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WAVELENGTH OF OSCILLATIONS ALONG TRANSMISSION LINES AND ANTENNAS

By

DR. ERNEST M. SIEGEL

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By

DR. ERNEST M. SIEGEL
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The benefits of education and of useful knowledge, generally diffused through a community, are essential to the preservation of a free government.

Sam Houston

Cultivated mind is the guardian genius of Democracy, and while guided and controlled by virtue, the noblest attribute of man. It is the only dictator that freemen acknowledge, and the only security which freemen desire.

Mirabeau B. Lamar

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SUMMARY

It is a well-known experimental fact that the wavelength of an oscillation along an antenna or an open-ended transmission line differs from the wavelength of the same oscillation in free space, but there did not exist a basic scientific explanation of this phenomena until now.

The following paper deals with the above problem from a theoretical point of view and demonstrates that the shortening of the wavelength is caused by a terminal effect, due to the change of inductance and capacitance near the end of a transmission line or the top of an antenna.

The general differential equations for a transmission line of variable characteristic impedance are derived and solved for the first time for the particular case of the antenna and of the terminal zone of a transmission line.

By means of the results obtained, numerical values for the shortening of the first halfwave are computed which are in good agreement with the values known from experience.

The above derivations are of particular practical and theoretical importance for transmitting antenna design.

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I. INTRODUCTION

When Hertz published his reports concerning electrical waves in 1893 based on his famous experiments, he found that the wavelength of an oscillation along a wire in free space was a little shorter than the wavelength $\lambda = \frac{c}{f}$ of the same oscillation in free space. Afterwards he corrected this statement, and influenced by the results of other experiments, stated that the wavelength is the same in both cases. Since that time it has been regarded as an axiom that the distances of the consecutive resonance and antiresonance points from the open end of a transmission line, upon which is impressed a voltage varying sinusoidally in time, are $\frac{\lambda}{4}, \frac{\lambda}{2}, \frac{3\lambda}{4}$ and so on, if only the attenuation constant of the line is small enough. In other words: If a transmission line of certain length l has impressed upon it an oscillation of variable wavelength, the resonance and antiresonance points are supposed to appear at the points $\frac{l}{\lambda} = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$ and so on, where $\lambda = \frac{c}{f}$ means the wavelength in free space. The resonance points $\frac{\lambda}{4}, \frac{3\lambda}{4}$ . . . correspond to voltage nodes and the antiresonance points $\frac{\lambda}{2}, \lambda$ . . . to current nodes.

But when exact tests with antennas and antenna feeder lines have been made during recent years, it has been found that the current and voltage resonance points are usually located at shorter distances from the end of the line or the top of the antenna than expected. It has been found in fact that the actual length of a half-wave antenna has to be about 5 per cent shorter than the length calculated by means of the formula $l = \frac{\lambda}{2} = \frac{1}{2}\frac{c}{f}$. That means ap-

parently that the wavelength of the oscillation along an antenna is smaller than the wavelength of the same oscillation in free space. As the antenna according to all experience can be treated as a special case of the transmission line or is at least nearly related to it, the question arises whether the shift of the resonance points and the corresponding shortening of the wavelength, undoubtedly existing with antennas, is only a property of antennas or a principal property of all transmission lines.

The distribution of current and voltage along a transmission line commonly are determined as solutions of the telegraph or wave equations:

$$\frac{\delta^2 i}{\delta x^2} = LC\frac{\delta^2 i}{\delta t^2} + CR\frac{\delta i}{\delta t}$$

$$\frac{\delta^2 e}{\delta x^2} = LC\frac{\delta^2 e}{\delta t^2} + CR\frac{\delta e}{\delta t}.$$

As long as these equations are valid, the velocity of propagation of an oscillation sinusoidal in time is[1]

$$v = \frac{1}{\sqrt{\frac{LC}{2}\left(\sqrt{1+\left(\frac{R}{\omega L}\right)^2}+1\right)}} = \frac{1}{\sqrt{LC}}\left(1-\frac{1}{8}\left(\frac{R}{\omega L}\right)^2\right)$$

and the wavelength

$$\lambda = \frac{v}{f} = \frac{1}{f\sqrt{LC}}(1-K)$$

The correction factor K stands for $= \frac{1}{8}\left(\frac{R}{\omega L}\right)^2$

As $\frac{1}{\sqrt{LC}} = c$ for a line in free space and K is usually very small we find $\lambda = \frac{c}{f}$. Therefore the wavelength of the oscillation along the line and in free space ought to be nearly

[1]See Pierce, Electric Oscillations and Electric Waves, London, 1920.

identical. They should be entirely equal for a line consisting of wires of infinite conductivity.

When the deviation of the wavelength along an antenna from the wavelength in free space was demonstrated, the writer tried to explain this deviation by virtue of the correction factor K in the above expression for v, assuming that the radiation of an antenna causes an increased attenuation by increasing R and that therefore K is much larger with antennas than with usual transmission lines. But all computations based on this method demonstrated the insufficiency of the theory. The calculated K was always much too small as compared with the measured values.[2] For this reason it proved necessary to look for another explanation, going back to the telegraph equations and determining their validity for the whole extent of a transmission line.

II. THE TELEGRAPH EQUATIONS AND THE TERMINAL ZONE OF TRANSMISSION LINES

Where do the telegraph equations come from? What are the necessary and commonly used assumptions for their derivation?

If we start from the Maxwell equations and assume a two-wire line of negligible resistance, neither the electric nor the magnetic field in the space around the wires have components in the x — direction parallel to the line, and the vector potential a_x in this space is parallel to the line. The electromagnetic field around the wires is determined by the well known relations

$$\frac{c^2}{\epsilon\mu}\frac{\delta a_x}{\delta x} = -\frac{\delta \phi}{\delta t}, \quad -\frac{\delta a_x}{\delta t} = \frac{\delta \phi}{\delta x} \quad \ldots\ldots\ldots (1)$$

ϕ is the scalar potential, a_x the vector potential, both of which are instantaneous values.[3]

It is ordinarily assumed that the vector potential in the space around the wires is connected with the current flowing through the wires by the relation

---

[2]See Siegel and Labus, "Transmitting Antennas," *Hochfrequenztechnik und Elektroakustik,* Munich, 1937, Vol. 49, p. 90.

[3]It may be mentioned that in this article all quantities are expressed in the electromagnetic system of units.

$$a_x = \left(\frac{1}{10^9} 4\mu \ln \frac{\rho_1}{\rho_2}\right) i$$

In this formula ρ_1 and ρ_2 are the perpendicular distances of the reference point from the wires and i is the instantaneous value of the current flowing through the wires at the intersection of the wires with the plane perpendicular to the wires through the reference point. If we proceed to the surface of one wire, we have to substitute $\rho_1 = d$, spacing of the wires, and $\rho_2 = R$, radius of the wires. Therefore

$$a_x = \left(\frac{1}{10^9} 4\mu \ln \frac{d}{R}\right) i = Li \quad \ldots\ldots\ldots\ldots \quad (2)$$

Simultaneously the scalar potential becomes identical with the voltage between the wires,

$$\phi = e.$$

$L = \frac{1}{10^9} 4\mu \ln \frac{d}{R}$ is the inductance of the line in henries per centimeter length. Substituting Equations (2) and (2a) in Equation (1):

$$\frac{c^2}{\epsilon\mu} L \frac{\delta i}{\delta x} = -\frac{\delta e}{\delta t}, \quad -L \frac{\delta i}{\delta t} = \frac{\delta e}{\delta x}$$

Because

$$\frac{c^2}{\epsilon\mu} L = \frac{c^2}{10^9} \times \frac{4 \ln \frac{d}{R}}{\epsilon} = \frac{1}{C}$$

where $C = \frac{10^9}{c^2} \times \frac{\epsilon}{4 \ln \frac{d}{R}}$ is the capacitance of the line in farads per cm length, we find

$$\frac{\delta i}{\delta x} = -C \frac{\delta e}{\delta t}$$

$$-L\frac{\delta i}{\delta t}=\frac{\delta e}{\delta x}$$

which are the telegraph equations for a resistanceless line.

We conclude that if we start from the Maxwell equations there are two basic assumptions necessary for the derivation of the telegraph equations. First, the vector potential, a_x, and the current in the wires, i, are in direct proportionality corresponding to Equation 2. Second, the scalar potential, ϕ, at the surface of the wire, and the voltage, e, between the wires are identical corresponding to Equation (2a).

For further investigations we will make use of the telegraph equations in a somewhat modified form, introducing the characteristic impedance, z_0, of the line. As

$$\frac{1}{\sqrt{LC}}=\frac{c}{\sqrt{\epsilon\mu}},\ \sqrt{\frac{L}{C}}=z_0$$

we find for a transmission line in free space with $\epsilon=\mu=1$

$$z_0=\frac{c}{10^9}4\ln\frac{d}{R}=120\ln\frac{d}{R}\text{ ohms}$$

$$C=\frac{1}{cz_0},\ L=\frac{z_0}{c}$$

Therefore

$$-cz_0\frac{\delta i}{\delta x}=\frac{\delta e}{\delta t} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3a)$$

$$-\frac{z_0}{c}\frac{\delta i}{\delta t}=\frac{\delta e}{\delta x} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3b)$$

Assuming an oscillation varying sinusoidally in time and separating the variables we can write for the steady state

$$\frac{d^2I}{dx^2}=-\frac{\omega^2}{c^2}I \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4a)$$

$$\frac{d^2E}{dx^2}=-\frac{\omega^2}{c^2}E \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4b)$$

The well known solutions of Eq. 4a and 4b are

$$I = A_1 e^{jkx} + A_2 e^{-jkx}$$

$$E = z_0 [- A_1 e^{jkx} + A_2 e^{-jkx}]$$

$$\text{with } k = \frac{\omega}{c} = \frac{2\pi}{\lambda}.$$

In the special case of an open ended line we have to introduce the boundary condition; where $x = 0$, $I = 0$. Therefore

$$I = j I_m \sin kx,\ i = I_m \sin kx \sin \omega t \quad \ldots\ldots\ldots \quad (5)$$

$$E = - z_0 I_m \cos kx,\ e = z_0 I_m \cos kx \cos \omega t \quad \ldots \quad (6)$$

The charge q along the line is connected with the voltage by the relation $Ce = q$ and since $Cz_0 = \frac{1}{c}$,

$$q = \frac{1}{c} I_m \cos kx \cos \omega t$$

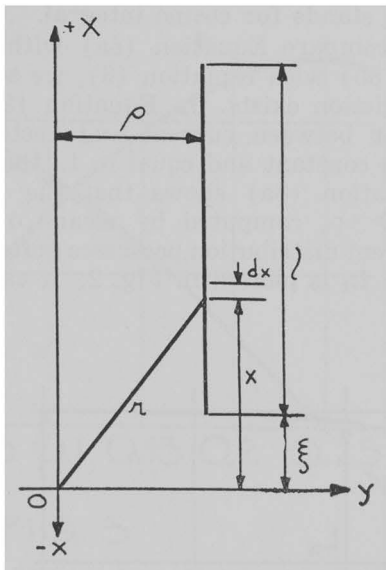
If we compute the vector and scalar potential of the line using this derived sinusoidal current and voltage distribution, and the well known relations (see Fig. 1 for notation)

$$- a_x = 10^{-9} \int_{x=\xi}^{x=l+\xi} \frac{i\left(t - \frac{r}{c}\right)}{r} dx =$$

$$= 10^{-9} \int_{x=\xi}^{x=l+\xi} \frac{I_m \sin k\,(x - \xi) \sin \omega \left(t - \frac{r}{c}\right)}{r} dx \,. \quad (7a)$$

$$\phi = \frac{c^2}{10^9} \int_{x=\xi}^{x=l+\xi} \frac{q\left(t - \frac{r}{c}\right)}{r} dx =$$

$$= \frac{c}{10^9} \int_{x=\xi}^{x=l+\xi} \frac{I_m \cos k\,(x - \xi) \cos \omega \left(t - \frac{r}{c}\right)}{r} dx \,. \quad (7b)$$



FIG·1·

Scalar and vector potential of a straight wire.

we find[4]

$$a_x = (10^{-9}\, p_x)\, I_m \sin k\xi \sin \omega t = L_x\, I_m \sin k\xi \sin \omega t,$$
$$A_x = jL_x\, I_m \sin k\xi \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (8a)$$

$$\phi = (30\, p_x)\, I_m \cos k\xi \cos \omega t = z_x\, I_m \cos k\xi \cos \omega t,$$
$$\Phi = -\, z_x\, I_m \cos k\xi \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (8b)$$

In these relations the factor p_x means

$$p_x = \left\{ 2 \ln \frac{d}{R} + \mathrm{Cik}(\xi + \sqrt{\xi^2 + d^2}) - \mathrm{Cik}(-\xi + \sqrt{\xi^2 + d^2}) - \right.$$
$$\left. - \mathrm{Cik}(\xi + \sqrt{\xi^2 + R^2}) + \mathrm{Cik}(-\xi + \sqrt{\xi^2 + R^2}) \right\} \ldots\ldots \quad (9)$$

[4]See Appendix I.

The symbol Ci stands for cosine integral.

If now we compare Equation (8a) with Equation (2) and Equation (8b) with Equation (6), we see immediately that a contradiction exists. In Equation (2), the proportionality factor between current and vector potential is supposed to be constant and equal to L, the inductance of the line. Equation (8a) shows that the proportionality factor $L_x = 10^{-9} p_x$ computed by means of the assumed sinusoidal current distribution possesses quite another shape than expected. L_x is plotted in Fig. 2. It can be seen that

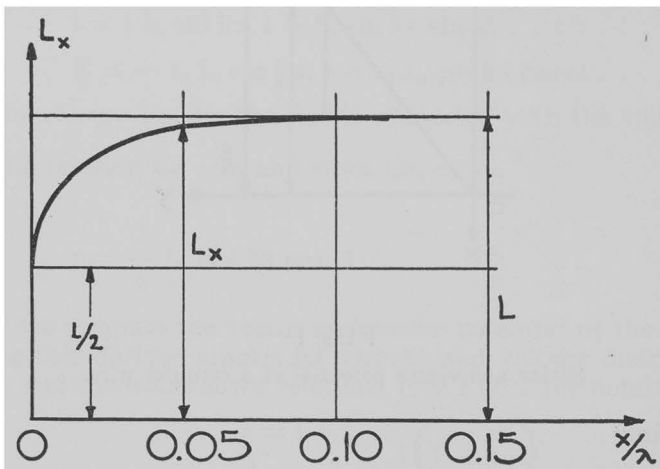


FIG. 2

Proportionality factor L_x near the open end of a transmission line.

L_x is not constant, at least not near the end of the line. L_x decreases to L/2 at the open end. However, it approaches the inductance L with increasing distance from the end of the line.

Comparing Equation (8b) with Equation (6) we see that scalar potential, ϕ, and voltage, e, are not identical either, and differ in exactly the same way as do a_x and i.

Fig. 3 shows the vector and scalar potentials A_x and Φ near the end of a transmission line, calculated by means of Equations (8a) and (8b)—dashed lines—and the solutions for the voltage and current distribution from the telegraph equations—solid lines—corresponding to Equations (5) and (6). They are in clear contradiction.

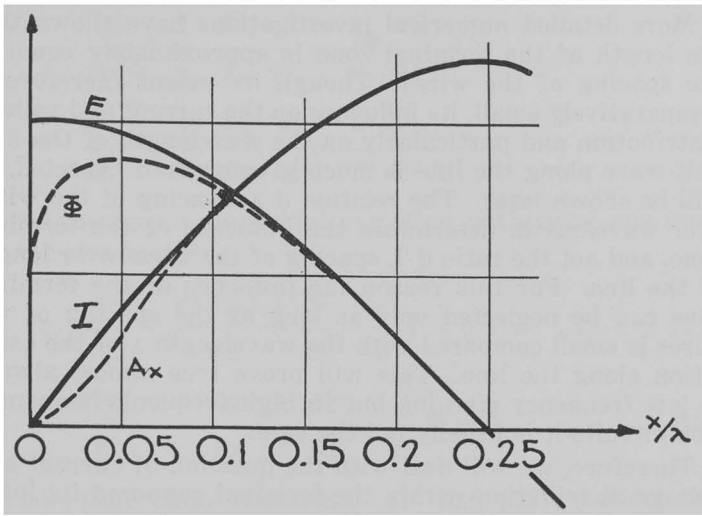


FIG. 3

Scalar and vector potential near the open end of a transmission line, current sinusoidally distributed.

From these results the conclusion can be drawn that the basic assumptions for the derivation of the telegraph equations, that is, proportionality between vector potential and current and identity of scalar potential and voltage, actually are wrong, at least near the end of the line, and that these equations are not valid within a certain "terminal zone" of the line.

However, there can be no doubt that the telegraph equations are valid for the normal zone of the line, that is at comparatively greater distances from the end of the line.

The physical explanation of the contradiction within the terminal zone is very simple: The telegraph equations require for their validity constant inductance, L, capacitance, C, and characteristic impedance, z_0, but these fundamental suppositions do not prove right within the terminal zone. There, inductance, L_x, capacitance, C_x, and characteristic impedance, z_x, are dependent upon the distance from the end. But they always satisfy the relations

$$\frac{1}{\sqrt{L_x C_x}} = c, \; z_x = \sqrt{\frac{L_x}{C_x}}$$

More detailed numerical investigations have shown that the length of the terminal zone is approximately equal to the spacing of the wires. Though its extent therefore is comparatively small, its influence on the current and voltage distribution and particularly on the wavelength of the first half-wave along the line is much greater than expected, as will be shown later. The relation d/λ, spacing of the wires over wavelength, determines the influence of the terminal zone, and not the ratio d/l, spacing of the wires over length of the line. For this reason the influence of the terminal zone can be neglected only as long as the spacing of the wires is small compared with the wavelength λ of the oscillation along the line. This will prove true almost always in low frequency circuits, but in high frequency transmission circuits it is usually not the case.

Therefore, we will deal with the question of current and voltage distribution within the terminal zone and its influence on current and voltage along the normal zone in a more detailed way in the next part of this paper.

III. THE DIFFERENTIAL EQUATIONS OF THE TERMINAL ZONE AND THEIR SOLUTION

Considering the fact that the current and voltage distribution near the end of a transmission line, derived from the telegraph equations, leads to a fundamental contradiction with the assumptions upon which these equations are based, one comes to the conclusion that apparently the telegraph equations must be modified for the terminal zone. Considering that apparently within this zone the characteristic impedance, z_x, cannot be regarded as a constant but must be treated as a function of the distance x from the open end of the line, one may write, instead of the equations (3) and (3a):

$$-cz_x \frac{\delta i}{\delta x} = \frac{\delta e}{\delta t} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (10a)$$

$$-\frac{z_x}{c} \frac{\delta i}{\delta t} = \frac{\delta e}{\delta x} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (10b)$$

or, separating the variables

$$\frac{\delta^2 i}{\delta x^2} + \frac{1}{z_x}\frac{\delta z_x}{\delta x}\frac{\delta i}{\delta x} = \frac{1}{c^2}\frac{\delta^2 i}{\delta t^2}$$

$$\frac{\delta^2 e}{\delta x^2} - \frac{1}{z_x}\frac{\delta z_x}{\delta x}\frac{\delta e}{\delta x} = \frac{1}{c^2}\frac{\delta^2 e}{\delta t^2}$$

If one assumes a sinusoidal variation with time, one finds for the steady state

$$\frac{d^2 I}{dx^2} + \frac{d(\log z_x)}{dx}\frac{dI}{dx} + \frac{\omega^2}{c^2} I = 0 \quad \ldots\ldots\ldots \quad (11)$$

$$\frac{d^2 E}{dx^2} - \frac{d(\log z_x)}{dx}\frac{dE}{dx} + \frac{\omega^2}{c^2} E = 0 \quad \ldots\ldots\ldots \quad (12)$$

These differential equations, (11) and (12), are our new equations valid for the whole extent of the terminal zone. It is remarkable that the two equations for current, I, and voltage, E, are not identical but differ in the sign of the second term.

In accordance with the shape of the curve

$$L_x = \frac{z_x}{c} = 10^{-9}\, p_x$$

which is plotted in Fig. 2, one can assume for the variable characteristic impedance of the terminal zone

$$z_x = \frac{z_0}{2}\left(\frac{x_0 + x}{x_0}\right)^m \quad \ldots\ldots\ldots\ldots\ldots \quad (13)$$

with

$$x_0 = \frac{d}{2^{1/m} - 1}$$

This assumption seems rather daring but it will be seen later that current and voltage along the line, computed by virtue of formula (13), satisfy the condition that scalar and vector potential, calculated by means of them, agree very closely with the current and voltage distribution.

From (13) it will be seen that for $x = 0$ one obtains $z_x = \frac{z_0}{2}$ and for $x = d$, end of the terminal zone, $z_x = z_0$.

Now from equation (13):

$$\log z_x = \log \frac{z_0}{2x_0^m} + m \log (x + x_0), \text{ and } \frac{d(\log z_x)}{dx} = \frac{m}{x + x_0}$$

Substituted in (11) and (12),

$$\frac{d^2I}{dx^2} + \frac{m}{x + x_0}\frac{dI}{dx} + k^2 I = 0 \quad \ldots\ldots\ldots \quad (14)$$

$$\frac{d^2E}{dx^2} - \frac{m}{x + x_0}\frac{dE}{dx} + k^2 E = 0 \quad \ldots\ldots\ldots \quad (15)$$

The solution of these differential equations[5] can be obtained in Bessel functions of the orders $\pm \frac{1 - m}{2}$ and $\pm \frac{-1 - m}{2}$. With $m = \frac{1}{5}$, assumed for the particular case of a line with $\frac{d}{R} = 10$, one finds

$$I = (x + x_0)^{2/5} \{AI_{-2/5}(k(x + x_0)) + BI_{2/5}(k(x + x_0))\}$$

$$E = (x + x_0)^{3/5} \{CI_{-3/5}(k(x + x_0)) + DI_{3/5}(k(x + x_0))\}$$

If we introduce the boundary conditions for an open ended line, $x = 0$, $I = 0$, and consider the connection between I and E according to equation 10a,

$$-cz_x \frac{dI}{dx} = j\omega E$$

we find finally

$$I = N\left(1 + \frac{x}{x_0}\right)^{2/5} \left\{ I_{-2/5}(k(x + x_0)) - \right.$$

$$\left. - \frac{I_{-2/5}(kx_0)}{I_{2/5}(kx_0)} I_{2/5}(k(x + x_0)) \right\} \quad \ldots\ldots\ldots \quad (16)$$

[5]See Appendix II.

$$E = -jN\frac{z_0}{2}\left(1+\frac{x}{x_0}\right)^{3/5}\left\{I_{3/5}(k(x+x_0))+\right.$$

$$\left.+\frac{I_{-2/5}(kx_0)}{I_{2/5}(kx_0)}I_{-3/5}(k(x+x_0))\right\}. \quad \ldots\ldots\ldots \quad (17)$$

$I_{2/5}$, $I_{-2/5}$, $I_{3/5}$, and $I_{-3/5}$ stand for the Bessel functions of order 2/5, — 2/5, 3/5, and — 3/5, respectively, $k(x+x_0)$ and kx_0) are the arguments of these functions.

The charge in the terminal zone is

$$Q = C_x E = \frac{1}{c\frac{z_0}{2}\left(1+\frac{x}{x_0}\right)^m}E =$$

$$=\frac{1}{c}(-jN)\left(1+\frac{x}{x_0}\right)^{2/5}\left\{I_{3/5}(k(x+x_0))+\right.$$

$$\left.+\frac{I_{-2/5}(kx_0)}{I_{2/5}(kx_0)}I_{-3/5}(k(x+x_0))\right\} \quad \ldots\ldots\ldots \quad (18)$$

I, E, and Q, are current, voltage, and charge within the terminal zone, leading into the corresponding values of the normal zone of the line at the point $x = d$. Current and voltage along this normal zone are, as is well known from the theory of the transmission line,

$$I_1 = C_1 \cos kx + j\, C_2 \sin kx$$

$$E_1 = -z_0(C_2 \cos kx + j\, C_1 \sin kx).$$

They are connected with the corresponding values of the terminal zone by the boundary conditions, when $x=d$, $I_1=I$, $E_1=E$. Finally

$$I_1 = 4\,N\{Y_1 \cos k(x-d) - Y_2 \sin k(x-d)\} \quad \ldots\ldots \quad (19)$$

$$E_1 = -j\,4\,z_0\,N\{Y_1 \sin k(x-d) + Y_2 \cos k(x-d)\} \quad (20)$$

Since $Q_1 = C\,E_1$, the charge is

$$Q_1 = -j\frac{1}{c}\,4\,N\{Y_1 \sin k((x-d) + Y_2 \cos k(x-d)\} \quad \ldots \quad (21)$$

which is directly proportional to E_1.

In these equations

$$Y_1 = I_{-2/5}(k(d + x_0)) - \frac{I_{-2/5}(kx_0)}{I_{2/5}(kx_0)} I_{2/5}(k(d + x_0))$$

$$Y_2 = I_{3/5}(k(d + x_0)) + \frac{I_{-2/5}(kx_0)}{I_{2/5}(kx_0)} I_{-3/5}(k(d + x_0))$$

In Fig. 4 curves are drawn showing in solid lines current, voltage, and charge distribution along an open circuited transmission line upon which is impressed a sinusoidal oscillation whose wavelength is ten times longer than the spacing between the wires ($d/\lambda = 1/10$). This means that the terminal zone — marked T Z — is equal to one-tenth of the wavelength. Current, voltage, and charge within the terminal zone are calculated from equations (16), (17), and (18), the corresponding quantities for the normal zone from equations (19), (20), and (21). For comparison the sinusoidal waves are plotted in dashed lines, which follow for voltage and current according to the ordinary solutions of the telegraph equations disregarding the influence of the terminal zone.

Fig. 4 shows this influence very clearly; the accumulation of the charge at the open end, the shortening of the first halfwave and as a consequence of this shortening, the shift of the consecutive resonance and antiresonancepoints toward the open end of the line.

More detailed investigations have shown that an increase of the term d/R has but little influence on the curves plotted in Fig. 4. For this reason the results of the above derivations are valid for transmission lines within a very large range of d/R, as long as the spacing of the wires is not too small. If $d/R, \leqq 5$, the influence of the terminal zone diminishes rapidly.

From Fig. 4, it may be seen that the first resonance point is not located at a distance $x_1 = 0.25\ \lambda$ from the open end of the line but at $x_1 = 0.231\ \lambda$. The second or antiresonancepoint at $x_2 = 0.481\ \lambda$ instead of $x_2 = 0.5\ \lambda$, and so on. The percentage shift of the former is therefore 7.6%, the shift of the latter, 3.8%.

We have still to demonstrate the adequacy of our supposition that according to equation (13),

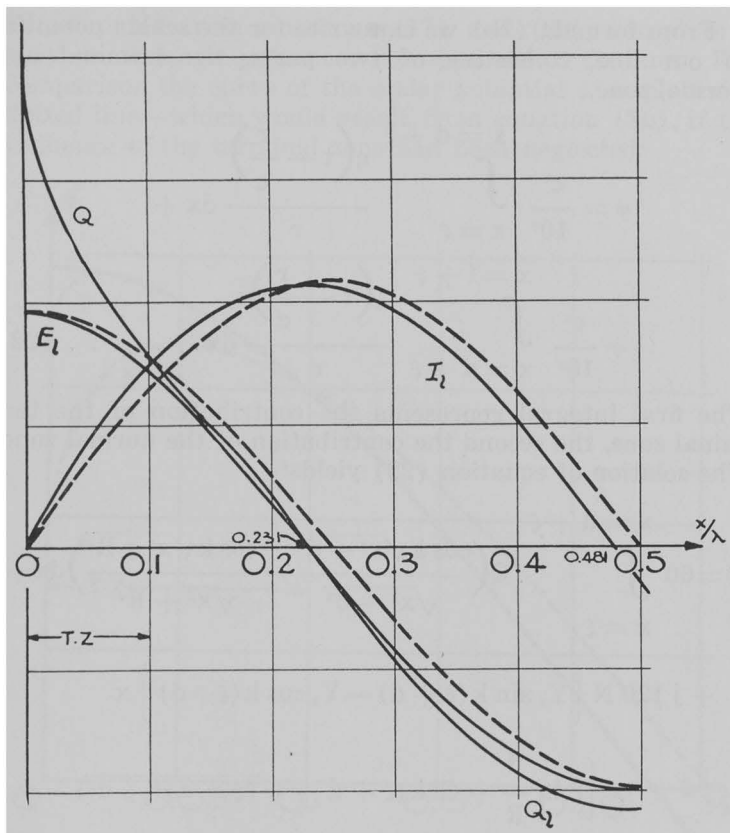


FIG. 4

Actual current, voltage and charge distribution near the end of a transmission line.

$$z_x = \frac{z_0}{2}\left(1+\frac{x}{x_0}\right)^m$$

with m = 1/5 in our particular case. This will be proved if we are able to show that the scalar potential of the line, calculated by virtue of the charge distribution derived by means of equation (13), and according to expressions (18) and (21), is identical with the voltage along the line determined by equations (17) and (20).

From formula (7b) we can write for the scalar potential of our line, consisting of two parts, the terminal and normal zone,

$$\phi = \frac{c^2}{10^9} \int_{x=\xi}^{x=d+\xi} \frac{q\left(t-\frac{r}{c}\right)}{r}\,dx + \frac{c^2}{10^9} \int_{x=d+\xi}^{x=l+\xi} \frac{q_1\left(t-\frac{r}{c}\right)}{r}\,dx \quad \ldots\ldots \quad (22)$$

The first integral represents the contribution of the terminal zone, the second the contribution of the normal zone. The solution of equation (22) yields[6]

$$\Phi = 60 \int_{x=\xi}^{x=d+\xi} Q\left(\frac{\cos k\sqrt{x^2+d^2}}{\sqrt{x^2+d^2}} - \frac{\cos k\sqrt{x^2+R^2}}{\sqrt{x^2+R^2}}\right)dx +$$

$$+ j\,120\,N\left\{Y_1 \sin k(\xi+d) - Y_2 \cos k(\xi+d)\right\} x$$

$$x\left\{2\ln\frac{d}{R} + Ci\,k(\xi+d+\sqrt{(\xi+d)^2+d^2}) - \right.$$

$$- Ci\,k(-\xi-d+\sqrt{(\xi+d)^2+d^2}) -$$

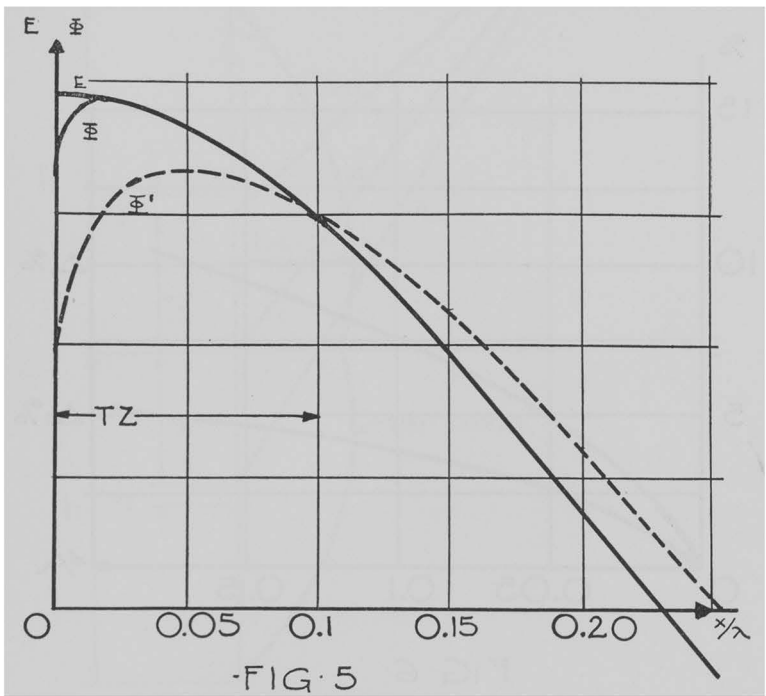
$$- Ci\,k(\xi+d+\sqrt{(\xi+d)^2+R^2}) +$$

$$\left. + Ci\,k(-\xi-d+\sqrt{(\xi+d)^2+R^2})\right\}. \quad \ldots\ldots \quad (23)$$

The writer of this paper has not been able to solve the first integral analytically; therefore, it has been determined graphically. Fig. 5 shows the result of the calculation according to formula (23). The scalar potential Φ determined in this way is in very good accord with the voltage,

[6]See Appendix III.

E, computed from equations (17) and (20), except in a very small range quite near the open end of the line. For comparison the curve of the scalar potential ϕ' is plotted—dotted line—which would result from equation (8b), if the influence of the terminal zone had been neglected.



Voltage and calculated scalar potential—solid lines—and scalar potential resulting from sinusoidal current distribution—dashed line.

Fig. 5 demonstrates that the assumption that the characteristic impedance of a transmission line varies approximately according to equation (13) is at least very near the truth, and in every case much more reliable than the ordinarily used supposition of constant characteristic impedance along the whole of the line, including the terminal zone near the end.

In a similar way it can be verified that the vector potential of the line calculated by means of the current distribution according to equations (16) and (19) is directly proportional to the current.

In Fig. 6 a curve is plotted showing the percentage shift of the first two resonance points $\triangle_1$ and $\triangle_2$ for variable ratio d/λ. The shift is still perceptible (about 1%) even with a ratio $d/\lambda = 0.005$.

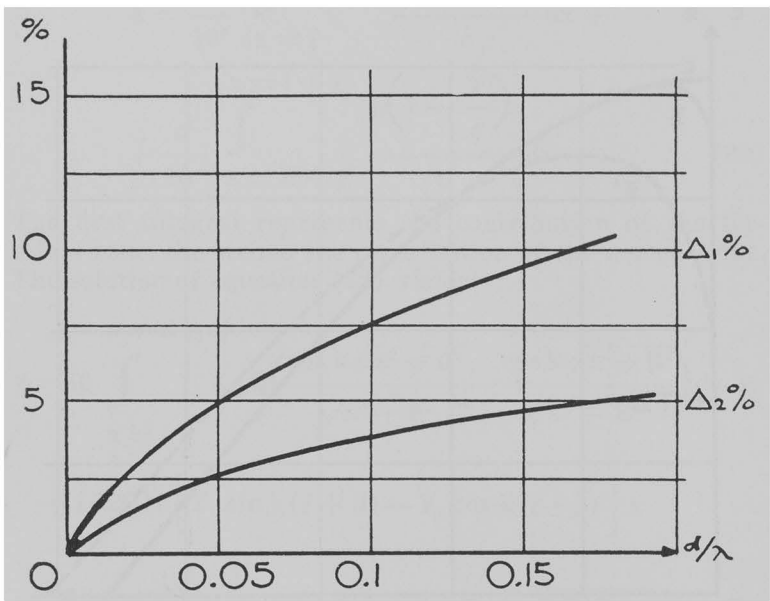


FIG. 6

Percentage shift of the first and second resonance point along a transmission line.

IV. APPLICATION OF THE THEORY TO THE ANTENNA

If we consider an antenna, erected perpendicular to a perfectly conducting earth, and assume that the current is sinusoidally distributed along the antenna, we can compute its vector and scalar potential by substituting for the influence of the ground an image of the antenna (See Fig. 7). Applying the equations (7a) and (7b) and writing

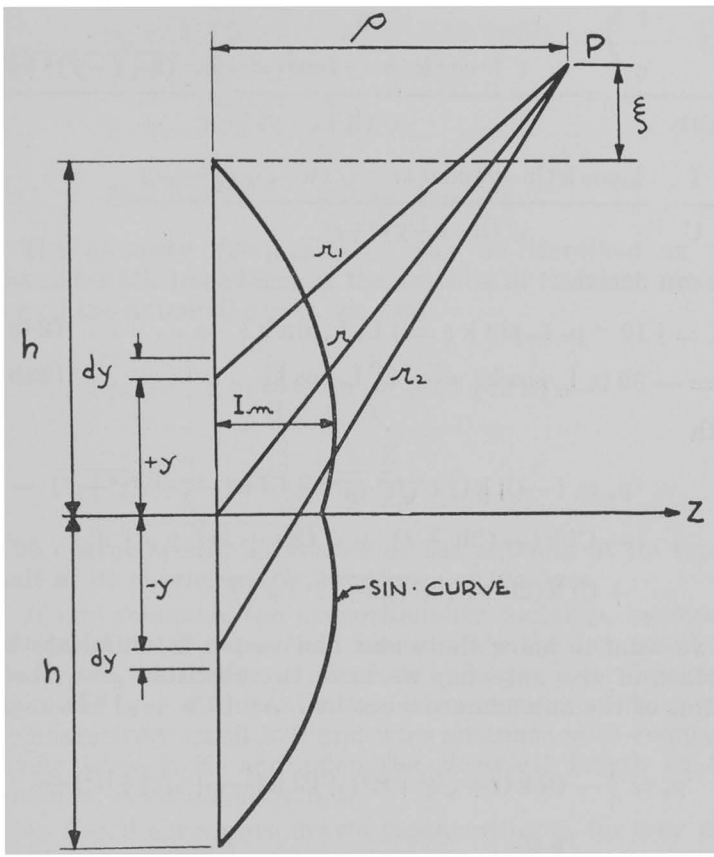


FIG. 7

Antenna with sinusoidal current distribution.

$$\frac{i\left(t-\frac{r}{c}\right)}{r}=\frac{I_m \sin k(h-y)\sin(\omega t-k\sqrt{(h+\xi-y)^2+\rho^2}}{\sqrt{(h+\xi-y)^2+\rho^2}}+$$

$$+\frac{I_m \sin k(h-y)\sin(\omega t-\sqrt{h+\xi+y)^2+\rho^2})}{\sqrt{(h+\xi+y)^2+\rho^2}}$$

$$\frac{q\left(t-\frac{r}{c}\right)}{r} = -\frac{1}{C}\,\frac{I^m \cos k(h-y)\cos(\omega t - k\sqrt{(h+\xi-y)^2+\rho^2}}{\sqrt{(h+\xi-y)^2+\rho^2}} +$$

$$+\frac{1}{C}\cdot\frac{I_m \cos k(h-y)\cos(\omega t-\sqrt{(h+\xi+y)^2+\rho^2})}{\sqrt{(h+\xi+y)^2+\rho^2}}$$

we can derive

$$A_{xa} = j\,10^{-9}\,p_a\,I_m \sin k\,\xi = j\,L_x\,I_m \sin k\,\xi \quad \ldots\ldots \quad (24a)$$

$$\Phi_a = -30\,p_a\,I_m \cos k\xi = -Z_x\,I_m \cos k\xi \quad \ldots\ldots\ldots (24b)$$

with

$$p_a = \{-Ci\,k(\xi+\sqrt{\xi^2+\rho^2}) + Ci\,k(-\xi+\sqrt{\xi^2+\rho^2}) -$$
$$- Ci\,k(-(2h+\xi)+\sqrt{(2h+\xi)^2+\rho^2}) +$$
$$+ Ci\,k(2h+\xi+\sqrt{(2h+\xi)^2+\rho^2})\}$$

If we want to know the scalar and vector potential at the surface of the antenna, we have to substitute $\rho = R =$ radius of the antenna cross section. As $(2h+\rho) \gg R$,

$$p_a = \left\{-Ci\,k(\xi+\sqrt{\xi^2+R^2}) + Ci\,k(-\xi+\sqrt{\xi^2+R^2}) - \right.$$
$$\left. - Ci\,\frac{k\,R^2}{2(2h+\xi)} + Ci\;2k(2h+\xi)\right\}. \quad \ldots\ldots\ldots \quad (25)$$

This expression is not quite exact but can be regarded as a fair approximation within the range of practical interest.

It is evident that the proportionality factor p_a, valid for a grounded single wire antenna, can not be identical with the proportionality factor p_x of a transmission line as can be seen from equation (9), owing to the lack of the second wire in whose place the image of the antenna plays an important role in the derivation.

At the base of the antenna for $\xi = -h$, equation (25) yields

$$p_{ab} = 2\left\{2\ln\frac{h}{R} - \ln\frac{h}{\lambda} + \mathrm{Ci}\,(2kh) - \right.$$

$$\left. - \ln\pi - E\right\} = \frac{1}{30} Z_{0b} \quad \ldots\ldots\ldots\ldots\ldots (25a)$$

$E = 0.5772\ldots$, Eulers constant

The quantity $(30\, p_{ab}) = z_{ob}$ may be identified as the characteristic impedance of the antenna at its base. At the top of the antenna, $\xi = 0$, we find

$$p_{at} = \left\{ 2\ln\frac{h}{R} - \ln\frac{h}{\lambda} + \mathrm{Ci}\,(4k\,h) - \right.$$

$$\left. - \ln\frac{\pi}{2} - E\right\} = \frac{1}{2}\,\frac{Z_{0b}}{30}$$

The characteristic impedance of the antenna at its top is half of its characteristic impedance at the base.

If one computes the proportionality factor p_a by means of equation (25) one finds immediately the important influence of the two quantities h/R and h/λ; p_a is dependent upon the type of the antenna, such as tower antennas with comparatively small h/R and wire antennas with comparatively large h/R, and upon the electrical length of the antenna, determined by h/λ.

In Fig. 8 curves are drawn representing p_a for four particular cases: Quarter-wave tower and quarter-wave wire in solid lines, 1 and 2 respectively, and half-wave tower and half-wave wire in dashed lines 1a and 2a respectively. p_a is expressed in per cent of the characteristic impedance at the base z_{ob} of each type respectively. From Fig. 8 we see that conditions along an antenna of any type are very similar to the conditions along the terminal zone of a transmission line. Vector and scalar potential calculated by virtue of the assumed sinusodial current distribution are in clear contradiction with this assumption. Therefore, the sinusoidal current distribution is physically impossible and must be modified, which leads to the conclusion that the characteristic impedance of an antenna is not constant

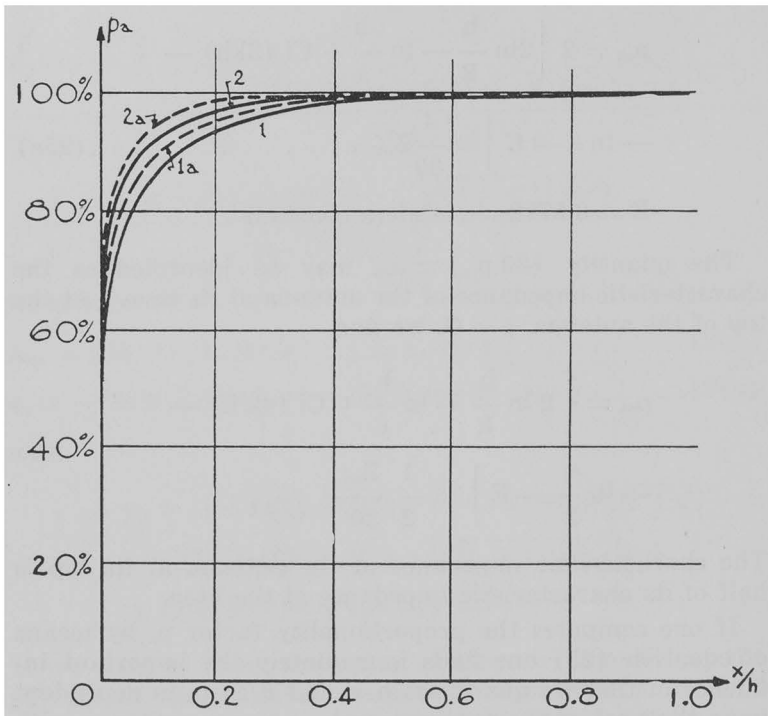


FIG. 8

Antenna proportionality factor for half and quarter-wave tower and wire antenna.

along the antenna, even with antennas of constant cross section.

The differential equations describing the distribution of sinusoidal oscillations along antennas are therefore the same as for the terminal zone of transmission lines. We will repeat these equations in their form for the steady state:

$$\frac{d^2 I}{dx^2} + \frac{d(\log z_x)}{dx}\frac{dI}{dx} + \frac{\omega^2}{c^2} I = 0 \quad \ldots\ldots \quad (11)$$

$$\frac{d^2 E}{dx^2} - \frac{d(\log z_x)}{dx}\frac{dE}{dx} + \frac{\omega^2}{c^2} E = 0 \quad \ldots\ldots \quad (12)$$

As shown in the above derivations we can solve these equations by means of a suitable supposition for the characteristic impedance z_x. We shall assume again

$$z_x = \frac{z_{0b}}{2}\left(1 + \frac{x}{x_0}\right)^m \quad \ldots\ldots\ldots\ldots\ldots \quad (26)$$

with

$$x_0 = \frac{h}{2^{1/m} - 1}$$

and z_{ob} as the characteristic impedance of the antenna at its base. With regard to the antenna proportionality factor p_a as compared to the factor p_x for the terminal zone of a transmission line we have to select different values of the power m. Detailed investigations and numerical computations have shown that m is dependent on the geometrical configuration of the antenna. We can assume for average tower antennas of about 200 up to 240 ohms characteristic impedance $m = 1/9$. For wire antennas of about 500 ohms characteristic impedance $m = 1/12$. The quantity h/λ has no essential influence on the size of m. So the same m can be used for the same antenna type no matter what its electrical length.

We can compute the average characteristic impedance of an antenna z_o according to

$$z_0 = \frac{1}{h}\int_0^h z_x\, dh = \frac{1}{h}\cdot\frac{z_{0b}}{2}\int_0^h \left[1 + (2^{1/m} - 1)\frac{x}{h}\right]^m dx =$$

$$= z_{0b}\frac{(2^{(1/m+1)} - 1)}{2(1+m)(2^{1/m} - 1)} = \frac{z_{0b}}{1+m}$$

Substituting the expression for z_{ob} from equation (25_a) into this formula we find

$$z_0 = 60\left\{\ln\frac{h}{R} - \frac{1}{2}\ln\frac{h}{\lambda} + \frac{1}{2}\,\mathrm{Ci}(2k\pi) - \right.$$

$$\left. - 0.861\right\}\frac{1}{1+m}\ \text{ohms} \quad \ldots\ldots\ldots\ldots \quad (27)$$

This formula for the characteristic impedance of an antenna is in very good numerical coincidence with the formula

$$z_0 = 60 \left\{ \ln \frac{h}{R} - \frac{1}{2} \ln \frac{2h}{\lambda} - 1 \right\}$$

derived in an earlier publication of the writer of this paper in a different way.[7]

The solution of our differential equations (11) and (12) are—according to the results of the third part and appendix II of this article—for a tower antenna with $m = 1/9$

$$I = N\left(1 + \frac{x}{x_0}\right)^{4/9} \left\{ I_{-4/9}(k(x + x_0)) - \right.$$

$$\left. - \frac{I_{-4/9}(k\,x_0)}{I_{4/9}(k\,x_0)} I_{4/9}\,(k(x + x_0)) \right\}$$

$$E = - j \frac{z_{0b}}{2} N\left(1 + \frac{x}{x_0}\right)^{5/9} \left\{ I_{5/9}(k(x + x_0)) + \right.$$

$$\left. + \frac{I_{-4/9}(k\,x_0)}{I_{4/9}(k\,x_0)} I_{-5/9}(k(x + x_0)) \right\}$$

$I_{4/9}$, $I_{-4/9}$, $I_{5/9}$, and $I_{-5/9}$ are the Bessell functions of the orders 4/9, — 4/9, 5/9, and — 5/9. For an average wire antenna the corresponding solutions for I and E contain the Bessel functions $I_{11/24}$, $I_{-11/24}$, $I_{13/24}$, and $I_{-13/24}$.

The charge along the antenna is

$$Q = C_x E_x = - j N\left(1 + \frac{x}{x_0}\right)^{4/9} \left\{ I_{5/9}(k(x + x_0)) + \right.$$

$$\left. + \frac{I_{-4/9}(k\,x_0)}{I_{4/9}(k\,x_0)} I_{-5/9}(k(x + x_0)) \right\}$$

In Fig. 9 curves are plotted showing current, I, voltage, E, and charge, Q, computed from equations (28), (29), and (30) respectively—solid lines—along an antenna tower of

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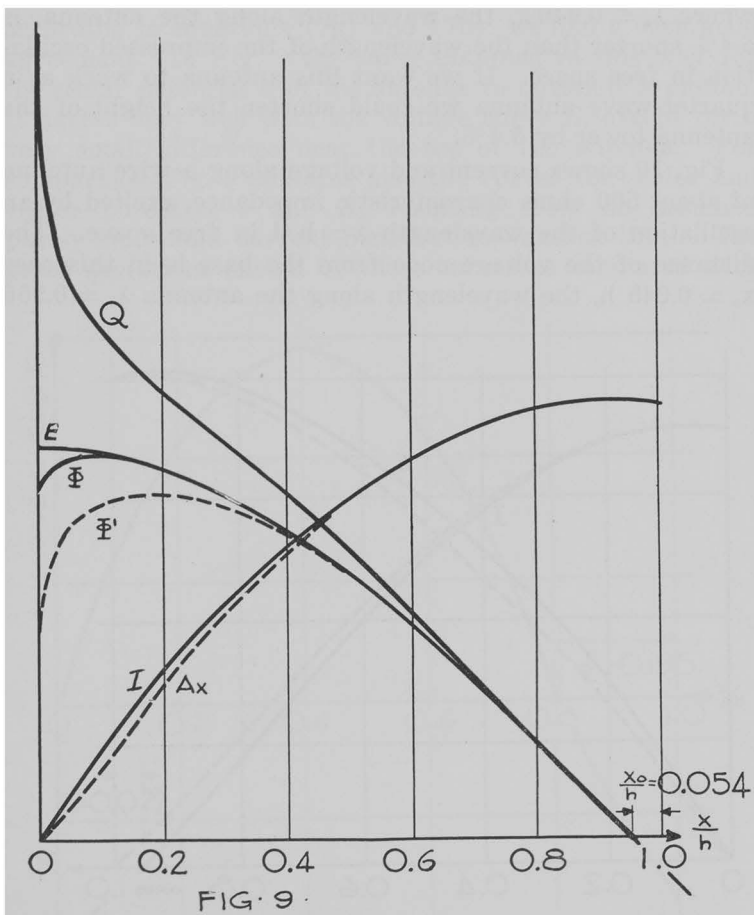
[7]Impedance of Antennas, Siegel and Labus, *Hochfrequenztechnik und Elektroakustik*, Munich, 1934, Vol. 43, p. 167.

215 ohms characteristic impedance upon which is impressed an oscillation whose wavelength in free space is four times the antenna height. The curves demonstrate very clearly that the electrical length of the antenna is more than a quarter wave. The voltage node is located not at the base of the antenna but at a distance $x_o = 0.054$ h above



Current, voltage and charge distribution and calculated scalar potential Φ along a quarter wave tower antenna—solid lines, scalar and vector potential derived from sinusoidal current distribution—dashed lines.

the antenna base. Apparently this antenna does not work as a quarter-wave antenna where $h/x = 1/4$ but as an antenna of the electrical length

$$\frac{h}{\lambda_a} = \frac{1}{0.946} \times \frac{1}{4} = 0.264 > 0.25,$$

where $\lambda_a = 0.946\,\lambda$, the wavelength along the antenna, is 5.4% shorter than the wavelength of the impressed oscillation in free space. If we want this antenna to work as a quarter-wave antenna we could shorten the height of the antenna tower by 5.4%.

Fig. 10 shows current and voltage along a wire antenna of about 500 ohms characteristic impedance, excited by an oscillation of the wavelength $\lambda = h/4$ in free space. The distance of the voltage node from the base is in this case $x_o = 0.045$ h, the wavelength along the antenna $\lambda_a = 0.955$

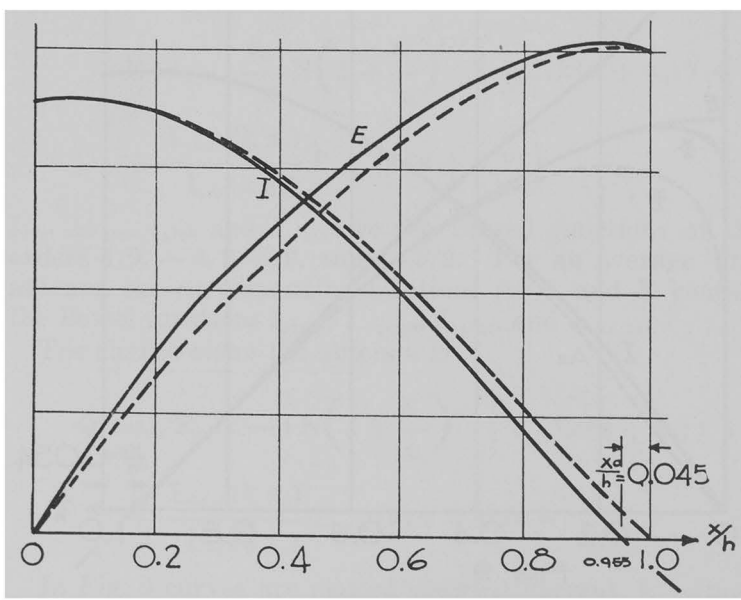


FIG. 10

Actual current and voltage distribution along a quarter-wave wire antenna, solid lines. Current and voltage sinusoidally distributed, dashed lines.

λ is shortened 4.5%. For comparison the sinusoidal current and voltage are drawn in Fig. 10—dashed curves—computed without regard to the shortening of the wavelength.

If we calculate by virtue of the current, voltage, and charge distribution determined by equations (28), (29), and (30), the scalar and vector potential of the antenna according to equations (7a) and (7b), we find a very good agreement. In Fig. 9 the curve obtained in this way for the scalar potential Φ of the antenna in question is plotted. It agrees very well with the voltage curve E, except for a very small difference near the top of the antenna. For comparison Fig. 9 contains also the curves for scalar and vector potential Φ and A_x resulting from an assumed purely sinusoidal current distribution—dashed lines. The great deviations shown by these curves as compared with

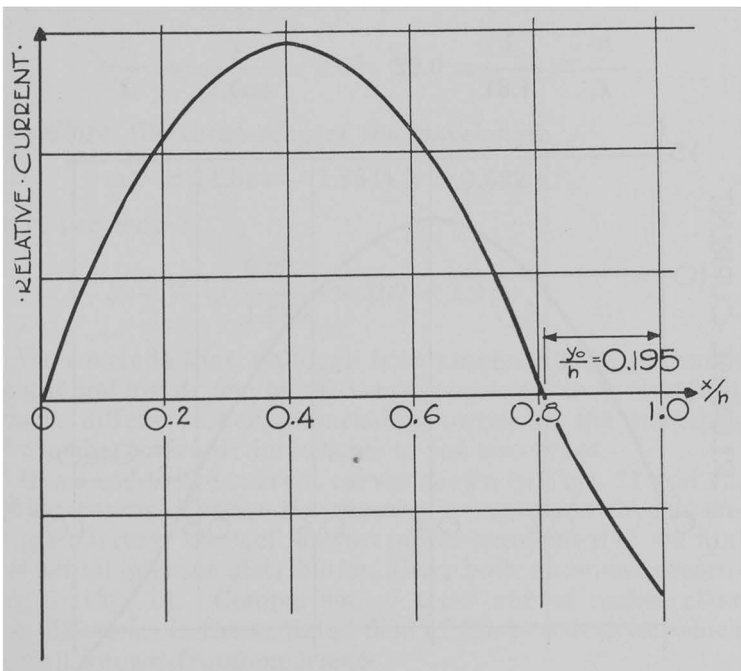


FIG. 11

Tower antenna $h/\lambda = 0.59$, current distribution.

the I and E curves and the improvement due to the method applied in this paper is evident. They prove the necessity of considering the variation of the characteristic impedance along an antenna.

In Figs. 11 and 12 curves are drawn, for a tower antenna of about 215 ohms characteristic impedance, and a wire antenna of about 500 ohms characteristic impedance, upon which are impressed oscillations of the wavelength $\lambda = 1.696$ h. This value corresponds to the electrical height of a modern broadcast antenna with reduced radiation at high angles with $h/\lambda = 0.59$. Fig. 11 shows the current distribution along the tower, Fig. 12 along the wire. We find for the former case the current node at a distance $y_o = 0.195$ h from the base. The wavelength along the antenna is therefore

$$\lambda_a = (1 - 0.195)\ 2\,h = 1.61\ h$$

and the actual electrical length of the tower

$$\frac{h}{\lambda_a} = \frac{1}{1.61} = 0.62$$

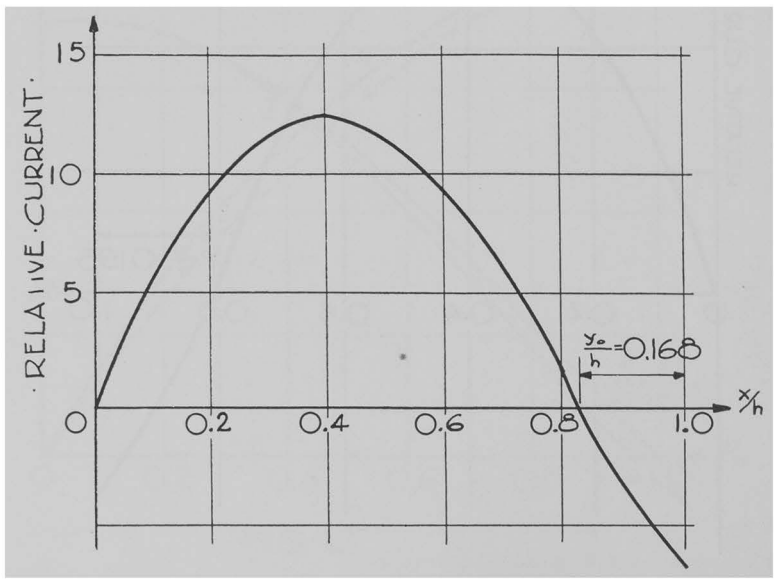


FIG. 12.
Wire antenna $h/\lambda = 0.59$, current distribution.

As the wavelength in free space is 1.696h, the shortening of the wavelength along the tower antenna follows.

$$\triangle \lambda = (1.696 - 1.61)\ h = 0.086\ h$$

Expressed in per cent,

$$\triangle \lambda\ \% = \frac{0.086}{1.696} \times 100 = 5.1\%$$

From Fig. 12 we find in the same way for the wire antenna of the same height, $h = 0.59\ \lambda$, the distance of the current node from the base

$$y_o' = 0.168\ h,$$

and the wavelength of the oscillation along the wire

$$\lambda_a' = (1 - 0.168)\ 2\ h = 1.664\ h,$$

and finally the electrical length of the wire antenna

$$\frac{h}{\lambda_a'} = \frac{1}{1.664} = 0.60$$

Therefore, the shortening of the wavelength

$$\triangle\lambda^1 = (1.696 - 1.664)\ h = 0.032\ h$$

or in per cent

$$\triangle \lambda'\ \% = \frac{0.032}{1.696} \times 100 = 1.9\%$$

We conclude that, although both antennas have the same height and are driven by the same frequency, their electrical length differs rather remarkably, owing to the influence of the characteristic impedance of the two types.

If we add to the current curves shown in Figs. 11 and 12, which represent only the reactive components of the antenna current, the well known power components,[8] we find the actual current distribution along both antennas according to Fig. 13. Comparison of these curves makes clear the difference in the radiated field of the two devices, which is well known from experience.

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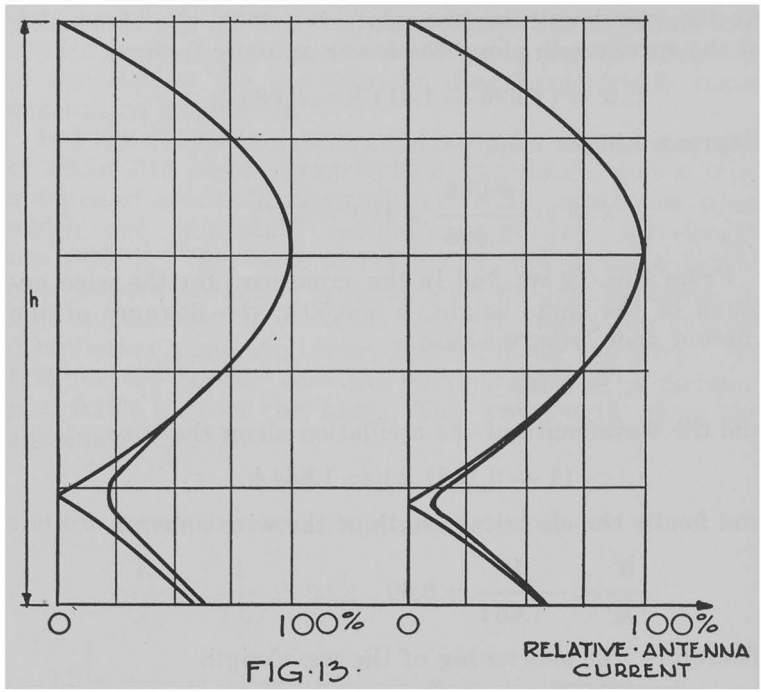
[8]Transmitting Antennas, Siegel and Labus, *Hochfrequenztechnik und Elektroakustik*, Munich, 1937, Vol. 49, p. 87.



Tower and wire antenna $h/\lambda = 0.59$: Current distribution considering the current power component.

To sum up the results of the above calculations together with more detailed investigations, the following table has been computed, showing the numerical values of the shortening of the wavelength for average tower and wire antennas of different electrical height.

h/λ	Tower antenna $z_0 = 220$ ohms	Wire antenna $z_0 = 500$ ohms
¼	5.4%	4.5%
⅜	5.3%	4.3%
½	5.2%	2.2%
0.59	5.1%	1.9%

V. CONCLUSION

From the above investigations the conclusion can be drawn that the velocity of propagation of electromagnetic waves along transmission lines in free space and along antennas is always equal to the velocity of light or at least very close to it. But the decrease of the characteristic impedance near the end of a line or near the top of an antenna is followed by an accumulation of charge within this zone and a very rapid increase of the current, exceeding the increase of a sinusoidal wave. It causes, therefore, a shortening of the first halfwave of the impressed oscillation. As a consequence all the consecutive current and voltage resonance and antiresonance points along the line or antenna are shifted in the direction towards the end of the line or the top of the antenna.

With a transmission line, the size of the terminal effect depends on the spacing of the wires, it disappears if the wires are very closely spaced. With antennas the terminal effect depends to a certain extent on the characteristic impedance z_o of the antenna, determined by its cross section. The smaller z_0 or the larger the cross section the greater the shift of the resonance points, as is very well known from experience.

As the current and voltage distribution along an antenna, computed in this article, is much too complicated for general use in antenna radiation calculations, it seems to be reasonable to calculate with sinusoidal distribution but to consider the terminal effect by virtue of the introduction of the wavelength along the antenna λ_a which is smaller than the wavelength λ of the same oscillation in free space. The size of the shortening can be taken from the table reproduced at the end of part IV of this paper. The deviation of the current distribution according to Equation (28) from a sinusoidal distribution of reduced wavelength $\lambda_a < \lambda$ is not very important.

It is possible without great difficulty to introduce into the differential equations (11) and (12), which are valid only for a resistanceless line and a non-radiating antenna, the influence of the radiation. This leads to Bessel functions with complex arguments, corresponding to the

reactive and resistive component of the current. But as the assumption of attenuated sinusoidal waves for these current components is in very good agreement with experience, if one considers the shortening of the wavelength, this method seems to be unnecessarily complicated for computation purposes.

The above derivations are valid only with antennas of constant cross section. Since the variation of the cross section causes an effect very similar to the terminal effect treated in this paper, it seems that it should be possible to consider the influence of the variable cross section on current and voltage distribution along an antenna in a similar way, and more exactly than hitherto.[9] [10]

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[9]"Current and Voltage Distribution along Tower Antennas," Siegel, *Hochfrequenztechnik und Elektroakustik*, Munich, 1936, Vol. 48.

[10]"Computation of Cantilever Tower Antennas," Siegel and Wiechowski, *Hochfrequenztechnik und Elektroakustik*, Munich, 1938, Vol. 51.

APPENDIX I

Scalar and Vector Potential of a Two-Wire Line Having Sinusoidal Current Distribution

If the current along a wire corresponding to Fig. 14 is distributed sinusoidally with a node at $x = \xi$ (end of the wire), we can write:

$$i = I_m \sin 2\pi \frac{x - \xi}{\lambda} \sin \omega t = I_m \sin k(x - \xi) \sin \omega t$$

$$e = z_0 I_m \cos k(x - \xi) \cos \omega t$$

$$q = Ce = \frac{1}{c z_0} e = \frac{1}{c} I_m \cos k(x - \xi) \cos \omega t$$

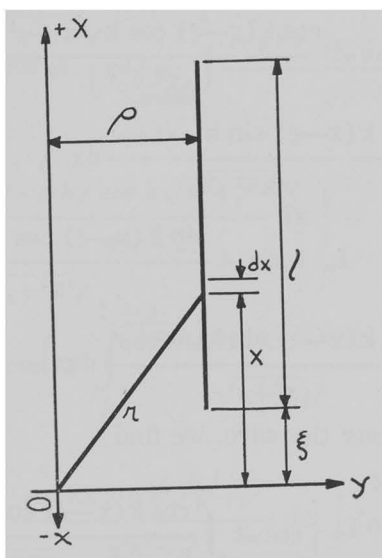


FIG. 14

Notations to Appendix I

The contributions of an element dx to the scalar and vector potential are

$$d\phi = \frac{c^2}{10^9} \cdot \frac{q\,dx}{r} = \frac{c}{10^9} \cdot \frac{I_m \cos k(x-\xi) \cos \omega \left(t - \frac{r}{c}\right)}{r} dx =$$

$$= 30 \frac{I_m \cos k(x-\xi) \cos(\omega t - k\sqrt{x^2+\rho^2})}{\sqrt{x^2+\rho^2}} dx.$$

$$-da_x = 10^{-9} \frac{i\,dx}{r} = 10^{-9} \frac{I_m \sin k(x-\xi) \sin \omega \left(t - \frac{r}{c}\right)}{r} dx =$$

$$= 10^{-9} \frac{I_m \sin k(x-\xi) \sin(\omega t - k\sqrt{x^2+\rho^2}}{\sqrt{x^2+\rho^2}} dx$$

or

$$d\phi = 30\, I_m \left\{ \cos \omega t \frac{\cos k(x-\xi) \cos k\sqrt{x^2+\rho^2}}{\sqrt{x^2+\rho^2}} dx + \right.$$

$$\left. + \sin \omega t \frac{\cos k(x-\xi) \sin k\sqrt{x^2+\rho^2}}{\sqrt{x^2+\rho^2}} dx \right\}$$

$$-d\,a_x = 10^{-9}\, I_m \left\{ \sin \omega t \frac{\sin k(x-\xi) \cos k\sqrt{x^2+\rho^2}}{\sqrt{x^2+\rho^2}} dx - \right.$$

$$\left. - \cos \omega t \frac{\sin k(x-\xi) \sin k\sqrt{x^2+\rho^2}}{\sqrt{x^2+\rho^2}} dx \right\}$$

Integrating along the wire, we find

$$\phi = \int_{x=\xi}^{x=l+\xi} d\rho = 30\, I_m \left\{ \cos \omega t \int_{\xi}^{l+\xi} \frac{\cos k(x-\xi) \cos k\sqrt{x^2+\rho^2}}{\sqrt{x^2+\rho^2}} dx + \right.$$

$$\left. + \sin \omega t \int_{\xi}^{l+\xi} \frac{\cos k(x-\xi) \sin k\sqrt{x^2+\rho^2}}{\sqrt{x^2+\rho^2}} dx \right\}$$

$$-a_x = \int_{x=\xi}^{x=l+\xi} -d\,a_x =$$

$$= 10^{-9}\, I_m \left\{ \sin \omega t \int_{\xi}^{l+\xi} \frac{\sin k(x-\xi)\, \cos k\sqrt{x^2+\rho^2}}{\sqrt{x^2+\rho^2}}\, dx - \right.$$

$$\left. - \cos \omega t \int_{\xi}^{l+\xi} \frac{\sin k(x-\xi)\, \sin k\sqrt{x^2+\rho^2}}{\sqrt{x^2+\rho^2}}\, dx \right\}$$

and finally,

$$\phi = 30\, I_m \left\{ \cos \omega t \left[\cos k\xi \int_{\xi}^{l+\xi} \frac{\cos kx \cos k\sqrt{x^2+\rho^2}}{\sqrt{x^2+\rho^2}}\, dx + \right.\right.$$

$$\left. + \sin k\xi \int_{\xi}^{l+\xi} \frac{\sin kx \cos k\sqrt{x^2+\rho^2}}{\sqrt{x^2+\rho^2}}\, dx \right] +$$

$$+ \sin \omega t \left[\cos k\xi \int_{\xi}^{l+\xi} \frac{\cos kx \sin k\sqrt{x^2+\rho^2}}{\sqrt{x^2+\rho^2}}\, dx + \right.$$

$$\left.\left. + \sin k\xi \int_{\xi}^{l+\xi} \frac{\sin kx \sin k\sqrt{x^2+\rho^2}}{\sqrt{x^2+\rho^2}}\, dx \right] \right\}$$

$$-a^x = 10^{-9}\, I_m \left\{ \sin \omega t \left[\cos k\xi \int_{\xi}^{l+\xi} \frac{\sin kx \cos k\sqrt{x^2+\rho^2}}{\sqrt{x^2+\rho^2}}\, dx - \right.\right.$$

$$- \sin k\xi \int_{\xi}^{l+\xi} \frac{\cos kx \cos k\sqrt{x^2+\rho^2}}{\sqrt{x^2+\rho^2}}\, dx \Big] -$$

$$- \cos \omega t \Big[\cos k\xi \int_{\xi}^{l+\xi} \frac{\sin kx \sin k\sqrt{x^2+\rho^2}}{\sqrt{x^2+\rho^2}}\, dx -$$

$$- \sin k\xi \int_{\xi}^{l+\xi} \frac{\cos kx \sin k\sqrt{x^2+\rho^2}}{\sqrt{x^2+\rho^2}}\, dx \Big] \Big\}$$

Detailed investigation shows that the contribution of all these integrals is zero or negligible, except those containing the term $\cos kx \cos k\sqrt{x^2+\rho^2}$. Therefore, we find:

$$\phi = 30\, p_x\, I_m \cos k\xi \cos \omega t = z_x\, I_m \cos k\xi \cos \omega t$$

$$a_x = 10^{-9}\, p_x\, I_m \sin k\xi \sin \omega t = L_x\, I_m \sin k\xi \sin \omega t$$

with

$$p_x = \int_{\xi}^{l+\xi} \frac{\cos kx \cos k\sqrt{x^2+\rho^2}}{\sqrt{x^2+\rho^2}}\, dx =$$

$$= \frac{1}{2} \int_{\xi}^{l+\xi} \frac{\cos k(x+\sqrt{x^2+\rho^2})}{\sqrt{x^2+\rho^2}}\, dx +$$

$$+ \frac{1}{2} \int_{\xi}^{l+\xi} \frac{\cos k(-x+\sqrt{x^2+\rho^2})}{\sqrt{x^2+\rho^2}}\, dx =$$

$$= \frac{1}{2} \Big\{ \mathrm{Ci}\, k(l+\xi+\sqrt{(l+\xi)^2+\rho^2}) -$$

$$- \operatorname{Ci} k(-l-\xi+\sqrt{(l+\xi)^2+\rho^2}) -$$

$$- \operatorname{Ci} k(\xi+\sqrt{\xi^2+\rho^2}) + \operatorname{Ci} k(-\xi+\sqrt{\xi^2+\rho^2}) \Big\}$$

In this expression the symbol Ci stands for the cosine integral.

If we assume a two-wire line and calculate the potentials at the surface of one of the wires

$$\begin{aligned} p_x = \Big\{ & - \operatorname{Ci} k(l+\xi+\sqrt{(l+\xi)^2+d^2}) + \\ & + \operatorname{Ci} k(-l-\xi+\sqrt{(l+\xi)^2+d^2}) + \\ & + \operatorname{Ci} k(\xi+\sqrt{\xi^2+d^2}) - \operatorname{Ci} k(-\xi+\sqrt{\xi^2+d^2}) - \\ & + \operatorname{Ci} k(l+\xi+\sqrt{(l+\xi^2)+R^2}) - \\ & - \operatorname{Ci} k(-l-\xi+\sqrt{\xi(l+\xi^2)+R^2}) - \\ & - \operatorname{Ci} k(\xi+\sqrt{\xi^2+R^2}) + \operatorname{Ci} k(-\xi+\sqrt{\xi^2+R^2}) \Big\}, \end{aligned}$$

where d is the spacing of the wires and R their radius.

Since within the zone near the end of the wire $(\xi+l) >> d$, $(\xi+l) >> R$:

$$\begin{aligned} p_x = \Big\{ & + 2 \ln \frac{d}{R} + \operatorname{Ci} k(\xi+\sqrt{\xi^2+d^2}) - \\ & - \operatorname{Ci} k(-\xi+\sqrt{\xi^2+d^2}) - \operatorname{Ci} k(\xi+\sqrt{\xi^2+R^2}) + \\ & + \operatorname{Ci} k(-\xi+\sqrt{\xi^2+R^2}) \Big\} \end{aligned}$$

corresponding to formula (9) of the text.

APPENDIX II

Solution of the Differential Equations (14) and (15)

To solve the differential equation (14) we put (see Bessel Functions for Engineers, N. W. McLachlan, Oxford 1934, IV. 13)

$$y = x + x_0 :$$

$$\frac{d^2 I}{dy^2} + \frac{m}{y}\frac{d I}{dy} + k^2 I = 0$$

and substitute $I = \frac{z}{y^{-\nu}}$. Then

$$\frac{d^2 z}{dy^2} + (m - 2\nu)\frac{1}{y}\frac{dz}{dy} + \left(k^2 + \frac{\nu^2 + \nu - \nu m}{y^2}\right) z = 0$$

Putting $m - 2\nu = 1$ or $m = 1 + 2\nu$, one obtains

$$\nu^2 + \nu - \nu m = -\nu^2$$

and

$$\frac{d^2 z}{dy^2} + \frac{1}{y}\frac{dz}{dy} + \left(k^2 - \frac{\nu^2}{y^2}\right) z = 0$$

This differential equation is a well known normal form of Bessel equation. Its solution is

$$z = A\, I_\nu(ky) + B\, I_{-\nu}(ky),$$

where $I_{\pm\nu}$ stands for the Bessel function of the orders $\pm\nu$ with the argument ky. As z is known we find

$$I = z y^{-\nu} = \frac{1}{(x+x_0)^\nu}\left\{ A\, I_\nu\,(k(x+x_0)) + B\, I_{-\nu}\,(k(x+x_0))\right\}$$

To solve (15) we proceed in the same way,

$$\frac{d^2E}{d\,y^2} - \frac{m}{\underset{-\delta}{y}} \cdot \frac{dE}{dy} + k^2\,E = 0$$

Putting $E = z\,y$,

$$\frac{d^2z}{dy^2} - (m+2\,\delta)\,\frac{1}{y} \cdot \frac{dz}{dy} +$$

$$+\left(k^2 + \frac{\delta^2+\delta+\delta m}{y^2}\right) = 0$$

with . . . . $m + 2\,\delta = -1$ . . . . . $m = -1 - 2\,\delta$

we find $\delta^2 + \delta + \delta m = -\delta^2$,

$$\frac{d^2z}{dy^2} + \frac{1}{y} \cdot \frac{dz}{dy} + \left(k^2 - \frac{\delta^2}{y^2}\right) = 0$$

which may be solved as above.

$$z = C\,I_{\delta}(ky) + D\,I_{-\delta}(ky),$$

$$E = z\,y^{-\delta} = \frac{1}{(x+x_0)^{\delta}} \cdot \left\{ C\,I_{\delta}(k(x+x_0)) + \right.$$

$$\left. + D\,I_{-\delta}(k(x+x_0)) \right\}$$

The voltage E is represented by Bessel functions of the orders $\pm\,\delta$, the current by Bessel functions of the orders $\pm\,\nu$. δ and ν are connected by the relations

$$m = 1 + 2\,\nu,\ \nu = \frac{m}{2} - \frac{1}{2}$$

$$m = -1 - 2\,\delta,\ \delta = -\frac{m}{2} - \frac{1}{2}$$

and therefore

$$\nu + \delta = -\,1$$

In the special case of $m = 1/5$, one finds

$$\nu = 1/10 - 1/2 = -2/5, \quad -\nu = +2/5,$$

$$\delta = -1/10 - 1/2 = -3/5, \quad -\delta = +3/5$$

In order to compute the integration constants in the above expressions for I and E, one has to introduce the boundary condition valid for the open end of the line, where $x = 0$, $I = 0$.

$$B = -A \frac{I_\nu(k x_0)}{I_{-\nu}(k x_0)}$$

and

$$I = A \frac{1}{(x+x_0)^\nu} \cdot \left\{ I_\nu(k(x+x_0)) - \right.$$

$$\left. - \frac{I_\nu(k x_0)}{I_{-\nu}(k x_0)} I_{-\nu}(k(x+x_{0\nu})) \right\}$$

As voltage and current are connected by the relation

$$-c z_x \frac{dI}{dx} = j \omega E$$

one can derive using the recurrence formulae

$$I'_\nu(z) - \frac{\nu}{z} I_\nu(z) = -I_{\nu+1}(z)$$

$$I'_\nu(z) + \frac{\nu}{z} I_\nu(z) = I_{\nu-1}(z)$$

$$E = -j A \frac{z_{0b}}{2} \frac{1}{(x+x_0)^\delta} \left\{ I_{-\delta}(k(x+x_0)) + \right.$$

$$\left. + \frac{I_\nu(k x_0)}{I_{-\nu}(k x_0)} I_\delta(k(x+x_0)) \right\}$$

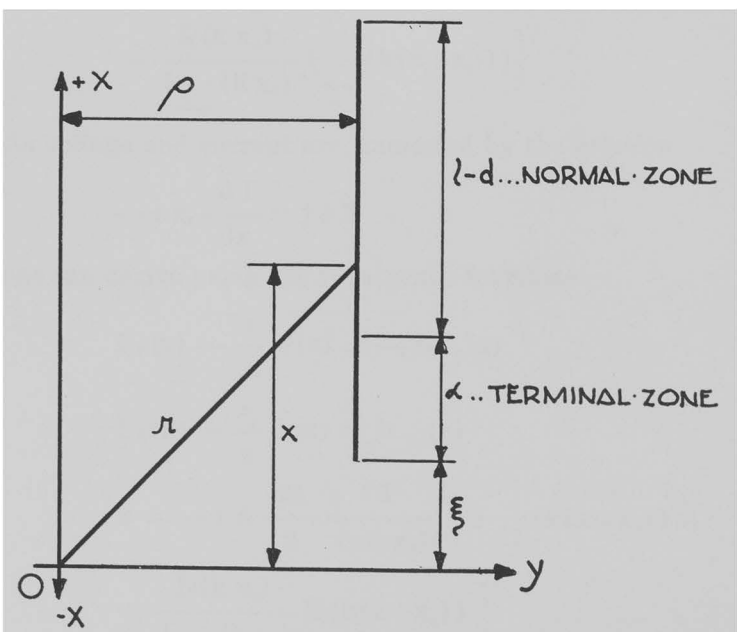
With $m = 0$, $\nu = \delta = -1/2$ the Bessel functions of the above solutions for I and E lead into Bessel functions of the orders $\pm 1/2$. Because of the well known relationship

among these Bessel functions and the circular functions the expressions for I and E become identical with the generally used solutions of the telegraph equations. They also do so for higher values of x or larger distances from the open end of the line where the second terms of the differential equations (14) and (15) vanish gradually with increasing x.

APPENDIX III

Scalar Potential of a Two-Wire Line Having Non-sinusoidal Current Distribution Within the Terminal Zone

In computing the scalar or vector potential of a transmission line considering the influence of the terminal zone, there will be (see Fig. 15) two parts: the contribution of the terminal zone and the contribution of the normal zone. It is not necessary to derive the expressions for both potentials, for as is shown in Appendix I the calculation is identical in both cases. Therefore, only the formula for the scalar potential will be calculated.



FIG·15·

Notations to Appendix III.

In accord with formula (7b) and Fig. 15 we can write:

$$\phi_1=\phi_1+\phi_2=\frac{c^2}{10^9}\int_{x=\xi}^{x=d+\xi}\frac{q\,(t-r/c)}{r}\,dx+\frac{c^2}{10^9}\int_{x=d+\xi}^{x=l+\xi}\frac{q1\,(t-r/c)}{r}\,dx,$$

where q, the charge along the terminal zone, is determined by Eq. (18) and q_1, the charge along the normal zone, by Eq. (21). Substituting these expressions we find for a two-wire line

$$\phi_1=\frac{c}{10^9}\cos\omega t\int_{x=\xi}^{x=d+\xi}q\left(\frac{\cos k\sqrt{x^2+d^2}}{\sqrt{x^2+d^2}}-\frac{\cos k\sqrt{x^2+R^2}}{\sqrt{x^2+R^2}}\right)dx$$

The root mean square value, writing $c/10^9 = 30$, will be

$$\Phi_1=j\,30\int_{x=\xi}^{x=d+\xi}q\left(\frac{\cos k\sqrt{x^2+d^2}}{\sqrt{x^2+d^2}}-\frac{\cos k\sqrt{x^2+R^2}}{\sqrt{x^2+R^2}}\right)dx$$

The contribution of the normal zone will be

$$\phi_2=-\frac{c}{10^9}4\,N\int_{x=d+\xi}^{x=l+\xi}\frac{[Y_1\sin k\,(x-d-\xi)+Y_2\cos k\,(x-d-\xi)\,]}{\sqrt{x^2+\rho^2}}\,\frac{\cos\omega\,(t-r/c)}{}\,dx$$

This equation leads to a sum of four integrals, of which only one, containing the integrand

$$\cos kx\cos k\sqrt{x^2+\rho^2}$$

must be considered,

$$\phi_2=\frac{c}{10^9}4\,N\cos\omega t\,[Y_1\sin k\,(d+\xi)-$$

$$-Y_2 \cos k(d+\xi)] \int_{x=d+\xi}^{x=l+\xi} \frac{\cos kx \cos k\sqrt{x^2+\rho^2}}{\sqrt{x^2+\rho^2}} dx$$

The root mean square value is

$$\Phi_2 = j\ 30x\ 4\ N\ [Y_1 \sin k(d+\xi) -$$

$$- Y_2 \cos k(d+\xi)] \int_{x=d+\xi}^{x=l+\xi} \frac{\cos kx \cos k\sqrt{x^2+\rho^2}}{\sqrt{x^2+\rho^2}} dx$$

Finally, for a two-wire line in agreement with appendix I,

$$\Phi_2 = j\ 120\ N\ p_x\ [Y_1 \sin k(d+\xi) - Y_2 \cos k(d+\xi)],$$

where p_x stands for

$$p_x = \left\{ 2 \ln \frac{d}{R} + Ci\, k(d+\xi+\sqrt{(d+\xi)^2+d^2}) - \right.$$

$$- Ci\, k(-d-\xi+\sqrt{(d+\xi)^2+d^2} -$$

$$- Ci\, k(d+\xi+\sqrt{(d+\xi)^2+R^2}) +$$

$$\left. + Ci\, k(-d-\xi+\sqrt{(d+\xi)^2+R^2}) \right\}$$

The scalar potential of the line is

$$\Phi = \Phi_1 + \Phi_2 = j\ 60 \int_{x=\xi}^{x=d+\xi} q\left(\frac{\cos k\sqrt{x^2+d^2}}{\sqrt{x^2+d^2}} - \right.$$

$$\left. - \frac{\cos k\sqrt{x^2+R^2}}{\sqrt{x^2+R^2}}\right) dx + j\ 120\ N\ p_x\ (y_1 \sin k(d+\xi) -$$

$$- y_2 \cos k(d+\xi)$$

The first term of this solution has been solved graphically, for the analytical solution proved too complicated.

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