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**Optimal investment with high-watermark fee in a
multi-dimensional jump diffusion model**

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Dedicated to my family.

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Optimal investment with high-watermark fee in a multi-dimensional jump diffusion model

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This dissertation studies the problem of optimal investment and consumption in a market in which there are multiple risky assets. Among those risky assets, there is a fund charging high-watermark fees and many other stocks, with share prices given exogenously as a multi-dimensional geometric Lévy process. Additionally, there is a riskless money market account in this market. A small investor invests and consumes simultaneously on an infinite time horizon, and seeks to maximize expected utility from consumption. Utility is taken to be constant relative risk aversion (CRRA). In this setting, we first employ the Dynamic Programming Principle to write down the Hamilton-Jacobi-Bellman (HJB) integro-differential equation associated with this stochastic control problem. Then, we proceed to show that a classical solution of the HJB equation corresponds to the value function of the stochastic control problem, and hence the optimal strategies are given in feedback

form in terms of the value function. Moreover, we provide numerical results to investigate the impact of various parameters on the investor's strategies.

Table of Contents

Acknowledgments	v
Abstract	vii
List of Tables	xi
List of Figures	xii
Chapter 1. Introduction	1
Chapter 2. Model	8
2.1 A general model of dynamic investment with high-watermark fees	8
2.2 Optimal investment and consumption in a special model	14
Chapter 3. Dynamic programming and main results	22
3.1 Formal derivation of the Hamilton-Jacobi-Bellman(HJB) equation	22
3.2 Dimension reduction	26
3.2.1 The case when paying no fee, $\lambda = 0$	30
3.3 Main results	32
3.3.1 Existence of a viscosity solution	35
3.3.2 Smoothness of the viscosity solution	48
3.3.3 Optimal policies and verification	55
3.4 Certainty equivalent analysis	65
Chapter 4. Numerics	67
Chapter 5. Conclusion	82
Appendix	84

Appendix 1. Appendix	85
1.1 Proof of Proposition 3.3.4	85
Bibliography	87
Index	92

List of Figures

2.1	Negative paper jumps ($\theta_t^F \mathbf{J}^E(\eta) < 0$)	18
2.2	Positive small paper jumps ($\theta_t^F \mathbf{J}^E(\eta) < Y_{t-}$)	19
2.3	Positive large paper jumps ($\theta_t^F \mathbf{J}^E(\eta) > Y_{t-}$)	20
4.1	Relative certainty equivalent initial wealth.	69
4.2	Relative certainty equivalent zero fee return.	70
4.3	Relative investment proportions and consumption proportion when $\mu^F < \mu_0^S$	71
4.4	Relative investment proportions and consumption proportion when $\mu^F > \mu_0^S$	72
4.5	Investment proportions and consumption proportion when vary- ing ρ	73
4.6	Value functions with and without jumps ($\mathbf{q}_1, \mathbf{J}_1$).	74
4.7	Investment proportion in hedge fund with and without jumps ($\mathbf{q}_1, \mathbf{J}_1$).	75
4.8	Investment proportion in stock with and without jumps ($\mathbf{q}_1, \mathbf{J}_1$).	76
4.9	Consumption proportion with and without jumps ($\mathbf{q}_1, \mathbf{J}_1$).	77
4.10	Value functions with and without jumps ($\mathbf{q}_2, \mathbf{J}_2$).	77
4.11	Investment proportion in hedge fund with and without jumps ($\mathbf{q}_2, \mathbf{J}_2$).	78
4.12	Investment proportion in stock with and without jumps ($\mathbf{q}_2, \mathbf{J}_2$).	78
4.13	Consumption proportion with and without jumps ($\mathbf{q}_2, \mathbf{J}_2$).	79
4.14	Value functions with and without jumps ($\mathbf{q}_3, \mathbf{J}_3$).	79
4.15	Investment proportion in hedge fund with and without jumps ($\mathbf{q}_3, \mathbf{J}_3$).	80
4.16	Investment proportion in stock with and without jumps ($\mathbf{q}_3, \mathbf{J}_3$).	80
4.17	Consumption proportion with and without jumps ($\mathbf{q}_3, \mathbf{J}_3$).	81

Chapter 1

Introduction

The portfolio optimization problem in continuous time has been studied extensively in the mathematical finance literature. One classical problem of this type is proposed by Merton [23], [24]. The basic setup of the Merton problem is as follows: a long-term investor decides, at each time, how much of his/her wealth should be allocated into a risky asset (for example, a stock), how much of his/her wealth should be allocated into a riskless asset (for example, a bank account) and how much he/she should consume each day in order to maximize his/her cumulative utility gained from consumption, with a possible subjective discounting. The subjective discounting means that the investor values utility of today more than that of the future. This problem exemplifies the technique of stochastic control in mathematical finance. Since Merton proposed this problem, various extensions have been studied in academia. Those extensions of the Merton problem include but are not limited to: (i) flexible retirement age can be considered [6]; (ii) transaction costs can be incorporated [9], [32], [10], [30], [25]; (iii) bankruptcy can be introduced [18], [31]. The classical Merton problem assumes the market is frictionless. More realistic models would consider markets with frictions, and hence there is a large literature considering market imperfections. Transaction costs just mentioned

are primary examples of market frictions. Similarly, the *high-watermark fees* introduced below can also be thought of as a kind of market friction.

In addition to a stock, a risky asset in the Merton problem can be a hedge fund share. Hedge fund managers charge fees for their service. The fees usually consist of *proportional fees*, in which the investor pays a fixed proportion of his/her total investment in the fund, and *high-watermark fees*, in which the investor pays a given percentage of his/her profit made from investing in the fund. *High-watermark* has the meaning of historic maximum up to today, and *high-watermark fees* are charged whenever the high-watermark exceeds the previously attained historic maximum. In the hedge fund industry, people often see a “2/20 rule”, meaning a combination of a 2% proportional fee and a 20% high-watermark fee charged for the investor. Proportional fees can be easily incorporated into the Merton problem, because the effect of proportional fees is equivalent to that of a reduced mean return of the hedge fund share price. On the other hand, high-watermark fees pose interesting problems mathematically. From a modeling perspective, one needs to keep track of not only the hedge fund price but also its historic maximum to account for the times when high-watermark fees are charged. This results in different dynamics of the state process and potentially much more challenging stochastic control problems. Along the line of extending Merton problem by adding the feature of high-watermark fees, Janeček and Sírbu in [17] proposed an infinite horizon optimal investment and consumption problem, where the risky asset is a hedge fund charging high-watermark fees at a rate λ , the riskless asset

is a bank account charging zero interest, and utility function is chosen to be power utility. This modified Merton problem yields a model in which the state process is a two-dimensional reflected diffusion. The authors were able to show that the value function of this problem is a classical solution of the corresponding Hamilton-Jacobi-Bellman (HJB) equation. Moreover, the optimal investment and consumption strategies can in turn be written in feedback form as functions of the value function and its derivatives. However, unlike the classical Merton problem, the HJB in this modification of Merton problem can not be solved closed-form. Therefore, [17] also provided numerical results to understand quantitatively the impact of high-watermark fees on the investor's behavior. More recently, there were some extensions of [17]: Kontaxis studied in his dissertation [22] asymptotic results of this modified Merton problem when λ is small; Lin, Wang and Yao in [34] built a model where the investor is an insurer who was subject to insurance claims, modelled as a compound Poisson process. Note that the model in [17],[22],[34] all assume that the investor can trade continuously in and out of the fund.

The problem of optimal investment with high-watermark fees in [17] is *technically related* to the problem of *optimal investment with draw-down constraints* in [15], [8], [29] and [11]. However, with *consumption present in the running maximum*, the problem in [17] does not have a closed-form solution, as opposed to that in [29] and [11]. Hence, in order to prove that the HJB equation has a classical solution, [17] used Perron's method to obtain existence of a viscosity solution and then upgraded its regularity.

It is also worth noting that research on high-watermark fees in the literature is not limited to the context of Merton problem. In [1] and [4], the authors argue that the high-watermark fees serve as incentives for the fund manager to seek long-term growth that is in line with the investor's objective. Panageas and Westerfield [27] studied the problem of maximizing present value of future fees from the perspective of a risk-neutral fund manager. Goetzmann, Ingersoll and Ross [14] derived a closed-form formula for the value of a high-watermark contract as a claim on the investor's wealth. Guasoni and Obłój [16] formulated a utility maximization problem also from the perspective of a hedge fund manager, rather than an investor. In [16], the stochastic differential equation governing the evolution of the hedge fund share price has a similar pathwise solution to the state equation describing the dynamics of the investor's wealth in the present work. However, the stochastic control problem is different.

The present work generalizes the model of [17] in two aspects. First, we allow for multiple risky assets including a hedge fund charging high-watermark fees and a large number of possibly correlated stocks, and the riskless asset can have non-zero interest rate. Second, those multiple risky assets can in general have jumps. The main goal is to understand, in our very general model, how high-watermark fees affect the investor's behaviour, compared to the case with no fees and all else being equal. Moreover, the generalization of [17] to include multiple assets and jump processes allows us to see how various model parameters impact the investor's strategies. One can ask questions like:

what role does correlation between different assets play? what is the effect of jumps in asset prices on the investor's behavior?

Compared to [17], care must be taken to assess the high-watermark fees at the time of a jump. This will be seen in Chapter 2 when we write down stochastic differential equations of the state process. Another modelling challenge is that, to generalize [17], it is not clear, at first sight, how many state variables we need to keep track of in our model for the purpose of utility maximization, given that there are many risky assets. Somewhat surprisingly, a state process consisting of the cumulative wealth X and the “distance to pay high-watermark fees” Y is enough for our purpose. Again, this will be explained in detail in Chapter 2.

Mathematically, this model leads to a multi-dimensional HJB equation that is in general hard to solve. However, if we consider power utility, the homogeneity property allows us to reduce the dimension of the HJB to one, in a similar fashion as in [17]. So we only discuss power utility in our work, and the investigation on this model with general utility seems to be a difficult problem and remains to be our future work.

After dimension reduction, the HJB equation becomes an ordinary differential-integral equation, in terms of one variable that is the ratio between the “distance to pay high-watermark fees” and the cumulative wealth. We show that a classical solution of the ordinary differential-integral equation corresponds to a solution to our stochastic control problem. The analysis is a generalization of that in [17], which is based primarily on viscosity solution

techniques. An outline of the analysis is as follows:

1. we construct a viscosity solution using Perron's method taking into account boundary conditions;
2. we prove smoothness of the solution using properties of viscosity solutions as well as convexity;
3. we finish with a verification argument.

For an introduction to viscosity solution as well as Perron's method, we refer the readers to [7], [12], [20]. In particular, viscosity solutions applied to integro-differential equations are discussed in [2], [3], [28], [5]. Moreover, for stochastic control problems with jumps, our references include [33], [26].

Because there is no closed-form solution to the stochastic control problem, we must rely again on numerical approximations to understand how various model parameters affect the investor's behaviour, and to compare this model with the classical Merton problem without high-watermark fees. We employ an iterative method of solving the associated integro-differential equation. While making comparison between models with and without high-watermark fees, we use some certainty equivalent analysis.

The main findings of the present work can be summarized as follows. First, in our general model (with multiple risky assets modeled by jump diffusion processes), the optimal investment/consumption problem can be formulated as a stochastic control of a two-dimensional reflected jump diffusion.

Solving this control problem can be reduced to finding a classical solution of the associated HJB equation. Second, the numerical results show that in a scenario of one hedge fund and one stock, the comparison with the case of no fee (the classical Merton problem) is as follows : (i) if the return of the fund is bigger than that of the stock, then the high-watermark fees would make the investor invest *more* in the hedge fund when the high-watermark is close to being reached; (ii) if the return of the fund is smaller than that of the stock, then the high-watermark fees would make the investor invest *less* in the hedge fund when the high-watermark is close to being reached; (iii) in either case, when the investor is far away from paying high-watermark fees, the investment and consumption strategies are close to those in the case of no fee. Note the third comparison result above is also proved analytically. Moreover, the numerics regarding the correlation between the hedge fund and the stock would demonstrate the benefit of diversification, as expected; the effect of jumps in risky assets can be seen as increased volatilities. The details of numerics will be presented in Chapter 4.

Chapter 2

Model

2.1 A general model of dynamic investment with high-watermark fees

We consider a hedge fund with share price F_t and a benchmark asset with share price B_t at time t , where F_t and B_t are strictly positive semimartingales. An investor chooses to invest θ_t^F units of wealth in the hedge fund at time t (right before the jump). For convenience, we use hats to denote quantities which are computed *before* any fees are assessed, which we call *paper* quantities. With this convention, the accumulated paper profits of the investor are given by

$$\widehat{P}_t = \int_0^t \theta_t^F dF_t / F_{t-},$$

or, in differential notation,

$$\begin{cases} d\widehat{P}_t = \theta_t^F \frac{dF_t}{F_{t-}}, \\ \widehat{P}_{0-} = 0. \end{cases}$$

Since this is rather heuristic, we impose no precise conditions yet.

Now, the *realized* profit P is subject to both high-watermark and hurdle provisions. In our model, the realized profit is reduced by a ratio $\lambda > 0$ of the excess (realized) profit over the strategy of investing in the benchmark, as shown in (2.1) below.

In order to impose the hurdle provision, the profit accumulated by an identical hypothetical investment in the benchmark asset is computed as follows,

$$P_t^B = \int_0^t \theta_s^F dB_t / B_{t-}.$$

If the investor is given an initial high-watermark $y \geq 0$ for her profits (in practical applications, we have $y = 0$), the fees paid to the hedge fund manager amount to λdM_t in the infinitesimal interval dt , where the process M is the so called *high-watermark*

$$M_t \triangleq \sup_{0 \leq s \leq t} \{ (P_s - P_s^B) \vee y \},$$

i.e., M is the running maximum of the excess *realized* accumulated profit from the investment over the profit from investing in the benchmark. Similarly, we can define the *paper* high-watermark as

$$\widehat{M}_t \triangleq \sup_{0 \leq s \leq t} \{ (\widehat{P}_s - P_s^B) \vee y \},$$

Remark 2.1.1. Note that λ can be greater than 1, because if we convert the implicit equations in (2.1) to explicit equations as in Proposition 2.1.1 below, we can see that λdM_t is equal to $\frac{\lambda}{1+\lambda} d\widehat{M}_t$.

With these notations, the *realized* accumulated profit P of the investor evolves as

$$\begin{cases} dP_t = \theta_t^F \frac{dF_t}{F_{t-}} - \lambda dM_t, & P_{0-} = 0, \\ M_t = \sup_{0 \leq s \leq t} \{ (P_s - P_s^B) \vee y \}. \end{cases} \quad (2.1)$$

Equation (2.1) is implicit, so the existence and uniqueness of the solution should be analyzed carefully. Fortunately, we can solve (P, M) closed form pathwise, as shown in Proposition 2.1.1 below.

Proposition 2.1.1. *Assume that the hedge fund share price process F_t and the benchmark asset price B_t are strictly positive semi-martingales, and F_{t-} , B_{t-} are positive for all t . Assume also that the predictable process θ_t^F is such that the accumulated excess profits corresponding to the trading strategy θ_t^F , in case no profit fees are imposed, namely*

$$I_t = \int_0^t \theta_t^F \left(\frac{dF_t}{F_{t-}} - \frac{dB_t}{B_{t-}} \right), \quad 0 \leq t < \infty,$$

is well defined. Then (2.1) has a unique solution, which can be represented pathwise by

$$P_t = \widehat{P}_t - \frac{\lambda}{1 + \lambda} \left(\widehat{M}_t - y \right), \quad 0 \leq t < \infty, \quad (2.2)$$

$$M_t = y + \frac{1}{1 + \lambda} \left(\widehat{M}_t - y \right), \quad 0 \leq t < \infty, \quad (2.3)$$

with

$$\begin{aligned} \widehat{P}_t &= \int_0^t \theta_t^F \frac{dF_t}{F_{t-}}, \quad 0 \leq t < \infty, \\ \widehat{M}_t &= y + \sup_{0 \leq s \leq t} [I_s - y]^+, \quad 0 \leq t < \infty, \end{aligned}$$

Proof. Equation (2.1) can be rewritten as

$$\left((P_s - P_s^B) - y \right) + \lambda \sup_{0 \leq s \leq t} \left[(P_s - P_s^B) - y \right]^+ = I_t - y, \quad 0 \leq t < \infty.$$

Taking the positive part and the supremum on both sides, we get

$$(1 + \lambda)(M_t - y) = (1 + \lambda) \sup_{0 \leq s \leq t} \left[(P_s - P_s^B) - y \right]^+ = \sup_{0 \leq s \leq t} [I_s - y]^+, \quad 0 \leq t < \infty.$$

Then (2.3) follows from the definition of \widehat{M}_t . Substituting (2.3) back into (2.1), we have (2.2). And we've established both existence and uniqueness. \square

Remark 2.1.2. An alternative and better explanation of the above proposition is connected to the famous Skorokhod equation (see [19], page 210 for the continuous case and [21] for the discontinuous case); given $i \geq 0$ and a right continuous with left limits function $f : [0, \infty) \rightarrow \mathbb{R}$ with $f(0-) = 0$, there exists a unique right continuous with left limits function k such that

1. $g(t) = i + f(t) + k(t) \geq 0$ for all t ;
2. k is non-decreasing with $k(0-) = 0$;
3. $\int_0^t \mathbf{1}_{\{g(s) > 0\}} dk(s) = 0$ for all t .

Explicitly, the solution is given by

$$k(t) = \sup_{0 \leq s \leq t} [-f(s) - i]^+.$$

Set $Y_t \triangleq M_t - (P_t - P_t^B)$. In other words, Y_t is the “distance to paying high-watermark fees”. Note that $Y_t \geq 0$, and Y_t satisfies the equation

$$\begin{cases} dY_t = -dI_t + (1 + \lambda) dM_t, \\ Y_{0-} = y. \end{cases}$$

We also have

$$\int_0^t \mathbf{1}_{\{Y_s > 0\}} dM_s = 0 \text{ for all } t.$$

Therefore, $(1 + \lambda)(M_t - y)$ is the solution k to the Skorokhod equation above, with $f(t) = -I_t$ and $i = y$.

Our model can be multi-dimensional in general. In addition to the hedge fund, the investor can also invest in n possibly correlated stocks whose share prices are given by S_i , $i = 1, \dots, n$. The investor chooses to invest θ_t^i units of her wealth in stock i at time t , and also to consume at a rate γ_t per unit of time. The remaining wealth sits in a bank account paying interest rate r . With (P, M) denoting the solution to (2.1), the total wealth of the investor evolves as

$$\begin{cases} dX_t = r(X_t - \theta_t^F - \sum_{i=1}^n \theta_t^i) dt - \gamma_t dt + \sum_{i=1}^n \theta_t^i \frac{dS_t^i}{S_{t-}^i} + \underbrace{\theta_t^F \frac{dF_t}{F_{t-}} - \lambda dM_t}_{dP_t}, \\ X_{0-} = x. \end{cases} \quad (2.4)$$

As seen from Proposition 2.1.1, the state equation (2.4) can be solved *pathwise closed-form* in terms of (θ^i, γ) , $i = F, 1, \dots, n$ provided all stochastic integrals involved are well defined. In addition, for $x > 0$ and $y \geq 0$ we impose the constraints on the set of controls (θ^i, γ) , $i = F, 1, \dots, n$ such that neither shorting selling of hedge fund shares or stocks, nor borrowing from money market is allowed (see Remark 2.1.3 below), and we will address admissibility in detail when we talk about a special model in the next subsection.

Remark 2.1.3. Note that admissible strategies can be equivalently represented in terms of the proportions $\pi = \theta/X_-$ and $c = \gamma/X_-$. In that case, we will impose the constraint that $\pi_t^i \geq 0$, $i = F, 1, \dots, n$ and $\pi_t^F + \sum_{i=1}^n \pi_t^i \leq 1$ for all times t , which means there is neither short selling of hedge fund shares or stocks, nor borrowing from money market. In other words, $\pi \in \Delta = \{\pi^i \geq 0, i = F, 1, \dots, n \text{ and } \pi^F + \sum_{i=1}^n \pi^i \leq 1\}$.

We intend to solve this problem using dynamic programming arguments. Therefore, we'd like to reformulate the model in terms of a controlled reflected Markov process with jumps. In addition, in order to keep the analysis tractable we wish to find such a state process with minimal dimension. Recall from Remark 2.1.2 that, we denote by

$$Y_t \triangleq M_t - (P_t - P_t^B),$$

“the distance to paying high-watermark fees”. With this notation, the crucial observation is that the two dimensional process (X, Y) , with X defined in (2.4) is, indeed, a state process. More precisely, the evolution equation for the process Y is

$$\begin{cases} dY_t = -\theta_t^F \left(\frac{dF_t}{F_{t-}} - \frac{dB_t}{B_{t-}} \right) + (1 + \lambda) dM_t, \\ Y_{0-} = y, Y_t \geq 0. \end{cases}$$

Now, the state process is a two-dimensional process in $D = \{x > 0, y \geq 0\}$, that is,

$$\begin{cases} dX_t = rX_t dt + \theta_t^F \left(\frac{dF_t}{F_{t-}} - r dt \right) + \sum_{i=1}^n \theta_t^i \left(\frac{dS_t^i}{S_{t-}^i} - r dt \right) - \gamma_t dt - \lambda dM_t, \\ dY_t = -\theta_t^F \left(\frac{dF_t}{F_{t-}} - \frac{dB_t}{B_{t-}} \right) + (1 + \lambda) dM_t \\ \int_0^t \mathbf{1}_{\{Y_s > 0\}} dM_s = 0. \end{cases} \quad (2.5)$$

and the initial conditions are given by

$$\begin{cases} X_{0-} = x > 0, \\ Y_{0-} = y \geq 0. \end{cases}$$

2.2 Optimal investment and consumption in a special model

So far, this is a general model of investment and consumption in a hedge fund and n stocks. In what follows, we choose a particular model for which we can solve the problem of optimal investment and consumption by dynamic programming. More precisely, we assume the hedge fund share price and stock prices evolve as a multi-dimensional geometric Lévy process,

$$\begin{bmatrix} \frac{dF_t}{F_{t-}} \\ \frac{dS_t^1}{S_{t-}^1} \\ \vdots \\ \frac{dS_t^n}{S_{t-}^n} \end{bmatrix} = \begin{bmatrix} \mu^F \\ \mu^1 \\ \vdots \\ \mu^n \end{bmatrix} dt + \sigma d\mathbf{W}_t + \int_{\mathbb{R}^l} \mathbf{J}(\eta) \mathcal{N}(d\eta, dt), \quad (2.6)$$

where $\sigma\sigma^T > 0$, \mathbf{W} is a d -dimensional Brownian motion, $\mathcal{N}(d\eta, dt)$ is a Poisson random measure on $\mathbb{R}^l \setminus \{0\} \times [0, \infty)$, with intensity $\mathbf{q}(d\eta)dt$, where \mathbf{q} is σ -finite, all the vectors are of appropriate dimensions. All $W_i, i = 1, \dots, d$ are independent. Both the Brownian motion \mathbf{W} and the counting process $\int_0^t \int \mathcal{N}(d\eta, dt)$ are defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$. The filtration $\{\mathcal{F}_t\}_{t \geq 0}$ is assumed to satisfy the usual conditions.

In order for the price processes to stay positive, we require that

$$\frac{\Delta F_t}{F_{t-}} > -1, \frac{\Delta S_t^i}{S_{t-}^i} > -1.$$

In other words, \mathbf{J} and \mathbf{q} must satisfy

$$\mathbf{q}\left(\eta | \mathbf{J}(\eta) \in ((-1, \infty)^{n+1})^c\right) = 0 \quad (2.7)$$

Moreover, we assume that

$$\int_{\mathbb{R}^l} |\mathbf{J}(\eta)| \mathbf{q}(d\eta) < \infty. \quad (2.8)$$

and

$$\int_{\mathbb{R}^l} |\mathbf{J}(\eta)|^2 \mathbf{q}(d\eta) < \infty. \quad (2.9)$$

where $|\cdot|$ denotes any vector norm since all norms of \mathbb{R}^{n+1} are equivalent.

Remark 2.2.1. Note that (2.8) implies that the jumps are Lévy processes of finite variation paths, this is not necessary as long as we compensate all the jumps. Still we assume (2.8) in order to simplify our discussion about Ito's formula and HJB equation, as well as the discussion on viscosity solutions in the next chapter.

Additionally, for technical reasons, we also assume that \mathbf{J} and \mathbf{q} satisfy

$$\begin{aligned} \max_{\pi \in \Delta} \int_{\mathbb{R}^l} \left| (1 + \pi^T \mathbf{J}(\eta))^{1-p} - 1 \right| \mathbf{q}(d\eta) < \infty, \\ \max_{\pi \in \Delta} \int_{\mathbb{R}^l} \left| (1 + \pi^T \mathbf{J}(\eta))^{-p} \pi^T \mathbf{J}(\eta) \right| \mathbf{q}(d\eta) < \infty. \end{aligned} \quad (2.10)$$

This assumption above ensures that the integral term of the HJB equation (which we will see later) is well-defined, and will also be used in the proof of verification later.

The benchmark asset evolves as

$$\frac{dB_t}{B_{t-}} = \mu^B dt + \sigma^B d\mathbf{W}_t + \int_{\mathbb{R}^l} J^B(\eta) \mathcal{N}(d\eta, dt).$$

For convenience, we set $\mu^E = \mu^F - \mu^B$, $\sigma^E = \sigma^F - \sigma^B$, $J^E(\eta) = J^F(\eta) - J^B(\eta)$ and we also denote by $\theta_t \triangleq (\theta_t^F, \theta_t^1, \dots, \theta_t^n)^T \in \mathbb{R}^{n+1}$ the complete investment strategy at time t . We define $\alpha \triangleq (\alpha_F, \alpha_1, \dots, \alpha_n)^T = (\mu_F - r, \mu_1 - r, \dots, \mu_n - r)^T \in \mathbb{R}^{n+1}$.

Remark 2.2.2. Note that α cannot be interpreted as the vector of excess returns because the vector of excess returns is indeed $\alpha + \int_{\mathbb{R}^l} \mathbf{J}(\eta) \mathbf{q}(d\eta)$.

With these notations, we now solve for the process (X, Y) . It turns out that the pathwise representation in Proposition 2.1.1 can be easily translated into a pathwise solution for (X, Y) . More precisely, we have the following proposition, whose proof is a direct consequence of Proposition 2.1.1, so we omit it.

Proposition 2.2.1. *Assume that the predictable processes (θ, γ) satisfy the following integrability property:*

$$\begin{aligned} \mathbb{P} \left(\int_0^t (|\theta_u|_2^2 + \gamma_u) du < \infty \quad \forall 0 \leq t < \infty \right) &= 1, \\ \mathbb{P} \left(\int_0^t \left(\int_{\mathbb{R}^l} |\theta_u^F \mathbf{J}^E(\eta)|^2 \mathbf{q}(d\eta) \right) du < \infty \quad \forall 0 \leq t < \infty \right) &= 1, \\ \mathbb{P} \left(\int_0^t \left(\int_{\mathbb{R}^l} |\theta_u^T \mathbf{J}(\eta)|^2 \mathbf{q}(d\eta) \right) du < \infty \quad \forall 0 \leq t < \infty \right) &= 1. \end{aligned}$$

Denote by

$$\begin{aligned} I_t &= \int_0^t \theta_u^F \left(\mu^E du + \sigma^E d\mathbf{W}_u + \int_{\mathbb{R}^l} \mathbf{J}^E(\eta) \mathcal{N}(d\eta, du) \right), \\ N_t &= \int_0^t \theta_u^T \left(\alpha du + \sigma d\mathbf{W}_u + \int_{\mathbb{R}^l} \mathbf{J}(\eta) \mathcal{N}(d\eta, du) \right), \\ C_t &= \int_0^t \gamma_u du, \quad 0 \leq t < \infty, \end{aligned}$$

the excess accumulated profit process from the hedge fund and the accumulated profit process from all assets corresponding to the trading strategy θ , in case no profit fees are imposed, and the accumulated consumption. Then, equation (2.5) has a unique solution (X, Y) , which can be represented by

$$\begin{aligned} X_t &= x + \int_0^t rX_s ds + N_t - C_t - \lambda(M_t - y), \quad 0 \leq t < \infty, \\ Y_t &= y - I_t + (1 + \lambda)(M_t - y), \quad 0 \leq t < \infty. \end{aligned} \quad (2.11)$$

where the high-watermark is computed as

$$M_t = y + \frac{1}{1 + \lambda} \sup_{0 \leq s \leq t} [I_s - y]^+. \quad (2.12)$$

The state process (X, Y) is a controlled two-dimensional reflected jump-diffusion. More precisely, the investor uses the strategy (θ, γ) to control the jump-diffusion (X, Y) given by (2.11) in its domain

$$D = \{(x, y) : x > 0, y \geq 0\}.$$

The diffusion part of (X, Y) is reflected on the line $\{y = 0\}$ in the direction given by the vector

$$\kappa \triangleq \begin{pmatrix} -\lambda \\ 1 + \lambda \end{pmatrix}.$$

The reflection is at the rate dM^c , where M is the high-watermark process and M^c denotes its continuous part. The reflection of jumps of (X, Y) happens only when the jump size of the accumulated paper profit is large enough to cause (X, Y) be out of its domain. At the time of such a large jump, high-watermark fees will be immediately deducted so that the (after-fees) process

(X, Y) will be pulled back to the line $\{y = 0\}$ in the direction κ as well. To illustrate jumps in different scenarios and possible reflections of the jump diffusion process (X, Y) , we present the following three figures,

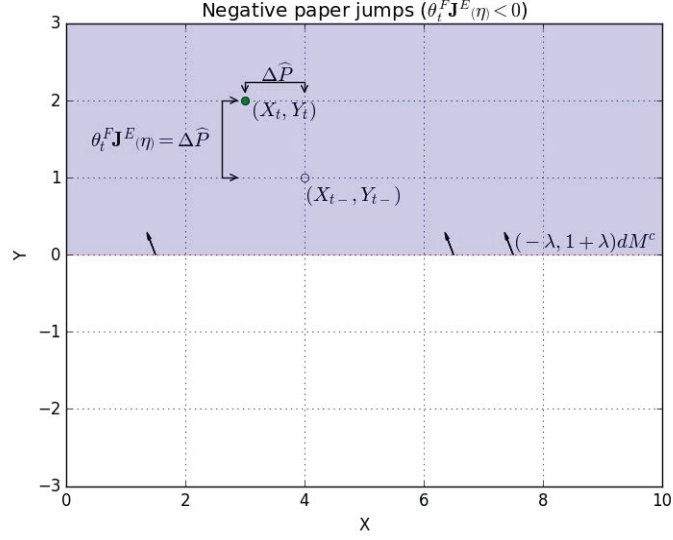


Figure 2.1: Negative paper jumps $(\theta_t^F \mathbf{J}^E(\eta) < 0)$

Remark 2.2.3. We observe that

$$\int_0^t \mathbf{1}_{\{Y_{s-} \neq 0\} \cup \{Y_s \neq 0\}} dM_s^c = 0,$$

which means dM_t^c is a measure only supported on $\{Y_{s-} = Y_s = 0\}$. This is true because even if there are diffusion reflections immediately after jump reflections, there are a countable number of jumps.

Remark 2.2.4. Note that Figure 2.1-2.3 only show the jumps coming from the hedge fund. There may be other simultaneous jumps coming from the other

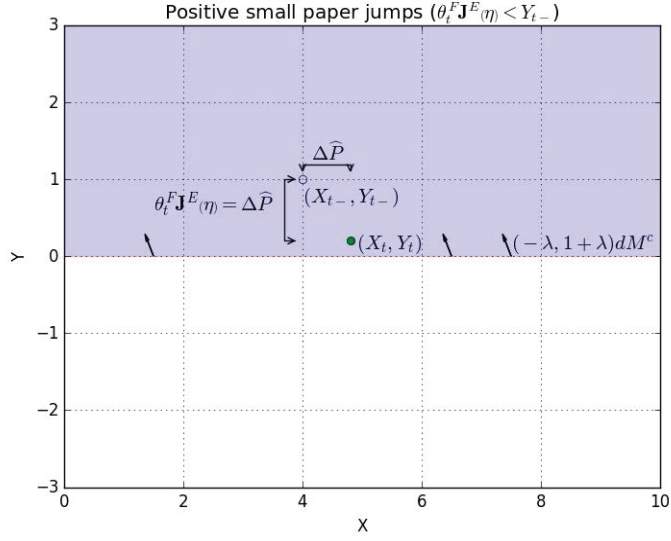


Figure 2.2: Positive small paper jumps ($\theta_t^F \mathbf{J}^E(\eta) < Y_{t-}$)

stocks. We can think of simultaneous jumps as a sequence: jump from fund, and immediately jumps from stocks. It is straightforward that jumps from stocks would cause a shift of X while having no effect on Y .

We model the preferences of the investor by the well-known concept of *expected utility* from consumption. Namely, we consider a concave utility function $U : (0, \infty) \rightarrow \mathbb{R}$ to define the expected utility from consumption $E \left[\int_0^\infty e^{-\beta t} U(\gamma_t) dt \right]$. The discount factor $\beta > 0$ accounts for the urgency of the investor to consume now rather than later. In this model, the problem of optimal investment and consumption accounts to finding, for each (x, y) , (the) optimal (θ, γ) in the optimization problem

$$V(x, y) \triangleq \sup_{(\theta, \gamma) \in \mathcal{A}(x, y)} E \left[\int_0^\infty e^{-\beta t} U(\gamma_t) dt \right], \quad x > 0, y \geq 0. \quad (2.13)$$

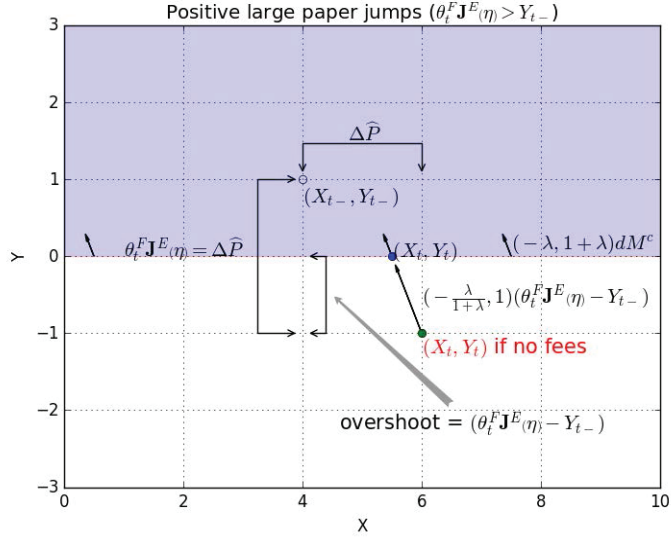


Figure 2.3: Positive large paper jumps ($\theta_t^F \mathbf{J}^E(\eta) > Y_{t-}$)

where

$$\mathcal{A}(x, y) = \left\{ \begin{array}{l} (\theta, \gamma) : \\ \text{Predictable processes satisfying integrability in Prop 2.2.1;} \\ \pi_t = \frac{\theta_t}{X_{t-}} \in \Delta; \\ \gamma_t \geq 0, X_t > 0 \text{ for all } t \geq 0. \end{array} \right\}$$

The function V defined above is called *value function* and $\mathcal{A}(x, y)$ is the *admissible set* of (θ, γ) . Recall from Remark 2.1.3 that $\Delta = \{\pi^i \geq 0, i = F, 1, \dots, n \text{ and } \pi^F + \sum_{i=1}^n \pi^i \leq 1\}$ and $\pi_t = \frac{\theta_t}{X_{t-}} \in \Delta$ is imposed to guarantee that there is neither short selling of hedge fund shares or stocks, nor borrowing from money market.

Remark 2.2.5. In order for X to stay positive all the time, in general we would

need π to satisfy

$$\mathbf{q}(\pi^T \mathbf{J}(\eta) \leq -1) = 0. \quad (2.14)$$

However, this constraint of π depends on the choice of \mathbf{q} and \mathbf{J} . In our model, we impose the universal constraint $\pi \in \Delta$. Because $\pi \in \Delta$ together with our assumption in (2.7) is sufficient for (2.14) to hold for any \mathbf{q} and \mathbf{J} .

We further assume that the investor has homogeneous preferences, meaning that the utility function U has the particular form

$$U(\gamma) = \frac{\gamma^{1-p}}{1-p}, \quad \gamma > 0,$$

for some $p > 0, p \neq 1$ called the relative risk-aversion coefficient.

Remark 2.2.6. In our model, we can easily incorporate the case when, in addition to the proportional high-watermark fee λ , the investor pays a continuous proportional fee with size $\nu > 0$ (percentage of wealth under investment management per unit of time). In order to do this we just need to reduce the size of α by the proportional fee to $\alpha - \nu$ in the evolution of the fund share price.

Chapter 3

Dynamic programming and main results

3.1 Formal derivation of the Hamilton-Jacobi-Bellman(HJB) equation

By applying Ito's lemma, we have the following lemma.

Lemma 3.1.1. *Let (X, Y, M) denote the solution of the state equation (2.11) for $(\theta, \gamma) \in \mathcal{A}(x, y)$, and let $b : \mathbb{R}^2 \rightarrow \mathbb{R}^{n+1}$ and $A : \mathbb{R}^{2,2} \rightarrow \mathbb{R}^{(n+1),(n+1)}$ denote the functions*

$$\mathbf{b} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \triangleq x_1 \alpha - x_2 \mu^E \mathbf{e}_F$$

$$\mathbf{A} \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} \triangleq y_{11} \sigma \sigma^T - y_{12} \sigma \sigma^E \mathbf{e}_F^T \\ - y_{21} \mathbf{e}_F (\sigma^E)^T \sigma^T + y_{22} \mathbf{e}_F (\sigma^E)^T \sigma^E \mathbf{e}_F^T$$

where \mathbf{e}_F denotes a column vector of dimension $n + 1$ with a one in the first coordinate, and set

$$\kappa \triangleq \begin{pmatrix} -\lambda \\ 1 + \lambda \end{pmatrix} \in \mathbb{R}^2.$$

If v is a C^2 (up to the boundary) function on $\{(x, y) \in \mathbb{R}^2 : x > 0, y \geq 0\}$, and assuming integrability condition below in (3.1), then

$$\begin{aligned}
& \int_0^t e^{-\beta s} U(\gamma_s) ds + e^{-\beta t} v(X_t, Y_t) \\
&= v(x, y) \\
&+ \int_0^t e^{-\beta s} \left\{ \begin{array}{l} -\beta v(X_{s-}, Y_{s-}) + U(\gamma_s) \\ + (rX_{s-} - \gamma_s + \theta_s^T \alpha) v_x(X_{s-}, Y_{s-}) \\ + \frac{1}{2} \theta_s^T \sigma \sigma^T \theta_s v_{xx}(X_{s-}, Y_{s-}) \\ + (-\theta_s^F \mu^E) v_y(X_{s-}, Y_{s-}) \\ + \frac{1}{2} (\theta_s^F)^2 (\sigma^E)^T \sigma^E v_{yy}(X_{s-}, Y_{s-}) \\ - (\theta_s^T \sigma) \theta_s^F \sigma^E v_{xy}(X_{s-}, Y_{s-}) \end{array} \right\} ds \\
&+ \int_0^t e^{-\beta s} \{-\lambda v_x(X_{s-}, Y_{s-}) + (1 + \lambda) v_y(X_{s-}, Y_{s-})\} dM_s^c \\
&+ \int_0^t e^{-\beta s} \{v_x(X_{s-}, Y_{s-}) \theta_s^T \sigma - v_y(X_{s-}, Y_{s-}) \theta_s^F (\sigma^E)^T\} d\mathbf{W}_s \\
&+ \int_0^t \int_{\mathbb{R}^l} e^{-\beta s} \left\{ v \left(\begin{array}{l} X_{s-} + \theta_s^T \mathbf{J}(\eta) \\ -\frac{\lambda}{1+\lambda} [\theta_s^F \mathbf{J}^E(\eta) - Y_{s-}]^+, \\ Y_{s-} - \theta_s^F \mathbf{J}^E(\eta) \\ + [\theta_s^F \mathbf{J}^E(\eta) - Y_{s-}]^+ \end{array} \right) - v(X_{s-}, Y_{s-}) \right\} \mathcal{N}(d\eta, ds) \\
&= v(x, y) \\
&+ \int_0^t e^{-\beta s} \left\{ \begin{array}{l} -\beta v(X_{s-}, Y_{s-}) + U(\gamma_s) + (rX_{s-} - \gamma_s) v_x(X_{s-}, Y_{s-}) \\ + \mathbf{b}^T (\mathbf{D}v(X_{s-}, Y_{s-})) \theta_s + \frac{1}{2} \theta_s^T \mathbf{A} (\mathbf{D}^2 v(X_{s-}, Y_{s-})) \theta_s \end{array} \right\} ds \\
&+ \int_0^t e^{-\beta s} \{\kappa^T \mathbf{D}v(X_{s-}, Y_{s-})\} dM_s^c \\
&+ \int_0^t e^{-\beta s} \{v_x(X_{s-}, Y_{s-}) \theta_s^T \sigma - v_y(X_{s-}, Y_{s-}) \theta_s^F (\sigma^E)^T\} d\mathbf{W}_s \\
&+ \int_0^t \int_{\mathbb{R}^l} e^{-\beta s} \left\{ v \left(\begin{array}{l} X_{s-} + \theta_s^T \mathbf{J}(\eta) \\ -\frac{\lambda}{1+\lambda} [\theta_s^F \mathbf{J}^E(\eta) - Y_{s-}]^+, \\ Y_{s-} - \theta_s^F \mathbf{J}^E(\eta) \\ + [\theta_s^F \mathbf{J}^E(\eta) - Y_{s-}]^+ \end{array} \right) - v(X_{s-}, Y_{s-}) \right\} \mathcal{N}(d\eta, ds)
\end{aligned}$$

where

$$\begin{aligned} & \int_0^t \int_{\mathbb{R}^l} e^{-\beta s} \left\{ v \left(\begin{array}{c} X_{s-} + \theta_s^T \mathbf{J}(\eta) \\ -\frac{\lambda}{1+\lambda} [\theta_s^F \mathbf{J}^E(\eta) - Y_{s-}]^+, \\ Y_{s-} - \theta_s^F \mathbf{J}^E(\eta) \\ + [\theta_s^F \mathbf{J}^E(\eta) - Y_{s-}]^+ \end{array} \right) - v(X_{s-}, Y_{s-}) \right\} \mathcal{N}(d\eta, ds) \\ & - \int_0^t \int_{\mathbb{R}^l} e^{-\beta s} \left\{ v \left(\begin{array}{c} X_{s-} + \theta_s^T \mathbf{J}(\eta) \\ -\frac{\lambda}{1+\lambda} [\theta_s^F \mathbf{J}^E(\eta) - Y_{s-}]^+, \\ Y_{s-} - \theta_s^F \mathbf{J}^E(\eta) \\ + [\theta_s^F \mathbf{J}^E(\eta) - Y_{s-}]^+ \end{array} \right) - v(X_{s-}, Y_{s-}) \right\} \mathbf{q}(d\eta) ds \end{aligned}$$

is a local martingale, in the case that the following condition holds,

$$\int_0^t \int_{\mathbb{R}^l} e^{-\beta s} \left| v \left(\begin{array}{c} X_{s-} + \theta_s^T \mathbf{J}(\eta) \\ -\frac{\lambda}{1+\lambda} [\theta_s^F \mathbf{J}^E(\eta) - Y_{s-}]^+, \\ Y_{s-} - \theta_s^F \mathbf{J}^E(\eta) \\ + [\theta_s^F \mathbf{J}^E(\eta) - Y_{s-}]^+ \end{array} \right) - v(X_{s-}, Y_{s-}) \right| \mathbf{q}(d\eta) ds < \infty, \text{ a.s.} \quad (3.1)$$

Remark 3.1.1. Note that in our model we do not compensate the jump term, i.e., we use $\int_{\mathbb{R}^l} \mathbf{J}(\eta) \mathcal{N}(d\eta, dt)$ instead of $\int_{\mathbb{R}^l} \mathbf{J}(\eta) (\mathcal{N}(d\eta, dt) - \mathbf{q}(d\eta)dt)$ in (2.6). This is because we assume in (2.8) that $\int_{\mathbb{R}^l} |\mathbf{J}(\eta)|_1 \mathbf{q}(d\eta) < \infty$, which allows us to separate the jump part from the diffusion part in the Ito's lemma. As previously mentioned, the assumption (2.8) is not necessary as long as we compensate the jump term, meaning that we replace $\int_{\mathbb{R}^l} \mathbf{J}(\eta) \mathcal{N}(d\eta, dt)$ by $\int_{\mathbb{R}^l} \mathbf{J}(\eta) (\mathcal{N}(d\eta, dt) - \mathbf{q}(d\eta)dt)$ in (2.6). If we compensate the jump term, we would introduce extra derivative terms, i.e., v_x, v_y , in the non-local part in the HJB equation (3.6). Considering the fact that the Ito's lemma above is already quite complicated, we insist on not compensating the jump term for simplicity. However, our analysis based on viscosity solutions would still apply

even if we remove the assumption (2.8) and compensate the jump term, with an appropriate definition of viscosity solutions.

Recall that M was explicitly defined in (2.12). Taking into account that dM_t^c is a measure with support on the set of times $\{t \geq 0 : Y_{t-} = Y_t = 0\}$, we can *formally* write down the HJB equation:

$$\sup_{\gamma \geq 0, \frac{\theta}{x} \in \Delta} \left\{ -\beta v + U(\gamma) + (rx - \gamma)v_x + \mathbf{b}^T(\mathbf{D}v)\theta + \frac{1}{2}\theta^T \mathbf{A}(\mathbf{D}^2v)\theta \right. \\ \left. + \int_{\mathbb{R}^l} \left\{ v \left(\begin{array}{c} x + \theta^T \mathbf{J}(\eta) \\ -\frac{\lambda}{1+\lambda} [\theta^F \mathbf{J}^E(\eta) - y]^+ \\ y - \theta^F \mathbf{J}^E(\eta) \\ + [\theta^F \mathbf{J}^E(\eta) - y]^+ \end{array} \right) - v(x, y) \right\} \mathbf{q}(d\eta) \right\} = 0, \\ x > 0, y > 0,$$

with the boundary condition

$$\kappa^T \mathbf{D}v = 0, \\ x > 0, y = 0.$$

If we can find a smooth solution for the HJB equation above, then the optimal consumption will be given in feedback form by

$$\hat{\gamma}(x, y) = I(v_x(x, y)), \quad (3.2)$$

where $I \triangleq (U')^{-1}$ is the inverse of marginal utility. In addition, we expect the

optimal investment strategy $\hat{\theta}$ to be given by

$$\begin{aligned} & \hat{\theta}(x, y) \\ &= \arg \max_{\frac{\theta}{x} \in \Delta} \left\{ + \int_{\mathbb{R}^l} \left\{ v \left(\begin{array}{c} \mathbf{b}^T (\mathbf{D}v) \theta + \frac{1}{2} \theta^T \mathbf{A} (\mathbf{D}^2 v) \theta \\ x + \theta^T \mathbf{J}(\eta) \\ -\frac{\lambda}{1+\lambda} [\theta^F \mathbf{J}^E(\eta) - y]^+, \\ y - \theta^F \mathbf{J}^E(\eta) \\ + [\theta^F \mathbf{J}^E(\eta) - y]^+ \end{array} \right) - v(x, y) \right\} \mathbf{q}(d\eta) \right\}. \end{aligned} \quad (3.3)$$

and the smooth solution of the HJB equation is indeed the value function, namely, that

$$v(x, y) = V(x, y), \quad x > 0, y \geq 0,$$

where V was defined in (2.13).

3.2 Dimension reduction

We expect that the solution of the HJB equation is the value function for the optimization problem (2.13). Therefore, we can use the *homogeneity property* for the power utility function to reduce the number of variables. More precisely, we expect that

$$v(x, y) = x^{1-p} v\left(1, \frac{y}{x}\right) \triangleq x^{1-p} u(z) \quad \text{for } z \triangleq \frac{y}{x}.$$

In addition, instead of looking for the optimal amounts $\hat{\theta}(x, y)$ and $\hat{\gamma}(x, y)$ in (3.3) and (3.2) we look for the *proportions*

$$\hat{c}(x, y) = \frac{\hat{\gamma}(x, y)}{x} = \frac{I(v_x(x, y))}{x}, \quad (3.4)$$

and

$$\widehat{\pi}(x, y) = \frac{\widehat{\theta}(x, y)}{x} \in \Delta. \quad (3.5)$$

Since

$$v_x(x, y) = ((1 - p)u(z) - zu'(z)) \cdot x^{-p},$$

$$v_y(x, y) = u'(z) \cdot x^{-p},$$

$$v_{xx}(x, y) = (-p(1 - p)u(z) + 2pzu'(z) + z^2u''(z)) \cdot x^{-1-p},$$

$$v_{yy}(x, y) = u''(z) \cdot x^{-1-p},$$

$$v_{xy}(x, y) = (-pu'(z) - zu''(z)) \cdot x^{-1-p},$$

we define the following differential operators on the function $u(z)$

$$D_x[u](z) = (1 - p)u(z) - zu'(z),$$

$$D_y[u](z) = u'(z),$$

$$D_{xx}[u](z) = -p(1 - p)u(z) + 2pzu'(z) + z^2u''(z),$$

$$D_{yy}[u](z) = u''(z),$$

$$D_{xy}[u](z) = D_{yx}[u](z) = -pu'(z) - zu''(z).$$

We therefore get the *one-dimensional* HJB equation for $u(z)$:

$$\begin{aligned}
& -\beta u + r \cdot D_x[u] \\
& + \sup_{c \geq 0} \left\{ \frac{c^{1-p}}{1-p} - c \cdot D_x[u] \right\} \\
& + \sup_{\pi \in \Delta} \left\{ \begin{array}{c} \mathcal{B}^T[u] \pi + \frac{1}{2} \pi^T \mathcal{A}[u] \pi \\ + \int_{\mathbb{R}^l} \left\{ \begin{array}{c} \left(\frac{1 + \pi^T \mathbf{J}(\eta)}{-\frac{\lambda}{1+\lambda} [\pi^F \mathbf{J}^E(\eta) - z]^+} \right)^{1-p} \\ \cdot u \left(\frac{z - \pi^F \mathbf{J}^E(\eta) + [\pi^F \mathbf{J}^E(\eta) - z]^+}{1 + \pi^T \mathbf{J}(\eta) - \frac{\lambda}{1+\lambda} [\pi^F \mathbf{J}^E(\eta) - z]^+} \right) \\ - u(z) \end{array} \right\} \mathbf{q}(d\eta) \end{array} \right\} \\
& = 0, \quad z > 0, \\
& -\lambda(1-p)u(0) + (1+\lambda)u'(0) \\
& = 0, \tag{3.6}
\end{aligned}$$

where

$$\mathcal{B}[u] \triangleq \mathbf{b} \begin{pmatrix} D_x[u] \\ D_y[u] \end{pmatrix},$$

and

$$\mathcal{A}[u] \triangleq \mathbf{A} \begin{pmatrix} D_{xx}[u] & D_{xy}[u] \\ D_{yx}[u] & D_{yy}[u] \end{pmatrix}.$$

Recall that \mathbf{b} and \mathbf{A} were defined in Lemma 3.1.1.

Remark 3.2.1. Note that, in the case of power utility, the integrability condi-

tion on v as in (3.1) is equivalent to the integrability condition on u below,

$$\int_0^t e^{-\beta s} X_{s-} \int_{\mathbb{R}^l} \left| \begin{array}{c} \left(\frac{1 + \pi^T \mathbf{J}(\eta)}{-\frac{\lambda}{1+\lambda} \left[\pi^F \mathbf{J}^E(\eta) - \frac{Y_{s-}}{X_{s-}} \right]^+} \right)^{1-p} \\ \cdot u \left(\frac{\frac{Y_{s-}}{X_{s-}} - \pi^F \mathbf{J}^E(\eta) + \left[\pi^F \mathbf{J}^E(\eta) - \frac{Y_{s-}}{X_{s-}} \right]^+}{1 + \pi^T \mathbf{J}(\eta) - \frac{\lambda}{1+\lambda} \left[\pi^F \mathbf{J}^E(\eta) - \frac{Y_{s-}}{X_{s-}} \right]^+} \right) \\ -u \left(\frac{Y_{s-}}{X_{s-}} \right) \end{array} \right| \mathbf{q}(d\eta) ds < \infty, \text{ a.s.}$$

Remark 3.2.2. Note that each element of the matrix $\mathcal{A}[u]$ is increasing in u'' , this observation will be used several times in our analysis.

We also expect that

$$\lim_{z \rightarrow \infty} u(z) = \frac{1}{1-p} c_0^{-p},$$

with c_0 given by (3.11) below. The optimal investment proportion in (3.5) could therefore be expressed (provided we can find a smooth solution for the reduced HJB equation (3.6)) as

$$\begin{aligned} & \widehat{\pi}(z) \\ &= \arg \max_{\pi \in \Delta} \left\{ \mathcal{B}^T[u] \pi + \frac{1}{2} \pi^T \mathcal{A}[u] \pi \right. \\ & \quad \left. + \int_{\mathbb{R}^l} \left\{ \begin{array}{c} \left(\frac{1 + \pi^T \mathbf{J}(\eta)}{-\frac{\lambda}{1+\lambda} \left[\pi^F \mathbf{J}^E(\eta) - z \right]^+} \right)^{1-p} \\ \cdot u \left(\frac{z - \pi^F \mathbf{J}^E(\eta) + \left[\pi^F \mathbf{J}^E(\eta) - z \right]^+}{1 + \pi^T \mathbf{J}(\eta) - \frac{\lambda}{1+\lambda} \left[\pi^F \mathbf{J}^E(\eta) - z \right]^+} \right) \\ -u(z) \end{array} \right\} \mathbf{q}(d\eta) \right\}. \end{aligned} \quad (3.7)$$

If $\widehat{\pi}(z)$ lies in the interior of Δ , we can also use the first order condition to get

that $\widehat{\pi}(z)$ satisfies

$$\begin{aligned} & \mathcal{B}[u] + \mathcal{A}[u] \pi & (3.8) \\ & + \int_{\mathbb{R}^l} \nabla \left\{ \left(\begin{array}{c} 1 + \pi^T \mathbf{J}(\eta) \\ -\frac{\lambda}{1+\lambda} [\pi^F \mathbf{J}^E(\eta) - z]^+ \end{array} \right)^{1-p} \right. \\ & \left. \cdot u \left(\frac{z - \pi^F \mathbf{J}^E(\eta) + [\pi^F \mathbf{J}^E(\eta) - z]^+}{1 + \pi^T \mathbf{J}(\eta) - \frac{\lambda}{1+\lambda} [\pi^F \mathbf{J}^E(\eta) - z]^+} \right) \right\} \mathbf{q}(d\eta) \\ & = 0. \end{aligned}$$

And the optimal consumption proportion \widehat{c} in (3.4) would be given by

$$\widehat{c}(z) = (D_x[u])^{-\frac{1}{p}} = ((1-p)u(z) - zu'(z))^{-\frac{1}{p}}. \quad (3.9)$$

3.2.1 The case when paying no fee, $\lambda = 0$

This is the classical Merton problem with jumps, except that we impose constraints, $\pi_0 \in \Delta$, such that no short selling of risky assets or borrowing from money market is allowed. The optimal investment and consumption proportions are constants. We can take the solution from [13], or solve our equation (3.6) and then use (3.7) and (3.9) to obtain the same results.

More precisely, for $\lambda = 0$, the optimal investment proportions π_0 and the optimal consumption proportion c_0 are given by

$$\pi_0 \triangleq \begin{cases} \arg \max_{\pi \in \Delta} \left\{ \begin{array}{l} (1-p) \alpha^T \pi + \frac{1}{2} (-p(1-p)) \pi^T \sigma \sigma^T \pi \\ + \int_{\mathbb{R}^l} \left\{ (1 + \pi^T \mathbf{J}(\eta))^{1-p} - 1 \right\} \mathbf{q}(d\eta) \end{array} \right\}, & p < 1, \\ \arg \min_{\pi \in \Delta} \left\{ \begin{array}{l} (1-p) \alpha^T \pi + \frac{1}{2} (-p(1-p)) \pi^T \sigma \sigma^T \pi \\ + \int_{\mathbb{R}^l} \left\{ (1 + \pi^T \mathbf{J}(\eta))^{1-p} - 1 \right\} \mathbf{q}(d\eta) \end{array} \right\}, & p > 1, \end{cases} \quad (3.10)$$

$$c_0 = \frac{1}{p} \left(\begin{array}{c} \beta - r(1-p) - (1-p) \alpha^T \pi_0 + \frac{1}{2} p(1-p) \pi_0^T \sigma \sigma^T \pi_0 \\ - \int_{\mathbb{R}^l} \left\{ (1 + \pi_0^T \mathbf{J}(\eta))^{1-p} - 1 \right\} \mathbf{q}(d\eta) \end{array} \right) \quad (3.11)$$

Remark 3.2.3. Note that because of the constraint $\pi_0 \in \Delta$, π_0 may well be obtained on the boundary, $(\pi_0)_i = 0$ for some $i \in \{F, 1, \dots, n\}$ or $\sum_i (\pi_0)_i = 1$. This is different from the classical Merton problem in which $\pi_0 \in \mathbb{R}^{n+1}$.

Remark 3.2.4. Recall that the assumption in (2.10) guarantees that the integral with respect to \mathbf{q} above is well-defined.

$$u_0 = \frac{1}{1-p} c_0^{-p} \quad (3.12)$$

Since u_0 in (3.12) is constant, we know that (3.10) and (3.11) are compatible with the feedback formulas (3.7) and (3.9).

We can also see from above that an additional constraint needs to be imposed on the parameters in order to obtain a finite value function. This is equivalent to c_0 in (3.11) being strictly positive, which translates to the following assumption

$$\begin{aligned} \beta &> r(1-p) + (1-p)\alpha^T \pi_0 - \frac{1}{2}p(1-p)\pi_0^T \sigma \sigma^T \pi_0 \\ &+ \int_{\mathbb{R}^l} \left\{ (1 + \pi_0^T \mathbf{J}(\eta))^{1-p} - 1 \right\} \mathbf{q}(d\eta). \end{aligned}$$

In order to compare with the case where there is no investment and only consumption, we also make the following assumption:

$$w_* \triangleq \frac{1}{1-p} \left(\frac{\beta}{p} - r \frac{1-p}{p} \right)^{-p} < u_0, \quad (3.13)$$

which is equivalent to

$$\pi_0^i > 0 \text{ for at least one } i \in \{F, 1, \dots, n\}. \quad (3.14)$$

because otherwise we would have $w_* = u_0$. The intuition behind this assumption is that we only consider a portfolio of risky assets worth investing, including the hedge fund share.

3.3 Main results

For fixed $c \geq 0$ and $\pi \in \Delta$, we denote by

$$\begin{aligned} \mathcal{L}_{c,\pi}[u](z) &\triangleq -\beta u + r \cdot D_x[u] + \left\{ \frac{c^{1-p}}{1-p} - c \cdot D_x[u] \right\} \\ &+ \left\{ + \int_{\mathbb{R}^l} \left\{ \begin{array}{c} \mathcal{B}^T[u] \pi + \frac{1}{2} \pi^T \mathcal{A}[u] \pi \\ 1 + \pi^T \mathbf{J}(\eta) \\ \left(-\frac{\lambda}{1+\lambda} [\pi^F \mathbf{J}^E(\eta) - z]^+ \right)^{1-p} \\ \cdot u \left(\frac{z - \pi^F \mathbf{J}^E(\eta) + [\pi^F \mathbf{J}^E(\eta) - z]^+}{1 + \pi^T \mathbf{J}(\eta) - \frac{\lambda}{1+\lambda} [\pi^F \mathbf{J}^E(\eta) - z]^+} \right) \\ -u(z) \end{array} \right\} \mathbf{q}(d\eta) \right\} \end{aligned}$$

The HJB equation for u can therefore be formally rewritten (with the implicit assumption that $D_x[u] > 0$) as

$$\left\{ \begin{array}{l} \sup_{c \geq 0, \pi \in \Delta} \mathcal{L}_{c,\pi} u = -\beta u + r \cdot D_x[u] + \tilde{V}(D_x[u]) \\ + \sup_{\pi \in \Delta} \left\{ \begin{array}{l} \mathcal{B}^T[u] \pi + \frac{1}{2} \pi^T \mathcal{A}[u] \pi \\ 1 + \pi^T \mathbf{J}(\eta) \\ \left(-\frac{\lambda}{1+\lambda} [\pi^F \mathbf{J}^E(\eta) - z]^+ \right)^{1-p} \\ \cdot u \left(\frac{z - \pi^F \mathbf{J}^E(\eta) + [\pi^F \mathbf{J}^E(\eta) - z]^+}{1 + \pi^T \mathbf{J}(\eta) - \frac{\lambda}{1+\lambda} [\pi^F \mathbf{J}^E(\eta) - z]^+} \right) \\ -u(z) \end{array} \right\} \mathbf{q}(d\eta) \right\} = 0, \\ z > 0, \\ -\lambda(1-p)u(0) + (1+\lambda)u'(0) = 0, \quad \lim_{z \rightarrow \infty} u(z) = \frac{1}{1-p} c_0^{-p}, \end{array} \right. \quad (3.15)$$

where $\tilde{V}(y) = \frac{p}{1-p} y^{\frac{p-1}{p}}$, $y > 0$.

Recall that w_* was defined in (3.13), and it is easy to see that w_* is the unique nontrivial solution to the equation

$$-\beta w + r(1-p)w + \tilde{V}((1-p)w) = 0.$$

This w_* plays an important role. We have

$$-\beta w + r(1-p)w + \tilde{V}((1-p)w) < 0, \quad w_* < w \leq u_0.$$

Then next theorem shows that the reduced HJB equation (3.15) has a classical solution which satisfies some additional properties.

Theorem 3.3.1. *There exists a strictly increasing function u which is C^2 on $[0, \infty)$, satisfies the condition $u(0) > w_*$ and*

$$(1-p)u - zu' > 0, \quad z \geq 0,$$

together with

$$u(z) \rightarrow u_0, \quad zu'(z), \quad z^2u''(z) \rightarrow 0 \quad \text{as } z \rightarrow \infty,$$

and is a solution to (3.15).

The proof of this above theorem is deferred to subsections 3.3.1 and 3.3.2. In subsection 3.3.1 we prove the existence of a viscosity solution using Perron's method, and in subsection 3.3.2 we upgrade its regularity.

The proposition below shows that the so-called closed-loop equation has a unique global solution, with its proof deferred to subsection 3.3.3.

Proposition 3.3.2. Fix $x > 0, y \geq 0$. Consider the feedback proportions $\hat{\pi}(z)$ and $\hat{c}(z)$ defined in (3.7) and (3.9), where u is the solution in Theorem 3.3.1. Define the feedback controls

$$\hat{\theta}(x, y) \triangleq x\hat{\pi}(y/x), \quad \hat{\gamma}(x, y) \triangleq x\hat{c}(y/x) \text{ for } x > 0, y \geq 0.$$

The closed-loop equation

$$\left\{ \begin{array}{l} X_t = x + \int_0^t rX_s ds + \int_0^t \hat{\theta}^F(X_{s-}, Y_{s-}) \left(\frac{dF_s}{F_{s-}} - r ds \right) \\ + \int_0^t \sum_{i=1}^n \hat{\theta}^i(X_{s-}, Y_{s-}) \left(\frac{dS_s^i}{S_{s-}^i} - r ds \right) - \int_0^t \hat{\gamma}(X_{s-}, Y_{s-}) ds - \lambda M_t, \\ Y_t = y - \int_0^t \hat{\theta}^F(X_{s-}, Y_{s-}) \left(\frac{dF_s}{F_{s-}} - \frac{dB_s}{B_{s-}} \right) + (1 + \lambda) M_t \\ \int_0^t \mathbf{1}_{\{Y_s > 0\}} dM_s = 0. \end{array} \right.$$

has a unique strong global solution (\hat{X}, \hat{Y}) such that $\hat{X} > 0$ and $\hat{Y} \geq 0$.

The next theorem addresses the optimality of the feedback controls, and its proof is deferred to subsection 3.3.3.

Theorem 3.3.3. Consider the solution u in Theorem 3.3.1. For each $x > 0, y \geq 0$, the feedback proportions $(\hat{\pi}, \hat{c})$ in (3.7) and (3.9) are optimal and

$$u\left(\frac{y}{x}\right) x^{1-p} \triangleq v(x, y) = V(x, y) \triangleq \sup_{(\theta, \gamma) \in \mathcal{A}(x, y)} \mathbb{E} \left[\int_0^\infty e^{-\beta t} U(\gamma_t) dt \right].$$

In addition to the results above in this section, we also give a proposition below which characterizes the properties of the feedback controls. More precisely, it analyzes the proportions $\hat{\pi}$ and \hat{c} , defined in (3.7) and (3.9), based on the solution u of the HJB given by Theorem 3.3.1. Proposition 3.3.4 is also used to prove the existence and uniqueness of the solution to the closed-loop equation in Proposition 3.3.2. To keep the presentation more streamlined, we relegate the proof of this proposition below to the Appendix.

Proposition 3.3.4. *The feedback controls $\widehat{\pi}$ and \widehat{c} satisfy*

$$0 < \widehat{c}(z) \rightarrow c_0, \quad 0 < \widehat{\pi}(z) \rightarrow \pi_0, \quad z \rightarrow \infty, \quad (3.16)$$

and

$$z\widehat{c}'(z) \rightarrow 0, \quad z\widehat{\pi}'(z) \rightarrow 0, \quad z \rightarrow \infty. \quad (3.17)$$

In addition,

$$\widehat{c}(z) > c_0 \text{ for } z \geq 1 \text{ if } p < 1 \text{ and } \widehat{c}(1) < c_0 \text{ if } p > 1. \quad (3.18)$$

3.3.1 Existence of a viscosity solution

As previously mentioned, the seminar paper [7] provides a good introduction to viscosity solutions for *local* equations. For the *non-local* equation (3.6) arising from our model, we adopt a definition of viscosity solutions from [5], though our definitions are slightly less general than that given in [5], since our value function is bounded. To start our definition of viscosity solutions, we consider the general equations written under the form

$$F(x, u, \Delta u, D^2u, I[x, u]) = 0 \text{ in a open domain } \Omega, \quad (3.19)$$

where F is a continuous function satisfying the local and non-local degenerate ellipticity conditions below in (3.20). The non-local term $I[x, u]$ can be quite general as seen in [5], a typical form of $I[x, u]$ is

$$I[x, u] = \int_{\mathbb{R}^d} (u(x+z) - u(x) - \nabla u(x) \cdot z \mathbf{1}_B(z)) \mu(z)$$

for some Lévy measure μ and some ball B centered at 0.

The *ellipticity assumption* of F means that: for any $x \in \Omega, u \in \mathbb{R}, p \in \mathbb{R}^d, M, N \in \mathbb{S}_d, l_1, l_2 \in \mathbb{R}$

$$F(x, u, p, M, l_1[x, u]) \leq F(x, u, p, N, l_2[x, u]) \text{ if } M \geq N, l_1 \geq l_2. \quad (3.20)$$

where \mathbb{S}_d denotes the space of real $N \times N$ symmetric matrices. Note that, apart from the usual ellipticity assumption for local equation, $F(x, u, p, M, l)$ is nondecreasing in the non-local operator l .

Let us now give a definition of viscosity solutions for the equation (3.19).

Definition 3.3.1. An upper semi-continuous and bounded function u is a viscosity subsolution of (3.19) if, for any bounded test function $\phi \in C^2(\Omega)$, if x is a *global* maximum point of $u - \phi$, then

$$F(x, u(x), \Delta\phi(x), D^2\phi(x), I[x, \phi]) \leq 0.$$

A lower semi-continuous and bounded function u is a viscosity supersolution of (3.19) if, for any bounded test function $\phi \in C^2(\Omega)$, if x is a *global* minimum point of $u - \phi$, then

$$F(x, u(x), \Delta\phi(x), D^2\phi(x), I[x, \phi]) \geq 0.$$

A function u is a viscosity solution of (3.19) if it is both a subsolution and supersolution.

It is also worth mentioning that boundary conditions used throughout our analysis can be interpreted in the classical sense.

Now we turn our attention back to the HJB equation (3.6). We observe that if $u(z) = u_0$ (defined in (3.12) in Remark 3.2.1), then

$$\begin{aligned}
& -\beta u_0 + r(1-p)u_0 + \tilde{V}((1-p)u_0) \\
& + \sup_{\pi \in \Delta} \left\{ \begin{aligned} & (1-p)u_0 \alpha^T \pi + \frac{1}{2} (-p(1-p)u_0) \pi^T \sigma \sigma^T \pi \\ & + \int_{\mathbb{R}^l} \left\{ \left(\left(1 + \pi^T \mathbf{J}(\eta) - \frac{\lambda}{1+\lambda} [\pi^F \mathbf{J}^E(\eta) - z]^+ \right)^{1-p} - 1 \right) u_0 \right\} \mathbf{q}(d\eta) \end{aligned} \right\} \\
& \leq -\beta u_0 + r(1-p)u_0 + \tilde{V}((1-p)u_0) \\
& + \sup_{\pi \in \Delta} \left\{ \begin{aligned} & (1-p)u_0 \alpha^T \pi + \frac{1}{2} (-p(1-p)u_0) \pi^T \sigma \sigma^T \pi \\ & + \int_{\mathbb{R}^l} \left\{ \left(1 + \pi^T \mathbf{J}(\eta) \right)^{1-p} - 1 \right\} u_0 \right\} \mathbf{q}(d\eta) \\ & = 0,
\end{aligned} \right.
\end{aligned}$$

and moreover u_0 is actually a classical supersolution of the HJB equation (3.6), which reads

$$\begin{aligned}
& \sup_{c \geq 0, \pi \in \Delta} \mathcal{L}_{c, \pi} u = -\beta u + r \cdot D_x[u] + \tilde{V}(D_x[u]) \\
& + \sup_{\pi \in \Delta} \left\{ \begin{aligned} & \mathcal{B}^T[u] \pi + \frac{1}{2} \pi^T \mathcal{A}[u] \pi \\ & + \int_{\mathbb{R}^l} \left\{ \begin{aligned} & \left(\frac{1 + \pi^T \mathbf{J}(\eta)}{-\frac{\lambda}{1+\lambda} [\pi^F \mathbf{J}^E(\eta) - z]^+} \right)^{1-p} \\ & \cdot u \left(\frac{z - \pi^F \mathbf{J}^E(\eta) + [\pi^F \mathbf{J}^E(\eta) - z]^+}{1 + \pi^T \mathbf{J}(\eta) - \frac{\lambda}{1+\lambda} [\pi^F \mathbf{J}^E(\eta) - z]^+} \right) \\ & - u(z) \end{aligned} \right\} \mathbf{q}(d\eta) \end{aligned} \right\} \leq 0, \\
& \quad z > 0, \\
& -\lambda(1-p)u(0) + (1+\lambda)u'(0) \leq 0, \quad \lim_{z \rightarrow \infty} u(z) \geq \frac{1}{1-p} c_0^{-p}.
\end{aligned} \tag{3.21}$$

For technical reasons we also need a subsolution with certain properties.

We remind the reader that the critical value w_* was defined in (3.13).

Proposition 3.3.5. *There exists a value $z_* \in (0, \infty)$ and a function*

$$u_s \in C^1[0, \infty) \cap C^2(0, z_*] \cap C^2[z_*, \infty)$$

such that

$$w_* - \xi \leq u_s \leq u_0$$

for some $\xi > 0$ (which can be arbitrarily small) and u_s is a strict viscosity subsolution for (3.15). More precisely, u_s satisfies

$$\sup_{c \geq 0, \pi \in \Delta} \mathcal{L}_{c, \pi} u_s > 0$$

in the viscosity sense on $(0, \infty)$, and

$$-\lambda(1-p)u_s(0) + (1+\lambda)u'_s(0) > 0, \quad \lim_{z \rightarrow \infty} u(z) < \frac{1}{1-p}c_0^{-p}.$$

Proof. For

$$\frac{\lambda}{1+\lambda}(1-p)w_* < a < (1-p)w_*,$$

we consider the function

$$u_s(z) = \begin{cases} w_* - \xi + az - \frac{2a}{1+\varepsilon}z^{1+\varepsilon}, & 0 \leq z \leq \left(\frac{1}{2}\right)^{\frac{1}{\varepsilon}} \\ w_* - \xi + a\frac{e}{1+\varepsilon}\left(\frac{1}{2}\right)^{\frac{1}{\varepsilon}}, & z \geq \left(\frac{1}{2}\right)^{\frac{1}{\varepsilon}} \end{cases}$$

for some small $\xi > 0$. Since

$$\begin{aligned} & \tilde{V} \left((1-p) \left(w_* - \xi + az - \frac{2a}{1+\varepsilon}z^{1+\varepsilon} \right) - a(z - 2z^{1+\varepsilon}) \right) \\ & \geq \tilde{V}((1-p)(w_* - \xi)) + \tilde{V}'((1-p)(w_* - \xi)) \\ & \cdot \left((1-p) \left(az - \frac{2a}{1+\varepsilon}z^{1+\varepsilon} \right) - a(z - 2z^{1+\varepsilon}) \right). \end{aligned}$$

We notice that

$$\begin{aligned}
& \sup_{c \geq 0, \pi \in \Delta} \mathcal{L}_{c, \pi} u_s(z) \\
& \geq -\beta(w_* - \xi) + r(1-p)(w_* - \xi) + \tilde{V}((1-p)(w_* - \xi)) \\
& - \beta \left(az - \frac{2a}{1+\varepsilon} z^{1+\varepsilon} \right) + r \left((1-p) \left(az - \frac{2a}{1+\varepsilon} z^{1+\varepsilon} \right) - a(z - 2z^{1+\varepsilon}) \right) \\
& + \tilde{V}'((1-p)(w_* - \xi)) \left((1-p) \left(az - \frac{2a}{1+\varepsilon} z^{1+\varepsilon} \right) - a(z - 2z^{1+\varepsilon}) \right) \\
& + \sup_{\pi \in \Delta} \left\{ \begin{array}{c} \mathcal{B}^T[u_s] \pi + \frac{1}{2} \pi^T \mathcal{A}[u_s] \pi \\ \left(\begin{array}{c} 1 + \pi^T \mathbf{J}(\eta) \\ -\frac{\lambda}{1+\lambda} [\pi^F \mathbf{J}^E(\eta) - z]^+ \end{array} \right)^{1-p} \\ + \int_{\mathbb{R}^l} \left\{ \begin{array}{c} u_s \left(\frac{z - \pi^F \mathbf{J}^E(\eta) + [\pi^F \mathbf{J}^E(\eta) - z]^+}{1 + \pi^T \mathbf{J}(\eta) - \frac{\lambda}{1+\lambda} [\pi^F \mathbf{J}^E(\eta) - z]^+} \right) \\ - u_s(z) \end{array} \right\} \mathbf{q}(d\eta) \end{array} \right\} \\
& \geq -\beta(w_* - \xi) + r(1-p)(w_* - \xi) + \tilde{V}((1-p)(w_* - \xi)) \\
& - Cz - Dz^{1+\varepsilon}
\end{aligned}$$

where the last inequality follows from setting $\pi = \mathbf{0}$, and C, D are some constants. For $\xi > 0$ fixed,

$$-\beta(w_* - \xi) + r(1-p)(w_* - \xi) + \tilde{V}((1-p)(w_* - \xi)) > 0.$$

So, if ε is sufficiently small,

$$\begin{aligned}
& -\beta(w_* - \xi) + r(1-p)(w_* - \xi) + \tilde{V}((1-p)(w_* - \xi)) \\
& - C(z-1) - D(z-1)^{1+\varepsilon} \\
& \geq -\beta(w_* - \xi) + r(1-p)(w_* - \xi) + \tilde{V}((1-p)(w_* - \xi)) \\
& - |C|(z-1) - |D|(z-1)^{1+\varepsilon} \\
& > 0 \text{ for } \forall 0 < z \leq \left(\frac{1}{2}\right)^{\frac{1}{\varepsilon}}.
\end{aligned}$$

Therefore, for such an ε we will have

$$\sup_{c \geq 0, \pi \in \Delta} \mathcal{L}_{c, \pi} u_s(z) > 0, \quad 0 < z \leq \left(\frac{1}{2}\right)^{\frac{1}{\varepsilon}}.$$

Since u_s is constant for $z \geq \left(\frac{1}{2}\right)^{\frac{1}{\varepsilon}}$ and is extended to be C^1 we obtain

$$\sup_{c \geq 0, \pi \in \Delta} \mathcal{L}_{c, \pi} u_s(z) > 0, \quad z > 0,$$

in the viscosity sense and actually in the classical sense for any $z \neq z_* \triangleq \left(\frac{1}{2}\right)^{\frac{1}{\varepsilon}}$. \square

We now construct a viscosity solution of the HJB equation (3.15) using Perron's method. More precisely, we denote by \mathcal{S} the set of functions

$$h : [0, \infty) \rightarrow \mathbb{R},$$

which satisfy the following properties:

1. h is continuous on $[0, \infty)$.
2. The function $(x, y) \rightarrow x^{1-p}h(y/x)$ is both concave and nondecreasing in the direction $\kappa = \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ (from upper left to lower right) within its domain $x > 0, y \geq 0$; for fixed x , the function $y \rightarrow x^{1-p}h(y/x)$ is concave and nondecreasing in $y \geq 0$.
3. h is a viscosity supersolution of the HJB equation on the open interval $(0, \infty)$.
4. $-\lambda(1-p)h(0) + (1+\lambda)h'(0) \leq 0$.

5. $u_s \leq h \leq u_0$.

Remark 3.3.1. Note that 2 and 5 above would imply that $h(z)$, $h(z-)$, $h(z+)$ are bounded. Together with the technical assumption in (2.10), it ensures that when plugging h into the HJB equation, the integral term is well-defined.

Theorem 3.3.6. *Define*

$$u \triangleq \inf \{h, h \in \mathcal{S}\}.$$

Then, $u_s \leq h \leq u_0$ is continuous on $[0, \infty)$, is a viscosity solution of the HJB equation on the open interval $(0, \infty)$, and satisfies $-\lambda(1-p)h(0) + (1+\lambda)h'(0) = 0$. In addition, The function $(x, y) \rightarrow x^{1-p}u(y/x)$ is concave and nondecreasing in the direction $\kappa = \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ (from upper left to lower right) within its domain $x > 0, y \geq 0$; for fixed x , the function $y \rightarrow x^{1-p}u(y/x)$ is concave and nondecreasing in $y \geq 0$, and $u(1) > w_$.*

Remark 3.3.2. As a consequence of our construction in Theorem 3.3.6, we have $u(z) \leq u_0(z)$ and therefore $v(x, y) \leq v_0(x)$. This means that, with high-watermark fees, the value function is always smaller than the value function of the Merton problem without fees ($\lambda = 0$). This is also expected from the financial intuition: high-watermark fee is a kind of market friction, thereby reducing the maximum expected utility an investor can achieve.

Proof. We follow the ideas of the proof of Proposition 1 in [2], with necessary modifications to take into account the boundary condition at $z = 0$ and to keep track of the convexity properties.

1. By construction, as an infimum of concave nondecreasing functions, we have that the function $(x, y) \rightarrow x^{1-p}u(y/x)$ is concave and nondecreasing in the direction $\kappa = \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ (from upper left to lower right) within its domain $x > 0, y \geq 0$, and for fixed x , the function $y \rightarrow x^{1-p}u(y/x)$ is concave and nondecreasing in $y \geq 0$.
2. Since $x \rightarrow x^{1-p}u(y/x)$ is concave and nondecreasing in the direction $\kappa = \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ we conclude that $x \rightarrow x^{1-p}u(y/x)$ is continuous in the direction $\kappa = \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$, which translates to that u is continuous in $[0, \infty)$.
3. We suppose that a C^2 function φ touches u from below at an interior point $z \in (0, \infty)$. For fixed c, π , each $h \in \mathcal{S}$ is a viscosity supersolution of $\mathcal{L}_{c,\pi}h \leq 0$, so by taking the infimum over $h \in \mathcal{S}$ we still get a supersolution, according to Proposition 1 in [2]. In other words, $(\mathcal{L}_{c,\pi}\varphi)(z) \leq 0$, and then we can take the supremum over (c, π) to get that u is a supersolution of the HJB equation.
4. By construction, $u_s \leq u \leq u_0$.
5. For each $h \in \mathcal{S}$, we have

$$h'(0) \leq \frac{\lambda}{1+\lambda} (1-p) h(0).$$

which translates in terms of $g(x, y) \triangleq x^{1-p}h(y/x)$ as

$$\nabla_{\kappa} g(1, 0) = \frac{\sqrt{2}}{2} ((1-p)h(0) - h'(0)) \geq \frac{\sqrt{2}}{2} \frac{1}{1+\lambda} (1-p)h(0).$$

Taking into account the concavity of $g(x, y)$ along the line $x - y = 1$ within its domain $x > 0, y \geq 0$, this is equivalent to

$$\begin{aligned} & g(1 - \xi, \xi) \\ &= (1 - \xi)^{1-p} h(\xi / (1 - \xi)) \\ &\leq \frac{\sqrt{2}}{2} \frac{1}{1 + \lambda} (1 - p) h(0) \cdot \sqrt{2}\xi + h(0), \quad 0 \leq \xi < 1. \end{aligned} \quad (3.22)$$

Since (3.22) holds for each $h \in \mathcal{S}$ the same inequality will hold for the infimum, which means that u satisfies (3.22), which reads

$$u'(0) \leq \frac{\lambda}{1 + \lambda} (1 - p) u(0).$$

Let us show that u is a viscosity subsolution. We start by making the following simple observation on the function u : By construction, the function $(x, y) \rightarrow x^{1-p}u(y/x)$ is concave in the direction $\kappa = \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ within its domain $x > 0, y \geq 0$. We denote the one-sided directional derivative by $\nabla_{\kappa-}v$ ($\nabla_{\kappa+}v$) when (x, y) approaching from upper left to lower right (lower right to upper left). Since

$$\begin{aligned} \nabla_{\kappa-}v(x, y) &= \frac{\sqrt{2}}{2} x^{-p} \cdot ((1 - p) u(z) - (z + 1) u'(z+)), \\ \nabla_{\kappa+}v(x, y) &= \frac{\sqrt{2}}{2} x^{-p} \cdot ((1 - p) u(z) - (z + 1) u'(z-)), \end{aligned}$$

we obtain

$$(1 - p) u(z) - (z + 1) u'(z+) \geq (1 - p) u(z) - (z + 1) u'(z-), \quad z > 0,$$

which of course means that $u'(z-) \geq u'(z+)$ for $z > 0$. Suppose, on the contrary, that for some $z_0 > 0$ we have $(1 - p) u(z) - (z + 1) u'(z-) = 0$.

Then, we have that $\nabla_{\kappa+} v \left(\frac{1}{z_0}, 1 \right) = 0$, which, together with the fact that $\xi \rightarrow v \left(\frac{1}{z_0} + \xi, 1 - \xi \right)$ is concave and nondecreasing for $\xi \in [0, 1]$, shows that $v \left(\frac{1}{z_0} + \xi, 1 - \xi \right) = v \left(\frac{1}{z_0} + 1, 0 \right)$ for $\xi \in [0, 1]$. This means that

$$\nabla_{\kappa} v \left(\frac{1}{z_0} + 1, 0 \right) = \frac{\sqrt{2}}{2} \left(\frac{1}{z_0} + 1 \right)^{-p} \cdot ((1-p)u(0) - u'(0)) = 0,$$

which is a contradiction to the boundary condition

$$u'(0) \leq \frac{\lambda}{1+\lambda} (1-p)u(0).$$

Therefore, for any $z > 0$ we have

$$(1-p)u(z) - (z+1)u'(z+) \geq (1-p)u(z) - (z+1)u'(z-) > 0. \quad (3.23)$$

Assume now that a C^2 function φ touches u from above at some interior point $z \in (0, \infty)$. If $u(z) = u_s(z)$ we can use the test function u_s (which is a strict subsolution) for the supersolution u to obtain a contradiction. The contradiction argument works even if $z = (1/2)^{1/\varepsilon}$ is the only exceptional point where u_s is not C^2 . Therefore, $u(z) > u_s(z)$. From (3.23) we can easily conclude that

$$(1-p)\varphi(z) - (z+1)\varphi'(z) > 0.$$

Assume now that u does *not* satisfy the subsolution property, which translates to

$$\sup_{c \geq 0, \pi \in \Delta} (\mathcal{L}_{c,\pi}\varphi)(z) < 0. \quad (3.24)$$

Since $(1-p)\varphi(z) - (z+1)\varphi'(z) > 0$ we can conclude that

$$(1-p)\varphi(z) - z\varphi'(z) > 0, \quad (3.25)$$

and

$$+ \sup_{\pi \in \Delta} \left\{ \begin{array}{l} -\beta\varphi + r \cdot D_x[\varphi] + \tilde{V}(D_x[\varphi]) \\ \mathcal{B}^T[\varphi] \pi + \frac{1}{2} \pi^T \mathcal{A}[\varphi] \pi \\ \left(\begin{array}{l} 1 + \pi^T \mathbf{J}(\eta) \\ -\frac{\lambda}{1+\lambda} [\pi^F \mathbf{J}^E(\eta) - z]^+ \end{array} \right)^{1-p} \\ \cdot \varphi \left(\begin{array}{l} z - \pi^F \mathbf{J}^E(\eta) + [\pi^F \mathbf{J}^E(\eta) - z]^+ \\ 1 + \pi^T \mathbf{J}(\eta) - \frac{\lambda}{1+\lambda} [\pi^F \mathbf{J}^E(\eta) - z]^+ \end{array} \right) \\ -\varphi(z) \end{array} \right\} \mathbf{q}(d\eta) \left\} < 0 \quad (3.26)$$

where $D_x[\varphi] = (1-p)\varphi(z) - z\varphi'(z)$.

Because the supremum is taken over a compact set and thus the left-hand side of the above equation is continuous in z , relations (3.25) or (3.26) actually hold in a small neighborhood $(z-\delta, z+\delta)$ of z , not just at z . Considering an even smaller δ we have that $u(\omega) < \varphi(\omega)$ for $\omega \in [z-\delta, z+\delta]$ if $\omega \neq z$. Now, for ε small enough, which means at least as small as

$$\varepsilon_0 \triangleq \min_{\frac{\delta}{2} \leq |\omega-z| \leq \delta} \varphi(\omega) - u(\omega),$$

but maybe much smaller, we define the function

$$\tilde{u}(\omega) \triangleq \begin{cases} \min\{u(\omega), \varphi(\omega) - \varepsilon\}, & \omega \in [z-\delta, z+\delta], \\ u(\omega), & \omega \notin [z-\delta, z+\delta]. \end{cases}$$

We note that if ε is indeed small enough, we have that $\tilde{u} \in \mathcal{S}$ and \tilde{u} is strictly smaller than u (around z), which is a contradiction.

6. From above, we already know that

$$u'(0) \leq \frac{\lambda}{1+\lambda} (1-p) u(0).$$

Let us now prove the above inequality is actually an equality. Assume now that the inequality above is strict. Since $u'_s(0) = a > \frac{\lambda}{1+\lambda} (1-p) u_s(0)$ and $u \geq u_s$, this rules out the possibility that $u(0) = u_s(0)$, so we have $u(0) > u_s(0)$. Also because $u_s(0) = w_* - \xi$ for arbitrarily small $\xi > 0$, we have $u(0) > w_*$. This implies

$$-\beta u(0) + r(1-p)u(0) + \tilde{V}((1-p)u(0)) < 0.$$

Recall that for fixed x , the function $y \rightarrow x^{1-p}u(y/x)$ is concave, this means u is concave and therefore two times differentiable on a dense set of $(0, \infty)$. Then, we can find $z_0 \in (0, \infty)$ very close to 0 such that

$$-\beta u(z_0) + r \cdot D_x[u(z_0)] + \tilde{V}(D_x[u(z_0)]) < 0$$

and $u(z)$ solves the HJB equation (3.15) at $z = z_0$ in the classical sense.

More precisely, we have

$$+\sup_{\pi \in \Delta} \left\{ \begin{array}{l} -\beta u(z_0) + r \cdot D_x[u(z_0)] + \tilde{V}(D_x[u(z_0)]) \\ \mathbf{B}^T[u(z_0)]\pi + \frac{1}{2}\pi^T \mathcal{A}[u(z_0)]\pi \\ \left(\begin{array}{l} 1 + \pi^T \mathbf{J}(\eta) \\ -\frac{\lambda}{1+\lambda} [\pi^F \mathbf{J}^E(\eta) - z_0]^+ \end{array} \right)^{1-p} \\ \cdot u \left(\frac{z_0 - \pi^F \mathbf{J}^E(\eta) + [\pi^F \mathbf{J}^E(\eta) - z_0]^+}{1 + \pi^T \mathbf{J}(\eta) - \frac{\lambda}{1+\lambda} [\pi^F \mathbf{J}^E(\eta) - z_0]^+} \right) \\ -u(z_0) \end{array} \right\} \mathbf{q}(d\eta) \Bigg\} = 0.$$

and the supremum part of the above equation is strictly positive, which means $\hat{\pi}(z_0) \neq \mathbf{0}$. With $u'(z_0)$ being very close to $u'(0)$, we can find a

number a' such that

$$u'(z_0) \leq u'(0) < a' < \frac{\lambda}{1+\lambda} (1-p) u(0).$$

and without loss of generality, we also choose a' to be very close to $u'(z_0)$.

Moreover, we observe that the left-hand side of the above equation is continuous in both u' and u'' , and increasing in u'' given $\widehat{\pi}(z_0) \neq \mathbf{0}$, since each element of $\mathcal{A}[u]$ is increasing in u'' . This allows us to choose a (possibly very large) $b > 0$ together with a' above such that the function

$$\psi(z) = u(0) + a'z - \frac{1}{2}bz^2$$

is a classical strict supersolution at $z = z_0$. More precisely, we have

$$+ \sup_{\pi \in \Delta} \left\{ \begin{array}{l} -\beta\psi(z_0) + r \cdot D_x[\psi(z_0)] + \widetilde{V}(D_x[\psi(z_0)]) \\ \mathcal{B}^T[\psi(z_0)]\pi + \frac{1}{2}\pi^T \mathcal{A}[\psi(z_0)]\pi \\ \left(\frac{1 + \pi^T \mathbf{J}(\eta)}{-\frac{\lambda}{1+\lambda} [\pi^F \mathbf{J}^E(\eta) - z_0]^+} \right)^{1-p} \\ \cdot \psi \left(\frac{z_0 - \pi^F \mathbf{J}^E(\eta) + [\pi^F \mathbf{J}^E(\eta) - z_0]^+}{1 + \pi^T \mathbf{J}(\eta) - \frac{\lambda}{1+\lambda} [\pi^F \mathbf{J}^E(\eta) - z_0]^+} \right) \\ -\psi(z_0) \end{array} \right\} \mathbf{q}(d\eta) < 0.$$

Then, continuity would imply that $\psi(z)$ is actually a classical strict supersolution in a small neighborhood $(0, \delta)$ of $z = 0$. In addition, it satisfies $\psi'(0) = a' < \frac{\lambda}{1+\lambda} (1-p) \psi(0)$. Thus, if δ is small enough, we have that $u(z) < \psi(z)$ on $(0, \delta)$, and

$$(1-p)\psi(z) - z\psi'(z) > 0, \quad z \in [0, \delta].$$

Now, for a very small ε , at least as small as

$$\varepsilon_0 \triangleq \min_{z \in [1+\frac{\varepsilon}{2}, 1+\delta]} \psi(z) - u(z)$$

but possibly even smaller, we have that the function

$$\tilde{u}(z) \triangleq \begin{cases} \min \{u(z), \psi(z) - \varepsilon\}, & z \in [0, \delta], \\ u(z), & z \in [\delta, \infty), \end{cases}$$

is actually an element of \mathcal{S} , contradicting with the assumption that u is the infimum over \mathcal{S} .

□

3.3.2 Smoothness of the viscosity solution

Theorem 3.3.7. *The function u in Theorem 3.3.6 is C^2 on $[0, \infty)$ and satisfies the conditions*

$$(1 - p)u(z) - zu'(z) > 0, \quad z \geq 0.$$

Moreover, it is a solution of the equation

$$\left\{ \begin{array}{l} \sup_{c>0, \pi \in \Delta} \mathcal{L}_{c, \pi} u = -\beta u + r \cdot (D_x [u]) + \tilde{V}(D_x [u]) \\ \mathcal{B}^T [u] \pi + \frac{1}{2} \pi^T \mathcal{A} [u] \pi \\ + \sup_{\pi \in \Delta} \left\{ + \int_{\mathbb{R}^l} \left\{ \begin{array}{l} \left(\frac{1 + \pi^T \mathbf{J}(\eta)}{-\frac{\lambda}{1+\lambda} [\pi^F \mathbf{J}^E(\eta) - z]^+} \right)^{1-p} \\ \cdot u \left(\frac{z - \pi^F \mathbf{J}^E(\eta) + [\pi^F \mathbf{J}^E(\eta) - z]^+}{1 + \pi^T \mathbf{J}(\eta) - \frac{\lambda}{1+\lambda} [\pi^F \mathbf{J}^E(\eta) - z]^+} \right) \\ -u(z) \\ z > 0, \\ -\lambda(1-p)u(0) + (1+\lambda)u'(0) = 0. \end{array} \right\} \mathbf{q}(d\eta) \right\} = 0, \end{array} \right.$$

Proof. First, we point out that the dual function $\tilde{V}(y)$ is defined for all values of y , not only $y > 0$. More precisely,

$$\tilde{V}(y) = \begin{cases} \frac{p}{1-p} y^{\frac{p-1}{p}}, & y > 0, \\ +\infty, & y \leq 0 \end{cases} \quad \text{for } p < 1, \quad \tilde{V}(y) = \begin{cases} \frac{p}{1-p} y^{\frac{p-1}{p}}, & y \geq 0, \\ +\infty, & y < 0 \end{cases} \quad \text{for } p > 1.$$

Let $z_0 > 0$ such that $u'(z_0-) > u'(z_0+)$. For each $u'(z_0+) < a < u'(z_0-)$ and $b > 0$ very large we use the function

$$\psi(z) \triangleq u(z_0) + a(z - z_0) - \frac{1}{2}b(z - z_0)^2$$

as a test function at $z = z_0$ for the viscosity subsolution property, so

$$\sup_{c \geq 0, \pi \in \Delta} \mathcal{L}_{c, \pi} \psi \geq 0.$$

Since $(1 - p)u(z_0) - z_0a > (1 - p)u(z_0) - z_0u'(z_0-) > 0$ the above equation can be rewritten as

$$-\beta u(z_0) + r((1 - p)u(z_0) - z_0a) + \tilde{V}((1 - p)u(z_0) - z_0a) + \sup_{\pi \in \Delta} \left\{ \begin{array}{l} \mathbf{B}^T [\psi(z_0)] \pi + \frac{1}{2} \pi^T \mathcal{A} [\psi(z_0)] \pi \\ \int_{\mathbb{R}^l} \left\{ \begin{array}{l} \left(\frac{1 + \pi^T \mathbf{J}(\eta)}{-\frac{\lambda}{1+\lambda} [\pi^F \mathbf{J}^E(\eta) - z_0]^+} \right)^{1-p} \\ \cdot \psi \left(\frac{z_0 - \pi^F \mathbf{J}^E(\eta) + [\pi^F \mathbf{J}^E(\eta) - z_0]^+}{1 + \pi^T \mathbf{J}(\eta) - \frac{\lambda}{1+\lambda} [\pi^F \mathbf{J}^E(\eta) - z_0]^+} \right) \\ -u(z_0) \end{array} \right\} \mathbf{q}(d\eta) \end{array} \right\} \geq 0.$$

We note that the above inequality holds even when $b \rightarrow \infty$ and the left-hand side is decreasing in b given $\hat{\pi}(z_0) \neq \mathbf{0}$, hence we must have $\hat{\pi}(z_0) = \mathbf{0}$. This implies

$$\begin{aligned} & -\beta u(z_0) + r((1 - p)u(z_0) - z_0a) \\ & + \tilde{V}((1 - p)u(z_0) - z_0a) \\ & \geq 0, \quad a \in (u'(z_0+), u'(z_0-)). \end{aligned}$$

It is easy to see that the function $g(a) \triangleq -\beta u(z_0) + r((1 - p)u(z_0) - z_0a) + \tilde{V}((1 - p)u(z_0) - z_0a)$ is not flat on any nontrivial interval within its domain,

we must have

$$\begin{aligned}
& -\beta u(z_0) + r((1-p)u(z_0) - z_0 a) \\
& + \tilde{V}((1-p)u(z_0) - z_0 a) \\
& > 0 \text{ for some } a \in (u'(z_0+), u'(z_0-)).
\end{aligned}$$

and we can also assume, without loss of generality, that a is very close to $u'(z_0-)$. So

$$-\beta u(z_0) + r((1-p)u(z_0) - z_0 u'(z_0-)) + \tilde{V}((1-p)u(z_0) - z_0 u'(z_0-)) > 0.$$

Since $u'(z-)$ is left continuous, and the function u is two times differentiable on a dense set $\mathcal{D} \subset (0, \infty)$ by convexity, there exists $z > 0$ very close to z_0 such that $z \in \mathcal{D}$, and

$$-\beta u(z) + r((1-p)u(z) - zu'(z)) + \tilde{V}((1-p)u(z) - zu'(z)) > 0.$$

However, this would contradict with the viscosity supersolution property at z , which reads

$$\begin{aligned}
& -\beta u(z) + r((1-p)u(z) - zu'(z)) + \tilde{V}((1-p)u(z) - zu'(z)) \\
& + \sup_{\pi \in \Delta} \left\{ + \int_{\mathbb{R}^l} \left\{ \begin{aligned} & \mathcal{B}^T[u(z)]\pi + \frac{1}{2}\pi^T \mathcal{A}[u(z)]\pi \\ & \left(\frac{1 + \pi^T \mathbf{J}(\eta)}{-\frac{\lambda}{1+\lambda} [\pi^F \mathbf{J}^E(\eta) - z]^+} \right)^{1-p} \\ & \cdot \psi \left(\frac{z - \pi^F \mathbf{J}^E(\eta) + [\pi^F \mathbf{J}^E(\eta) - z]^+}{1 + \pi^T \mathbf{J}(\eta) - \frac{\lambda}{1+\lambda} [\pi^F \mathbf{J}^E(\eta) - z]^+} \right) \\ & -u(z) \end{aligned} \right\} \mathbf{q}(d\eta) \right\} \leq 0,
\end{aligned}$$

since the supremum part of the left-hand side above is always non-negative.

We obtained a contradiction, so we have proved that

$$u'(z_0-) = u'(z_0+) \quad \forall z_0 > 0.$$

In other words, u' is well defined and continuous on $[0, \infty)$. In addition, $(1-p)u(z) - zu'(z) > 0$ for $z \geq 0$. Applying again the viscosity solution property at a point where u is two times differentiable we obtain

$$-\beta u(z) + r((1-p)u(z) - zu'(z)) + \tilde{V}((1-p)u(z) - zu'(z)) \leq 0, \quad z \in \mathcal{D}.$$

Using continuity and the density of \mathcal{D} , we get

$$\begin{aligned} f(z) & \triangleq \beta u(z) - r((1-p)u(z) - zu'(z)) \\ & \quad - \tilde{V}((1-p)u(z) - zu'(z)) \\ & \geq 0, \quad z \geq 0. \end{aligned} \tag{3.27}$$

The function f defined in (3.27) is continuous. Also, $u(0) > w_*$ as seen in Theorem 3.3.1, and u is nondecreasing since $y \rightarrow x^{1-p}u(y/x)$ is nondecreasing in $y \in [0, \infty)$, it follows that $u(z) > w_*$ for all $z \geq 0$. Hence, $\hat{\pi} \neq \mathbf{0}$ on $[a, b]$ for any open interval $(a, b) \subset [0, \infty)$. Therefore, $f(z) > 0$ on $[a, b]$ due to the HJB equation. Now, rewrite the HJB equation (3.15) in the following form,

$$H(z, u'') = 0$$

where H is continuous and strictly increasing in its second variable. Note that H depends on its first variable z through $u(z)$ and $u'(z)$, which are continuous. This implies that the HJB equation can further be rewritten as

$$u'' = h(z)$$

where h is continuous. Now, u is a viscosity solution of the equation

$$u - u'' = u - h(z), \quad z \in (a, b) \subset [0, \infty),$$

and the right-hand side is continuous in z on $[a, b]$. Comparing to the classical solution of this equation with the very same right-hand side and Dirichlet boundary conditions at a and b , we get that u is C^2 on $[a, b]$. We point out that the comparison argument between the viscosity solution and the classical solution is straightforward and does not involve any doubling argument.

Therefore u is C^2 on $(1, \infty)$ and satisfies the HJB equation. Since $u(1) > w_*$ which reads $f(1) > 0$, for f defined in (3.27), we can then use continuity and pass to the limit in the HJB equation for $z \searrow 1$ to conclude that u is C^2 in $[1, \infty)$ and the HJB equation is satisfied at the boundary as well. \square

Lemma 3.3.8. *The function u is strictly increasing on $[0, \infty)$ and*

$$\lim_{z \rightarrow \infty} u(z) = u_0, \quad \lim_{z \rightarrow \infty} zu'(z) = 0, \quad \lim_{z \rightarrow \infty} z^2 u''(z) = 0.$$

Proof. Recall that $y \rightarrow x^{1-p}u(y/x)$ is concave and nondecreasing in $y \in [0, \infty)$, this means u is concave and nondecreasing. Since u is nondecreasing and bounded, there exists

$$u(\infty) \triangleq \lim_{z \rightarrow \infty} u(z) \in (-\infty, \infty).$$

Now, since u is bounded and u' is continuous we conclude, by contradiction, that there exists a sequence $z_n \nearrow \infty$ such that

$$z_n u'(z_n) \rightarrow 0, \quad n \rightarrow \infty.$$

(Otherwise we would have $zu'(z) \geq \varepsilon$ for some ε for large z , which contradicts boundedness.) We let

$$0 \geq A := \liminf_{z \rightarrow \infty} zu'(z) \leq \limsup_{z \rightarrow \infty} zu'(z) =: B \geq 0.$$

For fixed $C \in \mathbb{R}$, denote by

$$f_C(z) = Cu + zu', \quad z \geq 1.$$

The function f_C is continuous and

$$\liminf_{z \rightarrow \infty} f_C(z) = Cu(\infty) + A \leq Cu(\infty) + B = \limsup_{z \rightarrow \infty} f_C(z).$$

Assume, on the contrary, that $0 < B \leq \infty$. Since $\lim_{n \rightarrow \infty} f_C(z_n) = Cu(\infty) < Cu(\infty) + B$, we can choose the points $\eta_n \in (z_n, z_{n+1})$ (interior points, and eventually for a subsequence n_k rather than for each n) for which f_C attains the maximum on $[z_n, z_{n+1}]$ such that $f_C(\eta_n) \rightarrow Cu(\infty) + B$, which is the same as $\eta_n u'(\eta_n) \rightarrow B$. Since f_C attains the interior maximum on each interval at η_n , we have $f'_C(\eta_n) = (1 + C)u'(\eta_n) + \eta_n u''(\eta_n) = 0$. Recall that $x \rightarrow x^{1-p}u(y/x)$ is concave in $x \in (0, n]$, which implies that

$$-p(1-p)u(\eta_n) + 2p\eta_n u'(\eta_n) + \eta_n^2 u''(\eta_n) \leq 0,$$

or

$$-p(1-p)u(\eta_n) + (2p-1-C)\eta_n u'(\eta_n) \leq 0.$$

Passing to the limit, we obtain that

$$-p(1-p)u(\infty) + (2p-1-C)B \leq 0$$

for each $C \in \mathbb{R}$, which means that $B = 0$. Similarly, we obtain $A = 0$ so $zu'(z) \rightarrow 0$. Now, since zu' is bounded and $(zu)'$ is continuous we conclude, by contradiction, that there exists a sequence $z_n \nearrow \infty$ such that

$$(z_n)^2 u''(z_n) \rightarrow 0, \quad n \rightarrow \infty.$$

Passing to the limit along z_n 's in the HJB equation, we obtain

$$\begin{aligned} & -\beta u(\infty) + r(1-p)u(\infty) + \tilde{V}((1-p)u(\infty)) \\ & + \sup_{\pi \in \Delta} \left\{ (1-p)u(\infty) \alpha^T \pi + \frac{1}{2}(-p(1-p)u(\infty)) \pi^T \sigma \sigma^T \pi \right. \\ & \quad \left. + \int_{\mathbb{R}^l} \left\{ \left((1 + \pi^T \mathbf{J}(\eta))^{1-p} - 1 \right) u(\infty) \right\} \mathbf{q}(d\eta) \right\} \\ & = 0. \end{aligned} \tag{3.28}$$

As already pointed out, the above equation has a unique solution $u(\infty)$ in $[w_*, u_0]$, namely, $u(\infty) = u_0$ so $u(z) \rightarrow u_0$ as $z \rightarrow \infty$. Going back to the ODE for all $z \rightarrow \infty$ and not only along the subsequence, we obtain $z^2 u''(z) \rightarrow 0$ as well.

Now we show that u is strictly increasing. Suppose otherwise, since u is nondecreasing and concave, it is only possible that $u(z) = u(\infty)$ for $z \geq z_0$ for some $z_0 > 0$. Plugging $u(z_0) = u(\infty)$ into the HJB equation we have

$$\begin{aligned} & -\beta u(\infty) + r(1-p)u(\infty) + \tilde{V}((1-p)u(\infty)) \\ & + \sup_{\pi \in \Delta} \left\{ (1-p)u(\infty) \alpha^T \pi + \frac{1}{2}(-p(1-p)u(\infty)) \pi^T \sigma \sigma^T \pi \right. \\ & \quad \left. + \int_{\mathbb{R}^l} \left\{ \left(\left(\frac{1 + \pi^T \mathbf{J}(\eta)}{-\frac{\lambda}{1+\lambda} [\pi^F \mathbf{J}^E(\eta) - z_0]^+} \right)^{1-p} - 1 \right) u(\infty) \right\} \mathbf{q}(d\eta) \right\} \\ & = 0, \end{aligned}$$

which is a contradiction with (3.28). Therefore, u is strictly increasing. \square

3.3.3 Optimal policies and verification

Proposition 3.3.9. *Let $\theta(x, y)$ and $\gamma(x, y)$ be two Lipschitz functions in both arguments on the two-dimensional domain*

$\{(x, y) \in \mathbb{R}^2; x > 0, y \geq 0\}$. The closed-loop state equation (2.11) corresponding to $\theta_s = \theta(X_{s-}, Y_{s-})$ and $\gamma_s = \gamma(X_{s-}, Y_{s-})$, which means the SDE

$$\left\{ \begin{array}{l} X_t = x + \int_0^t r X_s ds + \int_0^t \theta^F(X_{s-}, Y_{s-}) \left(\frac{dF_s}{F_{s-}} - r ds \right) \\ \quad + \int_0^t \sum_{i=1}^n \theta^i(X_{s-}, Y_{s-}) \left(\frac{dS_s^i}{S_{s-}^i} - r ds \right) - \int_0^t \gamma(X_{s-}, Y_{s-}) ds - \lambda M_t, \\ Y_t = y - \int_0^t \theta^F(X_{s-}, Y_{s-}) \left(\frac{dF_s}{F_{s-}} - \frac{dB_s}{B_{s-}} \right) + (1 + \lambda) M_t \\ \quad \int_0^t \mathbf{1}_{\{Y_s > 0\}} dM_s = 0. \end{array} \right.$$

has a unique strong solution (X, Y) .

Proof. Consider the operator

$$(N, L) \rightarrow (X, Y)$$

defined by

$$\left\{ \begin{array}{l} X_t \triangleq x + \int_0^t r X_s ds + \int_0^t \theta^F(N_{s-}, L_{s-}) \left(\frac{dF_s}{F_{s-}} - r ds \right) \\ \quad + \int_0^t \sum_{i=1}^n \theta^i(N_{s-}, L_{s-}) \left(\frac{dS_s^i}{S_{s-}^i} - r ds \right) - \int_0^t \gamma(N_{s-}, L_{s-}) ds - \lambda M_t, \\ Y_t \triangleq y - \int_0^t \theta^F(N_{s-}, L_{s-}) \left(\frac{dF_s}{F_{s-}} - \frac{dB_s}{B_{s-}} \right) + (1 + \lambda) M_t \\ \quad \int_0^t \mathbf{1}_{\{Y_s > 0\}} dM_s = 0. \end{array} \right.$$

In other words, we obtain (X, Y) from (N, L) by solving the state equation (2.11) for $\theta_s = \theta(N_{s-}, L_{s-})$ and $\gamma_s = \gamma(N_{s-}, L_{s-})$. According to Proposition

2.2.1, the solution (X, Y) is given by

$$X_t = e^{rt} \left(\begin{array}{c} x + \int_0^t e^{-rs} \theta^T(N_{s-}, L_{s-}) (\alpha ds + \sigma d\mathbf{W}_s + \int_{\mathbb{R}^l} \mathbf{J}(\eta) \mathcal{N}(d\eta, ds)) \\ - \int_0^t e^{-rs} \gamma(N_{s-}, L_s) ds \\ - \frac{\lambda}{1+\lambda} \int_0^t e^{-rs} d \left(\sup_{0 \leq u \leq s} \left[\begin{array}{c} \int_0^u \theta_\tau^F(N_{\tau-}, L_{\tau-}) \\ \mu^E d\tau \\ + \sigma^E d\mathbf{W}_\tau \\ + \int_{\mathbb{R}^l} \mathbf{J}^E(\eta) \mathcal{N}(d\eta, d\tau) \\ - y \end{array} \right]^+ \right) \end{array} \right)$$

and

$$Y_t = y - \int_0^t \theta^F(N_{s-}, L_{s-}) \left(\mu^E ds + \sigma^E d\mathbf{W}_s + \int_{\mathbb{R}^l} \mathbf{J}^E(\eta) \mathcal{N}(d\eta, ds) \right) + \sup_{0 \leq s \leq t} \left[\int_0^s \theta^F(N_{u-}, L_{u-}) \left(\mu^E du + \sigma^E d\mathbf{W}_u + \int_{\mathbb{R}^l} \mathbf{J}^E(\eta) \mathcal{N}(d\eta, du) \right) - y \right]^+.$$

Now we can use the usual estimates in the Ito's theory of SDEs to obtain

$$\mathbb{E} \left[\sup_{0 \leq s \leq t} \|(X_s^1 - X_s^2, Y_s^1 - Y_s^2)\|^2 \right] \leq C^*(T) \int_0^t \mathbb{E} \left[\|(N_s^1 - N_s^2, L_s^1 - L_s^2)\|^2 \right] ds,$$

as long as $0 \leq t \leq T$ for each fixed $T > 0$, where $C^*(T) < \infty$ is a constant depending on the Lipschitz constants of θ and γ , and quantity $\int_{\mathbb{R}^l} |\mathbf{J}(\eta)|_2^2 \mathbf{q}(d\eta) < \infty$ by assumption, as well as the time horizon T . This allows us to prove path-wise uniqueness using Gronwall's inequality and also to prove existence using a Picard iteration. \square

Proof of Proposition 3.3.2. From Proposition 3.3.4 we can see that

$$\widehat{\theta}(x, y) \triangleq \begin{cases} x\widehat{\pi}(x, y), & x > 0, y \geq 0, \\ 0, & x \leq 0, y \geq 0, \end{cases}$$

and

$$\widehat{\gamma}(x, y) \triangleq \begin{cases} x\widehat{c}(x, y), & x > 0, y \geq 0, \\ 0, & x \leq 0, y \geq 0, \end{cases}$$

are globally Lipschitz in the domain $x \in \mathbb{R}, y \geq 0$. Therefore, according to Proposition 3.3.9 the equation has a unique solution $(\widehat{X}, \widehat{Y}) \in \mathbb{R} \times [0, \infty)$. It only remains to prove that $\widehat{X} > 0$ in order to finish the proof of Proposition 3.3.2, and this is shown in the next proposition. \square

Proposition 3.3.10. *Let $x > 0, y \geq 0$. Assume that the predictable process (π, c) satisfies the integrability condition*

$$\begin{aligned} \mathbb{P} \left(\int_0^t (|\pi_u|_2^2 + c_u) du < \infty \quad \forall 0 \leq t < \infty \right) &= 1, \\ \mathbb{P} \left(\int_0^t \left(\int_{\mathbb{R}^l} |\pi_u^F \mathbf{J}^E(\eta)|^2 \mathbf{q}(d\eta) \right) du < \infty \quad \forall 0 \leq t < \infty \right) &= 1, \\ \mathbb{P} \left(\int_0^t \left(\int_{\mathbb{R}^l} |\pi_u^T \mathbf{J}(\eta)|^2 \mathbf{q}(d\eta) \right) du < \infty \quad \forall 0 \leq t < \infty \right) &= 1. \end{aligned}$$

If (X, Y) is a solution to the equation

$$\begin{cases} X_t = x + \int_0^t r X_s ds + \int_0^t \pi^F X_{s-} \left(\frac{dF_s}{F_{s-}} - r ds \right) \\ \quad + \int_0^t \sum_{i=1}^n \pi^i X_{s-} \left(\frac{dS_s^i}{S_{s-}^i} - r ds \right) - \int_0^t c X_{s-} ds - \lambda M_t, \\ Y_t = y - \int_0^t \pi^F X_{s-} \left(\frac{dF_s}{F_{s-}} - \frac{dB_s}{B_{s-}} \right) + (1 + \lambda) M_t \\ \int_0^t \mathbf{1}_{\{Y_s > 0\}} dM_s = 0. \end{cases}$$

then

$$X_t > 0, Y_t \geq 0, \quad 0 \leq t < \infty.$$

Proof. Denote by $\tau \triangleq \{t \geq 0 : X_t = 0\}$. We can apply Ito's formula to $N_t = \log(X_t)$ and take into account that $Y_t \geq 0$ (also $Y_{t-} \geq 0$) and $Y_t = Y_{t-} = 0$ on

the support of dM^c to obtain

$$\begin{aligned}
N_t &= \log x + R_t - \lambda \int_0^t \frac{dM_s^c}{X_{s-}} \\
&\quad + \int_0^t \int_{\mathbb{R}^l} \log \left(1 + \pi_s^T \mathbf{J}(\eta) - \frac{\lambda}{1+\lambda} \left(\left[\pi^F \mathbf{J}^E(\eta) - \frac{Y_{s-}}{X_{s-}} \right]^+ \right) \right) \mathcal{N}(ds, d\eta) \\
&= \log x + R_t - \frac{\lambda}{1+\lambda} \int_0^t \frac{1}{X_{s-}} d \left(\sup_{0 \leq u \leq s} \left[\int_0^u \theta_\tau^F (\mu^E d\tau + \sigma^E d\mathbf{W}_\tau) - y \right]^+ \right) \\
&\quad + \int_0^t \int_{\mathbb{R}^l} \log \left(1 + \pi_s^T \mathbf{J}(\eta) - \frac{\lambda}{1+\lambda} \left(\left[\pi^F \mathbf{J}^E(\eta) - \frac{Y_{s-}}{X_{s-}} \right]^+ \right) \right) \mathcal{N}(ds, d\eta) \\
&= \log x + R_t - \frac{\lambda}{1+\lambda} \int_0^t \pi_s^F 1_{\{dM^c > 0\}} (\mu^E ds + \sigma^E d\mathbf{W}_s) \\
&\quad + \int_0^t \int_{\mathbb{R}^l} \log \left(1 + \pi_s^T \mathbf{J}(\eta) - \frac{\lambda}{1+\lambda} \left(\left[\pi^F \mathbf{J}^E(\eta) - \frac{Y_{s-}}{X_{s-}} \right]^+ \right) \right) \mathcal{N}(ds, d\eta)
\end{aligned}$$

where

$$R_t \triangleq \int_0^t \left(r + \pi_s^T \alpha - c_s - \frac{1}{2} \pi_s^T \sigma^T \sigma \pi_s \right) ds + \int_0^t \pi_s^T \sigma d\mathbf{W}_s, \quad t \geq 0.$$

We observe that

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}^l} \log \left(1 + \pi_s^T \mathbf{J}(\eta) - \frac{\lambda}{1+\lambda} \left(\left[\pi^F \mathbf{J}^E(\eta) - \frac{Y_{s-}}{X_{s-}} \right]^+ \right) \right) \mathcal{N}(ds, d\eta) \\
&\geq \int_0^t \int_{\mathbb{R}^l} \log(1 + \pi_s^T \mathbf{J}(\eta)) \mathcal{N}(ds, d\eta) \\
&> \int_0^t \int_{\mathbb{R}^l} \pi_s^T \mathbf{J}(\eta) \mathcal{N}(ds, d\eta_i) - \int_0^t \int_{\mathbb{R}^l} \pi_s^T \mathbf{J}(\eta)^2 \pi_s \mathcal{N}(ds, d\eta_i) \\
&> -\infty,
\end{aligned}$$

according to the assumption about jumps in (2.9). And because

$$\lim_{t \nearrow \tau} R_t > -\infty \quad \text{on } \{\tau < \infty\},$$

we can then obtain that $\tau = \infty$. □

Proof of Theorem 3.3.3. First we verify that the condition in (3.1) is satisfied,

because

$$\begin{aligned}
& \int_0^t \int_{\mathbb{R}^l} e^{-\beta s} \left| v \begin{pmatrix} X_{s-} + \theta_s^T \mathbf{J}(\eta) - \frac{\lambda}{1+\lambda} [\theta_s^F \mathbf{J}^E(\eta) - Y_{s-}]^+, \\ Y_{s-} - \theta_s^F \mathbf{J}^E(\eta) + [\theta_s^F \mathbf{J}^E(\eta) - Y_{s-}]^+ \\ -v(X_{s-}, Y_{s-}) \end{pmatrix} \right| \mathbf{q}(d\eta) ds \\
&= \int_0^t \int_{\mathbb{R}^l} e^{-\beta s} \left| \begin{pmatrix} (X_{s-} + \theta_s^T \mathbf{J}(\eta) - \frac{\lambda}{1+\lambda} [\theta_s^F \mathbf{J}^E(\eta) - Y_{s-}]^+)^{1-p} \\ \cdot u \left(\frac{Y_{s-} - \theta_s^F \mathbf{J}^E(\eta) + [\theta_s^F \mathbf{J}^E(\eta) - Y_{s-}]^+}{X_{s-} + \theta_s^T \mathbf{J}(\eta) - \frac{\lambda}{1+\lambda} [\theta_s^F \mathbf{J}^E(\eta) - Y_{s-}]^+} \right) - (X_{s-})^{1-p} u \left(\frac{Y_{s-}}{X_{s-}} \right) \end{pmatrix} \right| \mathbf{q}(d\eta) ds \\
&= \int_0^t \int_{\mathbb{R}^l} e^{-\beta s} (X_{s-})^{1-p} \left| \begin{pmatrix} \left(1 + \pi_s^T \mathbf{J}(\eta) - \frac{\lambda}{1+\lambda} [\pi_s^F \mathbf{J}^E(\eta) - \frac{Y_{s-}}{X_{s-}}]^+ \right)^{1-p} \\ \cdot u \left(\frac{\frac{Y_{s-}}{X_{s-}} - \pi_s^F \mathbf{J}^E(\eta) + [\pi_s^F \mathbf{J}^E(\eta) - \frac{Y_{s-}}{X_{s-}}]^+}{1 + \pi_s^T \mathbf{J}(\eta) - \frac{\lambda}{1+\lambda} [\pi_s^F \mathbf{J}^E(\eta) - \frac{Y_{s-}}{X_{s-}}]^+} \right) - u \left(\frac{Y_{s-}}{X_{s-}} \right) \end{pmatrix} \right| \mathbf{q}(d\eta) ds \\
&\leq \max_{z \geq 0} |u(z)| \max_{\pi \in \Delta} \int_{\mathbb{R}^l} \left| (1 + \pi^T \mathbf{J}(\eta))^{1-p} - 1 \right| \mathbf{q}(d\eta) \cdot \int_0^t e^{-\beta s} (X_{s-})^{1-p} ds \\
&+ \max_{z \geq 0} |u'(z)| \max_{\pi \in \Delta} \int_{\mathbb{R}^l} \left| (1 + \pi^T \mathbf{J}(\eta))^{-p} \pi^T \mathbf{J}(\eta) \right| \mathbf{q}(d\eta) \cdot \int_0^t e^{-\beta s} (X_{s-})^{1-p} ds \\
&+ \max_{z \geq 0} |u'(z)| \max_{\pi \in \Delta} \int_{\mathbb{R}^l} \left| (1 + \pi^T \mathbf{J}(\eta))^{-p} \pi^T \mathbf{J}(\eta) \right| \mathbf{q}(d\eta) \cdot \int_0^t e^{-\beta s} (X_{s-})^{-p} Y_{s-} ds \\
&< \infty \text{ a.s.}
\end{aligned}$$

where the last inequality follows from the assumption (2.10) and u, u' are bounded, and that X_{s-}, Y_{s-} are left continuous with right limits. The second

to last inequality holds true since

$$\begin{aligned}
& \left(1 + \pi_s^T \mathbf{J}(\eta) - \frac{\lambda}{1+\lambda} \left[\pi_s^F \mathbf{J}^E(\eta) - \frac{Y_{s-}}{X_{s-}} \right]^+ \right)^{1-p} \\
& \cdot u \left(\frac{\frac{Y_{s-}}{X_{s-}} - \pi_s^F \mathbf{J}^E(\eta) + \left[\pi_s^F \mathbf{J}^E(\eta) - \frac{Y_{s-}}{X_{s-}} \right]^+}{1 + \pi_s^T \mathbf{J}(\eta) - \frac{\lambda}{1+\lambda} \left[\pi_s^F \mathbf{J}^E(\eta) - \frac{Y_{s-}}{X_{s-}} \right]^+} \right) - u \left(\frac{Y_{s-}}{X_{s-}} \right) \\
& = \left(\left(1 + \pi_s^T \mathbf{J}(\eta) - \frac{\lambda}{1+\lambda} \left[\pi_s^F \mathbf{J}^E(\eta) - \frac{Y_{s-}}{X_{s-}} \right]^+ \right)^{1-p} - 1 \right) \cdot u \left(\frac{Y_{s-}}{X_{s-}} \right) \\
& + \left(1 + \pi_s^T \mathbf{J}(\eta) - \frac{\lambda}{1+\lambda} \left[\pi_s^F \mathbf{J}^E(\eta) - \frac{Y_{s-}}{X_{s-}} \right]^+ \right)^{1-p} \\
& \cdot \left(\frac{\frac{Y_{s-}}{X_{s-}} - \pi_s^F \mathbf{J}^E(\eta) + \left[\pi_s^F \mathbf{J}^E(\eta) - \frac{Y_{s-}}{X_{s-}} \right]^+}{1 + \pi_s^T \mathbf{J}(\eta) - \frac{\lambda}{1+\lambda} \left[\pi_s^F \mathbf{J}^E(\eta) - \frac{Y_{s-}}{X_{s-}} \right]^+} - \frac{Y_{s-}}{X_{s-}} \right) u'(\xi) \\
& = \left(\left(1 + \pi_s^T \mathbf{J}(\eta) - \frac{\lambda}{1+\lambda} \left[\pi_s^F \mathbf{J}^E(\eta) - \frac{Y_{s-}}{X_{s-}} \right]^+ \right)^{1-p} - 1 \right) \cdot u \left(\frac{Y_{s-}}{X_{s-}} \right) \\
& + \left(1 + \pi_s^T \mathbf{J}(\eta) - \frac{\lambda}{1+\lambda} \left[\pi_s^F \mathbf{J}^E(\eta) - \frac{Y_{s-}}{X_{s-}} \right]^+ \right)^{-p} \\
& \cdot \left(\begin{array}{c} -\pi_s^F \mathbf{J}^E(\eta) + \left[\pi_s^F \mathbf{J}^E(\eta) - \frac{Y_{s-}}{X_{s-}} \right]^+ \\ - \left(\pi_s^T \mathbf{J}(\eta) - \frac{\lambda}{1+\lambda} \left[\pi_s^F \mathbf{J}^E(\eta) - \frac{Y_{s-}}{X_{s-}} \right]^+ \right) \frac{Y_{s-}}{X_{s-}} \end{array} \right) u'(\xi) \\
& = \left(\left(1 + \pi_s^T \mathbf{J}(\eta) - \frac{\lambda}{1+\lambda} \left[\pi_s^F \mathbf{J}^E(\eta) - \frac{Y_{s-}}{X_{s-}} \right]^+ \right)^{1-p} - 1 \right) \cdot u \left(\frac{Y_{s-}}{X_{s-}} \right) \\
& - \left(1 + \pi_s^T \mathbf{J}(\eta) - \frac{\lambda}{1+\lambda} \left[\pi_s^F \mathbf{J}^E(\eta) - \frac{Y_{s-}}{X_{s-}} \right]^+ \right)^{-p} \\
& \cdot \left(\pi_s^F \mathbf{J}^E(\eta) - \left[\pi_s^F \mathbf{J}^E(\eta) - \frac{Y_{s-}}{X_{s-}} \right]^+ \right) u'(\xi) \\
& - \left(1 + \pi_s^T \mathbf{J}(\eta) - \frac{\lambda}{1+\lambda} \left[\pi_s^F \mathbf{J}^E(\eta) - \frac{Y_{s-}}{X_{s-}} \right]^+ \right)^{-p} \\
& \cdot \left(\pi_s^T \mathbf{J}(\eta) - \frac{\lambda}{1+\lambda} \left[\pi_s^F \mathbf{J}^E(\eta) - \frac{Y_{s-}}{X_{s-}} \right]^+ \right) \frac{Y_{s-}}{X_{s-}} u'(\xi)
\end{aligned}$$

Now, according to Lemma 3.1.1, the process

$$V_t = \int_0^t e^{-\beta s} U(\gamma_s) ds + e^{-\beta t} v(X_t, Y_t), \quad t \geq 0,$$

is a local supermartingale for each admissible control and a local martingale for the feedback control $(\hat{\theta}, \hat{\gamma})$.

1. If $p > 1$, then for a sequence of stopping times τ_k we have

$$\begin{aligned} v(x, y) &= \mathbb{E} \left[\int_0^{\tau_k} e^{-\beta s} U(\hat{\gamma}_s) ds + e^{-\beta \tau_k} v(\hat{X}_{\tau_k}, \hat{Y}_{\tau_k}) \right] \\ &\leq \mathbb{E} \left[\int_0^{\tau_k} e^{-\beta s} U(\hat{\gamma}_s) ds \right]. \end{aligned}$$

Letting $k \rightarrow \infty$ and using monotone convergence theorem, we get

$$v(x, y) \leq \mathbb{E} \left[\int_0^{\infty} e^{-\beta s} U(\hat{\gamma}_s) ds \right].$$

Now, let $(\theta, \gamma) \in \mathcal{A}(x, y)$ be admissible controls. It is easy to see from Proposition 2.2.1 that $(\theta, \gamma) \in \mathcal{A}(x + \varepsilon, y)$, and the wealth X corresponding to (θ, γ) starting at $x + \varepsilon$ with high-watermark y satisfies $X > \varepsilon$. Using the local supermartingale property along the solution (X, Y) starting at $(x + \varepsilon, y)$ with controls (θ, γ) , we obtain

$$v(x + \varepsilon, y) \geq \mathbb{E} \left[\int_0^{\tau_k} e^{-\beta s} U(\gamma_s) ds + e^{-\beta \tau_k} v(X_{\tau_k}, Y_{\tau_k}) \right].$$

However, since $X > \varepsilon$ we obtain

$$|v(X, Y)| \leq C\varepsilon^{1-p},$$

where C is a bound on $|u|$. Therefore, we can again let $k \rightarrow \infty$ and use monotone convergence theorem together with the bounded convergence theorem (respectively for the two terms on the right-hand side) to obtain

$$v(x + \varepsilon, y) \geq \mathbb{E} \left[\int_0^\infty e^{-\beta s} U(\gamma_s) ds \right]$$

for all $(\theta, \gamma) \in \mathcal{A}(x, y)$. This means that

$$v(x + \varepsilon, y) \geq \sup_{(\theta, \gamma) \in \mathcal{A}(x, y)} \mathbb{E} \left[\int_0^\infty e^{-\beta s} U(\gamma_s) ds \right] = V(x, y)$$

and the conclusion follows from letting $\varepsilon \searrow 0$.

2. Let $p < 1$. Then by the local supermartingale property we obtain

$$\begin{aligned} v(x, y) &\geq \mathbb{E} \left[\int_0^{\tau_k} e^{-\beta s} U(\gamma_s) ds + e^{-\beta \tau_k} v(X_{\tau_k}, Y_{\tau_k}) \right] \\ &\geq \mathbb{E} \left[\int_0^{\tau_k} e^{-\beta s} U(\gamma_s) ds \right]. \end{aligned}$$

Letting $k \rightarrow \infty$ we get

$$v(x, y) \geq \mathbb{E} \left[\int_0^\infty e^{-\beta s} U(\gamma_s) ds \right]$$

for each $(\theta, \gamma) \in \mathcal{A}(x, y)$.

Now, for the optimal $(\hat{\pi}, \hat{c})$ (in *proportion* form) we have

$$v(x, y) = \mathbb{E} \left[\int_0^{\tau_k} e^{-\beta s} U(\hat{c}_s \hat{X}_s) ds + e^{-\beta \tau_k} v(\hat{X}_{\tau_k}, \hat{Y}_{\tau_k}) \right].$$

If we can show that

$$\mathbb{E} \left[e^{-\beta \tau_k} v(\hat{X}_{\tau_k}, \hat{Y}_{\tau_k}) \right] \rightarrow 0, \quad (3.29)$$

then we use monotone convergence theorem to obtain

$$v(x, y) = \mathbb{E} \left[\int_0^\infty e^{-\beta s} U(\widehat{c}_s \widehat{X}_s) ds \right]$$

and finish the proof. Let us now prove (3.29). The value function $v_0(x, y) \triangleq u_0 x^{1-p}$ corresponding to $\lambda = 0$ is a supersolution of the HJB equation since the constant function u_0 is a supersolution to (3.21). Using Lemma 3.1.1 for the function v_0 and denoting by

$$Z_t \triangleq e^{-\beta t} v_0(\widehat{X}_t, \widehat{Y}_t) = u_0 e^{-\beta t} (\widehat{X}_t)^{1-p},$$

we obtain

$$\begin{aligned} & Z_t + \int_0^t e^{-\beta s} \frac{(\widehat{c} \widehat{X}_s)^{1-p}}{1-p} ds \\ &= Z_t + \int_0^t Z_s \frac{(\widehat{c})^{1-p}}{(1-p) u_0} ds \\ &\leq \int_0^t (1-p) (\widehat{\pi}_s)^T \sigma Z_s d\mathbf{W}_s \\ &+ \int_0^t Z_{s-} \int_{\mathbb{R}^l} \left\{ \left(\frac{1 + \widehat{\pi}_s^T \mathbf{J}(\eta)}{-\frac{\lambda}{1+\lambda} [\widehat{\pi}_s^F \mathbf{J}^E(\eta) - \frac{Y_{s-}}{X_{s-}}]^+} \right)^{1-p} - 1 \right\} \widetilde{\mathcal{N}}(d\eta, ds) \\ &\leq \int_0^t (1-p) (\widehat{\pi}_s)^T \sigma Z_s d\mathbf{W}_s \\ &+ \int_0^t Z_{s-} \int_{\mathbb{R}^l} \left\{ (1 + \widehat{\pi}_s^T \mathbf{J}(\eta))^{1-p} - 1 \right\} \widetilde{\mathcal{N}}(d\eta, ds). \end{aligned}$$

Recall that from Proposition 3.3.4 we have that $\widehat{c} \geq c_0$. This means that if we denote by

$$\delta \triangleq \frac{c_0^{1-p}}{(1-p) u_0} > 0,$$

then we have

$$\begin{aligned} & Z_t + \int_0^t \delta Z_s ds \\ & \leq \int_0^t (1-p) (\widehat{\pi}_s)^T \sigma Z_s d\mathbf{W}_s + \int_0^t Z_{s-} \int_{\mathbb{R}^l} \left\{ (1 + \widehat{\pi}_s^T \mathbf{J}(\eta))^{1-p} - 1 \right\} \widetilde{\mathcal{N}}(d\eta, ds). \end{aligned}$$

By the well-known comparison principle we get

$$\begin{aligned} & Z_t \\ & \leq Z_0 e^{-\delta t} \exp \left\{ \begin{aligned} & \int_0^t (1-p) (\widehat{\pi}_s)^T \sigma d\mathbf{W}_s \\ & - \frac{1}{2} \int_0^t (1-p)^2 (\widehat{\pi}_s)^T \sigma \sigma^T \widehat{\pi}_s ds \\ & + \int_0^t \int_{\mathbb{R}^l} (1-p) \ln(1 + \widehat{\pi}_s^T \mathbf{J}(\eta)) \mathcal{N}(d\eta, ds) \\ & - \left(\int_{\mathbb{R}^l} \left\{ (1 + \widehat{\pi}_s^T \mathbf{J}(\eta))^{1-p} - 1 \right\} \mathbf{q}(d\eta) \right) t \end{aligned} \right\}, \\ & \triangleq \overline{Z}_t. \end{aligned}$$

We observe that for $k > 1$,

$$\overline{Z}_t^k = e^{-k\delta t} \exp \left\{ \begin{aligned} & \int_0^t k(1-p) (\widehat{\pi}_s)^T \sigma dW_s \\ & - \frac{1}{2} k^2 \int_0^t (1-p)^2 (\widehat{\pi}_s)^T \sigma \sigma^T \widehat{\pi}_s ds \\ & + \frac{1}{2} k(k-1) \int_0^t (1-p)^2 (\widehat{\pi}_s)^T \sigma \sigma^T \widehat{\pi}_s ds \\ & + \int_0^t \int_{\mathbb{R}^l} k(1-p) \ln(1 + \widehat{\pi}_s^T \mathbf{J}(\eta)) \mathcal{N}(d\eta, ds) \\ & - \left(\int_{\mathbb{R}^l} \left\{ (1 + \widehat{\pi}_s^T \mathbf{J}(\eta))^{k(1-p)} - 1 \right\} \mathbf{q}(d\eta) \right) t \\ & + \left(\int_{\mathbb{R}^l} \left\{ (1 + \widehat{\pi}_s^T \mathbf{J}(\eta))^{k(1-p)} - 1 \right\} \mathbf{q}(d\eta) \right) t \\ & - k \left(\int_{\mathbb{R}^l} \left\{ (1 + \widehat{\pi}_s^T \mathbf{J}(\eta))^{1-p} - 1 \right\} \mathbf{q}(d\eta) \right) t \end{aligned} \right\},$$

and if k is sufficiently close to 1, $\{\overline{Z}_t\}_{t \geq 0}$ is \mathcal{L}^k -bounded, and hence uniformly integrable. Therefore, $\{Z_t\}_{t \geq 0}$ is uniformly integrable. Now taking into account that

$$e^{-\beta t} v(X_t, Y_t) \leq Z_t \rightarrow 0 \text{ a.s. for } t \rightarrow \infty,$$

we obtain (3.29) and the proof is complete. □

3.4 Certainty equivalent analysis

We evaluate the quantitative impact of paying proportional high-watermark fee λ on the initial wealth of the investor. The size of the value function does not provide any intuitive interpretation. A useful method is to compute the so-called certainty equivalent wealth. By definition, the certainty equivalent wealth is such a size of initial bankroll \tilde{x} that the agent would be indifferent between \tilde{x} when paying zero fees and wealth x when paying proportional high-watermark fees λ , all other parameters being the same.

From Remark 3.2.1 we infer the proper transformation by equating $v_0(\tilde{x})$ and $v(x, y) = x^{1-p}u(z)$. We solve for quantity

$$\frac{\tilde{x}(z)}{x} = \left(\frac{u(z)}{u_0} \right)^{\frac{1}{1-p}} = ((1-p)c_0^p u(z))^{\frac{1}{1-p}}, \quad z \geq 0,$$

which is the relative amount of wealth needed to achieve the same utility if no fee is paid (which also quantifies the proportional loss of wealth).

It is also useful to evaluate the size of the proportional fee (percentage per year, as in Remark 2.2.6) that would cause the same loss in utility as the current high-watermark performance fee. More precisely, we want to find the certainty equivalent $\tilde{\alpha} < \alpha$ so that the value function obtained by using $\tilde{\alpha}$ and no fee is equal to the value function when the return is α but the high-watermark performance fee is paid.

Keeping all other parameters the same, the value function for zero high-

watermark performance fee corresponding to $\tilde{\alpha}$ is given by

$$\tilde{u}_0(\tilde{\alpha}) = \frac{1}{1-p} \tilde{c}_0(\tilde{\alpha})^{-p}, \quad z \geq 0.$$

where \tilde{c}_0 is defined as in (3.11) with α being replaced by $\tilde{\alpha}$.

Therefore, we are looking for the solution to the equation

$$\tilde{u}_0(\tilde{\alpha}(z)) = u(z),$$

In general, this equation above is difficult to solve analytically. However, in the particular case where the jump term vanishes, i.e., $\mathbf{q} = \mathbf{0}$, then

$$\tilde{c}_0(\tilde{\alpha}) \triangleq \frac{\beta}{p} - r \frac{1-p}{p} - \frac{1}{2} \frac{1-p}{p^2} \cdot \tilde{\alpha}^T (\sigma \sigma^T)^{-1} \tilde{\alpha}.$$

and $\tilde{\alpha}$ is implicitly given by

$$\tilde{\alpha}^T (\sigma \sigma^T)^{-1} \tilde{\alpha} = \frac{2p^2}{1-p} \left(\frac{\beta}{p} - r \frac{1-p}{p} - ((1-p)u(z))^{-\frac{1}{p}} \right), \quad z \geq 0.$$

Because $\tilde{\alpha}$ and α differ only in their first element ($\tilde{\alpha}_F$ and α , respectively), the above is a quadratic equation of $\tilde{\alpha}_F$. Once we get $\tilde{\alpha}_F$, the relative size of the certainty equivalent $\tilde{\alpha}_F$ is then $\tilde{\alpha}_F/\alpha$.

Chapter 4

Numerics

To the best of our knowledge, there is no closed-form solution for our optimization problem at hand. In order to understand the impact of the high-watermark fees on the investor, we need to resort to numerics. The paper [17] gave numerical results for the case in which there is only a single risky asset, the hedge fund, the interest rate is zero and the fund share price is a continuous process. Specifically, the authors numerically solve the HJB equation for the value function using an iterative method, then use the results to describe the optimal investment/consumption proportions, as well as the certainty equivalent wealth and the certainty equivalent $\tilde{\alpha}$ (which we defined in the last chapter). Our numeric experiment generalizes the result of [17] in two ways:

1. In addition to a hedge fund F , we introduce another stock S , possibly correlated with F , and investigate the value function, the optimal investment/consumption proportions, as well as the certainty equivalent wealth and the certainty equivalent $\tilde{\alpha}$ in this multiple-asset case.
2. On top of the multiple-asset case described above, we incorporate jumps

into the processes of F and S , and study the effect of jumps by comparison.

We follow [17] and set our *benchmark parameters* as follows,

$$p_0 = 7, \beta_0 = 5\%, \mu_0^F = 20\%, \mu_0^S = 10\%, r_0 = 4\%, \\ \sigma_0^F = 20\%, \sigma_0^S = 20\%, \rho_0 = 0, \lambda_0 = 25, \mathbf{q} = \mathbf{0}.$$

The Merton values for these parameters are:

$$\pi_0^F = 0.571, \pi_0^S = 0.214, c_0 = 0.0861$$

We keep the size of the volatilities σ^F, σ^S to be fixed. This is actually not restrictive since a model with given α and σ has an identical value function as a scaled model with return $k\alpha$ and standard deviation $k\sigma$, while investment proportion scales by $1/k$. Since we draw most of our graphs for the relative investment proportion (compared to Merton case), this would actually not change at all, even by scaling.

First, from a certainty equivalence perspective, we present two graphs when varying λ , each representing, respectively

- the *relative* size of the certainty equivalent initial wealth (which means the proportion $\tilde{x}_0(z)/x, z \geq 0$);
- the *relative* size of the certainty equivalent excess return (which means $\tilde{\alpha}^F(z) - \alpha^F, z \geq 0$).

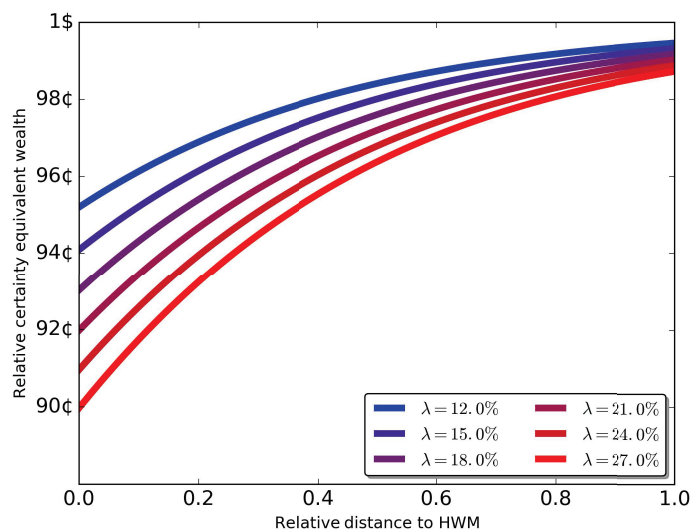


Figure 4.1: Relative certainty equivalent initial wealth.

Recall from Remark 2.2.2 that $\tilde{\alpha}^F(z)$ may not be interpreted as the excess return of the hedge fund share, but $\tilde{\alpha}^F(z) - \alpha^F$ may well be interpreted as the *relative* excess return of the hedge fund share, since the jump term is the same with or without fees. Note that the horizontal axis is the variable z , the “relative distance to pay HWM fees”, and this applies to all subsequent graphs as well.

Next, we present two graphs, each representing the size of the *relative* optimal investment proportion $\hat{\pi}(z)/\pi_0$, $z > 0$ (for both the fund and the stock), and the size of the relative optimal consumption proportion $c(z)/c_0$, $z > 0$ in two different cases:

- small average return on hedge fund meaning that $\mu^F < \mu_0^S$;

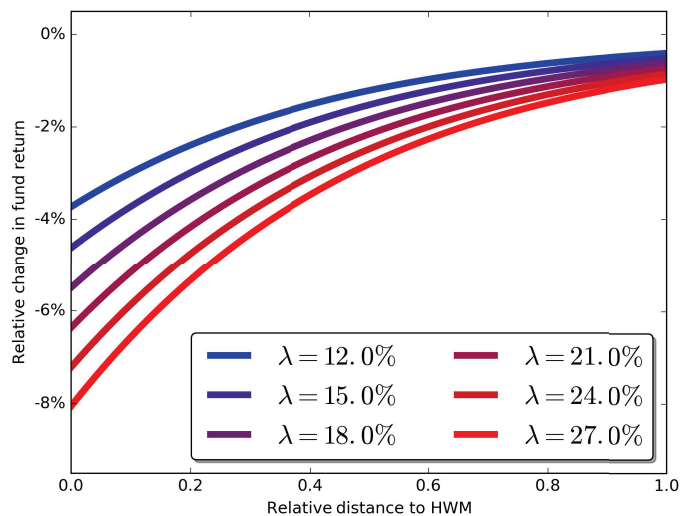


Figure 4.2: Relative certainty equivalent zero fee return.

- large average return on hedge fund meaning that $\mu^F > \mu_0^S$

We remind the reader that the values for zero high-watermark fee are obtained for $z \nearrow \infty$. This means that all the *relative* quantities presented below approach one as $z \nearrow \infty$.

Then, we present a graph representing

- absolute optimal investment proportions and consumption proportion when varying ρ

Lastly, we present figures comparing the value function, the optimal investment proportion in the hedge fund, the optimal investment proportion

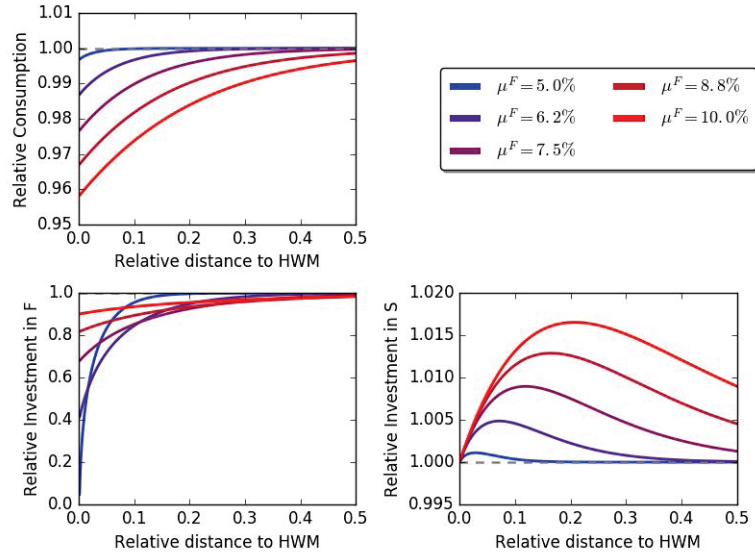


Figure 4.3: Relative investment proportions and consumption proportion when $\mu^F < \mu_0^S$.

in the stock and the optimal consumption with and without jumps. For illustration purpose, we experiment with several discrete measures \mathbf{q} and functions \mathbf{J} :

- independent jumps:

$$\mathbf{q}_1 = 0.001 \cdot \frac{1}{4} [\delta_{[0.8,0]} + \delta_{[0,0.8]} + \delta_{[-0.8,0]} + \delta_{[0,-0.8]}],$$

$$\mathbf{J}_1(\eta) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \eta$$

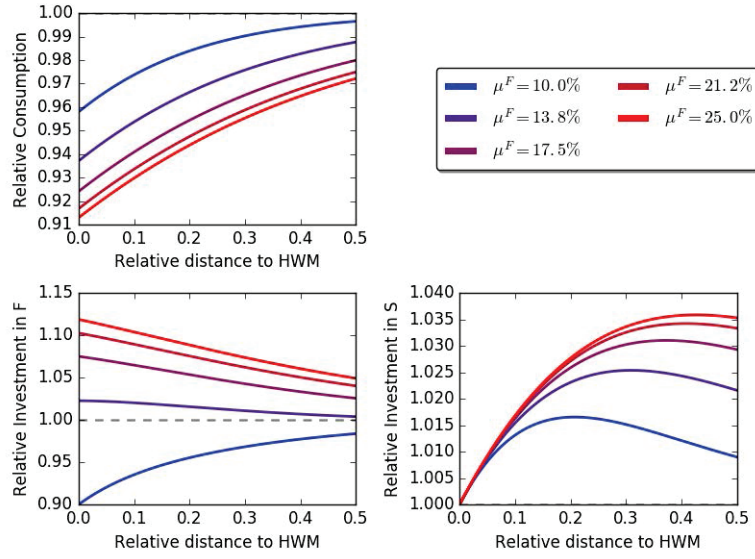


Figure 4.4: Relative investment proportions and consumption proportion when $\mu^F > \mu_0^S$.

- simultaneous jumps and some correlation:

$$\mathbf{q}_2 = 0.001 \cdot \frac{1}{4} [\delta_{[0.5,0.5]} + \delta_{[0.5,-0.5]} + \delta_{[-0.5,0.5]} + \delta_{[-0.5,-0.5]}],$$

$$\mathbf{J}_2(\eta) = \begin{bmatrix} 1 & 0.3 \\ 0.3 & 1 \end{bmatrix} \eta$$

- more jump sizes, some closer to zero:

$$\mathbf{q}_3 = 0.001 \cdot \frac{1}{8} \left[\begin{array}{c} \delta_{[0.9,0]} + \delta_{[0,0.9]} + \delta_{[-0.9,0]} + \delta_{[0,-0.9]} + \\ \delta_{[0.009,0]} + \delta_{[0,0.009]} + \delta_{[-0.009,0]} + \delta_{[0,-0.009]} \end{array} \right],$$

$$\mathbf{J}_3(\eta) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \eta$$

Remark 4.0.1. 1. From the perspective of certainty equivalent analysis, the high-watermark fees have the effect of either reducing the initial wealth

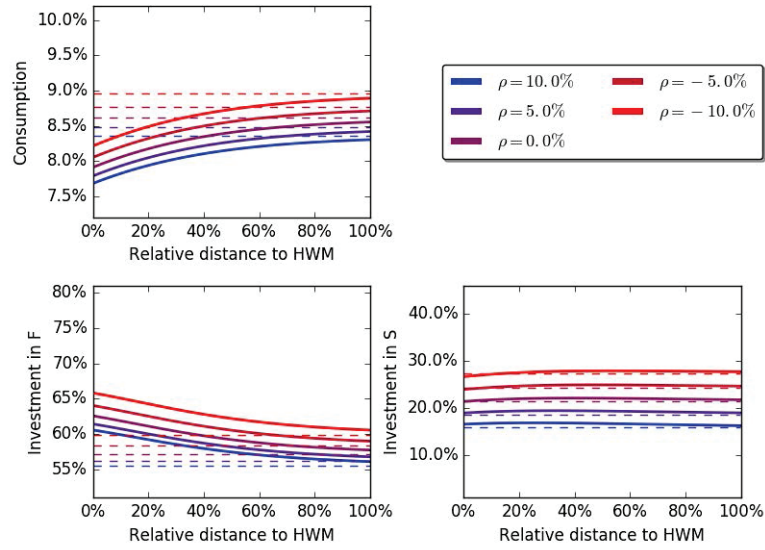


Figure 4.5: Investment proportions and consumption proportion when varying ρ .

of the investor or reducing the excess return of the fund. From Figure 4.1 and 4.2, we can have a more intuitive understanding of this effect for different values of λ . As expected, certainty equivalent initial wealth and certainty equivalent zero-fee return decrease as λ increases.

- From Figure 4.3-4.4 we can see that, when the hedge fund return is bigger than the stock return, the optimal investment proportions at the high-watermark level $\hat{\pi}^F(0)$ is greater than its Merton counterpart π_0^F . The intuitive explanation for this feature is that the investor wants to play the “local time game” at the boundary. When making a high investment proportion for a short time the loss in value due to over-investment is

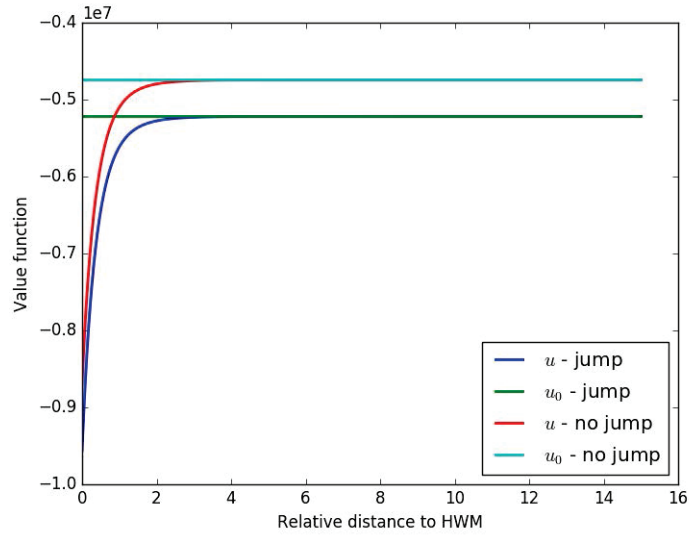


Figure 4.6: Value functions with and without jumps ($\mathbf{q}_1, \mathbf{J}_1$).

small, while the investor is able to push the high-watermark a little bit extra and benefit from an increased high-watermark in the future. This additional increase in high-watermark can be also interpreted as hedging. On the other hand, when the hedge fund return is smaller than or equal to the stock return, the optimal investment proportions at the high-watermark level $\hat{\pi}^F(0)$ is less than its Merton counterpart π_0^F .

3. In Figure 4.3-4.4, people may wonder why varying μ^F has an effect on the investment in the stock, given that the fund and the stock are independent in this case? The reason is that: varying μ^F increases the value function u , and the investment in the stock depends on the value function u and its derivatives (up to second order) as given in (3.7). Hence,

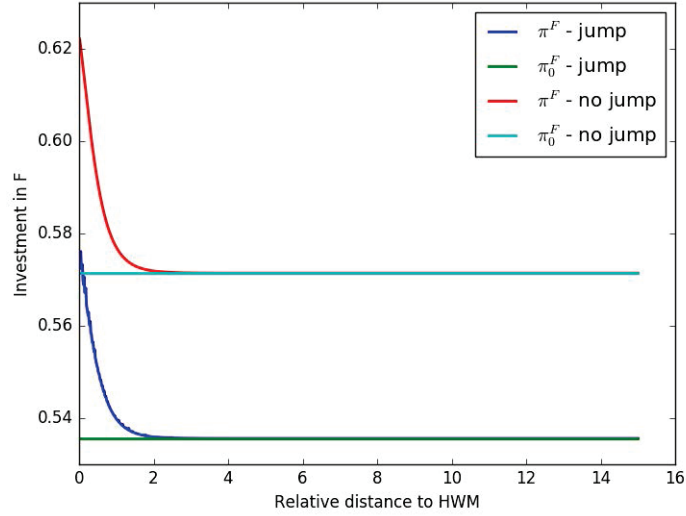


Figure 4.7: Investment proportion in hedge fund with and without jumps $(\mathbf{q}_1, \mathbf{J}_1)$.

varying μ^F indirectly changes the investment in the stock. Also in Figure 4.3-4.4, we observe that the graph of the investment in the stock is non-monotone with respect to the horizontal axis (i.e., the relative distance to paying HMW fees). This is because in (3.7) the investment in stock depends on z in a complex and non-monotone manner, through the value function u and its derivatives (up to second order). We don't have a very intuitive explanation of this non-monotonicity observation.

4. When we investigate the effect of correlation, we can clearly see the “diversification benefit” in Figure 4.5 (where the quantities are in absolute sense). This means, more specifically, the more negatively correlated of

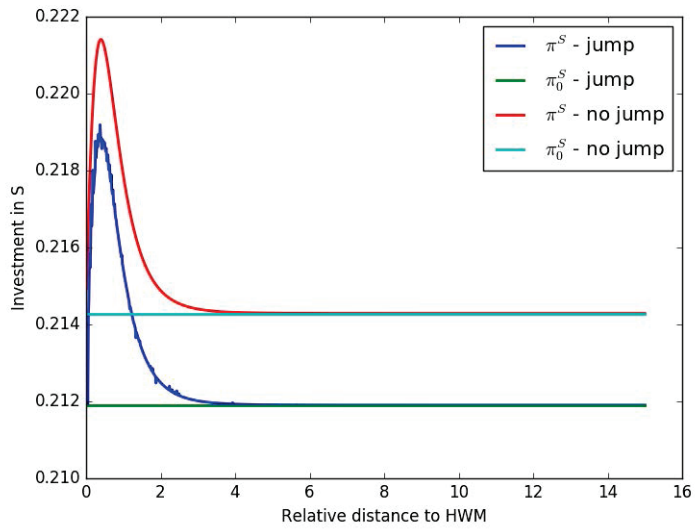


Figure 4.8: Investment proportion in stock with and without jumps ($\mathbf{q}_1, \mathbf{J}_1$).

the hedge fund and the stock, the more the investor would invest and consume.

- From Figure 4.6-4.17, we can see that the introduction of nontrivial zero-mean jumps has an effect of increasing volatility, as expected. Thus, it results in lower value function, lower optimal investment and consumption compared to no-jump case.

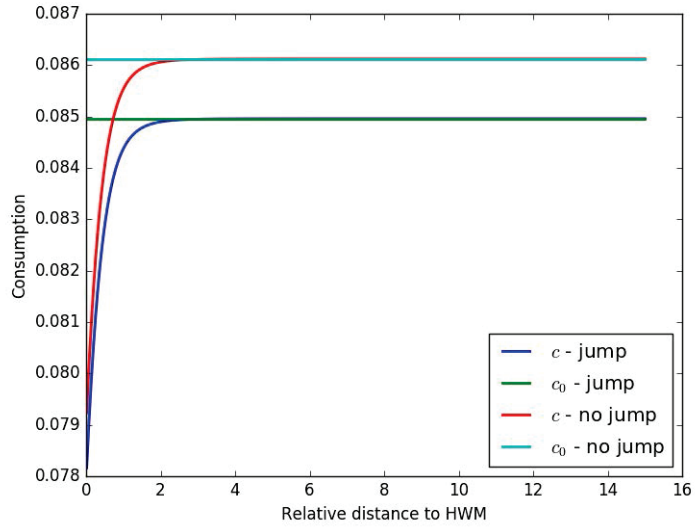


Figure 4.9: Consumption proportion with and without jumps ($\mathbf{q}_1, \mathbf{J}_1$).

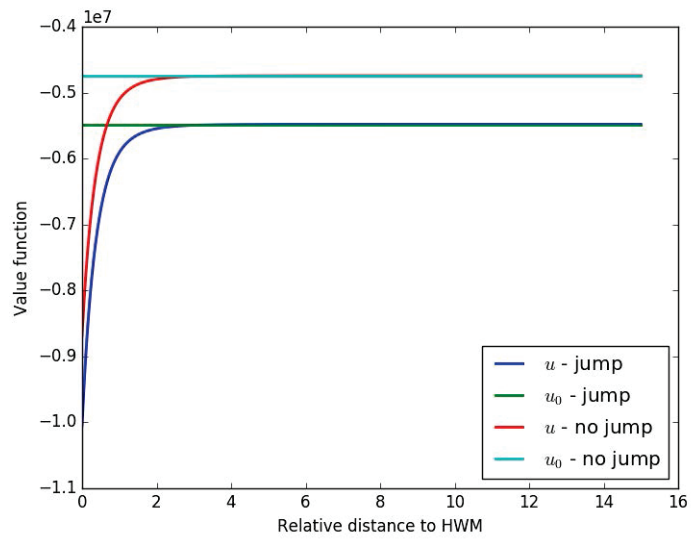


Figure 4.10: Value functions with and without jumps ($\mathbf{q}_2, \mathbf{J}_2$).

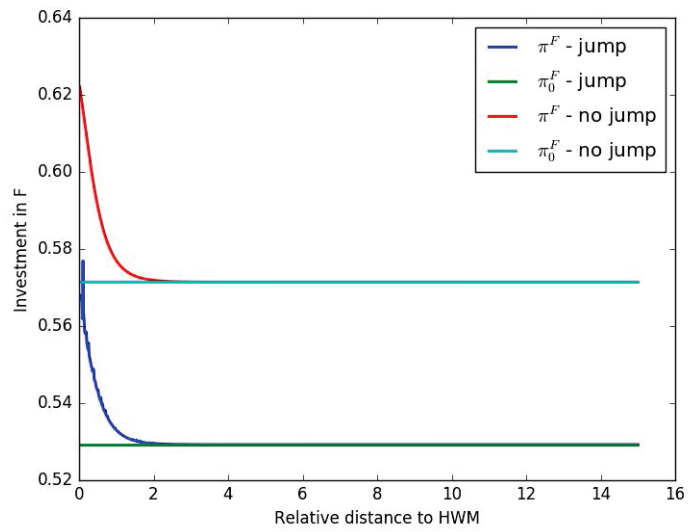


Figure 4.11: Investment proportion in hedge fund with and without jumps ($\mathbf{q}_2, \mathbf{J}_2$).

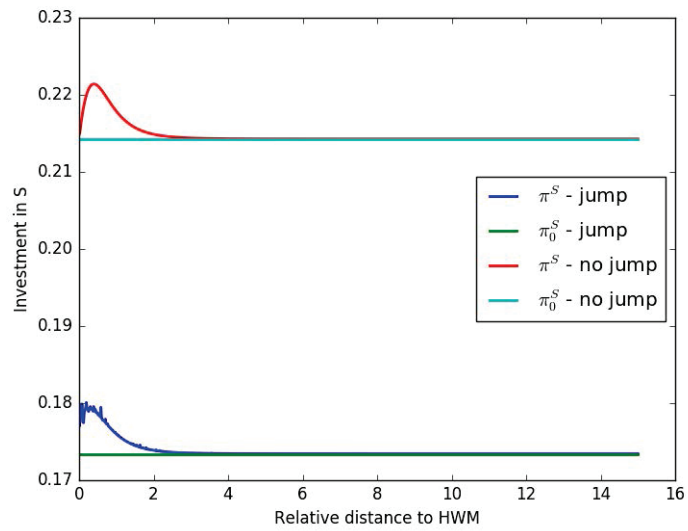


Figure 4.12: Investment proportion in stock with and without jumps ($\mathbf{q}_2, \mathbf{J}_2$).

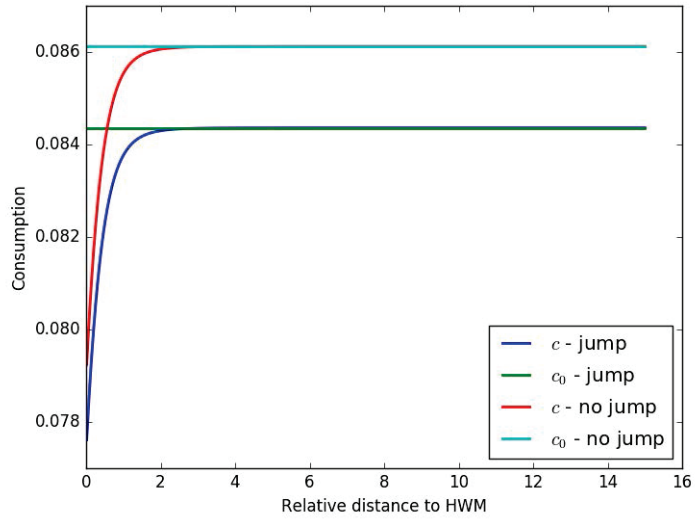


Figure 4.13: Consumption proportion with and without jumps ($\mathbf{q}_2, \mathbf{J}_2$).

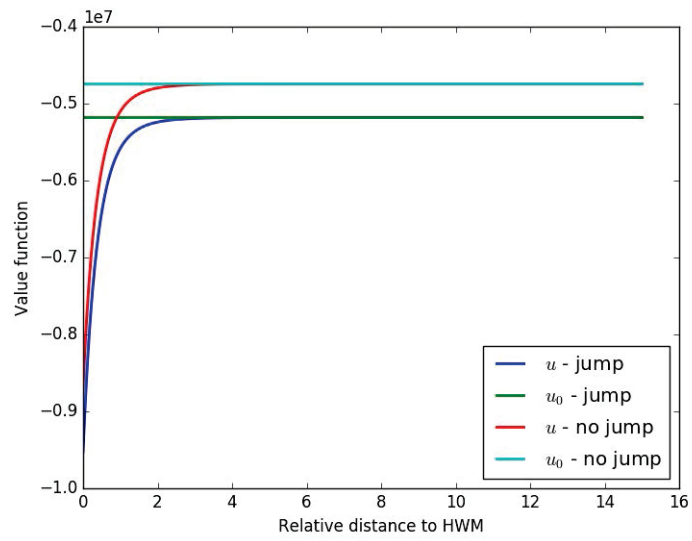


Figure 4.14: Value functions with and without jumps ($\mathbf{q}_3, \mathbf{J}_3$).

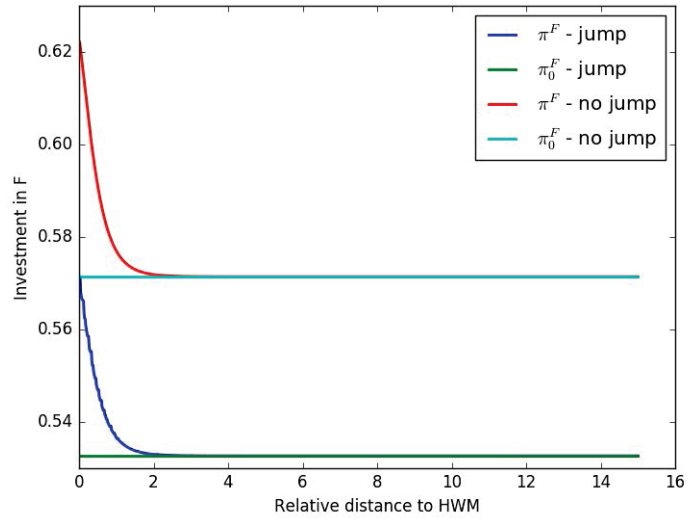


Figure 4.15: Investment proportion in hedge fund with and without jumps ($\mathbf{q}_3, \mathbf{J}_3$).

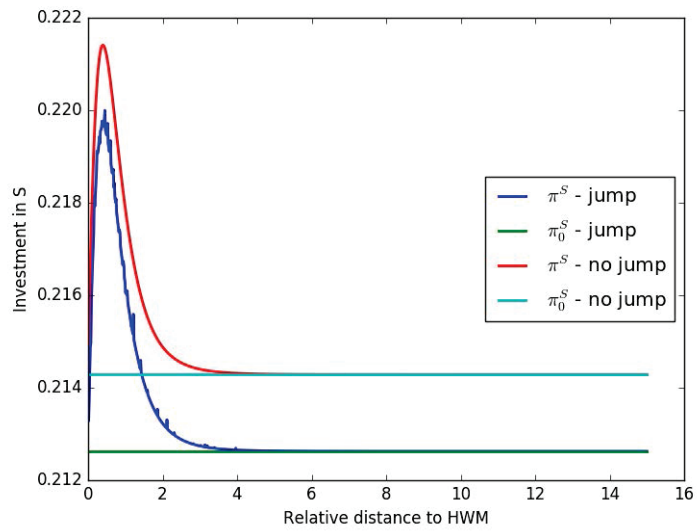


Figure 4.16: Investment proportion in stock with and without jumps ($\mathbf{q}_3, \mathbf{J}_3$).

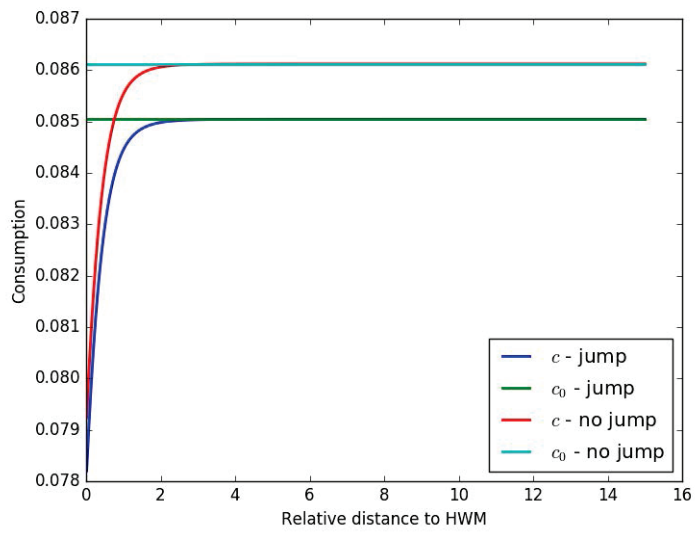


Figure 4.17: Consumption proportion with and without jumps ($\mathbf{q}_3, \mathbf{J}_3$).

Chapter 5

Conclusion

From a finance perspective, we built a general model of optimal investment and consumption when one of the investment opportunities is a hedge-fund charging high-watermark performance fees. Our model is a significant generalization of the previous model in [17] so that it can be applied in a market with more assets and richer dynamics (meaning jump price processes).

Mathematically, our approach illustrated a direct way of solving the problem of stochastic control of jump processes, by finding a classical solution to the associated HJB equation and then proving verification. This procedure can be carried out for many other stochastic control problems in different contexts.

Numerically, our iterative procedure of solving non-linear ODEs proved to be effective when dealing with ODEs of the HJB type, even when the ODEs are non-local and the boundary conditions are of different types (Dirichlet, Neumann or mixed). Also, our numerical experiment provided a variety of ways of understanding the impact of the high-watermark fees, as well as other parameters, on the behavior of the investor both qualitatively and quantitatively.

Some of the extensions and future directions are:

- The utility function in our model is limited to be power utility, to allow for dimension reduction. A natural extension is to consider general utility function. Then, we would probably need a mixture of viscosity and probabilistic techniques to solve the much more technical problem of general utility in our general model.
- In our model, we only consider one hedge fund charging high-watermark fees among all risky assets. It would be interesting to extend it to a model with multiple hedge funds each charging its own high-watermark fees. This would yield a genuine multi-dimensional control problem with reflection. However, at this moment, it's not clear to us if this much more general model is tractable.
- Our model does not address the behavior of the hedge fund manager. If the hedge fund manager can also adjust the rate of the fees and/or invest in opportunities that may or may not be accessible to normal investors, then the fund manager also faces her own utility maximization problem. In that case, we have both the investor and the fund manager trying to maximize their own expected utility, which depends on both of their strategies. We can formulate a differential game between the investor and the hedge fund manager. This is also an interesting future direction.

Appendix

Appendix 1

Appendix

1.1 Proof of Proposition 3.3.4

Proposition. *The feedback controls $\hat{\pi}$ and \hat{c} satisfy*

$$0 < \hat{c}(z) \rightarrow c_0, \quad 0 < \hat{\pi}(z) \rightarrow \pi_0, \quad z \rightarrow \infty, \quad (1.1)$$

and

$$z\hat{c}'(z) \rightarrow 0, \quad z\hat{\pi}'(z) \rightarrow 0, \quad z \rightarrow \infty. \quad (1.2)$$

In addition,

$$\hat{c}(z) > c_0 \text{ for } z \geq 1 \text{ if } p < 1 \text{ and } \hat{c}(1) < c_0 \text{ if } p > 1. \quad (1.3)$$

Proof. The limits of \hat{c} and $\hat{\pi}$ as $z \rightarrow \infty$ are direct consequences of their representations in (3.9) and (3.7) together with the fact that $zu'(z), z^2u''(z) \rightarrow 0$ as $z \rightarrow \infty$. Using the implicit characterization of $\hat{\pi}(z)$ in (3.8) and implicit function theorem, we can obtain $\hat{\pi}'(z)$. Then taking derivative of the HJB in (3.6) with $\hat{\pi}(z)$ plugged in, and using again $zu'(z), z^2u''(z) \rightarrow 0$ as $z \rightarrow \infty$, we conclude that we also have

$$z^3u'''(z) \rightarrow 0 \text{ as } z \rightarrow \infty.$$

We can now use the representations (3.9) and (3.7) and take their derivatives (meaning implicit function theorem for (3.7)) to obtain (1.2). The qualitative description of consumption (1.3) is obtained from the representation (3.9) using the fact that $u' \geq 0$ and $u \leq u_0$. \square

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Index

- Abstract, vii
- Acknowledgments*, v
- Appendix
 - Appendix*, 85
 - Appendix*, 84
- Bibliography*, 91
- Conclusion*, 82
- Dedication*, iv
- Dynamic programming and main results*, 22
- Introduction*, 1
- Model*, 8
- Numerics*, 67