,1,0)}\) is also in \(\Lambda\). We then go one step southeast to the vertex \((n-2,1,1)\) and repeat this process, starting with the diagonally oriented vector and following up with the horizontally oriented vector, to find that \(e^*_{(n-2,1,1)}\) and \(e^*_{(n-2,2,0)}\) are in \(\Lambda\) as well. Continuing southeast gives every \(e^*_{(a,b,c)}\) with \(c = 0\) or 1. Now we’ll push toward the southwest corner using rhombi of the only remaining orientation. For each vertex \((a,b,2)\) along the \(c = 2\) line, there is a single rhombus having this as one of its vertices, two vertices with \(c = 1\), and one vertex with \(c = 0\). Combining the vector associated to this rhombus with \(e^*_{(a,b,c)}\) for \(c = 0\) or 1, we find that \(e^*_{(a,b,2)}\) is also in \(\Lambda\). Repeat this for \(c = 3\), then 4, and so on out to \(n\). So each \(e^*_{(a,b,c)}\) is in \(\Lambda\), and \(\Lambda = \mathbb{Z}^\mathcal{H}\), completing the proof.

\[\square\]

4.c Discussion of \(H^{X^3}\) action and the weight map

Let \(K\) be the kernel of \(p_*^2\) – the composition \(N \xrightarrow{p} M \rightarrow M/N_{uf}^\perp\). The inclusion \(K \subset N\) induces an inclusion of tori \(T_K \subset T_N\), and so an action of \(T_K\) on \(T_N\). Furthermore, it induces a map \(T_M = \text{Spec}(k[N]) \rightarrow T_K = \text{Spec}(k[K])\). Since \(p^*\) commutes with mutation,

1. it defines a map \(p : \text{Conf}_{X^3}^X(A) \rightarrow \text{Conf}_{X^3}^X(A)^{\vee}\),
2. the action of \(T_K\) on \(T_N\) extends to an action on \(\text{Conf}_{X}^X(A) = \bigcup_s T_N; s\), and
3. it gives a map \(\text{Conf}_{X^3}^X(A)^{\vee} = \bigcup_s T_{M; s} \rightarrow T_{K}^*\).

This is discussed in greater detail in [GHK15a, Section 2].

Here I’ll identify the action of \(T_K\) on \(\text{Conf}_{X}^X(A)\) with the \(H^{X^3}\) action, and the tropicalization of \([3]\) with the map \(w\) sending \(\partial_p \in \text{Conf}_{X}^X(A)^{\vee}(\mathbb{Z}^T)\) to its \(H^{X^3}\)-weight \(w(\partial_p)\).

The \(H^{X^3}\) action scales the decorations \((x_\bullet, y_\bullet, z_\bullet)\). We decompose \(h \in H^{X^3}\) as

\[
h = ((h_{x_1}, \ldots, h_{x_n}), (h_{y_1}, \ldots, h_{y_n}), (h_{z_1}, \ldots, h_{z_n})),
\]

39
where, e.g., $h_{x_i}$ is the scale factor for $x_i$. Each component defines a one-parameter subgroup of $T_N$, which we’ll show is in fact contained in $T_K$. For instance, take $n = 5$. Then the scaling coming from $h_{x_3}$ can be represented by

$$h_{x_3}$$

$$h_{x_3} \quad h_{x_3}$$

$$h_{x_3} \quad h_{x_3} \quad h_{x_3}$$

$$1 \quad 1 \quad 1 \quad 1$$

$$1 \quad 1 \quad 1 \quad 1 \quad 1$$

$$1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1$$.

For arbitrary $n$, $h_{x_i}$ corresponds to the cocharacter

$$n_{x_i} := \sum_{(a,b,c) \in H, a \geq i} e_{(a,b,c)},$$

$h_{y_i}$ to

$$n_{y_i} := \sum_{(a,b,c) \in H, b \geq i} e_{(a,b,c)},$$

and $h_{z_i}$ to

$$n_{z_i} := \sum_{(a,b,c) \in H, c \geq i} e_{(a,b,c)}.$$

**Proposition 24.**

$$K = \text{span}_\mathbb{Z}\{n_{x_i}, n_{y_i}, n_{z_i}\}_{1 \leq i \leq n}$$

**Proof.** We’ll show that

$$\text{span}_\mathbb{Z}\{p^*(n_{x_i}), p^*(n_{y_i}), p^*(n_{z_i})\}_{1 \leq i \leq n} = N_{\text{aff}}^\perp.$$

This implies containment in $K$, and unimodularity of $p^*$ boosts this containment to an equality.
We simply compute.

\[
p^* \left( n_{x_i} \right) = p^* \left( \sum_{(a,b,c) \in H, a \geq 1} e_{(a,b,c)} \right) = p^* \left( e_{(n,0,0)} \right) + \sum_{a=i}^{n-1} p^* \left( e_{(a,0,n-a)} \right) + \sum_{b=1}^{n-a} p^* \left( \sum_{a=i}^{n-a} e_{(a,b,n-a-b)} \right)
\]

\[
= e_{(n,0,0)} - e_{(n-1,0,1)} + \sum_{a=i}^{n-1} e_{(a,0,n-a)} - e_{(a,1,n-a-1)} - e_{(a-1,0,n-a-1)} + e_{(a-1,1,n-a)}
\]

Similarly,

\[
p^* \left( n_{y_i} \right) = e_{(n-i,i,0)} - e_{(n-i+1,i-1,0)}
\]

and

\[
p^* \left( n_{z_i} \right) = e_{(0,n-i,i)} - e_{(0,n-i+1,i-1)}.
\]

Then clearly

\[
\text{span}_\mathbb{Z} \left\{ p^* \left( n_{x_i} \right), p^* \left( n_{y_i} \right), p^* \left( n_{z_i} \right) \right\}_{1 \leq i \leq n} = N_{\text{aff}}^+, (\text{recall that } N = \mathbb{Z}^H / 1_H \cdot \mathbb{Z}) \text{ and so }
\]

\[
K = \text{span}_\mathbb{Z} \left\{ n_{x_i}, n_{y_i}, n_{z_i} \right\}_{1 \leq i \leq n}.
\]

To see that (3) tropicalizes to the weight map, first restrict to tori for a fixed seed $s$. Then $\vartheta_p$ is a finite sum of characters on $T_{N,s}$, and since $\vartheta_p$ is an $H^{\times 3}$ eigenfunction, each of these characters has the same $H^{\times 3} = T_K$ weight. Let one of the characters be $z^m$. Then the weight of $\vartheta_p$ under the $T_K$ action is the map $z^k \mapsto z^{(k,m)}$ for $z^k \in T_K$. In other words, the $T_K$ weight of $z^m$ is $(m \mod K^\perp) \in K^*$. (Note that the bending parameters for broken lines are in $K^\perp$, so all of the summands of $\vartheta_p$ do indeed have the same weight. See [GHKK16, Section 3] for a discussion of broken lines.) The map $m \mapsto m \mod K^\perp$ dualizes the inclusion $K \hookrightarrow N$, so for each seed $s$ the weight of $\vartheta_p$ is the tropicalization of (3). Since this holds when we restrict to every torus, it holds for all of $\text{Conf}_3^x \left( \mathcal{A} \right)^\vee$. 

\[
\square
\]
As alluded to previously, there is a related action on $A_{\text{prin}}$. Let $\tilde{K}$ be the kernel of

$$ N \oplus M \to M/N_{\text{aff}}^\perp $$

$$(n, m) \mapsto p_2^*(n) - m. $$

The surjection $\pi : A_{\text{prin}} \to T_M$ is $T_{\tilde{K}}$-equivariant. The fact that the exchange matrix is full rank implies that $\pi$ is isomorphic to the trivial bundle $\text{Conf}_3^\times (A) \times T_M$. This is used to translate basis results for $A_{\text{prin}}$ to $\text{Conf}_3^\times (A)$.\[GHKK16, Lemma B.7\] This is used in GHKK16, Proof of Corollaries 0.20 and 0.21, page 110]

4.d Ray representation of $\Xi$

In Mag15, I showed that for the base affine space, the cone $\Xi_A$ is generated by the $g$-vectors for Plücker coordinates. This is probably the most natural generating set for the homogeneous coordinate ring of the flag variety, and I wonder if the generators of $\Xi$ can fill this role for $\text{Conf}_3^\times (B)$. This is necessarily vague and subjective. I am primarily asking if the $\vartheta$-functions corresponding to generators of $\Xi$ in the initial seed have a simple explicit description. In this subsection, I describe the rays generating $\Xi$ and give partial results relating these rays to functions. Since $\Xi$ is not strictly convex for $G = \text{GL}_n$, we'll temporarily restrict to $\text{SL}_n$.

**Proposition 25.** The $g$-vectors of all frozen variables generate edges of $\Xi$.

**Proof.** These $g$-vectors have a single non-zero entry. The frozen variable $A_{(a,b,c)}$ has a 1 in the $(a, b, c)$ position, which is on a boundary edge of the triangle in Figure 1. For example, in $\text{SL}_4$ the entries of the $g$-vector for $A_{(2,0,2)}$ are

$$
\begin{bmatrix}
0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & .
\end{bmatrix}
$$

Figure 24: Pictorial representation of $A_{(2,0,2)}$’s $g$-vector.

We can include the boxes and arrows representing $\Xi$’s defining inequalities:
Figure 25: Pictorial representation of $A_{(2,0,2)}$'s $g$-vector and $\Xi$'s inequalities. Boxes and arrows representing strict inequalities are red.

The line spanned by such a $g$-vector is the intersection of the hyperplanes defined by the boxes for the remaining frozen variables and the arrows parallel to the boundary edge containing $v_{(a,b,c)}$. So for $A_{(2,0,2)}$, we get the line described by the following picture:

The ray $R_{\geq 0} \cdot g \left( A_{(a,b,c)} \right)$ contained in this line satisfies the remaining inequalities and is an edge of $\Xi$.  

**Proposition 26.** The $g$-vectors of all initial seed variables generate edges of $\Xi$.

**Proof.** Proposition 25 took care of the frozen variables. For the unfrozen variable $A_{(a,b,c)}$, there is a 1 in the interior of our triangle with every other entry 0. For starters, take the hyperplanes defined by each box. Then fix a box and add as many consecutive arrows in the string emanating from it as possible without hitting $v_{(a,b,c)}$. Doing this for all boxes gives enough hyperplanes to determine the line given by a free parameter in position $v_{(a,b,c)}$ and 0 elsewhere. It really gives more hyperplanes than needed, but that’s not a problem. For example, for $A_{(2,1,1)}$ we would have the following picture:

The ray $R_{\geq 0} \cdot g \left( A_{(a,b,c)} \right)$ contained in this line satisfies the remaining inequalities and is an edge of $\Xi$.  

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The functions associated to the remaining edges take more effort to describe. We’ll figure out what these edges actually are before worrying what functions they might correspond to. Consider for a moment the inequalities represented by arrows. These are described in Figure 15. Staring at this for a little while suggests the following picture for some of the remaining rays of \( \Xi \). Take an interior vertex \( v(a,b,c) \). There are three subquivers \( Q_L \) (from Proposition 16) starting at a frozen vertex and ending at \( v(a,b,c) \). Draw a line segment through the vertices of each of these subquivers. Here’s an example:

![Figure 26: Here we’ve chosen the interior vertex \((2,1,1)\). The relevant subquivers \(Q_L\) are black instead of gray. The line segments of interest are dotted orange.](image)

Now put a “−1” in the \((a, b, c)\) position, a single “1” along each of the three segments, and “0” elsewhere. The idea is that each of the three arrows pointing to \( v(a,b,c) \) indicate that the sum of the entries along one of the three segments should be non-negative. If we make any entry negative, we’ll get something outside of the cone generated by vectors in Propositions 25 and 26. The construction described is the simplest way to achieve this without violating any inequalities.

![Figure 27: This vector generates a ray of \( \Xi \). Strict inequalities are blue. See Proposition 27 for the proof.](image)

Let’s call such a picture a trivalent vertex.

**Proposition 27.** The vectors associated to trivalent vertices generate edges of \( \Xi \).

**Proof.** Consider all of the hyperplanes defining faces of \( \Xi \). It is easiest to say which to exclude from our intersection. Essentially, we want to intersect all of the hyperplanes associated to inequalities that should reduce to equalities for the vector in question, and only these hyperplanes. So in Figure 27, we would remove exactly the blue boxes and arrows corresponding to strict inequalities, and the intersection of the remaining hyperplanes is a line containing the given vector. To do this for an arbitrary vector associated to a trivalent vertex, we start by going to the position of the entry 1 along each line segment. If this isn’t on the boundary,
then the box at the end of the line segment and all arrows leading to this entry apart from the last one give equalities. But the arrow whose tip hits this entry gives a strict inequality, as do the arrows coming after it, until we get to the arrow whose tip hits the $-1$ entry. So given the segment

$$-1\overrightarrow{-1}1\overrightarrow{-1},$$

Figure 28: Example line segment.

we would not include the following hyperplanes in the intersection:

$$-1\overrightarrow{-1}1\overrightarrow{-1}.$$

Figure 29: The blue arrows give strict inequalities, so we exclude the corresponding hyperplanes from the intersection that will yield the span of our vector.

Do this for all three line segments and then intersect all remaining hyperplanes. The result is the real span of the given vector, and its $\mathbb{R}_{\geq 0}$ span also satisfies the inequalities that have been omitted from the intersection.

The next thing to notice is that we can overlay two trivalent vertices, and as long as none of the line segments are colinear we’ll have two line segments intersecting at a vertex. If we make the entry of this vertex 1, we’ll get a vector outside the span of the vectors previously described. For example, take

Figure 30: Two trivalent vertices overlaid.

which gives the vectors
Consider either of these two vectors. If it were the sum of vector $s$ described previously, we’d have to take at least two vectors associated to trivalent vertices to account for the two minus signs. Then the sum of all entries must be at least 4, but it is in fact only 3. So it is indeed outside of the span of the vectors described previously, and it clearly lies in our cone. The proof that it generates an edge of $\Xi$ is basically identical to the proof of Proposition 27.

Next, there is no reason to limit ourselves to only overlaying two trivalent vertices. We can overlay as many as we want. Say we overlay $k$ of them. Then we are describing vectors for which $k$ entries are $-1$. As long as we place our 1’s in such a way that our vector cannot be a positive combination the vectors associated to $k - 1$ or fewer trivalent vertices, we will get an edge of $\Xi$ by the argument used above. In particular, if we ensure that the sum of the entries is less than could be achieved with such positive combinations, our vector must be an edge of $\Xi$.

**Proposition 28.** Every edge of $\Xi$ is generated by an initial seed $g$-vector, a trivalent vertex, or overlaid trivalent vertices.

**Proof.** The initial seed $g$-vectors already generate the entire positive orthant, so any additional edge of $\Xi$ must have some negative entry. Negative entries must be at interior vertices on account of the box inequalities. Say the entry at $(a, b, c)$ is negative, with value $-x$. Then the three incoming arrows at $v_{(a,b,c)}$ indicate that the sum of the remaining entries along each of the three line segments leading to $v_{(a,b,c)}$ must be at least $x$. 

---

**Figure 31:** A vector associated to the overlaid trivalent vertices of Figure 30. Strict inequalities are blue.

**Figure 32:** A vector associated to the overlaid trivalent vertices of Figure 30. Strict inequalities are blue.
However, if the sum is more than $x$, we would be able to realize this vector as a sum of vectors from trivalent vertices and vectors in the positive orthant. You can see this by restricting to a line segment first. It’s easy to see for this restriction, and the result transfers over directly as the trivalent vertices are made up of three line segments, and the position of a 1 along one line segment is completely independent of the other two line segments. It follows that if some edge of $\Xi$ lies outside of the positive orthant, it must be generated by a trivalent vertex or a collection of overlaid trivalent vertices. So this is indeed all of the edges of $\Xi$.

**Proposition 29.** Consider the trivalent vertex $p$ having 1’s in positions $(a_1, b_1, c_1)$, $(a_2, b_2, c_2)$, and $(a_3, b_3, c_3)$, labeled such that the top left 1 is in position $(a_1, b_1, c_1)$, the rightmost 1 is at $(a_2, b_2, c_2)$, and the bottom left 1 is $(a_3, b_3, c_3)$. Now take the triangle $\triangle$ with sides $b = b_1 + 1$, $c = c_2 + 1$, and $a = a_3 + 1$. Then $\vartheta_p$ is obtained by performing the maximal green sequence of Proposition 10 on the subquiver with vertices $\triangle$.

Let’s illustrate the claim first. Suppose we take

Then $\vartheta_p$ is produced by the mutation sequence
Proof. Each vertex of mutation in this sequence is green, so when mutating at \( v_k \) the terms coming from arrows emanating from \( v_k \) vanish on the central fiber of \( A_{\text{prin},s_0} \). Only those arrows terminating at \( v_k \) contribute to the \( g \)-vector. Let \( A_{(a,b,c)} \) be the cluster variable obtained after the \( k \)th mutation at \( v_{(a,b,c)} \) in the sequence. Then, using the quivers from the proof of Proposition 10, \( g_{s_0} (A_{(a,b,c)}_k) \) is given by

\[
g_{s_0} (A_{(n-b_1-c_2-k,b_1,c_2+k)}) + g_{s_0} (A_{(a-k+1,b+k,c_2)}) + g_{s_0} (A_{(a-1,b,c+1)_{k-1}}) - g_{s_0} (A_{(a,b,c)_{k-1}})
\]

if \( c = c_2 + 1 \), and

\[
g_{s_0} (A_{(a+1,b,c-1)_{k}}) + g_{s_0} (A_{(a-1,b,c+1)_{k-1}}) - g_{s_0} (A_{(a,b,c)_{k-1}})
\]

otherwise. I claim that the for each \( v_{(a,b,c)} \in \Delta \),

\[
g_{s_0} (A_{(a,b,c)}_k) = e^*_t (n-b_1-c_2-k,b_1,c_2+k) + e^*_t (n-b-c_2-k,b+k,c_2) + e^*_t (a-k,b,c+k) - e^*_t (n-b-c_2-k,b,c_2+k).
\]

For \( k = 1 \), we have \( g_{s_0} (A_{(a,b,c)}_1) \) is given by

\[
e^*_t (n-b_1-c_2-1,b_1,c_2+1) + e^*_t (a,b+1,c_2) + e^*_t (a-1,b,c+1) - e^*_t (a,b,c)
\]

if \( c = c_2 + 1 \), in agreement with the claim, and

\[
g_{s_0} (A_{(a+1,b,c-1)_1}) + e^*_t (a-1,b,c+1) - e^*_t (a,b,c)
\]

otherwise. When \( c - 1 = c_2 + 1 \), we have

\[
g_{s_0} (A_{(a+1,b,c+2)_1}) = e^*_t (n-b_1-c_2-1,b_1,c_2+1) + e^*_t (a+1,b+1,c_2) + e^*_t (a,b,c) - e^*_t (a+1,b,c_2+1);
\]

so

\[
g_{s_0} (A_{(a,b,c+2)_1}) = e^*_t (n-b_1-c_2-1,b_1,c_2+1) + e^*_t (a+1,b+1,c_2) + e^*_t (a-1,b,c_2+3) - e^*_t (a+1,b,c_2+1);
\]
which again agrees with the claim. Assume the claim holds for all $c' < c$. Then
\[
g_{s_0}(A_{(a,b,c)}_1) = \left(e^{*}_{(n-b_1-c_2-1,b_1,c_2+1)} + e^{*}_{(n-b-c_2-1,b+1,c_2)} + e^{*}_{(a,b,c)} - e^{*}_{(n-b-c_2-1,b,c_2+1)}\right)
\[
+ e^{*}_{(a-1,b,c+1)} - e^{*}_{(a,b,c)}
\[
= e^{*}_{(n-b_1-c_2-1,b_1,c_2+1)} + e^{*}_{(n-b-c_2-1,b+1,c_2)} + e^{*}_{(a-1,b,c+1)} - e^{*}_{(n-b-c_2-1,b,c_2+1)},
\]

which proves the claim for $k = 1$. Now suppose it holds for all $k' < k$. Then $g_{s_0}(A_{(a,b,c+1)}_k)$ is given by
\[
e^{*}_{(n-b_1-c_2-k,b_1,c_2+k)} + e^{*}_{(a-k+1,b+k,c_2)}
\[
+ \left(e^{*}_{(n-b_1-c_2-k+1,b_1,c_2+k-k)} + e^{*}_{(n-b-c_2-k+1,b+1,k+1,c_2)} + e^{*}_{(a-k,b,c_2+k+1)} - e^{*}_{(n-b-c_2-k+1,b,c_2+k-k)}\right)
\[- \left(e^{*}_{(n-b_1-c_2-k+1,b_1,c_2+k-k)} + e^{*}_{(n-b-c_2-k+1,b+1,k+1,c_2)} + e^{*}_{(a-k+1,b,c_2+k+1)} - e^{*}_{(n-b-c_2-k+1,b,c_2+k-k)}\right)
\[
= e^{*}_{(n-b_1-c_2-k,b_1,c_2+k)} + e^{*}_{(a-k+1,b+k,c_2)} + e^{*}_{(a-k,b,c_2+1+k)} - e^{*}_{(a-k+1,b,c_2+k)},
\]

in agreement with the claim, and $g_{s_0}(A_{(a,b,c)}_k), c \neq c_2 + 1$, is given by
\[
g_{s_0}(A_{(a+1,b,c-1)}_k)
\[
+ \left(e^{*}_{(n-b_1-c_2-k-1,b_1,c_2+k-k)} + e^{*}_{(n-b-c_2-k+1,b+1,k-1,c_2)} + e^{*}_{(a-k,b,c_2+k+1)} - e^{*}_{(n-b-c_2-k+1,b,c_2+k-k)}\right)
\[- \left(e^{*}_{(n-b_1-c_2-k-1,b_1,c_2+k-k)} + e^{*}_{(n-b-c_2-k+1,b+1,k-1,c_2)} + e^{*}_{(a-k+1,b,c_2+k+1)} - e^{*}_{(n-b-c_2-k+1,b,c_2+k-k)}\right)
\[
= g_{s_0}(A_{(a+1,b,c-1)}_k) + e^{*}_{(a-k,b,c_2+k)} - e^{*}_{(a-k+1,b,c_2+k-1)}.
\]

As before, when $c = c_2 + 1$, we have
\[
g_{s_0}(A_{(a+1,b,c+1)}_k) = e^{*}_{(n-b_1-c_2-k,b_1,c_2+k)} + e^{*}_{(a-k+2,b,k,c_2)} + e^{*}_{(a+1-k,b,c_2+1+k)} - e^{*}_{(a-k+2,b,c_2+k)},
\]

which agrees with the claim. So suppose it holds for all $c' < c$. Then
\[
g_{s_0}(A_{(a,b,c)}_k) = g_{s_0}(A_{(a+1,b,c-1)}_k) + g_{s_0}(A_{(a-1,b,c+1)}_{k-1}) - g_{s_0}(A_{(a,b,c)}_{k-1})
\[
= \left(e^{*}_{(n-b_1-c_2-k,b_1,c_2+k)} + e^{*}_{(n-b-c_2-k,b+k,c_2)} + e^{*}_{(a-k+1,b,c_2+k-1)} - e^{*}_{(n-b-c_2-k,b,c_2+k)}\right)
\[
+ \left(e^{*}_{(n-b_1-c_2-k+1,b_1,c_2+k-k)} + e^{*}_{(n-b-c_2-k+1,b+1,k+1,c_2)} + e^{*}_{(a-k,b,c_2+k+1)} - e^{*}_{(n-b-c_2-k+1,b,c_2+k-k)}\right)
\[- \left(e^{*}_{(n-b_1-c_2-k+1,b_1,c_2+k-k)} + e^{*}_{(n-b-c_2-k+1,b+1,k+1,c_2)} + e^{*}_{(a-k+1,b,c_2+k+1)} - e^{*}_{(n-b-c_2-k+1,b,c_2+k-k)}\right)
\[
= e^{*}_{(n-b_1-c_2-k,b_1,c_2+k)} + e^{*}_{(n-b-c_2-k,b+k,c_2)} + e^{*}_{(a-k,b,c_2+k)} - e^{*}_{(n-b-c_2-k,b,c_2+k)}.
\]

This proves the claim.

Now take $k = c_1 - c_2$, and take $(a,b,c) = (a_3 - c_1 - c_2, b_3, c_2 + a_2 - a_3)$. Then
\[
g_{s_0}(A_{(a,b,c)}_k) = e^{*}_{(a_1,b_1,c_1)} + e^{*}_{(n-b_3-c_1,b_3+c_1-c_2,c_2)} + e^{*}_{(a_3,b_3,a_2-a_3+c_1)} - e^{*}_{(n-b_3-c_1,b_3,c_1)}
\[
= e^{*}_{(a_1,b_1,c_1)} + e^{*}_{(a_2,b_2,c_2)} + e^{*}_{(a_3,b_3,c_3)} - e^{*}_{(a_2,b_3,c_1)},
\]

proving the proposition.
Taking $\triangle = \triangle_{n-2}$ leads to a nice observation about the maximal green sequence of Proposition 10. Call the final seed of the sequence $s$.

**Corollary 30.** Let $\mu_{\text{MGS}}$ be the automorphism of $\text{Conf}_3(A)$ induced by the maximal green sequence of Proposition 10 along with the indicated permutation of frozen vertices. Then the assignment

$$A_{(a,b,c),s_0} \mapsto \mu_{\text{MGS}}^* (A_{(a,b,c),s_0})$$

induces an automorphism of $\mathcal{O} (\text{Conf}_3(A))$ sending $(V_\alpha \otimes V_\beta \otimes V_\gamma)^G$ to $\left( V_\alpha^* \otimes V_\gamma^* \otimes V_\beta^* \right)^G$.

**Proof.** Since the only difference between $Q_s$ and $Q_{s_0}$ is the overall orientation, the variables of $s$ and $s_0$ have the same relations– given the relation

$$r \left( A_{(a_1,b_1,c_1),s_0}, \ldots, A_{(a_i,b_i,c_i),s_0} \right) = 1,$$

the relation

$$r \left( \mu_{\text{MGS}}^* (A_{(a_1,b_1,c_1)}), \ldots, \mu_{\text{MGS}}^* (A_{(a_i,b_i,c_i)}) \right) = 1$$

must hold as well. By Theorem 1, every $\vartheta_p \in \Xi (Z^T)$ is a Laurent polynomial in either of these two collections of variables. Since $\Xi (Z^T)$ generates $\mathcal{O} (\text{Conf}_3(A))$ and the relations between the two (identical but reordered) generating sets $\Xi (Z^T)$ match, this assignment gives an automorphism of $\mathcal{O} (\text{Conf}_3(A))$.

Now the claim is that if some $\vartheta_p$ has $H^{x_3}$-weight $(\alpha, \beta, \gamma)$, then its image $\vartheta_p'$ under this automorphism has $H^{x_3}$-weight $(-w_0(\alpha), -w_0(\gamma), -w_0(\beta))$. It’s sufficient to show that this holds for the cluster variables of $s_0$. The weight of $A_{(a,b,c),s_0}$ is

$$((h_{x_1}, \ldots, h_{x_{n-1}}), (h_{y_1}, \ldots, h_{y_{n-1}}), (h_{z_1}, \ldots, h_{z_{n-1}})) \mapsto h_{x_1} \cdots h_{x_a} h_{y_1} \cdots h_{y_b} h_{z_1} \cdots h_{z_c}.$$

For the weight of $\mu_{\text{MGS}}^* (A_{(a,b,c),s_0})$, take $k = a$ and $b_1 = c_2 = a_3 = 0$ in the proof of Proposition 29. This yields

$$g_{s_0} \left( A_{(a,b,c)} \right) = e_{(n-a,0,a)}^* + e_{(n-b-a,0,b)}^* + e_{(0,b,c+a)}^* - e_{(n-b-a,b,a)}^*$$

$$= e_{(n-a,0,a)}^* + e_{(c,n-c,0)}^* + e_{(0,b,n-b)}^* - e_{(c,b,a)}^*,$$

so the weight is

$$((h_{x_1}, \ldots, h_{x_{n-1}}), (h_{y_1}, \ldots, h_{y_{n-1}}), (h_{z_1}, \ldots, h_{z_{n-1}})) \mapsto h_{x_1} \cdots h_{x_a} h_{y_1} \cdots h_{y_b} h_{z_1} \cdots h_{z_c}.$$

This proves the claim.

It isn’t clear yet whether the rays of $\Xi$ correspond to a “simple” collection of functions, but I hope that observations in this subsection provide a foundation for addressing this question.

\[19\text{Recall that } G = \text{SL}_n \text{ here.}\]
5 Recovering $\mathcal{A}$ and $U$ from $\text{Conf}_3(\mathcal{A})$

The base affine space $\mathcal{A}$ is a partial minimal model for the double Bruhat cell $G^{e,w_0}$. Fix $B_+, B_- \subset G$ to be the subgroups of upper and lower triangular matrices, and take $V^+_\bullet$ and $V^-\bullet$ to be their fixed flags. Then $G^{e,w_0} \subset \mathcal{A}$ is the subset whose underlying flags $F_\bullet$ intersect both $V^+_\bullet$ and $V^-\bullet$ generically. That is, $F = (F_\bullet, f_\bullet)$ is in $G^{e,w_0}$ if and only if each subspace $F_i$ intersects both $V^+_i$ and $V^-_i$ transversely.

This description, while satisfyingly simple, involves a choice—fixing the pair $(B_+, B_-)$. It would be philosophically more appealing to have a description that avoids such choices. So, instead of choosing a single pair, we’ll choose all pairs at once, and later we’ll mod out to identify all of these choices. First notice that we could recover the usual cluster structure for this space by defining a new collection of variables, say $A_{(a,b,c)} := A_{(a,b,c)}/A_{(0,n−c,c)}$. But the point here is to avoid making choices, so we won’t do that.

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20 As in Remark 8 using this condition we could recover the usual cluster structure for this space by defining a new collection of variables, say $\tilde{A}_{(a,b,c)} := A_{(a,b,c)}/A_{(0,n−c,c)}$. But the point here is to avoid making choices, so we won’t do that.
as a sum of rational $GL_n$ representations, which include duals of polynomial representations. Essentially, $\det$ is an invertible function on $GL_n$ and this gives $\Xi_A$ a 1-dimensional linear subspace. That said, the cone defined by Gelfand and Tsetlin in [GT50, Equation 3] encodes rational representations and is the cone identified with $\Xi_A$ by $p^*$.

Using [GS14, Figure 32], the inequalities defining $\Xi_A$ and the Gelfand-Tsetlin cone are identified via $p^*$ as follows:

Figure 33: Correspondence between inequalities defining $\Xi_A$ in the initial seed and the Gelfand-Tsetlin cone for $n = 4$. For the Gelfand-Tsetlin cone, an arrow indicates that the entry at the tail is at least as large as the entry at the tip.

The agreement of $p^* W$ with the potential $f$ of Berenstein-Kazhdan [BK06] is given in the appendix of [GS14].

To describe $U$ in this way, we view $B$ as the subset of $\text{Conf} (A, B, B)$ with $(A_1, B_3)$ generic, $U$ as the subset where $(A_1, B_2)$ is also generic, and the cluster variety $\hat{U}$ in $U$ is $\text{Conf}^\times (A, B, B)$. Usually $\hat{U}$ is described as the subset of $U$ (upper triangular unipotent matrices) where the minors $\Delta_{n-i+1,...,n}$ are non-vanishing. Here we are getting to $U$ from $B$ by requiring $(A_1, B_2)$ to be generic, and we get to $\hat{U}$ from $U$ by requiring $(B_2, B_3)$ to be generic.

A basis for $\mathcal{O} \left( \hat{U} \right)$ is given by taking the slice of $\text{Conf}^\times (A) (Z^T)$ with $H_y \times H_z$ weight $(\beta, \gamma) = 0$. When we partially compactify to $U$, the divisors that we add are $D_{(0, i, n-i)}$. The corresponding inequalities are the solid (as opposed to dashed) boxes and arrows of Figure 33. Then $\Xi_U$ is a simplicial cone of dimension $\binom{n}{2}$. 

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References


