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Jair Javier Aguilar Batista

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The Dissertation Committee for Jair Javier Aguilar Batista Certifies that this is the approved version of the following dissertation:

MODELING THROUGH MODEL-ELICITING ACTIVITIES: A COMPARISON AMONG STUDENTS AT DIFFERENT PERFORMANCE LEVELS

Committee:

Flávio S. Azevedo, Supervisor

Jill A. Marshall

Catherine Riegle-Crumb

Maura Borrego

María de los Ángeles Domínguez Cuenca

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by

Jair Javier Aguilar Batista

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Dedication

To my wife Julia, who is my strength, partner, lover, and twin soul. Te amo!

To my sons Diego and Dante, who are my motors and reasons to keep going.

To my parents Boris and Angie, because without your endless support and example, I wouldn't be here.

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Jair Javier Aguilar Batista, PhD

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Supervisor: Flávio S. Azevedo

Abstract:

My dissertation investigated students' thinking, understanding, and mathematical development when solving mathematics classroom open-ended, real-life context tasks in which they were required to model a solution. Specifically, thought-revealing activities known as Model-Eliciting Activities (MEAs) (Lesh et al., 2000) were used to compare and contrast the reasoning and solutions of 11th grade students at different levels of performance in a northeastern Mexican private institution. Two aspects of student work were analyzed: The quality of the students' final and intermediate product-solutions and the patterns of group discourse that emerge while they engage such process. Seventy-two students at different level of performance (e.g., low, average, and high performance), as measured by tests and classroom activities, participated in the study. Students formed teams of three members for a total of 24 teams. A case study methodology was used to study three MEAs adapted to the context of the target population. The questions addressed in this research included: To what extent do students' products and solution processes differ across performance levels (low, average, and high-performance) in

Model-Eliciting Activities? What are the mathematical elements, models types, and strategies that students used during the problem-solving process, and how do these differ among students labeled as low, average, and high performance (as measured by classroom's grades and test scores)? The findings reveal low-performing students' ability to propose and develop adequate solutions, comparable to the average- and high-performance peers. Furthermore, solutions provided by "average" teams were not so different from the ones provided by the low performers. In addition, although some high-performance teams used more sophisticated elements in their strategies and final and intermediate product-solutions, some teams in this category had incomplete and not well-developed solutions. Finally, transcripts of students' work and discourse patterns reveal that students engaged in a pattern of argumentation discourse providing claims, evidence, and justifications for their thoughts. In addition, students engaged in describing and explaining type of discourse. Specifically, low-performance teams engaged in an argumentation type of discourse almost double the time in comparison to their average- or high-performance peers. In contrast, average- and high-performance teams engaged mostly in a describing or explaining discourse type, aligning most of the time with each other. In the end, teams at all three performance levels proposed solutions that were adequate and comparable, but also rich and diverse. However, more evidence is needed to test the conjecture that MEAs can indeed level the playing field among students of different levels of achievement.

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Chapter 1: Introduction

In this dissertation, I investigated how students developed their mathematical reasoning skills as they worked to understand and solve open-ended, real-life context problems by modeling a solution. To do this, I first video-recorded 11th grade students engaged in thought-revealing activities known as Model-Eliciting Activities (MEAs) (Lesh, Hoover, Hole, Kelly, & Post, 2000). Then, I compared and contrasted their reasoning and solutions across different performance levels — low, average, and high, as measured by standardized test scores — and across the dimensions of (1) products and (2) processes. The products are students' model-solutions and how well they help solve a problem. Processes are the set of steps needed to create the product, including elements or components of mathematical constructs (e.g., graphs, variables, tables, lists, inscriptions, ideas) and strategies. My hypothesis is that MEA-based activities, given their open-ended yet structured nature, help students at all levels achieve equally adequate good solutions and develop their mathematical reasoning skills. Overall, the aim of my research was to find evidence that support my claims.

It is not uncommon for students across all performance levels to struggle when solving a traditional textbook mathematical problem, but low-performing students tend to struggle even more, often because their teachers often underestimate their abilities (Madon, Jussim, & Eccles, 1997). What I aim to demonstrate is that, by using MEAs, teachers can help *all* students without leaving "low-performing" students behind, since these activities enable equally good outcomes across all performance levels. The main difference between MEAs and traditional classroom tasks (Alsup & Sprigler, 2003; Dewey, 1998; Beck, 1956) is that, in MEAs, the problem-solver's goal is to find a solution to a problem by focusing on the solution process (Lesh & Doerr, 2003) rather than looking for a single answer using predetermined mathematical algorithms. That is to say, the solution to a MEA is the *model* the problem-solver produces to provide the best answer to the problem, not the answer itself. MEAs thus enable students to interpret, invent, and find solutions in ways that “jump” the barriers of “achievement stereotype” by focusing on process instead of more rigid strategies. For example, Carmona and Greenstein (2007) compared the solutions that a group of 3rd graders and a group of post-graduate math and science students brought to the same MEA. The activity required the problem-solvers (i.e., the students) to rank 12 teams of players presented on a two-dimensional coordinate system. Working in small groups, the students were required to model a solution to rank the top five teams with the most wins. Each group worked for approximately two hours developing a solution to the problem. By building on their previous knowledge and experiences, the problem-solvers needed to decide which solution-paths led to the best outcome, and which mathematical ideas worked best in developing their constructs.

Each group of students was very different in their respective mathematical skills, age, and grade level, yet Carmona and Greenstein showed that both the elementary and post-graduate students developed adequate solutions. As expected, the post-graduates’ solutions were more mathematically advanced (e.g., they used vector concepts and Pythagorean formulae) and better justified. However, the elementary students’ models were as adequate — in terms of the process and model developed — as the post-graduates’ for providing accurate answers to the proposed problem. If we consider the elementary students as “low-performing” and post-graduates as “high-performing,” then we can conclude two things: first, that MEAs can work to “level the playing field” across performance levels, and I seek to verify and demonstrate that students at these different levels can indeed provide equally adequate solutions to activities like MEAs;

second, that the degree of difficulty any particular problem poses for each performance level is "better determined by the solutions the problem solver produced" (Carmona & Greenstein, p. 253) than by the ability (or lack thereof) teachers presuppose students to have attained. In contrast to Carmona and Greenstein, my research shows that different performance levels within the same grade level work the same way: although low-performing students often use different processes than their average- or high-performing peers, they attain equally adequate outcomes, and in some instances even better in terms of ideas' justification, reasoning, and explanation. Further, I show that the types of interaction "low-performing" students engaged in enhanced their group and individual learning in comparison to their average- and high-performing peers. I explain and detail this in chapters three and four.

1.1. Model-Eliciting Activities: A Brief Introduction

Model eliciting activities (MEAs) are open-ended, real-life mathematical problems in which students work in small teams to model a solution (Lesh & Doeer, 2003). They follow six principles of design (Lesh et al., 2000), which I discuss in detail in chapter two. To illustrate their potential as both research and instructional tools at the outset, in this section I describe the *Historic Hotel* problem, one of three MEAs implemented in my research. The activity describes a decades-old hotel that has suffered financial and management problems and striven to stay in business for many years. It is based on an economic problem (Aliprantis, 1999) commonly used in post-secondary calculus courses to illustrate economic and mathematical concepts like price, equilibrium, cost, profit, variable recognition, maximization, and product of linear relations (Aliprantis & Carmona, 2003).

The classroom implementation of the *Historic Hotel* activity consists of three main parts: the warm-up day, day one work, and day two work. The warm-up day contains two steps, assigned as homework. First, to contextualize the problem, students are assigned a newspaper article related to historic hotels (see Appendix A). This reading imposes a "fairly superficial demand" on them (Chamberlin & Coxbill, 2013) since no mathematics is involved. However, it leads to the second warm-up step, which represents a more cognitively demanding task: answering a set of comprehension questions about the article, forcing the students to reflect on the topic and the problem to be solved. Students are to answer four to five such comprehension questions. For the *Historic Hotel* problem, some of the questions may include, "What do hotels have to accomplish in order to be recommended by the National Trust Historic Hotels of America?", "How many people have owned the hotel since it opened?", or "What are some of the responsibilities that a hotel manager might have?" Although the warm-up does not require students to get involved in any mathematical work, it connects the content of the task to real-world problems by introducing the topic and getting the students to engage with it.

At the beginning of day one work, the teacher discusses the homework assignment and the comprehension question with the students in order to connect the reading with the problem-solving activity that is to follow. Then, students receive the *Historic Hotel* problem statement, which defines the mathematical problem they are required to model:

> Mr. Frank Graham has just inherited a historic hotel. He would like to keep the hotel, but he has little experience in hotel management. The hotel has 80 rooms, and Mr. Graham was told by the previous owner that all of the rooms are occupied when the daily rate is $60 per room. He was also told that for every dollar increase in the daily $60 rate, one less room is rented. So, for example, if he charged $61 dollars per room, only 79 rooms

> would be occupied. If he charged $62, only 78 rooms would be occupied. Each occupied room has a $4 cost for service and maintenance per day.
>
> Mr. Graham would like to know how much he should charge per room in order to maximize his profit and what his profit would be at that rate. Also, he would like to have a procedure for finding the daily rate that would maximize his profit in the future even if the hotel prices and the maintenance costs change. Write a letter to Mr. Graham telling him what price to charge for the rooms to maximize his profit and include your procedure for him to use in the future (Carmona, 2003).

Students engage with the problem statement for approximately 60 minutes. They spend ten minutes reading the problem statement and identifying the elements of the problem formulation, then work on developing a model[1] in small teams of three or four. The teacher has previously formed the teams, or has asked them to do so.

Some elements of the problem's formulation include a client for whom students develop their models, the client's needs, and the reasons for the client's needs. For example, in the *Historic Hotel* problem, the client (Mr. Graham) is the owner of the hotel, his need is a model that will help him maximize profits, and the reason for this need is that he wants to keep the hotel, regardless of his lack of experience.

During day two work, teams present and share their findings to the rest of the class. For approximately 60 minutes, each team explains how they constructed their model-solution, what mathematical elements they used, and what factors they considered. After each presentation, the teacher and students have time to ask questions of each other, provide interactive feedback, and

[1] Model refers to a mathematical solution (i.e., conceptual systems composed of elements and relations that are

engage in substantive mathematical argument and discussion. While students communicate their solutions, they are externalizing their thoughts, which leaves a trail of documentation (Lesh & Doerr, 2003) teachers and researchers can use to have a better understanding of students' thought processes. Students also leave a similar trail when they write letters of advice to the client.

1.2. RATIONALE FOR THE STUDY

Scholars have researched MEAs across a variety of contexts, educational settings, and grade levels, from elementary school to college (Carmona, 2004; Mousoulides & English, 2011; English & Mousoulides, 2011; Moore & Diefes-Dux, 2004; Lesh & Doerr, 2003; Hernández, Cantú, Domínguez, 2011). However, at least two research areas have been neglected in these studies. First, only a few scholars have tested MEAs in pre-college courses (e.g., high school levels), as I do here. According to Yu and Chang (2009), some reasons for this may be teachers' lack of training or professional development in implementing MEAs, insufficient knowledge in adapting curricular content to MEAs, uncertainties about the time required to implement MEAs, or simply because they have other priorities (e.g., preparing students for tests).

Second, most studies have focused on *products* rather *processes*, and even fewer have investigated the patterns of discourse that emerge during group work. Researchers have considered the students' models, with their elements and strategies, rather than the problem-solving group interactions that obtain during the development of the model. In my research, I examined the product and also the process through which the students obtained the product. In particular, I focused on the patterns of discourse students involved while working on the process-solution. These helped me to have a better understanding of the richness of the their reasoning across all levels.

I carried out my research in Mexico, where very little research implementing MEAs among high-school students has been conducted. Hernandez et al. (2011) studied student abilities and the learning outcomes MEAs produced in Guadalajara, but no one has contrasted and compared MEA results across performance levels, and few have studied high school students.

Applying MEAs as a mathematical, open-ended task at the high school level in Mexico is important for three reasons. First, a majority of Mexican high school students demonstrate low performance on the *Programme for the International Student Assessment* (PISA). Second, the most recent standardized assessment in Mexico, the National Plan for the Assessment of Learning (PLANEA for its acronym in Spanish) shows that almost 50 percent of high school students can perform only simple mathematical operations. Third, the core competencies and standards of the national high school system in Mexico mandate that students should be able to apply mathematical principles to solve real-life problems by the end of their secondary education, yet the majority of Mexican students are not.

In the long run, I believe MEAs can help rectify all of these deficiencies among Mexican high-school students. My hypothesis is that MEAs provide an alternative teaching and learning method to help level the playing field between low-, average- and high-performing students, i.e., to help all students (regardless of their performance level) achieve adequate results that solve a proposed situation. And if this is true, I believe they can also help raise performance levels on PISA and PLANEA, since these assessments test students' ability to model real-life problems, just as MEAs are designed to do (PISA, 2012b). Thus, my research in this dissertation is a first step in developing further research on how to raise overall math achievement levels among high school students.

1.3. MEAs as Case Study

In carrying out my research, I took on the dual role of co-teacher and ethnographic researcher collecting data in the classroom. Because my primary aim was to compare and contrast the solutions of students across different performance levels, I adopted a case study methodology in order to focus in on the specific phenomenon (Merriam, 2009, as cited by Harrison, Birks, Franklin, & Mills, 2017, p. 5) of the products and processes students used to solve the MEAs. Case studies allow researchers to interpret both participants' perspectives (Simmons, 2009) in solving mathematical activities, as well as how these are informed by external factors such social stereotypes or academic achievements.

As previously mentioned, one of my main goals was to analyze students' discourse patterns that emerged during group work, making a case study methodology a perfect fit. In case studies, an "inductive reasoning and interpretation rather than testing [an] hypothesis take[s] priority" (Harrison et al., p. 9), the aim being "to provide a rich, holistic description that illuminates [one's] understanding of the phenomena" (Merriam, 1998). The type of case study method I implemented in this research is both descriptive in nature (Yin, 2003) and multiple in its application. It is descriptive in that it describes "an intervention [in] the real-life context in which it occurred" (Baxter & Jack, 2008), and multiple in application in that it "enabled the researcher [me] to explore differences within and between cases," drawing contrasts and comparisons (Yin, 2003 as cited by Baxter & Jack, 2008, p. 549).

As a qualitative method, case studies are naturalistic, i.e., “descriptive, inferential, and based on evidence” (Gillhanm, 2000, p. 10). By applying them to the analysis of MEAs, I was able to (1) connect the students’ thought processes with educational theories of model and modeling (Lesh & Doerr, 2003), (2) draw inferences about students’ products, processes, and discourse patterns, and thus (3) generate evidence-based claims that enable me to help educators to improve their understanding of how students learn mathematics, and thereby enhance their mathematical pedagogical knowledge (Barab & Squire 2004).

1.4. Summary and Research Questions

More and better strategies for teaching high school students how to solve real-life problems effectively, creatively, and mathematically are needed. I have presented MEAs in this research to a number of ends. First, to illustrate a possible alternative for presenting instructional models in ways that enhance and elicit mathematical literacy for high school students at different performance levels; second, as a way to provide a research-based instructional tool that teachers might use to motivate and engage their students; third, to make mathematics curriculum relevant for real-life contexts (since MEAs by design simulate real-life situations), particularly on topics requiring students to model solutions; fourth, to counter the stereotype that low- and (perhaps) average-performing students cannot solve open-ended mathematics activities (like the one presented here) as adequately as students stereotyped as high-performing can, and even to show they can “jump” the boundaries that define these stereotypes; fifth, to illustrate the richness of the students’ solutions and reveal the types of discourse that emerged while students solved the

MEAs. Finally, that students at any performance level can engage in a type of interaction that elicits a mathematical solution.

I used a case-study methodology that documents the interaction of the participants (students and teacher) and researcher as participant-observer, to ethnographically capture and experience the process of model-development the students went through when solving the MEAs.

I propose MEAs be implemented with students at different performance levels (low, average, high) in order to compare and contrast their mathematical reasoning and solutions (i.e., products and processes), to find evidence that MEAs can indeed *level the playing* field across performance levels, and that although students' products and processes across those levels might be different, they are adequate, creative, unique, and acceptable to provide a solution to a problem and solve the clients' needs. To this end, my research answered the following questions:

1. To what extent do students' *products* and *processes* differ both within and across performance levels? More specifically:
   1.1. What are the mathematical strategies, type of models, and elements students used during the problem-solving process of their models?
   1.2. How adequate are the students' solutions?
   1.3. What is the quality of student products based on their letter to the "client"?
   1.4. How do students' problem-solving processes effectively correlate to their products?
   1.5. What type of discourse patterns did students engage in the solving process of MEAs?
2. To what extent does the quality of the students' products change from the second MEA to the third based on the Quality Assessment Guide?

Chapter 2: Literature Review

Exposing students to mathematical modeling activities helps them to grasp mathematics and its applications in a real-world context (Asempapa, 2015). It is easier to motivate and encourage students to learn mathematics when they see a connection between something they already know and a new task (Protheroe, 2007). Modeling activities close "the gap between students' real-life experiences and mathematics…and provide the means to understand real-life situations" (Blomhøj, 2004, as cited by Chan, 2008, p. 49). In addition, models are "powerful tools for solving problems" (Chan, p. 61), and if students are trained in how to develop and apply models and modeling in authentic settings, it may be easier for them to overcome the challenges of the modern world.

MEAs are designed to simulate real-life situations. They engage students in collaborative efforts (i.e., working in small teams) to mathematize a real-life problem; in other words, to engage in a process of problem solving that is repeatedly reviewed, tested, and reconstructed (Doeer & English, 2003).

2.1. What are Model-Eliciting Activities?

MEAs arose from educators' and researchers' wishes to implement problem-solving activities that were self-adapting (i.e., encouraging understanding and interpreting), self-documenting (i.e., leaving a trail of documentation), self-monitoring (i.e., requiring observation, revision, and refinement of solutions), and knowledge-sharing (collaborative sharing of ideas) (Lesh et al., 2001). Lesh and Doerr (2003) then combined earlier forms of MEA construction with the concepts of models and modeling to create and define MEAs that motivate students to mathematize real-life problems by "quantifying, coordinatizing, categorizing, algebratizing, and systematizing" (Lesh & Doerr, 2003, p. 2) its elements so as to produce a model-solution. They define models as:

> Conceptual systems (consisting of elements, relations, operations, and rules governing interactions) that are expressed using external notation systems, and that are used to construct, describe, or explain the behaviors of other system(s), perhaps so that the other system can be manipulated or predicted intelligently (p.10).

In a classroom context, models are a "student-generated way of organizing their mathematical activity with physical and mental tools" (Zandieh & Rasmussen, 2010, p.68). Modeling, then, is the process through which students construct or adapt and structure conceptual systems — i.e., models — in order to build to solve real-life problems (Zandieh & Rasmussen, 2010; Ekmekci, 2013; Lesh et al., 2000).

MEAs are open-ended problem-solving activities in which students working in small teams are encouraged to develop and generate useful solutions (i.e., conceptual systems or models) to real-world problems by repeatedly communicating, testing, refining, and extending their thoughts (Lesh et al., 2000). These thought processes often involve several modeling cycles.

2.1.1. Modeling Cycles

Within MEAs, a modeling cycle is the process by which students describe, manipulate, predict, and verify their mathematical constructs, then adapt, modify, and/or refine their own knowledge and ideas (Lesh & Doerr, 2003). Often, they must go through several cycles of modeling to interpret and improve their products (Kaput, 1998) in ways that go beyond "just providing an answer" to offering unique solutions to a problem (Lesh & Doerr, 2003). Multiple cycles of modeling enhance students' learning because students "have multiple opportunities to invent, revise, and then compare the explanatory adequacy of different models" (Lehrer & Schauble, 2006, p. 382).

There is a basic four-step process for the modeling cycle (Lesh & Doerr, 2003):

1. Description: Students describe the real world in terms of a mathematical model to make sense of the situation. For example, in the *Historic Hotel* MEA, "profit" represents the relationship of the real world to the mathematical-model world, and in their several implementations of this MEA, the students frequently defined profit as:

P=(booking price) • (rooms booked) – (Maintenance fee) • (rooms booked)

2. Manipulation: Students manipulate the mathematical models they have created in order to verify and refine them to make predictions about the intended solution.

3. Translation: Students transfer their mathematical model to the real world in order to make predictions and test their results against the situation or problem. For example, according to the model for the *Historic Hotel* shown above (no. 1), it is common to see that students test their model using different prices and number of rooms booked. (see example Table 2.1). As the table illustrates, students translated from the mathematical world (i.e., the model representation through symbols and letters) to the real contextual

world (presented in the MEA) using numbers and operations that could make sense to them.

Table 2.1. Example of a cycle of modeling

Price	Rooms	Maintenance	Profit (P)
600	80	$40 x 80	($600 x 80) – ($40 x 80) $48,000 - $3,200 = $44,800
700	70	$40 x 70	($700 x 70) – ($40 x 70) $49,000 - $2,800 = $46,200

Table 2.1. Example of a cycle of modeling where students test their model and predict what the hotel's profits would be. As explained above, students translated the profit from their model world to the real world by predicting what would be the profit at different prices and rooms. They still would need to verify if those prices are the ones that maximize the profit.

4. Verification: Students verify, examine, and validate the usefulness of the models and predictions, so it makes sense for the context of the situation.

Developing a productive interpretation of a complex problem-solving situation normally requires engagement with many modeling cycles. To compel such engagement, MEAs must follow a design that includes six specific principles, which I describe in the following section, that ensure the activities elicit and reveal students' mathematical "big ideas."

2.2.2. Principles of Model-Eliciting Activities

There are six principles guiding the design of MEAs to ensure that their implementation contributes to students' understanding of mathematical concepts and develops their problem-solving, communication, and teamwork skills (Yildirim, Shuman, Besterfield-Sacre, & Yildirim, 2010). Lesh et al. (2000) described the six principles as follows:

2.1.2.1. Personal Meaningfulness Principle

This principle — also known as the reality principle — encourages students to draw on their background and past experiences to make sense of the real-world situation and motivates them to reflect on whether the situation presented could happen in real life. The reality principle "stipulates that the problem must be meaningful and relevant to the students" (DelMas, Garfield & Zieffler, 2009, as cited by Chamberlin & Coxbill, 2013, p. 172). The aim is to engage students and increase their interest in solving the problem. For example, in the *Historic Hotel* MEA, the warm-up reading and problem statement are contextualized in the setting of a hotel with many years of history, adapted to the students' reality, so they recognize or identify a hotel in their own city, or past experiences.

2.1.2.2. Model Construction Principle

This principle encourages students to recognize the necessity of applying a model-solution to solve a problem in which "patterns and rules" (delMas, et al., 2009) govern the relationship between the elements of the model. An MEA requires students to construct, describe, explain, manipulate, predict, and control a relevant conceptual system. For example, the problem statement of the *Historic Hotel* requires students to model a solution that would maximize the hotel's profit, i.e., to explain in detail what price per room should be charged and how many rooms should be booked. The mathematical model solutions created according to this principle reveal how students "interpret the activity and the type of mathematical quantities, relationship, operations, and patterns that they take into account" (Chamberlin, 2002, p. 49).

2.1.2.3. Self-Evaluation Principle

This principle states that students should assess the usefulness and appropriateness of their solutions, and how these might solve the real-life situation proposed. Students must judge whether their solution is good enough to solve the problem, and if not, they might refine, revise, or modify it. For example, in the *Historic Hotel*, students assess the usefulness of their model — i.e., verifying whether the profit has been maximized — by comparing and contrasting booking rooms and prices.

2.1.2.4. Model-Externalization Principle

This principle is also called the "Model Documentation Principle" requiring students to explain their thoughts in detail. Students reveal their thoughts — i.e., externalize their thinking — in many different ways: when they engage in a discussion through the process of model development, while they collect and analyze data in which they have to perform some operations, when writing their solution report (i.e., the letter to the client), or when developing their final presentation or sharing their solution with their peers. These are the reason MEAs are also called thought-revealing activities. The fact that MEAs are thought-revealing benefits both teachers and researchers — teachers because it enables them to assess students' work (i.e., understanding and reasoning) in order to better target their needs, and researchers because it enables them to investigate the details of students' developing mathematical reasoning. It also helps students by serving as a form of self-assessment to verify their model-solution (Lester, 2007). In a pilot study in which I implemented the *Historic Hotel Problem,* students created flip-chart presentations (See Figure 2.1) to externalize and share their solutions. The image below shows how they interpreted the MEA, the type of operations they undertook, and the solutions they obtained. This way of externalizing ideas served not only to show the students' work, ideas, and reasoning, but also to mediate students' conversation regarding different solution paths.

Figure 2.1. Example of an externalization.

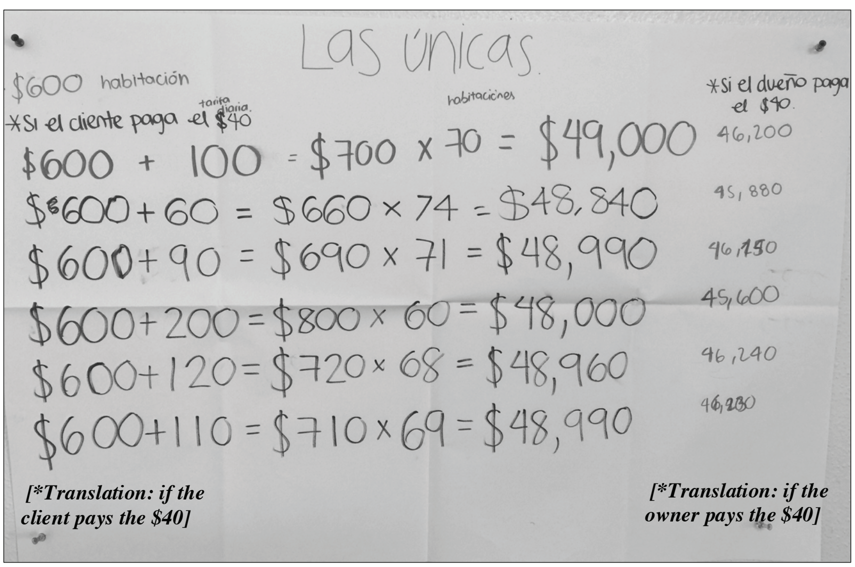


Figure 2.1. In this image, students depicted a model in which the hotel's profit changes based on the number of booked rooms and the price of each. Additionally, the students contrasted the hotel's profit considering whether its clients or its owner pays the maintenance fee.

2.1.2.5. Simple Prototype Principle

This principle requires teachers or researchers to present students with an activity that is relatively simple, yet mathematically significant and capable of making them think and create meaningful models. The model the students will create must be easy to interpret and understand by others. In the *Historic Hotel* MEA, the problem statement is clear and simple: Mr. Graham would like to know how much he should charge per room in order to maximize his profit and what his profit would be at that rate. In many cases, students use tables as a representation to visualize prices, rooms booked, and profits.

2.1.2.6. Model Generalization Principle

This principle requires researchers to design MEAs so students' predictions and solutions are reusable, modifiable, and applicable to many analogous situations. For example, in the problem statement of the *Historic Hotel*, students are asked to include a procedure that could be to used in the future. A group of students solving this MEA created a model for calculating the hotel profit "P" based on the number of rooms that remained unbooked (i.e., "x") and the price of the room per night (i.e., "y"):

$$P = [(80 - x)\ (y)] - [(80 - x)\ (40)];\ y = 600 + 10x.$$

This model might be used in similar situations (e.g., other hotels or rental properties) where a quantity needs to be maximized.

The six principles, when properly applied, not only guarantee that every MEA complies with the academic objectives and characteristics of its design; additionally, it ensures that students engage in the problem-solving process mathematically and externalize their ideas (Chamberlin & Moon, 2005) so others can understand their thought processes. This externalization process, which involves reflecting and communicating, leads students to compare and contrast their own ideas with other points of view (Hiebert, Carpenter, Fennema, Fuson, Wearne, & Murray,1997).

2.1.3. Multiple Perspectives in MEAs

Because MEAs are collaborative activities in which students work in small teams, each team member can use the others' knowledge and expertise to discover various solutions to a problem beyond the limits of his or her own experiences (Huxham & Vangen, 2013). Students can then test, refine, or reject them to discover the best solution to a problem. MEAs, therefore, provide students with a problem-solving process that incorporates many perspectives and gives them an opportunity to learn from these different perspectives on the same problem (Figure 5). This is called the multiple perspective principle[2] (Lesh, Cramer, Doerr, Post, & Zawojewski, 2003), and it is the cornerstone of MEA design. A well-designed MEA will provoke students to engage in this collaborative process, forcing them to take multiple perspectives on a problem. This makes the *process of solving the problem* more important than any specific solution to an MEA.

[2] The Multiple Perspective Principle is a conceptual part of the model and modeling perspective, and should not be confused with the MEA's six design principles.

Figure 2.2. Single Perspective vs. Multiple Perspectives

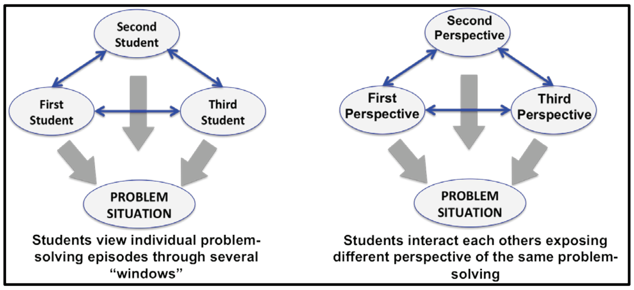


Figure 2.2: This image depicts different perspectives on the same problem (Lesh et al., 2001). On the left, students' ideas are viewed as isolated possibilities to solve a situation. Perhaps a process of reflection occurs (Hiebert et al., 1997). In contrast, on the right, students work collaboratively by communicating with each other (Hiebert et al., 1997) to find a solution to a problem. In this process, students reveal ideas they may then re-consider, revise, reject, or develop.

Previous scholars have addressed how developing multiple perspectives during the problem-solving process is beneficial for the problem-solver. Guilford (1984) states that the more alternatives generated during the process of solving a problem, the better the solution can be. Carr and Borkowski (1987) state that multiple perspectives are important when thinking about the solution to a problem, or when striving to foster students' reflection and communication. Similarly, Hiebert et al., (1997) define reflection as a personal development of ideas that takes place internally, and communication as a collaborative process where the individual shares, argues, and discusses his/her ideas and others' ideas. These cognitive processes drive students to generate a number of possible models, since they will be "evaluating their ideas and deciding which one is more effective or relevant" (Wang, Dogan, & Lin, 2006, p. 813).

2.1.4. MEAs as an Assessment Tool

MEAs are thought-revealing activities because students leave a trail of documentation that "reveals their thinking naturally" (Lester, 2007, p. 796) as they work to solve a problem, as I have addressed above in the model-externalization principle. This characteristic allows researchers and teachers to use MEAs as an assessment tool to evaluate students' mathematical "knowledge, insights, understanding, skills, achievement, performance, and capabilities" (Niss, 1998, as cited by Pegg, 2003, p.228). Assessing the mathematical understanding of students solving MEAs is important for teachers and researchers in that it helps them: (1) verify which concepts students have acquired and which need to be reinforced, (2) uncover students' skills in the use of models and modeling, and (3) identify abilities that may otherwise not have been identified, such as creativity in solving a problem (Coxbill, Chamberlin, & Weatherford, 2013; Shuman, Besterfield-Sacre, Sieworiek, Bursic, & Vidic, 2011).

In my research, I assessed the products (i.e., models) and processes (i.e., the path students followed to get the product) encouraged by MEAs. My intention was to attain a better understanding of students' mathematical thought and reasoning in order to compare and contrast their solutions (i.e., the products and the processes) at different performance levels in two dimensions —deductively and inductively, as explained in Chapter Three.

2.2. ALTERNATIVES TO MODEL-ELICITING ACTIVITIES

In this study, I present MEAs as an instructional and research tool useful in revealing students' mathematical reasoning when solving real-life problems that require them to construct a conceptual model for a specific "client" or purpose. Although many characteristics of MEAs are beneficial for students, teachers, and researchers, MEAs are not the only problem-solving, open-ended activity type that can be implemented in a classroom setting. In fact, some characteristics of MEAs characteristics are similar to those of other instructional problem-solving activities such Inquiry-Based Learning, Benchmark Lessons, or Problem-Based Learning. In the following paragraphs, I outline each of these strategies and compare them to MEAs in order to unveil their similarities and differences and to show some reasons why MEAs fit the setting and context of this research.

2.2.1. Inquiry-Based Learning

Researchers and teachers use Inquiry-Based Learning Activities (IBLA) to motivate students to learn through "inquiry and engagement" (Georgette, 2013, p. 9). Their design encourages students to predict possible outcomes of a scenario, document events as they would unfold, and draw conclusions about the achieved results.

According to Georgette (2003), in IBLAs, students are encouraged not just to believe everything an instructor says, rather — because "physical experimentation communicates relevant information to the learner" (p. 9) — they are pushed to consider reality (i.e., real-life factors) and to find their own answers to the problem posed. Further, they create their own “experiments and engage in the learning process, where they develop conceptual understanding” (Georgette, 2003, p.9). Similarly, Prince and Vigeant (2006) define Inquiry-Based Learning Activities as activities in which “students pose and answer questions through physical experience and direct observation rather than by listening to lectures or following a highly prescribed laboratory experience” (p. 4). This definition implies a combination of several elements, also addressed by Laws, Sokoloff and Thornton (1999): (a) the use of peer instruction and collaborative work, (b) the use of activity-based guided-inquiry curricular materials, (c) the use of a learning cycle beginning with predictions, (d) an emphasis on conceptual understanding, (e) the necessity to let the physical world be the authority, (f) the evaluation of students’ understanding, (g) the appropriate use of technology, and (h) moving from the specific to the general.

Compared to traditional problem-solving approaches, IBLAs seem to benefit both students and teachers — students in solving problems and teachers in assessing their students’ problem solving-processes. For instance, Thacker, Kim, Trefz, and Lea (1994) compared IBLAs to traditional methods of teaching physics to undergraduates and found that students exposed to the IBLAs performed “significantly better than the ones not exposed to IBLA” (p. 631). However, unlike MEAs, they do not specifically require students to develop a conceptual tool (i.e., a model) that is sharable (Lesh et al., 2000). They do not require students to predict what the solution will be or what it will look like, nor do they motivate an interactive cycle of modeling to develop a mathematical solution (Lesh & Doerr, 2003). Additionally, “inquiry and engagement” may not motivate students sufficiently to foster learning, as Kirschner, Sweller, and Clark (2006) have shown. Finally, in IBLAs students have no context for the problems they are asked to solve, whereas in MEAs they begin to work on an activity by reading an article describing the situation they are tasked to address.

2.2.2. Benchmark Lessons

Benchmark Lessons (BL) are instructional and assessment strategies used to improve students' conceptualizations by encouraging them to share productive ideas. They are designed to foster "sense-making activities" that help students connect existing ideas with what they already know from their context-world so they may develop those ideas further (Boaler, 2011; Arcavi, 1994; Silver, 1994). According to diSessa and Minstrell (1998), the most important characteristics of BLs are that they: (a) are memorable, (b) possess content and epistemological and social goals, (c) represent the beginning of an extended process, (d) focus on important issues and familiar ideas, (e) foster progress across multiple dimensions without restraining students' ideas, participation, or contribution, (f) develop both teachers' and students' skills and attitudes, and (g) are difficult to learn and to implement, but are highly rewarding.

The BL teaching method is based on continual discourse between teachers and students. When teachers implement BL activities, they engage in rich discussions aimed at exploring, understanding, reflecting upon, and enhancing students' mathematical or scientific ideas (diSessa & Minstrell, 1998), and at reviewing the principles, concepts, and skills students need to develop their classroom projects. For this reason, BL activities should be designed to create a safe environment in which students are encouraged to share their ideas and take ownership of their contributions, and teachers can then draw upon these ideas to use students' prior understanding as a "scaffold that provides a reference marker in the process of learning" (diSessa & Minstrell,

p. 157). BLs thereby entail “a central commitment to students’ learning of important subject matter as a natural continuation of their own ideas and sense-making capabilities” (diSessa & Minstrell 1998, p. 166). Such a discourse-focused method encourages teachers to pay closer attention to students’ ideas than other teaching methods demand. In the process, students develop a mathematical or scientific perspective in which the learning process is the result of a trail of “exchange and reflection” (diSessa & Minstrell 1998, p. 181).

BL activities thus encourage students to engage in a well-designed discussion, which the instructor moderates while drawing on students’ previous knowledge as a scaffold for developing their skills and understanding. For example, Azevedo, Martalock, and Keser (2015) conducted research in which students developed a solution to an open-ended activity, wherein the teacher was responsible for “orchestrating substantive and productive discursive practice among all participants” (p. 296).

MEAs, on the other hand, require students to invest time working in small teams in order to find a solution or develop a conceptual tool. In this process, students reflect, revise, reject, and evaluate all their ideas, and enter into the cycle of modeling as many times as needed.

2.2.3. Problem-Based Learning

Problem-Based Learning (PBL) was developed in the 1960s at the University of Canada’s McMaster’s Medical School (Lohfeld, Neville, & Norman, 2005) to help instructors assess their “students’ ability to function as practicing doctors, and to aid students in seeing real-life scenarios” (Chamberlin & Moon, 2008, p. 3).

PBL activities require students to go through eight steps (Fogarty, 1997): (1) to be presented with the problem they have to work on, (2) to define and understand the problem, (3) to gather all the facts and relevant information about the problem, (4) to hypothesize a solution, (5) to research and get to know more about the problem, (6) to rephrase the problem, (7) to generate as many alternative solutions as possible, (8) and to recommend the best solution.

PBLs are mathematical problem-solving activities with many similarities to MEAs and also a few differences (Chamberlin & Moon, 2008). They are similar because both are based on solving real-life problems and open-ended tasks, requiring students to get involved in higher-order thinking, using the instructor as a facilitator, striving to develop students' skills (e.g., self-directed learning, self-assessment, and group work), and can be interconnected across multiple disciplines or implemented at different grade levels. They are different in two ways: first, PBLs tend to be much more demanding, requiring much more time for implementation; second, because they do not always require constructing a mathematical model, the product might be anything that provides a solution the problem posed.

Researchers and teachers should choose whether to implement an MEA or a PBL activity based on the curricular and instructional objectives of the course. For example, MEAs require fewer materials and logistics than PBLs, since they require fewer cycles and potentially cover a smaller number of topics. Therefore, if instructors have limited time, but are interested in developing the construction of models, they should choose MEAs. However, if their intention is maximize the learning process, PBLs would be the better approach, though these would require much more time and effort.

In conclusion, a researcher's choice of the specific problem-solving strategy to be used as an instructional-research tool will depend on his or her preferences, research design, and objectives. I used MEAs as my research and assessment tool because they respond best to the goals and limitations of my study. It is not my purpose in this dissertation to explore other strategies.

2.3. Constraints and Limitations of MEAs

In addition to the differences shown in the preceding section, I want to address some of the constraints and limitations of MEAs that might also be common in other complex instructional tools, like those previously mentioned.

One of most important limitations in problem-solving activities like MEAs is the teachers' lack of knowledge and experience in implementation. This often affects the level and quality of guidance and feedback teachers are able to give to students. To minimize the effect of this, it is necessary to provide professional development to teachers in both the theoretical concepts and implementation (Mousoulides, 2009).

Another important limitation is the time allotted for students to generate a solution. In MEAs, although many studies have let students work for no more than two sessions of seventy minutes, there are always those who need more time to complete their work (Yildirim, Shuman, & Besterfield-Sacre, 2010). In the end, what instructors and researchers want is to potentiate the students' learning process and for all students to complete their tasks so their work can be assessed, studied, and understood.

Third, if an MEA has not been well designed — i.e., if it does not follow the six principles detailed above — it may not elicit the problem-solving skills intended. Therefore, it is warranted to implement activities that have already been tested and piloted.

As to constraints, the number and size of teams must be taken into consideration when implementing an MEA. For example, Yildirim et al. (2010) argue that teams should contain no more than three or four members. If teams are too large, difficulties in both the MEA's implementation and the students' work may arise. Likewise, if the number of teams surpasses the number an instructor can manage, the quality of instruction (i.e., guidance and feedback) can be compromised.

2.4. Research on Model-Eliciting Activities

In this section I summarized additional examples of different MEAs that have been implemented with diverse purposes and participants, and in several contexts and situations. Many studies have focused on different aspects of the problem-solving process — e.g., problem formulation during the solution-process (Diefes-Dux & Salim, 2009); communication skills and collaborative working (Zawojewski, Lesh, & English, 2003); students' engagement, attitude, beliefs, and motivations (Di Martino & Zan, 2011); and meta-cognitive developments (Ekmekci, 2013). However, for my research I have selected only those focused on the mathematical domain.

Implementing the "University Cafeteria" MEA with 6th and 8th grade students, Mousoulides, Pittalis, Christou, and Sriraman (2010) showed that MEAs enable both low- and high-performing students to arrive at equally adequate solutions to a problem. The students were required to select the best of three full-time and three part-time food vendors based on data provided (tables with the hours worked and money collected). The researchers aimed to compare and contrast students' modeling and mathematization process when solving a modeling task (e.g., an MEA). They found that both groups of students were able to develop a mathematical construct by solving the activity "through a meaningful problem-solving" (p. 119) process. Each group approached the solution differently, but both groups demonstrated the ability to enter into the model-cycling process, improve through every cycle, and arrive at a solution. Thus, the "University Cafeteria" MEA showed both groups to have equal ability in arriving at a solution, even though the level of sophistication that each used may have been different (Mousoulides et al., 2010).

In 2006, Iversen and Larson set out to answer two questions posed the previous year by Lesh and Srirman (2005): "Why do students who score well on traditional standardized tests often perform poorly in more complex 'real life' situations where mathematical thinking is needed?" and "Why do students who have poor records of performance in school often perform exceptionally well in relevant 'real life' situations?" Iversen and Larson (2006) use a design-based research method called the "multi-tiered teaching experiment" to implement their MEAs for solving these problems. A multi-tiered teaching experiment is a design in which three levels of participants (students, teachers, and researchers) (Lesh & Doerr, 2003) interact in a natural setting (classroom) in which solutions are not of the type correct/incorrect, but rather are the multiple strategies students use to obtain a results; students are engaged and encouraged to play an active role in their own learning; enhance communication (i.e., discussion) and reflection; and that may be multi-disciplinary (Resnick, 1996, as cited by Azevedo et al., 2014, p. 289).

Iversen and Larson (2006) implemented both a standardized test and an MEA called the "The Penalty Throw Problem" with a group of first-year engineering students. This MEA, inspired by "The Volleyball Problem" (Lesh & Doerr, 2003) but designed specifically for this study, required students to advise a coach in selecting the best possible players for a handball team based on player statistics. After correlating the results of both assessments, they confirmed that standardized test scores were no indication of how well either level of student was able to perform the MEA. In fact, the high performers did not perform as well as the “high-performing” label would imply, while the low-performing students performed above and beyond expectations. Thus, standardized tests were not able to predict students’ mathematical thinking abilities in solving open-ended activities such as MEAs. Iversen and Larson (2006) suggest one possible reason for this might be related to the attitudes, beliefs, and motivations students had toward modeling. Another might be the fact that standardized tests require students to use “simple thinking in complex mathematics problems” while the MEA demand “complex thinking in simple mathematics” (p. 290).

Finally, in a study based on the “Summer Job” MEA, Larson (2010) demonstrated the cycle of modeling that MEAs require helps to enhance students’ quantitative and qualitative reasoning skills. She implemented the MEA in an advanced algebra course at the college level, tasking students to help the owner of a concession business hire workers based on hourly wage rates and the number of vendors needed. Using this MEA, the students were able to develop their quantitative reasoning skills by identifying the mathematical elements involved in determining a solution. They also demonstrated the importance of quantitative reasoning as “the central mechanism for an iterative refinement” (p. 117) — i.e., cycles of modeling — in the construction of powerful solutions to the MEA. That is, every time they entered a new cycle, they were able to improve the solution to the problem.

The research of Iversen and Larson (2006), and the ideas of Lesh et al. (2000), and Lesh and Sriraman (2005), Carmona and Greenstein (2010), Larson (2010), and Mousoulides et al., (2010), show that MEAs demonstrate that both low- and high-performing students can have similar abilities in solving real-life problems, that standardized tests cannot predict students' performance on MEAs, and that MEAs can enhance students' understanding in both the qualitative and quantitative domains. None of this research included implementing problem-solving activities at the high school level or analyzing the patterns of discourse of the students' interaction. My research addresses this gap.

2.4. Theoretical Framework

Solving real-life problems in the modern world is neither about following a certain set of rules nor working in isolation to find an answer (CORD 1999). Rather, people today often need to adapt problem-solving strategies to unique and complex situations and to develop collaborative solutions to difficult problems (Lombardi, 2007). However, when students are taught how to solve problems in school, they are often presented with unrealistic situations (e.g., textbook problems) that require them to follow particular rules or strategies (already practiced and mastered) to solve a problem (usually individually) that is seldom similar to any real-life situation. Further, the concepts they are tasked to learn are limited to grasping and/or memorizing a set of rules and passing standardized tests about topics that are usually quickly forgotten. In mathematics classrooms in particular, it is common to find this type of traditional problem-solving perspective — learning to follow specific steps in order to solve formulaic problems. However, this rarely helps students solve real-life problems that require a level of abstraction.

By contrast, in this research project, I took an approach in which students can work collaboratively to interpret situations in multiple ways, evaluate possible solution paths, and enter into a cycle of description, explanation, and prediction as they solve a problem; and in which the development of a useful and powerful solution is beneficial not only to students who excel in memorizing mathematical solution-steps or have attained a high level of achievement in school, but to everyone. This approach is called the Model and Modeling perspective (MM) (Lesh & Doerr, 2003).

Within the MM perspective, models are conceptual tools used to mathematize a real situation and modeling is the process in which a model is adapted or constructed to provide a solution to a problem (Lesh & Doerr, 2003; Zandieh & Rasmussen, 2010; Lesh & Lehrer, R., 2003). Following MM, the model-construction process is an interactive leaning cycle in which students collaboratively (1) examine a situation, (2) identify the problem to be solved and the variables involved, (3) formulate a model, (4) test the model, (5) interpret the results, (6) validate their model solution's applicability to the original situation, and (7) apply the model to other similar situations to test its usefulness (Kang & Noh, 2012). Moreover, since MM emphasizes teamwork when solving a problem, students not only reflect on their own thoughts individually, but also communicate their ideas in ways that other team members can evaluate, reject, or accept. In general, each member learns from the different perspectives that emerge through collaboration (Lesh, Cramer, Doerr, Post, & Zawojewski, 2003).

The MM perspective supports and advocates *unstructured*, collaborative learning, since learning occurs in the social context (Vygotsky, 1962) of student interaction, during which team members mutually "negotiate goals, define problems, develop procedures, and produce socially constructed knowledge in small groups" (Springer, Stanne, & Donovan, 1999, p. 24).

Many theories and theorists have addressed the benefits of learning in small groups (Springer et al., 1999): Piaget (1926) and Vygotsky (1978) in cognitive psychology, Deutsch

(1949) and Lewin (1935) in social psychology, Dewey (1943) in experiential education, and Belenky, Clinchy, Goldberger, and Tarule (1986) in humanist and feminist theory. However, the collaborative learning in the MM perspective is primarily rooted in the developmental theories of Piaget (1926) and Vygotsky (1978) in which "face-to-face work on open-ended tasks—projects with several possible paths leading to multiple acceptable solutions—facilitate cognitive growth" (Springer et al., 1999, p. 25). Based on this principle, MM recognizes that (1) it is crucial for students not simply to argue and discuss their opinions, but to share each other's ideas and perspectives when working collaboratively — i.e., an idea similar to the multiple-perspective principle of Lesh et al. (2003) and (2) that students can uncover their inadequate reasoning as disagreements arise out of their discussing, and that working out these disagreements enhances the understanding of all (Springer et al., 1999).

MM thus proposes types of problem-solving activities that afford learners a space to collaborate with others, to try out, reflect on, and re-enact theirs own and others' ideas — and MEAs are such an activity. In MEAs, the social and communal construction of knowledge takes place (Vygotsky, 1978; Tangney, FitzGibbon, Savage, Mehan, & Holmes, 2001) and "the learners not only construct their own knowledge while interacting with their environment [and other people] but are also actively engaged in the process of constructing knowledge for their learning community" (e.g., while working in teams) (Tangney et al., 2001, p. 3114). In collaborative groups, individuals merge their knowledge to strengthen and broaden their skills while achieving a common goal. This escalates their motivation toward an interest in problem-solving (Zawojewski, Lesh, & English, 2003). It also increases the possibility for students to create and invent more sophisticated and powerful solutions using mathematical representations (e.g., artifacts and constructs) and inscriptions — e.g., graphs, tables, diagrams — to mediate their thought processes and reasoning.

To encourage collaborative learning, researchers must thus ensure that an MEA's design (Lesh et al., 2000) is able to trigger students' collaboration in a sequence of modeling cycles that will result in the construction of a conceptual system (i.e., a model). As discussed above, a well-designed MEA follows six principles of design (Lesh et al., 2000), which "map neatly onto [students'] self-system [i.e., background and experiences]" (Middleton, Lesh, & Heger, 2003, p. 414) for the purpose of engaging and encouraging them to develop powerful mathematical solutions for real-life complex problems. This is particularly important, because if students' engagement is limited, it is less likely that they will develop a powerful solution or enhance their individual and community learning. According to Hidi and Renninger (2006) "students' relatively natural and competent initial engagement with an activity is potentially encouraging and might raise their confidence and willingness to further contribute to the activity" (as cited by Azevedo & Sherin, 2012, p. 283).

The MM theories and concepts I draw on throughout this research project consider that students develop models with the purpose of "creating relevant interpretations" (Lesh & Doerr, 2003, p. 533) of real-world problem situations in a context of collaboration (Vygotsky, 1978). The students' background and experiences (e.g., attitudes, feelings, beliefs, skills, or abilities) influence these interpretations, which students express through some sort of media. Thus, in my research, I studied the variety of models that students created (i.e., *the product*), and the discourse that emerged from the *process* by which these models were created, to interpret real-life situations.

Chapter 3: Research Design and Methodology

In this chapter, I discuss the methodology used for my research and analysis. I describe the characterization of discourse in MEAs, the design of the study, the participants, the pilot study I used to test the MEAs implemented in this study, the implementation of these MEAs, and the methods of collecting and analyzing my data.

3.1. Discourse Analysis in MEA

By design, MEAs encourage students not only to work in collaboration, but also to develop model solutions that are adequate to provide accurate solutions to a problem (i.e. The model construction principle), and also to deeply analyze, evaluate, and critique those solutions. In this process, students working in small groups externalize their ideas and thoughts (that are analyzed, evaluated and critiqued) with the intention of providing a solution to the problem. The ideas that emerge are discussed with the intention of finding the best model-solution. The process through which the students "negotiated" and chose the solution that they consider most adequate, included each one sharing and constructing their individual and the team's collaborative knowledge (Zawojewski, Lesh, & English, 2003): a cycle of discourse through which their disagreements are solved by demonstrating their rationale and reasons (Springer et al., 1999). Similarly, students deepen their understanding by reasoning about their own and others' ideas, and articulating, justifying, and explaining the rationale for their perspectives (Carpenter, Franke, & Levi, 2003). Through these processes, the students enhanced their conceptual understanding and "short and long-term retention" (Bentea, 2008, pg. 3). In terms of the type of discourse that emerges while students work through MEAs and how I characterize it herein, I must first define what discourse is and how it is related to the process-solution of MEAs.

Little prior research has focused on the type and characterization of discourse that occurs when students work in small teams in a mathematical problem-solving activity like MEAs in a natural classroom setting (Yackel, Cobb, & Wood, 1991; Zahner & Moschkovich, 2010). Rather, most existing research addresses the benefits and advantages of whole-class mathematical discussions as a strategy for teaching and learning (Goos, 2004; Lamper, 1990 as cited by Zahner & Moschkovich, 2010). By contrast, in MEAs, students work in small teams, and one of the two objectives of this research project is to characterize the type of discussions that arises during this problem-solving process. To define what discourse is, I borrowed, combined, and adapted the characterizations of discourse and argumentation that different prominent researchers have proposed to occur during MEAs (Azevedo et al., 2015; McNeill & Pimentel, 2010; Wertsch, 1991; Toulmin, 2003).

According to the NCTM (1991), discourses are "ways of representing, thinking, talking, agreeing, and disagreeing…, [ways of] how ideas are exchanged and what the ideas entail; and [of] how [ideas] are being shaped by the tasks in which students engage as well as by the nature of the learning environment" (p. 34). Therefore, discourse is the process through which a speaker sends a message to a listener, and in response the listener can agree or disagree with the speaker's message. Wertsch (1991) has categorized this process in two ways: *univocal* and *dialogic* (as cited by Knuth & Peressini, 2001). Univocal discourse refers to a type of communication in which a listener receives an idea/message in the "exact" way the sender intends. The communication ends once the idea is received. On the other hand, dialogic discourse is a type of communication characterized by a "give-and-take" between members of a group (Knuth & Peressini, p. 321), in which each participant contributes meaning and understanding, and in which both the listener(s) and speaker(s) contribute to the discussion. For instance, the following small excerpt of an MEA implementation shows how a discourse between two students occurs; how a message sent by Student 1 is not only received by Student 2, but also responded to, and in some way challenged:

> **Student 1**: Look, what we can do is to set a base price, so it won't keep increasing (the price)… do you understand? For example, let's say they charge a daily price of $MX700.00
>
> **Student 2**: but… what if they decide to charge less than $MX600.00 ?
>
> **Student 1**: you mean… like MX$500.00?

MEAs encourage students to reveal their perspectives and ideas, which in return generate meaningful discussions in which students evaluate, accept, or reject ideas with the intention of developing a model (Carlson, Larsen, & Lesh, 2003).

I have characterized the discourse episodes in the MEAs I implemented during this research project following a discourse characterization Azevedo et al. (2015) made while implementing a series of design-based activities called Inventing Graphing (IG). In these activities, students were required to create a model — an "artifact, or process to solve a given problem" (p. 287) — while entering into a cycle of modeling (Lesh & Doeer, 2003; Azevedo et al., 2015) as many times as necessary. The authors asked middle and high school students to represent a motion problem about a driver in a desert stopping to get water and then slowly continuing to drive (Sherin, 2000) in whatever way seemed most appropriate to them. Azevedo et al. categorized discourse practices in three ways, combining methods and theories of different influential researchers in two steps. First, they worked to "capture the degree of direct interchanges between students" (p. 10) dialogically, using McNeill & Pimentel's (2010) concept of measuring the level of responsiveness among students' talk-interaction and their ability to set, reason, and articulate each other's positions and perspectives. Further, they considered an adapted version of the Toulmin model of argumentation (Toulmin, 2003) to analyze the structure and quality of students' utterances. This served as a framework to characterize the epistemological discourse practices as describing, explaining, and arguing. Describing discourse refers to students' presentation of ideas and perspectives, so the audience — the members of the collaborative team — can gain a better sense or understanding of the "big picture." Explaining discourse is a type of discourse practice in which a team member explicates an idea/perspective/rationale in a deeper way and seeks to verify the audience's understanding.

Azevedo et al. defined arguing discourse as type of discourse practice consisting of shared features of scientific argumentation (McNeill & Pimentel, 2010) and critical discussion (Andriessen & Baker, 2014). In scientific argumentation, "students use evidence and reasoning to justify their claims" (p. 203) while "coordinating evidence and theory to support or refute…a model or prediction" (Osborne, Erduran, & Simon 2004, p. 995, as cited by Azevedo et al., p. 287). In critical discussion, students begin with a difference of opinion, establish a common goal, exchange persuasive opinions about how to get to that goal, and agree upon a path to that goal.

The type of scientific argumentation Azevedo et al. considered, and that is central to the type of activities implemented in this project, differs from that of McNeill and Pimentel, in which students spend more time debating opposing ideas or theories. For Azevedo et al., and the type of design-based activities they considered, students develop their model-solutions by arguing over competing (i.e.. perspectives), but not necessarily by debating ideas that are "opposite and mutually excusive" (p. 305).

For the purpose of this dissertation, and taking into account the above, I have characterized the types of discourse in which students engaged as "arguing discourse" and "non-arguing discourse," in which the latter includes "describing" and "explaining" discourse.

For me, as for Azevedo et al., the utterances in which students argue over ideas or perspectives become the primary process-solution through which students develop their model-products. The arguing episodes thus become my main focus in analyzing each team's iterations, and my way of determining the richness of each "student's contribution across contexts [of the MEAs'] implementation" (Azevedo et al., p.300). Azevedo et al. considered the Toulmin model as a framework to evaluate and study the structure of each team's arguing episodes.

Toulmin's model can be used not only to construct a claim, but also to examine the quality, structure, and validity of others' claims (Rumsey, 2013). Researchers in different academic disciplines like science and mathematics education and language arts (Pedemonte, 2007, as cited by Rumsey, 2013) have implemented Toulmin's model as both a methodological and a theoretical framework. According to Toulmin (2003), an argument is composed of six interconnected elements that work together to prove an idea or perspective, or to convince or persuade: Claims, Grounds, Warrants, Backings, Qualifiers, and Rebuttals. Claims are the thesis ideas, or the reason for an argumentation. Grounds are the facts, the data that back up and support the claims, which provide the "foundations for the claims" (Karbach, 1987, p. 83). Warrants are the justification for linking a claim and the grounds; these can be suggested but not directly expressed or stated (Karbach, 1987; Azevedo et al., 2015). Karbach (1987) believes only the first three are usually present in an argument. The other three elements of the Toulmin model can be added as necessary. For him, Backings have the function merely of supporting Warrants, and in general represent further reasons to justify Warrants. Qualifiers generally appear only as part of an argumentation episode and are expressed in terms of working out the claim. Finally, Rebuttals are ideas or perspectives that might invalidate a claim.

Following consideration, I used the first two of Toulmin's elements: Claims and Grounds. Then, following Azevedo et al., I combined the elements of Warrants and Backings into Reasoning, which they borrow from McNeill and Pimentel (2009). In Reasoning, students may choose to use "either or both the warrants and backings to justify their move from claims to grounds" (Azevedo et al., p. 306). Finally, I used Toulmin's last element of Rebuttals, also used by Acevedo et al., who term it counter-arguments (p. 305).

3.2. Research Design

At the most basic level, the research was conceived as a series of case studies, which were then compared and contrasted along various dimensions (Thomas, 2011; Goodrick, 2014; Hayes, 2000). Case studies are especially well-suited to my purposes because they allow me to observe both single individuals and participant groups and to study the students' solving processes as they unfolded, with a higher level of detail than if I were examining a larger population (McLeod, 2008). Case study is an empirical research method that "investigates a contemporary phenomenon within its real-life context…in which multiple sources or evidence are used" (Yin, 1984) with the intention of "identifying variables, structure, forms and orders of interaction [among] participants…to assess the performance of work or progress in development" (Mesec, 1998, as cited by Starman, 2013, p. 31).

In this dissertation, I adopted Creswell, Hanson, Plano, and Morales' definition of case studies as "a qualitative approach in which the investigator explores a case or multiple cases over time through detailed, in-depth data collection involving multiple sources of information" (2007, 245). The source of data could be video, interviews, questionnaires, or other means. In addition, given my focus on students' mathematical reasoning *processes* (i.e., the discourse of the group interaction), relying on a few in-depth cases was meant to afford a detailed account of the moment-by-moment unfolding of the students' interaction while working to solve the MEAs.

Case studies can take on many forms (Yin, 1984; Merriam, 1998; Stake, 1995) and cases can be as rigidly or flexibly defined as one might want (Starman, 2013; Gerring, 2004; Sturman, 1997; Simons, 2009). For the current research, I defined and chose empirical (Yin, 1984) and analytical (Stake, 2005) cases prior to data collection (Lesh et al., 2000; Greenstein, 2008, Aliprantis et al., 2003), according to (1) the core goals of the research, (2) the inherent organization of MEAs, and (3) the specificities of the data collection site and classroom:

1. Core goals of the research: Because my aim is to assess the relative performance of students at different mathematical performance-levels (i.e., low, average, and high), cases/units could conceivably be individual students or groups of students.

2. The organization of MEAs: Because MEAs are designed and enacted as small group (Lesh & Doerr, 2003) activities (three to four students per group), student groups (or teams) emerged as natural candidate units.

3. The specifics of the data collection site: As I will explain in detail later, I collected data in a mathematics classroom setting in which the teacher strongly favored group learning, collaboration, and working due to the benefits that encompass (Gokhale, 1995). As such, she

created a “natural experiment” in which low-, average-, and high-performance groups of students formed without her influence or enforcement.

The cases were, therefore, the ability groups (i.e., low-, average, high-performance) that formed prior to implementing the classroom activities (i.e., MEAs). Once the groups were formed, I (as researcher and co-teacher) determined the cases that were the focus of analysis. The units of analysis were, first, each team as a whole, then each individual in the group, taken as a factor that influenced the whole.

3.3. The Setting

The target population for this study was 11th grade high school students at different performance levels at a private Catholic school located in northeastern Mexico. The school is 65 years old with a student population of approximately 800, divided into four departments: Pre-K, Elementary, Middle School, and High School. The high school program lasts two years — 10th and 11th grade — and the 11th grade class contains approximately 74 students divided into three sections of 22 to 28. In a conversation with the high school department’s principal, who is also the head of the mathematics department, I found out that students engaged collaboratively in small groups in activities and projects one or two every month. When forming these groups, teachers at the high school usually considered one of the following two routines: grouping students randomly (less common) or allowing students to choose their own teams (most common). Further, the principal stated that when choosing their own teams, students naturally group together based on various shared similarities and common interests. As she remarked, sometimes ability groupings naturally emerge (e.g., low-, average, and high-performers), and these coexist with other affinity groupings (e.g., sports-minded students, other students with a common interest).

Following the team-forming routine at the high school, the classroom teacher and I (as a co-teacher and co-leader) asked the students to form teams of two or three members without forcing or imposing any type of categorization, randomization, or selection rule (i.e., they formed these groupings "naturally."). These naturally formed teams represent the cases and focus of this study and analysis.

Later, the classroom teacher categorized each team as low-, average, and high-performance based on the students' individual achievements in class (as measured by their grades, test scores, and classroom activities). Mixed teams were categorized based on the majority achievement of the students. For example, if a team had two members the teacher considered as low-performing, and the third member was average-performing, then the team was classified as low-performance. In the end, 24 teams were formed: 14 average-, 5 low-, and 5 high-performance, from which 9 were selected to be video recorded. Further detail on the teams' selection can be fund in the data collection section below.

I chose to carry out my research in this high school because of my prior affiliation and experience there, having worked for five years as a computer science teacher and communications manager. This facilitated access to the data I needed to collect.

All the teams, regardless of the teacher's classification, sat wherever they wanted in the classroom. Only the groups to be video recorded (i.e., one in each performance level) were asked to sit at the end of the classroom where the cameras were placed (see Figure 3.1). The image below highlighted (as an example) how teams sat: The non-recorded teams in blue, and the video-recorded in gray.

Figure 3.1. Example of Teams' sitting arrangement

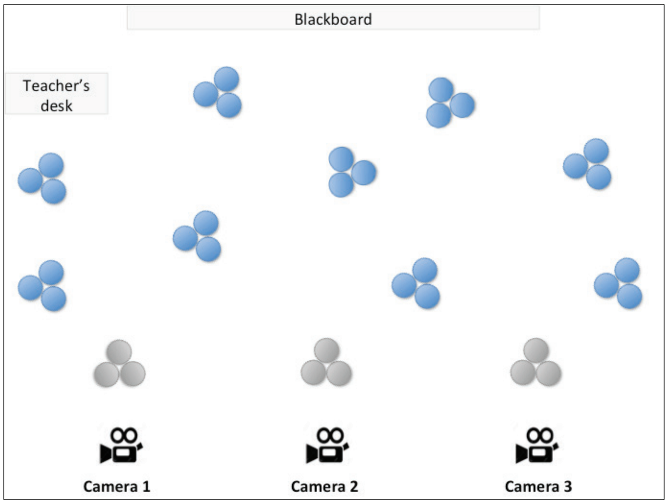


Figure 3.1. The image shows an example of how the teams were sitting in the classroom. In general, all teams (represented in blue) decided where they wanted to sit. Only the selected teams to be video recorded (represented in grey) were asked to sit in from of the camera.

The video-recording equipment and accessories (see Figure 3.2) were placed at the end of the classroom (as shown in Figure 3.1) for technical reasons. All the cameras were placed to capture the students at work and in interaction during the moments of solving process. I used three Canon RFX600 cameras to video-record the students, with three professional Shure CVB-B/O omnidirectional microphones plugged into the cameras. These microphones were connected to a power-supplier that provided the necessary electrical power. A tripod to mount the camera, an audio cable transferring the audio from the power-supplier to the camera, and an audio adapter to the cable output were also required.

Figure 3.2. Example of the video-recording equipment

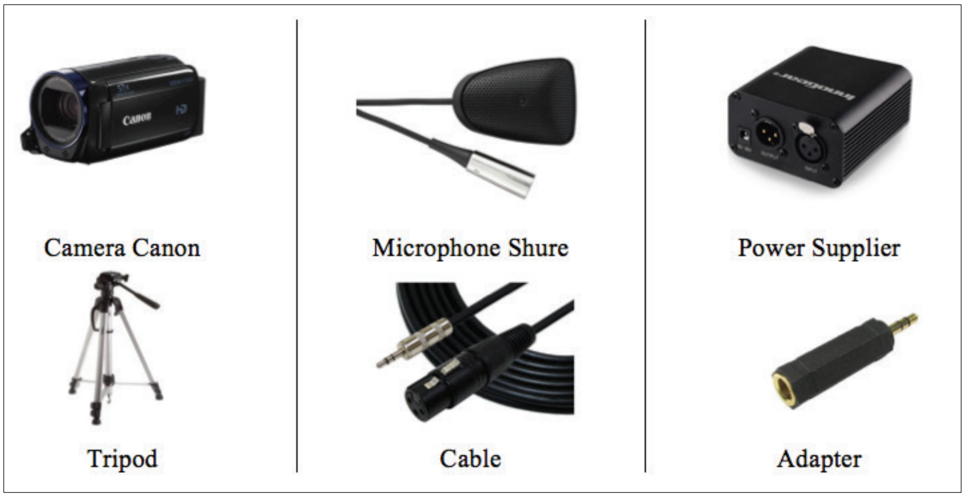


Figure 3.2. The necessary equipment to record the students' work and interactions during the MEAs. All the equipment was tested during the pilot study and before the recording began.

3.4. DATA COLLECTION

The data for this dissertation project was ethnographically informed (Agar, 1982), since the primary source of data was collected through fieldwork (Whitehead, 2005). I was interested in mapping the culture of the classroom, norms, and values, and the type of discourse students engaged in — and what was considered valid — when working in collaboration (Azevedo et al., 2015). As a private, Catholic institution, the school strives to foster respect for self and others, critical thinking, proactivity, teamwork, and reflection in their students. Accordingly, students are trained to express their ideas and defend their positions, but also to accept when their ideas are either rejected or wrong, all in a context of reflexivity.

Following ethnographic methods, I took on the role of a *participant observer* (Kawulich, 2015) during data collection, which entails certain unique ontological and epistemological perspectives (Whitehead, 2005). Here ontology refers to the nature of what is being studied and how it is affected by external variables, such as social stereotypes or experiences, while epistemology refers to the "intersubjectivity product of the relation between [me as researcher] and the study population" (Guba, & Lincoln, 1994 as cited by Whitehead, p. 4). Participant observation thus describes a process that allows the researcher (i.e., me) to learn first-hand "about the activities of the people under study in the natural setting" (DeWalt & DeWalt, 2002). It is a "process of learning through exposure to or involvement in the day-to-day or routine activities of participants in the researcher setting" (Schensul, Schensul, & LeCompte, 1999, p. 91). Engaging in my research through participant observation, therefore, greatly enhanced my ability to understand and interpret the students' experiences as engaged in the collaborative effort of solving MEAs.

My simultaneous role as co-teacher and researcher allowed me both a sensibility and opportunity to (1) co-prepare the mathematical activities implemented, (2) observe the students while they worked in solving the problems, (3) respond and solve students' doubts in a formative-assessment way, and (4) collect field notes and data through the video-recording of the selected teams.

The fieldwork for this study took place mid- to late-spring 2016 and was divided into three phases. In the first week, I implemented *The Team Ranking Problem*, the following week, *The Hybrid vs. Gas Car* problem, and two to three weeks later *The Historic Hotel* problem. During the two weeks prior to the first implementation, I met with the teacher of the three course sections in which I implemented the MEAs to explain the details of my research, provide training, select the order in which the MEAs were to be implemented based on the curriculum, and prepare the sessions. I also engaged with the students to begin establishing a good rapport with them. In addition, I collected the IRB consent and assent forms from all participants.

In implementing this as an ethnographically inspired study, I sought to collect data that would allowed me to focus my analysis — detailed here and deeply analyzed in chapter 4 — on the students' *products* (materials, representations, worksheets, artifacts, and final presentations) and *processes* (the discourse patterns that emerged while students engaged in the modeling solving-process). For all the three MEAs, I had three information sources: I gathered all the students' paperwork and artifacts, observed and took notes during the project time, and most importantly, I video-recorded all the students' interactions, presentations, and discussions for nine selected teams.

From the 24 teams, the teacher and I selected nine groups (three in each classroom section, 1 group from each performance level) to be video recorded for the cases of study. The nine groups were selected among the groups in which students had similar achievement in class based on the teachers' categorization (i.e., all nine groups were non-mixed teams). Given the time-consuming nature of analyzing group discourse at the utterance level, I chose three focal groups—one for each performance level—for detailed transcription and analysis. These groups were selected because their intermediate and final products seemed to be the richest among all 9 video recorded groups.

The importance of using video as a data source is threefold: (1) it allowed me to capture the richness of the students' behavior and interactions (Clement, 2000, as cited by Stemler, 2001, p. 407); (2) "it extend[ed] and enhance[d] the possibilities of observational research by capturing moment-by-moment unfolding, subtle nuances in speech and non-verbal behavior" (Martin, 1999, p. 79), and (3) it "enable[d] me to revisit the data [videos] as many times as necessary" (Stemler, 2001) to deepen my understanding of the students' mathematical reasoning adequately for full analysis. For the purpose of this study, I only video-recorded the work of one team per performance level in each section: i.e., only three teams per section, or nine in total.

Students' paperwork and artifacts included all the work the students did individually and in teams when developing their models, and the letter to the "client" each team wrote explaining and detailing their solution or mathematical construct. Observations included notes about the students' mathematical ideas taken by me and the classroom teacher.

3.5. ACTIVITY IMPLEMENTATION

I, as the researcher, was co-teaching and co-leading the implementation of the activities. Aware of the importance of the classroom teacher's participation in this research, I encouraged her involvement and participation in certain ways. For example, I asked the classroom teacher to observe and take notes on students' mathematical development and ideas, walk among groups, avoid intervening in the students' work and team discussions, and refrain from answering students' questions that "may provide students with ideal procedures that could assist them in solving the problem" (Aliprantis & Carmona, 2003).

In my previous experience from the pilot study, I learned that, even though I co-taught the students when implementing the MEAs, classroom teachers who lack experience, knowledge, and practice in MEAs might unintentionally hamper the experiment. To avoid this, as mentioned above, the weeks before implementing the MEAs, I met with the classroom teacher for approximately two to three hours to prepare her for the study by (1) discussing each activity with her, (2) familiarizing her with the basic theory, use, and implementation of MEAs, (3) explaining her role during the activity's implementation, and (4) addressing any questions or doubts she might have in regards to the study.

Also, I recognized the importance of building a good rapport with the students to "encourage involvement, commitment, and interest" (Ramsden, 2003) in engaging in activities with me as their co-teacher. Since this was my first time interacting with these high school students, the weeks before implementing the MEAs, I spent time at the school interacting with the students in several ways. For example, I observed their classes (particularly in mathematics), helped the classroom teachers, and talked to the students during recess and in between classes.

In this study, I implemented three MEAs. Each MEA took approximately 180 minutes divided into three classroom sessions of 60 minutes each; the three MEAs together total about 540 minutes of interaction. Each MEA has the following stages or phases:

The day before each implementation, I briefly introduced the students to the topic of the next day's activity (i.e., *The Ranking Team*, *The Historic Hotel,* or *The Hybrid vs. Gas Car*) and assigned them an article and four to six readiness questions related to it as homework. This gave them time to reflect on and think about the topic they would work on the following day.

On the first day of implementation, we (the classroom teacher and I, the co-teacher/researcher) reviewed the reading and readiness questions with the students to assess their understanding of the homework. In this regard, Aguilar (2012) stated it is relevant for classroom teachers "to know the level of understanding of their students" (p. 3) in relation to an activity or task. This took approximately five to ten minutes.

Next, we asked each student to read the problem statement and answer three further readiness questions individually: Who is the client? What are the client's needs? What are the reasons for these needs? The readiness questions of the problem statement were intended to encourage students to identify and reflect on the main aspects of the problem they were to solve. By identifying who the client is, what is needed, and the reason for these needs, students realized the necessity of developing a model — as stated by the model-construction principle (Lesh et al., 2000). They had 10 to 15 minutes to complete this task.

In the pilot study, I found that some students at each performance level struggled either to understand the problem statement or to answer the readiness questions. During the implementation, I had to decide if we would spend some time (no more than 10 minutes) helping students make sense of the activity's problem statement. It is important to note that this is not part of the steps of the MEA's implementation. However, in my role as researcher, I am aware that this type of modification or adaptation might be needed for the study's success (Cobb et al., 2004; Collin et al., 2004).

Finally, we asked the students to form their teams, which would remain the same for each MEA, to work on solving the problem statement for the remainder of the classroom session time. We asked students to choose a name for their teams in an effort to create cohesion and a sense of belonging needed to encourage individual team-members to work toward the common goal of developing a model to solve a problem (Osterman, 2000). Each group was provided with pencils, paper, a "basic" small calculator, and a ruler. Students had the freedom of using any other tool they had. (e.g., cellphones, TI-84, tablets/iPads). This was the time during which students started to develop their mathematical model as posited in the MEA. At the end of the session, we collected each team's work in individual folders labeled by team name.

In the pilot study, I found many students required more time to complete their model-solutions. Therefore, on the second day of implementation, we asked each team to reconvene and continue developing the previous day's work for approximately 50 minutes. During this time, the classroom teacher and I continued observing, taking notes on the students' work, and answering any question students might have, in a formative way, i.e., answering their questions with another question to help them enhance their understanding.

Next, we provided each team with transparencies and markers to create a presentation of their findings (e.g., model, representations, procedures, ideas). The rationale for this was to encourage students to further externalize their findings — beyond creating the letter to the client, in their worksheets, or while interacting within their teams — in ways they could share with their peers (i.e., explaining in detail their solutions and the rationale of it). These presentations were shared during day three. As with day one, we ended the day two sessions by collecting each team's work and adding it to the folders labeled by team name.

On the third day of implementation, students reconvened their teams. We gave them 10 minutes to revise and finish their presentations and get ready to present their findings to the rest of the class. Each team had approximately five minutes to explain their model, procedures, ideas, and inscriptions (e.g., tables, graphs, diagrams, draws). The rationale behind these presentations was that they elicited and encouraged students to externalize the mathematical processes and reasoning they followed and used in the solving-process. This provided me (as both researcher and co-teacher) a deeper understanding of the students' mathematical understanding, reasoning, and development when working on the MEAs. In addition, while students explained and justified their solutions, we orchestrated a class discussion in which we, both the classroom teacher and I as co-teacher, and the student teams provided feedback to the group that was presenting.

Finally, at the end of the presentations, the classroom teacher and I conducted a final class discussion about the activity, the mathematical ideas that emerged and evolved during the solving process, and the way those ideas connected to real-life contexts and the mathematical curriculum.

During the day three implementations, I made an informal analysis of how the activities unfolded in case any change or modification was needed for the next MEA to be implemented.

3.6. Classroom Activities

In this study, I implemented open-ended, mathematical problem-solving activities designed following the six principles for MEA design outlined above (Lesh et al., 2000). The MEAs I used were *The Ranking Team* (Greenstein et al., 2008), *The Historic Hotel* (Aliprantis & Carmona, 2003) and *Hybrid vs. Gas Cars* (Elliott, 2014) problems. I translated all MEAs into Spanish and adapted them to the reality and context of the high school students in order to maintain the reality principle of MEAs.

3.6.1. The Historic Hotel

As described in the introduction, this MEA, developed by Aliprantis and Carmona (2003), is based on an economics problem (Aliprantis, 1999) that asks and encourages students to develop a mathematical model from a real-life type situation. In *The Historic Hotel* MEA, students are required to maximize the profit of a hotel. The hotel has 80 rooms with a daily rate of MX$600, and a maintenance cost of MX$40 for each booked room. The hotel has a price/booking rule stating that for each MX$10 the rate is increased, one less room is booked. An adequate possible solution-path would include a table (see Table 3.1) in which students show the price's increase versus the number of unbooked rooms. In addition, students would show some type of graph, and model solutions detailing how they found the number of rooms and price needed to maximize the hotels' profit.

Table 3.1. Example of price and rooms booked

Number of rooms booked	Daily room price (MX$)
80	600
79	610
78	620
77	630
76	640
75	650
74	660
73	670
72	680
71	690
70	700
69	710
68	720

Table 3.1. The daily price of the hotel's room increases by MX$10, which results in having one less room booked.

The activity has been validated and tested in many different educational settings, with different participants. For example, Aliprantis and Carmona (2003) implemented it in middle schools, Dominguez (2010) also applied it in a calculus class while acting as visiting professor at a university in Texas, and Ekmekci (2013) implemented it when working with pre-service mathematics and science teachers. Though implemented in many different contexts, this MEA has not been implemented in a high school or with the intention of comparing and contrasting the model-solutions of low, average, and high-performing students. The mathematical ideas this activity addresses cover the topics of patterns, variables, and parabola (i.e., quadratic function), among others. Thus, the activity is appropriate for the students' level, age, and background, since they would have already learned most of these skills.

In the pilot study of this dissertation, I showed that many solution paths could be taken to propose an adequate model-solution that solves this problem. There could be two possible routes that students could take when solving this MEA. First, students might consider providing maintenance service to all booked rooms of the hotel. In this case, the hotel room price that would maximize the profit (P) would be MX$720, having 68 rooms booked. Students could use a general model to determine the profit of the hotel:

P = (Booked rooms) x ($Price) – (Booked rooms) x (Maintenance cost)

When factorizing this equation, students would find a simple version:

Profit = ($Price-Maintenance cost) x (Booked rooms)

As in the pilot study, students would create a set of alternate models— other than just using a table — to determine the number of booked rooms (BR) and the maintenance cost. For example, to determine the rooms needed for a specific price, the following model could be considered:

BR= Hotel Rooms –[(Desired Price – Initial Daily Price)/10]

If the data mentioned above in Table 4.3 were input into the previous booking-room model, the equation would provide the number of rooms needed. For example, let's consider the ideal booking price of MX$720 :

$$BR = 80 - \left[\left(\frac{720-600}{10}\right)\right]$$

$$BR = 68$$

Using this methodological-process approach, and the profit model, would generate a final total profit after the maintenance cost of:

$$P= MX\$(720-40)\ (68)$$

$$P= MX\$46{,}240$$

The second approach is considering providing maintenance service to all rooms, regardless as to whether they were booked or not. This means the maintenance cost would be a constant variable of MX$3200 — i.e., the product of 80 rooms times the daily maintenance fee of MX$40. If students decide to follow this path, then the number of rooms and price that would maximize the hotel's profit would be 70 and MX$700, respectively. Using a similar profit-model, the final profit would be:

$$P= (MX\$700)(70) - MX\$3200$$

$$P= MX\$45{,}800$$

Although I have shown two possible solution-model routes that students could follow to find the hotel room and price that would maximize the hotel's profit, MEAs are open-ended activities that allow for multiple path solutions (Dominguez, 2010). This means that students could creatively develop other possible adequate solutions not previously considered.

3.6.2. Hybrid vs. Gas Cars

The *Hybrid vs. Gas Cars* problem (See Appendix B) is an MEA developed by Elliott (2014) and reviewed and approved by his fellow educators and subject experts from the Curriculum Planning and Learning Management System (CPALMS). This group ensures sure all the MEAs they create are designed following the six principles Lesh et al. (2000) describe. My rationale for selecting this MEA is that the activity is an extension of the mathematical ideas covered in the *Historic Hotel* (i.e., functions and modeling). For example, in the *Historic Hotel* MEA, the main mathematical ideas were related to the development of the quadratic function. In the *Hybrid vs. Gas Cars* problem, the main ideas are related to the development of linear functions (e.g. of the form $y = mx + b$), ordered pairs on a coordinate grid, and decision making. The activity is appropriate for the academic level of the students since they have already learned these curricular materials.

In the *Hybrid vs. Gas Cars* MEA, students needed to graph data of quantitative variables on a scatter plot, determine how these are related, and find a pattern that helps them create a function to predict the price of gas in the future. In addition, students needed to evaluate the trade-offs and constraints of "alternative strategies for solving a specific societal problem by comparing a number of different costs and benefits, such as human, economic, and environmental" (Elliott, 2014, p. 2).

The activity posits a taxi company's need to buy several new cars, and requires students to develop a solution for making the best economic decision when selecting which cars to buy. Their choice is between a regular gas car or a hybrid-electric vehicle (HEV), and their decision should consider all known variables — like estimated city and highway consumption, factors affecting the car's performance (e.g., weight, engine size, etc.), estimated miles driven annually, and the length of time the company expects to keep the cars — as well as unknown ones, like the future price of gasoline, which requires them to make a projection based on averages over the last 10 years, which will be provided. In the *Hybrid vs. Gas Cars* problem, students needed to make important decisions when detailing and selecting their solutions based on real information. In addition, the activity calls for students to combine real information with data that students need to analyze in advance in order to transform it into relevant information that helps them select their best possible construct (i.e., model).

The company needs to buy 10 cars that will be used for five years. The cars have an average annual use of 32,000 km, of which 75% are spent on highways and the rest in the city. The gasoline car has an estimated performance of 40 km in the city and 56 km on the highway. The hybrid has an estimated performance of 150 km in the city (in electric mode), and 80 km on the highway. Students were provided with the cost of gasoline for the last 10 years so they could estimate the price for the next five years. In addition, students were given the price and characteristics of both the gasoline and hybrid cars.

One possible solution-path for this *Hybrid vs. Gas Cars* MEA might include a table (see Table 3.2) in which students would show how the price would increase in five years, following the pattern of the ten previous years. Table 3.2 shows the price of gasoline over time and the difference between each year, from which students might find a pattern and estimate the price for the following years.

Table 3.2. Gas prices

Year	Price per liter (MX$)	Difference (MX$)
2005	7.54	----
2006	8.12	0.58
2007	8.71	0.59
2008	9.31	0.60
2009	9.92	0.61
2010	10.54	0.62
2011	11.17	0.63
2012	11.81	0.64
2013	12.46	0.65
2014	13.12	0.66
2015	13.79	0.67

Table 3.2. The price of gasoline increased every year following an average pattern of cents. Considering this information, students would need to estimate the price of gasoline through 2020.

As shown above, there is a difference of MX$0.58 when subtracting the price of gasoline from 2005 to 2006 (i.e., 8.12 minus 7.54). Doing the same computation with the following years will result in a pattern in which the difference increases one cent every year through 2015. This information is critical in order to estimate the price of the following five years (see Table 3.3).

Table 3.3. Price Estimation

Year	Estimated increase	Price per liter (MX$)
2016	0.68	14.47
2017	0.69	15.16
2018	0.70	15.86
2019	0.71	16.57
2020	0.72	17.29

Table 3.3. This table shows the estimated price of gasoline from 2016 to 2020 based on the increasing pattern of the cost for the first 10 years.

Once students have estimated the price of gasoline, they might then determine how much the company would spend if it decided to buy either the hybrid or the gasoline car. However, there are considerations that would need to be taken into account before students could make a recommendation to the company. For example, the cost of maintenance, depreciation, potential resale price, environmental factors, among others, could make the students' decision deviate to one side or the other.

To calculate the annual cost (AC_X) — in which "x" means the year considered — for any type of car, a model that considers kilometers traveled and the cost of gasoline both in the city and on highways, would provide an estimate of a general annual cost. To achieve this, students might use the rate of kilometers on the highways (KH) or the city (KC) versus the car performance (CP_Y), in which “y” means the type of car, multiplied by the estimated annual cost of gasoline (GP_X):

General model:

$$AC_X = \left[\left(\frac{KH}{CP_Y}\right) \bullet GP_X\right] + \left[\left(\frac{KC}{CP_Y}\right) \bullet GP_X\right]$$

Factorizing the equation above, the new model would be simplified as:

$$AC_X = GP_X \left(\frac{KH}{CP_Y} + \frac{KC}{CP_Y}\right)$$

This last model might be used to estimate the annual cost of gasoline for either type of car based on the estimated number kilometers driven annually, recalling that the company needs ten cars expected to be used for five years. Therefore, obtaining the annual cost of gasoline would be just one step further toward the main goal of finding the total cost, which would also include the cost of each vehicle. Next, I show what the annual cost of gasoline of each type of car would be, and the total cost of considering either the gasoline or hybrid car.

3.6.2.1. Gasoline Cars

Considering the general factored model above, now I will input the data for each variable and year to show the gasoline cost for each year from 2016 to 2020:

$$AC_{16} = MX\$14.47\left(\frac{24000\ Km.}{56\ \text{Km./L.}} + \frac{8000\ Km.}{40\ \text{Km./L.}}\right)$$

AC_{16} = MX14.47 • 628.57 L. = MX$9,095.40

Since the number of kilometers traveled is estimated, which means that number of liters needed is also, I will calculate the cost of gasoline considering the same number of liters for 2016 through 2020.

AC_{17} = MX15.16 • 628.57 L. = MX$9,529.12

AC_{18} = MX15.86 • 628.57 L. = MX$9,969.12

AC_{19} = MX16.57 • 628.57 L. = MX$10,415.40

AC_{20} = MX17.27 • 628.57 L. = MX$10,867.97

When adding all gasoline costs for each year, the total gasoline cost for the five years is:

MX$ 49,877.01

Now it is necessary to consider the cost of the cars multiplied by 10, which is the number of cars the company needs, to obtain a total of:

Cars cost = 10 x MX$ 230,235.00

=MX$ 2,302,350.00

Finally, when adding the total cost of gasoline and the cost for the ten cars, the taxi company would end up investing the following amount in gasoline cars:

MX$ 2,235,227.01

This final cost would need to be compared with that of the hybrid cars, and students would need to determine which type of car was the best investment.

3.6.2.2. Hybrid Cars

Similar to the procedure followed with the gasoline cars, for the hybrid cars the same general model might be used, recalling that AC_X represents the annual cost— in which "x" means the year considered —, KH and KC the rate of kilometers on the highways or the city versus the car performance (CP_Y), and GP_X the cost of gasoline. However, there is one particular relevant fact that students would need to consider. Hybrid cars have the capability to use their electric mode in the city. This represents a significant savings in the total liters of gasoline needed for both the highway and the city — i.e., the liters of gasoline for the city would be zero — which will also affect the total cost of gasoline. In the following computations, I show the total cost of gasoline for and the final cost of the hybrid cars.

$$AC_X = \left[\left(\frac{KH}{CP_Y}\right) \bullet GP_X\right] + \left[\left(\frac{KC}{CP_Y}\right) \bullet GP_X\right]$$

$$AC_{16} = \left(\frac{24000\ Km.}{80\ \text{Km./L.}} \bullet MX\$14.47\right) + \left(\frac{8000\ Km.}{150\ \text{Km./L.}} \bullet 0\right)$$

$$AC_{16} = 300\text{ L.} \bullet \text{MX}\$14.47 = \text{MX}\$\ 4{,}341$$

Using the same rational for all the other years, the gasoline's cost for each following year would be:

AC_{17} = MX$15.16 • 300 L. = MX$4,548

AC_{18} = MX$15.86 • 300 L. = MX$4,758

AC_{19} = MX$16.57 • 300 L. = MX$4,971

AC_{20} = MX$17.29 • 300 L. = MX$5,187

The total cost for gasoline using hybrid cars for the five years is:

MX$23,805

The cost of the hybrid car multiplied by 10 (the number of cars needed) is:

Cars cost = 10 x MX$ 350,715

=MX$ 3,507,150

Finally, the combined cost of gasoline and hybrid cars is:

MX$3,530,955.00

The possible solutions presented above for the *Hybrid vs. Gas Cars* MEA seem to have a straightforward solution path, which might fail to be in compliance with the generalizable principle of MEAs (Lesh et al., 1999). However, when students would subsequently write their written report to the client, they would have to decide which type of car is the most economic investment for the company. On one hand, gasoline cars have a higher cost in gasoline overall, but the price of the car is cheaper than the hybrid cars. This makes the whole gasoline-car investment more attractive. On the other hand, hybrid cars require barely half the gasoline required by gas cars. However, hybrid cars are more expensive, and the total investment is significantly higher for these cars.

Based on all the data and external variables — e.g., depreciation, environment, maintenance, taxes — students would need to discuss the best solution for the client. They would also need to state their reasons for such solutions and their rationale for selecting either the hybrid or the gas car.

3.6.3. The Ranking Team

The *Ranking Team* (See Appendix C) is an MEA developed by Greenstein (2008) while working as a member of a research group at The University of Texas at Austin. It has been implemented and tested in different settings (Carmona & Greenstein, 2010), as detailed above (see chapter 1). The activity addresses concepts of distance, slope, Pythagorean theorem, triangles, and measurements in a Cartesian plane, and requires students to rank 12 soccer teams (as adapted for this research) based on the number of games won or lost by each. For the students participating in this study, the *Ranking Team* was their first experience solving a mathematical activity of this type; therefore, I did not analyze this MEA as part of this study, since their lack of experience in solving these types of open-ended activities may have contributed to inaccurate data. Thus, only the *Hybrid vs. Gas Cars* (second implementation) and *The Historic Hotel* (last implementation) are presented here.

3.7. Qualitative Analysis

I followed the taxonomy of the analytical model developed by Powell, Francisco, and Maher (2003) to analyze the video recording of the nine groups of students — three from each class, with one at each performance level, as mentioned above. This technique for analyzing data is a proven strategy developed particularly to analyze video-recorded data of students' work. Its objective is to study the complex and non-linear processes of students' development of mathematical ideas and reasoning. Further, it focuses especially on students' mathematical work when they are working to solve open-ended problems in small groups in which they discuss their ideas and create inscriptions (Powell et al., 2003).

According to Powell et al.(2003), the analytical model has seven phases, which can be carried out in a non-linear way: (1) closely viewing the video data, (2) describing the video data, (3) identifying critical events, (4) transcribing the recording, (5) coding the processes, (6) constructing a storyline, and (7) composing a narrative. In the following paragraph, I explain in detail each one of the phases of the analytical model and how I carried it out.

First, according to Powell et al. (2003), the researcher has to carefully watch and listen to the video several times to become as familiar, as much as possible, with its content. For this step, it is important for the researcher to view the videos without any analytical lens in mind. Second, the researcher has to describe several episodes of two to three minutes' duration from each video so that anyone who reads the descriptions will have a general idea of its content. The researcher must include the time-code of each episode in the description. At this stage, it is important that the researcher make only time-coded descriptions and not interpretations or conjectures.

Due to the amount of data collected and analyzed, and to time constraints, I combined these first two steps — observing and describing the videos — into one phase. That is, I wrote the description of each episode as I was watching the videos.

The third phase of the analytical model is to identify significant moments, or *critical events,* in the learning process, which I considered here to be the steps of the problem-solving process (Vidic, Ozaltin, Besterfield-Sacre, & Shuman, 2014) described later. According to Maher (2002), an event is considered critical when “it demonstrates a significant advance from previous understanding, or a conceptual leap in earlier understanding, or the identification of a cognitive obstacle” (p. 34). Identifying these critical events is crucial because once one such event is identified, it is likely that other critical events, prior or following, can be seen as well, showing a set of critical events that enables researchers to acquire a deeper understanding of the students’ mathematical reasoning and ideas. This set of critical events is defined as a “pivotal strand” (Kiczek, 2000), which, according to Powell et al. (2003) points to “the mathematical ideas and forms of reasoning learners develop that are key in building learners’ mathematical understanding” (p. 418). The identification of critical events is not only identifiable in the video, but can also be discerned in the students’ inscriptions, artifacts or written reports.

In my research, I looked for those critical events in which students discuss or provide any type of mathematical explanation, in both the video recording and all the artifacts the students used to support their solutions (e.g., the written letter to the client, worksheets, flip-charts). One possible example of a set of critical events that may occur in implementing the MEAs is when students identify the problem and subsequently begin to create a model.

The fourth phase of the analytical model is transcribing the students’ discussions in each video recording. Powell et al. (2003) suggest it is not always necessary to transcribe the video, but they recommend four reasons why it may be important. First, it facilitates the coding process (the next step in the analytical model). Second, it enables researchers to understand what the students' discourse reveals about their mathematical solution as they develop it. Third, transcripts become "permanent records" (p. 422) that researchers can revisit as many times as necessary to reveal categories that might not be easily identified by looking only at the video. Finally, the transcription provides proof (if needed) of the students' critical events in their own voice. For all of these reasons, I fully transcribed the videos — of all the selected teams — to better understand and identify the critical events that occur outside of the research questions and objectives but may not be easily identified without the physical transcriptions.

The fifth phase is the coding process. Powell et al. (2003) explain that this task helps researchers interpret the data collected by helping them identify the students' "mathematical ideas, mathematical explanations or arguments… and features and functions of discourse" (p. 424) that might not be identified otherwise. This is the stage at which I began to apply a theoretical framework for analyzing the data, taking the deductive approach entailed by the Model and Modeling perspective on problem solving (Lesh & Doerr, 2003). A deductive approach "is required when the researcher has a strong theory and clear research questions" (p. 10). Researchers follow a deductive approach when they systematically select samples from their collected data "to examine specific research questions" (p. 10). Considering that the *critical events* as moments when students realize they understand something or have to "jump" a certain cognitive barrier, I used an adapted coding scheme from the problem-solving categories developed by Vidic et al. (2014): reading and making sense of the problem statement, planning, collecting data, analyzing data, formulating/re-formulating a model-solution, testing, evaluating/testing the solution, and reporting/documenting the solution. Through these categories, I aimed to deepen my understanding of the students' mathematical ideas and reasoning when working toward a model-solution.

In addition, I inductively (Miles, Huberman, & Saldaña, 2014) performed a second round of coding of the moments on which the students spent most of their solving time — i.e., the stages of the solving process adapted from Vidic et al. — to analyze the students' interaction in a group-level dimension. According to Miles et al., inductive coding refers to a set of codes that "emerge progressively" (p. 81) during data collection or analysis, and serve to uncover relevant aspects of the topic being researched. The coding scheme I considered for this second cycle was: explanations to/from, question to/from, individual turns, ideas generation/contribution, reactions to ideas (heard or ignored), and argumentation/discussion. As detailed above, I analyzed the discussions and/or argumentation moments following Azevedo et al.'s (2015) discourse characterization, which I later adapted as arguing and non-arguing discourse. In the next chapter, I detail the students' types of discourse moments and make inferences about their importance in the student's learning.

I double-coded all coding processes, whether deductive (Vidic et al., 2014) or inductive (Miles et al., 2014), and had them verified by a second rater with many years of experience in mathematics education. The codes had an inter-rater agreement of 85 percent and ended in 88 percent after solving the disagreements.

Phase six is the construction of a storyline. Through coding the relevant data, the researcher makes sense of it, making it possible to elaborate a storyline about the students' problem-solving process. This normally requires the researcher to revisit the transcripts and code in a non-linear process (i.e., to return to previous steps before proceeding to the next step). According to Powell et al. (2003), researchers make inferences and interpretations during this process in order to “come up with insightful and coherent organizations of the critical events, [which] often involve complex flowcharting” (p. 430), and I used this method to map the students’ problem-solving processes. This provided me a graphical representation of how students constructed their model, how their reasoning emerged, what steps they followed to develop their solution, and how much time they spent in each step of the process.

The last step is creating a narrative, which I wrote in the form of a report (chapter 4). At this final stage, I “decomposed this whole into smaller segments, interpreting the smaller segments in light of the whole and then recompos[ing] the whole in light of a storyline and explor[ing] a particular interpretation of the whole using data as evidence, thereby producing a written narrative” (Erickson, 1992, as cited by Powell et al., 2003, p. 431). Specifically, I showed in detail the connections among the transcripts, critical events, codes, and storyline, and made inferences and interpretations about all the data collected. Most importantly, I provided a response to the research questions of the study.

3.7.1. Quality of the Students’ Solution

This research is mainly qualitative, and I have graded the quality of the students’ solutions as represented in their written reports (i.e., the letter to the client) following the Quality Assessment Guide (QAG). The QAG (See Table 3.4) is an instrument Lesh and Clarke (2000) developed to assess the quality of solutions students had produced in a specific MEA. The QAG rates the quality of a solution considering five levels of performance[3], i.e., those that: (1) require

[3] I have used the term performance level throughout this proposal to refer to students’ level of achievement, i.e. low, average and high. In this section, it refers specifically to the quality of the students’ written reports, as labeled in the chart by Lesh and Clarke (2000) cited above.

redirection, (2) require major extensions or refinements, (3) require only minor editing (4) are useful for the specific data given and (5) are sharable or reusable.

Table 3.4. Quality Assessment Guide

Performance level	How useful is the product?	What might the client say?
Requires Redirection	The product is on the wrong track. Working longer or harder won't work. The students may require some additional feedback from the teacher.	"Start over. This won't work. Think about it differently. Use different ideas or procedures."
Requires Major Extensions or Refinements	The product is a good start toward meeting the client's needs, but a lot more work is needed to respond to all of the issues.	"You're on the right track, but this still needs a lot more work before it'll be in a form that's useful."
Requires Only Minor Editing	The product is nearly ready to be used. It still needs a few small modifications, additions, or refinements.	"This is close to what I need. You just need to add or change a few small things."
Useful for this Specific Data Given	No changes will be needed to meet the immediate needs of the client.	"This will work well as it is. I won't even need to do any editing."
Sharable or Reusable	The tool not only works for the immediate situation, but it also would be easy for others to modify and use it in similar situations.	"Excellent, this tool will be easy for me to modify or use in other similar situations – when the data is slightly different."

Table 3.4. Quality Assessment Guide. This table is a rubric for scoring the students' written report. Adapted from "Formulating Operational Definitions of Desired Outcomes of Instruction in Mathematics and Science Education" by Lesh and Clark. (2000) In A. Kelly, R. Lesh (Eds.), Research Design in Mathematics and Science Education. (p. 145). Lawrence Erlbaum Associates, Mahwah, New Jersey.

In implementing the MEAs I used for my research, I rated the students' product-solution myself. But to avoid bias, two independent reviewers scored the students' products as well. One is a doctoral student in engineering with more than 20 years of teaching experience, the other holds a Ph.D. in mathematics education. They rated the students' work in a blind process. The initial inter-rater agreement was 88 percent, which ended in 92 percent after resolving for differences.

In addition, I compared the performance quality of each group on the second and third implementations, the *Hybrid vs. Gas Cars* and *Historic Hotel*, in order to assess any possible differences over time to see if there were any gains in the quality of the students' responses after the first implementation. Finally, it is relevant to highlight that because only 24 teams were formed and their product-solutions analyzed, I was only able to perform some descriptive statistics and no inferential statistics.

3.8. Pilot Study

Pilot studies are conducted to test the design of a research project and to assess its feasibility (Glesne & Peshkin, 1992; Ary, Jacobs, Sorensen, C., & Walker, D., 2013). During the summer and fall of 2015, I conducted my first pilot study for this research project. As a co-teacher in collaboration with the classroom teacher, I implemented two Model-Eliciting Activities with 11th grade students previously determined to perform at specific levels (low, average, high) at a Private University, high school division. I adapted each activity to the students' context and reality[4] in accordance with the MEAs' design principles (Lesh et al., 2000). The MEAs I implemented were the *Historic Hotel* problem and the *Gas vs. Hybrid Cars* problem (see Appendix B), but to illustrate how the MEA unfolded, I detail here only the students' artifacts and worksheets of the implementation of the *Historic Hotel* problem.

[4] For instance, the currency showed in the problem statement is in Mexican Pesos (MX$).

Sixty-one students participated in the study, divided in three classes in the following way: one classroom of 26 low-performing students, another of 22 average-performing students, and one more of 13 high-performing students. Each class was further broken into teams of three or four members — eight teams for the low- and average-, and five for the high-performing students, totaling 21 teams. For the purpose of this pilot study, low-performing students were classified as those who were retaking a mathematics course during the summer of 2015, average students those who were enrolled in a regular mathematics course during the upcoming fall, and high-performing students as those taking advanced mathematics courses in the International Baccalaureate Diploma program[5], also in the upcoming fall.

Each MEA allotted approximately two 70-minute sessions for students to develop their solutions; in total it required approximately 280 minutes to complete both MEAs. Each MEA comprised three main parts, as addressed in the introduction of this proposal: the warm-up day, day one work, and day two work.

On the warm-up day, we, the classroom teacher and I (as a co-teacher), gave students, as homework, an article related to the next day's MEA, with four to six questions related to that MEA's topic. For the day one work, we discussed the homework reading with the students for approximately 10 minutes to introduce them to the activity, to engage, motivate, and encourage them to think more deeply about the topic, and to assess their understanding and comprehension of the reading. Next, we gave the students 10 minutes to read the problem statement and gather in their assigned teams of three or four. They then worked on solving the problem statement for approximately 45 minutes. During this time, the teacher and I walked among the teams observing, listening, providing guidance, and taking field notes to learn more about their

[5] The Diploma program is offered as an advance program endorsed by the International Baccalaureate Organization.

mathematical strategies and thoughts, collaborative learning, and modeling cycles (Zawojewski, Lesh, & English, 2003).

For the day two work, we gave the students approximately 30 minutes to finish the previous day's work, write the letter to the client explaining their procedures, and prepare a brief presentation of their findings, thoughts, and solutions in flip-chart format. The letter and flip-chart provided us a trail that documented how they developed their models and externalized their mathematical thoughts (Lesh et al., 2001). Then, for approximately the next 30 minutes, each team gave a five-minute presentation explaining their solutions and findings to the rest of the class, the teacher, and me. A rich learning feedback-discourse emerged, in which the students, the classroom teacher, and I each shared our opinions and thoughts about the solutions presented.

To demonstrate some of this pilot study's findings, I present next some of the similarities and differences between the various solutions students provided to the *Historic Hotel* problem, taken from their artifacts and presentations.

3.8.1. Findings Within the Pilot Study

Twenty-one teams developed solutions to the *Historic Hotel* problem, but space allows me to include only five of them here. My aim is to show that students at each different performance level developed good solutions for MEAs. The students' solutions across different performance levels were not as different as might be expected. In fact, I found several similarities in the students' answers regardless of how they were categorized (low, average, or high-performing).

As indicated in the problem statement, students wrote a letter to the “client” explaining how to maximize profits. They were tasked with determining what number of hotel rooms would have to be booked to attain the greatest economic benefit, after considering the maintenance fee and the unbooked rule. Many teams explained in their letters that 70 booked rooms were required. Several of these teams created tables to visualize the data and make computations.

For example, one team among the low-performing category called “Los Soles” made a table (see Table 3.5) showing that increasing the price per room decreased the number of booked rooms.

Table 3.5. Establishing the booking rate. Team “Los Soles”

Rate to maximize the profit (MX$)	Rooms not booked
600	0
610	1
620	2
630	3

Table 3.5. Team “Los Soles” depicted an inversely proportional interpretation of the price vs. rooms. For example, with a price of MX$600, there are no rooms that remain unbooked.

This team determined that to maximize the profit of the hotel, they would need to multiply the hotel rate times the number of rooms booked, minus the maintenance fee times the number of rooms booked (See Figure 3.3).

Figure 3.3. Profit computation. Team “Los Soles”

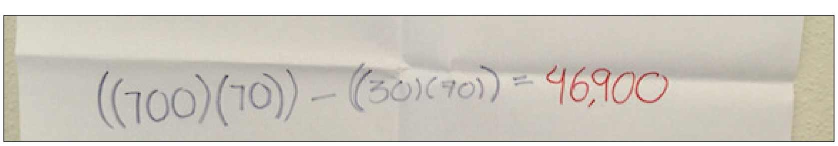


Figure 3.3. Team “Los Soles” showing how they compute a profit considering a price of MX$700, 70 rooms, and a maintenance fee of MX$30 (MX$10 less of what it should be).

Mathematically, the profit equation would be: P = (rbr) (rb) – (m)(rb), where P is the profit, rbr the room booking rate, rb the rooms booked, and m the maintenance fee.

In a similar but more elaborate way, another team categorized as average in performance called "Mate" also used a table (Table 3.6) to organize their data and visualize the number of rooms needed to maximize the profit.

Table 3.6. Team "Mate" profit computations

Rooms	Price (MX$)	Maintenance	Profit (MX$)
80	600	$40 x 80	$48,000 - $3,200 = $44,800
75	650	$40 x 75	$48,750 - $3,000 = $45,750
70	700	$40 x 70	$49,000 - $2,800 = $46,200
60	800	$40 x60	$48,000 - $2,400 = $45,600
40	1000	$40 x 40	$40,000 - $1,600 = $38,400

Table 3.6. Team "Mate" showing how they compute profit considering the number of rooms, price, and the maintenance cost. Differently from team "Los Soles," they consider the rules of the problem statement as they were given.

"Mate" used the same mathematical algorithm (P = (rbr) (rb) – (m)(rb)) as the low-performing team "Los Soles," in which the number of rooms booked was multiplied by the price of booking, minus the maintenance fee times the number of rooms booked. After computing and comparing their findings, this team also concluded that 70 rooms at $700 would be required, totaling only $46,200 instead of "Los Soles'" $46,900. The reason for this disparity in profits is addressed below in the differences section.

Finally, a team called "Matebrios," considered high-performing, also created a table (Table 3.7) showing the number of rooms booked and the price.

Table 3.7. Team “Matebrios. Rate vs. rooms booked

Rate (MX$)	Rooms booked
600	80
610	79
620	78
630…	77…
700	70

Table 3.7. Team “Matebrios” shows the number of rooms booked. They consider only the rooms to be booked and not the rooms unbooked as the team “Los Soles” did.

Similar to the other two teams (low- and average-performing), the “Matebrios” considered the same mathematical algorithm to calculate the final profit, and like the average-team (Mate), they concluded that the profit would be $46,200.

In the following section, I address the differences between each performance level’s solutions to show how MEAs, as problem-solving activities, serve not only to enhance and elicit mathematical reasoning for students at different levels, but also that students can develop good solutions regardless of how they have been categorized.

3.8.2. Differences within the pilot study

The MEAs implemented in the pilot study, with their open-ended yet structured characteristics, allowed students the freedom to develop solutions from different perspectives and to go beyond what was expected. In particular, many of the low-performing teams were able to develop strategies and processes to solve the problem. In what follows, I present some of these solutions, showing different approaches and strategies that are equally effective.

As discussed above, the problem statement stipulates that the hotel has 80 rooms, charges MX$600 per day for each, incurs a MX$40 per room maintenance fee, and that every MX$10 increase in room rates results in one less room being booked. The students' task was to determine the optimal number of rooms needing to be booked to maximize profit at the given rates. Most of the teams' solutions show this number to be approximately 70 rooms. They use various strategies to arrive at this conclusion. Three teams, 2 high- and 1 average-performing group arrived at a more precise number—68 rooms.

I begin with the low-performing team "Los Soles", who used a table to visualize their data. At the outset, they decided one way to maximize profit would be to cut the hotel's costs, and so decided to cut the maintenance fee from the MX$40 given in the problem statement to MX$ $30. This is significant because this team felt that they could change the terms of the problem, an indication that the team was interpreting the problem in a more real-world way. Then, they determined that 70 rooms booked at MX$700 would maximize the hotel's profit at MX$46,900 per day: (700)70 – 70(30) = MX$46,900[6].

Another team called "Los Rellis", also in the low-performing class, chose not to reduce maintenance costs; instead they implemented a trial-and-error strategy to find the number of rooms needed. For every number of rooms they tried, they computed the profit using the same mathematical algorithm as "Los Soles"—multiplying the number of rooms booked by the price of booking, minus the number of booked rooms times the maintenance fee, considering 65, 70,

6 This group also created a solution of what to do with the unused rooms: build a restaurant. Though they did not consider the added cost that would have to be factored into their optimal booked room number, they did consider the additional revenue this might generate. It is thus worth noting how the MEA provoked a creative idea beyond the scope of the problem, one whose feasibility was actually in their grasp to develop further.

and 75 rooms. Although they determined 70 to be the correct number of rooms, when computing the profit they made an error, obtaining a final profit of MX$47,800: (700)70 – 70(40) = MX$46,200. In other words, they used the correct algorithm, but made a calculation error.

The average-perfoming team called “Azules” also decided to use a trial and error strategy, applying the same mathematical algorithm as the other groups. They computed profit P considering 65, 70, 75, and 80 rooms and determined that having between 65 and 70 rooms booked would maximize the profit. Subsequently, they computed the profit with 67, 68, and 69 rooms and determined 68 rooms at $720 per night to optimize a profit of MX$46,240: 720(68) – 40(68) = MX$46,240. Although this team used a trial-and-error approach like the other teams, they explored the possibilities in a deeper way (considering different numbers of rooms and prices) until they were able to find the optimal number of rooms to maximize profit.

Other teams used different mathematical strategies and algorithms. For example, a team in the average-performing class called “Los Cookies” calculated profit P by factoring the variables, then organizing their data in a table considering different numbers of rooms (see figure 3.4).

Figure 3.4. Team “Los Cookies”. Profit calculation and pattern

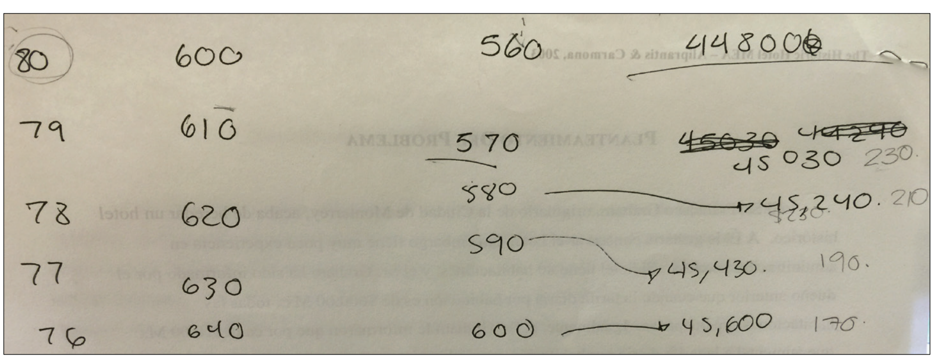


Figure 3.4. Team “Los Cookies” showing how they calculated profit. Importantly, this team found a pattern on the profit (i.e., MX$230, MX$210… MX$170).

Mathematically, this team used the following algorithms to factor the variables representing the number of rooms booked (rb):

P = (rbr) (rb) – (m)(rb)

P = rb (rbr – m)

In Table 3.8 below, I detail the way the team "Los Cookies" created their table, factored the rooms booked, and showed the profit.

Table 3.8. Team "Los Cookies". Factorizing the profit

Rooms	Price (MX$)	Factorizing strategy	Profit (MX$)
80…	$600…	P = (80)($600)-(80)($40) P = 80($600-$40) P =80($560)	$44,800
76	$640	P = (76)($640)-(76)($40) P = 76($640-$40) P =80($600)	$45,750

Table 3.8. Team "Los Cookies" showing how they factored the profit by manipulating the rooms, price, and profit.

In addition, this team found a pattern among the profits they computed. As can be seen above in Figure 3.4, they subtracted the following profit minus the previous profit to obtain MX$230, MX$210, MX$190, and MX$170. Although, they could have used this idea to find the point at which it would no longer be necessary to continue looking for the price that would maximize profit, they decided not to elaborate further on this idea or make any comment about it.

In the end, this group did not determine a particular rate or number of rooms to be booked, but I include their example to show that they were able to implement a strategy that could determine this result. When using MEAs, helping students determine a feasible process is more important than their specific solutions (Lesh & Doerr, 2003).

Teams in the high-performing class used more sophisticated mathematical elements in their solutions, as expected. For example, the "Matebrios" team elaborated a graph (see Figure 3.5) to visualize the change of the booking price vs. booked rooms.

Figure 3.5. Team "Matebrios" graph of Rooms vs. Price

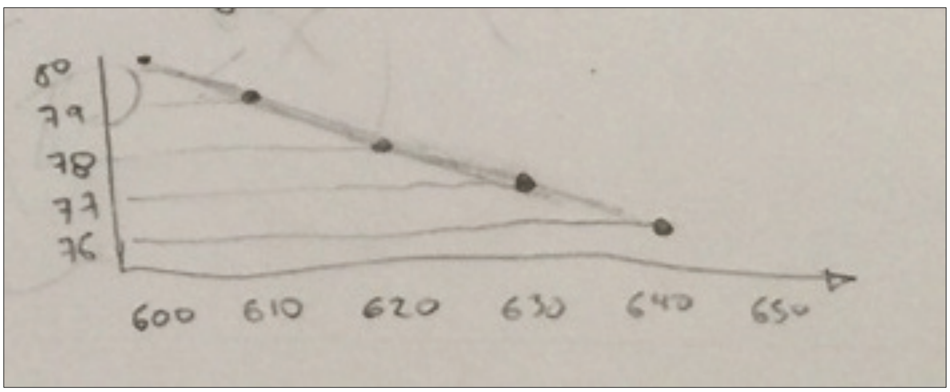


Figure 3.5. This image depicts a graph created by team "Matebrios", responding to a booking rule given in the problem statement. They show the number of rooms booked decreased as the room rate increased.

They also explored the problem using the slope-point formula– i.e., $y - y_1 = m (x - x_1)$– and defined variables (e.g., x = price, y= number of rooms). However, they were not able to develop any further solutions. Like the low- and average-performing teams, they demonstrated a booked room rate of 70 rooms at MX$700 per night. Thus, even though they used more sophisticated mathematical elements (i.e., formulae, variables, graphing), they arrived at a similar conclusion as the other groups.

Another team in the high-performing class called “The LDJP” used the concept of variables and first derivate to calculate the number of rooms *not to be booked* to maximize the profit.

This team defined “x” as the number of times the booking price was increased, which also represents the number of rooms not to be booked, and “y” as the booking price. They entered in a cycle of modeling two times. During the first cycle they did not consider the maintenance fee but used concepts of calculus to find the first derivative, aiming to find the local maximum of the equation (Figure 3.6).

Figure 3.6. Team “LDJF” First Cycle of Modeling

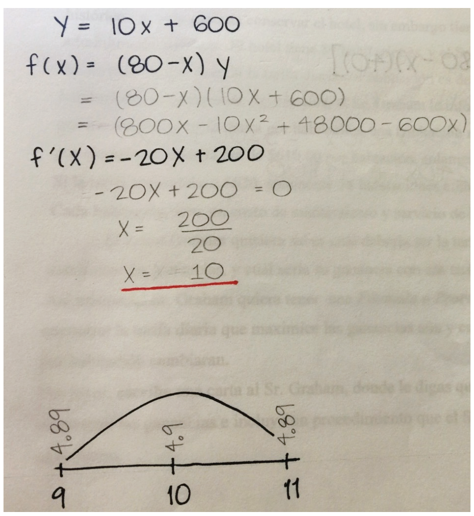


Figure 3.6. This image shows the first attempt of the team “LDJP” to calculate profit. During this first cycle of modeling, they realized they had a mistake.

However, they realized that something was either wrong or incomplete. As a result, they entered into a second cycle of modeling (Figure 3.7) in which they were able to find the correct number of rooms not to be booked.

Figure 3.7. Team “LDJF” Second Cycle of Modeling

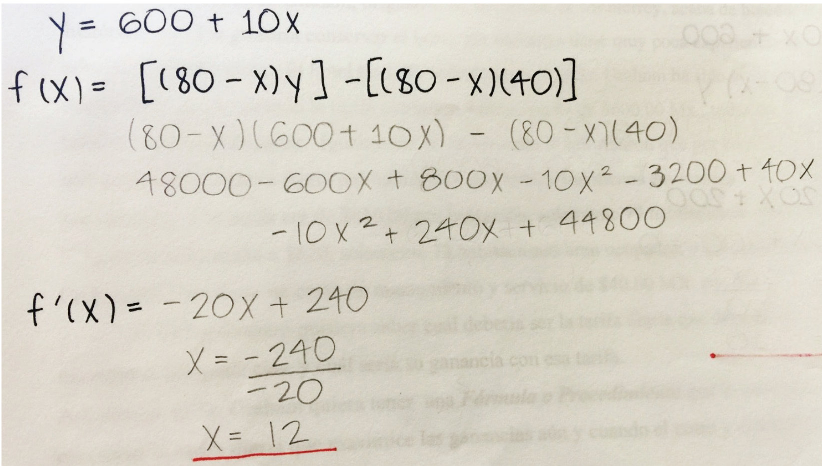


Figure 3.7. This image shows team “LDJP”s second cycle of modeling. In this cycle, they were able to successfully find the number of rooms that remained unbooked (i.e., 12 rooms).

This group determined the number of rooms to remain empty rather than to be booked. With 12 unbooked rooms, their solution is effectively the same as “Los Azules”, who calculated 68 booked rooms as optimal. The team then proposed a final formula that could be used in the future to calculate the profit P as:

$$[(80 - x)\ y] - [(80 - x)\ (40)].$$

The solutions of the different teams at different performance-levels were certainly different in both quality and sophistication, but in the end the students’ mathematical reasoning was comparable. For example, two different teams—one in the low-performing class (Los Soles) and another in the high-performing (LDJP)—addressed the same idea of considering rooms that remain empty instead of rooms booked. Although the “LDJP” team developed the idea further and “Los Soles” decided to follow a different solution path, the mathematical ideas were the same during the initial interpretation of the activity for these two “different” groups of students.

In conclusion, the *Historic Hotel* MEA required students to find the number of rooms to be booked in order to maximize the profit. Many teams concluded that 70 rooms at a price of

$700 was the correct solution, but 68 booked rooms at a price of $720 (as the team "Azules" of the average-performing class showed) or 12 unbooked at the same price (as the high-performing "LDJP" team demonstrated) seems to be more optimal. These two groups arrived to the right solution using different strategies; one was more sophisticated than the other, but in the end, both valid. Again, the results are comparable, and the solution-path these two teams followed was not determined by how they have been categorized (low-, average- or high-performing), but rather how they understood, analyzed, and reasoned through the problem.

What is important, however, is that students *across all performance levels* arrived at roughly the same conclusions while employing many different strategies (Dominguez, 2010), showing their abilities to create, invent, and elaborate mathematical elements to provide a solution.

3.8.3. Limitations of the Pilot Study

I faced several challenges when conducting this pilot study: limited time in the classroom, lack of teacher training in MEA implementation, and failure to video-record students while they worked to develop their model-solutions.

First, although The Institution had approved the study and given me authorization to conduct my research, they allowed me to be in the classroom for no more than two consecutive 70-minute sessions for each MEA implemented. As a result, the time the students had to develop their solutions, the opportunities to have a deeper formative discussion with the students, and the length of the activities were compromised.

Second, because of the time constraint, I could not provide any training to the students' teachers. As a consequence, they were unwilling to concede more class time that may have enhanced the results since they lacked sufficient understanding of MEAs and their implementation.

Third, though I set up cameras to record the classroom as a whole, I did not consider video-recording the work of any specific team but only collected audio recordings of the students as they worked. These turned out to be insufficient during analysis because they did not allow me to see the whole picture of the students' interaction and development process. Consequently, my analysis is mostly limited to evaluating the students' written work (i.e., the written letter to the client, artifacts, and flipchart paper presentations) and my observation notes. A more sophisticated understanding of how their mathematical thought developed requires a more complete record of the processes when implementing MEAs.

During the data collection of the dissertation study, I changed the way I implement the MEAs in several ways: I will allowed more time for each MEA, trained the teachers to understand my aims in implementing MEAs, and most importantly, I will collected not just audio but also video of specific teams at different performance-levels so I could analyze their mathematical thoughts, strategies, and development during the problem-solving process more deeply.

3.9. Summary

In sum, in carrying out my research, I used an ethnographically informed case study methodology, in which (1) cases comprised teams of students (Gillham, 2000), one in each category, stereotyped and classified as low-, average-, and high-performance; (2) the researcher was involved as a participant and observer; (3) little was known or had been studied about the students' processes of reasoning (i.e., the types of discourse that emerged during the students' interactions); (4) the activity unfolded in a naturalistic setting; and (5) three different types of participants were involved (students, classroom teachers and I, the researcher as co-teacher).

The study took place in northeastern Mexico, where I implemented three MEAs with 11th grade mathematics students at a private Catholic high school, making both audio and video recordings of the students carrying out the MEA and collecting their written work. I analyzed the data from the model-product and model-process dimensions with the Model and Modeling Perspective (Mousoulides & Sriraman, 2008; English, 2005; Lesh & Doerr, 2003; Lehrer & Lesh, 2003). My aim was to seek an alternative way to teach and learn mathematical problem solving by having a better understanding not only of the product students generated (as others have done), but also by deeply analyzing the interactional process that emerged and in which students engaged while developing their product.

Chapter 4: Analysis and Results

To assess whether MEAs can "level the playing field," I look at two different dimensions of students' group work during MEAs implementations — namely the mathematical products they generated and the discourse patterns of the processes that they went through when creating such products. As indicated throughout this dissertation, in MEAs students describe and explain their understanding of a problem in order to construct a model to solve it. The description and explanation are part of a *process* that generates a *product*, the model (Lesh & Doerr, 2003). Products are important because they inform me about students' progress toward canonical goals of mathematics education and curriculum (Gravemeijer et al., 2017). In terms of MEAs, researchers have been concerned with analysis of students' products (Ekmekci, 2013; Larson, 2010; Zawojewski, Lesh, & English, 2003; Lesh & Harel, 2003), but few have considered the discourse patterns that emerge from the process by which students have gone about developing their model-products-solutions. The processes, and in particular the patterns of talk that students engage in, are crucial because they point to forms of participation that are more or less conducive to learning.

In this chapter, I provide evidence of both the product-model-solutions students created while solving a series of MEAs, and the epistemological discourses and content development that occurred during the process of working-solution.

4.1. Mapping the Chapter

Seventy-four students participated in this study, out of which sixty-nine percent were female and the rest male. They formed teams of two or three for a total of twenty-four teams. These teams worked for approximately 600 minutes solving three MEAs. Each MEA required around 180-200 minutes of work, divided into sessions of about 60-70 minutes. For the purpose of the analysis, I considered only the last two MEAs implemented (i.e., *Hybrid vs. Gas Cars* and *The Historic Hotel*). The activities were analyzed in the following way: First, I show what could be an "ideal-general" acceptable solution for both *The Hybrid vs. Gas Cars* (Elliott, 2014) and *The Historic Hotel* MEAs (Aliprantis & Carmona, 2003). Second, I consider the Quality Assessment Guide (Lesh & Clarke, 2000) as a framework to "roughly measure" the quality of the full samples' final product-solutions for both *The Hybrid vs. Gas Cars* and *The Historic Hotel* MEAs. This helps me determine their improvement, or lack thereof, over time, from the implementation of the *Hybrid vs. Gas Cars* to the *Historic Hotel* MEA. Third, I use the last MEA implemented, *The Historic Hotel*, to analyze in detail the intermediate-product-solutions, strategies, and components/elements all groups created and used (i.e., the 24 groups). Fourth, from the nine video recorded groups, three were selected (i.e., one in each performance-level) to be fully transcribed and coded to study and analyze the process of model development aiming to discover, compare, and categorize the type of discourse that emerged during their model-solution process. Fifth, I compare and contrast the product-process of the three selected teams to better understand the richness of their solutions, and the differences and similarities of each solution-path.

4.2. MEAs' Solution Paths

In the previous chapter, both *The Historic Hotel* and *Hybrid vs. Gas Cars* MEAs were detailed and explained. But, to remember what are the possible solutions paths students' teams might provide to the MEAs, I show in the next paragraph a brief summary of the MEAs' model-solutions.

4.2.1. The Historic Hotel

As detailed before, *The Historic Hotel* MEA is a mathematical activity that requires the students — working in small teams — to provide a model-solution that maximize the profit of a hotel. The hotel has 80 rooms, an initial daily booking price of MX$600, and a daily maintenance cost of MX$40 per room. In addition, the number of booked room varies depending on how much the initial daily price is increased. For example, if the daily rate increases MX$10 (i.e., to MX$610), the 79 rooms will be booked. If the price is increased to MX$620, then 78 rooms are booked.

To maximize the hotel's profit, students would need to find the right number of rooms to be booked, the rooms booking price, and an adequate model for the profit. In addition, students would need to create tables, list, charts, or graphs that help them visualize their solutions.

Previous implementations of *The Historic Hotel* demonstrated that most students have found the hotel's maximum profit (P) by creating a table (see Table 4.1) of the booking price vs. the number of booked rooms, and by considering the following model:

$$P = (\text{Booked rooms}) \times (\$\text{Price}) - (\text{Booked rooms}) \times (\text{Maintenance cost})$$

Table 4.1. Booking Rooms vs. Price

Number of rooms booked	Daily room price (MX$)
80	600
79	610
…	…
69	710
68	720

Table 4.1. The hotel's daily booking price is increased by MX$10, while the number of booked rooms decrease by one.

Once students have developed their model that would maximize the profit, they proceed to test their model by input the number of rooms and prices from Table 4.1 in the model. For example, if a team selects to test their model considering 69 rooms with a booking price of MX$710, and only providing maintenance service to booked rooms, their model would be:

P = (Booked rooms) x (MX$Price) – (Booked rooms) x (Maintenance cost)

P = [(69) x (MX$710)] – [(69) x (MX$40)]

P = MX$46,230

Another possible solution path might be considering providing service to all rooms regardless of the number of rooms booked. If this is the case, then the maintenance cost is taken as a constant:

Maintenance = 80 x MX$40

Maintenance = MX$3200

As result, the profit would be calculated slightly differently:

P = (Booked rooms) x (MX$Price) – (Booked rooms) x (Maintenance cost)

P = [(69) x (MX$710)] – MX$3200

P = MX$46,230

P = MX$45,790.

In either case, students would need to continue looking for the right number of rooms and booking price that maximize the profit. In the end, MEAs are open-ended activities that allow for multiple solution paths and perspectives (Dominguez, 2010; Lesh & Doerr, 2003).

4.2.2. The Hybrid vs. Gas Car

The activity requires students to help a taxi company decide what type of car to buy: either a gasoline or a hybrid car. Students need to consider several variables such the cost of each car, past (from 2005 to 2015) and estimated future price of the gasoline (from 2016 to 2020), characteristics of the cars, and other external variables like the resale price or environmental factors. The company needs 10 cars that would be used for five years. In addition, they would need to consider the average annual use of the cars at 32,000 km, of which 75 % are spent on highways and 25% in the city.

One possible solution path for this MEA would be to first estimate the gasoline's price. In Table 4.2, I show the estimated gasoline price for five years, which would serve to later determine the annual cost of gasoline and eventually the final cost of each type of car.

Table 4.2. Price Estimation for five years

Year	Price per liter (MX$)
2016	14.47
2017	15.16
2018	15.86
2019	16.57
2020	17.29

Table 4.2. Here the estimated projected price of gasoline from 2016 to 2020 is shown.

Once students determined the estimated price of the gasoline, they could develop a model to determine the annual cost of gasoline (for any type of car) for a certain year, based on the annual average kilometers traveled in the city and highways:

Annual cost = [(Rate of kilometer on the highway vs. Cars' Performance) (Cost of Gasoline)] + [(Rate of kilometer on the city vs. Cars' Performance) (Cost of Gasoline)].

The performance and general characteristics of each car (including the price) are provided as part of the activity; students use this information to determine the annual cost of gasoline for each type of car, and the final cost considering all 10 cars for five years. One of the main characteristics that students need is the performance of the cars (i.e., the fuel economy), which is required to estimate the annual cost of gasoline. For example, for the gasoline cars, the performance is 56 km and 40 km per liter in the highway and city respectively. Hybrid cars get 80 km/liter on the highway and zero in the city (i.e., in electric mode). When students input all these data into their model, they obtain the annual cost, which is different for each type. Students need to repeat the computations every year. For instance, in Table 4.3, I show what would be the gasoline cost for 2016 for either the gas or hybrid car:

Table 4.3. Example of the cost of gasoline for a single year

Gasoline Cars	Hybrid Cars
Cost = $\left(\frac{24000\ Km.}{56\ \text{Km./ltr.}} \bullet \text{MX\$14.47}\right) + \left(\frac{8000\ Km.}{40\ \text{Km./ltr.}} \bullet MX\$14.47\right)$	Cost = $\left(\frac{24000\ Km.}{80\ \text{Km./ltr.}} \bullet \text{MX\$14.47}\right) + \left(\frac{8000\ Km.}{150\ \text{Km./ltr.}} \bullet 0\right)$
Cost = MX$9,095.40	Cost = MX$ 4,341

Table 4.3. The cost of the gasoline for 2016 is calculated here considering the estimated price of gasoline and the cars' performance.

Once students have calculated the cost of gasoline for every year and multiply the cost of each type of car by ten (the number of cars needed), they would need to add it all to obtain a final cost of MX$ 2,235,227.01 for the gasoline cars and MX$3,530,955.00 for the hybrid cars. Although the hybrid cars' final cost is higher than the gasoline car, and the answer seemed to be straightforward, teams may choose to propose the hybrid cars as the best ideal solution considering other variables like price of resale, environmental factors, etc. Once the cost of the cars is added to the cost of gasoline needed for the five years, when students write their final report to the client (i.e., the letter to the client), they must justify why they recommend either type of car and the factors that lead them to make such a decision.

4.3. THE PRODUCT

In this section I analyze the teams' final and intermediate models that students created while working *The Hybrid vs. Gas Cars* and *Historic Hotel* MEAs. Both the final and the intermediate products show students' group work from different perspectives: the quality of the solution based on the clients' needs, and the mathematics created and externalized in worksheets, report, final presentation, and artifacts.

4.3.1. Final products

After briefly showing[7] what would be a possible expected model-product-solution from the last two MEAs implemented (i.e., *The Hybrid vs. Gas Cars* and *The Historic Hotel*), I now consider The Quality Assessment Guide (QAG) in order to evaluate the quality of the models the teams constructed (i.e., their final products) based on whether the model solves the client's problem.

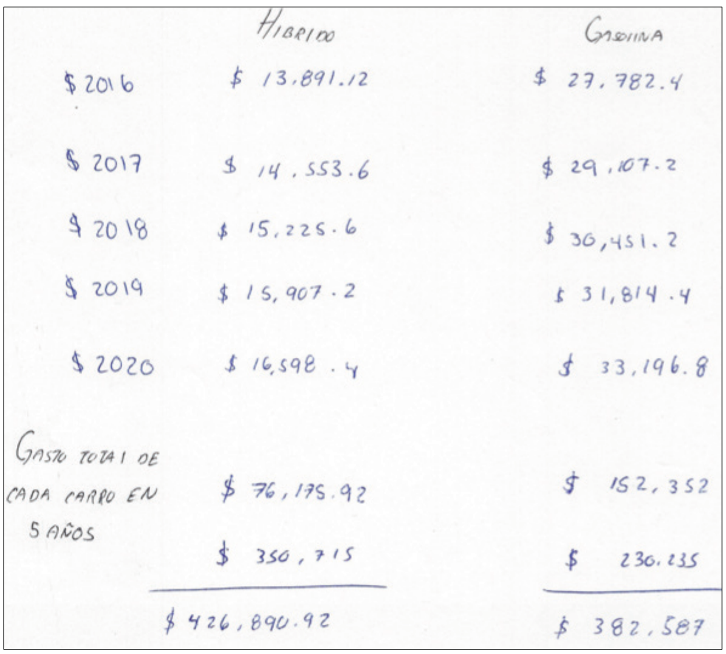
Recalling from previous chapters, the QAG is a rubric that scores the teams' solutions considering five levels of performance ("performance" here refers to evaluation of students' work, not the students' achievement), where the lower level represents the "lowest" quality solution and the higher the "highest." In the following paragraph, I briefly recall what each level represents and exemplify each one with teams' solutions from either *The Hybrid vs. Gas Cars* or *The Historic Hotel.*

---

[7] Full detailed explanation of each MEA presented here can be found in chapter 3.

At the first level, students reconsider the effort, since the model does not respond to the client's need and it is on the wrong track toward a solution. The student needs to start over and rework the problem. In Figure 4.1, I show an example of a solution that falls into this category from *The Hybrid vs. Gas Cars* MEA. The LGA team developed a solution based mainly on assumptions and not the information provided in the problem statement.

Figure 4.1. LGA's solution



	Hibrido	Gasolina
$ 2016	$ 13.891.12	$ 27.782.4
$ 2017	$ 14.553.6	$ 29.107.2
$ 2018	$ 15,225.6	$ 30,451.2
$ 2019	$ 15,907.2	$ 31,814.4
$ 2020	$ 16,598.4	$ 33,196.8
Gasto total de cada carro en 5 años	$ 76,175.92	$ 152,352
	$ 350,715	$ 230,235
	$ 426,890.92	$ 382,587

[Translation:
Hybrid Gasoline

Total cost by car for five years]

Figure 4.1. The image shows the LGAs' solution in which no model was developed.

The LGA team developed a solution that did not consider the elements stated in the problem statement. For example, the activity established that a driver traveled approximately 32,000km annually, of which 75% are on highways (24,000km) and the rest in the city (8,000km). But, the team ignored these aspects. In addition, the team made several assumptions not mentioned in the problem statement, like considering that both cars (i.e., the hybrid and gasoline) have a fuel capacity of 40 liters, that the gasoline lasts no more than one week, or that a year is composed exactly of 48 weeks (to calculate the annual gasoline consumption). Although the team does not mention any model, they calculate the annual cost of gasoline for a gasoline car in 2016 using the following calculation:

Cost= (Car's gas capacity) x (Weeks/year) (Cost of gasoline)

= 40 ltrs. • 48 • MX$14.47 = MX$27,782.

Then, to obtain the cost of gasoline for the hybrid car, they simply divided the total cost of gasoline calculated above for gasoline cars by two, assuming that hybrid cars have a gasoline composition of exactly half of what gasoline cars have.

Comparing the solution of the LGA team with the possible solution path showed above and in chapter 3, it is clear that the team must completely reexamine their approach, following what the problem statement states and considering all the information provided.

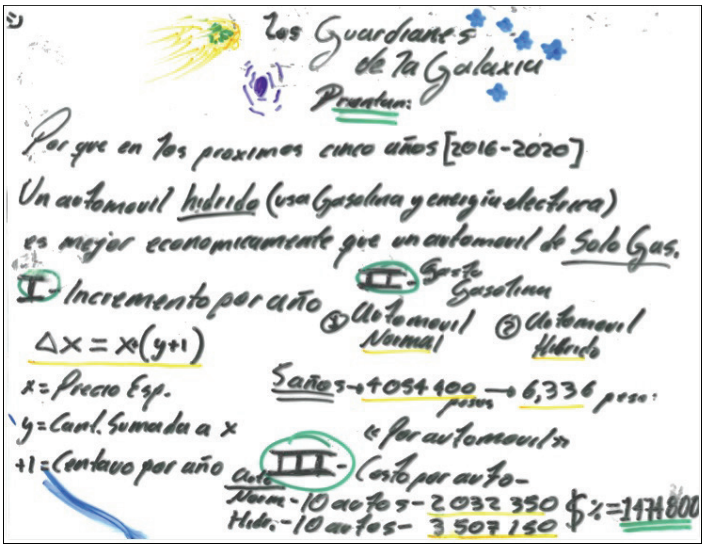
At the second level, the student has made a good start but requires more work to produce an adequate solution. For example, the team CFN developed a solution for *The Historic Hotel* that considered increasing the hotel's booking price by five, instead of by ten as stated in the problem statement. In addition, they created a general model to calculate the hotel's profit:

Profit=(# Rooms • $Booking Price) – (# Rooms • $Cost of Maintenance)

But, they fail to compute and find the hotel's number of rooms that would maximize the profit, as stated in the problem statement. The CFN team proposed that 80 rooms at a price of MX$605 would be the ideal price to maximize the hotel profit. However, as shown above and in chapter 3, a price that would possible maximize the profit would be MX$720 with 68 rooms.

At the third level, the model-solution requires some refinements in order to respond to the client's requirements. For example, in Figure 4.2, I show the team Guardianes de la Galaxias' solution. They created a model for *Hybrid vs. Gas Cars* that included variables, mathematical concepts, and an explanation of their results. However, they failed to include the computations for the consumption of gasoline. In addition, their solution is not in compliance with the sixth principle of MEAs, which is the generalizable principle (Lesh et al., 1999).

Figure 4.2. Guardianes de la Galaxias' solution



[Translation:
Because in the next four years [2016-2020]
a hybrid car (use gasoline and electric energy) is better economically than a car that only uses gas.
I. Annual increase
II. Gasoline cost

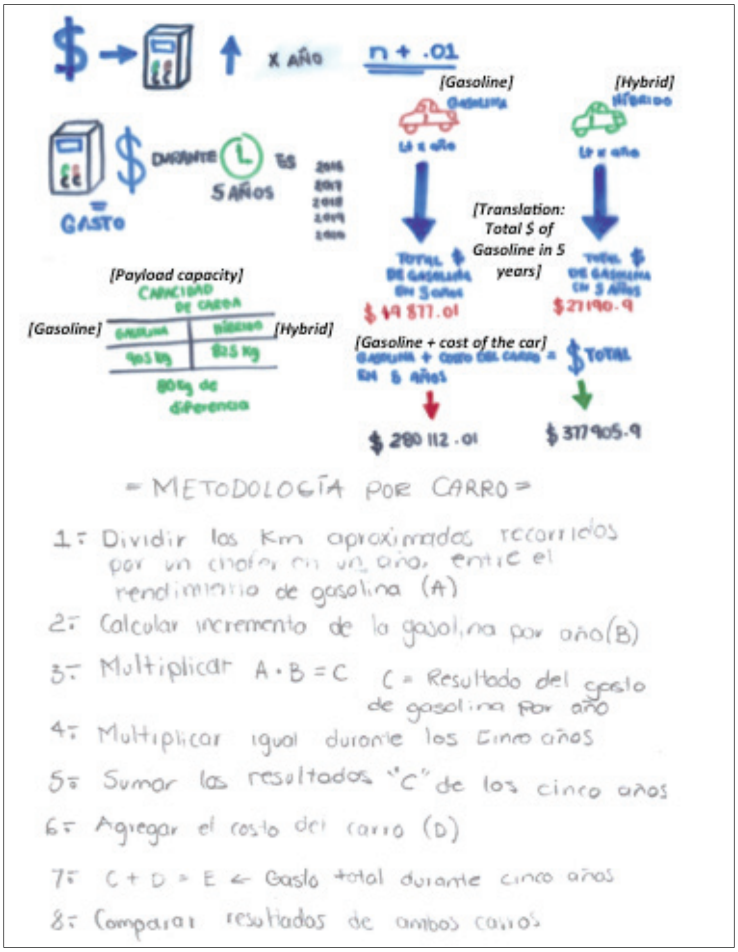
x = Expected price
y = Amount added to x
+1 = Annual Penny

III. Cost by Type of Car
Normal – 10 cars
Hybrid – 10 cars]

Figure 4.2. The team Guardianes de la Galaxia proposed that the hybrid cars are a better investment for the company despite being more expensive. They based their response on the relation of the cost of gasoline vs. the consumption, which is higher in gasoline cars and therefore more costly.

At the fourth level, the model responds to the client's needs, but only considers a specific set of data. For instance, a team called Dinamita developed a methodology to obtain the total cost of gasoline when solving *The Hybrid vs. Gas Cars* MEA. In the Figure 4.3, I show the team's final presentation in which they externalized the way they interpreted the givens to create a model that solves the activity. The team proposed a more complete solution than other teams, but is not in compliance with the sixth principle (Lesh et al., 1999).

Figure 4.3. Team Dinamita's solution



[Translation:
= Methodology by Car =

1. Divide the annual approximate traveled km by the car performance. (A)

2. Calculate the annual increase of gasoline. (B)
3. Multiply A • B = C C = is the result of the annual cost of gasoline.

4. Compute the same multiplication for the five years.
5. Add all the "C" results of the five years.
6. Add the cost of the car. (D)

7. C + D = E 🡐 Total cost for the five years
8. Compare the result of both cars.]

Figure 4.4. Team Dinamita clearly explained how they developed their solution for *The Hybrid vs. Gas Cars.*

Finally, at the fifth level, which represents the maximum possible score, a higher quality of a generalizable product responds completely to the client's need. In Figure 4.4, I show an example of team MA^2. They developed a solution for *The Historic Hotel* in which their model is not only developed for the given and collected data, but also allows to be used in similar situations. The solution created by this team is one of the multiple solution paths detailed above and in chapter 3 for *The Historic Hotel.*

Figure 4.4. Team MA^2 set of model

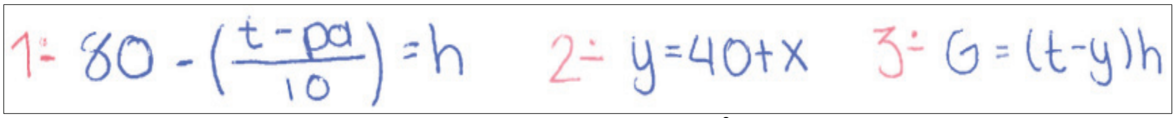


Figure 4.4. The image shows a set of model that the MA^2 team developed to obtain the ideal number rooms (h) and the maintenance cost (y) that would maximize the hotel profit (G).

The 24 teams' final products for the *Hybrid vs. Gas* and *the Historic Hotel* MEAs were analyzed by two reviewers with experience in mathematics education at different grade levels. The reviewers scored the students' final model-products based on the levels of performance of the QAG. The inter-rater agreement score, which represents the percentage of agreement between raters (Tinsley & Weiss, 2000; Gwet, 2014), was approximately 88%. After discussing and resolving the differences, the inter-rater agreement score was 92%.

The MEA and QAG scores for each team are shown in the following table.

Table 4.4. Teams' Quality Assessment Scores

Team	Level	MEA QAGs Score	
		Hybrid vs. Gas	**Historic Hotel**
Guardianes de la Galaxia	LP	3	4
Dinamita	LP	4	2
VACAOS	LP	2	3
LGH	LP	1	3
LNA*	LP	3	3
Mateatletas	HP	3	2
Carrojan	HP	2	2
Kam Girls	HP	2	3
SEJUSA	HP	3	5
MA^2 *	HP	4	5
Princesas	AP	2	3
Minions	AP	2	3
Los Compadres	AP	3	2
APS	AP	2	1
CEZAMO	AP	2	3
Chetos	AP	3	2
FC Valle	AP	3	4
Aguilas Doradas	AP	3	3
CFN*	AP	1	2
OP	AP	2	3
Fleurs	AP	2	3
ICA	AP	4	3
Enchiladas	AP	2	3
GYF	AP	2	2

Table 4.4. The teams' scores are shown based on the QAG. In the table, teams are organized by their performance level (average, high, and low), and in the order in which the MEAs were implemented (i.e., The *Hybrid vs. Gas Cars* was implemented first, then *The Historic Hotel*).

The QAG is a general instructional approach to scoring teams' work (Hjalmarson, Moore, & Delmas, 2011; Lesh & Clarke, 2000). It only considers the written reports and/or presentations the students created and generated as results of their model-construction process (Clement, 2008). As a rubric that scores the quality of the students' work, it provides an overall idea on how the students' model solves the client's problem and, at some point, a way to

* Represent the groups that were selected among the video-recorded groups.

compare the final products based on the client's needs. For example, Table 4.4 shows that most students, regardless of their performance level, had a score of 2 or 3, meaning that they might need either more time to work on the problem or more practice solving MEAs. In fact, it shows almost all students had a higher score from MEA2 (*Hybrid vs. Gas*) to MEA3 (*The Historic Hotel*). The general mean for both MEAs was 2.5 and 2.9, respectively. In terms of comparing the quality of final solutions among performance-level teams, Table 4.5 shows all three categories of performance level improved from MEA2 to MEA3. In addition, note that low-performance teams had a higher score mean in MEA3 when compared to their average peers, meaning that they developed better solutions for the client based on the QAG, and a very similar mean in contrast to their high-performing peers.

Table 4.5. Student's QAG means' score by performance level.

Student's Performance Level	**Mean**	
	MEA2	**MEA3**
Low	2.61	3.21
High	2.80	3.40
Average	2.36	2.64

Table 4.5. The means of students' solutions categorized by performance level. All three types of performances have improved over time, from MEA1 to MEA2.

The QAG is a fairly simple way to understand the teams' final solutions in a very typical manner, by scoring and providing a grade. But it fails to consider a deeper analysis of the type of model the students constructed, the mathematical components/elements they considered for their models, and the types of strategies they used in developing their models. In the end, the QAG does not provide full evidence of the richness of the students' solutions. It is not possible to properly define how good a solution is just by looking at a number from one to five. Certainly, a deeper way to characterize, compare, and contrast the teams' work is required, rather than simply considering the QAG as a unique method of evaluation and analysis of the students' models-solutions. Therefore, in the next section I analyze the intermediate product-solutions that students generated aiming to have a deeper understanding of their models.

4.3.2. Intermediate Products

As mentioned above, considering only the QAG for scoring and evaluating the students' models is not sufficient. Because of this, and to attain a deeper understanding of the model-construction, I have analyzed the students' intermediate-products considering several characteristics (Dominguez, 2010; English, 2010; Aliprantis & Carmona, 2003; Ekmekci, 2013; Greenstein & Carmona, 2013):

a. The type of model-solution (Lesh & Zawojewski, 2007) created,

b. The strategies (Kent et al., 2015) followed to obtain their model, and

c. The components (Lohse, Biolsi, Walker, & Rueter, 1994) — also called elements of the visual representation — used to represent the mathematical construct and data.

To deeply analyze and study the intermediate-products, I considered only the last MEA implemented, *The Historic Hotel*. All 24 teams' model-intermediate-products solutions were coded based on the characteristics of model-solutions discussed above and in chapter 3, which takes into account the written report to the client, all the worksheets and artifacts generated, and the final presentation transparency.

4.3.2.1. Model-Solution types

I have categorized all 24 teams' solutions for *The Historic Hotel* into five general forms of models the students created. In a general sense, most solutions considered the profit — "Ganancia" in Spanish (G) —, maintenance cost ($MC) —"Mantenimiento" —, the booked rooms (#Rooms), and the room's price ($Price).

In the first model, the profit is the result of subtracting the maintenance cost of the rooms from the cost of all booking rooms. Two subcategories emerged from this model-type solution: One that considers providing maintenance only to booked rooms (1A), and another that considers providing maintenance service to all rooms, regardless of whether maintenance is needed (1B). In this second subcategory, the maintenance cost is the result of multiplying the daily cost of maintenance ($DCM) — MX$40 — by the total number of rooms (80):

1A. G=(# Rooms • $Price) – (# Rooms • $DCM)

1B. G=(# Rooms • $Price) – $MC $MC=80 • 40 = MX$3200

The second model-type is a factored version (2A) of the original profit equation 1A in which the variable #Rooms is a common factor in all terms. In addition, another subcategory-model emerged that considers a set of alternate models that model the number of rooms to be booked and the maintenance cost (2B). In the former model the initial booking price ($IBP) — MX$600 — and the increased desired price ($IDP) are considered main variables to determine the number of booked rooms (#Rooms). Furthermore, a variable "x" represents any increment in the maintenance cost:

2A. $G=(\$ Price – \$DCM) \bullet (\# Rooms)$

2B. $G=(\$ Price – \$DCM) \bullet (\# Rooms)$, where

$\#Rooms = 80 – [(\$IDP - \$IBP) / 10]$

$\$DCM = 40 + x$

The third model-type that emerged in the students' work considers both the number of booked (#Rooms) and unbooked rooms (#UBRooms) as a strategy to determine the increase in the initial booking price ($IBP):

3. $G = (\# Rooms) \bullet [(\$IBP + (\# UBRooms \bullet 10)] – (\# Rooms \bullet \$DCM)$

The fourth model-solution, which is the least sophisticated model in comparison with the other three models-type showed above, does not take into account the cost of maintenance. The rationale of the teams that decided this solution-path was that the maintenance cost would be included in the booking price, so it was not necessary to include it in the model. The model is only composed of the number of booked rooms and the booking price:

4. $G = (\# Rooms \bullet \$Price)$

Finally, the fifth category clusters all models of category 1 of the QAG. These require major reconsiderations, which may reveal a lack understanding of the problem statement of the MEA. Few teams fall into this category, however, and none of the low-performing teams fail to propose a model-solution showing a profit. In fact, only three teams fall into this category: two "average," and one "high-performing."

In Table 4.6, I show the type of model that each team in the different performance level created. Teams created different model types, and in many cases the type of model is the same regardless of the level of performance.

Table 4.6. Model type by team

Team	Level of performance	Model
Guardianes de la Galaxia	LP	3
Dinamita	LP	1A
VACAOS	LP	1A
LGH	LP	1A
LNA	LP	1B
MateAtletas	HP	5
Carrojan	HP	1A
Kam Girls	HP	1A
SEJUSA	HP	2B
MA^2	HP	2B
Princesas	AP	3
Minions	AP	4
Los Compadres	AP	5
APS	AP	5
CEZAMO	AP	1A
Chetos	AP	1A
FC Valle	AP	1A
Aguilas Doradas	AP	1A
CFN	AP	1A
OP	AP	1A
Fleurs	AP	1B
ICA	AP	2A
Enchiladas	AP	2A
GYF	AP	2A

Table 4.6. Teams at different performance levels elaborated a model that fell into one of the five categories detailed above.

From Table 4.6 I made a rough count to obtain percentages of the different types of models the teams created with the intention of knowing how many model-types fell into each category. For example, from the table presented above, I can demonstrate that 46% of the teams constructed the same type of solution (1A) to model the hotel's profit: G[8] = (#Rooms • $Price) –

[8] G=Profit "Ganancia"

(#Rooms • $DCM[9]). From this percentage, 55% belong to the average-performing teams, and 27% and 18% to the low- and high-performing teams, respectively. In addition, 13% of the teams used a reduced factored version (2A) to model the profit, and only 8% considered providing service to all the rooms — model 1B.

Comparing solutions among teams, 60% of low-performing teams created a model that fell into the 1A category, which is the most common model created among all teams. Most (80%) considered following strategy A to model their solution, and all or almost all of them used lists, charts, and tables as components to represent their mathematical construct. In contrast, only 40% of high-performing teams used the model-type A, and another 40% used the 2B model, which is the more mathematically sophisticated model as it considers an extra set of models to determine the number of rooms to be booked and the maintenance cost. For the average-performing teams, less than half created a model-type A, 20% used the factored model-version 2A, and a small percentage fell under the model-category type 5, in which the model needs to be restructured and reconsidered.

4.3.2.2. Model-Strategies

To obtain their models, the teams needed to evaluate the strategy they would follow considering all the characteristics and information provided in *The Historic Hotel* problem statement. In this section, I describe the three types of strategies the teams followed when developing their model-solution.

---

[9] DCM= Daily Cost of Maintenance

The first strategy-type (A) implies systematically increasing the initial price by 10, and decreasing the number of booked rooms by 1. For example, if the initial price of MX$600 for a booking room is increased by MX$10, then only 79 rooms get booked, rather than the 80 originally booked.

The second strategy (B) considers increasing the initial price by MX$5–instead of by the MX$10 stated in the problem statement and decreasing the number of booked rooms by one *only* when the initial price had been increased by a multiple of 10. This strategy requires increasing the price more times than other strategies and booking fewer rooms, but it generates a higher profit in the long term. For example, increasing the initial price from MX$600 to MX$605, would still generate the profit of 80 booked rooms. However, increasing the price to MX$610 would reduce the number of booked rooms to 79.

In the third strategy-type (C), the students considered systematically increasing the initial price by MX$100, and decreasing the number of booked rooms by 10.

One more general strategy (D) includes the combination of any of the above strategies, mixing either the first and second, the first and third, or the second and third.

In Table 4.7, I show the strategies used by each team, which fell into one of the three different strategies, and in some cases a combination of any of the strategy's types.

Table 4.7. Teams strategy to model-creation

Team	Level of performance	Strategy
Guardianes de la Galaxia	LP	A
Dinamita	LP	A
VACAOS	LP	A,C
LGH	LP	A
LNA	LP	A
MateAtletas	HP	A
Carrojan	HP	B
Kam Girls	HP	A
SEJUSA	HP	A
MA^2	HP	A
Princesas	AP	A
Minions	AP	A
Los Compadres	AP	A
APS	AP	C
CEZAMO	AP	A
Chetos	AP	A
FC Valle	AP	A,B
Aguilas Doradas	AP	A
CFN	AP	B
OP	AP	B
Fleurs	AP	A
ICA	AP	A
Enchiladas	AP	A
GYF	AP	A

Table 4.7. The table shows that teams used different strategies to develop their model for the *Historic Hotel* MEA.

Just as I did with the model-types, I made a rough count of the type of strategies that teams used in order to know how many strategies fell into each category. For example, just as with the low-performing teams, 80% of high-performing teams considered following strategy A. Similarly, almost 80% percent of the average-performing teams considered also following strategy A. Only 13% of teams considered this strategy or a combination with any other strategy. Finally, strategy C was only considered by 4% of the teams. Most of the teams, regardless of performance level, considered using the same type of strategy, which once again shows that MEAs might serve to level the playing field.

4.3.2.3. Components/Elements of the mathematical constructs

In the definition of models I adopt in this research project, conceptual tools (models) are composed of elements that allow the mathematization of a real-life situation with the ultimate purpose of predicting how that real-life situation would behave (Lesh & Doerr, 2003; Blumm & Ferri, 2009; Mousoulides, 2011; Blomhøj, 2004). For such predictions, the components or elements of the model and the strategies become essential in helping to make connections from real life to the mathematics world and vice versa, to visually represent the collected data, and ultimately predict a situation.

While working on their model-product-solution, students created and used different types of components that helped them to reach a better understanding of the situation they were modeling–in this case the maximization of a hotel's profit. The components of the mathematical constructs that emerged in the students' work were: charts (C), tables (T), lists (L), and graphs (G). Examples of how the students used these types of components will be given below, but to provide a basic idea here, I present some examples in Figure 4.5:

Figure 4.5. Mathematical components of the written solution

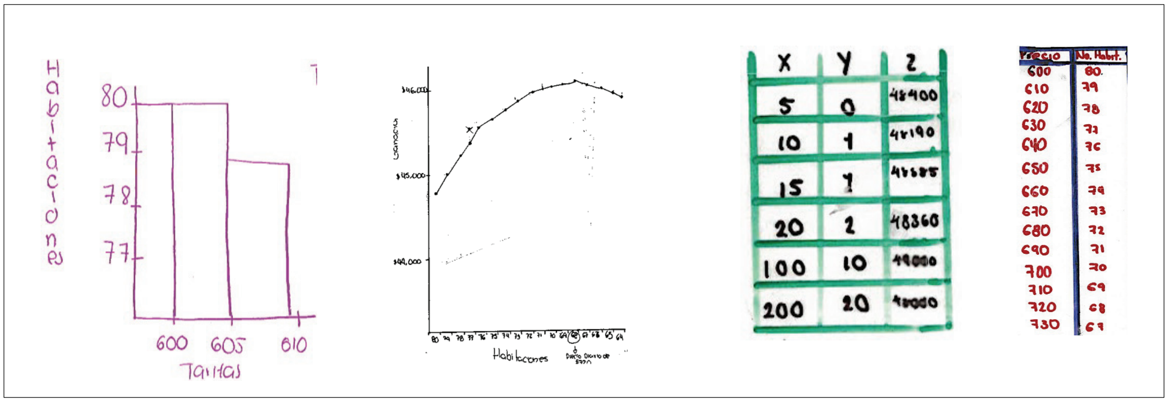


Figure 4.5. The figures above are examples of the types of components created and used during the process of model-development.

Similar to what I did before with the models and strategy types, I also made a rough count of the component teams decided to create and use when developing their solution. As can be seen in the summary Table 4.8, 80% of the low-performing teams used graphs (G) as a type of component to represent their data and mathematical constructs, and most of them used lists (L) and tables (T). Similarly, all the high-performing teams used graphs (G) and most of them considered using lists (L) and tables (T). In contrast, few average-performing teams considered creating graphs or charts, instead more than half used tables or lists as a way to represent their constructs and data.

I have detailed and explained above the three characteristics of the students' intermediate products — i.e., the model, the strategies, and the components — based on what they showed in their written reports, worksheets, artifacts, and final presentations. In Table 4.8, I show a summary of the models, strategies, and components each team created and used during their model-solving process.

Table 4.8. Summary of the model-solution characteristics by team

Team	Level of performance	Type of Model	Type of Strategy	Mathematical component used/created
Guardianes de la Galaxia	LP	3	A	G,Ch,L
Dinamita	LP	1A	A	T,G,L
VACAOS	LP	1A	A,C	L
LGH	LP	1A	A	L,T,Ch
LNA*	LP	1B	A	Ch,L
MateAtletas	HP	5	A	G
Carrojan	HP	1A	B	G,L
Kam Girls	HP	1A	A	T,L
SEJUSA	HP	2B	A	Ch,T,G
MA^2 *	HP	2B	A	G,T,L
Princesas	AP	3	A	T
Minions	AP	4	A	T,Ch,L
Los Compadres	AP	5	A	L
APS	AP	5	C	T
CEZAMO	AP	1A	A	L
Chetos	AP	1A	A	ND
FC Valle	AP	1A	A,B	T,L
Aguilas Doradas	AP	1A	A	T
CFN*	AP	1A	B	T
OP	AP	1A	B	G,T
Fleurs	AP	1B	A	T
ICA	AP	2A	A	T,L
Enchiladas	AP	2A	A	G,L
GYF	AP	2A	A	L

Table 4.6. Students were categorized according to their performance level, showing that teams at different performance levels considered similar models, strategies, and components to represent their models.

The summary of the intermediate products (Table 4.6) is useful to get a better idea, in comparison to only considering the QAG, of how students construct their model-solutions and as

* Selected groups.

a measure of comparison among all 24 teams. Only high-performing teams created, used, and developed solutions that required using more than a single model (e.g., model 2B), but in some aspects (e.g., strategies and components), high-performing teams' final and intermediate solutions were similar across performance levels.

Analyzing the teams' products provides a general perspective on how comparable these solutions are. But, it is still not sufficient to provide the whole picture of how the products and processes used to obtain these solutions differed among teams. Therefore, in the next section, I analyze in detail the process that the three selected teams followed to get their models and strategies. For these, I considered each step the students followed in their problem-solving process, the discussions that emerged, and the content that they developed as a result of their interaction.

4.4. The Process

In the previous section I analyzed all groups' final and intermediate products comparing and contrasting their solutions. This served to provide an initial idea of how similar or different the students' models for solving the MEAs were. As discussed above, I found many similarities between the students' solutions — i.e., the models, strategies, and components. However, as mentioned earlier, looking only at the students' products, which include their written reports to the client, all created artifacts, worksheets, and final presentations, does not fully provide evidence that MEAs might "level the playing field" among performance levels, or that MEAs elicit and enhance students' learning when solving a problem (Shuman et al., 2011) without leaving any student behind.

To show this, I focus here on the process the three selected teams (i.e., one in each performance level) engaged in their group interactions. To this end, I have fully transcribed and coded the video recording of each of the three selected teams as they worked to solve the *Historic Hotel* MEA using the MAXQDA software. The coding was composed of a double-round coding process. For the first round, I deductively (Hesse-Biber, 2010) captured the segments or phases of the process that teams took in solving the problem. For this, I used an adapted version of the coding schemed developed by Vidic et al. (2014), which they developed to study the phases of the problem-solving process in MEAs. The coding scheme adapted from Vidic et al. included reading and making sense of the problem statement, planning, collecting data, analyzing data, formulating/re-formulating a model-solution, evaluating/testing the solution, and reporting/documenting the solution. The second round of coding was inductively (Miles, Huberman, & Saldaña, 2014) performed to capture line by line the group-interaction moments of the model-process phases in which each team spent most of its time, i.e., data collection and data analysis. During these moments, I coded for instances in which students were requesting explanations or questioning to/from each other, generating or contributing ideas, and discussing a problem-related idea. In the next section, I explain in detail and provide examples of both rounds of coding and the outcomes of this part of the analysis.

4.4.1. Steps of the process of model development

To study the path the students followed in the modeling-process for obtaining a model-product-solution for *The Historic Hotel*, I fully transcribed their interactions using an adapted version of the seven coding categories of Vidic et al. (2014): reading and making sense of the problem statement, planning, collecting data, analyzing data, formulating/re-formulating a model-solution, evaluating/testing the solution, and reporting/documenting the solution. Students went through this process of model-development in a non-linear way, which means that they could "jump" to any of the phases at any moment during the solving process.

The reading and making sense of the problem statement represent the moments in which students were either reading or understating what was stated in the problem statement of the MEA. Teams revisited the problem statement several times during the solving process with the intention of recalling what they were asked to do, or to deeper their comprehension of the problem. For instance, in the following excerpt (Transcript 1), I show a low performing team reading the problem statement as part of the solving process and making sense of the instructions:

Video 83-5.07 (Transcript[10] 1)[11]

1. Elena: …((reading)) please write a letter where you say…
2. Saira: What do we have to do?
3. Maria: … ((reading)) find the booking rate that maximize the profit even when…
4. Elena: It says, Mr. Francisco Graham…
5. … ((Unrelated talk))
6. Elena: Born and raised in Saltillo ((…continued reading the PS)). The hotel has 80 rooms… ((Wrote it in her notebook)). When the booking rate per room is MX$600… ((Wrote it in her notebook)) all 80 rooms are booked. Same, for every MX$10... which means MX$610, we have to take out one room. For MX$620…
7. Maria: //78
8. Elena: //but, every room has a maintenance cost of MX$40. He ((Mr. Graham)) wants to maximize the profit, and what would be the profit with that booking rate.

In the transcript presented above, the members of the team read the problem statement to deepen their understanding of what they are asked to do in the activity. The group made inferences about how the rules should be applied. For example, Elena (Turn 6) mentioned that every time the booking price is increased, one less room is booked, and when she mentioned MX$620, Maria (Turn 7) responded with the number of rooms that should be booked.

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[10] All the students' names have been changed to protect their privacy.
[11] Refer to the Appendix for transcription conventions.

The planning phase is the moment when a team is deciding what to do and how to interpret the givens. For example, in the following excerpt (Transcript 2), a high perfrming teams team is planning how they would proceed once they have fully understood the problem statement:

Video 51 – 4.44 (Transcript 2)

1. Marlen: So, he wants to know the daily booking rate that would maximize the profit. Then, he also wants to have a formula that would allow him to find the daily booking price to maximize the profit even if the daily maintenance fee changes.
   So he wants the formula and then we have to write a letter where we say the right price to maximize the profit including a procedure.
   Then, what we can do is to… look the price…is "X," and the daily prices would be 40 Y ((referring to the maintenance cost)).
   ((Silence))
2. Amaris: Ok, so to calculate the change in price, it could be (40+X) times N, where N is the number of rooms per clients, or the number of rooms? Mmm…..
3. Marlen: Look, here it says that the price varies depending on the number of rooms. Then, why we don't…instead of the price change… we make the price the "main variable"?
   So, why we don't make that depending of the number of booked rooms is how the price changes? Do you understand?

This team is planning how they would proceed and interpret the givens in order to provide a solution to the client. For example, Marlen (Turn 1), after reading the problem statement, proposed to consider "X" as the daily price and 40Y as the maintenance cost. But Amaris in Turn 2 proposed that the maintenance cost could have a formula of 40 +X. Finally, Marlen proposed to use the price as the "main variable" and that it changed depending on the number of rooms.

The data collection and data analysis moments are one of the most important parts of the process-solution. Teams have already interpreted the givens and they are either generating the data based on the information provided, or organizing it. Then, as result of these, they make inferences based on the data already collected. In the following transcript (Transcript 3), I show an example of a team that is creating a table (i.e., collecting data) of the prices versus the hotel profit, and that as result they find a pattern of how the profit increases every time that the number of rooms booked decreases and the price is increased by MX$10.

Video 61- 14.45 (Transcript 3)

1. Marlen: So, we said MX$44,800?
2. Amaris: ((Responded yes moving her head))
3. Marlen: Now, the other one.
4. Ana: MX$45,600
5. Marlen: mmm…let me do it in my cellphone ((the computations))
6. Ana: MX$790 – MX$40 is equal to… times 61 ((the number of booked rooms))
7. Marlen: ((Make the same computation in her cellphone)) MX$45,750

8. Marlen: 45 thousands what?

9. Ana: 750!

10. Amaris: Now…eh, MX$610 right? Minus MX$40 times 79…MX$45,030

11. Ana: The other one MX$45,880 ((this is the result of multiplying [(780 - 40) x 62]))

12. Marlen: Now MX$620 - MX$40 times 78

13. Amaris: Then, it is not MX$720 right? Because here at MX$800…

14. Marlen: //but, remember that it was increasing, and then here, it is when it starts decreasing.

15. Amaris: And here how much it is?

16. Marlen: MX$45 thousands… and then here it will continue increasing.

Look these are the same 45, 45, 46, 46…until it gets here in the middle

((Marlen explained to Amaris how with different opposing booking prices the profit would be the same))

17. Amaris: Oh yes, it is right!

((The team continued calculating profits))

18. Ana: When it is 66… MX$700 x 66, MX$46,200.

19. Marlen: Ey wait, STOP! It will keep increasing, isn't it?

Don't do any other computation because it will be the same…

Here it is MX$45,600, then here MX$47,750.

Ohhhh…wow! We got the same… there is a pattern of proportionality!

20. Amaris: Yes, it happens every time!

In the transcript above, I show how the team computed operations to calculate the profit (Turns 1–15) using different booking prices and booking rooms (i.e., they were collecting data). Because they considered a broad range of rooms available (e.g., from 80 to 60) they were able to identify a pattern in the profit computations (Turn 16 & 19). These allowed them to have a better understanding of how increasing the booking price and decreasing the number of booking rooms affects the hotel's profit, which can be considered data analysis.

The model formulation stage represents the instances when teams generate their either final or partial model solutions. For example, in Transcript 2, the team (Turn 2) created a model to calculate the maintenance cost (40 + X). The model evolved from the previous model "40Y" proposed by Marlen in Turn 1. In the next section, I will show how this team used a set of models as part of their solution that includes the maintenance model showed above.

The evaluation or test of the solution is the stage in which teams verify that their solution effectively responds to the client's needs. Sometimes, when teams are testing their models, disagreements may occur among the team members. For example, in the following excerpt (Transcript 4), I show the low performing team that is verifying their model and at the same time disagreeing about whether the solution is adequate:

Video 97 – 21.1 (Transcript 4)

1. Elena: Look at this one, I told you it is like a pyramid! This is the biggest ((Referring to the profit at 70 rooms at a price of MX$700, which is MX$49,000 with a constant maintenance cost)).
   It goes down, down, down ((Like a pyramid shape)). THAT'S WHY it is MX$700. If you subtract MX$3200 ((maintenance cost)), it still the biggest!

2. Maria: But, what about this one ((pointing at MX$660 x 74))?
3. Elena: ((Elena takes the notebook, the calculator and redoes the computations for a price of MX$660)). The problem is that you are multiplying MX$660 x 79 instead of by 74. The biggest is still MX$700, that's why it is MX$700.
4. Maria: ((Maria takes the notebook))
5. Elena: Because, even if you increase the price by MX$10, which means that you will lose rooms, you will be making profit too.
6. Maria: Then?
7. Elena: Well, we don't want to lose more than 10 rooms, because it will make us lose profit, even if the booking price is higher… do you understand me?

In Transcript 4, Elena (Turn 1 & 3) tries to explain to Maria why MX$700 is the ideal price. But Maria (Turn 2) is not convinced of Elena's explanation and requested that she clarify her position. Elena deepens her justifications (Turn 5 & 7) to clarify how the model adequately maximizes the profit.

The final step of the process of model-solution is to report the findings, externalizing (Lesh & Doerr, 2003) by way of a letter to the client and creating a final presentation, how the model solves the clients' needs. For example, in the following excerpt of the high-performing team, (Transcript 5) I show a team working on reporting their findings in a letter to the client.

Video 52 – 9.57 (Transcript 5)

1. Marlen Ok, so what it his name? But first, the date! Should I use something like Dear Mr.?
2. Amaris: Oh yes!

3. Amaris: Dear Mr.

4. Marlen: We hope this letter finds you well. We are writing to inform you that we found an easy way to…

5. Amaris: To obtain…

6. Marlen: ((writing the letter))

7. Amaris: a maximized profit…

8. Marlen: To obtain maximized profit in all your…

9. Amaris: In your hotel!

10. Marlen: Business?

11. Ana: Yes!

12. Marlen: In your hotel business.

13. Marlen: First, we need to explain you the variables.
Which one should we include first?

14. Amaris: The one of MX$600 maybe?

15. Marlen: Wait, what about the pattern instead?

16. Amaris: //of the price with the number of rooms, and then we add this equation: $H = 80 - ((T - Pa)/10)$, in which Pa is always MX$600, and T is the booking price.

The transcript above shows how the team worked to write a letter in which they strive to explain the client how they got their model. The team started writing the letter in a very formal way (Turns 1–12). Then, Marlen (Turn 13) proposed that they explain the variables used during the process of model-solution, but she then changed her mind (Turn 15) and proposed explaining the pattern of the booking price versus the number of booked rooms, which Amaris seconded (Turn 16), and also proposed to add the profit equation.

I designated a color for every code (see Figure 4.6) to represent each step in the process to provide a visual display of the qualitative data. According to Verdinelli and Scagnoli (2013), visualized data facilitates the process of making "inferences and conclusions and [of] represent[ing] ways of organizing, summarizing, simplifying, or transforming data" (p. 359).

Figure 4.6. Coding Scheme System

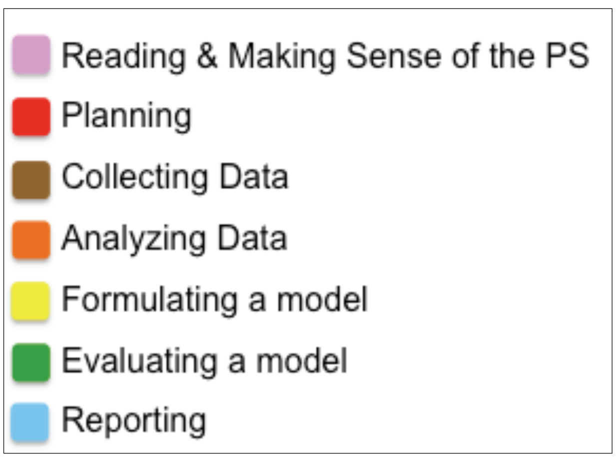


Figure 4.6. The system used to code the students' MEA processes with the color designated to each theme.

The figure above shows all seven codes I used to code the transcripts and the colors that each represents. The colors were particularly useful to me in finding patterns in the places where students spent most of their working-process time. For example, the next image (see Figure 4.7) shows that students spent most of their time either collecting (brown) or analyzing data (orange).[12]

[12] Although colors vary in darkness of tone across performance level in this chart, this is due to technical difficulties in color-coding and does not signify any change in the code's meaning.

Figure 4.7. Coding Color Pattern across performance level

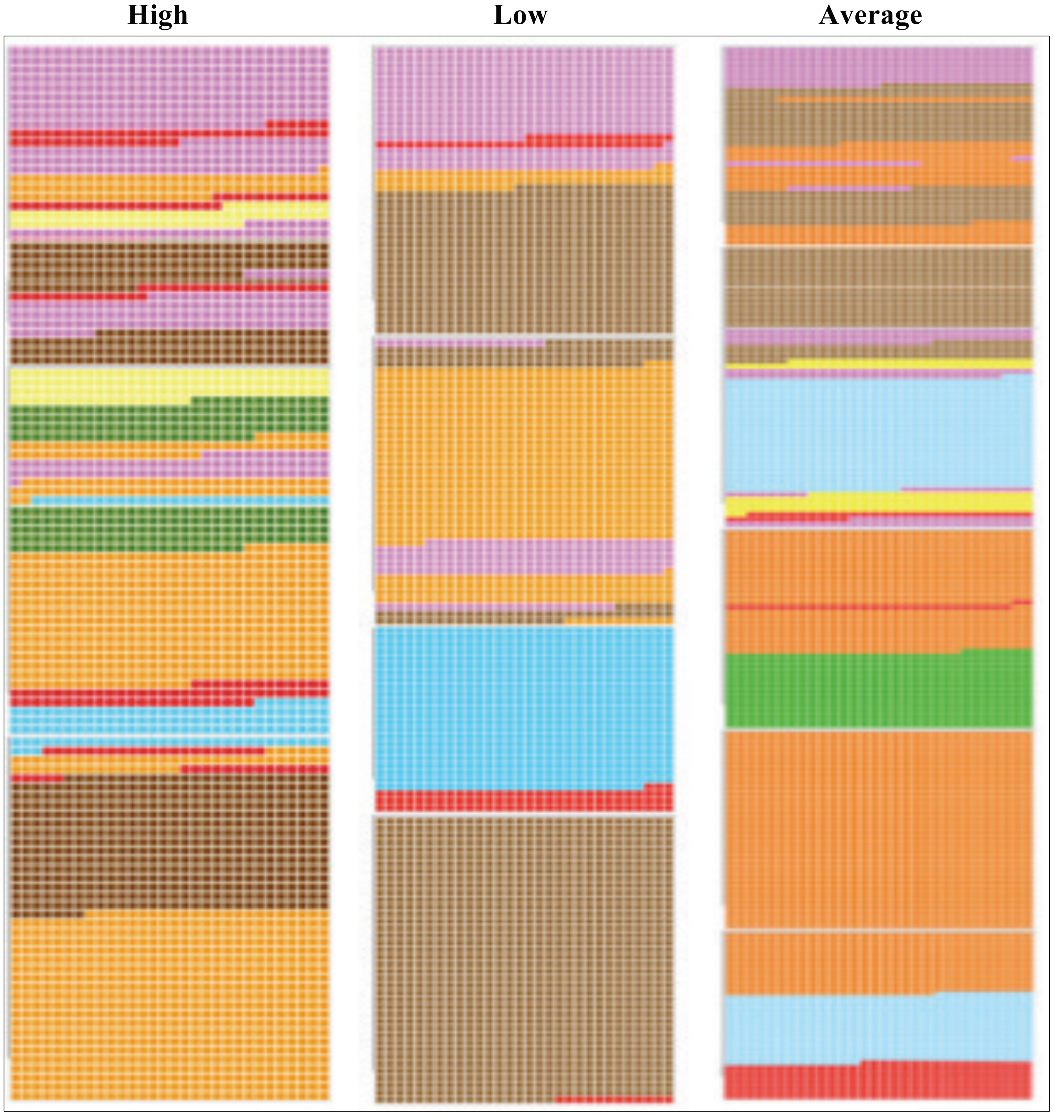


Figure 4.7. Colors represent patterns in the amount of time in minutes that students spent while working on their solution. The left-most diagram represents the high-performance team, the middle the low-performance team, and the right the average-performance team.

Image 4.7 reveals that the average- and high-performing groups went through all the stages or segments of the modeling process (Vidic et al., 2014). However, apparently the low-performing team failed to engage in the process of creating or evaluating their model solution (i.e., yellow and green color), but these can be justified as follows: First, the low-performing group decided to spend most of their time working on collecting or analyzing data, which hampered the time they had to physically and literally write down their model, but this does not mean they did not create or test their model. Second, the low-performing group wrote, tested, and verified their model and its accuracy while reporting or externalizing (Lesh et al., 2000) their solution (i.e., the blue segment). The process section below reveals how the low-performing teams created a final presentation in which they depict and explain their solution path, and the number of rooms and price that would maximize the hotel's profit.

Collecting or analyzing data were two of the most important MEA-process moments for the students to make sense of the data, in which they had the opportunity to uncover patterns, relationships, and trends. For example, in the following excerpt from the high-performance team (Transcript 6), the team is making sense of the problem and analyzing calculation of the profit with respect to different numbers of rooms and booking prices.

Video 60-17.57 (Transcript 6)

1. Amaris: All right, let's find it! Look, so far we have until $700. How much is it ((the profit)) at MX$600?
2. Marlen: How much was it with MX$600?
3. Amaris: MX$540 x 80?
4. Marlen: MX$600 - MX$600? Wait, no::
5. Amaris: 80…well, but you still have 80.
6. Marlen: (MX$600 - 40) x 80… MX$44,800.
7. Amaris: //and with MX$700? Let's jump to MX$700 to see…
8. Marlen: let's see… MX$700… ((Marlen used the calculator))
   mmm… ((Aloud)) MX$700 times 70
   // MX$46,200.

9. Amaris: With MX$1000, when I computed considering MX$1000, I got a lower profit…

10. Marlen: Let's see…

11. Amaris: How much would it be with $800? Let's see how much we get.

12. Marlen: 1000 minus 40… times 40. ((This is the number of booked rooms))

13. Amaris: Wow, we get a lower profit…

The example above shows how the team looked both to find the profit for several different booking prices (Turn 6 and Turn 7), and at the same time understanding — i.e., making sense of the information gathered once the computations were done — the difference among the sets of data that emerged from their computations. After analyzing part of the data, they realized that when the hotel's room booking price increased, the profit decreased (Turn 13). These data were essential to accurately determine the right number of rooms and price that would maximize the profit.

The moments of analyzing and collecting data required the students to explain and justify their rationale as their various perspectives and ways of understanding emerged, and to resolve their disagreements. Therefore, I focused my attention on the interactions among team members, because when students face disagreements, they pursue extra information and approach the already existing data from new perspectives (Webb, 1982). This allows them to either accept, reject, or work around the new ideas, which in return might change their opinions or understandings of a certain situation. For example, in Transcript 6, Amaris (Turn 13) realized — and expressed her surprise — that higher booking prices would affect the profit negatively, which resulted in the team testing a set of lower booking prices (other than the MX$1000 proposed). Therefore, in the following section, I explore how students interacted within their teams (line by line), and how their interactions positively and/or negatively affected their model and final outcomes — i.e., their written report, the model's components, and the final presentation.

4.4.2. Group-level interaction

As I began my inductive-analysis (Miles et al., 2014) in the two particular moments when teams spent most of their process-time (i.e., data analysis and data collecting), several relevant patterns of interaction quickly became evident. I identified how team members reacted when they needed to explain themselves to each other, when a different idea or a different perspective was proposed, or when disagreements emerged. For example, in the table below (Table 4.9), I show the frequency of the codes for the students' interactions as they worked in developing their solutions during the moments of data analysis and data collection.

Table 4.9. Frequency of group-level codes

Code	Performance Level		
	High	Average	Low
Explanations to and from	17	26	40
Ideas generated/contributed/proposed	124	95	78
Discourse/discussion problem related	19	37	18
Questions to/from	14	14	24

Table 4.8. Students at different performance levels interacted with different frequency depending on the type of interaction.

The table above reveals several interesting and surprising pieces of information from which many inferences can be made. First, the low performance team asked each other to explain concepts, ideas, or strategy almost twice as many times than their average- or high-performance peers. This was also true of the number of problem-related questions. I believe this is due to the type of discourse the low-performance team engaged in, which I will address below.

Second, the high-performance team proposed between 30 and 55 percent more ideas than their average- or low- performance peers. Again, this might be related to the type of discourse they had. However, from among these many more ideas, a larger percentage were acknowledged but not followed up on than between the other two teams. For example, when comparing the number of ideas asserted in each group, the low-performance students acknowledged and developed 78%, whereas the high-performance students only did so only 67% of the time.

Finally, both the low-and high-performance teams had almost the same number of discussions, in contrast with the average-performance team, who had twice as many as their peers. These numbers, however, do not sufficiently explain why the teams' interactions differed in this way. To determine that requires a discourse analysis, which I turn to in the next section to provide a clear indication of the type of discussion students engaged in.

4.4.2.1. Discourse Analysis of the process

In this section, I analyze the types of discourse that the three selected teams engaged in while solving *The Historic Hotel* MEA. As noted above in chapter 3, I have adapted Azevedo et al.'s (2015) epistemological discourse practices of describing, explaining, and argumentation discourse, which I have reduced to the two categories of argumentation and non-argumentation, the last comprising the explaining and describing discourse types (see Table 4.10). For the argumentation type of discourse, I adapted Toulmin's (2003) argumentation framework to identify the arguing elements: claims, grounds, reasoning (McNeill & Pimentel, 2009), and rebuttals. As elaborated in chapter 3, claims represent the motivations for arguing. Grounds (i.e., evidence), are the data that would support the claims. Reasoning, which is the combination of warrants and backing, is the use of warrants and/or backings to justify the jump from the claims to the grounds. Lastly, rebuttals represent the ideas that either oppose or refute the claim.

Table 4.10. Percentage time of students' discourse

Type of Discourse	**Performance-Level**		
	Low	**High**	**Average**
Argumentation	45%	9%	20%
Non-Argumentation	55%	91%	80%

Table 4.10. Students' discourses for all performance levels showing that students were not arguing most of the time.

As mentioned above, I have characterized the type of discourse as argumentation and non-argumentation. The table above indicates that, in general, teams interacted more in a non-arguing discourse than in argumentation. However, the low-performing team spent almost half of their interacting time — during the moments of data analysis and data collection — argumenting about their ideas, disagreements, and models. In the next section, I explain and exemplify how teams interacted in both the arguing and non-arguing discourse type.

4.4.2.1.1. Argumentation discourse

All three performance level teams spent less time in a argumentation than in a non-argumenting type of discourse, but the low-performance team spent the most time argumenting. As noted above in Table 4.9, they are also the team that asked each other for the most justification of their ideas. These two facts are related. The disagreements that emerged during these arguing discourse mathematical episodes are comparable to the practice of doing real and "authentic mathematics and science" (Lampert, Rittenhouse, & Crumbaugh, 1996, as cited by Horn, 1999, p. 99), in which students must explain their thoughts and rationale and need to justify their positions. In addition, from a constructivist standpoint, when students argue over a disagreement, learning is elicited through the externalization of their thoughts, reflection upon others' and their own ideas, and through the reconstruction of knowledge (Kanselaar, Erkens, Andriessen, Prangsma, Veerman, & Jaspers, 2003; Lesh et al., 1990). For example, in the following excerpt (Transcript 7) of the low-performance team's video, several utterances of the students' argumentation evoked moments in which new knowledge not only emerged, but also was constructed, with all team members being forced to deeply explain and justify their ideas.

Video 97-9.04 (Transcript 7)

1. Elena: ((Talking to Maria)) Let see…MX$700, why not MX$800?
2. Maria: //or why not MX$1000?
3. ((…silence))
4. Maria: or why not $670?
5. Saira: ((Asked a question, but no one listens))
6. Elena: or why not MX$600? Well, because the profit would decrease… look do it yourself…

7. Maria: No wait! It would be… would be MX$800…

8. Elena: Eh? DO IT! one by one ((Elena is asking Maria to compute the hotel's profit, considering a different booking process from MX$600 to MX$800))

9. Maria: //Wait a little! ((Maria starts doing some computations in her worksheet))

10. Elena: Start with MX$700, so you'll see that the profit would begin to decrease from $48…

11. Maria: ((Maria continues doing computations…))

12. Elena: ((Elena looks at what Maria is doing) but why MX$800 x 70? If the hotel would be losing rooms?

13. Maria: oh…then it would be MX$800 x 79…

14. Elena: No! why are you multiplying by 79?

15. Maria: By 80?

16. Elena: For every MX$10 pesos ((The booking price increases)), you lose 1 room… If with MX$600…

17. Maria: [it will be… Wait!

18. Elena: If by 80…. I mean, if you have a booking price of MX$600 pesos, then all the rooms would be booked, if you charge MX$610, that is, you increase the price by MX$10… then you lose 1 room…

19. Maria: And why not $700 and something?

20. Elena: Do it! ((Meaning to make the computations))

21. Maria: …and why not MX$710?

22. Elena: DO IT!

23. Maria: Why are you multiplying by 70, 60, 69?

24. Elena: It is not a random number! It depends on the number of rooms that you lose ((Once the price has increased))

25. Maria: 79? The hotel already lost 1 room!

26. Elena: But why are you considering 79?

27. Maria: Because you have 80 rooms, you lost 1, then you have 79!

28. Elena: Look, when the price is MX$600, you'll have all the rooms booked. If the price increases to MX$610, then you'll book 79!
How is that by having a booking price of MX$710 you are booking 79 rooms?
You should had increased the price 10 by 10 by 10 by10…
((.....))

29. Maria: [If you want me to do as you say, you need to explain me first why it is MX$700, if you are losing $100 anyway…

30. Elena: I'm telling you, if I increase the booking price by MX$10, it will decrease ((the number of booked rooms))
Look, it is like MX$700 is the top, if you increase the price to MX$710, you will have fewer rooms, and the profit will change, it will decrease,
It is like a pyramid!

In the excerpt above, the team struggled to justify why MX$700 was the right price and not other (e.g., MX$800, or MX$670). As a result, they had an arguing discourse (Azevedo et al., 2015) that lasted several minutes. In the following lines, I adapted Toulmin's model (2003) to analyze the argumentation of the low-performance team. The patterns of argumentation in this episode do not follow Toulmin's sequence of argumentation in precisely the same way as found in scientific argumentation (McNeil & Pimentel, 2009); however, I show how a similar but different pattern emerged from the students' utterances, in which some — and sometimes all — components of Toulmin's model of argumentation appeared.

Elena started the discussion (Turn 1) when she claimed that the ideal price was MX$700 and asked why the price would not be MX$800. She then provided grounds for her claim of MX$700 by saying that any other booking price would result in decreasing the profit (Turn 6). However, for Maria, a booking price of MX$800 seemed more ideal for maximizing profit (Turn 7). Elena then responded to Maria with a rebuttal (Turn 8), offering a type of reasoning by asking Maria to start doing computations ("DO IT! One by one"). Here Elena is trying to get Maria — by getting her to do the work herself — to realize that a price of MX$700 would maximize the profit. When Maria began the computations (Turn 9), Elena expressly stated: "start at $700" (Turn 10) — which can be considered a reasoning (warrant) for asking Maria to start in MX700) — so Maria would discover for herself that a higher price would decrease the profit.

Maria continued her computations, writing "MX$800 x 70," which Elena questioned (Turn 12). The way Elena questioned Maria can be viewed as a rebuttal that she immediately supports by saying that the hotel "would be losing rooms." Following the problem statement's rule that for every MX$10 price increase, one less room is booked, Elena was trying to say that if the price is increased from MX$600 (the initial price with 80 rooms) to MX$800, then only 60 rooms would be booked. Therefore, for Elena, having 70 rooms at a price of MX$800 did not make sense.

The discussion then continued around the number of booked rooms that would maximize profit. Elena (Turn 16 and 18) expanded her ideas from the previous turns and offered Maria more evidence for her reasoning with a deeper justification of why the ideal price should be MX$700.

Maria implicitly rebutted Elena's claims and reasoning by asking questions (Turns 19 & 23). Her assumption seems to be that the hotel should be booking all 80 rooms. (Maria's reasoning as a warrant is suggested but not directly expressed). Elena (Turn 24) again backed her claim trying to explain to Maria what she already clarified. But, Maria's ground for her rebuttal was that the hotel had already lost one room (Turns 25 & 27) and that 79 rooms were left.

Elena, frustrated with Maria's misunderstanding and confusion, tried to clarify her reasoning by explaining the MEA's rules to Maria one more time (Turn 28), and by trying to help Maria make sense of her rationale. But Maria kept asking for more explanations (Turn 29) and backed her rationale by saying that the hotel had already lost MX$100 by having a price of MX$700 instead of MX$800.

Finally, Elena made one more attempt (Turn 30) to graphically provide evidence and justification for her reasoning. As already seen in Figure 4.4, Elena represented (Clement, 2004) her claim with a pyramid shape. She indicated that by either increasing or decreasing the hotel's booking price and number of rooms (other than MX$700 and 70 rooms), the profit would decrease. Thus, she proved that the ideal way to maximize the hotel's profit, when considering providing service to all the hotel rooms, would be with a price of MX$700 and 70 rooms.

In this transcript, the students' disagreements forced them both to reveal their misunderstandings and explain their rationales. On one hand, Maria showed her confusion and misinterpretation of the givens in the MEA, which caused her to mistakenly propose an inadequate solution. On the other hand, Elena was forced to justify her rationale, construct her own knowledge and that of her team (e.g., by creating a pyramid chart to explain her thoughts), and externalize her thoughts in a way that made sense to her peers.

Toulmin's framework served to characterize the argumentation-discourse episode presented above. Although the students' discussions did not follow a pattern of scientific discourse (McNeil & Pimentel), all the elements of the adapted version of Toulmin (Claims, Grounds, Reasoning, and Rebuttals) appeared here, allowing me to identify how the argumentation type of discourse unfolded among students of the low-performing team.

4.4.2.1.2. Non-Argumentation discourse

In the discourse episodes of *The Historic Hotel,* students at different performance levels took part in several forms of discourse. As I showed above, students in the low-performing team argued about the number of rooms and booking price that would maximize the profit. This made them go further in justifying their ideas to the point where they reconstructed and generated knowledge to solve their disagreements.

As explained earlier, disagreements elicit the construction of "new evidence and the development of alternative approaches" (Souza, 2016, p. 185), which aim, through several mechanisms, to reach an agreement (when possible).

However, not all episodes of *The Historic Hotel* MEA resulted in an arguing discourse. In fact, as already shown in Table 4.10, the average- and high-performance teams spent more than half their time in a discourse type characterized as explaining or describing, which is a type of non-argumentation discourse I call an aligning discourse, i.e., a type of discourse in which students mostly accept and/or agree with what has been proposed, or show little or no rebuttal.

For example, in the following excerpt (Transcript 8), the high-performance team MA^2 worked to solve *The Historic Hotel*. Having plotted changes in price vs. profit, they recognized the parabolic shape of the graph and set out to obtain the equation that matched that shape. This demonstrates a high level of content knowledge, and would suggest they would make arguments based on this understanding. However, they do not argue over their ideas. Instead, they either accept each other's ideas or simply ignore them. Only rarely did they discuss or refute them.

Video 97-9.04 (Transcript 8)

1. Amaris: ((Amaris is trying to graph the equation using graphic software on her cellphone.))
   If you want it negative ((the function)), the negative sign should be there…because when we used… ((When they used a positive sign the quadratic they found was inverted.))
   Because, you will have it in (0,0), right? The vertex? But because we haven't used a variable…
2. Ana: Then, should we add MX$600? ((Ana is ignored.))
3. Marlen: Look, this is how it looks ((the graph))…
4. Amaris: Now, we just need to move it to the right ((referring to the "x" axis))
   I believe that is how it should be…now, we only need to move it to the right…
5. Marlen: Let's see what would happen if you add plus 6?
6. Ana The graph ((meaning the parabola)) opens up, not down!
7. Marlen: //Exactly!
8. Amaris: But…you can change it. You can multiply the x^2 by a number, for

example now it's multiplied by negative 1, the x^2

9. Marlen: Let's see… minus 6 to the square ((-6^2)).

10. Amaris: Try to graph $y = -6x^2$ and let see what you get.

11. Amaris: Awesome! It's changing ((meaning the graph is transforming))

12. Marlen: plus… ((Inaudible))

13. Ana: The thing is, the changes have to be in the "x," so it moves over here… ((to the right, Ana refers to changing the value of the h in the vertex form equation of a quadratic)).

((…..))

14. Amaris: What if we add 10? Like $x^2 + 10$, would we get the same?

15. Marlen: No, lets see…where? Then…

16. Ana: It has to be negative! ((meaning that it is $- x^2 + 10$))

17. Marlen: //Perhaps it is minus "y" ? Try!... ((Ana does as Marlen says)), no no, but put it in the beginning… oh no! forget it!

18. Marlen: So, the "y" is normal ((meaning that it is positive))

19. Marlen: [Look!

20. Ana: It opens more if… it opens more!

21. Marlen: Yes, if we use a denominator… look, it opens if it is dividing…

22. Ana Yes, it opens if you have a division ((meaning that if the x-variable is multiply by a fraction)

23. Marlen: Ey guys, why don't we just write the general equation with the number, and that's it? Mmm… in that case, it would be 80 - ((700 - 100)/10).

24. Ana: The problem is that this app does not let me graph...mmm…

25. Marlen: Well, in that case, use only MX$600…

26. Amaris: The thing is, you will always need to have an "x" and a "y" ((meaning that Ana was writing the equation without using y =))
You won't be able to graph if you don't have an "x" and a "y" variable.

27. Marlen: Let's see… something like this?
In that case, this will be my profit equation, right?

28. Marlen: Add the "x"…no:: forget it!
So, y = (600 - 40)x

29. Amaris: The problem is that you'll get something like this, because this represents the number of rooms… ((Amaris refers to the x))

30. Marlen: What if you add a square to the x, like x^2?

31. Amaris: Look, what you have to do is to write (x - 40) (# Rooms).

32. Marlen: x – 40… But it should be with this one: $y = -x^2$, to get what we need. So, maybe it's $y = 40x^2$.

33. Ana: It's a line?… and it goes up!

34. Marlen: Oh wait, use $y= - 40x^2$, with the negative sign at the beginning.

In the excerpt above, the team had many discourse moments, most of which can be categorized as either describing or explaining discourse (Azevedo et al., 2015) since all team members mainly align in their ideas. As defined in the previous chapter, for the purpose of this research project, describing discourse refers to the moments in which a student presents his/her ideas.

On the other hand, explaining discourse refers to those moments when students explicate their rationales, justifications, perspectives, or ideas, or when a team member seeks to verify that his/her ideas were comprehended. For instance, Amaris (Turn 1), tried to explain to her peers that the quadratic function they are getting does not face down the way they want it to. She continued by saying that the equation is missing a negative sign. In addition, she explained that the parabola (which they knew to be the shape that represents a quadratic function) had not moved from the vertex (0,0) because they had not used a variable. (What Amaris means is that in the vertex equation of the parabola, the "h" and "k" have no value, or is zero, which causes the parabola to have a vertex in Cartesian plane in (0,0).)

Then, Ana and Marlen (Turns 2 & 3) tried to show they had understood what Amaris explained earlier: Ana by proposing adding 600 to the equation, and Marlen by trying to graph an equation following Amaris's idea to use a negative sign, which Marlen does with the support of Amaris (Turn 5). Ana (Turn 6) then mentions that the graph opened up, and not down. Although they are trying to make the function face downwards, when Ana found otherwise, Marlen (Turn 7) simply accepted it, i.e., aligned, without providing any kind of rebuttal or objection.

Amaris deepened her explanation (Turn 8), saying that the variable can be multiplied by a number in order to obtain the type of graph that they had previously and manually done with the collected data. Marlen (Turn 9) aligned with the idea, but mistakenly used a negative number, instead of using a number and a variable. Amaris (Turn 10) asked Marlen to use a specific equation that included a negative sign, a number and a variable. This is a describing discourse, since Amaris showed Marlen how to graph the equation using "-6x2," and proposed a change in the “x,” which refers to making an adjustment in the quadratic equation’s variable that transforms the function.

Then, all three members (Turns 13-20) of the team put forth several different ideas aiming to find the equation that would simulate the hotel’s profit (see Figure 4.10). For example, Amaris (Turn 14) proposed adding a number to the variable “x” (i.e., x^2 +10), but Ana (Turn 16) expressed concern because the equation was missing the negative sign on the variable “x,” which would make the quadratic open facing the negatives y direction. The team then aligned in how to use the signs and how to write the equation to be graphed.

Furthermore, Ana and Marlen (Turns 21 & 22) explained how multiplying the variable x by a fraction opened the parabola even more. Marlen (Turn 23) then suggested (descriptive discourse) writing the quadratic equation in a different form in an effort to get the ideal equation that simulated the profit’s maximization.

Ana endorsed Marlen’s suggestion and tried to graph the new equation (Turn 24), but without considering an “x” or “y” variable, which generated an error message on the software they we using to graph the equation. Amaris (Turn 26) then explained to her peers that both an “x” and “y” variable were needed when graphing an equation, and certainly when using the software. The team continued working in an aligning way (Turns 27-31), examining what the quadratic equation should be, until Marlen and Ana (Turns 32 & 34) figured it out.

This high-performance team’s interaction when solving *The Historic Hotel* unfolded with little argument or arguing discourse. Instead, the team aligned in most of their interactions, engaging in either a describing or explaining discourse. As seen in the excerpt above, a pattern emerged in an aligning-interaction way. Most of the time, a describing discourse followed an explaining discourse or vice-versa, which reinforces the idea that the students strove either to express their thoughts and perspectives, or corrected and explained as needed, but did not to argue over their ideas.

So far in this chapter, I have provided and analyzed examples of argumentation and non-argumentation discourse. In these different episodes, I contrasted how the students on the low-performance team (The LNA) had strong disagreements versus the students on the high-performance team (The MA^2), who hardly argued about their ideas, aligning instead. In the end, whether the teams argued about their perspectives is important because MEAs strive to elicit student learning and provoke knowledge generation. It is not that when students do not argue knowledge fails to emerge. It is rather that they are not forced to justify and defend their thoughts. On the other hand, when disagreements arose, and students needed to prove their points and perspectives, they had to explore different solution-paths in order to help their peers comprehend and understand their approaches. In fact, when teams debated about their ideas and perspectives on how to solve a problem, it became more likely they would produce and explore more alternative solutions than when they aligned. Sutton (2012) stated that when a team faces persisting episodes of argumentation and debate, such as in the way I have showed with the low-performing team, could mean that there is a competition to “develop and test as many ideas as possible” (p. 85). Debates encourage and enhance the production of ideas more often in teams that argue and debate than in teams that do not (Nemeth & Wachtler, 1983).

The examples showed about the teams’ type of interaction reveal that MEAs somehow trigger students’ discussion skills in either an argumenting or non-argumenting way. However, it only shows the type of mathematical interaction (i.e., the discourse that occurs during the solving process) that students engage while working as a team, and not the content of the mathematics produced during these moments. Therefore, in the next section, and as part of the process, I show the product-process that the three selected teams went through while solving *The Historic Hotel* MEA.

4.4.3. Students' products – The content of the process

As mentioned before, I have selected three teams — one in each category — to be deeply analyzed when solving *The Historic Hotel* MEA. The analysis included not only comparing and contrasting their model-solutions, but also breaking down all the characteristics, elements, and components of each part of their product-process, so I can understand the students' work. It is worth mentioning that each team used a different path-solution to maximize the profit; however, what is important here is how students were able to create solutions that respond to the client's need considering different perspectives (Dominguez, 2010).

4.4.3.1. Low-performing team

I have selected a low-performing team called "LNA" from among the teams that I video-recorded. Then I analyzed all their work, i.e., written reports, worksheets, presentations, and artifacts, in order to understand their model-construction process and compare their solution with the other two selected teams, one each from the average- and high-performing categories. To analyze LNA's solution, I first looked at their initial cycle of modeling, in which they generated a model that considered providing maintenance service only to booked rooms. Then, I moved to their second modeling cycle, in which they mutually agreed to consider the maintenance cost as a constant, i.e., provide maintenance service to all rooms. Finally I examined their written report and final presentation, where they detailed their final model-solution and presented elements of their mathematical construct.

The LNA team started solving *The Historic Hotel* MEA by first understanding the problem statement and what they were asked to solve. After several disagreements on how to interpret the MEA, the team agreed to follow Model 1A, where G=(# Rooms • $Price) – (# Rooms • $DCM[13]), and strategy A (i.e., for every MX$10 the booking price increases, one less room is booked). For example, in the following excerpt (Transcript 9), one of the team members explained to the others her rationale for the strategy she was developing. Then, they continued developing their model solution by multiplying different booking prices and rooms.

Video 83-15:06 (Transcript 9)

1. Elena: Then in total, if he ((the owner of the hotel)) charges MX$600 per room
he would have a profit of 4000… ehh well only if we would increase…
MX$44,800// ((Represent the result of Model 1A, where she calculated
(600 x 80) - (40 x 80)))
//Multiplied by MX$610… means having only 79 rooms…
Multiply ((Elena asked Maria)) MX$610 by 79….
2. Maria ((Computing operations in the calculator)) 59?
3. Elena: //79::
4. Saira: ((Looking at what Maria is doing in the calculator)) You used percentage!
((Instead of multiplication))
5. Maria: //69?
6. Elena and 79!

[13] DCM refers to the daily cost of maintenance.

Saira:

7. Maria: Four… eight…

8. Elena: Now… 79 x MX$40?

9. Maria: ((Taking out the calculator again to make some computations.))

10. Saira: I'll do the computations.

11. Maria: //MX$3160.

12. Elena: Look… adding MX$10 will have a daily profit of MX$45030.

It's easy…let's see… multiply ((Elena asks Maria)) MX$640… in that case he already lost 4 rooms…

13. Maria: MX$640 multiply by what?

14. Elena: //76… MX$640 x 76.

15. Maria: MX$48 thousand…

16. Elena: //MX$48 What?

17. Maria: MX$48640.

18. Elena: MX$48640//

Now, multiply 76 by MX$40…

19. Maria: …MX$3040.

20. Elena: The hotel will have a profit of $45600, if we increase the daily price by MX$40.

21. //…Silence.

22. Elena: If we say that MX$50 ((referring to how much they will increase the price)), then there will be 75 rooms…
MX$670…74…73…
Please, (Elena ask Maria) multiply MX$700 times 70…

In the transcript above, the LNA team was collecting a set of data considering increasing the price and having one less room every time the price was increased. The data the team collected went from considering all the rooms all the way down to only 70 — i.e., one less room was booked for every price increase of MX$10. It was clear to Elena (Turn 20) that each time the room price increased, the hotel's profit increased and the number of booking rooms simultaneously went down. This is the reason Elena proposed (Turn 22) to keep doing computations all the way down to only 70 rooms and a booking price of MX700. The team also elaborated a table (see Figure 4.8 and Figure 4.9), in which they showed how they were collecting the data they used later when they decided whether to continue on that path or to change to a similar but different one.

Figure 4.8 Booking price, rooms, and daily profit

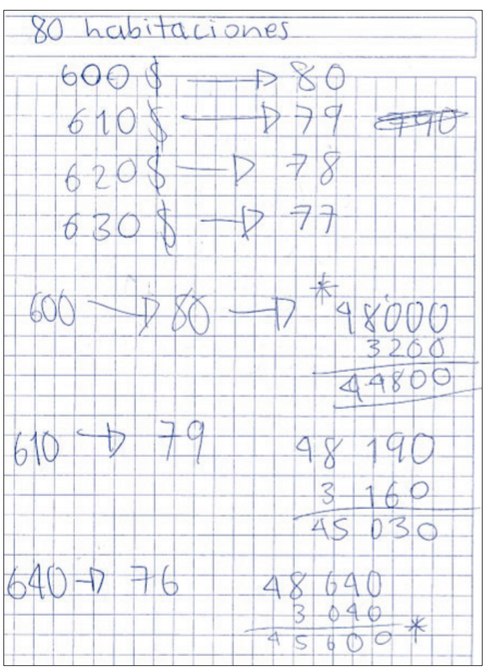


Figure 4.8. Sample of the team's computation of the profit

Figure 4.9 The list of booking price and rooms.

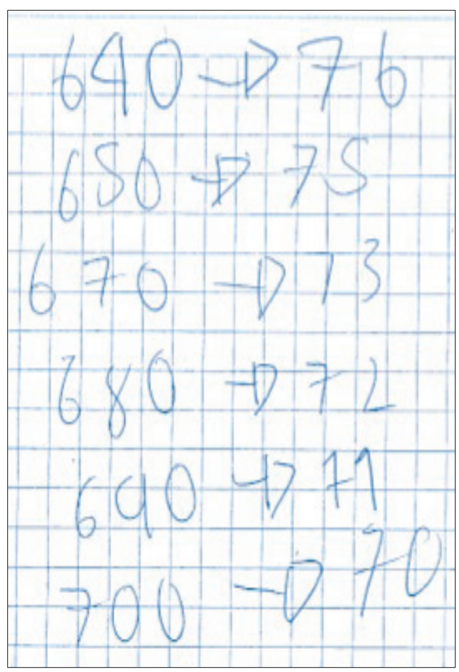


Figure 4.9. The low-performing teams' list of rooms and prices: the number of booked rooms decrease while the price increase.

For Elena, continuing with this solution path was the most correct and easiest way to find the hotel's booking price and number of rooms to maximize the profit. However, for the rest of the team, Elena's solution was missing a key component (i.e., the maintenance of the unbooked rooms). For Maria and Saira, it was very important to keep providing a cleaning service to all rooms, so every single room in the hotel would be in the condition to be booked. The team had several disagreements about whether they should consider providing maintenance only to the booked rooms, or to all rooms of the hotel. In the following excerpt (Transcripts 10), the team discussed what to do, and at the end decided to follow the path of providing service to all rooms:

Video 84-01:50 (Transcript 10)

1. Maria: But it would lose rooms and money, because it loses maintenance.
2. Elena: //No… no.
3. Maria: But the maintenance is MX$40 anyway.
4. Elena: But… what if…and…
5. Maria: Look, think about it… you are losing rooms, but you have to keep the rooms clean.
6. Elena: But, if you are not using it ((a room)), what is the point of cleaning it?
   Do you think they ((hotels in general)) clean it? ((Elena ask Saira))
   Would you clean a room that it is already clean?
7. Saira … (silence)
8. Maria: //YES:: plus if you are cleaning one room, then you have to clean the other one too.
9. Elena: //NO:: because it's already clean…
10. Maria: But, you are losing anyway.

11. Maria: But how do you know that… you are using a system ((i.e., a procedure)) where you're always increasing MX$10… then how do you know that you would always have clean rooms?... what if someone booked a room, you think the room is clean, and then POOM:: you lose money because nobody would pay for an uncleaned room…

Would you pay for a room without maintenance? ((Maria asked Saira))

12. Saira: My opinion is that all rooms should receive maintenance.

13. Maria: Yes:: see?

14. Elena: Daily?

15. Saira: ((Saira move her head up and down, saying yes))

//That happens in all hotels. ((She is connecting her idea with her real-world experiences.))

16. Elena: All right! We will have a very-clean, well-maintained hotel! ((Elena is being sarcastic))

In Transcript 10, Maria tried to convince Elena that with her proposed procedure, the hotel would lose money because even if a room is not booked, it needs maintenance. When Saira got involved in the discussion, she provided a strong objection, connecting her idea to something that seems to be part of her real experience. This idea, justified with something real ("That happen in all hotels"), was what finally convinced Elena to accept her team's idea. Elena is clearly frustrated with her other two team members' proposal to clean all the rooms (i.e., Strategy 1B), but at the end she agreed and the team continued developing their solution based on the idea of providing maintenance to all rooms.

The team continued looking for the right price and number of rooms to maximize profit. This was a fairly easy task for them since they had already calculated this profit when considering the maintenance cost of only the booked rooms. That means they only needed to replace the cost of the maintenance variable, i.e., the total maintenance cost dependent on the number of rooms booked, with a constant of MX$3200 (the result of multiplying MX$40 x 80). Although the team had some disagreements, they were able to resolve them and agree on the model and strategy that would maximize the hotel's profit. The team had two relevant cycles of modeling: the first one in which they considered maintenance only for booked rooms, and the second in which they would clean all rooms, requiring them to explain, accept, reject, and modify some of their individual and team's ideas (Lesh & Doerr, 2003; Zawojewski, Lesh, & English, 2003). Using the image below (see Figure 4.10), the team presented their solutions on a transparency, showing their model (1B), in which they subtracted the constant maintenance cost from the multiplication of the number of booked rooms times the price: G=(# Rooms • $Price) – $MC). They indicated the type of strategy (A) they used, i.e., increasing the initial price by 10, and decreasing the number of booked rooms by 1. Furthermore, they used a chart with the shape of a pyramid to present their data. This shape, similar to a real object, has been shown to represent an introspective visualization (Phillips, Norris, & Macnab, 2010) — a strong asset I will discuss further, below.

Figure 4.10. Low-Performance team LNA's transparency presentation

[Translation: Maintenance and service…

Then, the right price is 700]

Figure 4.10. The LNA team presented their solution, in which two main relevant aspects deserve to be highlighted: room maintenance as a constant (MX$3200), and the pyramid shape as an introspective visualization in which the top represents the best price and number of booked rooms (MX$700 and 70 respectively).

4.4.3.2. Average-performing team

I selected to fully analyze a team called CFN from among the video-recorded teams. As with the other team, I considered this team's written solution, worksheets, artifacts, and final transparency presentation of *The Historic Hotel* for analysis. The team's model-solution was unique in the way they created and thought about it. The team interpreted the problem statement in a way that enabled them to "trick" the "system," without breaking any of the MEA's rules, to maximize the hotel profit. However, they had serious difficulties in the early stages of interpreting the problem statement. For example, in the following excerpt (Transcript 11), two team members misinterpreted the givens. They were considering the increasing rule — for every MX$10 that the booking price increased, one less rooms is booked— as the initial price.

Video 77- 12.30 (Transcript 11)

1. Carmen: ((Using the calculator))

   80 x MX$40… are MX$3200.

   Now, from the MX$48000, you take away the MX$3200, and you will have a profit of MX$44800, because you are already covering the maintenance service cost, that is the MX$40 of maintenance for every booked room.

2. Nancy: But, he wants to know the "booking price that would maximize the profit even when the booking price and the number of rooms changes."

   And, what if the maximum booking price is MX$5?

3. Fabi: But the price

4. Carmen: // (Interrupting Fabi). WAIT, what are you considering as the price?

5. Fabi and Nancy: The MX$10…

6. Fabi: That is the price!

7. Nancy: It says, ((the problem statement)) "for every MX$10 the daily price is increased," here it is the problem!

8. Carmen: The daily price is MX$600!

The team faced problems in how to interpret the MEA (Turns 5, 6, and 7), but this actually opened a door for a completely different way of interpreting the givens, not only for solving the problem, but also in figuring out how to maximize the hotel's profit. What seemed to be a failure in how to interpret the problem statement became an advantage for this team (Derek, 2015).

The CFN team combined the model type that Carmen was proposing (Turn 1) with Fabi and Nancy's misunderstanding to create and implement a new model strategy that was completely different from what the other teams proposed. Team CFN team used model 1A, in which the number of booked rooms times the daily maintenance cost is subtracted from the booking price times the number of rooms:

$$G=(\#\text{ Rooms} \bullet \$\text{Price}) - (\#\text{ Rooms} \bullet \$\text{DCM}).$$

However, when using strategy B, they increased the price by only MX$5 rather than by the MX$10 specified in the problem statement, in which the number of booked rooms decreases only when the booking price is a multiple of 10. For example, according to the problem statement, for a booking price of MX$600, the number of booked rooms would be 80, but decrease to 79 when the booking price increased to MX$610. However, when Team CFN increased the booking price to only MX$605, the number of booked rooms turned out to remain the same (i.e., 80 rooms). In this way, the team's error revealed a path to a solution that was not available based on the data given.

This strategy has advantages and disadvantages for developing a model-solution. One main advantage is that implementing this model generates a higher net profit for the hotel owner than when only increasing the price by multiples of MX$10. To the contrary, though, it requires increasing the initial booking price more times to find the ideal number of rooms and booking price that would maximize the profit. That is, it requires making many more calculations and increases the amount of time spent on the activity. The image below (see Figure 4.11) shows how the Team CFN used model 1A and strategy B to calculate hotel profit. However, it also shows that they tested only a limited number of price increases and decreasing number of rooms, which does not represent in the end the numbers that would actually maximize the profit.

Figure 4.11. Team CFN's computation table

Tarifa	Habitaciones	Ganancia Anual
605	80	16,814,400.00 Mx
610	79	16,435,950.00 M
615	79	16,550,125.00 M
620	78	16,512,600.00 Mx
625	78	16, 654, 950.00
630	77	16,581,950.00 Mx

[Translation: Price | Rooms | Annual profit]

Figure 4.11. The team makes an effort to find the number of rooms and price that would maximize the profit, increasing the price by MX$5, instead of by MX$10, and calculating an annual revenue.

By looking at CFN's table, it seems that by considering a booking price of MX$605 and having all 80 rooms would maximize the profit of the hotel. However, this contradicts the fact that profit would increase if the number of booked rooms decreased and the booking price increased. Because of the calculations the team performed and the variables they considered for booking price and number of rooms booked, the annual profits they determined do not successfully represent the best measure of comparison. In the following paragraph, I analyze CFN's solution in order to reveal their disagreements in computing the profit.

The team decided they would start with MX$605 and 80 rooms, and calculate profit annually. However, they argued about whether to calculate annual costs based on 365 days or 12 months. In the following excerpt (Transcript 12), after a disagreement, the team concluded that they would consider calculating profit by month, and each month would have 31 days.

Video 86-14.15 (Transcript 12)

1. Carmen: Ok, do it for a year…oh that is for a year.
   Well, do it monthly, do it monthly…
2. Fabi: …but, there are months that have…what month do you want me to calculate? ((Fabi means that some months have a different number of days, 28, 29, 30, or 31.))
3. Carmen: Look, do it considering 31 days, it is only 3 days. That does not make any difference
4. Fabi: //Yes:: it makes a difference!
5. Carmen: It is only a little more than a thousand… ((Mexican pesos))
6. Fabi: That's not true! Because it is MX$605 times 80…
7. ((Carmen and Nancy reacted to Fabi's argument about the number of days, so they agreed to do it as Carmen said.))
8. Fabi: Only for one month?
9. Carmen: Look, do it by month and then…
10. Fabi: 40…
11. Carmen: Only check the result and that's all, got it?

Fabi realized that different months have a different number of days, which in the end would make a difference when calculating profit. Although she tried to explain and justify her rationale, the rest of the team (i.e., Carmen and Nancy) disagreed with her. At that point, Fabi's only path was just to accept what the other two team members were saying (Turn 6). This is the reason why the CFN team had a higher profit with a lower booking price (MX$605). The way they calculated annual profit was as follows:

$$G_{annual}= [(\# \text{ Rooms} \bullet \$\text{Price}) – (\# \text{ Rooms} \bullet \$\text{DCM})] \bullet (\text{Month's days})(\text{Year's month})$$

$$G_{annual}= [(80)\ (\text{MX}\$650) – (80)(40)] \bullet (31)(12)$$

$$G_{annual}=\text{MX}\$16{,}814{,}400$$

For computing profit at other price levels, (i.e., from MX$610 to MX$630), the team used a different strategy. Carmen and Nancy finally listened to what Fabi was saying and considered making the calculation by multiplying the profit by the 365 days of a year, instead of the 31 days of a month and then by the 12 months (as previously done). The way the strategy was done is exemplified below, considering a booking price of MX$630 and having 77 rooms booked:

$$G_{annual}= (\# \text{ Rooms} \bullet \$\text{Price})(\text{Year's days}) – (\# \text{ Rooms} \bullet \$\text{DCM})(\text{Year's days})$$

$$G_{annual}= (77\bullet 630)(365) – (77\bullet 40)(365)$$

$$G_{annual}= \text{MX}\$16{,}581{,}950$$

The annual profits calculated by the CFN team shown above are still far from the ideal final profit they would have found if they had continued working on their computations. Finally, although the team calculated the hotel's profit considering a unique strategy (i.e., by increasing the price by MX$5), they did not go further in exploring and computing the profit with a higher price and a lower number of rooms. In that case, they would have continued making computations and found that the number of rooms and booking price that would maximize profit would be 68 and MX$725 respectively (see Table 4.11).

Table 4.11. Annual profit for a booking price that increase by MX$5

Daily Rate (MX$)	Rooms	Total Annual Profit (MX$)
630	77	16,581,950
635	77	16,722,475
640	76	16,644,000
645	76	16,782,700
….	….	….
710	69	16,873,950
715	69	16,999,875
720	68	16,877,600
725	***68***	***17,001,700***
730	67	16,873,950
735	67	16,996,225
740	66	16,863,000
745	66	16,983,450

Table 4.11. The booking price and number of rooms to maximize profit is shown here for a final annual profit of MX$17,001,700.

In the end, the CFN team went through several cycles of modeling before deciding to explore the solution path shown above. Although their solution was incomplete, they made a decision based on the information they collected. If they had carried it out further following the same computation, they would probably have noticed that MX$605 and 80 rooms did not maximize profit. Thus, even though they did not solve the problem correctly themselves, the team did develop a solution that was unique and different that would have led to a different outcome if they had carried it out more fully. This emphasizes the idea that MEAs allow students to explore multiple routes for the same problem (Dominguez, 2010).

4.4.3.3. High-performing team

I selected to fully analyze Team MA^2 from among the high-performing teams that I video-recorded. As with their low- and average-performing peers, I considered their written report, worksheets, artifacts, and final presentation for *The Historic Hotel* as part of the analysis. The team worked around two cycles of modeling in which they created and developed a model-solution and strategy, used different components in their modeling product-process, and proposed a solution using the givens of the MEA.

The team started by reading and understanding the problem statement. They made sure to comprehend what was asked and began making notes and computations. One of the first tasks they completed was finding a model they could use to determine the number of rooms based on a profit-maximizing price. After a series of trial and error, they developed a set of models (see Figure 4.12) to obtain the number of rooms, the profit and the maintenance cost, considering that it could vary.

Figure 4.12. The MA^2 team's set of models

1: $80-\left(\frac{t-pa}{10}\right)=h$ 2: $y=40+x$ 3: $G=(t-y)h$

Figure 4.12. The team developed a set of models in which "h" represents the #Room, "t" the desired price, "pa" the initial price of MX$600, "y" the maintenance cost, "x" the increase or decrease of the daily maintenance cost, and "G" is the profit.

The set of models that MA^2 used corresponds to the model-type 2B, in which the main model-solution to obtain the profit is factored:

G=($ Price – $DCM) • (# Rooms)

Once the team found their models — the first cycle of modeling — they elaborated a table of price versus number of rooms from 80 to 70 (see Figure 4.13), and in which they considered that a price of MX$700 and 70 rooms would maximize the hotel's profit. In addition, as part of their written report, the team provided an example (see Figure 4.14) showing how their models would work, and what the price to maximize the profit would be. However, the fact that this team created a set of models to obtain the number of rooms, the maintenance cost, and the profit, does not provide a solution to the client's problem, i.e., to maximize the hotel's profit.

Figure 4.13. Table of booking price versus rooms

Tarifa		habitaciones
600	-	80
610	-	79
620	-	78
630	-	77
640	-	76
650	-	75
660	-	74
670	-	73
680	-	72
690	-	71
700	-	70

[Translation: Rate | Rooms]

Figure 4.13. The table shows how the MA^2 team created a list of booking rate versus number of rooms and arbitrarily considered the desired price and number of rooms by following strategy A– i.e., for every MX$10 the initial price is increased, one less room is booked.

Figure 4.14. MA^2's example models

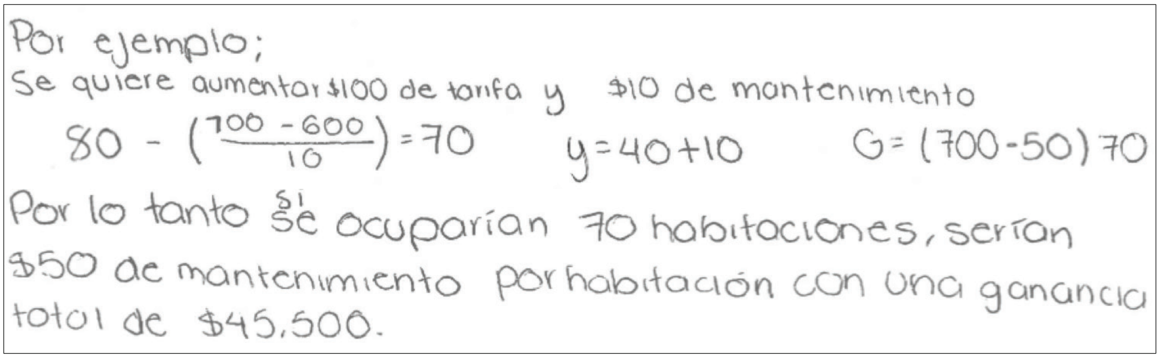

Por ejemplo;
Se quiere aumentar $100 de tarifa y $10 de mantenimiento
$80 - \left(\frac{700-600}{10}\right) = 70$ $y = 40+10$ $G = (700-50)70$
Por lo tanto si se ocuparían 70 habitaciones, serían
$50 de mantenimiento por habitación con una ganancia
total de $45,500.

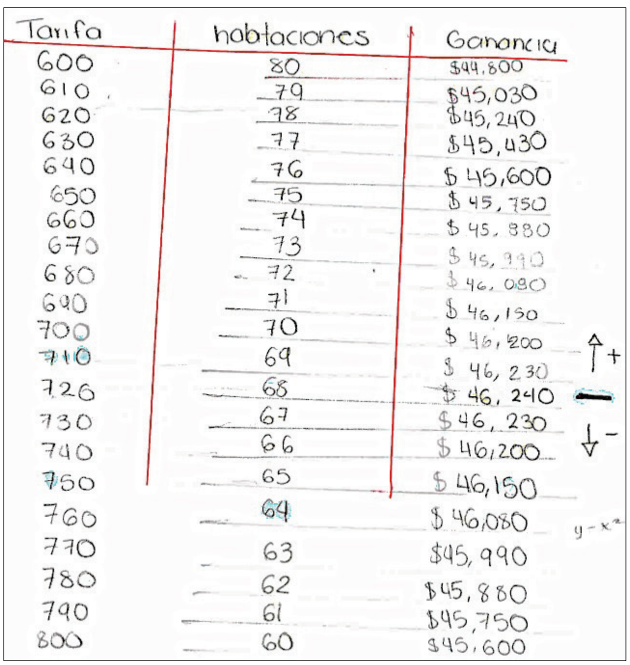
Figure 4.14. The team shows how to use and apply their set of models, and stress the fact that by having 70 rooms, a maintenance cost of MX$50, and a booking price of MX700 would show a profit of MX$45,500.
[Translation: for example, if you would like to increase the initial price by $100 and the maintenance cost by $10 (Computations) Then, you will have 70 rooms booked, with a maintenance cost of $50 per room and a profit of $45,500.]

The two figures above show that the team was able to create a set of models, but also that they failed to justify why 70 rooms would be the right number to maximize profit. Likewise, the team does not provide any evidence to justify the desired price of MX$700 as the ideal booking rate. In fact, in Figure 4.14, the team only mentions that the hotel would see a profit of MX$45,500, but does not clarify whether that amount represents the maximum profit, which in fact it does not.

Even though during their first cycle of modeling the team presented a model-solution that does not maximize the hotel's profit, in their second modeling cycle they were able not to only to justify their result, i.e., both the number of rooms to be booked and the booking price, but also to propose a solution that would actually maximize the hotel's profit. This happened while the team worked on their presentation. The fact that they had to present and justify their product-solution in front of their peers triggered their attention and efforts into why considering 70 rooms at

MX$700 would maximize profit. To answer this question, the team discussed and created a table (see Figure 4.15) in which they showed booking price, number of rooms, and profit. By doing this, they were able to compare and visualize profit maximization. In addition, the team created a graph for their presentation (see Figure 4.16) in which they contrasted profit and price, and determined the type of model equation that emerged from the activity.

Figure 4.15. Table of price, rooms and, profit



Tarifa	habitaciones	Ganancia	
600	80	$44,800	
610	79	$45,030	
620	78	$45,240	
630	77	$45,430	
640	76	$45,600	
650	75	$45,750	
660	74	$45,880	
670	73	$45,990	
680	72	$46,080	
690	71	$46,150	
700	70	$46,200	
710	69	$46,230	↑ +
720	68	$46,240	—
730	67	$46,230	↓ -
740	66	$46,200	
750	65	$46,150	
760	64	$46,080	y - x²
770	63	$45,990	
780	62	$45,880	
790	61	$45,750	
800	60	$45,600	

Figure 4.15. The table shows the profit for different booking prices and number of booked rooms from MX$600 with 80 to MX$800 with 60 rooms. The table facilitates the visualization of higher profit, which occurs when considering a price of MX$720 with 68 rooms.

Figure 4.16. Model graph of price versus profit

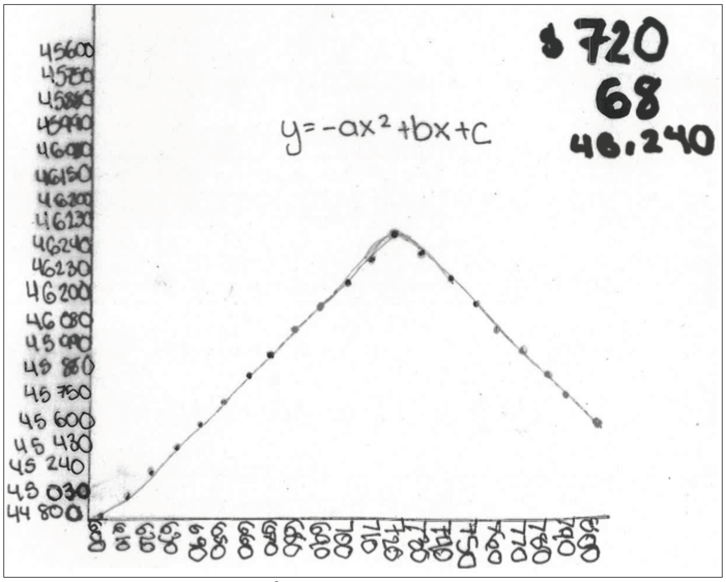
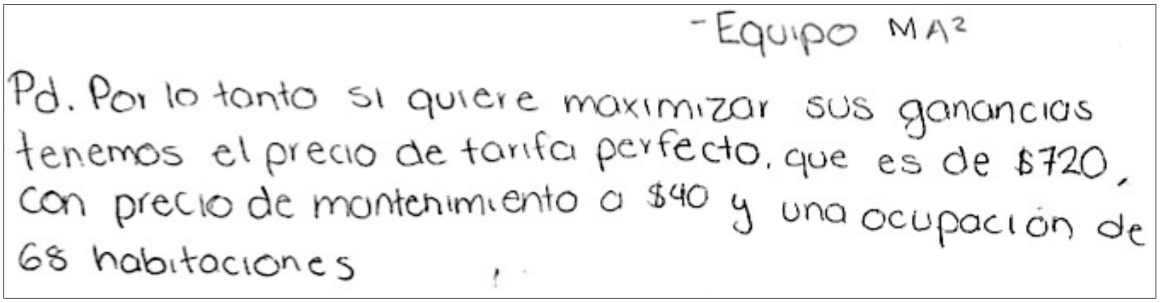


Figure 4.16. The MA^2 team created a graph to show the type of mathematical function that emerged when graphing profit and price. The team concluded that a quadratic equation would be the function that best fit the shape, where the maximum profit represents the vertex of the parabola.

In the figures above, the team related profit and price and were able to extrapolate their findings into a graph, which later served to reconfirm their findings. Once the team was able create the table, they were able to identify the number of booked rooms and price that maximized profit. These are relevant findings, since in the team's first cycle of modeling they were not able to justify or deeply explain their rationale, but during their second cycle, they had more elements to compare and thus finally propose a better solution for the client.

In general, this high-performing team was able to find the best ideal solution for the client. They created tables, lists, a set of models that, combined, determined the profit, and a graph that helped to validate their findings. In their written report, they added a final remark (see Figure 4.17) emphasizing the best numbers for maximizing the hotel's profit.

Figure 4.17 MA²'s final remark in the written report


-Equipo MA²

Pd. Por lo tanto si quiere maximizar sus ganancias tenemos el precio de tarifa perfecto, que es de $720, con precio de mantenimiento a $40 y una ocupación de 68 habitaciones

Figure 4.17. The team wrote their report during their first cycle, adding this final observation after their second cycle of modeling, to stress the ideal number of rooms, booking price, and profit.
[Translation: PS. Therefore, if you would like to maximize your profit, we have the perfect booking price, which is $720, with a maintenance cost of $40 and with 68 rooms booked]

4.4.4. Comparison across process-products

MEAs allow students to propose various solutions using different strategies and perspectives that sometimes go beyond what teachers and researchers anticipate. They let students gain new knowledge, both individually and as a community, in a context in which ideas are accepted, rejected, reformulated, or elaborated (Cramer, Lesh, Doerr, Post, & Zawojewski, 2003). In particular, they enable all members of a group to enhance and elicit each other's understanding by juxtaposing their individual perspectives. Above, I analyzed three teams solving *The Historic Hotel* MEA (one in each performance level — the LNA, CFN, and MA²); each proposed model-solutions that would maximize the hotel's profit applying and considering different perspectives on the same situation. In the following paragraphs, I address some of the similarities and differences between the three teams' product-model-solution for *The Historic Hotel.*

The average-performance team CFN modeled a solution in which they creatively increased the price by MX$5 rather than by the MX$10 in the MEA's problem statement. The team did not carry their procedures through sufficiently and therefore ended up proposing a solution that would not ideally maximize profit. However, if they had, their proposal would have in fact shown a maximized profit higher than what other teams proposed. In term of the type of model and strategy, this team developed the model 1A (i.e., G=(# Rooms • $Price) – (# Rooms • $DCM))–the most common model among teams, and in fact used by the other two teams. The difference was the CFN team decided to multiply the daily profit by 365 days to obtain an annual profit, instead of a daily profit as others teams did. As mentioned before, their strategy (type B) was unique in terms of increasing the price by MX$5. However, it is also similar to what the other two teams did when decreasing the number of booked rooms every time the hotel booking rate was increased by MX$10.

The low- and high-performance teams' solutions have several contrasting characteristics. First**,** the model the two teams created came from the original model type 1A (also used by the average-performing team), with the particular difference that the low-performance team considered providing service to all rooms — i.e., G=(# Rooms • $Price) – $MC. This means the maintenance cost became a constant. On the other hand, the high-performance team factored the model to have a shorter version–i.e., G=($ Price – $DCM) • (# Rooms). In terms of strategy, both teams used strategy A, i.e., when increasing the price by MX$10, one less room is booked. In addition, both teams decided to use almost the same type of components to supplement their solutions (e.g., tables, lists, graphs, and charts).

Second, both the low- and high-performance teams entered into two cycles of modeling while solving the MEA. During the first cycle, both teams arrived at the point of considering that

70 rooms at a booking price of MX$700, and only providing maintenance to the booked rooms, maximized the profit. However, the teams deviated from considering 70 rooms for very different reasons. In the case of the Team LNA (low-performance), a disagreement forced them to consider a different solution-path. This drove them to enter a second cycle of modeling, providing a final model-solution that took maintenance of all the hotel's rooms into account. On the other hand, during their first cycle of modeling, the high-performance team did not provide much rationale of why they selected 70 rooms as their ideal number of booked rooms. However, while working on their final presentation, they realized that 70 rooms and MX$700 were not the numbers that would maximize profit. As a result, they entered into a second cycle of modeling, in which they compared different profits at different booking rooms and prices. By considering the correct number of rooms and price they determined they would maximize profit based on the model and strategy they had generated. In the end, both teams proposed a final model and solution from different perspectives, but both calculated adequate solutions that maximized the client's profit.

Third, comparing the two teams' representations of their data shows a less and more sophisticated form, a chart and a graph (see Figure 4.18). The low-performance team created a chart with a pyramid shape, the high-performance team a graph with a parabola shape. The chart and the graph may be visualized as forms of a novice (low-performance) and expert (high-performance) interpretation of the same situation (Atman, Adams, Cardella, Turns, Mosborg, & Saleem, 2007). Nevertheless, the novice was able to generate a solution as powerful as that of the expert, since both were able to maximize the hotel's profit in accordance with the terms of the MEA's problem statement.

Figure 4.18. Comparison between chart and graph

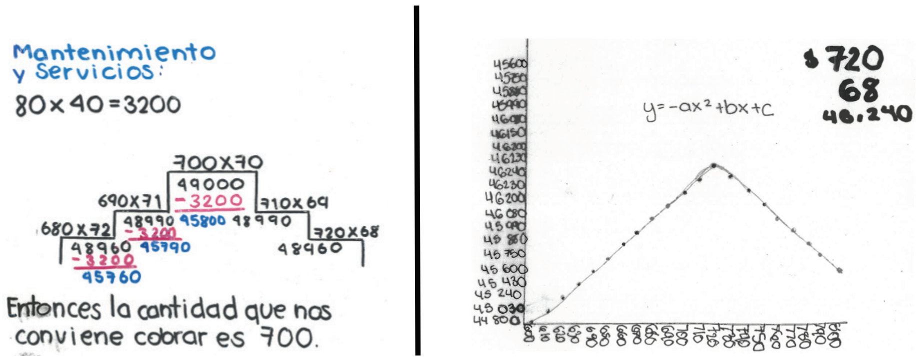


Figure 4.12. On the left, the low-performance team's chart simulates a pyramid shape. On the right, the high-performance team's graph represents a function with a parabolic shape.

In the end, the three teams' product-model-solutions were adequate and comparable, though also rich and varied. As expected, the high-performance team's solution is more elegant and sophisticated in terms of the mathematics they developed. But this does not make the solutions the average- and low- performance teams provided to be any less adequate than that of the high-performance team. Indeed, the richness and variety across groups' solutions suggest that MEAs may provide rich learning experiences to students across all performance levels.

Chapter 5: Conclusions, Implications, and Recommendations

In this study, high school students worked in teams of two or three members to solve open-ended mathematics activities known as Model-Eliciting Activities (Lesh & Doerr, 2003; Lesh et al., 1999). As they engaged in these activities, they "stretched and developed their conceptual understanding" (R. Capraro, M. Capraro, & Cifarelli, 2007, p. 124), and viewed the emerging path-solutions from many different perspectives (Dominguez, 2010; Greenstein & Carmona, 2007; Lesh, Cramer, Doerr, Post, & Zawojewski, 2003). My goal in implementing MEAs was to motivate and encourage students to collaborate to develop mathematical model-solutions, and to provide a platform for students — regardless of their academic performance in class — to exhibit the richness of the mathematical solutions and their competency in solving open-ended, real-life mathematics activities. Further, I aimed to find evidences that students at any performance-level might develop adequate model-solutions for MEAs.

In this chapter, I present my conclusions about the products and processes-solutions detailed in this dissertation. I also show some evidence that students at any performance level, in particular, the so-called "low-performance" students, could develop and model a solution equally as others in different performance levels. Finally, I address three other relevant aspects of my research: the limitations of this study and how they limit its scope, the implications of this project in the field of mathematics education, and future approaches and recommendations in terms of use and implementation of MEAs.

5.1. THE PRODUCTS AND MODELING PROCESS

My evaluation of the students' product-solutions based on the Quality Assessment Guide (Lesh & Clarke, 2000) of the *Hybrid vs. Gas Cars* MEA were similar in quality, both among teams at the same and across different performance levels. For example, between 60 and 80 percent of the teams across performance levels achieved a score between two and three out of five. This means that their solutions, according to the QAG and based on the client's need, were incomplete and required further development. In general, teams at all three performance levels — i.e., low, average, and high — obtained a QAG mean of 2.6, 2.36, and 2.8, respectively.

Although the solutions were incomplete, most of them were similar, relevant, and adequate given the recent exposure the students had to solving open-ended problems like MEAs, and to the type of mathematical concepts addressed in them (Chang, 2008). In addition, students were able to engage in a collaborative effort, aiming not only to mathematize a real-life situation, but also to find an adequate model-solution using their individual and shared knowledge while evaluating, proposing, or rejecting different ideas and perspectives (Doerr & English, 2003).

Based on the QAG, the last MEA implemented (i.e., *The Historic Hotel*) showed the students' solutions to have become not only more sophisticated, but also more acceptable regarding the client's needs. For example, all three performance-level teams obtained a higher mean of 3.0, 2.64, and 3.4 for this MEA. Also based on the QAG score, high-performance teams appear to have performed better in solving these MEAs; however, a deeper analysis of the students' model-solutions and process –i.e., three selected teams, one at each performance-level– shows constrasting results.

First, all three teams developed multiple modeling cycles (Lesh & Doerr, 2003) revising their previous models, refining their current model-solutions, and exploring different model-paths that responded better to the client's needs (Kaput, 1998; Lehrer & Schauble, 2006; Lesh & Doerr, 2003).

Second, *The Historic Hotel* MEA evoked and elicited different interpretations, in which the students externalized how to interpret the activity — i.e., the externalization principle. This allowed them to create relationships among the mathematical components, strategies, and models they developed — the model construction principle (Chamberlin, 2002; Lesh et al., 1999). For example, the high-performance team developed a solution, consisting of a set of three-models, for maximizing the hotel's profit. Their final model-solution is a factored version of the general profit model "G=(# Rooms • $Price) – (# Rooms • $DCM)," in which they suggest providing maintenance only to the booked rooms, for a final booking price of MX$720 and 68 rooms. The strategy they followed was to systematically increase the price by MX$10 and reduce the number of rooms booked by one. In addition, they created and used several mathematical components like tables, lists, and graphs.

Similarly, the low-performance team also developed a model-solution derived from the general profit-model. However, instead of considering cleaning only the booked rooms, they decided to provide maintenance service to all rooms, making the maintenance cost a constant variable within the profit model — i.e., G=(# Rooms • $Price) – $MC. This low-performance team also determined that the best strategy would be the one in which the price systematically increased every MX$10. In addition, they also developed lists, tables, and charts.

Finally, the average-performing team proposed a model-solution uniquely proposing increasing the booking price by MX$5 and reducing the number of rooms booked only when the price was a multiple of ten. They also used tables and lists as part of their mathematical components. Unfortunately, their solution was incomplete because they did not obtain the price that would maximize profit due to an error in their first computation.

All three teams were able to externalize their solutions and their individual reflections (i.e., individual learning) and their inter-group communications (i.e., collaborative learning) (Vygotsky, 1962; Piaget, 1926). These led them to find and consider the similarities and differences among their proposals, and use or discard ideas as needed (Hiebert et al., 1997). In addition, all teams had different approaches to solving the problem. Each was adequate to provide a solution that would maximize the hotel's profit (even the average-performing team would have had a solution if they that had completed their proposal), similarly to what occurred in the implementation of the "University Cafeteria" MEA (Mousoulides et al., 2010) detailed in previous chapters.

Furthermore, performance level as considered and established in this research project does not effectively determine the ability of students to generate solutions to open-ended mathematics activities like MEAs, which require complex thinking (Iversen & Larson, 2006; Lesh & Sriraman, 2005; Carmona & Greenstein, 2010). However, it was noticeable that high-performing teams developed solutions that were more sophisticated, mathematically speaking, than those of the low- and average performance teams. Therefore, although the open-ended nature of the MEAs allows students to consider different solution paths for the same problem, much more evidence would be needed to show that MEAs could indeed serve to level the playing field among students of different performance levels.

Third, after I transcribed, coded (Vidic et al., 2014), and analyzed all the videos (Powell et al., 2003) in what represented the first round of coding-analysis, a pattern was immediately identifiable. Namely, the students spent most of their time in the process of collecting or analyzing data. Regardless of their performance level, students devoted a similar amount of time to these two moments, in which they negotiated the main ideas and paths that they would follow to obtain their model-solutions (Zawojewski et al., 2003).

Thus, for these two process-moments, I did a second, inductive coding (Miles et al., 2014) to identify the group-level moments in which students discussed the main ideas of their model-process-path. During the moments, the students had several disagreements, which generated several different types of discourses. I categorized these following Azevedo et al.'s (2015) discourse typology of argumentation, explaining, and describing, which I reduced to two types, argumenting and non-argumenting, the latter combining the explaining and describing types. Looking at the process through which students solved their disagreements, if any, showed the low-performance students to have had almost twice as many disagreements as the average- and high-performance teams, leading them into in twice as many argumentation type of discourses. The high-performance team, on the other hand, engaged in the highest number of non-arguing discourse — i.e., explaining or describing. One of the reasons for this may be that the students felt too secure about what they already knew, without paying much attention to how they knew it (Reddy, 1979, as cited by Osborne, 2010, p.463).

From the standpoint of learning, when disagreement occurs and students are forced to argue over their ideas, they are also required to deepen their explanations of and justifications or reasons for their positions and perspectives (Lai, 2003), which enhances their knowledge generation and acquisition. For example, when the low-performance team argued over what number of rooms and booking price they should consider (see Transcript 7 in chapter 4), one of the team members, in an effort to explain her rationale and justify her perspectives, visualized her ideas and representations in the shape of a pyramid, which is a powerful way of mathematization (Phillips, et al., 2010; Lesh & Doerr, 2003). In addition, when students argued, they gave rebuttals, which required them to "compare, contrast, and distinguish different lines of reasoning" (Osborne, 2010, p.464).

Although all three teams' modeling-processes correlated effectively with what they presented as their final products — i.e., the client's letter or report and final presentation — my analysis of their processes revealed that low-performance students had more moments and opportunities for knowledge generation, while having more argumentation discourse moments when resolving their disagreements.

In addition, analyzing the process of modeling in combination with the product-solution revealed that all three teams were able to develop solutions that considered variables beyond what anyone could expect. For example, the high-performance team developed a set of models, the low-performance team considered providing maintenance to all rooms and related their ideas to a pyramid shape, and the average-performance team increased their prices in multiples of five rather than the 10 given in the problem statement. Similar to what Mousoulides et al. (2010) found when they implemented the *University Cafeteria* MEA, students considered "low-achievers" were able to model a solution by developing a meaningful product and a process, just as their high-achieving peers. Therefore, instructors and researchers should expect extraordinary achievements from students at any performance-level when developing model-solutions during a MEA (Harel & Lesh, 2003).

As with every skill, the more it is practiced, the more it will be used it accurately (Anderson, 1993). Students in this research project developed solutions to a series of three MEAs, showing slightly improved QAG scores for solving MEAs between the second and third, which were the focus of my study. Even though their improvement is relatively small, it suggests the students' solutions would likely become more sophisticated and better developed as they are more exposed MEAs. I address this further in the next section, as part of the limitations and recommendations.

5.2. Limitations of the Study

This research project is founded on my implementation of several MEAs during the spring of 2016 during class time in a real school setting. Although I was careful and deliberate in selecting which MEAs to implement, the unpredictability of "real time" classroom events constrained what I was able to accomplish. The first constraint was the time allotted for the MEAs' completion. School administrators gave me approximately 10 hours (i.e., 600 minutes) of class time to implement the MEAs, allowing approximately 200 minutes divided across several sessions for each of the three MEAs. Many students requested or required more time to finish the activities, demonstrating the time allotted was too short for them, particularly during *The Historic Hotel*. In addition, students had no experience in solving open-ended activities like MEAs. Therefore, the first MEA implemented (i.e., the Team Ranking Problem) was not considered as part of the analysis.

Second, the school authorities' consent for my research did not include time for me to interview or assess the students individually after implementing the MEAs. As result, I could not verify the degree of individual students' learning. Could I have done so, I might be able to make stronger inferences about the relationship between students' learning and MEAs. Thus, the time allotted was also too short for me to carry out my research to the fullest extent it might have been.

Third, students' group selection might have been affected by the selection procedure of the video recorded teams, in which the classroom teacher suggested the groups to be recorded. Although in the last stage of the analysis I tried to minimize the effect of the lack of randomization by selecting the three representative teams among the larger sample, it would be necessary to fully analyze three more teams (one in each category level) to find more and stronger evidence that MEAs could level the playing field.

Fourth, the fact that group's categorization (i.e., low, average, and high-performance) were based on the individual students' achievement in class comprising their test grades, homework, projects, and other assignments, and because these grades were assigned by the teacher, the groups' formation might have been inadvertently influenced by the teacher's own bias and perceptions toward the students.

Fifth, real classroom-settings have many uncontrollable variables that can either hamper the work time, or distract the students' from their work. For example, in one of the sessions, the school's principal inadvertently interrupted the session to address a disciplinary incident that had happened during the students' recess. This not only distracted the whole class, but the activity had to be stopped until the principal resolved the situation. This caused a reduction in the time allotted for the activity on that particular day, so that the activity had to continue in a different session, which interrupted the flow of work.

A sixth limitation was the sample size. Three sections of the same mathematics class participated in the study, with 74 students in total. Students were asked to forms teams of three, thus 24 in total, from which the teacher and I selected nine to be video-recorded, of which I analyzed only three (one in each performance level) deeply. Because of the size of the sample, I could not perform a statistical analysis in compliance with statistical assumptions. As a result, I performed only a descriptive analysis but did no inferential analysis whatsoever.

Seventh, most of the students participating in my research study, and all of the students on the selected teams, were girls. Because of the extent of data needing to be collected and analyzed to control for gender, I did not take this variable into consideration. However, the dynamics of mixed-gender teams could likely effect outcomes of collaborative activities. Further research would be needed to show whether gender might affect the outcomes.

Finally, my own time constraints in collating and analyzing the data limited my ability to make further comparisons that might provide evidence to support or refute my analysis here.

5.3. IMPLICATIONS

Many research projects have addressed the benefits and limitations of implementing MEAs as a teaching, learning, assessment, and research tool with students at different grade levels (Yildirim et al., 2010; Yu & Chang, 2009; Mousoulides & English, 2011; Moore & Diefes-Dux, 2004; Lesh & Doerr, 2003; Hernández et al., 2011; Chan, 2008; Mousoulides, 2009). However, few research projects have focused their attention on either high school students at different performance levels or the patterns of discourse that emerge while students solve MEAs, as I have done here. Therefore, this study has many implications for mathematics education in general, and for teachers, students, policy makers, and administrators, in particular.

First, in terms of the mathematics education, researchers have implemented MEAs mostly in elementary, middle, and post-secondary levels, but few have done so at the pre-college level. In this dissertation, I have addressed that gap in the field by contributing a study at the high school level. In addition, I have also compared and contrasted the solutions of students at different performance levels as measured through —classroom activities, grades, and assessments. This is something few previous studies have done, mostly due to concerns about equity. I tried to avoid this by asking the students to form teams of two or three based on a common interest, not creating the teams myself based on their academic achievement. Later, the classroom teacher classified each team as low-, average-, and high-performance, based on the team members' level of performance in mathematics. Furthermore, studies that involve implementing MEAs, and in particular those I have implemented in this project, have mostly focused on analyzing only the product-solution — i.e., the students' written report, worksheets, and final presentation. However, to better understand, compare, and contrast the students' models, I not only did this, but also examined the process and the patterns of discourse by which the students obtained their models. This involved looking at the students' interactions and utterances in working for a solution. Focusing on the process unveiled the types of discourse — i.e., argumentation or non-argumentation (describing and explaining) — the students used, as well as the moments of disagreements, including the resolving processes of knowledge generation and model description or explanation.

Second, MEAs have proven to be an assessment tool that instructors at any grade level from elementary through undergraduate and graduate levels can use. Specifically, high school students (like those in this study) at any achievement level can be offered an opportunity to demonstrate their true potential in problem solving without underestimating their abilities

(Madon, Jussim, & Eccles, 1997), or predisposing them to relying on memorized concepts or pre-learned mathematics algorithms that can be easily forgotten. In other words, MEAs have the potential to leave no students behind in the learning process.

In addition, MEAs help students develop their collaborative learning skills (Vygotsky, 1978; Piaget, 1926) and improve their self-reflection, communication, and management of criticism or different perspectives. In other words, students are also offered opportunities to externalize their ideas (Lesh et al., 1999), so they can contrast their thoughts with other team members, helping them to reflect on their own and others' thoughts (Hiebert et al., 1997).

Third, MEAs serve as instructional, assessment, and research tools (Stohlmann, Moore, Kim, Park, & Roehrig, 2011), and provide some evidence that they can level the playing field among students with different academic achievement levels—and in particular, for those considered as low performance based on "past situations that emphasized traditional textbooks, teaching, and standardized tests" (Harel & Lesh, 2003). MEAs can therefore be proposed to be part of professional development programs emphasizing the use of modeling in the classroom, for high school teachers in particular.

High school teachers acknowledge the importance of modeling and value what it adds to the curriculum. However, too few teachers have been involved or received any kind of training in modeling. As a result, teachers "feel unsure of how to [incorporate modeling in their classes]" (Hernández, Levy, Felton-Koestler, & Zbiek, 2017, p. 337). This study provides research-proven evidence to encourage administrators and policy-makers, not only to learn more about the benefits of modeling in mathematics, but to support and advocate for mathematics curricula and practices that "highlight the importance and relevance of mathematics in answering important questions [and that] help students gain transferable skills, such as habits of mind that are pervasive across subject matter" (Gaimme, 2016, p. 8).

Many aspects of modeling research still need to be investigated further, particularly for high schools. The research I have presented here is a first step in empowering low-performance students in mathematics, too often sent to remedial classes, and in breaking the stereotype of them as being unable to develop solutions to mathematical activities like MEAs as creatively, adequately, and powerfully as average- or high-performance students. In fact, it also contributes to breaking the stereotype of high-performance students performing outstandingly well.

5.4. Future Approaches and Recommendations

This study took place in a naturalistic setting involving many uncontrollable variables and demanded a great amount of time and effort to collect and code the many hours of data. Consequently, there are still areas I recommend to be developed by further research in different directions, to improve both upon my own research and upon the implementations of MEAs in mathematics education more generally.

In regards to my own research, I recommend first developing the theoretical frameworks for analyzing the discourse types students engage in solving MEAs. Many prominent researchers have studied argumentation and discourse, yet I found their categories inadequate to describe or analyze the discourse that emerged during the MEAs I implemented in this research project. As a result, I had to adapt existing characterizations of discourse to complete my analysis. Although I was able to develop a theory of the discourse patterns, more research is needed in this regard.

Second, more evidence is needed as to whether MEAs can level the playing field among students at different performance levels. Therefore, I would continue my research by analyzing the video recorded groups that were not considered to be transcribed and coded. In particular, one group in each performance level would be ideal to obtain more evidence of the above and to verify the patterns of discourse previously found.

Third, a longitudinal study is needed to verify and confirm the benefits of integrating MEAs at the high school level in the long run, and to determine the benefits of implementing MEAs to improve students' beliefs about and attitudes toward mathematics (i.e., before and after the interventions).

Fourth, I would recommend assessing individual students before and after the MEA implementations, either through interviews or other means, in order to determine the level of individual learning, which could then be correlated with the measure of the group.

Finally, a professional development program needs to be set in place to train pre- and in-service high school teachers in the creation, use, and development of MEAs. I also recommend a research project that documents teachers' use and implementation of MEAs in the classroom in order to help improve their pedagogic knowledge.

5.5. Final Words

I have presented in this research project an open-ended activity that allows students to model a solution in ways that are sometimes unexpected or out-of-the-box. Students at different performance levels developed solutions that were comparable, but at the same time different in scope and sophistication. While the high-performance teams' solutions were elegant and accurate, solutions of the low- and average-performing teams were mathematically rich and creative. In particular, the low-performing teams' group interactions reveal that they engaged in a pattern of argumentation discourse in a way that would enhance their opportunities for learning to a greater degree than their peers.

In the end, more and further research is needed to fully demonstrate that MEAs could indeed *level the playing field* among students of different performance levels.

APPENDICES

APPENDIX A: THE HISTORIC HOTEL

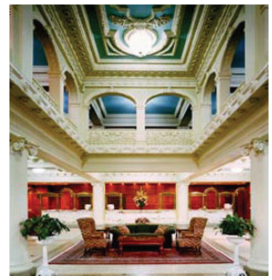
Newspaper Article: An Enchanting Vacation

Going on vacation is something that everyone looks forward to. But staying in a historic hotel transforms any vacation into an enchantment. Finding these charming places is a task to which the National Trust of Historic Hotels of America is committed.

To be recommended by the National Trust of Historic Hotels of America, hotels have to prove that they have faithfully maintained their historic architecture and ambience. Several of these hotels hold great pride in their stories, myths, and legends. For example, the French Lick Springs Resort and Hotel in Indiana was named after an early French outpost and its rich mineral springs that included a naturally produced salt lick (a lick is a deposit of exposed natural salt that is licked by passing animals). The hotel was originally built by Dr. William A Bowles, and it flourished during the mid-nineteenth century.

Whether for health reasons, or just for curiosity, visitors were compelled to visit the rich mineral springs, which were said to possess curative powers.

In 1897, the hotel burned down, and it was not rebuilt until 1902. The new owner, Thomas Taggart, built the new French Lick Springs Resort on the ruins of the original hotel. Mr. Taggart, mayor of Indianapolis, made the resort grow in size and reputation in the early decades of the 20th century. Surrounded by lush gardens and landscaping, the six-story hotel, with its sprawling sitting veranda, was a more than relaxing environment many wished to enjoy. Among the most interesting celebrities that frequented the resort were John Barrymore, Clark Gable, Bing Crosby, the Trumans, the



Reagans, Al Capone and President Franklin Roosevelt. In fact, Roosevelt even locked up the Democratic nomination for president in the hotel's Grand Colonnade Ballroom. Maintenance for a hotel like the French Lick Springs Resort, with all its services, is not an easy task. In 1929, Mr. Taggart died, and left it to his son — the only boy among six children — Thomas D. Taggart. With the Depression, however, the popular French Lick Springs began to decline. World War II brought a monetary revival, but in 1946 young Tom Taggart sold the hotel to a New York syndicate.

Today, French Lick Springs Resort rests on some 2,600 acres in the breathtaking Hoosier National Forest. Newly acquired by Boykin Lodging Company, the resort eagerly embraces a "New Beginning." It provides 470 rooms, full service spa, two golf courses, in-house bowling, a video arcade, indoor tennis center and outdoor courts, swimming, croquet, horseback riding, children's activities, skiing, boating, and fishing. Fine and casual dining are also available at a variety of restaurants. Two main meeting rooms, the Grand Colonnade Ballroom and the Exhibit Center, accommodate large-scale events.

Besides the French Lick Springs Resort, the National Trust of Historic Hotels of America has identified over 140 quality hotels located in 40 states, Canada, and Puerto Rico.

Readiness questions

1. What do hotels have to accomplish in order to be recommended by the National Trust Historic Hotels of America?

2. What are the main features of the French Lick Springs Resort?

3. How many owners has the French Lick Springs Resort had since it opened?

4. What are some responsibilities that a hotel manager might have?

Problem statement

Mr. Frank Graham, from Elkhart District in Indiana, has just inherited a historic hotel. He would like to keep the hotel, but he has little experience in hotel management. The hotel has 80 rooms, and Mr. Graham was told by the previous owner that all of the rooms are occupied when the daily rate is $60 per room. He was also told that for every dollar increase in the daily $60 rate, one less room is rented. So, for example, if he charged $61 per room, only 79 rooms would be occupied. If he charged $62, only 78 rooms would be occupied. Each occupied room has a $4 cost for service and maintenance per day.

Mr. Graham would like to know how much he should charge per room in order to maximize his profit and what his profit would be at that rate. Also, he would like to have a procedure for finding the daily rate that would maximize his profit in the future even if the hotel prices and the maintenance costs change. Write a letter to Mr. Graham telling him what price to charge for the rooms to maximize his profit and include your procedure for him to use in the future.

Problem Statement Questions

1. Who is the client ?

2. What are the client's needs?

3. Why does the client have those needs?

APPENDIX B: THE HYBRID VS. GAS CAR PROBLEM

Benefits and Considerations of Electricity as a Vehicle Fuel

From: http://www.afdc.energy.gov/fuels/electricity_benefits.html

Hybrid and plug-in electric vehicles can help increase energy security, improve fuel economy, lower fuel costs, and reduce emissions.

Energy Security

In 2013, the United States imported about 33% of the petroleum it consumed, and transportation was responsible for nearly three-quarters of total U.S. petroleum consumption. With much of the world's petroleum reserves located in politically volatile countries, the United States is vulnerable to price spikes and supply disruptions.

Using hybrid and plug-in electric vehicles instead of conventional vehicles can help reduce U.S. reliance on imported petroleum and increase energy security. Hybrid electric vehicles (HEVs) typically use less fuel than similar conventional vehicles, because they employ electric-drive technologies to boost efficiency. Plug-in hybrid electric vehicles (PHEVs) and all-electric vehicles (EVs) are both capable of using off-board sources of electricity, and almost all U.S. electricity is produced from domestic coal, nuclear energy, natural gas, and renewable resources.

Fuel Economy

HEVs typically achieve better fuel economy and have lower fuel costs than similar conventional vehicles. For example, the 2012 Honda Civic Hybrid has an EPA combined city-and-highway fuel economy estimate of 44 miles per gallon, while the estimate for the conventional 2012 Civic (four cylinder, automatic) is 32 miles per gallon.

PHEVs and EVs can reduce fuel costs dramatically because of the low cost of electricity relative to conventional fuel. Because they rely in whole or part on electric power, their fuel economy is measured differently than in conventional vehicles. Miles per gallon of gasoline equivalent (mpge) and kilowatt-hours (kWh) per 100 miles are common metrics. Depending on how they're driven, today's light-duty EVs (or PHEVs in electric mode) can exceed 100 mpge and can drive 100 miles consuming only 25-40 kWh.

The fuel economy of medium- and heavy-duty PHEVs and EVs is highly dependent on the load carried and the duty cycle, but in the right applications, they can maintain a strong fuel-cost advantage over their conventional counterparts as well.

Infrastructure Availability

PHEVs and EVs have the benefit of flexible fueling: Since the electric grid is available almost anywhere people park, PEVs can charge overnight at a residence (or a fleet facility), at a workplace, or at public charging stations. PHEVs have added flexibility, because they can also refuel with gasoline or diesel (or possibly other fuels in the future) when necessary.

Public charging stations are not as ubiquitous as gas stations, but charging equipment manufacturers, automakers, utilities, Clean Cities coalitions, municipalities, and government agencies are establishing a rapidly expanding network of charging infrastructure. The number of publicly accessible charging stations surpassed 8,800 in 2014, offering more than 21,000 outlets.

Costs

Although fuel costs for hybrid and plug-in electric vehicles are generally lower than for similar conventional vehicles, purchase prices can be significantly higher. However, prices are likely to decrease as production volumes increase. And initial costs can be offset by fuel cost savings, a federal tax credit, and state incentives. The federal Qualified Plug-In Electric Drive Motor Vehicle Tax Credit is available for PHEV and EV purchases through 2014 (or until manufacturers meet certain thresholds of vehicle sales). It provides a tax credit of $2,500 to $7,500 for new purchases, with the amount determined by the size of the vehicle and capacity of its battery.

Emissions

Hybrid and plug-in electric vehicles can have significant emissions benefits over conventional vehicles. HEV emissions benefits vary by vehicle model and type of hybrid power system. EVs produce zero tailpipe emissions, and PHEVs produce no tailpipe emissions when in all-electric mode.

The life cycle emissions of an EV or PHEV depend on the sources of electricity used to charge it, which vary by region. In geographic areas that use relatively low-polluting energy sources for electricity production, plug-in vehicles typically have a life cycle emissions advantage over similar conventional vehicles running on gasoline or diesel. In regions that depend heavily on conventional fossil fuels for electricity generation, PHEVs and EVs may not demonstrate a strong life cycle emissions benefit.

Batteries

Like the engines in conventional vehicles, the advanced batteries in plug-in electric vehicles are designed for extended life but will wear out eventually. Several manufacturers of plug-in vehicles are offering 8-year/100,000 mile battery warranties. Test and simulation *(PDF)* results from the National Renewable Energy Laboratory indicate that today's batteries may last 12 to 15 years in moderate climates (eight to 12 years in extreme climates).

Readiness questions

The teacher will assess student performance using the provided rubric.

The teacher can also ask the following questions for support:

1. What could increase the energy security of the US?

2. What is the relationship between a car's fuel economy and the price of gas? (As the price of gas increases, the importance of owning a fuel-efficient vehicle increases.)

3. If owning a conventional vehicle is more cost-effective, would you still consider owing a HEV? How does being environmentally conscious factor into your decision?

4. How important is battery life and vehicle emissions when purchasing a HEV?

Griswold, Inc.
555 Cavanaugh Drive
Vacation City, FL

Griswold, Inc.

June 10, 2014

Dear Students,
Griswold Inc. is a rapidly growing medical supply business and we are in need of your help. We are looking to purchase a fleet of ten vehicles for our sales force. Our dilemma is whether we should purchase conventional gasoline-powered vehicles or hybrid electric vehicles (HEV). We would like your help in making this decision.

We have narrowed our search down to two vehicles. I will provide you with the specifications sheet for each located on Data Sheet 1. We are planning on owning the vehicles for five years. Our sales force drives an average of 20,000 miles per year. 75% of those miles are highway miles. The sales team will only be transporting equipment and not clients.

The price of gasoline is a critical factor in your decision. I am also providing average U.S. gasoline prices from 2003 through 2013. You will need to use this data to predict the price of gasoline in five years.

After researching this topic, please draft a letter stating whether you recommend our company purchase a fleet of conventional gasoline-powered vehicles or hybrid electric vehicles. Explain in detail your response, and how it could be used in the future.

We look forward to hearing from you and we appreciate your time.

Sincerely,

Clark W. Griswold
CEO, Griswold, Inc.

Problem Statement Questions

1. Who is the client ?

2. What are the client's needs?

3. Why does the client have those needs?

Historic Gasoline Prices

(http://www.eia.gov/dnav/pet/hist/LeafHandler.ashx?n=pet&s=emm_epm0_pte_nus_dpg&f=a)

Year	U.S. Average Price Per Gallon ($)
2003	1.603
2004	1.895
2005	2.314
2006	2.618
2007	2.843
2008	3.299
2009	2.406
2010	2.835
2011	3.576
2012	3.68
2013	3.575

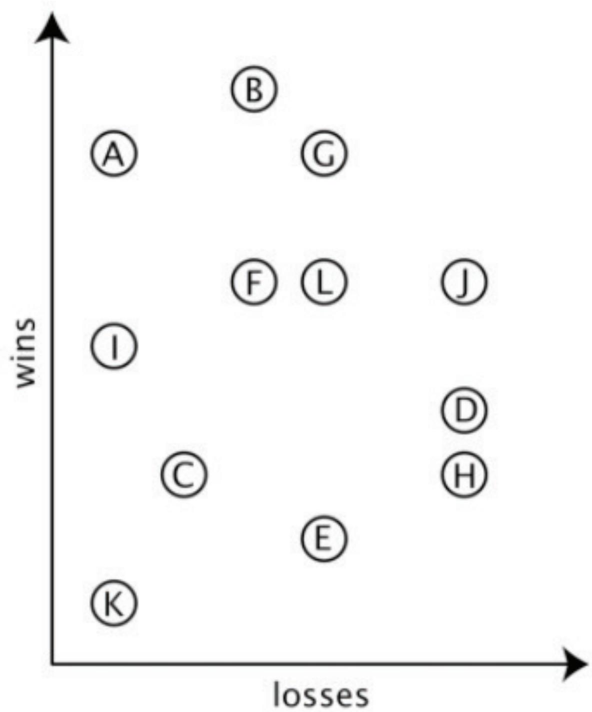
Spec Sheet

Conventional Gas-Powered Vehicle	
Price	$23,235
Fuel Economy (city/hwy)	25/35
Doors	4
Body Style	Sedan
Cylinders	4
Horsepower	178@6000
Torque	170@4100
Transmission	Automatic
Payload capacity	905
Seating capacity	5
Airbags	Standard
Power doors, locks, windows	Standard
Bluetooth	Standard
Powertrain warranty (miles/months)	60,000/60
Basic warranty (miles/months)	36,000/36

Hybrid Electric (plug in) Vehicle	
Price	$35,715
Fuel Economy (city/hwy)	95*/50 (*Electric mode)
Doors	4
Body Style	Hatchback
Cylinders	4
Horsepower	98@5200
Torque	105@4000
Transmission	Automatic
Payload capacity	825
Seating capacity	5
Airbags	Standard
Power doors, locks, windows	Standard
Bluetooth	Standard
Powertrain warranty (miles/months)	60,000/60
Basic warranty (miles/months)	36,000/36

APPENDIX C: THE TEAM RANKING PROBLEM

Each point in the figure below represents the win-loss record of each of the 12 elementary school soccer teams in Flat Mountain School District.



READINESS QUESTIONS

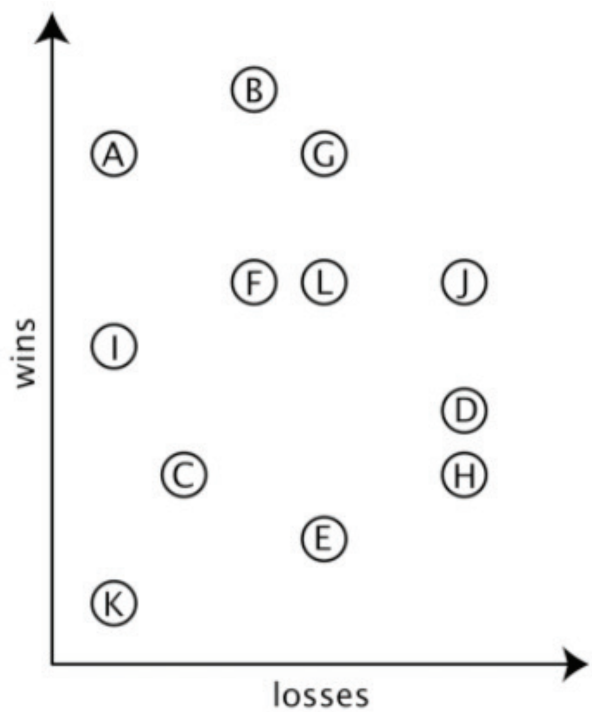
1. Which team(s) won the most games? How do you know?

2. Which team(s) lost the most games? How do you know?

3. Did all teams play the same number of games? How do you know?

THE PROBLEM

Your task is to develop a method for determining which of the 12 teams wins the first, second, third, fourth, and fifth place trophies. Make sure you explain very clearly why the teams should be ranked this way, so that the kids on all the teams will understand that your system is fair.



There are 12 teams in the Flat Mountain School District, but many, many more in the whole state! At the end of the season, the state soccer officials will need to find the top five teams for the state. Write a letter to the soccer officials where you explain how your system works and why it is fair. Make sure the officials will understand how to use your ranking system for any number of teams.

APPENDIX D: TRANSCRIPT CONVENTIONS

The following conventions were used in the transcription of students' interaction (adapted from Hall & Stevens 1995, cited by Azevedo, 2014):

. . .	Ellipses show a pause of less than three seconds
…..	Interaction among students that continue for more than three turns, but not relevant for the analysis.
::	Extended vowel sound (e.g., No::)
(( ))	Authors' comments or description of activity
[	Beginning of overlapping talk
//	Indicates no interval between the end of a prior and start of a next piece of talk unit. Unintelligible talk, with duration indicated if equal to, or greater than one second
Caps	Emphatic talk

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